Proof Theory for Admissible Rules

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Abstract

Admissible rules of a logic are those rules under which the set of theorems of the logic is closed. In this paper, a Gentzen-style framework is introduced for analytic proof systems that derive admissible rules of non-classical logics. While Gentzen systems for derivability treat sequents as basic objects, for admissibility, the basic objects are sequent rules. Proof systems are defined here for admissible rules of classes of modal logics, including K4, S4, and GL, and also Intuitionistic Logic IPC. With minor restrictions, proof search in these systems terminates, giving decision procedures for admissibility in the logics.

Key words: Admissible Rules, Proof Theory, Intuitionistic Logic, Modal Logic.

1 Introduction

Investigations of logical systems usually concentrate on the derivability of theorems. However, it is also interesting to "move up a level" and consider the admissible rules of the system, that is, to investigate under which rules the set of theorems

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is closed. Algebraically, this corresponds to the study of quasivarieties generated by free algebras, while from a computational perspective the investigation is significant since adding further (admissible) rules to a system may improve proof search. In Classical Logic CPC the situation is straightforward: admissible rules are also derivable (CPC is *structurally complete*). However, in Intuitionistic Logic IPC, and many other non-classical (e.g. modal and intermediate) logics this is no longer the case. Providing characterizations of admissibility for these logics becomes an interesting and challenging task.

In recent years, one successful approach to characterizing admissible rules has been via *bases*, which may be viewed (roughly speaking) as axiomatizations for sets of rules. More precisely, a basis for admissible rules in a logic is a set of admissible rules that when added to the logic allows all the admissible rules to be derived. That the set of admissible rules of IPC has no finite basis but is nevertheless decidable was proved by Rybakov [17], answering a problem originally posed by Friedman [4]. Moreover, it was conjectured by de Jongh and Visser that such a basis is provided by the "Visser rules":

$$(V_n) \ \ (C \to (A_{n+1} \lor A_{n+2})) \lor D \ / \ (\bigvee_{j=1}^{n+2} C \to A_j) \lor D$$

for $n = 1, 2, \ldots$ where $C = \bigwedge_{i=1}^{n}(A_i \to B_i)$. This was confirmed independently by Iemhoff [9] and Rozière [16], the former making key use of Ghilardi's work on unification and projective approximations [5]. Related characterizations have since been obtained for intermediate logics by Iemhoff [12], and the Visser rules have been used to define a first basic provability logic for IPC [19, 10].

Bases have also been found for classes of modal logics. Based again on Ghilardi's work on unification [6], Jeřábek [14] has introduced the following "generalized" (multiple conclusion) rules (where $\boxdot A$ is defined as $\Box A \land A$):

$$(A^\bullet) \qquad \Box A \to \bigvee_{i=1}^{n} \Box B_i \ / \ \{\boxdot A \to B_i\}_{i=1}^{n}$$

$$(A^\circ) \qquad \bigwedge_{j=1}^{m}(A_j \leftrightarrow \Box A_j) \to \bigvee_{i=1}^{n} \Box B_i \ / \ \{\bigwedge_{j=1}^{m} \boxdot A_j \to B_i\}_{i=1}^{n}.$$

Admissible rules of extensible modal logics are captured using $(A^\bullet)$ for non-reflexive logics (e.g. GL), (A°) for non-irreflexive logics (e.g. S4), and both for logics that are neither reflexive nor irreflexive (e.g. K4).

Although decision procedures for admissibility are described (or implicit) in the works of Rybakov [17], Ghilardi [5, 6], and Jeřábek [14, 15], a systematic presentation of analytic proof systems for deriving admissible rules has been lacking. Such a presentation is important not only for developing systems that reason directly about rules, but also for investigating relationships between admissibility in different logics, obtaining decidability and complexity results, and studying the use of admissible rules in proofs, e.g. which rules are needed, their effect on speeding up proofs, etc. A first step in this direction was taken by Iemhoff in [11] where an analytic proof system was defined for admissibility in IPC based partly on an algorithm by Ghilardi for projectivity [7]. However, this system makes use of a number of inelegant syntactic divisions and semantic checks, and is unsuitable for generalization to other logics.

In this work we develop a general framework for defining Gentzen-style proof systems for admissible rules. The key idea is to give a uniform proof-theoretic characterization of admissibility by generalizing proof calculi at the theorem level. For derivability, the basic objects are typically sequents, not formulas. Similarly, for admissibility, we take the basic objects of our systems to be not rules, but *sequent rules*. Rules (now one level up) of these systems thus have sequent rules as premises, and a sequent rule as the conclusion. Each logical connective is characterized by four rules: the connective can occur either on the left or the right of a sequent, and the sequent itself can occur either as a premise or a conclusion of a sequent rule. Our systems also include rules that allow sequents to interact: an anti-cut rule corresponding to the admissibility of cut for the logic, a projection rule reflecting the fact that derivability implies admissibility, and one or two extra rules capturing key facts of admissibility in the logic.

We begin, following the work of Jeřábek [14] and Ghilardi [6], by considering a wide class of (so-called extensible) modal logics extending $\mathsf{K4}$, treating as particular case studies $\mathsf{K4}$, $\mathsf{S4}$, and Gödel-Löb Logic GL. Analytic systems for admissibility in these logics are obtained as uniform extensions of systems for derivability. The extra rules depend on whether the logic can be characterized as non-reflexive and/or non-irreflexive. We then provide a system for the fundamental (and most studied) case of IPC, making essential use of theorems by Ghilardi [5]. With minor modifications, all these systems terminate, and hence provide the basis for decision procedures for deriving admissible rules in the corresponding logics.

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2 Preliminaries

We treat a logic L as a consequence relation based upon a propositional language with binary connectives $\wedge$, $\vee$, $\to$, a constant $\bot$, and sometimes also a modal connective $\Box$. Other connectives are then defined as:

$$\neg A =_{\text{def}} A \to \bot \qquad A \leftrightarrow B =_{\text{def}} (A \to B) \wedge (B \to A)$$
$$\top =_{\text{def}} \neg\bot \qquad \boxdot A =_{\text{def}} \Box A \wedge A.$$

We denote (propositional) variables by $p, q, \ldots$, formulas by $A, B, \ldots$, and sets of formulas by $\Gamma, \Pi, \Sigma, \Delta, \Theta, \Psi$. Formulas $p \to q$, $p \leftrightarrow q$, and $\Box p$ are called *variable implications*, *variable equivalences*, and *boxed variables*, respectively. We adopt the convention of writing $\bigvee \Gamma$ and $\bigwedge \Gamma$ where $\bigvee \emptyset =_{\text{def}} \bot$ and $\bigwedge \emptyset =_{\text{def}} \top$ for iterated disjunctions and conjunctions of formulas in a finite set Γ. We also make use of the following abbreviations:

$$\Box\Gamma =_{\text{def}} \{\Box A : A \in \Gamma\}$$
$$\boxdot\Gamma =_{\text{def}} \Gamma \cup \Box\Gamma$$
$$\Gamma \equiv \Box\Gamma =_{\text{def}} \{A \leftrightarrow \Box A : A \in \Gamma\}.$$

Finally, for brevity we sometimes write $\{f(x)\}_{x\in\Gamma}$ for the set $\{f(x) : x \in \Gamma\}$, reserving ordinary brackets (and) for clarification.

For further details on modal logics and Kripke frame semantics, we refer to [2].

2.1 Generalized Rules and Admissibility

Rules are usually asymmetric, having many premises but just one conclusion. However, it is convenient when considering admissibility to treat "generalized" rules having also many conclusions. Intuitively, a generalized rule is admissible for a logic L if whenever a substitution makes all the premises theorems of L, it also makes one of the conclusions a theorem.

Definition 1 *A* generalized rule *is an ordered pair of finite sets of formulas, written:*

$$A_1, \ldots, A_n \vartriangleright B_1, \ldots, B_m.$$

An L-unifier *for a formula A is a substitution σ such that $\vdash_{\mathsf{L}} \sigma A$.*
A generalized rule $\Gamma \vartriangleright \Delta$ is L-admissible*, written $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$, if each L-unifier for all $A \in \Gamma$, is an L-unifier for some $B \in \Delta$.*

In developing proof systems for derivability in a logic it is helpful to consider sequents, defined and interpreted as follows:

Definition 2 *A* sequent *S is an ordered pair of finite sets of formulas, written $\Gamma \Rightarrow \Delta$. S is L-derivable, written $\vdash_{\mathsf{L}} S$, iff $\vdash_{\mathsf{L}} I(S)$ where $I(\Gamma \Rightarrow \Delta) =_{\text{def}} \bigwedge \Gamma \to \bigvee \Delta$.*

For admissibility, rather than deal with rules involving only formulas, we consider *sequent rules*, represented as follows:

Definition 3 *A* generalized sequent rule *(gs-rule for short)* $\mathtt{R}$ *is an ordered pair of finite sets of sequents, written:*

$$\{\Gamma_i \Rightarrow \Delta_i\}_{i=1}^n \rhd \{\Pi_j \Rightarrow \Sigma_j\}_{j=1}^m.$$

$\mathtt{R}$ *is* L-admissible*, written* $\mathrel{|\!\!\sim}_{\mathsf{L}} \mathtt{R}$*, iff* $\{I(\Gamma_i \Rightarrow \Delta_i)\}_{i=1}^n \mathrel{|\!\!\sim}_{\mathsf{L}} \{I(\Pi_j \Rightarrow \Sigma_j)\}_{j=1}^m$.
$\mathtt{R}$ *is* L-derivable*, written* $\vdash_{\mathsf{L}} \mathtt{R}$*, iff* $\bigwedge_{i=1}^n I(\Gamma_i \Rightarrow \Delta_i) \vdash_{\mathsf{L}} \bigvee_{j=1}^m I(\Pi_j \Rightarrow \Sigma_j)$.

Note that crucially:

$$\mathrel{|\!\!\sim}_{\mathsf{L}} A_1, \ldots, A_n \rhd B_1, \ldots, B_m \text{ iff } \mathrel{|\!\!\sim}_{\mathsf{L}} (\Rightarrow A_1), \ldots, (\Rightarrow A_n) \rhd (\Rightarrow B_1), \ldots, (\Rightarrow B_m).$$

Hence a proof system for the admissibility of gs-rules is also a proof system for the admissibility of generalized rules, and of course, rules in the usual sense.

Rules (now at the next level up) for gs-rules consist of a set of premises $\mathtt{R}_1, \ldots, \mathtt{R}_n$ and a conclusion $\mathtt{R}$, rules with no premises being called *initial gs-rules*. Such rules are L-*sound* if whenever $\mathrel{|\!\!\sim}_{\mathsf{L}} \mathtt{R}_i$ for $i = 1 \ldots n$, then $\mathrel{|\!\!\sim}_{\mathsf{L}} \mathtt{R}$, and L-*invertible*, if whenever $\mathrel{|\!\!\sim}_{\mathsf{L}} \mathtt{R}$, then $\mathrel{|\!\!\sim}_{\mathsf{L}} \mathtt{R}_i$ for $i = 1 \ldots n$.

Example 4 *As an illustration of these ideas, consider the disjunction property, written as the generalized rule* $p \vee q \rhd p, q$*. This rule is* L*-admissible iff the following gs-rule is* L*-admissible:*

$$(\Rightarrow p \vee q) \ \rhd \ (\Rightarrow p), (\Rightarrow q)$$

Observe now that if σ *is an* IPC*-unifier for* $p \vee q$*, i.e.* $\vdash_{\mathsf{IPC}} \sigma p \vee \sigma q$*, then* σ *must be an* IPC*-unifier for* p *or* q*, i.e. either* $\vdash_{\mathsf{IPC}} \sigma p$ *or* $\vdash_{\mathsf{IPC}} \sigma q$*. However, this does not hold for* CPC*. For example, let* $\sigma(p) = p$ *and* $\sigma(q) = \neg p$*; plainly* $\vdash_{\mathsf{CPC}} p \vee \neg p$*, but* $\nvdash_{\mathsf{CPC}} p$ *and* $\nvdash_{\mathsf{CPC}} \neg p$*.*

2.2 Projectivity and the Extension Property

Admissibility and derivability do not coincide in general for non-classical logics. However, for certain families of normal modal logics and intermediate logics, Ghilardi [5, 6] has identified classes of "projective" formulas where if A is projective, then the relationship "$A \mathrel{|\!\!\sim}_{\mathsf{L}} B$ iff $A \vdash_{\mathsf{L}} B$" holds for all formulas B.

