PSYCHOMETRIKA—VOL. 82, NO. 3, 648–659
SEPTEMBER 2017
DOI: 10.1007/s11336-016-9538-5

LOWER BOUNDS TO THE RELIABILITIES OF FACTOR SCORE ESTIMATORS

DAVID J. HESSEN

UTRECHT UNIVERSITY

Under the general common factor model, the reliabilities of factor score estimators might be of more interest than the reliability of the total score (the unweighted sum of item scores). In this paper, lower bounds to the reliabilities of Thurstone's factor score estimators, Bartlett's factor score estimators, and McDonald's factor score estimators are derived and conditions are given under which these lower bounds are equal. The relative performance of the derived lower bounds is studied using classic example data sets. The results show that estimates of the lower bounds to the reliabilities of Thurstone's factor score estimators are greater than or equal to the estimates of the lower bounds to the reliabilities of Bartlett's and McDonald's factor score estimators.

Key words: reliability, classical test theory, common factor model, factor scores.

1. Introduction

In the psychometric literature, the description and the assessment of the reliability of the total score (the unweighted sum of the item scores) has received a lot of attention (e.g., ten Berge & Sočan, 2004; Zinbarg, Revelle, Yovel, & Li, 2005; Sijtsma, 2009; Bentler, 2009; Revelle & Zinbarg, 2009). For the assessment of the reliability of the total score, many coefficients have been proposed (for an overview, see Revelle and Zinbarg, 2009). All these coefficients are mathematical lower bounds to the reliability of the total score. One of these coefficients is the popular coefficient alpha (Guttman, 1945; Cronbach, 1951). Coefficient alpha is widely used but other coefficients have been shown to be greater lower bounds to the reliability of the total score. Therefore, Sijtsma (2009) advised against using coefficient alpha and suggested to report the coefficient referred to as the greatest lower bound (Woodhouse & Jackson, 1977; ten Berge, Snijders, & Zegers, 1981). Bentler (2009), however, made some critical observations about the greatest lower bound and suggested to consider reliability coefficients based on a factor or structural equation model. Revelle and Zinbarg (2009) showed that for some classic example data sets the estimate of the greatest lower bound was systematically less than the estimate they obtained using the factor model-based coefficient ω (Heise & Bohrnstedt, 1970; McDonald, 1978).

Under a factor model, however, the reliability of the total score might be of less interest than the reliabilities of factor score estimators. Factor score estimators are the random variables used to estimate the true values of the unobserved common factors assumed to underlie the item scores. The estimated values of the common factors are called factor scores. When factor scores are used for diagnostic purposes, or as inputs to subsequent analyses, then the reliabilities of the factor score estimators are of interest.

Many factor score estimators have been proposed (for an overview, see Grice, 2001). Three well-known types of factor score estimators that are defined in terms of the parameters of a factor model are the regression estimators proposed by Thurstone (1935), the weighted least squares estimators proposed by Bartlett (1937), and the correlation-preserving estimators proposed by McDonald (1981). In this paper, lower bounds to the reliabilities of these three types of factor

Correspondence should be made to David J. Hessen, Department of Methodology and Statistics, Utrecht University, Padualaan 14, PO Box 80.140, 3508 TC Utrecht, The Netherlands. Email: D.J.Hessen@uu.nl

score estimators are derived and conditions under which they are equal are provided. Subsequently, classic example data sets are used to study the relative performance of the derived lower bounds. First, however, to introduce notation and to relate the general common factor model to the classical test theory model, a general decomposition of item scores is presented in the next section.

2. A General Decomposition

The reliability of a test score is defined in classical test theory (Lord & Novick, 1968). The reliability of a test score is defined as the squared correlation between the test score and its true score, which is defined as the test score minus its associated random error score. The factor score estimators of interest in this paper are test scores constructed from the item scores and factor model parameters. The reliabilities of these factor score estimators are the squared correlations between the factor score estimators and their true scores. In order to derive lower bounds to the reliabilities of these factor score estimators in terms of factor model parameters, the true scores of these factor score estimators and the true scores of the item scores must be made explicit in terms of factor model concepts. In this section, therefore, the general common factor model is related to the classical test theory model.

Let $\mathbf{X}$ be a random vector of k item scores for a randomly sampled individual from a population and let $\mathbf{x}$ be a realization. It is assumed that the item scores are indicators of q common factors, where $q \leq k$. Lord and Novick (1968) gave the following generalization of the general common factor model:

$$\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\Lambda}\mathbf{F} + \boldsymbol{\Gamma}\mathbf{S} + \mathbf{E}, \tag{1}$$

where $\boldsymbol{\mu}$ is the mean vector of $\mathbf{X}$, $\boldsymbol{\Lambda}$ is a $k \times q$ matrix of constant factor loadings, $\mathbf{F}$ is a random vector of q common factors, $\boldsymbol{\Gamma}$ is a $k \times k$ diagonal matrix of constants, $\mathbf{S}$ is a random vector of k specific factors, and $\mathbf{E}$ is a random vector of k random error scores. The difference between $\mathbf{S}$ and $\mathbf{E}$ is that the values of the elements of $\mathbf{S}$ are only assumed to vary between individuals, whereas the values of the elements of $\mathbf{E}$ are assumed to vary between and within individuals. All within-person means of the elements of $\mathbf{E}$ are assumed to be zero. As a consequence, the means of all elements of $\mathbf{E}$ are equal to zero in the population of individuals. In addition, the means of all elements of $\mathbf{F}$ and $\mathbf{S}$ are only assumed to be zero in the population of individuals. Furthermore, all elements of $\mathbf{F}$ are assumed to be uncorrelated with all elements of $\mathbf{S}$ and $\mathbf{E}$, and all elements of $\mathbf{S}$ are assumed to be uncorrelated with all elements of $\mathbf{E}$. Finally, the specific factors are assumed to be independent of each other and the random error scores are assumed to be independent of each other, so that cov($\mathbf{S}$) and cov($\mathbf{E}$) are diagonal matrices. It follows that

$$\text{cov}(\mathbf{X}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Gamma}\text{cov}(\mathbf{S})\boldsymbol{\Gamma} + \text{cov}(\mathbf{E}), \tag{2}$$

where $\boldsymbol{\Psi} = \text{cov}(\mathbf{F})$.

In the classical test theory model (Lord & Novick, 1968), the random vector of item scores is defined as $\mathbf{X} = \mathbf{T} + \mathbf{E}$, where $\mathbf{T} = \boldsymbol{\mu} + \boldsymbol{\Lambda}\mathbf{F} + \boldsymbol{\Gamma}\mathbf{S}$ is a random vector of the so-called true item scores. Since all elements of $\mathbf{T}$ are assumed to be uncorrelated with all elements of $\mathbf{E}$, it follows that $\text{cov}(\mathbf{X}) = \text{cov}(\mathbf{T}) + \text{cov}(\mathbf{E})$, where $\text{cov}(\mathbf{T}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Gamma}\text{cov}(\mathbf{S})\boldsymbol{\Gamma}$.

