A Hexagon-Shaped Stable Kissing Unit Disk Tree

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1 Introduction

A *disk contact graph* is a graph that can be represented by a set of interior-disjoint disks in the plane, where each disk represents a vertex and an edge between two disks exists if and only if the disks touch (or *kiss*). Many studies have been conducted to classify the types of graphs that can be represented as disk contact graphs as well as to design algorithms to find a set of disks that represent the graph (or to determine if this is even possible). A fundamental results in this area is Koebe's theorem, which states that every planar graph can be represented as a contact graph of disks [5].

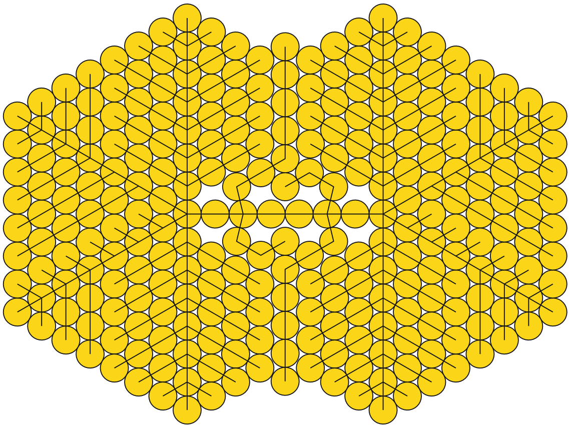
The same question can be asked for unit disk graphs (or *coin graphs)*. Breu and Kirkpatrick show that recognizing unit disk graphs is NP-hard [2]. Bowen et al. [1] study the problem of recognizing unit disk *trees*, and show that it is NP-hard to determine if a given tree can be represented as a unit-disk contact graph with a given embedding—that is, given the cyclic order of neighbors around each vertex. They claim the main obstacle in proving NP-hardness of unit disk tree recognition is creating a *stable* tree for which all embeddings are similar so the embedding can be used as a building block for an NP-hardness reduction.

A graph G is *ε-stable* if for any two embeddings of G as contact graphs of unit disks, there is a rigid transformation of one such that there is a matching between the resulting embeddings where the distance between matched disks is at most ε. Note that although each disk represents a tree-node, its matched node may not be the same node in the tree. Here, we show that arbitrarily large ε-stable trees exist.

2 The Construction

Since the embedding is free, our strategy will be to create a configuration that is stable because of its density: the circles will be so tightly packed, that there is simply no room to significantly change the embedding.

Y. Hu and M. Nöllenburg (Eds.): GD 2016, LNCS 9801, pp. 628–630, 2016.
DOI: 10.1007/978-3-319-50106-2



Tóth proved that a hexagonal lattice is the densest of all possible plane packings [6] (see also Chang and Wang [3]). This suggests that a hexagon-shaped graph, consisting of a central hub and six "feathers" growing out of it, would be very stable. However, we cannot use it, because its dual would not be a tree. We need enough room to move the disks slightly such as to make sure the dual graph is a tree.

To do this, we place *two* disks (the *hubs*) at distance d from each other, and center two hexagonal circle packings around them. If we choose d to be an integer which is slightly larger than an integer multiple of $\sqrt{3}$, the two packings will almost, but not quite, fit each other. Then we connect the hubs with a straight path to make sure they cannot drift further than d from each other. We then fill the space in the heart of the construction with as many disks as fit.

Concretely, we choose $d = 7$. Our construction (see above) consists of:

- two *hub*s: vertices of degree 5, connected each other by a *spine* of length 7;
- two times four *feathers*, connected to the hubs;
- two *arms*: paths of length k, connected to the 2nd and 5th spine-vertex;
- two *stubs*: paths of length 3, also connected to the 2nd and 5th spine-vertex.

Here, a *feather* consists of a *shaft* of length k: a path of mostly degree 4 vertices, with two *barbs* connected to each vertex of the shaft. A barb is a path of degree 2 vertices ending in a degree 1 vertex; the lengths of the barbs may vary. Due to space constraints, we sketch the main steps of the proof of stability.

- We show that it is possible to perturb the disks slightly so that their contact graph is indeed the correct tree.
- We show that the hubs must be placed at distance at least $4\sqrt{3}$ (otherwise, there is not enough space to fit all the disks).
- We show that in any valid embedding of G, all shaft vertices must be straight: they must have their barbs attached on opposite sides.
- We show that we cannot make enough room for any path to be compressed by "zigzagging" (because $7 - 4\sqrt{3}$ is smaller than $1 - \frac{1}{2}\sqrt{3}$).

We expect the ε-stable trees can be used to create a so-called logic engine [4] to show NP-hardness of unit disk tree recognition.

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