Definition 5 *A formula* A *is* L-projective *if there exists a substitution* σ*, called an* L-projective unifier *for* A*, such that:*

$$\vdash_{\mathsf{L}} \sigma A \quad \textit{and} \quad A \vdash_{\mathsf{L}} \sigma(p) \leftrightarrow p \textit{ for all variables } p.$$

Lemma 6 *Let* L *be a normal modal logic or* IPC*:*

(a) If A *is* L*-projective, then* $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$ *iff* $A \vdash_{\mathsf{L}} B$ *for some* $B \in \Delta$*.*
(b) If $A_1, \ldots, A_n$ *are* L*-projective, then* $\bigvee_{i=1}^n A_i \mathrel{|\!\sim}_{\mathsf{L}} B$ *iff* $\bigvee_{i=1}^n A_i \vdash_{\mathsf{L}} B$*.*

Proof. (a) The right-to-left direction is immediate. For the other direction, suppose that $A \mathrel{|\!\sim}_{\mathsf{L}} \Delta$. Since A is L-projective, there exists an L-projective unifier σ of A such that $\vdash_{\mathsf{L}} \sigma B$ for some $B \in \Delta$. Using the fact that σ is an L-projective unifier, $A \vdash_{\mathsf{L}} \sigma B \to B$. Hence, by modus ponens, $A \vdash_{\mathsf{L}} B$. (b) Again, the right-to-left direction is immediate. For the other direction, suppose that $\bigvee_{i=1}^n A_i \mathrel{|\!\sim}_{\mathsf{L}} B$. Then $A_i \mathrel{|\!\sim}_{\mathsf{L}} B$ for $i = 1 \ldots n$. By (a), $A_i \vdash_{\mathsf{L}} B$ for $i = 1 \ldots n$ and hence $\bigvee_{i=1}^n A_i \vdash_{\mathsf{L}} B$. $\square$

What makes projective formulas particularly interesting (and useful) is the fact that for certain logics they can also be characterized in terms of *Kripke models*. First for Intuitionistic Logic:

Definition 7 *Two Kripke models* K_1*,* K_2 *are* (modal) variants *of each other if they have the same set of nodes and order (or accessibility in the case of modal variants) relation, and their forcing relations agree on all nodes except possibly the root.*

Definition 8 *For Kripke models* $K_1, \ldots, K_n$*, let* $(\sum_i K_i)'$ *denote the Kripke model obtained by attaching one new node below all nodes in* $K_1, \ldots, K_n$ *where no propositional variables are forced. A class of Kripke models* $\mathcal{K}$ *has the* extension property *if for every finite family of models* $K_1, \ldots, K_n \in \mathcal{K}$*, there is a variant of* $(\sum_i K_i)'$ *in* $\mathcal{K}$*.*

Theorem 9 (Ghilardi [5]) *A formula is* IPC*-projective iff its class of Kripke models has the extension property.*

Example 10 *It is not difficult to see that the formulas* p*,* $\neg p$*,* $\neg p \to (q \wedge r)$*, and* $p \to A$ *are* IPC*-projective (e.g. for* p *and* $\neg p$ *the constant substitutions* $\top$ *and* $\bot$ *are* IPC*-projective unifiers), while* $\neg p \to (q \vee r)$ *and* $\neg p \vee \neg\neg p$ *are not. In particular, each disjunct of the conclusions of atomic instances of the Visser rules (given in the introduction) is* IPC*-projective, but not the disjuncts in the premises of the form* $C \to (A_{n+1} \vee A_{n+2})$*.*

De Jongh and Bezhanishvili in [1, 3] have given characterizations of the classes of IPC-projective formulas in a fixed number of propositional variables.

Ghilardi [6] has also extended this characterization to a wide range of modal logics (we follow here the terminology of [14]). Recall that for any normal modal logic L, an L-frame is a frame such that every model on that frame is a model of L, and an L-model is a model based on an L-frame. Moreover, L has the *finite model property* if every refutable formula is refutable on a finite L-frame. Let us also recall a couple of other common notions: K_k denotes the Kripke model K restricted to the domain $\{l : kRl \text{ or } k = l\}$ and the *root* of K is the cluster $\{k : \forall l \neq k(kRl)\}$.

Definition 11 *For frames $F_1, \ldots, F_n$, $(\sum F_j)^i$ and $(\sum F_j)^r$ are obtained by adding, respectively, an irreflexive or a reflexive node beneath (connected to all nodes of) the disjoint sum of $F_1 \ldots F_n$. A normal extension* L *of* $\mathsf{K4}$ *with the finite model property is* extensible *if for all finite* L*-frames $F_1, \ldots, F_n$:*

(1) $(\sum F_j)^i$ *is an* L*-frame unless* L *is reflexive;*
(2) $(\sum F_j)^r$ *is an* L*-frame unless* L *is irreflexive.*

In particular, the one node irreflexive (reflexive) frame is an L*-frame unless* L *is reflexive (irreflexive).*

Definition 12 *A class of finite modal models $\mathcal{K}$ has the* modal extension property *if for every model K, if $K_k \in \mathcal{K}$ for all k not in the root of K, then there is a variant of K in $\mathcal{K}$.*

Theorem 13 (Ghilardi [6]) *For every normal extension* L *of* $\mathsf{K4}$ *with the finite model property (in particular if* L *is extensible), a formula is* L*-projective iff its class of* L*-models has the modal extension property.*

Example 14 *Using this theorem it is not difficult to see that for each extensible modal logic* L*, the formulas $\boxdot p$, $\neg p$, and $p \to A$ are* L*-projective, while e.g. $\boxdot p \to (q \vee r)$ is not. In particular, the conclusions of the atomic instances of the rules $(A^\bullet)$ and (A°) given in the introduction are* L*-projective, but not the premises.*

Note that in the semantic characterization of projective formulas in Theorem 13 we can restrict the models K to those with a root consisting of one node. Given a reflexive model K, let us denote by K^1 the model obtained from K by replacing the root by one reflexive node and forcing no variables in this node. Observe that in irreflexive models the root cannot contain more than one node. If for any model K such that K_k is a model of A for all k not in the root, there is a variant of K^1 that is a model of A, then A is projective; i.e. its class of L-models has the modal extension property. For suppose K is such an L-model and let r be its root. Then there is a variant K' of K^1 in which A holds. Now define a variant of K in which A holds by forcing in every node of r, the same variables as in the root of K'. This shows that the class of L-models of A has the modal extension property.

3 Modal Logics

In this section we define Gentzen-style calculi for deriving admissible gs-rules of extensible modal logics. We begin with systems for the paradigmatic cases $\mathsf{K4}$, $\mathsf{S4}$, and GL. Then we use these systems to show that any calculus for derivability in an extensible modal logic can be extended to a proof system for admissibility.

Proof systems for admissibility are constructed in much the same way as proof systems for derivability: we give rules for connectives on the left and right of sequents. The difference here is that the sequents themselves occur either on the left or the right, that is, as premises or conclusions of a gs-rule, doubling the number of rules required. For sequents occurring on the right, we adapt rules from calculi for derivability by adding variables $\mathcal{G}$ and $\mathcal{H}$ standing for arbitrary sets of sequents. For sequents occurring on the left, we make use of invertibility properties of the rules on the right. The proof systems are completed by adding various rules that allow sequents to interact.

The core modal rules for admissibility in extensible modal logics are presented in Fig. 1. Proof systems for the paradigmatic cases of $\mathsf{K4}$, $\mathsf{S4}$, and GL are obtained by extending this core set with further rules from Fig. 2 as follows:

> **GAK4** consists of the core modal rules plus $\rhd(\Box)_{\mathsf{K4}}$, $(\mathrm{AC}_\Box)$, (V^i), and (V^r);
> **GAS4** consists of the core modal rules, $\rhd(\Box)_{\mathsf{K4}}$, $\rhd(\Box)_{\mathsf{S4}}$, $(\mathrm{AC}_\Box)$, and (V^r);
> **GAGL** consists of the core modal rules, $\rhd(\Box)_{\mathsf{K4}}$, $\rhd(\Box)_{\mathsf{GL}}$, and (V^i).

Let us now explain these rules in some detail.

The *Core Left and Right Logical Rules* are ($(\Box\Rightarrow)\rhd$ and $(\Rightarrow\Box)\rhd$ excepted) taken directly from a standard calculus for Classical Logic (see e.g. [18]) but formulated in terms of gs-rules. That is, for any logical sequent rule of such a calculus:

$$\frac{S_1 \quad \dots \quad S_n}{S}\ (\mathrm{R})$$

where n is 1 or 2, we have a corresponding logical rule on the right:

$$\frac{\mathcal{G} \rhd S_1, \mathcal{H} \quad \dots \quad \mathcal{G} \rhd S_n, \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}}\ \rhd(\mathrm{R})$$

Moreover, using the invertibility of (R) in Classical Logic, we also obtain the following sound rule on the left:

$$\frac{\mathcal{G}, S_1, \dots, S_n \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}}\ (\mathrm{R})\rhd$$

Weakening and contraction are built into the right logical rules but this is not strictly necessary; in fact, any calculus for derivability in the logic can be used as a template for the right logical rules. However, as noted, for $\wedge$, $\vee$, and $\to$, the rules given here are easily "inverted" to obtain corresponding rules on the left by replacing the conclusion of the original sequent rule with the premises of that rule. This approach fails in the case of the (non-invertible) modal rules. Instead the rules $(\Box\Rightarrow)\rhd$ and $(\Rightarrow\Box)\rhd$ decompose modal formulas on the left by replacing the formula A in $\Box A$

Initial GS-Rules

$$\frac{}{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow A, \Delta), \mathcal{H}} \text{ (ID)}$$

Structural Rules

$$\frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}} \text{ (W)}\triangleright \qquad \frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G} \,\triangleright\, S, \mathcal{H}} \triangleright\text{(W)}$$

Anti-Cut and Projection Rules

$$\frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma), (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \,\triangleright\, \mathcal{H}} \text{ (AC)} \qquad \frac{\mathcal{G}, S \,\triangleright\, (\Gamma, \boxdot I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \,\triangleright\, \mathcal{H}} \text{ (PJ)}$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

Right Logical Rules

$$\frac{}{\mathcal{G} \,\triangleright\, (\Gamma, \bot \Rightarrow \Delta), \mathcal{H}} \triangleright(\bot\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow \bot, \Delta), \mathcal{H}} \triangleright(\Rightarrow\bot)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \wedge B, \Delta), \mathcal{H}} \triangleright(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, A, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \wedge B \Rightarrow \Delta), \mathcal{H}} \triangleright(\wedge\Rightarrow)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \vee B \Rightarrow \Delta), \mathcal{H}} \triangleright(\vee\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \vee B, \Delta), \mathcal{H}} \triangleright(\Rightarrow\vee)$$

$$\frac{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \,\triangleright\, (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma, A \to B \Rightarrow \Delta), \mathcal{H}} \triangleright(\to\Rightarrow) \qquad \frac{\mathcal{G} \,\triangleright\, (\Gamma, A \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \,\triangleright\, (\Gamma \Rightarrow A \to B, \Delta), \mathcal{H}} \triangleright(\Rightarrow\to)$$

Left Logical Rules

$$\frac{\mathcal{G} \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \,\triangleright\, \mathcal{H}} (\bot\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \,\triangleright\, \mathcal{H}} (\Rightarrow\bot)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}} (\wedge\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \,\triangleright\, \mathcal{H}} (\Rightarrow\wedge)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \,\triangleright\, \mathcal{H}} (\Rightarrow\vee)\triangleright \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}} (\vee\Rightarrow)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, B \Rightarrow \Delta), (\Gamma \Rightarrow A, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \,\triangleright\, \mathcal{H}} (\to\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow B, \Delta) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \,\triangleright\, \mathcal{H}} (\Rightarrow\to)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, \Box p \Rightarrow \Delta), (A \Rightarrow p) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma, \Box A \Rightarrow \Delta) \,\triangleright\, \mathcal{H}} (\Box\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Box p, \Delta), (p \Rightarrow A) \,\triangleright\, \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \Box A, \Delta) \,\triangleright\, \mathcal{H}} (\Rightarrow\Box)\triangleright$$

where p does not occur in $\mathcal{G}$, $\mathcal{H}$, Γ, Δ, A in $(\Box\Rightarrow)\triangleright$ and $(\Rightarrow\Box)\triangleright$.

Fig. 1. Core Modal Rules

by a new propositional variable p. The soundness of these rules follows from the fact that any substitution for the conclusion can be extended (since p does not occur there) by substituting A for p.