In the general common factor model, the random vector of item scores is defined as $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\Lambda}\mathbf{F} + \mathbf{U}$, where $\mathbf{U} = \boldsymbol{\Gamma}\mathbf{S} + \mathbf{E}$ is a random vector of unique factors. Since all elements of $\mathbf{F}$ are assumed to be uncorrelated with all elements of $\mathbf{U}$, it follows that $\text{cov}(\mathbf{X}) = \boldsymbol{\Sigma} = \boldsymbol{\Omega} + \boldsymbol{\Theta}$, where $\boldsymbol{\Omega} = \text{cov}(\boldsymbol{\Lambda}\mathbf{F}) = \boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}'$ and $\boldsymbol{\Theta} = \text{cov}(\mathbf{U}) = \boldsymbol{\Gamma}\text{cov}(\mathbf{S})\boldsymbol{\Gamma} + \text{cov}(\mathbf{E})$ is a diagonal matrix. A special case of the general common factor model is McDonald's (1999) hierarchical factor model given by $\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{\lambda}_1 F_1 + \mathbf{B}\mathbf{G} + \mathbf{U}$, where F_1 is a general factor (common to all k items), $\boldsymbol{\lambda}_1$ is the vector of general factor loadings, $\mathbf{G}$ is a $(q-1)\times 1$ vector of group factors (common to some but not all k items), and $\mathbf{B}$ is a $k\times(q-1)$ matrix of group factor loadings. Note that $\mathbf{F}=[F_1\ \mathbf{G}']'$ and $\mathbf{\Lambda}=[\boldsymbol{\lambda}_1\ \mathbf{B}]$.

Without additional restrictions, the general common factor model is not identified. In exploratory factor analysis, it is customary to resolve this indeterminacy by setting $\mathbf{\Psi}=\mathbf{I}$ and restricting $\mathbf{\Lambda}'\mathbf{\Theta}^{-1}\mathbf{\Lambda}$ to be diagonal. These identification restrictions are however arbitrary and other restrictions are possible. In confirmatory factor analysis, identification is usually achieved by setting equal to zero more than $q-1$ elements in each column of $\mathbf{\Lambda}$ and by setting equal to one either all factor variances or one factor loading for each factor.

Estimates of $\boldsymbol{\mu}$, $\mathbf{\Lambda}$, $\mathbf{\Psi}$, and $\mathbf{\Theta}$ are often obtained using a least squares criterion or a maximum likelihood criterion. The estimate of $\boldsymbol{\mu}$ is typically the observed sample mean vector $\bar{\mathbf{x}}$ and the estimates of $\mathbf{\Lambda}$, $\mathbf{\Psi}$, and $\mathbf{\Theta}$ are usually obtained by numerical procedures. In what follows, irrespective of the estimation criterion, estimates of $\mathbf{\Lambda}$, $\mathbf{\Psi}$, and $\mathbf{\Theta}$ are denoted by $\hat{\mathbf{\Lambda}}$, $\hat{\mathbf{\Psi}}$, and $\hat{\mathbf{\Theta}}$, respectively.

3. A Linear Combination of Item Scores

In this section, a lower bound to the reliability of a general linear combination of the item scores is presented. Let $\boldsymbol{\alpha}$ be a nonzero vector of constants (weights). Substitution from Equation 1 into the linear combination $Y=\boldsymbol{\alpha}'(\mathbf{X}-\boldsymbol{\mu})$ yields

$$Y=C+S+E, \tag{3}$$

where $C=\boldsymbol{\alpha}'\mathbf{\Lambda}\mathbf{F}$, $S=\boldsymbol{\alpha}'\mathbf{\Gamma}\mathbf{S}$, and $E=\boldsymbol{\alpha}'\mathbf{E}$. The reliability of Y is defined as the squared correlation between Y and its true score $C+S$. The reliability of Y is the proportion of the variance of Y explained by its true score $C+S$, and can be seen as a measure of the extent to which Y is free of random measurement error. It can be shown that the reliability of Y equals

$$\rho^2_{Y(C+S)}=\rho^2_{YC}+\rho^2_{YS}, \tag{4}$$

where ρ^2_{YC} is the proportion of the variance of Y explained by the common factors (the communality of Y) and ρ^2_{YS} is the proportion of the variance of Y explained by the specific factors. Without additional information, $\mathbf{S}$ and $\mathbf{E}$ cannot be separated from each other and only ρ^2_{YC} is estimable. Since $\rho^2_{YS}\geq 0$, it follows from Eq. 4 that $\rho^2_{Y(C+S)}\geq\rho^2_{YC}$, that is, the proportion of the variance of Y explained by the common factors is a lower bound to the reliability of Y. ρ^2_{YC} is equal to the reliability of Y if and only if all specific factors are random measurement errors. Now, since under the general common factor model $\mathrm{cov}(Y,C)=\mathrm{var}(C)=\boldsymbol{\alpha}'\mathbf{\Omega}\boldsymbol{\alpha}$ and $\mathrm{var}(Y)=\boldsymbol{\alpha}'\mathbf{\Sigma}\boldsymbol{\alpha}$, it follows that a lower bound to the reliability of the linear combination Y is given by

$$\rho^2_{YC}=\frac{\{\mathrm{cov}(Y,C)\}^2}{\mathrm{var}(Y)\mathrm{var}(C)}=\frac{\mathrm{var}(C)}{\mathrm{var}(Y)}=\frac{\boldsymbol{\alpha}'\mathbf{\Omega}\boldsymbol{\alpha}}{\boldsymbol{\alpha}'\mathbf{\Sigma}\boldsymbol{\alpha}}. \tag{5}$$

Since variances are nonnegative, it follows that $0\leq\rho^2_{YC}\leq 1$. If $\rho^2_{YC}=1$, then $\rho^2_{Y(C+S)}=1$ and Y is perfectly reliable. Here, it is stated without proof that $\rho^2_{YC}=1$ if and only if the variances of the unique factors are all equal to zero.

A special case of the coefficient in Eq. 5 is $\omega=\mathbf{1}'\mathbf{\Omega}\mathbf{1}/\mathbf{1}'\mathbf{\Sigma}\mathbf{1}$ (Heise & Bohrnstedt, 1970; McDonald, 1978), where $\mathbf{1}$ is a vector of ones. Coefficient ω is a lower bound to the reliability of the special linear combination $\mathbf{1}'(\mathbf{X}-\boldsymbol{\mu})$. The estimate of ω is $\hat{\omega} = \mathbf{1}'\hat{\boldsymbol{\Omega}}\mathbf{1}/\mathbf{1}'\hat{\boldsymbol{\Sigma}}\mathbf{1}$, where $\hat{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Psi}}\hat{\boldsymbol{\Lambda}}'$ and $\hat{\boldsymbol{\Sigma}} = \hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Psi}}\hat{\boldsymbol{\Lambda}}' + \hat{\boldsymbol{\Theta}}$. Another special linear combination of the item scores that might be of interest in practice is $\boldsymbol{\beta}'(\mathbf{X}-\boldsymbol{\mu})$, where $\boldsymbol{\beta}$ is the eigenvector of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$ corresponding to the largest eigenvalue of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$. For this linear combination, the coefficient in Eq. 5 attains its maximum value of $\delta = \boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta}/\boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{\beta}$. The estimate of δ is $\hat{\delta} = \hat{\boldsymbol{\beta}}'\hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\beta}}/\hat{\boldsymbol{\beta}}'\hat{\boldsymbol{\Sigma}}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is the eigenvector of $\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\Omega}}$ corresponding to the largest eigenvalue of $\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\Omega}}$. Other special linear combinations of the item scores that might be of interest in practice are factor score estimators. In the following section, lower bounds to the reliabilities of three well-known types of factor score estimators are derived.

