On topological models of **GLP**

Lev Beklemishev*, Guram Bezhanishvili, Thomas Icard

July 31, 2009

Abstract

We develop topological semantics of a polymodal provability logic **GLP**. Our main result states that the bimodal fragment of **GLP**, although incomplete with respect to relational semantics, is topologically complete. The topological (in)completeness of **GLP** remains an interesting open problem.

1 Introduction

In this paper we initiate a study of topological models of an important polymodal provability logic **GLP** due to Japaridze [21, 22]. This system describes in the style of provability logic all the universally valid schemata for the reflection principles of restricted logical complexity in arithmetic. Thus, it is complete with respect to a very natural kind of arithmetical semantics.

The logic **GLP**, and its restricted bimodal version **GLB**, have been extensively studied in the early 1990s by Ignatiev [19, 20] and Boolos, who simplified and extended Japaridze's work. Boolos incorporated a very readable treatment of **GLB** into his popular book on provability logic [11]. More recently, interesting applications of **GLP** have been found in proof theory and ordinal analysis of arithmetic. In particular, **GLP** gives rise to a natural system of ordinal notation for the ordinal ϵ_0. Based on this system and the use of **GLP**, the first author of this paper gave a simple proof of consistency of Peano Arithmetic à la Gentzen and formulated a new independent combinatorial principle. This stimulated further interest towards **GLP** (see [3, 4] for a detailed survey).

The main difficulty in the modal-logical study of **GLP** comes from the fact that it is incomplete with respect to its relational semantics; that is, **GLP** is the logic of no class of *frames*. On the other hand, a suitable class of relational *models* for which **GLP** is sound and complete was developed in [5]. However, these models are sufficiently complicated to warrant a search for an alternative and simpler kind of semantics.

*Supported by the Russian Foundation for Basic Research (RFBR), Dutch Organization for Scientific Research (NWO), Russian Presidential Council for Support of Leading Scientific Schools, and the ECO-NET cooperation project "Computational interpretations of modal logic."

Many standard modal logics enjoy a natural topological interpretation. Topologically, propositions are interpreted as subsets of a topological space and boolean connectives correspond to the standard set-theoretic operations. For logics containing the reflection axiom $\varphi \to \Diamond\varphi$, one usually interprets the modal $\Diamond$ as the closure operator of a topological space. However, provability logics fall outside this class due to the presence of Löb's axiom which contradicts reflection. For these logics one takes a different approach that reads $\Diamond$ as the derived set operator d mapping a set A to the set of limit points of A. The study of this interpretation was suggested in the Appendix of [25], and was developed by Esakia (see [13, 14] and [7] for a survey). In particular, Esakia noticed that a topological space satisfies Löb's axiom iff it is *scattered*. The concept of a scattered space goes all the way back to Cantor. Typical examples of scattered spaces are ordinals (in the interval topology). In fact, it was shown independently by Abashidze [2] and Blass [10] that the provability logic **GL** is complete with respect to any ordinal $\alpha \geq \omega^\omega$.

When generalizing topological interpretation to several modalities we deal with *polytopological spaces*; that is, sets equipped with several topologies τ_0, τ_1, ... The corresponding derived set operators d_0, d_1, ... then interpret the diamond modalities $\langle 0 \rangle$, $\langle 1 \rangle$, ... of our language in the usual way. The axioms of **GLP** impose restrictions on the relevant class of polytopological spaces, which leads to the concept of a **GLP**-*space* (or, of a **GLB**-*space* for the language with just two modalities).

It is well known that **GL** is complete with respect to its relational semantics; in fact, **GL** is the logic of finite irreflexive transitive trees (see, e.g., [11]). In contrast, **GLB** is incomplete with respect to its relational semantics. But the main result of this paper states that **GLB** is topologically complete. Thus, **GLB** appears to be the first naturally occurring example of a modal logic which is topologically complete but incomplete with respect to its relational semantics (artificial examples of this kind have already been known; see, e.g., [15, 16]). It is also worth pointing out that in [26] it was stated as an open problem whether there existed a topologically complete but relationally incomplete finitely axiomatizable modal logic. The question was stated for the case of modal logics with one modality and in this stronger form it still remains open. Nevertheless, since **GLB** is finitely axiomatizable, our results provide an answer to the bimodal version of the problem.

Our technique (which is based on the construction in [5]) does not obviously extend to the case with three or more modalities. Therefore, the topological completeness of **GLP** remains an interesting open problem. At the end of the paper we discuss some negative results indicating that the situation here could be significantly more complicated and the question of topological completeness of **GLP** might be independent of the axioms of Zermelo–Fraenkel set theory ZFC with the axiom of choice.