The *Structural Rules* permit weakening of sequents occurring as premises and conclusions of sequent rules. If we were to use multisets of sequents rather than sets, Additional Logical Rules

$$\frac{\mathcal{G} \rhd (\boxdot\Gamma \Rightarrow A), \mathcal{H}}{\mathcal{G} \rhd (\Box\Gamma, \Pi \Rightarrow \Box A, \Delta), \mathcal{H}} \rhd(\Box)_{\mathsf{K4}} \quad \frac{\mathcal{G} \rhd (\boxdot\Gamma, \Box A \Rightarrow A), \mathcal{H}}{\mathcal{G} \rhd (\Box\Gamma, \Pi \Rightarrow \Box A, \Delta), \mathcal{H}} \rhd(\Box)_{\mathsf{GL}} \quad \frac{\mathcal{G} \rhd (\boxdot\Gamma, \Pi \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \rhd (\Box\Gamma, \Pi \Rightarrow \Delta), \mathcal{H}} \rhd(\Box)_{\mathsf{S4}}$$

Anti-Cut for Boxed Formulas

$$\frac{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma), (\Gamma, \Pi, A \leftrightarrow \Box A \Rightarrow \Sigma, \Delta) \rhd \mathcal{H}}{\mathcal{G}, (\Gamma, \Theta \Rightarrow \Delta), (\Pi \Rightarrow \Psi, \Sigma) \rhd \mathcal{H}} \text{(AC}_\Box\text{)}$$

where $(\Theta \cup \Psi) \subseteq \{A, \Box A\}$ and $\Theta, \Psi \neq \emptyset$

Visser Rules

$$\frac{[\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow A) \rhd \mathcal{H}]_{A \in \Delta}}{\mathcal{G}, (\Box\Gamma \Rightarrow \Box\Delta) \rhd \mathcal{H}} \text{(V}^i\text{)} \qquad \frac{[\mathcal{G}, (\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow A) \rhd \mathcal{H}]_{A \in \Delta}}{\mathcal{G}, (\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta) \rhd \mathcal{H}} \text{(V}^r\text{)}$$

Fig. 2. Additional Modal Rules

then we could also make use of contraction rules:

$$\frac{\mathcal{G}, S, S, \rhd \mathcal{H}}{\mathcal{G}, S \rhd \mathcal{H}} \text{(C)}\rhd \qquad \frac{\mathcal{G} \rhd S, S, \mathcal{H}}{\mathcal{G} \rhd S, \mathcal{H}} \rhd\text{(C)}$$

The *Projection Rule* (PJ) allows sequents on the left to be used as modal implications on the right, corresponding to the fact that derivability implies admissibility. [3] Just notice that if $I(\Gamma \Rightarrow \Delta) \vdash_{\mathsf{L}} I(\Pi \Rightarrow \Sigma)$, then assuming the completeness of the right logical rules for derivability in L, we can derive any gs-rule of the form:

$$\mathcal{G} \rhd (\Pi, \bigwedge \Gamma \to \bigvee \Delta \Rightarrow \Sigma), \mathcal{H}\,.$$

Hence we can use (PJ) to obtain the derivation:

$$\frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd (\Pi, \bigwedge \Gamma \to \bigvee \Delta \Rightarrow \Sigma), \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd (\Pi \Rightarrow \Sigma), \mathcal{H}} \text{(PJ)}$$

It follows that any gs-rule containing the same sequent on both sides (i.e. as a premise and as a conclusion) is derivable using (PJ). Indeed, generalizing a little, the following gs-rule may be taken as a useful derived initial gs-rule:

$$\frac{}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \rhd (\Gamma, \Pi \Rightarrow \Sigma, \Delta), \mathcal{H}} \text{(SID)}$$

Note that in the special case where the right hand side $\mathcal{H}$ of the gs-rule is empty, (PJ) still allows us to "project" a sequent from the left to the right, but this time into the "empty sequent" $\Rightarrow$.

The *Anti-Cut Rule* (AC) corresponds directly to the fact that the usual cut rule is admissible in the logic. Observe however that, unlike cut, (AC) and indeed all the

[3] In the particular cases of **GAK4**, **GAS4**, and **GAL**, $\boxdot I(S)$ in (PJ) can be replaced with $I(S)$, allowing sequents on the left to be used directly as implications on the right.

rules except $(\Rightarrow\Box)\triangleright$, $(\Box\Rightarrow)\triangleright$, and (AC$_\Box$), have the subformula property. That is, every formula occurring in a premise of such a rule occurs as a subformula of a formula in the conclusion. Note, moreover, that a suitable cut rule for admissibility would be of the form:

$$\frac{\mathcal{G}, S \ \triangleright\ \mathcal{H} \quad \mathcal{G}' \ \triangleright\ S, \mathcal{H}'}{\mathcal{G}, \mathcal{G}' \ \triangleright\ \mathcal{H}', \mathcal{H}}\ (\textsc{cut})$$

Rather than eliminate such a rule syntactically, here we obtain the admissibility of the rule indirectly via a (semantic) completeness proof.

Example 15 *Consider the following cut rule:*

$$\frac{\Gamma, A \Rightarrow \Delta \quad \Pi \Rightarrow A, \Sigma}{\Gamma, \Pi \Rightarrow \Sigma, \Delta}$$

The gs-rule version is derivable as follows:

$$\dfrac{\dfrac{}{(\Gamma, A \Rightarrow \Delta), (\Pi, A \Rightarrow \Sigma), (\Gamma, \Pi \Rightarrow \Sigma, \Delta)\ \triangleright\ (\Gamma, \Pi \Rightarrow \Sigma, \Delta)}\ (\textsc{sid})}{(\Gamma, A \Rightarrow \Delta), (\Pi, A \Rightarrow \Sigma)\ \triangleright\ (\Gamma, \Pi \Rightarrow \Sigma, \Delta)}\ (\textsc{ac})$$

The *Anti-Cut Rule for Boxed Formulas* corresponds to the admissibility of another cut-like rule used for logics that contain the reflexive Visser rule (V^r). For these logics, it enables derivations of such gs-rules as $(p, \Box p \Rightarrow \Box q, \Box r), (\Rightarrow p, \Box p, \Box q, \Box r)\ \triangleright\ (\boxdot p \Rightarrow q), (\boxdot p \Rightarrow r)$ as follows, omitting some of the sequents on the left for space reasons:

$$\dfrac{\dfrac{\dfrac{}{(\boxdot p \Rightarrow q)\ \triangleright\ (\boxdot p \Rightarrow q), (\boxdot p \Rightarrow r)}\ (\textsc{sid}) \qquad \dfrac{}{(\boxdot p \Rightarrow r)\ \triangleright\ (\boxdot p \Rightarrow q), (\boxdot p \Rightarrow r)}\ (\textsc{sid})}{(p \leftrightarrow \Box p \Rightarrow \Box q, \Box r)\ \triangleright\ (\boxdot p \Rightarrow q), (\boxdot p \Rightarrow r)}\ (\textsc{v}^r)}{(p, \Box p \Rightarrow \Box q, \Box r), (\Rightarrow p, \Box p, \Box q, \Box r)\ \triangleright\ (\boxdot p \Rightarrow q), (\boxdot p \Rightarrow r)}\ (\textsc{ac}_\Box)$$

Of course we could also build the rule (AC$_\Box$) into (V^r) but this would make the formulation of the latter less elegant.

The *Visser Rules* (V^i) and (V^r) are a little harder to understand but correspond directly to the rules $(A^\bullet)$ and (A°), respectively, given by Jeřábek in [14] (see the introduction). For non-reflexive logics, the gs-rule versions of $(A^\bullet)$ are derived using (V^i) as follows:

$$\dfrac{\dfrac{}{[(\Box\Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow A)\ \triangleright\ \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}]_{A\in\Delta}}\ (\textsc{sid})}{(\Box\Gamma \Rightarrow \Box\Delta)\ \triangleright\ \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}}\ (\textsc{v}^i)$$

and for non-irreflexive logics, (V^r) is used to derive the gs-rule versions of (A°):

$$\dfrac{\dfrac{}{[(\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta), (\boxdot\Gamma \Rightarrow A)\ \triangleright\ \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}]_{A\in\Delta}}\ (\textsc{sid})}{(\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta)\ \triangleright\ \{\boxdot\Gamma \Rightarrow A\}_{A\in\Delta}}\ (\textsc{v}^r)$$

Note that the cases for (V^i) and (V^r) where $\Delta = \emptyset$ have no premises and are therefore treated as initial gs-rules.

Extensible modal logics with a natural sequent calculus for derivability provide the most elegant examples of our systems for admissibility. However, all that we really require of the rules on the right is that they provide a sound and complete method for establishing derivability in the logic at hand. We can then expand this calculus with the core modal rules, plus (V^i) if the logic is not reflexive, and (V^r) and $(\mathrm{AC}_\Box)$ if the logic is not irreflexive. The result is a calculus that is sound and complete for admissibility in the logic.

Definition 16 *A calculus* $\mathbf{GAL}$ *is* L*-fitting for an extensible modal logic* L *if:*

(1) $\mathbf{GAL}$ *extends the core modal rules.*
(2) If L *is not reflexive, then* (V^i) *is a rule of* $\mathbf{GAL}$.
(3) If L *is not irreflexive, then* (V^r) *and* $(\mathrm{AC}_\Box)$ *are rules of* $\mathbf{GAL}$.
(4) If $\vdash_{\mathsf{L}} S$*, then* $\vdash_{\mathbf{GAL}} \rhd S$.
(5) If $\vdash_{\mathbf{GAL}} \mathtt{R}$*, then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathtt{R}$.

3.2 Soundness

We first show that the core modal rules and (where appropriate) the Visser rules are sound for extensible modal logics.

Proposition 17 *Let* L *be an extensible modal logic.*

(a) All the core modal rules are L*-sound.*
(b) If L *is not reflexive, then* (V^i) *is* L*-sound.*
(c) If L *is not irreflexive, then* (V^r) *and* $(\mathrm{AC}_\Box)$ *are* L*-sound.*

Proof. (a) The initial gs-rules and right logical rules (taken from a calculus for CPC in [18]) and the structural rules are clearly L-sound. For the left logical rules for $\wedge$, $\vee$, and $\to$, soundness follows directly from the CPC-invertibility of the rules on the right. For $(\Box\Rightarrow)\rhd$, suppose that the premise is L-admissible and let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Gamma, \Box A \Rightarrow \Delta)$. Since p does not occur in $\mathcal{G}, \mathcal{H}, \Gamma, \Delta, A$ we can extend σ by mapping p to A. It follows that σ is an L-unifier for $I(\Gamma, \Box p \Rightarrow \Delta)$ and $I(A \Rightarrow p)$. Hence, by the admissibility of the premise, σ is an L-unifier for some $S \in \mathcal{H}$ as required. The argument for $(\Box\Rightarrow)\rhd$ is very similar.

For (AC), suppose that the premise is L-admissible. Let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$, $I(\Gamma, A \Rightarrow \Delta)$, and $I(\Pi \Rightarrow A, \Sigma)$. By the admissibility of the usual cut rule for L, we get that σ is an L-unifier for $I(\Gamma, \Pi \Rightarrow \Sigma, \Delta)$ and the result follows using the L-admissibility of the premise. For (PJ), suppose that the premise is L-admissible and that σ is an L-unifier for $I(S')$ for all $S' \in \mathcal{G}$ and $I(S)$. It follows that σ is an L-unifier for either $I(\Gamma, \boxdot I(S) \Rightarrow \Delta)$ or $I(S')$ for some $S' \in \mathcal{H}$. In the latter case we are done. In the former case, since σ is an L-unifier for $I(S)$ it is an L-unifier for $\boxdot I(S)$ and hence also for $I(\Gamma \Rightarrow \Delta)$. Since there is no L-unifier for the empty sequent $\Rightarrow$, we have $(\Gamma \Rightarrow \Delta) \in \mathcal{H}$ as required.

(b) For (V^i), suppose that all the premises are L-admissible. Consider first the case where $\Delta = \emptyset$ and thus there are no premises. No sequent of the form $(\Box\Gamma \Rightarrow\ )$ is L-derivable (just consider the irreflexive one-node L-frame), so the rule is sound. Now consider the case where $\Delta \neq \emptyset$. Let σ be an L-unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Box\Gamma \Rightarrow \Box\Delta)$. If σ is a L-unifier for $I(\boxdot\Gamma \Rightarrow A)$ for some $A \in \Delta$, then the result follows immediately using the admissibility of the premises. Otherwise, since L has the finite model property, let K_A be a finite L-model refuting $\sigma(I(\boxdot\Gamma \Rightarrow A))$ for each $A \in \Delta$, and let F_A be the L-frame for K_A. The fact that L is extensible and not reflexive implies that $(\Sigma_{A\in\Delta}F_A)^i$ is also an L-frame. Then $\sigma(I(\Box\Gamma \Rightarrow \Box\Delta))$ is refuted at the root of any model on the frame $(\Sigma_{A\in\Delta}F_A)^i$ for which the forcing in all nodes except the root is the same as in the K_A. Since there exists at least one such model, this contradicts the assumption that $\sigma(I(\Box\Gamma \Rightarrow \Box\Delta))$ is L-derivable.