4. Factor Score Estimators

4.1. Thurstone's Factor Score Estimators

Under the assumption that $\mathbf{X}$ and $\mathbf{F}$ have a joint multivariate normal distribution, that is,

$$\begin{bmatrix}\mathbf{X}\\ \mathbf{F}\end{bmatrix} \sim N_{k+q}\left(\begin{bmatrix}\boldsymbol{\mu}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{\Sigma} & \boldsymbol{\Lambda}\boldsymbol{\Psi}\\ \boldsymbol{\Psi}\boldsymbol{\Lambda}' & \boldsymbol{\Psi}\end{bmatrix}\right),$$

it follows that the mean vector of $\mathbf{F}$ conditional on $\mathbf{x}$ equals $\mathbf{f}_1 = \mathbf{A}_1'(\mathbf{x}-\boldsymbol{\mu})$, where $\mathbf{A}_1' = \boldsymbol{\Psi}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}$ (Thurstone, 1935; Thomson, 1946; Thompson, 1993). The vector $\mathbf{f}_1$ can be seen as a realization of the random vector $\mathbf{D} = \mathbf{A}_1'(\mathbf{X}-\boldsymbol{\mu})$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Psi}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}$. The ith element of $\mathbf{D}$ is the ith Thurstone factor score estimator $D_i = \boldsymbol{\alpha}_{i1}'(\mathbf{X}-\boldsymbol{\mu})$, where $\boldsymbol{\alpha}_{i1}'$ is the ith row of $\mathbf{A}_1'$. It follows from Eq. 5 that a lower bound to the reliability of D_i is given by

$$\rho^2_{D_iC} = \frac{\nu_{ii}}{\nu_{ii} + \tau_{ii}}, \tag{6}$$

where $\nu_{ii} = \boldsymbol{\alpha}_{i1}'\boldsymbol{\Omega}\boldsymbol{\alpha}_{i1}$ is the ith diagonal element of the matrix $\mathbf{A}_1'\boldsymbol{\Omega}\mathbf{A}_1 = \boldsymbol{\Psi}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}$ and $\nu_{ii}+\tau_{ii} = \boldsymbol{\alpha}_{i1}'\boldsymbol{\Sigma}\boldsymbol{\alpha}_{i1}$ is the ith diagonal element of the matrix $\mathbf{A}_1'\boldsymbol{\Sigma}\mathbf{A}_1 = \boldsymbol{\Psi}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}$. The estimate of $\rho^2_{D_iC}$ is $\hat{\rho}^2_{D_iC} = \hat{\nu}_{ii}/(\hat{\nu}_{ii} + \hat{\tau}_{ii})$, where $\hat{\nu}_{ii}$ is the ith diagonal element of $\hat{\boldsymbol{\Psi}}\hat{\boldsymbol{\Lambda}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\Omega}}\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Psi}}$ and $\hat{\nu}_{ii} + \hat{\tau}_{ii}$ is the ith diagonal element of $\hat{\boldsymbol{\Psi}}\hat{\boldsymbol{\Lambda}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Psi}}$.

4.2. Bartlett's Factor Score Estimators

Bartlett (1937) proposed the estimate of the vector $\mathbf{f}$ that minimizes the sum of squares $(\mathbf{x}-\boldsymbol{\mu}-\boldsymbol{\Lambda}\mathbf{f})'\boldsymbol{\Theta}^{-1}(\mathbf{x}-\boldsymbol{\mu}-\boldsymbol{\Lambda}\mathbf{f})$, as a vector of estimates of the true factor scores of an individual with realization $\mathbf{x}$. The so-called weighted least squares estimate of $\mathbf{f}$ that minimizes this sum of squares is the vector $\mathbf{f}_2 = \mathbf{A}_2'(\mathbf{x}-\boldsymbol{\mu})$, where $\mathbf{A}_2' = (\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda})^{-1}\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}$. The vector $\mathbf{f}_2$ can be seen as a realization of the random vector $\mathbf{V} = \mathbf{A}_2'(\mathbf{X}-\boldsymbol{\mu})$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Psi} + (\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda})^{-1}$. The ith element of $\mathbf{V}$ is the ith Bartlett factor score estimator $V_i = \boldsymbol{\alpha}_{i2}'(\mathbf{X}-\boldsymbol{\mu})$, where $\boldsymbol{\alpha}_{i2}'$ is the ith row of $\mathbf{A}_2'$. It follows from Eq. 5 that a lower bound to the reliability of V_i is given by

$$\rho^2_{V_iC} = \frac{\psi_{ii}}{\psi_{ii} + \eta_{ii}}, \tag{7}$$

where $\psi_{ii} = \boldsymbol{\alpha}_{i2}'\boldsymbol{\Omega}\boldsymbol{\alpha}_{i2}$ is the ith diagonal element of $\mathbf{A}_2'\boldsymbol{\Omega}\mathbf{A}_2 = \boldsymbol{\Psi}$ and $\psi_{ii} + \eta_{ii} = \boldsymbol{\alpha}_{i2}'\boldsymbol{\Sigma}\boldsymbol{\alpha}_{i2}$ is the ith diagonal element of $\mathbf{A}_2'\boldsymbol{\Sigma}\mathbf{A}_2 = \boldsymbol{\Psi} + (\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda})^{-1}$. The estimate of $\rho^2_{V_iC}$ is $\hat{\rho}^2_{V_iC} = \hat{\psi}_{ii}/(\hat{\psi}_{ii} + \hat{\eta}_{ii})$, where $\hat{\psi}_{ii}$ is the ith diagonal element of $\hat{\boldsymbol{\Psi}}$ and $\hat{\psi}_{ii} + \hat{\eta}_{ii}$ is the ith diagonal element of $\hat{\boldsymbol{\Psi}} + (\hat{\boldsymbol{\Lambda}}'\hat{\boldsymbol{\Theta}}^{-1}\hat{\boldsymbol{\Lambda}})^{-1}$.