On the other hand, the third author of this paper established the topological completeness of the closed fragment of **GLP** (in the language with ω-many modalities) with respect to a natural polytopological space on the ordinal ϵ_0 (see [17, 18]). However, this space fails to be a **GLP**-space.

The paper is organized as follows. In Section 2 we introduce **GLP** and its bimodal fragment **GLB**, and discuss their relational, topological, and algebraic semantics. We also discuss Stone-like duality for **GLP**-algebras and the resulting descriptive frames. In Section 3 we prove topological completeness of **GLB** with respect to the class of **GLB**-spaces. We finish the paper with a discussion of some further results and remaining open questions.

2 Relational, topological, and algebraic semantics of GLP

2.1 GLP and its relational semantics

GLP is a propositional modal logic formulated in a language with infinitely many modalities [0], [1], [2], ... As usual, $\langle n\rangle\varphi$ stands for $\neg[n]\neg\varphi$.

Definition 2.1. **GLP** is given by the following axiom schemata and rules.

Axioms:

(i) Boolean tautologies;

(ii) $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;

(iii) $[n]([n]\varphi \to \varphi) \to [n]\varphi$ (Löb's axiom);

(iv) $[m]\varphi \to [n]\varphi$, for $m < n$;

(v) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, for $m < n$.

Rules:

(i) $\vdash \varphi,\ \vdash \varphi \to \psi\ \Rightarrow \vdash \psi$ (modus ponens);

(ii) $\vdash \varphi\ \Rightarrow \vdash [n]\varphi$, for each $n \in \omega$ (necessitation).

In other words, for each modality we have the Gödel-Löb Logic **GL**, and (iv) and (v) are the two axioms relating modalities to one another.

We denote by **GLB** the bimodal fragment of **GLP**, restricted to the language with only [0] and [1], and by $\mathbf{GLP}_0$ the letterless fragment of **GLP**, restricted to the language without variables (we assume propositional constants $\top$ and $\bot$ to be part of the language).

As usual, we would like to know what class of frames, if any, these logics define. Relational models of $\mathbf{GLP}_0$ have been studied extensively, first in [19] and [20], and later in [6]; see also [17, 18]. Unfortunately, for the fragments with variables, and already in the case of **GLB**, there is no single non-trivial frame for which we have soundness. To see this, we briefly recall relational semantics for **GL**.

A *(unimodal) frame* is a pair $\mathfrak{F} = \langle W, R\rangle$, where W is a nonempty set and R is a binary relation on W; $\mathfrak{F}$ is *transitive* if $wRvRu$ implies wRu for each $w, v, u \in W$ and *irreflexive* if wRw for no $w \in W$; a transitive frame $\mathfrak{F}$ is *dually well-founded* if for each nonempty subset U of W there exists $w \in U$ such that

wRu for no $u \in U$. In such a case we call R a *dually well-founded relation*. It is well known that $\mathfrak{F} \vDash \mathbf{GL}$ iff $\mathfrak{F}$ is dually well-founded. Typical examples of dually well-founded frames are finite transitive irreflexive frames, and in fact, **GL** is the logic of these (see, e.g., [11]).

Next we recall that a *(polymodal) frame* is a tuple $\mathfrak{F} = \langle W, \{R_n\}_{n\in\omega}\rangle$, where W is a nonempty set and each R_n is a binary relation on W. For $A \subseteq W$ let $-A$ denote the complement of A in W. We recall that a *valuation* is a map $v : \mathrm{Var} \to 2^W$ from the set of propositional variables to the powerset of W and that v extends to all formulas as follows:

- $v(\varphi \vee \psi) = v(\varphi) \cup v(\psi)$, $v(\neg\varphi) = -v(\varphi)$, $v(\top) = W$, $v(\bot) = \varnothing$,
- $v(\langle n\rangle\varphi) = \{x \in W : \exists y\, (xR_n y \;\&\; y \in v(\varphi))\}$
- $v([n]\varphi) = \{x \in W : \forall y\, (xR_n y \Rightarrow y \in v(\varphi))\}$.

We will write $\mathfrak{F}, x \vDash_v \varphi$ for $x \in v(\varphi)$. If v is fixed, we abbreviate $\mathfrak{F}, x \vDash_v \varphi$ by $\mathfrak{F}, x \vDash \varphi$ or even $x \vDash \varphi$. A formula φ is *valid in* $\mathfrak{F}$, denoted $\mathfrak{F} \vDash \varphi$, if $v(\varphi) = W$ for all v.

In order for $\mathfrak{F}$ to be a **GLP**-frame, each R_n should be a dually well-founded relation and in addition $\mathfrak{F}$ should validate axioms (iv) and (v). The next lemma, which is well-known, gives necessary and sufficient conditions for this.