(c) The fact that $(\mathrm{AC}_\Box)$ is L-sound follows easily from the fact that $\vdash_{\mathsf{L}} (A \leftrightarrow \Box A) \leftrightarrow ((A \wedge \Box A) \vee \neg(A \vee \Box A))$. For (V^r), suppose that all the premises are L-admissible. Consider first the case where $\Delta = \emptyset$ and thus there are no premises. No sequent of the form $(\Gamma \equiv \Box\Gamma \Rightarrow)$ is L-derivable (just consider the reflexive one-node L-frame), so the rule is sound. Now assume that $\Delta \neq \emptyset$. Arguing by contraposition, suppose that σ is not an L-unifier for $\mathcal{H}$. Then σ is not an L-unifier for $(\boxdot\Gamma \Rightarrow A)$ for any $A \in \Delta$. Since L has the finite model property, let K_A be a finite L-model refuting $\sigma(I(\boxdot\Gamma \Rightarrow A))$ for each $A \in \Delta$ and let F_A be the L-frame for K_A. Since L is extensible and not irreflexive, $(\Sigma_{A\in\Delta}F_A)^r$ is an L-frame with a reflexive root r. Using the reflexivity of r and the fact that each K_A forces the members of $\sigma(\boxdot\Gamma)$, it follows that $\sigma(C) \leftrightarrow \sigma(\Box C)$ is forced at r for all $C \in \Gamma$. But also $\sigma(\Box A)$ is not forced at r for each $A \in \Delta$, so $\sigma(\Gamma \equiv \Box\Gamma \Rightarrow \Box\Delta)$ is not L-derivable as required. $\Box$

In particular, using the fact that the rules on the right for $\mathbf{GAK4}$, $\mathbf{GAS4}$, and $\mathbf{GAGL}$ are sound and complete for $\mathsf{K4}$, $\mathsf{S4}$, and GL, respectively (see e.g. [8] for references), we obtain:

Corollary 18 $\mathbf{GAL}$ *is* L*-fitting for* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{GL}\}$.

3.3 Completeness

Our completeness proof consists of several stages. First we show completeness for a restricted class of gs-rules: L-derivable gs-rules with at most one sequent on the right, the idea being (to look ahead a little) to show that all L-admissible gs-rules are $\mathbf{GAL}$-derivable from gs-rules in this class. Let us assume in what follows that L is an extensible modal logic and that $\mathbf{GAL}$ is L-fitting.

Lemma 19 *If* $\vdash_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H}$ *where* $|\mathcal{H}| \leq 1$, *then* $\vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}$.

Proof. If $\mathcal{H} = \{\Gamma \Rightarrow \Delta\}$, or taking $\Gamma = \Delta = \emptyset$ if $\mathcal{H} = \emptyset$, then:

$$\{I(S)\}_{S \in \mathcal{G}} \vdash_{\mathsf{L}} I(\Gamma \Rightarrow \Delta).$$

But then using the modal deduction theorem (see e.g. [2] for details):

$$\vdash_{\mathsf{L}} (\bigwedge_{S \in \mathcal{G}} \boxdot I(S)) \to I(\Gamma \Rightarrow \Delta).$$

Hence easily, $\vdash_{\mathsf{L}} \Gamma, \{\boxdot I(S)\}_{S \in \mathcal{G}} \Rightarrow \Delta$, and since $\mathbf{GAL}$ is L-fitting:

$$\vdash_{\mathbf{GAL}} \ \rhd (\Gamma, \{\boxdot I(S)\}_{S \in \mathcal{G}} \Rightarrow \Delta).$$

So by repeated applications of (PJ) and (W)$\rhd$, $\vdash_{\mathbf{GAL}} \mathcal{G} \ \rhd \ \mathcal{H}$ as required. $\square$

The next step is to show that the left logical rules are invertible with respect to L-admissibility (in fact, the right logical rules for $\wedge$, $\vee$, and $\to$ are also invertible, but this is not needed). Each left logical rule replaces (working bottom to top) a sequent on the left in the conclusion with sequents on the left in the premise that have fewer connectives. Hence it follows by an easy induction that any L-admissible gs-rule is derivable from L-admissible gs-rules of a certain "irreducible" form.

Lemma 20 *The left logical rules,* (AC)*,* (AC$_\Box$)*,* (V^i)*, and* (V^r) *are* L*-invertible.*

Proof. The L-invertibility of (AC), (AC$_\Box$), (V^i), and (V^r) is immediate since all the sequents in the conclusion appear in the premises, while the cases of $\wedge$, $\vee$, and $\to$ follow from the L-soundness of the right logical rules. As an example, we consider $(\Rightarrow \wedge)\rhd$. Suppose that the conclusion is L-admissible and let σ be a unifier for $I(\Gamma \Rightarrow A, \Delta)$, $I(\Gamma \Rightarrow B, \Delta)$, and $I(S)$ for all $S \in \mathcal{G}$. In particular, $\vdash_{\mathsf{L}} \sigma(I(\Gamma \Rightarrow A, \Delta))$ and $\vdash_{\mathsf{L}} \sigma(I(\Gamma \Rightarrow B, \Delta))$. Hence, using the soundness of the usual conjunction right rule, $\vdash_{\mathsf{L}} \sigma((I(\Gamma \Rightarrow A \wedge B, \Delta))$, i.e. σ is a unifier for $I(\Gamma \Rightarrow A \wedge B, \Delta)$. It follows therefore by the admissibility of the conclusion, that σ is a unifier for $I(S)$ for some $S \in \mathcal{H}$.

For $(\Box \Rightarrow)\rhd$, suppose that the conclusion is L-admissible and let σ be an L-unifier for $I(\Gamma, \Box p \Rightarrow \Delta)$, $I(A \Rightarrow p)$, and $I(S)$ for all $S \in \mathcal{G}$. Since L is a normal modal logic, σ is an L-unifier for $I(\Box A \Rightarrow \Box p)$. Hence by the L-admissibility of the usual cut rule, σ is an L-unifier for $I(\Gamma, \Box A \Rightarrow \Delta)$ and the result follows using the L-admissibility of the conclusion. The case of $(\Rightarrow \Box)\rhd$ is very similar. $\square$

Definition 21 *A gs-rule* $\mathcal{G} \rhd \mathcal{H}$ *is:*

- modal-irreducible *if* $\mathcal{G}$ *contains only variables and boxed variables.*
- modal-semi-irreducible *if* $\mathcal{G}$ *contains only variables and boxed variables, and on the left of sequents possibly also variable equivalences (i.e. of the form* $p \leftrightarrow \Box p$*).*

Lemma 22 *Every* L*-admissible gs-rule is derivable from an* L*-admissible modal-irreducible gs-rule using the left logical rules.*

This means that it is sufficient to establish completeness for modal-irreducible L-admissible gs-rules. We achieve this (again, working bottom to top) by applying the relevant anti-cut rules (AC) and $(\text{AC}_\Box)$ and Visser rules (V^i) and (V^r) (and perhaps also $(\wedge\Rightarrow)\triangleright$ and $(\rightarrow\Rightarrow)\triangleright$) exhaustively until the gs-rules obtained (there can be more than one since the Visser rules can have more than one premise) are "full". In the presence of $(\text{AC}_\Box)$ we end up with a set of modal-semi-irreducible gs-rules, and otherwise, a set of just modal-irreducible gs-rules.

Definition 23 *A gs-rule $\mathcal{G} \triangleright \mathcal{H}$ is* full *with respect to a set of rules X if whenever $(\mathcal{G}_1 \triangleright \mathcal{H}_1), \ldots, (\mathcal{G}_n \triangleright \mathcal{H}_n)/(\mathcal{G} \triangleright \mathcal{H})$ is an instance of a rule in X, then $\mathcal{G}_i \subseteq \mathcal{G}$ and $\mathcal{H}_i \subseteq \mathcal{H}$ for some $i \in \{1, \ldots, n\}$.*

There is a finite number of different modal-semi-irreducible sequents for a fixed set of variables. Thus applying any subset of $\{(\text{V}^i), (\text{V}^r), (\text{AC}), (\text{AC}_\Box), (\wedge\Rightarrow)\triangleright, (\rightarrow\Rightarrow)\triangleright\}$ backwards to a modal-irreducible gs-rule will terminate with modal-semi-irreducible gs-rules full with respect to that set.

Lemma 24 *Let $X \subseteq \{(\text{V}^i), (\text{V}^r), (\text{AC}), (\text{AC}_\Box), (\wedge\Rightarrow)\triangleright, (\rightarrow\Rightarrow)\triangleright\}$ be rules of $\mathbf{GAL}$. Then every modal-irreducible L-admissible gs-rule is derivable using X from a set of modal-semi-irreducible L-admissible gs-rules that are full with respect to X. If X does not contain $(\text{AC}_\Box)$, then these gs-rules are modal-irreducible.*

Suppose now that $\mathcal{G} \triangleright \mathcal{H}$ is an L-admissible gs-rule that is full with respect to the appropriate rules of $\mathbf{GAL}$. Our aim is to show that the formula $\bigwedge_{S\in\mathcal{G}} I(S)$ is either inconsistent or L-projective, and hence that $\mathcal{G} \triangleright \mathcal{H}$ is $\mathbf{GAL}$-derivable. We use Ghilardi's characterization of L-projective formulas to show that the case where $\bigwedge_{S\in\mathcal{G}} I(S)$ is consistent and not L-projective cannot occur. We treat reflexive logics and irreflexive logics separately, starting with the considerably less difficult latter case (recommended to the reader wanting just the general idea behind the proofs).

Theorem 25 *If $\mathbf{GAL}$ is L-fitting for an irreflexive extensible modal logic L, then:*

$$\mathrel{|\!\sim}_{\mathsf{L}} \mathcal{G} \triangleright \mathcal{H} \quad \text{iff} \quad \vdash_{\mathbf{GAL}} \mathcal{G} \triangleright \mathcal{H}.$$

Proof. The right-to-left direction follows from the definition of L-fitting. For the other direction, it is sufficient by Lemma 24 to assume that $\mathcal{G} \triangleright \mathcal{H}$ is a modal-irreducible L-admissible gs-rule that is full with respect to (V^i) and (AC). Define:

$$C =_{\text{def}} \bigwedge_{S\in\mathcal{G}} I(S).$$

If C is inconsistent, then $\vdash_{\mathsf{L}} \mathcal{G} \triangleright$, and if C is L-projective, then using Lemma 6 (a), $\vdash_{\mathsf{L}} \mathcal{G} \triangleright S$ for some $S \in \mathcal{H}$. In both cases, using Lemma 19 and the structural rules of $\mathbf{GAL}$, $\vdash_{\mathbf{GAL}} \mathcal{G} \triangleright \mathcal{H}$.

Assume then that C is consistent and not L-projective: we show that this cannot occur by deriving a contradiction. Using Theorem 13 of Ghilardi, which tells us that C does not have the L-extension property, we obtain a non-empty L-model K such that:

1. $K_k \Vdash C$ for all k not in the root of K.
2. every variant of K refutes C.

We write $K' \Vdash A$ if $K_k \Vdash A$ for all k not in the root of K. Note that we can assume K' to be non-empty.

Let $M_1, \ldots, M_m$ be the variants of K. For $i = 1 \ldots m$, let $\Gamma_i \Rightarrow \Delta_i$ be a sequent in $\mathcal{G}$ for which $M_i \nVdash I(\Gamma_i \Rightarrow \Delta_i)$. So $M_i \Vdash \bigwedge \Gamma_i$ and $M_i \nVdash \bigvee \Delta_i$.

$\mathcal{G}$ is modal-irreducible, so each Γ_i and Δ_i contains only variables and boxed variables. Also, since L is irreflexive, so is K. Observe that:

$$\Box A \in \Gamma_i \implies K' \Vdash \boxdot A \quad \text{and} \quad \Box A \in \Delta_i \implies K' \nVdash A. \tag{1}$$

We define the (classical) formulas:

$$A_i =_{\text{def}} \bigwedge_{p \in \Gamma_i} p \wedge \bigwedge_{p \in \Delta_i} \neg p \quad \text{and} \quad A =_{\text{def}} \bigvee_{i=1}^{m} A_i,$$

where p ranges over variables. If $\Gamma_i \cup \Delta_i$ contains no atoms we have $\Gamma_i = \Box\Gamma_i'$ and $\Delta_i = \Box\Delta_i'$, $(\Box\Gamma_i' \Rightarrow \Box\Delta_i') \in \mathcal{G}$, and A_i is equivalent to $\top$. For this case, the reader should skip to the last paragraph of this proof to obtain the desired contradiction. Otherwise we proceed as follows. We will show that A is a tautology. Consider a valuation v on variables occurring in C, and define a variant M of K by forcing p at the root iff $v(p) = 1$. Suppose that M is the variant M_i. It is not difficult to see that $v(p) = 1$ for $p \in \Gamma_i$ and $v(p) = 0$ for $p \in \Delta_i$; i.e. $v(A_i) = 1$. So A is a tautology, and therefore $\neg A$ is inconsistent.