4.3. McDonald's Correlation-Preserving Factor Score Estimators

The covariance matrices of both $\mathbf{D}$ and $\mathbf{V}$ are not equal to $\text{cov}(\mathbf{F}) = \boldsymbol{\Psi}$. Factor score estimators with the covariance matrix equal to $\boldsymbol{\Psi}$ have been proposed by Anderson and Rubin (1956) for the special case of orthogonal factors, and by McDonald (1981) for the general case of oblique factors. ten Berge, Krijnen, Wansbeek, and Shapiro (1999) showed that the vector of factor scores proposed by McDonald (1981) satisfies $\mathbf{f}_3 = \mathbf{A}_3'(\mathbf{x} - \boldsymbol{\mu})$, where $\mathbf{A}_3' = \boldsymbol{\Psi}^{1/2}\{\boldsymbol{\Psi}^{1/2}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}^{1/2}\}^{-1/2}\boldsymbol{\Psi}^{1/2}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}$. The vector $\mathbf{f}_3$ can be seen as a realization of the random vector $\mathbf{Z} = \mathbf{A}_3'(\mathbf{X} - \boldsymbol{\mu})$ with mean vector $\mathbf{0}$ and covariance matrix $\boldsymbol{\Psi}$. The ith element of $\mathbf{Z}$ is the ith McDonald factor score estimator $Z_i = \boldsymbol{\alpha}_{i3}'(\mathbf{X} - \boldsymbol{\mu})$, where $\boldsymbol{\alpha}_{i3}'$ is the ith row of $\mathbf{A}_3'$. It follows from Eq. 5 that a lower bound to the reliability of Z_i is given by

$$\rho^2_{Z_iC} = \frac{\nu_{ii} + \tau_{ii}}{\psi_{ii}}, \tag{8}$$

where $\nu_{ii} + \tau_{ii} = \boldsymbol{\alpha}_{i3}'\boldsymbol{\Omega}\boldsymbol{\alpha}_{i3}$ is the ith diagonal element of $\mathbf{A}_3'\boldsymbol{\Omega}\mathbf{A}_3 = \boldsymbol{\Psi}\boldsymbol{\Lambda}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Lambda}\boldsymbol{\Psi}$ and $\psi_{ii} = \boldsymbol{\alpha}_{i3}'\boldsymbol{\Sigma}\boldsymbol{\alpha}_{i3}$ is the ith diagonal element of $\mathbf{A}_3'\boldsymbol{\Sigma}\mathbf{A}_3 = \boldsymbol{\Psi}$. The estimate of $\rho^2_{Z_iC}$ is $\hat{\rho}^2_{Z_iC} = (\hat{\nu}_{ii} + \hat{\tau}_{ii})/\hat{\psi}_{ii}$, where $\hat{\nu}_{ii} + \hat{\tau}_{ii}$ is the ith diagonal element of $\hat{\boldsymbol{\Psi}}\hat{\boldsymbol{\Lambda}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\Lambda}}\hat{\boldsymbol{\Psi}}$ and $\hat{\psi}_{ii}$ is the ith diagonal element of $\hat{\boldsymbol{\Psi}}$.

4.4. Equality Conditions

In the following theorem, conditions are given under which $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, and $\rho^2_{Z_iC}$ are equal.

Theorem 1. *If, under the general common factor model, both $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ are diagonal, then (a) $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, and $\rho^2_{Z_iC}$ are all equal to*

$$\delta_i = \frac{\psi_{ii}\sum_{j=1}^{k}\lambda^2_{ji}/\theta_{jj}}{1 + \psi_{ii}\sum_{j=1}^{k}\lambda^2_{ji}/\theta_{jj}}, \quad \textit{for } i = 1, \ldots, q, \tag{9}$$

where λ_{ji} is the loading of the jth item score on the ith common factor and $\theta_{jj} = var(U_j)$ is the variance of the jth unique factor, and (b) $\delta_1, \ldots, \delta_q$ are the nonzero eigenvalues of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$.

Proof. (a) Let $\boldsymbol{\Delta} = \boldsymbol{\Lambda}\boldsymbol{\Psi}^{1/2}$, so that $\boldsymbol{\Lambda}\boldsymbol{\Psi}\boldsymbol{\Lambda}' + \boldsymbol{\Theta} = \boldsymbol{\Delta}\boldsymbol{\Delta}' + \boldsymbol{\Theta}$. From Duncan's formula (Henderson & Searle, 1981), we have

$$(\boldsymbol{\Delta}\boldsymbol{\Delta}' + \boldsymbol{\Theta})^{-1} = \boldsymbol{\Theta}^{-1} - \boldsymbol{\Theta}^{-1}\boldsymbol{\Delta}(\boldsymbol{\Delta}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Delta} + \mathbf{I})^{-1}\boldsymbol{\Delta}'\boldsymbol{\Theta}^{-1}. \tag{10}$$

Premultiplying both sides of Eq. 10 by $\boldsymbol{\Delta}'$ and postmultiplying both sides of Eq. 10 by $\boldsymbol{\Delta}$ yields

$$\boldsymbol{\Delta}'(\boldsymbol{\Delta}\boldsymbol{\Delta}' + \boldsymbol{\Theta})^{-1}\boldsymbol{\Delta} = \mathbf{H} - \mathbf{H}(\mathbf{H} + \mathbf{I})^{-1}\mathbf{H}, \tag{11}$$

where $\mathbf{H} = \boldsymbol{\Delta}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Delta}$. Let $\boldsymbol{\Phi} = \mathbf{H} - \mathbf{H}(\mathbf{H} + \mathbf{I})^{-1}\mathbf{H}$. If both $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ are diagonal, then both $\mathbf{H}$ and $\boldsymbol{\Phi}$ are diagonal. If $\mathbf{H}$ is diagonal, then its ith diagonal element is given by $h_{ii} = \psi_{ii}\sum_{j=1}^{k}\lambda^2_{ji}/\theta_{jj}$. If $\boldsymbol{\Phi}$ is diagonal, then its ith diagonal element is given by $h_{ii} - h_{ii}(h_{ii} + 1)^{-1}h_{ii} = h_{ii}/(1 + h_{ii})$, which is equal to the right-hand side of Eq. 9. Coefficient $\rho^2_{D_iC}$ for the ith Thurstone factor score estimator is the ith diagonal element of