Lemma 2.2. *Let $m < n$. Then:*

1. *$\mathfrak{F} \vDash [m]\varphi \to [n]\varphi$ iff $wR_n v$ implies $wR_m v$.*
2. *$\mathfrak{F} \vDash \langle m\rangle\varphi \to [n]\langle m\rangle\varphi$ iff $wR_m v$ and $wR_n u$ imply $uR_m v$.*

Proof. See, e.g., [11]. $\dashv$

Remark 2.3. Let $\mathfrak{F}$ be a (polymodal) frame, R_n^{-1} denote the inverse of R_n, $R_n(U) := \{w \in W : \exists u \in U, uR_n w\}$, and $R_n^{-1}(U) := \{w \in W : \exists u \in U, wR_n u\}$. We call U an R_n-*upset* if it is upward closed with respect to R_n; that is, $u \in U$ and $uR_n w$ imply $w \in U$. (Similarly, we call U an R_n-*downset* if $u \in U$ and $wR_n u$ imply $w \in W$.) Then axiom (iv) states that $R_n \subseteq R_m$ and axiom (v) states that each set of the form $R_m^{-1}(U)$ is an R_n-upset.

We show that no non-trivial frame satisfies all of these requirements. Suppose for a contradiction that **GLB** is sound with respect to a frame $\mathfrak{F}$ with R_1 nonempty. Then there are $w, v \in W$ such that $wR_1 v$. By Lemma 2.2(1), $wR_0 v$, and by Lemma 2.2(2), $vR_0 v$, which contradicts to R_0 being dually well-founded. Consequently, if $\mathfrak{F} \vDash \mathbf{GLB}$ then $R_1 = \varnothing$, so $[1]\bot$ becomes valid. This obviously generalizes to **GLP**. Thus, we obtain:

Theorem 2.4. **GLP** *is incomplete with respect to its class of frames. In particular,* **GLP** *is not sound on any frame for which $R_n \neq \varnothing$ for $n > 0$.*

2.2 Algebraic semantics and descriptive frames

As we saw, **GLP** is incomplete with respect to relational semantics, and as we will see, topological completeness of **GLP** remains an open problem. Nevertheless, there is a semantics for which completeness of **GLP** is automatic, viz. algebraic semantics. Of course, algebraic semantics is not as informative as either relational or topological semantics, but completeness is straightforward through the well-known Lindenbaum construction. Moreover, Stone-like duality for **GLP**-algebras can be developed without much trouble.

We recall that a pair $\mathfrak{A} = \langle B, \delta \rangle$ is a **GL**-*algebra* (also known as a *diagonalizable algebra* or a *Magari algebra*) if B is a boolean algebra and $\delta : B \to B$ is a unary function on B such that $\delta 0 = 0$, $\delta(a \vee b) = \delta a \vee \delta b$, and $\delta a = \delta(a - \delta a)$. Given a **GL**-algebra $\mathfrak{A} = \langle B, \delta \rangle$, let $\tau a = -\delta(-a)$. It is well known that if we interpret formulas as elements of a **GL**-algebra $\mathfrak{A} = \langle B, \delta \rangle$, boolean connectives as boolean operations of B, and $\Diamond$ as δ (and hence $\Box$ as τ), then **GL**-algebras provide an adequate semantics for **GL**.

Definition 2.5. We call a tuple $\mathfrak{A} = \langle B, \{\delta_n\}_{n \in \omega} \rangle$ a **GLP**-*algebra* if

(i) $\langle B, \delta_n \rangle$ is a **GL**-algebra for each $n \in \omega$;

(ii) $\delta_n a \leq \delta_m a$ for each $m < n$ and $a \in B$;

(iii) $\delta_m a \leq \tau_n \delta_m a$ for each $m < n$ and $a \in B$.

In particular, a triple $\mathfrak{A} = \langle B, \delta_0, \delta_1 \rangle$ is a **GLB**-*algebra* if both $\langle B, \delta_0 \rangle$ and $\langle B, \delta_1 \rangle$ are **GL**-algebras, $\delta_1 a \leq \delta_0 a$, and $\delta_0 a \leq \tau_1 \delta_0 a$ for each $a \in B$.

A standard argument shows that **GLP**-algebras provide an adequate semantics for **GLP**, and **GLB**-algebras provide an adequate semantics for **GLB**. We give three types of examples of **GLP**-algebras.