Now let σ be the substitution mapping each variable p to $\neg p$. Then also $\neg(\sigma A)$ is inconsistent and equivalent (using DeMorgan laws) to:

$$\bigwedge_{i=1}^{m} \Big(\bigvee_{p \in \Gamma_i} p \vee \bigvee_{p \in \Delta_i} \neg p \Big).$$

Therefore there exists a resolution refutation starting with the clauses:

$$\{p : p \in \Gamma_j\} \cup \{\neg p : p \in \Delta_j\} \ \text{ for } j = 1 \ldots m$$

that ends in the empty clause $\emptyset$. Let $\Theta \cup \Psi'$ be any clause in the refutation, where Θ contains only variables and Ψ' contains only negated variables. Define $\Psi = \{p : \neg p \in \Psi'\}$. Observe that every cut on a variable p can be "mimicked" in $\mathcal{G}$ via a backwards application of (AC) on p. Since $\mathcal{G} \rhd \mathcal{H}$ is full with respect to (AC) this implies that there exists $(\Box\Gamma, \Theta \Rightarrow \Psi, \Box\Delta) \in \mathcal{G}$ for all such clauses $\Theta \cup \Psi'$. Also:

$$K' \Vdash \bigwedge \boxdot \Gamma \quad \text{and} \quad \Box B \in \Box\Delta \implies K' \nVdash B. \tag{2}$$

That (2) holds for the initial sequents $(\Gamma_i \Rightarrow \Delta_i)$ follows from (1). That it also holds for other sequents corresponding to clauses in the refutation follows inductively using the fact that they are all obtained via backward applications of $(\textsc{ac})$.

Now consider the empty clause $\emptyset$ and the corresponding sequent in $\mathcal{G}$. This sequent has to be of the form $(\Box\Gamma \Rightarrow \Box\Delta)$. Suppose first that $\Delta = \emptyset$, i.e. $(\Box\Gamma \Rightarrow) \in \mathcal{G}$. Then the fact that $K' \Vdash \boxdot \bigwedge \Gamma$ and $K' \Vdash C$, and hence $K' \Vdash (\bigwedge \Gamma \to \bot)$, leads to a contradiction. If $\Delta \neq \emptyset$, then closure under $(\textsc{v}^i)$ implies that $(\boxdot\Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$. Hence $K' \Vdash (\bigwedge \boxdot\Gamma \to q)$, since $K' \Vdash C$. But by (2), also $K' \Vdash \bigwedge \boxdot\Gamma$ and $K' \nVdash q$, a contradiction. Hence the assumption that C is consistent and not L-projective cannot occur. Since we started the proof by explaining that in the cases that C is inconsistent or L-projective the result holds, we are done. $\Box$

Theorem 26 *If* $\mathbf{GAL}$ *is* L*-fitting for a reflexive extensible modal logic* L*, then:*

$$\mid\!\sim_{\mathsf{L}} \mathcal{G} \rhd \mathcal{H} \quad \textit{iff} \quad \vdash_{\mathbf{GAL}} \mathcal{G} \rhd \mathcal{H}.$$

Proof. The right-to-left direction follows from the definition of L-fitting. For the other direction, we can assume using Lemma 24 that $\mathcal{G} \rhd \mathcal{H}$ is a modal-semi-irreducible L-admissible gs-rule that is full with respect to $(\textsc{v}^r)$, $(\textsc{ac})$, $(\textsc{ac}_\Box)$, $(\wedge\!\Rightarrow)\rhd$, and $(\to\Rightarrow)\rhd$, and obtained by applying these rules (backwards) to a modal-irreducible gs-rule. The first part of the proof up to the description of the model K is then completely the same as for the irreflexive case, the only difference being that now K is reflexive. So let us pick up the proof there, the idea being, as in the case for irreflexive logics, to derive a contradiction. Instead of $(\textsc{v}^i)$, the rule $(\textsc{v}^r)$ will play a crucial role here since the logic we consider is reflexive.

Let us write $K' \Vdash A$ if $K_k \Vdash A$ for all k that are not the root of the non-empty model K, and denote the set $\{A : K' \Vdash A\}$ also by K'. We let $M_1, \ldots, M_m$ be the variants of K where $(\Gamma_i \Rightarrow \Delta_i)$ for $i = 1 \ldots m$ are the sequents such that:

$$M_i \Vdash \bigwedge \Gamma_i \quad \text{and} \quad M_i \nVdash \bigvee \Delta_i.$$

Observe that we can choose each $(\Gamma_i \Rightarrow \Delta_i)$ in such a way that they contain only variables and boxed variables. That is, $\mathcal{G}$ may contain sequents that are not of this form (as we only know that $\mathcal{G}$ is modal-semi-irreducible), but we can choose the $(\Gamma_i \Rightarrow \Delta_i)$ so as to satisfy this constraint. For example, if $\Gamma_i = \Gamma \cup \{p \leftrightarrow \Box p\}$, then (since the gs-rule is full with respect to $(\wedge\!\Rightarrow)\rhd$ and $(\to\Rightarrow)\rhd$), also $(\Gamma, p, \Box p \Rightarrow \Delta_i)$ and $(\Gamma \Rightarrow p, \Box p, \Delta_i)$ belong to $\mathcal{G}$. Also, either $M_i \Vdash p \wedge \Box p$ or $M_i \Vdash \neg p \wedge \neg\Box p$. So in the first case we replace $(\Gamma_i \Rightarrow \Delta_i)$ with $(\Gamma, p, \Box p \Rightarrow \Delta_i)$, and in the second case with $(\Gamma \Rightarrow p, \Box p, \Delta_i)$.

Define $K'_c = \{A : K' \nVdash A\}$. Observe that for all variables $p \in K'$:

$$\begin{array}{ll} p \in \Gamma_i \Longrightarrow M_i \Vdash \boxdot p & \Box p \in \Gamma_i \Longrightarrow M_i \Vdash \boxdot p \\ p \in \Delta_i \Longrightarrow M_i \Vdash \neg p \wedge \neg\Box p & \Box p \in \Delta_i \Longrightarrow M_i \Vdash \neg p \wedge \neg\Box p. \end{array} \tag{3}$$

As in the irreflexive case, we obtain a contradiction via a resolution refutation. We associate a resolution refutation with the sequents in $\mathcal{G}$, and show that we end up with a sequent that corresponds to the empty clause, and to which we can apply the (V^r) rule. The input clauses will correspond to the sequents $(\Gamma_i \Rightarrow \Delta_i)$.

What complicates the description of the sequents that correspond to clauses in the resolution refutation is that we now want, given $p \in K'$, to cut on $\{p, \Box p\}$ for sequents of the form $(\Gamma, \Box p, p \Rightarrow \Delta)$ and $(\Gamma' \Rightarrow \Box p, p, \Delta')$. In such cases, we cannot step to the sequent $(\Gamma, \Gamma' \Rightarrow \Delta, \Delta')$. Instead we can associate the result of the cut with the sequent $(\Gamma, \Gamma', p \leftrightarrow \Box p \Rightarrow \Delta, \Delta')$, guaranteed to be in $\mathcal{G}$ by fullness with respect to ($\mathrm{AC}_\Box$). For these cuts we introduce new variables l_p corresponding to $\{p, \Box p\}$ for each $p \in K'$.

Define A_i to be the formula:

$$\bigwedge_{p\in\Gamma_i\setminus K'} p \wedge \bigwedge_{p\in\Gamma_i\cap K'} l_p \wedge \bigwedge_{\Box p\in\Gamma_i, p\in K'} l_p \wedge \bigwedge_{p\in\Delta_i\setminus K'} \neg p \wedge \bigwedge_{p\in\Delta_i\cap K'} \neg l_p \wedge \bigwedge_{\Box p\in\Delta_i, p\in K'} \neg l_p.$$

Note that the condition $p \in K'$ for $\Box p \in \Gamma_i$ is superfluous; it is added just to stress that these p are in K'. Note also that all formulas in the Γ_i "correspond" to a variable of A_i because the case $\Box p \in \Gamma_i\setminus K'$ cannot occur, while this is not true for Δ_i since it may contain formulas $\Box p \in \Delta_i\setminus K'$.

If all conjuncts of A_i are empty, and hence A_i is equivalent to $\top$, $(\Gamma_i \Rightarrow \Delta_i)$ has to be of the form $(\Rightarrow \Box\Delta)$, where $\Delta' \subseteq K_c$. Thus $(\Rightarrow \Box\Delta) \in \mathcal{G}$, which contradicts the fact that all sequents in $\mathcal{G}$ hold in K'. Thus this case cannot occur and we proceed as follows.

We show that $A = \bigvee_{i=1}^m A_i$ is a tautology. Consider a valuation v and define a variant M of K via:

$$\forall p \notin K' : \ M \Vdash p \ \Leftrightarrow\ v(p) = 1 \quad \text{and} \quad \forall p \in K' : \ M \Vdash p \ \Leftrightarrow\ v(l_p) = 1.$$

Suppose that M is the variant M_i. This implies that for each $p \notin K'$:

$$p \in \Gamma_i \implies v(p) = 1 \quad \text{and} \quad p \in \Delta_i \implies v(p) = 0.$$

For the l_p we have that for each $p \in K'$:

$$v(l_p) = 1 \Leftrightarrow M_i \Vdash p \wedge \Box p \quad \text{and} \quad v(l_p) = 0 \Leftrightarrow M_i \Vdash \neg p \wedge \neg\Box p. \tag{4}$$

By (3) we have that for each $p \in K'$:

$$(p \in \Gamma_i \text{ or } \Box p \in \Gamma_i) \implies v(l_p) = 1 \quad \text{and} \quad (p \in \Delta_i \text{ or } \Box p \in \Delta_i) \implies v(l_p) = 0.$$

So $v(A_i) = 1$ and hence A is a tautology. It follows that $\neg A$ is inconsistent, and so as in the irreflexive case (applying the substitution $\sigma(p) = \neg p$ for all variables p)

also the following formula obtained by swapping literals in $\neg A$ is inconsistent:

$$\bigwedge_{i=1}^{m}\Big(\bigvee_{p\in\Gamma_i\setminus K'} p\vee \bigvee_{p\in\Gamma_i\cap K'} l_p\vee \bigvee_{\Box p\in\Gamma_i, p\in K'} l_p\vee \bigvee_{p\in\Delta_i\setminus K'} \neg p\vee \bigvee_{p\in\Delta_i\cap K'} \neg l_p\vee \bigvee_{\Box p\in\Delta_i, p\in K'} \neg l_p\Big).$$

So there exists a resolution refutation of $\emptyset$ from the clauses (recalling that m is the number of variants of K):

$$\begin{aligned}&\{p : p\in\Gamma_i\setminus K'\}\cup\{l_p : p\in K',\text{ and } p\in\Gamma_i \text{ or } \Box p\in\Gamma_i\ \}\cup\\ &\{\neg p : p\in\Delta_i\setminus K'\}\cup\{\neg l_p : p\in K',\text{ and } p\in\Delta_i \text{ or } \Box p\in\Delta_i\ \}\ \text{ for } i=1,\dots,m.\end{aligned}$$

Let $\Theta\cup\Pi\cup\Psi\cup\Sigma$ be any clause in the refutation, where Θ contains only variables not in K', Π contains only variables of the form l_q, Ψ contains only negated variables not in K', and Σ contains only negated variables of the form $\neg l_q$.

Observe that no clause contains both p and l_p since only for $p\notin K'$ do we have variables p in a clause, and only for $p\in K'$ do we have variables of the form l_p in a clause. Note that in the starting sequents $(\Gamma_i\Rightarrow\Delta_i)$, no variable appears in both Γ_i and Δ_i (since the sequent is falsified in some variant). As usual, we can assume that no clause in the refutation contains both an variable and its negation. Let:

$$\begin{aligned}E\ &=_{\text{def}}\ \{p\leftrightarrow\Box p : p\in K'\}\\ R^+ &=_{\text{def}}\ \{\Box p, p : l_p\in R\}\cup\{p : p\in R\}\\ R^- &=_{\text{def}}\ \{\Box p, p : \neg l_p\in R\}\cup\{p : \neg p\in R\}.\end{aligned}$$

We show that for all clauses R in the refutation the property $\circ(R,\mathcal{G})$ holds, where:

$$\circ(R,\mathcal{G}) =_{\text{def}} \exists(\Gamma\Rightarrow\Delta)\in\mathcal{G}\ \Big(\Gamma\subseteq R^+\cup E, \Delta\subseteq R^-\cup\Box(K'_c)\Big).$$

Let us first see why this claim suffices to prove the theorem. Consider the empty clause $\emptyset$, and its corresponding sequent $(\Gamma\Rightarrow\Box\Delta)$ in $\mathcal{G}$. Since $\circ(\emptyset,\mathcal{G})$ holds, such a sequent of the described form must exist in $\mathcal{G}$ with $\Gamma\subseteq E$ and $\Box\Delta\subseteq\Box K'_c$. Also, Γ has to be of the form $\Pi\equiv\Box\Pi$ for some $\Pi\subseteq K'$. By the fullness of $\mathcal{G}$ with respect to (V^r), either $\Delta=\emptyset$ or $(\boxdot\Pi\Rightarrow p)\in\mathcal{G}$ for some $p\in\Delta$. In the first case, $(\Gamma\Rightarrow\,)\in\mathcal{G}$, and hence $K'\Vdash\bigwedge\Gamma\to\bot$, while also $K'\Vdash\boxdot\bigwedge\Pi$, a contradiction. In the second case, since $K'\Vdash C$, it follows that $K'\Vdash\bigwedge\boxdot\Pi\to p$. But also $K'\Vdash\bigwedge\boxdot\Pi$, so $K'\Vdash p$. However, $\Delta\subseteq K'_c$, so $K'\nVdash p$, a contradiction. Recall finally then that this means that C must be either inconsistent or L-projective, and hence the result holds.