$\boldsymbol{\Psi}^{1/2}\boldsymbol{\Phi}\boldsymbol{\Phi}\boldsymbol{\Psi}^{1/2}$ divided by the ith diagonal element of $\boldsymbol{\Psi}^{1/2}\boldsymbol{\Phi}\boldsymbol{\Psi}^{1/2}$. If both $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are diagonal, then $\rho^2_{D_iC} = \left[\psi_{ii}^{1/2}\{h_{ii}^2/(1+h_{ii})^2\}\psi_{ii}^{1/2}\right]/\left[\psi_{ii}^{1/2}\{h_{ii}/(1+h_{ii})\}\psi_{ii}^{1/2}\right] = h_{ii}/(1+h_{ii})$. Coefficient $\rho^2_{V_iC}$ for the ith Bartlett factor score estimator is the ith diagonal element of $\boldsymbol{\Psi}$ divided by the ith diagonal element of $\boldsymbol{\Psi} + (\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda})^{-1}$. If both $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ are diagonal, then $\rho^2_{V_iC} = \psi_{ii}/(\psi_{ii} + 1/\sum_{j=1}^{k}\lambda_{ji}^2/\theta_{jj})$, which is also equal to the right-hand side of Eq. 9. Coefficient $\rho^2_{Z_iC}$ for the ith McDonald factor score estimator is the ith diagonal element of $\boldsymbol{\Psi}^{1/2}\boldsymbol{\Phi}\boldsymbol{\Psi}^{1/2}$ divided by the ith diagonal element of $\boldsymbol{\Psi}$. If both $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ are diagonal, then $\rho^2_{Z_iC} = \left[\psi_{ii}^{1/2}\{h_{ii}/(1+h_{ii})\}\psi_{ii}^{1/2}\right]/\psi_{ii} = h_{ii}/(1+h_{ii})$. This completes the proof of part (a). (b) The nonzero eigenvalues of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$ are equal to the nonzero eigenvalues of $\boldsymbol{\Phi}$ because $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}\boldsymbol{\Delta}'$ and $\boldsymbol{\Phi} = \boldsymbol{\Delta}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Delta}$. If both $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ are diagonal, then $\boldsymbol{\Phi}$ is diagonal and its eigenvalues are given by $\delta_1, \ldots, \delta_q$. This completes the proof of part (b). □

Since in exploratory factor analysis the initial unrotated solution is usually obtained under the conditions that $\boldsymbol{\Psi} = \mathbf{I}$ and $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ is diagonal, it follows from part (a) of Theorem 1 that for this initial solution the coefficients $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, and $\rho^2_{Z_iC}$ are all equal to δ_i, for all i. In addition, it follows from part (b) of Theorem 1 that for this initial solution $max\{\delta_1, \ldots, \delta_q\} = \delta = \boldsymbol{\beta}'\boldsymbol{\Omega}\boldsymbol{\beta}/\boldsymbol{\beta}'\boldsymbol{\Sigma}\boldsymbol{\beta}$, where $\boldsymbol{\beta}$ is the eigenvector of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$ corresponding to the largest eigenvalue of $\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}$. Since in the one-factor model both $\boldsymbol{\Psi}$ and $\boldsymbol{\Lambda}'\boldsymbol{\Theta}^{-1}\boldsymbol{\Lambda}$ are scalars, it also follows from Theorem 1 that under the one-factor model, the coefficients $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, and $\rho^2_{Z_iC}$ are all equal to δ.

It is important to note that the coefficients $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, and $\rho^2_{Z_iC}$ are in general not invariant under factor rotation. To see this, let $\mathbf{f}_0$ be either $\mathbf{f}_1$, $\mathbf{f}_2$, or $\mathbf{f}_3$, let $\mathbf{A}$ be either $\mathbf{A}_1$, $\mathbf{A}_2$, or $\mathbf{A}_3$, let $\rho^2_{F_iC}$ be either $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, or $\rho^2_{Z_iC}$, and let $\mathbf{M}$ be a $q \times q$ invertible transformation matrix. It is a well-known fact that the rotated vector of factor scores is given by $\mathbf{M}^{-1}\mathbf{f}_0 = \mathbf{M}^{-1}\mathbf{A}'(\mathbf{X} - \boldsymbol{\mu})$. This means that the coefficient $\rho^2_{F_iC}$ is obtained by dividing the ith diagonal element of $\mathbf{M}^{-1}\mathbf{A}'\boldsymbol{\Omega}\mathbf{A}(\mathbf{M}^{-1})'$ by the ith diagonal element of $\mathbf{M}^{-1}\mathbf{A}'\boldsymbol{\Sigma}\mathbf{A}(\mathbf{M}^{-1})'$. So the value of $\rho^2_{F_iC}$ for the ith factor score estimator calculated before rotation is in general not the same as the value of $\rho^2_{F_iC}$ calculated after rotation. It is therefore recommended to calculate the estimates of the coefficients $\rho^2_{D_iC}$, $\rho^2_{V_iC}$, and $\rho^2_{Z_iC}$ after rotation. To do this, the presented formulas can be used where $\hat{\boldsymbol{\Lambda}}$ and $\hat{\boldsymbol{\Psi}}$ are the rotated matrices of estimated factor loadings and estimated factor covariances.

5. Factor Model Selection

The estimated value of any factor model-based lower bound to the reliability of a linear combination of the item scores depends on the specified number of common factors in the factor model used. In general, any lower bound estimate increases with the specified number of common factors. This means that when a lower bound estimate is unsatisfactory, a new factor can be added to the model to obtain a higher estimate. In principle, this can be repeated until the number of factors reaches its maximum, which is equal to the number of items. In the extreme and unrealistic situation where as many common factors are specified as the number of items, all unique variances are zero and the estimate of any factor model-based lower bound to the reliability of a linear combination of the item scores is one. Therefore, in assessing the reliability of any linear combination of item scores by means of a factor model-based lower bound coefficient, it is essential to use an interpretable common factor model that fits well to the data.

The ideal measurement situation in practice is the situation in which an available confirmatory factor model fits well to the data. When there is no confirmatory factor model available or when a confirmatory factor model does not fit well to the data, exploratory factor analysis can be used to find an appropriate model. However, exploratory factor analysis will not always yield an acceptable model. Even in the case of the overidentified model with the largest possible number of common factors, the fit to the data might be poor. For this situation, it is recommended to choose a model from the set of saturated common factor models with the smallest possible number of factors. Such a factor model is recommended because its interpretation and specification are in general simpler than those of other not underidentified models, and its selection prevents the artificial increase of the reliability lower bound estimate by adding common factors to the model when previous lower bound estimates are unsatisfactory.

To be able to specify a saturated common factor model with the smallest possible number of factors, this number of factors must determined. In the following theorem, the number of factors of a saturated model with the smallest possible number of factors is given for a fixed number of items.

Theorem 2. *Given a fixed k number of item scores, the number of factors in a saturated common factor model with the smallest possible number of factors is the ceiling given by*

$$q_c = min\left\{q \in \mathbb{Z} \,|\, q \geq k + \tfrac{1}{2} - \sqrt{2k + \tfrac{1}{4}}\right\}, \tag{12}$$

where $\mathbb{Z}$ is the set of all integers.

Proof. In the general common factor model, the total number of parameters is $kq+q(q+1)/2+k$. The number of restrictions needed for identification is q^2. Consequently, the number of estimable parameters is $kq - q(q-1)/2 + k$. Since $k(k+1)/2$ is the maximum number of estimable parameters, the degrees of freedom are given by $df = k(k-1)/2 + q(q-1)/2 - kq$. A necessary condition for a factor model to be saturated is the condition that $df = 0$. Multiplying out and collecting powers of q yields the quadratic function $df = \frac{1}{2}q^2 - \left(k + \frac{1}{2}\right)q + \frac{1}{2}k(k-1)$ of q given a fixed k. Since the coefficient of q^2 is $\frac{1}{2}$, the function has a minimum. Since the discriminant is $2k + \frac{1}{4} > 0$, for all k, the function has two roots. The smallest root is given by $q_0 = k + \frac{1}{2} - \sqrt{2k + \frac{1}{4}}$. If q_0 is a positive integer, then q_0 is the number of factors. If q_0 is a positive real number, then the nearest positive integer larger than q_0 is the number of factors. In general, the number of factors is given by the ceiling of q_0. This completes the proof. □

If q_0 is a positive real number, then $q_c > q_0$ and $df = k(k-1)/2 + q_c(q_c-1)/2 - kq_c < 0$. In this case, the model is not identified. The number of additional restrictions then needed to obtain a saturated model is given by $-df = kq_c - k(k-1)/2 - q_c(q_c-1)/2$. In Table 1, q_c and $-df$ are given for $k = 1, 2, \ldots, 100$.

There are many ways to specify a saturated common factor model with the smallest possible number of factors. One possible way is to set the $q_c \times q_c$ matrix $\mathbf{\Psi}$ equal to $\mathbf{I}$, the factor loadings λ_{ji}, for $j = 1, \ldots, q_c - 1$ and $i = j+1, \ldots, q_c$, equal to zero, and $kq_c - k(k-1)/2 - q_c(q_c-1)/2$ other factor loadings equal to zero.

As an example, consider the situation of four item scores and two common factors. If $\mathbf{\Psi} = \mathbf{I}$ and $\mathbf{\Theta}$ is diagonal, then two different saturated common factor models are given by the following two matrices of factor loadings:

$$\begin{bmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \\ \lambda_{31} & \lambda_{32} \\ \lambda_{41} & \lambda_{42} \end{bmatrix} \text{ and } \begin{bmatrix} \lambda_{11} & 0 \\ \lambda_{21} & \lambda_{22} \\ 0 & \lambda_{32} \\ \lambda_{41} & \lambda_{42} \end{bmatrix}.$$

TABLE 1.
The smallest possible number of factors q_c for a saturated factor model as a function of k, and the necessary number of additional identification restrictions $-df$.

k	q_c	$-df$	k	q_c	$-df$	k	q_c	$-df$	k	q_c	$-df$	k	q_c	$-df$
1	0	0	21	15	0	41	33	5	61	51	6	81	69	3
2	1	1	22	16	1	42	34	6	62	52	7	82	70	4
3	1	0	23	17	2	43	35	7	63	53	8	83	71	5
4	2	1	24	18	3	44	36	8	64	54	9	84	72	6
5	3	2	25	19	4	45	36	0	65	55	10	85	73	7
6	3	0	26	20	5	46	37	1	66	55	0	86	74	8
7	4	1	27	21	6	47	38	2	67	56	1	87	75	9
8	5	2	28	21	0	48	39	3	68	57	2	88	76	10
9	6	3	29	22	1	49	40	4	69	58	3	89	77	11
10	6	0	30	23	2	50	41	5	70	59	4	90	78	12
11	7	1	31	24	3	51	42	6	71	60	5	91	78	0
12	8	2	32	25	4	52	43	7	72	61	6	92	79	1
13	9	3	33	26	5	53	44	8	73	62	7	93	80	2
14	10	4	34	27	6	54	45	9	74	63	8	94	81	3
15	10	0	35	28	7	55	45	0	75	64	9	95	82	4
16	11	1	36	28	0	56	46	1	76	65	10	96	83	5
17	12	2	37	29	1	57	47	2	77	66	11	97	84	6
18	13	3	38	30	2	58	48	3	78	66	0	98	85	7
19	14	4	39	31	3	59	49	4	79	67	1	99	86	8
20	15	5	40	32	4	60	50	5	80	68	2	100	87	9

In both cases, the model is identified and the number of parameters to be estimated (six factor loadings and four unique variances) is equal to the maximum number of estimable parameters.

6. A Comparison of Lower Bounds

To get an indication of the relative performance of the proposed reliability lower bound coefficients for the different factor score estimators, estimates of these coefficients have been calculated for the same nine classic example data sets used by Revelle and Zinbarg (2009). The first six data sets have been taken from Sijtsma (2009) and are called S1, S2, S3, S4, S5, and S6. S1 consists of the scores of 828 respondents on eight rating scale items. S2 and S3 are independent subsets of S1 and each of these two data sets consists of the scores on four items. S4, S5, and S6 are three artificially created covariance matrices for six indicators each. The seventh data set has been taken from Lord and Novick (1968) and is called LN. LN is the covariance matrix that belongs to the scores of 1416 respondents on four items that measure English as a foreign language. The eighth data set has been taken from Warner, Meeker, and Eels (1960) and is called WM. WM is the correlation matrix that belongs to the scores on six indicators of social class. The ninth data set has been taken from De Leeuw (1983) and is called DL. DL is the correlation matrix that belongs to the scores on six political survey items.

Revelle and Zinbarg (2009) analyzed each of the nine data sets using a higher order factor analysis with a Schmid–Leiman transformation (Schmid and Leiman, 1957) and the subsequent estimation of the general factor saturation. They reported the estimates of ω calculated with the estimates from the higher order factor analyses and compared these estimates to the values of the glb and to the estimates of other lower bound coefficients for the reliability of the total score.

TABLE 2.
Estimates of the well-known reliability lower bounds λ_2, λ_3, λ_4, λ_6, μ_3, and the glb, and estimates of the factor model-based reliability lower bounds ω and δ calculated with the estimates from regular exploratory factor analysis.

Reliability lower bound	Data set								
	S1	S2	S3	S4	S5	S6	LN	WM	DL
λ_2	.789	.753	.655	.643	.585	$\underline{.533}$	.898	.943	.842
λ_3	.785	.748	.652	.533	.533	$\underline{.533}$	.894	.942	.840
λ_4	.849	$\underline{.820}$	$\underline{.696}$	$\underline{.889}$	.593	$\underline{.533}$	.909	.969	.883
λ_6	.785	.713	.593	.800	.571	.488	.880	.960	.830
μ_3	.791	.755	.658	.666	.592	$\underline{.533}$	.900	.943	.843
glb	.852	$\underline{.820}$	.683	$\underline{.889}$	.647	$\underline{.533}$	.920	.976	$\underline{.885}$
ω	.844	.755	.644	$\underline{.889}$	$\underline{.667}$	$\underline{.533}$	.900	.972	.871
δ	$\underline{.995}$	.807	.689	$\underline{.889}$	$\underline{.667}$	$\underline{.533}$	$\underline{.930}$	$\underline{.998}$	.883

For each data set, the greatest estimate is underlined.