It remains then to prove the claim. We write $\circ(R,S,\mathcal{G})$ if $S=(\Gamma\Rightarrow\Delta)$ is a witness of $\circ(R,\mathcal{G})$; i.e. if $S\in\mathcal{G}$ with $\Gamma\subseteq R^+\cup E$ and $\Delta\subseteq R^-\cup\Box(K'_c)$.

Claim 1 *For every clause R in the resolution refutation, $\circ(R,\mathcal{G})$ holds.*

Proof of the Claim. We proceed by induction on the length of the resolution refutation. Suppose first that R_i is an input clause:

$$\{p : p \in \Gamma_i \backslash K'\} \cup \{l_p : p \in K',\ \text{and}\ p \in \Gamma_i \text{ or } \Box p \in \Gamma_i\ \} \cup$$
$$\{\neg p : p \in \Delta_i \backslash K'\} \cup \{\neg l_p : p \in K',\ \text{and}\ p \in \Delta_i \text{ or } \Box p \in \Delta_i\ \}.$$

Then $\circ(R_i, \Gamma_i \Rightarrow \Delta_i)$ clearly holds, since we assumed that Γ_i and Δ_i consist only of variables and boxed variables. Thus the sequent can be written as $(\Gamma_i \Rightarrow \Sigma_i, \Box\Theta_i)$, where $\Gamma_i \subseteq R_i^+ \cup E$, $\Sigma_i \subseteq R_i^-$, and $\Theta_i \subseteq \Box K_c'$. For the presence of Θ_i see the remark after the definition of the A_i.

Cuts on p. For the induction step, first consider a cut on a variable p, with input clauses $R \cup \{p\}$ and $R' \cup \{\neg p\}$, and conclusion $R \cup R'$. Let $S = (\Gamma, p \Rightarrow \Delta)$ and $S' = (\Gamma' \Rightarrow p, \Delta')$ be such that $\circ(R \cup \{p\}, S, \mathcal{G})$ and $\circ(R' \cup \{\neg p\}, S', \mathcal{G})$ hold. The case where p does not belong to the antecedent of S or the succedent of S' is similar. By the rule (AC), the sequent $(\Gamma, \Gamma' \Rightarrow \Delta, \Delta')$ belongs to $\mathcal{G}$. Since no p occurs in the antecedent and succedent of the same sequent, it is not difficult to see that $\Gamma \cup \Gamma' \subseteq (R \cup R')^+ \cup E$ and $\Delta \cup \Delta' \subseteq (R \cup R')^- \cup \Box K_c'$, and thus that $\circ(R \cup R', (\Gamma, \Gamma' \Rightarrow \Delta, \Delta'), \mathcal{G})$ holds.

Cuts on l_p. For a cut on l_p the input clauses are $R \cup \{l_p\}$ and $R' \cup \{\neg l_p\}$ with conclusion $R \cup R'$. We have to show that $\circ(R \cup R', \mathcal{G})$. Let $S = (\Gamma, p, \Box p \Rightarrow \Delta)$ and $S' = (\Gamma' \Rightarrow p, \Box p, \Delta')$ be such that $\circ(R \cup \{l_p\}, S, \mathcal{G})$ and $\circ(R' \cup \{\neg l_p\}, S', \mathcal{G})$. The case where $\Box p$ or p does not belong to the antecedent of S or the succedent of S' is similar. By the rule ($\text{AC}_\Box$) and the fullness of $\mathcal{G}$, the sequent $(\Gamma, \Gamma', p \leftrightarrow \Box p \Rightarrow \Delta, \Delta')$ is in $\mathcal{G}$. It is not difficult to see that $\Gamma \cup \Gamma' \subseteq (R \cup R')^+ \cup E$ and $\Delta \cup \Delta' \subseteq (R \cup R')^- \cup \Box K_c'$, and thus that $\circ(R \cup R', (\Gamma, \Gamma' \Rightarrow \Delta, \Delta'), \mathcal{G})$ holds. This proves the claim, which as explained above, implies the theorem. $\Box$

Corollary 27 *If* $\mathbf{GAL}$ *is* L*-fitting for an extensible modal logic* L*, then:*

$$\mathrel{|\!\sim}_{\mathsf{L}} \mathrm{R} \quad \textit{iff} \ \vdash_{\mathbf{GAL}} \mathrm{R}.$$

Proof. Theorem 25 and Theorem 26 take care of the cases where L is reflexive or irreflexive. If L is neither reflexive nor irreflexive, then we start the proof as in the two theorems above. However, the root of the model K could now be reflexive or irreflexive. For the former case, we follow the proof of Theorem 26 and in the latter, the proof of Theorem 25. Since both (V^r) and (V^i) are rules of $\mathbf{GAL}$, the reasoning in the proofs of these theorems carries over to this case. $\Box$

In particular, using also Corollary 18, we obtain soundness and completeness results for $\mathbf{GAK4}$, $\mathbf{GAS4}$, and $\mathbf{GAL}$.

Corollary 28 *Let* $\mathsf{L} \in \{\mathsf{K4}, \mathsf{S4}, \mathsf{GL}\}$. *Then* $\mathrel{|\!\sim}_{\mathsf{L}} \mathrm{R}$ *iff* $\vdash_{\mathbf{GAL}} \mathrm{R}$.

Finally, it is easy to see from the preceding proofs that by adding some control to the application of rules, we obtain calculi for admissibility in our logics that are *terminating* in the sense that applying the rules backwards to any gs-rule terminates. We just insist that the left logical rules are applied exhaustively (backwards) to transform a gs-rule into a modal-irreducible gs-rule, and that then the appropriate rules from $\{(\text{V}^i), (\text{V}^r), (\text{AC}), (\text{AC}_\Box), (\wedge\Rightarrow)\triangleright, (\to\Rightarrow)\triangleright\}$ are applied exhaustively (backwards) to obtain modal-semi-irreducible gs-rules that are full with respect to these rules. Since for these gs-rules, admissibility and derivability coincide, we can then rely on any decision procedure we like for the modal logic in question, obtaining known decidability results from [17, 6, 14].

Corollary 29 *Admissibility is decidable for any extensible modal logic.*

4 Intuitionistic Logic

We turn our attention now to the historically most significant case of Intuitionistic Logic IPC, henceforth assuming that all notions of admissibility, derivability, projectivity, and so on refer exclusively to this logic. Our calculus $\mathbf{GAI}$ for admissibility in IPC, presented in Figure 4, is very similar to the calculi for modal logics. Indeed, using the well-known translation of IPC into $\mathsf{S4}$ (see e.g. [2]), we could have obtained a (rather inelegant) calculus for IPC directly from the calculus for $\mathsf{S4}$ presented above.

Instead the right logical rules are based here on a multi-succedent calculus for derivability in IPC taken from [18]. As in the modal case, left logical rules are obtained for $\bot$, $\wedge$, and $\vee$ directly from the invertible right rules for these connectives. However, unlike modal logics, the implication rules on the right are not invertible for IPC. $\mathbf{GAI}$ therefore includes not only the rule $(\to)\triangleright$ obtained directly from $\triangleright(\to\Rightarrow)^i$, but also rules $(\to\Rightarrow)\triangleright^i$ and $(\Rightarrow\to)\triangleright^i$ that, like the rules $(\Box\Rightarrow)\triangleright$ and $(\Rightarrow\Box)\triangleright$ for modal logics, use variables to decompose implications.

The *Visser Rule* (V) deserves more detailed explanation. It might be expected that such a rule would correspond directly to the Visser rules (V_n) displayed in the introduction, and be of the form:

$$\frac{[\mathcal{G}, (\Gamma \Rightarrow A) \ \triangleright \ \mathcal{H}]_{A\in\Delta\cup\Gamma^a}}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \tag{5}$$

where Γ contains only implications and $\Gamma^a = \{B : \exists C(B \to C) \in \Gamma\}$. However, "stronger versions" of the Visser rules (V_n) are also admissible; e.g.

$$(\Gamma, A \Rightarrow \Delta), (\Gamma, A \to B \Rightarrow A, \Delta) \ \triangleright \ \{(\Gamma, A \to B \Rightarrow D) : D \in \Delta \cup (\Gamma^a \backslash \{A\})\}. \tag{6}$$

Initial GS-Rules

$$\frac{}{\mathcal{G} \ \triangleright \ (\Gamma, A \Rightarrow A, \Delta), \mathcal{H}} \ (\textsc{id})$$

Structural Rules

$$\frac{\mathcal{G} \ \triangleright \ \mathcal{H}}{\mathcal{G}, S \ \triangleright \ \mathcal{H}} \ (\textsc{w})\triangleright \qquad \frac{\mathcal{G} \ \triangleright \ \mathcal{H}}{\mathcal{G} \ \triangleright \ S, \mathcal{H}} \ \triangleright(\textsc{w})$$

Right Logical Rules

$$\frac{}{\mathcal{G} \ \triangleright \ (\Gamma, \bot \Rightarrow \Delta), \mathcal{H}} \ \triangleright(\bot\Rightarrow) \qquad \frac{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow \bot, \Delta), \mathcal{H}} \ \triangleright(\Rightarrow\bot)$$

$$\frac{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \ \triangleright \ (\Gamma \Rightarrow B, \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow A \wedge B, \Delta), \mathcal{H}} \ \triangleright(\Rightarrow\wedge) \qquad \frac{\mathcal{G} \ \triangleright \ (\Gamma, A, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma, A \wedge B \Rightarrow \Delta), \mathcal{H}} \ \triangleright(\wedge\Rightarrow)$$

$$\frac{\mathcal{G} \ \triangleright \ (\Gamma, A \Rightarrow \Delta), \mathcal{H} \quad \mathcal{G} \ \triangleright \ (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma, A \vee B \Rightarrow \Delta), \mathcal{H}} \ \triangleright(\vee\Rightarrow) \qquad \frac{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow A, B, \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow A \vee B, \Delta), \mathcal{H}} \ \triangleright(\Rightarrow\vee)$$

$$\frac{\mathcal{G} \ \triangleright \ (\Gamma, A \to B \Rightarrow A, \Delta), \mathcal{H} \quad \mathcal{G} \ \triangleright \ (\Gamma, B \Rightarrow \Delta), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma, A \to B \Rightarrow \Delta), \mathcal{H}} \ \triangleright(\to\Rightarrow)^i \qquad \frac{\mathcal{G} \ \triangleright \ (\Gamma, A \Rightarrow B), \mathcal{H}}{\mathcal{G} \ \triangleright \ (\Gamma \Rightarrow A \to B, \Delta), \mathcal{H}} \ \triangleright(\Rightarrow\to)^i$$

Left Logical Rules

$$\frac{\mathcal{G} \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma, \bot \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \ (\bot\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow \bot, \Delta) \ \triangleright \ \mathcal{H}} \ (\Rightarrow\bot)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, A, B \Rightarrow \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma, A \wedge B \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \ (\wedge\Rightarrow)\triangleright \qquad \frac{\mathcal{G}, (\Gamma \Rightarrow A, \Delta), (\Gamma \Rightarrow B, \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \wedge B, \Delta) \ \triangleright \ \mathcal{H}} \ (\Rightarrow\wedge)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow A, B, \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \vee B, \Delta) \ \triangleright \ \mathcal{H}} \ (\Rightarrow\vee)\triangleright \qquad \frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, B \Rightarrow \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma, A \vee B \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \ (\vee\Rightarrow)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma, B \Rightarrow \Delta), (\Gamma, A \to B \Rightarrow A, \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \ (\to)\triangleright$$

$$\frac{\mathcal{G}, (\Gamma \Rightarrow p, \Delta), (p, A \Rightarrow B) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma \Rightarrow A \to B, \Delta) \ \triangleright \ \mathcal{H}} \ (\Rightarrow\to)\triangleright^i \qquad \frac{\mathcal{G}, (\Gamma, p \to q \Rightarrow \Delta), (p \Rightarrow A), (B \Rightarrow q) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \ (\to\Rightarrow)\triangleright^i$$

where p and q do not occur in $\mathcal{G}$, $\mathcal{H}$, Γ, and Δ in $(\to\Rightarrow)\triangleright^i$, $(\Rightarrow\to)\triangleright^i$.