Revelle and Zinbarg (2009) concluded that the estimate of ω is systematically higher than the value of the glb. In the present study, however, regular exploratory factor analyses have been carried out on the data sets. Assuming multivariate normality of the item scores, estimates of the parameters have been obtained by the method of maximum likelihood. The results of these exploratory factor analyses show that the one-factor model fits satisfactorily well to S3 and LN, and perfectly to S6. The two-factor model fits well to DL and perfectly to S5. Since the one-factor model does not fit to S2, a saturated two-factor model has been selected for S2 in the way described in the preceding section. The three-factor model fits reasonably well to S1 and perfectly to S4. Since the two-factor model does not fit well to WM, the saturated three-factor model has been selected for WM.

Although the main interest is in the relative performance of the reliability lower bound estimates for the different factor score estimators, it is also interesting to see whether the estimates of ω and δ calculated with the estimates from the regular exploratory factor analyses are systematically higher than the values of the glb and other well-known lower bound coefficients for the reliability of the total score. The other well-known lower bound coefficients of interest are Guttman's λ_2, λ_3 (coefficient alpha), λ_4, λ_6, and μ_3 (ten Berge & Zegers, 1978). The estimates of ω and δ have been calculated using self-written R code (R Core Team, 2015). The values of the glb and the other well-known reliability lower bound coefficients have been calculated using the R package **psych** (Revelle, 2014). The results are shown in Table 2.

The results in Table 2 show that for six of the nine data sets the estimate of δ takes on the highest value. For S1, LN, and WM, the estimate of δ is greater than the estimates of all other coefficients. For all data sets, the estimate of δ is greater than or equal to the estimates of λ_2, λ_3, λ_6, μ_3, and ω. For four of the nine data sets, the estimates of the glb and λ_4 take on the highest value. Only for S3 the estimate of λ_4 is greater than all other estimates, and only for DL the estimate of the glb is greater than all other estimates. For S2, the estimates of λ_4 and the glb are equal and their common value is greater than the estimates of all other coefficients. For all data sets, the estimates of the glb and λ_4 are greater than or equal to the estimates of λ_2, λ_3, λ_6, and μ_3. For S1, S3, S5, LN, and WM, the estimate of δ is greater than the value of the glb, for S4 and S6 the estimates of δ and the glb are equal, and for S2 and DL the estimate of δ is less than the value of the glb. For S1, S5, LN, and WM, the estimate of δ is greater than the estimate of λ_4, for S4, S6, and DL the estimates of δ and λ_4 are equal, and for S2 and S3 the estimate of δ is less than the estimate of λ_4. Only for S5 the estimate of ω is greater than the estimate of the glb,

TABLE 3.
Estimates of reliability lower bounds (communalities) of factor score estimators before factor rotation.

Factor	Estimator	Communality	Data set								
			S1	S2	S3	S4	S5	S6	LN	WM	DL
1	Thurstone	$\rho^2_{D_1C}$	.995	.700	.689	.889	.667	.533	.930	.998	.883
	Bartlett	$\rho^2_{V_1C}$	.995	.684	.689	.889	.667	.533	.930	.998	.883
	McDonald	$\rho^2_{Z_1C}$	.995	.692	.689	.889	.667	.533	.930	.998	.883
2	Thurstone	$\rho^2_{D_2C}$	.853	.762		.889	.667			.988	.611
	Bartlett	$\rho^2_{V_2C}$	.853	.745		.889	.667			.988	.611
	McDonald	$\rho^2_{Z_2C}$	.853	.754		.889	.667			.988	.611
3	Thurstone	$\rho^2_{D_3C}$	.577			.889				.947	
	Bartlett	$\rho^2_{V_3C}$	.577			.889				.947	
	McDonald	$\rho^2_{Z_3C}$	.577			.889				.947	

For each data set and each factor, the greatest estimate is underlined.

for S4 and S6 the estimates of ω and the glb are equal, and for S1, S2, S3, LN, WM, and DL, the estimate of ω is less than the estimate of the glb.

Next, the estimates of the lower bounds to the reliabilities of the factor score estimators before factor rotation have been calculated using self-written R code (R Core Team, 2015). The results are shown in Table 3.

For all data sets except S2, the values of the lower bound estimates in Table 3 are the same for the different factor score estimators. The explanation for this is given by the proof of Theorem 1. For S2, the values of the lower bound estimates are not the same for the different factor score estimators because in the fitted saturated factor model $\mathbf{\Lambda}'\mathbf{\Theta}^{-1}\mathbf{\Lambda}$ is not diagonal. The results for S2 show that the lower bound estimates to the reliabilities of Thurstone's factor score estimators are greater than those for Bartlett's and McDonald's factor score estimators and that the estimates for McDonald's factor score estimators are greater than those for Bartlett's factor score estimators.

Finally, the estimates of the lower bounds to the reliabilities of the factor score estimators after oblimin rotation have been calculated using self-written R code (R Core Team, 2015). The results are shown in Table 4.

The results in Table 4 show that the estimates of the lower bounds to the reliabilities of Thurstone's factor score estimators are greater than or equal to the estimates of the lower bounds to the reliabilities of Bartlett's and McDonald's factor score estimators. In addition, the estimates of the lower bounds to the reliabilities of McDonald's factor score estimators are greater than or equal to the estimates of the lower bounds to the reliabilities of Bartlett's factor score estimators. For S1 and WM, the estimates of the lower bounds to the reliabilities of the different factor score estimators seem to be equal but differences are found at higher decimal places.

7. Discussion and Conclusion

In this paper, lower bounds to the reliabilities of the elements of the random vector of Thurstone factor score estimators $\mathbf{D}$, the random vector of Bartlett factor score estimators $\mathbf{V}$, and the random vector of McDonald factor score estimators $\mathbf{Z}$ have been derived. These elements are, however, not exactly the factor score estimators used in practice. In practice, factor score estima-

TABLE 4.
Estimates of reliability lower bounds (communalities) of factor score estimators after oblimin rotation.

Factor	Estimator	Communality	Data set								
			S1	S2	S3	S4	S5	S6	LN	WM	DL
1	Thurstone	$\rho^2_{D_1C}$	$\underline{.995}$	$\underline{.711}$	$\underline{.689}$	$\underline{.889}$	$\underline{.667}$	$\underline{.533}$	$\underline{.930}$	$\underline{.993}$	$\underline{.849}$
	Bartlett	$\rho^2_{V_1C}$	.995	.693	$\underline{.689}$	$\underline{.889}$	$\underline{.667}$	$\underline{.533}$	$\underline{.930}$	.993	.820
	McDonald	$\rho^2_{Z_1C}$	.995	.701	$\underline{.689}$	$\underline{.889}$	$\underline{.667}$	$\underline{.533}$	$\underline{.930}$	.993	.836
2	Thurstone	$\rho^2_{D_2C}$	$\underline{.875}$	$\underline{.791}$		$\underline{.889}$	$\underline{.667}$			$\underline{.994}$	$\underline{.830}$
	Bartlett	$\rho^2_{V_2C}$	.860	.783		$\underline{.889}$	$\underline{.667}$			.994	.792
	McDonald	$\rho^2_{Z_2C}$	.868	.787		$\underline{.889}$	$\underline{.667}$			.994	.813
3	Thurstone	$\rho^2_{D_3C}$	$\underline{.756}$			$\underline{.889}$				$\underline{.978}$	
	Bartlett	$\rho^2_{V_3C}$	.684			$\underline{.889}$				.978	
	McDonald	$\rho^2_{Z_3C}$	.719			$\underline{.889}$				.978	