Anti-Cut and Projection Rules

$$\frac{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma), (\Gamma, \Pi \Rightarrow \Sigma, \Delta) \ \triangleright \ \mathcal{H}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Pi \Rightarrow A, \Sigma) \ \triangleright \ \mathcal{H}} \ (\textsc{ac}) \qquad \frac{\mathcal{G}, S \ \triangleright \ (\Gamma, I(S) \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, S \ \triangleright \ \mathcal{H}} \ (\textsc{pj})$$

where $(\Gamma \Rightarrow \Delta) \in \mathcal{H} \cup \{\Rightarrow\}$

Visser Rule

$$\frac{[\mathcal{G}, (\Gamma \Rightarrow \Delta), (\Gamma \Rightarrow A) \ \triangleright \ \mathcal{H}]_{A \in \Delta} \quad [\mathcal{G}, (\Gamma \Rightarrow \Delta) \ \triangleright \ (\Gamma^{\Pi}, \Pi \Rightarrow \Delta), \mathcal{H}]_{\emptyset \neq \Pi \subseteq \Gamma_\Delta}}{\mathcal{G}, (\Gamma \Rightarrow \Delta) \ \triangleright \ \mathcal{H}} \ (\textsc{v})$$

where Γ contains only implications, and:

1. $\Gamma^{\Pi} = \{A \to B \in \Gamma : A \notin \Pi\}$.
2. $\Gamma_\Delta = \{A \notin \Delta : \exists B\, (A \to B) \in \Gamma\}$.

That is, suppose that there is a counter-model for each member of:

$$\{(\Gamma, A \to B \Rightarrow D) : D \in \Delta \cup (\Gamma^a \backslash \{A\})\}.$$

Consider the model obtained by putting one node below these models. If the root forces A, then the model refutes $\Gamma, A \Rightarrow \Delta$. Otherwise, it refutes $\Gamma, A \to B \Rightarrow A$.

That the rule in (5) does not suffice to capture all admissible rules can be seen from the fact that it is not strong enough to derive the following admissible gs-rule:

$$(p \to q \Rightarrow p), (p \Rightarrow r, s) \ \triangleright\ (p \to q \Rightarrow r), (p \to q \Rightarrow s).$$

Since the Visser rules (V_n) form a basis for the admissible rules of IPC, they can be used to derive the formula version of (6). However, this derivation makes a detour via formulas more complicated than those occurring in the rule itself which would be undesirable in a proof system. One possible formulation of (6) in the setting of gs-rules could be:

$$\frac{[\mathcal{G}, (\Gamma, A \to B \Rightarrow D) \ \triangleright\ \mathcal{H}]_{D \in \Delta \cup (\Gamma^a \backslash \{A\})}}{\mathcal{G}, (\Gamma, A \Rightarrow \Delta), (\Gamma, A \to B \Rightarrow \Delta) \ \triangleright\ \mathcal{H}} \tag{7}$$

Instead, we reformulate (7) to fit better into a proof-theoretic framework as:

$$\frac{[\mathcal{G}, (\Gamma, A \to B \Rightarrow D) \ \triangleright\ \mathcal{H}]_{D \in \Delta \cup (\Gamma^a \backslash \{A\})} \quad \mathcal{G} \ \triangleright\ (\Gamma, A \Rightarrow \Delta), \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B \Rightarrow \Delta) \ \triangleright\ \mathcal{H}}$$

The same observations apply to more complicated admissible rules; e.g. let:

$$S_A = (\Gamma, A, C \to D \Rightarrow C, \Delta), \quad S_C = (\Gamma, A \to B, C \Rightarrow A, \Delta), \quad S_{AC} = (\Gamma, A, C \Rightarrow \Delta).$$

Then the admissible gs-rule:

$$(\Gamma, A \to B, C \to D \Rightarrow A, C, \Delta), S_A, S_C, S_{AC} \ \triangleright\ \{\Gamma, A \to B \Rightarrow E : E \in \Delta \cup (\Gamma^a \backslash \{A, C\})\}$$

is captured by the following rule:

$$\frac{[\mathcal{G}, (\Gamma, A \to B \Rightarrow D) \ \triangleright\ \mathcal{H}]_{D \in \Delta \cup (\Gamma^a \backslash \{A,C\})} \quad \mathcal{G} \ \triangleright\ S_A, \mathcal{H} \quad \mathcal{G} \ \triangleright\ S_C, \mathcal{H} \quad \mathcal{G} \ \triangleright\ S_{AC}, \mathcal{H}}{\mathcal{G}, (\Gamma, A \to B, C \to D \Rightarrow A, C, \Delta) \ \triangleright\ \mathcal{H}}$$

In general, (V) captures all admissible rules of the form:

$$\frac{\Gamma \Rightarrow \Delta \quad [\Gamma^\Pi, \Pi \Rightarrow \Delta]_{\emptyset \neq \Pi \subseteq \Gamma_\Delta}}{\{\Gamma \Rightarrow D : D \in \Delta\}}$$

Example 30 *Consider the admissible but non-derivable Kreisel-Putnam rule* $\neg A \to (B \vee C) \ \triangleright\ (\neg A \to B) \vee (\neg A \to C)$*, written in gs-rule format as:*

$$(\neg A \Rightarrow B, C) \ \triangleright\ (\Rightarrow \neg A \to B, \neg A \to C).$$

Since IPC *has the disjunction property, the following gs-rule is also admissible:*

$$(\neg A \Rightarrow B, C) \ \triangleright \ (\neg A \Rightarrow B), (\neg A \Rightarrow C).$$

We prove the second gs-rule in **GAI** *(a proof of the first is very similar), writing* $(\mathrm{V})'$ *or* $(\mathrm{PJ})'$ *for* (V) *or* (PJ) *combined with applications of* $(\mathrm{W})\triangleright$ *and* $\triangleright(\mathrm{W})$*:*

$$\dfrac{\dfrac{\dfrac{\triangleright (\neg A, \neg A \to A \Rightarrow B)}{(\neg A \Rightarrow A) \ \triangleright \ (\neg A \Rightarrow B)}\,(\mathrm{PJ})' \qquad (\neg A \Rightarrow B) \ \triangleright \ (\neg A \Rightarrow B) \qquad (\neg A \Rightarrow C) \ \triangleright \ (\neg A \Rightarrow C)}{(\neg A \Rightarrow A, B, C), (\bot \Rightarrow B, C) \ \triangleright \ (\neg A \Rightarrow B), (\neg A \Rightarrow C)}\,(\mathrm{V})'}{(\neg A \Rightarrow B, C) \ \triangleright \ (\neg A \Rightarrow B), (\neg A \Rightarrow C)}\,(\to)\triangleright$$

The two rightmost leaves in this proof tree are instances of (SID)*, while the derivability of the other leaf follows from the Right Logical Rules since* $(\neg A, \neg A \to A \Rightarrow B)$ *is derivable in Intuitionistic Logic for any* B*.*

Soundness for **GAI** is established similarly to the case of modal logics.

Theorem 31 *If* $\vdash_{\mathbf{GAI}} \mathrm{R}$*, then* $\mid\!\sim_{\mathsf{IPC}} \mathrm{R}$*.*

Proof. It is sufficient to show that each rule of **GAI** is sound, concentrating just on those cases different to Lemma 17. For $(\to\Rightarrow)\triangleright^i$, let σ be a unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Gamma, A \to B \Rightarrow \Delta)$. Since p and q do not occur in the conclusion of the rule, we can extend σ with $\sigma(p) = \sigma(A)$ and $\sigma(q) = \sigma(B)$. It follows immediately that σ is a unifier for $I(\Gamma, p \to q \Rightarrow \Delta)$, $I(p \Rightarrow A)$, and $I(B \Rightarrow q)$. Hence, if the premise is admissible, then σ is a unifier for $I(S)$ for some $S \in \mathcal{H}$ as required. The case of the rule $(\Rightarrow\to)\triangleright^i$ follows a similar pattern.

For (V), suppose that σ is a unifier for $I(S)$ for all $S \in \mathcal{G}$ and $I(\Gamma \Rightarrow \Delta)$, and let $\Delta = \{A_1, \ldots, A_n\}$ (including the case where $\Delta = \emptyset$). Using the right set of premises, σ is either a unifier for some $S \in \mathcal{H}$ or for $I(\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for all $\emptyset \neq \Pi \subseteq \Gamma_\Delta$. In the first case we are done, so assume the latter. It suffices now by the left set of premises to show that σ is a unifier for $I(\Gamma \Rightarrow A_i)$ for some $i \in \{1, \ldots, n\}$. Suppose, arguing contrapositively, that this is not the case. Then there exist counter-models $K_1, \ldots, K_n$ such that $K_i \Vdash \sigma(\bigwedge \Gamma)$ and $K_i \nVdash \sigma(A_i)$ for $i = 1 \ldots n$. Consider the model $K = (\Sigma_{i=1}^n K_i)'$ (a one-node model if $\Delta = \emptyset$). Let $\Pi = \{D \in \Gamma_\Delta : K \Vdash \sigma(D)\}$. Thus $K \Vdash \sigma(\bigwedge \Pi)$. Observe that for all $(B \to C) \in \Gamma$ such that $B \notin \Pi$, either $B \in \Delta$ or $K \nVdash \sigma(B)$. Note also that $B \in \Delta$ implies $K \nVdash \sigma(B)$. Hence for all $B \notin \Pi$ it follows that $K \nVdash \sigma(B)$, and so $K \Vdash \sigma(B \to C)$. It follows that $K \Vdash \sigma(\bigwedge(\Gamma^\Pi \cup \Pi))$. Note that if $\Pi = \emptyset$, then $\Gamma^\Pi = \Gamma$; i.e. $K \Vdash \sigma(\Gamma)$. But σ is a unifier for $I(\Gamma \Rightarrow \Delta)$ and, when $\Pi \neq \emptyset$, for $I(\Gamma^\Pi, \Pi \Rightarrow \Delta)$, so $K \Vdash \sigma(\bigvee \Delta)$, a contradiction. $\square$

For completeness, we need a series of lemmas corresponding to those used in the modal case. First we have, exactly as in Lemma 19 (except replacing the application of the modal deduction theorem with the usual deduction theorem), that derivable gs-rules with at most one sequent on the right are **GAI**-derivable.

Lemma 32 *If* $\vdash_{\mathsf{IPC}} \mathcal{G} \rhd \mathcal{H}$ *where* $|\mathcal{H}| \leq 1$*, then* $\vdash_{\mathbf{GAI}} \mathcal{G} \rhd \mathcal{H}$*.*

We then establish that admissible rules are **GAI**-derivable from admissible rules that are full (recalling Definition 23) with respect to $(\vee)$, $(\rightarrow)\rhd$, and (AC).

Lemma 33 *The left logical rules,* (AC)*, and* $(\vee)$ *are invertible.*

Proof. The invertibility of all these rules except for $(\rightarrow\Rightarrow)\rhd^i$ and $(\Rightarrow\rightarrow)\rhd^i$ follows either immediately from the fact that every sequent in the conclusion occurs in the premises, or from the soundness of the rules on the right. For $(\rightarrow\Rightarrow)\rhd^i$, suppose that the conclusion is admissible and assume that σ is a unifier for $I(S)$ for all $S \in \mathcal{G}$, $I(\Gamma, p \rightarrow q \Rightarrow \Delta)$, $I(p \Rightarrow A)$, and $I(B \Rightarrow q)$. Since $\vdash_{\mathsf{IPC}} I(A, A \rightarrow B \Rightarrow B)$ it follows that σ is a unifier for $I(p, A \rightarrow B \Rightarrow q)$, and hence for $I(A \rightarrow B \Rightarrow p \rightarrow q)$. So by the admissibility of cut for IPC, σ is a unifier for $I(\Gamma, A \rightarrow B \Rightarrow \Delta)$, and hence, by the admissibility of the conclusion, for $I(S)$ for some $S \in \mathcal{H}$. The case of $(\Rightarrow\rightarrow)\rhd^i$ is very similar. $\square$

Definition 34 *A gs-rule* $\mathcal{G} \rhd \mathcal{H}$ *is* implication-irreducible *if all sequents in* $\mathcal{G}$ *contain only variables on the right and variables and variable implications on the left.*

As for Lemmas 22 and 24, it is straightforward to show that applying the invertible left logical rules backwards reduces any gs-rule to an implication-irreducible gs-rule, and then that applying the rules $(\vee)$, $(\rightarrow)\rhd$, and (AC) exhaustively backwards terminates with a set of gs-rules full with respect to these rules.

Lemma 35 *Every admissible gs-rule is* **GAI**-*derivable from admissible implication-irreducible gs-rules that are full with respect to* $(\vee)$*,* $(\rightarrow)\rhd$*, and* (AC)*.*

We now use Ghilardi's characterization of projective formulas to establish completeness for **GAI**. First, we give a technical lemma showing that a crucial property of gs-rules is preserved "premise to conclusion" by the rule (AC). The completeness proof, as in the modal case, establishes the existence of a resolution refutation where certain sequents correspond to the clauses and (AC) corresponds to the resolution rule. The technical lemma establishes the existence of the required sequents.

Definition 36 *We define the following property* $*$ *on pairs consisting of a sequent* $(\Gamma \Rightarrow \Delta)$ *and a set of sequents* $\mathcal{G}$*:*

$$*((\Gamma \Rightarrow \Delta), \mathcal{G}) \quad \Leftrightarrow \quad \forall \Pi \subseteq \Gamma_\Delta (\exists \Gamma' \subseteq (\Gamma^\Pi \cup \Pi)\, \exists \Delta' \subseteq \Delta\, (\Gamma' \Rightarrow \Delta') \in \mathcal{G})$$

where $\Gamma^\Pi = \{A \rightarrow B \in \Gamma : A \notin \Pi\}$ *and* $\Gamma_\Delta = \{A \notin \Delta : \exists B\, (A \rightarrow B) \in \Gamma\}$*.*

Notice the similarity with the rightmost premises in the $(\vee)$ rule. If $*((\Gamma \Rightarrow \Delta), \mathcal{G})$ holds, then $\bigwedge_{S \in \mathcal{G}} I(S) \vdash_{\mathsf{IPC}} I(\Gamma^\Pi, \Pi \Rightarrow \Delta)$ for every $\Pi \subseteq \Gamma_\Delta$. Hence in this case, the gs-rule $\mathcal{G} \rhd (\Gamma^\Pi, \Pi \Rightarrow \Delta), \mathcal{H}$ is derivable.

Lemma 37 *Let* $\mathcal{G}, (\Gamma_1, p \Rightarrow \Delta_1), (\Gamma_2 \Rightarrow p, \Delta_2) \rhd \mathcal{H}$ *be a gs-rule full with respect* *to* (AC). *Then:*

$$*((\Gamma_1, p \Rightarrow \Delta_1), \mathcal{G}) \textit{ and } *((\Gamma_2 \Rightarrow p, \Delta_2), \mathcal{G}) \textit{ implies } *((\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2), \mathcal{G}).$$

Proof. Consider $\Pi \subseteq (\Gamma_1 \cup \Gamma_2)_{\Delta_1 \cup \Delta_2}$. We show that there exist $\Gamma' \subseteq \Gamma_1^\Pi \cup \Gamma_2^\Pi \cup \Pi$ and $\Delta' \subseteq \Delta_1 \cup \Delta_2$ such that $(\Gamma' \Rightarrow \Delta') \in \mathcal{G}$. First, assume $p \in \Pi$. By the hypothesis for $(\Gamma_1 \Rightarrow \Delta_1)$, there exist $\Gamma' \subseteq \Gamma_1^\Pi \cup \Pi$ and $\Delta' \subseteq \Delta_1$ such that $(\Gamma' \Rightarrow \Delta') \in \mathcal{G}$, and we are done. Second, assume $p \notin \Pi$. By hypothesis there exist $\Gamma_1' \subseteq \Gamma_1^\Pi \cup \Pi$, $\Gamma_2' \subseteq \Gamma_2^\Pi \cup \Pi$, $\Delta_1' \subseteq \Delta_1$, and $\Delta_2' \subseteq \Delta_2$ such that $(\Gamma_1', p \Rightarrow \Delta_1') \in \mathcal{G}$ and $(\Gamma_2' \Rightarrow p, \Delta_2') \in \mathcal{G}$. Hence also $(\Gamma_1', \Gamma_2' \Rightarrow \Delta_1', \Delta_2') \in \mathcal{G}$ by fullness. Moreover, $\Gamma_1' \cup \Gamma_2' \subseteq \Gamma_1^\Pi \cup \Gamma_2^\Pi \cup \Pi$ and $\Delta_1' \cup \Delta_2 \subseteq \Delta_1 \cup \Delta_2$, so we are done. $\square$

Theorem 38 $\mathrel{|\!\sim}_{\mathsf{IPC}} \mathcal{G} \rhd \mathcal{H}$ *iff* $\vdash_{\mathbf{GAI}} \mathcal{G} \rhd \mathcal{H}$.

Proof. The right-to-left direction is Theorem 31. For the other direction, it is sufficient using Lemma 35 to assume that $\mathcal{G} \rhd \mathcal{H}$ is an admissible implication-irreducible gs-rule that is full with respect to (AC), $(\to)\rhd$, and $(\vee)$. Let $C = \bigwedge_{S \in \mathcal{G}} I(S)$. If C is inconsistent, then $\vdash_{\mathsf{IPC}} \mathcal{G} \rhd$. If C is IPC-projective, then using Lemma 6 (a), $\vdash_{\mathsf{IPC}} \mathcal{G} \rhd S$ for some $S \in \mathcal{H}$. In both cases, by Lemma 32 and the structural rules, $\vdash_{\mathbf{GAI}} \mathcal{G} \rhd \mathcal{H}$.

Assume then that C is consistent and not projective. We use Ghilardi's key result, Theorem 9, which tells us that the class of models of C does not have the extension property, to show that $\mathcal{G} \rhd \mathcal{H}$ is $\mathbf{GAI}$-derivable.

First, unpacking the definition of the extension property, we obtain a model K such that $K \Vdash C$ and every variant of K' refutes C. We can assume K to be non-empty since if no one-node model were a model of C, C would be inconsistent.

Let $M_1, \ldots, M_k$ be all the variants of K' and fix sequents $(\Gamma_i \Rightarrow \Delta_i)$ in $\mathcal{G}$ for $i = 1 \ldots k$ such that $M_i \nVdash I(\Gamma_i \Rightarrow \Delta_i)$. We can assume that:

$$(p \to q) \in \Gamma_i \quad \Longrightarrow \quad p \in \Delta_i. \tag{8}$$

For suppose that $(p \to q) \in \Gamma_i$ and $p \notin \Delta_i$; we show that $\Gamma_i \Rightarrow \Delta_i$ can be replaced by a sequent $S \in \mathcal{G}$ that has property (8) where $M_i \nVdash I(S)$. Since $M_i \Vdash \bigwedge \Gamma_i$ and $M_i \nVdash \bigvee \Delta_i$, it follows that $M_i \Vdash p \to q$, and so either $M_i \nVdash p$ or $M_i \Vdash q$. This means that either $M_i \nVdash I(\Gamma_i \Rightarrow p, \Delta_i)$ or $M_i \nVdash I(\Gamma_i \backslash \{p \to q\}, q \Rightarrow \Delta_i)$. Since $\mathcal{G} \rhd \mathcal{H}$ is full with respect to $(\to)\rhd$, both of these sequents are in $\mathcal{G}$, and can replace $(\Gamma_i \Rightarrow \Delta_i)$.

The following argument is based on Ghilardi's proof that the algorithm for checking projectivity introduced in [7] is correct. First we define the set of variables:

$$P = \{p : p \text{ occurs in } C \text{ and } K \Vdash p\}.$$

Now define for $i = 1 \ldots k$:

$$A_i =_{\text{def}} \bigwedge_{p \in \Gamma_i \cap P} p \wedge \bigwedge_{p \in \Delta_i \cap P} \neg p \quad \text{and} \quad A =_{\text{def}} \bigvee_{i=1}^{k} A_i.$$

If the conjuncts in A_i are empty, and hence A_i is equivalent to $\top$, Γ_i consists of implications only, and all atoms in Δ_i are not in P. That this case leads to a contradiction is shown in the last two paragraphs of this proof. If this situation does not occur we proceed as follows.

We show that A is a classical tautology. Since $K \Vdash p$ for all $p \in P$, given a classical valuation v on P, we consider the variant M of K' defined at the root by:

$$M \Vdash p \;\;\Leftrightarrow\;\; v(p) = 1$$

where $M = M_j$ for some $j \in \{1, \ldots, k\}$. Observe that $M \Vdash p$ and hence $v(p) = 1$ for all variables $p \in \Gamma_j$. Also, $M \nVdash p$ and hence $v(p) = 0$ for all variables $p \in \Delta_j$. Thus $v(A_j) = 1$. It follows that A is a classical tautology and $\neg A$ is classically inconsistent. But then as in the modal cases, the negation of A where the literals are swapped (p to $\neg p$ and vice versa), is also classically inconsistent:

$$\bigwedge_{j=1}^{k} \Big(\bigvee_{p \in \Gamma_j \cap P} p \vee \bigvee_{p \in \Delta_j \cap P} \neg p \Big).$$

Hence there exists a resolution refutation starting with the clauses:

$$\{p : p \in \Gamma_j \cap P\} \cup \{\neg p : p \in \Delta_j \cap P\} \;\text{ for } j = 1 \ldots k$$

that ends in the empty clause $\emptyset$.

Let $\Theta \cup \Psi'$ be a clause in the refutation, where Θ contains only variables and Ψ' contains only negated variables. Define $\Psi = \{p : \neg p \in \Psi'\}$. Observe that every cut on a variable $p \in P$ can be "mimicked" in $\mathcal{G}$ via a backwards application of (AC). Since $\mathcal{G} \,\triangleright\, \mathcal{H}$ is full with respect to (AC), this implies that there exists $(\Gamma, \Theta \Rightarrow \Psi, \Delta) \in \mathcal{G}$ such that $\Delta \cap P = \Gamma \cap P = \emptyset$, and $K \Vdash \bigwedge \Gamma$ (where Θ and Ψ are the parts of the sequent that occur in the resolution refutation).

Observe that (8) implies that $*((\Gamma_i \Rightarrow \Delta_i), \mathcal{G})$ holds for $i = 1 \ldots k$ since $(\Gamma_i)_{\Delta_i} = \emptyset$. By multiple applications of Lemma 37 we also have $*((\Gamma, \Theta \Rightarrow \Psi, \Delta), \mathcal{G})$. In particular for the sequent $(\Gamma \Rightarrow \Delta)$ corresponding to the empty clause $\emptyset$ we have $*((\Gamma \Rightarrow \Delta), \mathcal{G})$. Note that since $(\Gamma \Rightarrow \Delta)$ corresponds to the empty clause, Δ contains only variables not in P, and Γ contains only implicational formulas (any variable in Γ must be in P). Also, $K \Vdash \bigwedge \Gamma$ since this holds for all sequents that correspond to clauses in the refutation.

Since $\mathcal{G} \,\triangleright\, \mathcal{H}$ is full with respect to (V), either $(\Gamma \Rightarrow q) \in \mathcal{G}$ for some $q \in \Delta$, or $(\Gamma^{\Pi}, \Pi \Rightarrow \Delta) \in \mathcal{H}$ for some $\emptyset \neq \Pi \subseteq \Gamma_{\Delta}$. In the first case, we get that $K \Vdash$

$(\bigwedge\Gamma \to q)$, since $K \Vdash C$. But $K \Vdash \bigwedge\Gamma$ so it follows that $K \Vdash q$, which implies $q \in P$, a contradiction. In the second case, using the fact that $*((\Gamma \Rightarrow \Delta), \mathcal{G})$, there exists a sequent $(\Gamma' \Rightarrow \Delta') \in \mathcal{G}$ for some $\Gamma' \subseteq \Gamma^{\Pi} \cup \Pi$ and some $\Delta' \subseteq \Delta$. Clearly, $\vdash_{\mathsf{IPC}} (\Gamma' \Rightarrow \Delta') \rhd (\Gamma^{\Pi}, \Pi \Rightarrow \Delta)$. By Lemma 32, $\vdash_{\mathbf{GAI}} (\Gamma' \Rightarrow \Delta') \rhd (\Gamma^{\Pi}, \Pi \Rightarrow \Delta)$. Hence using the weakening rules, also $\vdash_{\mathbf{GAI}} \mathcal{G} \rhd \mathcal{H}$ as required. $\Box$

As in the modal case, we can easily obtain a terminating proof system. We just insist that the left logical rules are applied exhaustively (backwards) to transform a gs-rule into an implication-irreducible gs-rule, and that then (AC), (V), and $(\to)\rhd$ are applied exhaustively (backwards) to obtain implication-irreducible gs-rules that are full with respect to these rules. Since for the gs-rules thus obtained, admissibility reduces to derivability, we can then rely on any decision procedure we like for IPC, obtaining Rybakov's result [17]:

Corollary 39 *Admissibility is decidable for Intuitionistic Logic.*

Finally, illustrating the flexibility of our approach, we note that proof systems for admissibility have been developed in a sequel to this paper [13] for both intermediate logics and a wider class of "mono-extensible" modal logics by extending the framework from sequent rules to hypersequent rules.

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