For each data set and each factor, the greatest estimate is underlined.

tors with realizations $\hat{\mathbf{f}}_1 = \hat{\mathbf{A}}_1(\mathbf{x} - \bar{\mathbf{x}})$, $\hat{\mathbf{f}}_2 = \hat{\mathbf{A}}_2(\mathbf{x} - \bar{\mathbf{x}})$, and $\hat{\mathbf{f}}_3 = \hat{\mathbf{A}}_3(\mathbf{x} - \bar{\mathbf{x}})$ are used, where the estimates $\hat{\boldsymbol{\Lambda}}$, $\hat{\boldsymbol{\Psi}}$, $\hat{\boldsymbol{\Theta}}$, and $\bar{\mathbf{x}}$ are taken as the true values of $\boldsymbol{\Lambda}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Theta}$, and $\boldsymbol{\mu}$. Under general regularity conditions, however, maximum likelihood estimates of $\boldsymbol{\Lambda}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Theta}$, and $\boldsymbol{\mu}$ are known to converge in probability to the true parameter values. Therefore, in using maximum likelihood estimates, it can be expected that the difference between a particular reliability lower bound estimate and the lower bound to the reliability of the corresponding estimator actually used in practice will become smaller when the sample size increases. In practice, maximum likelihood estimates are usually obtained under the assumption that the item scores have a multivariate normal distribution.

The decision to use either an overall test score or a factor score estimator should not be based on a comparison of reliability lower bound estimates. This decision should be based on a comparison of the validity of these scores for the purpose of measurement. However, the decision to use either Thurstone's, Bartlett's, or McDonald's factor score estimators should to some extent be based on a comparison of reliability lower bound estimates. Fortunately, the general common factor model provides a means to compare reliability lower bound estimates of different linear combinations of the item scores. The results in Tables 3 and 4 show that the reliability lower bound estimates for Thurstone's factor score estimators are at least as high as the estimates for Bartlett's and McDonald's factor score estimators. Therefore, Thurstone's factor score estimators can be recommended unless it is desirable to preserve the covariances among the factors. Then, of course McDonald's factor score estimators are recommended.

References

Anderson, R. D., & Rubin, H. (1956). Statistical inference in factor analysis. In *Proceedings of the third Berkeley symposium of mathematical statistics and probability* (Vol. 5, pp. 111–150).

Bartlett, M. S. (1937). The statistical conception of mental factors. *British Journal of Psychology*, *28*, 97–104.

Bentler, P. M. (2009). Alpha, dimension-free, and model-based internal consistency reliability. *Psychometrika*, *74*, 137–143.

Cronbach, L. J. (1951). Coefficient alpha and the internal structure of tests. *Psychometrika*, *16*, 297–334.

De Leeuw, J. (1983). Models and methods for the analysis of correlation coefficients. *Journal of Econometrics*, *22*(1–2), 113–137.

Grice, J. W. (2001). Computing and evaluating factor scores. *Psychological Methods*, *6*, 430–450.

Guttman, L. (1945). A basis for analyzing test-retest reliability. *Psychometrika*, *10*(4), 255–282.

Heise, D. R., & Bohrnstedt, G. W. (1970). Validity, invalidity, and reliability. In E. F. Borgatta (Ed.), *Sociological methodology 1970* (pp. 104–129). San Francisco: Jossey-Bass.

Henderson, H. V., & Searle, S. R. (1981). On deriving the inverse of a sum of matrices. *SIAM Review*, *23*, 53–60.
Lord, F. M., & Novick, M. R. (1968). *Statistical theories of mental test scores*. Reading, MA: Addison-Wesley.
McDonald, R. P. (1978). Generalizability in factorable domains: "Domain validity and generalizability": 1. *Educational and Psychological Measurement*, *38*, 75–79.
McDonald, R. P. (1981). Constrained least squares estimators of oblique common factors. *Psychometrika*, *46*, 337–341.
McDonald, R. P. (1999). *Test theory: A unified treatment*. Hillsdale, MI: Erlbaum.
R Core Team (2015). R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. http://www.R-project.org/.
Revelle, W. (2014). psych: Procedures for personality and psychological research (R package version 1.4.3).
Revelle, W., & Zinbarg, R. E. (2009). Coefficients alpha, beta, omega, and the glb: Comments on Sijtsma. *Psychometrika*, *74*, 145–154.
Schmid, J. J., & Leiman, J. M. (1957). The development of hierarchical factor solutions. *Psychometrika*, *22*(1), 83–90.
Sijtsma, K. (2009). On the use, the misuse, and the very limited usefulness of Cronbach's alpha. *Psychometrika*, *74*, 107–120.
ten Berge, J. M. F., Krijnen, W., Wansbeek, T., & Shapiro, A. (1999). Some new results on correlation-preserving factor scores prediction methods. *Linear Algebra and Its Applications*, *289*, 311–318.
ten Berge, J. M. F., Snijders, T. A. B., & Zegers, F. E. (1981). Computational aspects of the greatest lower bound to reliability and constrained minimum trace factor analysis. *Psychometrika*, *46*, 201–213.
ten Berge, J. M. F., & Sočan, G. (2004). The greatest lower bound to the reliability of a test and the hypothesis of unidimensionality. *Psychometrika*, *69*, 613–625.
ten Berge, J. M. F., & Zegers, F. E. (1978). A series of lower bounds to the reliability of a test. *Psychometrika*, *34*(4), 575–579.
Thompson, B. (1993). Calculation of standardized, noncentered factor scores: An alternative to conventional factor scores. *Perceptual and Motor Skills*, *77*(3), 1128–1130.
Thomson, G. H. (1946). *The factorial analysis of human ability* (2nd ed.). New York: Houghton Mifflin.
Thurstone, L. L. (1935). *The vectors of mind*. Chicago: University of Chicago Press.
Warner, W. L., Meeker, & Eels, (1960). *Social class in America: A manual of procedure for the measurement of social status*. New York: Harper.
Woodhouse, B., & Jackson, P. H. (1977). Lower bounds for the reliability of the total score on a test composed of non-homogeneous items: II: A search procedure to locate the greatest lower bound. *Psychometrika*, *42*, 579591.
Zinbarg, R. E., Revelle, W., Yovel, I., & Li, W. (2005). Cronbach's α, Revelles β, and McDonalds ω_H: their relations with each other and two alternative conceptualizations of reliability. *Psychometrika*, *70*, 123–133.

Manuscript Received: 3 JUL 2015
Final Version Received: 20 JAN 2016
Published Online Date: 6 OCT 2016