Example 2.6 (free algebras)**.** Free n-generated **GLP**-algebras, also known as Lindenbaum algebras, are obtained from the set of all formulas of **GLP** in the language with n propositional variables by identifying **GLP**-equivalent formulas and defining the boolean algebra operations by logical connectives. The modal operators δ_n map the equivalence class of a formula φ to the equivalence class of the formula $\langle n \rangle \varphi$. In particular, the free 0-generated algebra is the Lindenbaum algebra of the letterless fragment $\mathbf{GLP}_0$.

Another kind of **GLP**-algebras come from **GLP**-spaces (see next section).

Example 2.7. Let $\mathcal{X}$ be a **GLP**-space. The boolean algebra of all subsets of X enriched with the derived set operators d_n, for each $n \geq 0$, acting on 2^X is obviously a **GLP**-algebra.

Perhaps the most intriguing examples of **GLP**-algebras come from proof theory, where they have been introduced under the name of *graded provability algebras* [3].

Example 2.8 (provability algebras)**.** Let T be a first order arithmetical theory containing a sufficiently large fragment of Peano arithmetic PA. T is called *n-consistent* if the union of T and all true Π_n-sentences is consistent. If φ is an arithmetical sentence, let $\langle n\rangle_T\varphi$ denote a natural formalization of the statement that the theory $T+\varphi$ is n-consistent. (Such a formalization is equivalent to the so-called *uniform Σ_n-reflection principle* for $T+\varphi$.) This defines a function $\delta_n : \varphi \mapsto \langle n\rangle_T\varphi$, which is correctly defined on the equivalence classes of sentences modulo provable equivalence in T. The Lindenbaum algebra of T enriched with all the operators δ_n happens to be a **GLP**-algebra. This example plays a fundamental role in the proof-theoretic analysis of PA based on provability logic (see [3, 4]).

Of course, **GLP**-algebras (respectively, **GLB**-algebras) in general are rather abstract entities. Therefore, it is desirable to have a good representation for them. This is done through the well-known Stone construction.

Let X be a topological space. We recall that a subset A of X is *clopen* if A is both closed and open, and that X is *zero-dimensional* if clopen subsets form a basis for X. We also recall that X is a *Stone space* if it is compact, Hausdorff, and zero-dimensional.

It is a celebrated result of Stone that boolean algebras can be represented as the algebras of clopen subsets of Stone spaces. We recall that given a boolean algebra B, the dual Stone space X of B is constructed as the space of ultrafilters of B and that a topology on X is defined by declaring $\{\varphi(a) : a \in B\}$ to be a basis for the topology, where $\varphi(a) = \{x \in X : a \in x\}$. Let $\mathrm{Cp}(X)$ denote the set of clopen subsets of X. Then $\mathrm{Cp}(X)$ with set-theoretic operations $\cap, \cup, -$ is a boolean algebra, and $\varphi : B \to \mathrm{Cp}(X)$ is a boolean algebra isomorphism. This 1-1 correspondence between boolean algebras and Stone spaces extends to a categorical dual equivalence between the category of boolean algebras and boolean algebra homomorphisms and the category of Stone spaces and continuous maps.

This representation of boolean algebras was extended to a representation of **GL**-algebras by Magari [24] and by Esakia and Abashidze [1] (see also [12] and [8]). Let X be a Stone space and R a transitive relation on X. For a clopen $A \subseteq X$ we call $x \in A$ a *strongly maximal point* of A if xRy for no $y \in A$. In particular, a strongly maximal point is irreflexive. Now we call a pair $\langle X, R\rangle$ a *descriptive* **GL**-*frame* if X is a Stone space and R is a transitive binary relation on X such that $R(x)$ is closed for each $x \in X$, A clopen implies $R^{-1}(A)$ is clopen, and for each clopen A and $x \in A$, either x is strongly maximal or there exists a strongly maximal point $y \in A$ such that xRy.[1]

Let $\langle B, \delta\rangle$ be a **GL**-algebra and let X be the Stone space of B. We define R on X by xRy iff $a \in y$ implies $\delta a \in x$ for each $a \in B$. Since in each **GL**-algebra we have $\delta\delta a \leq \delta a$, it is easy to verify that R is transitive. It is also standard to show that $R(x)$ is closed for each $x \in X$, A clopen implies $R^{-1}(A)$ is clopen, and $\varphi(\delta a) = R^{-1}(\varphi(a))$. In fact, $\langle X, R\rangle$ is a descriptive **GL**-frame. This follows from the following lemma proved in [1].

[1] Descriptive **GL**-frames were called *strong transits* in [1].

Lemma 2.9. *If $\langle B, \delta\rangle$ is a* **GL***-algebra and $\langle X, R\rangle$ is the dual of $\langle B, \delta\rangle$, then $\langle X, R\rangle$ is a descriptive* **GL***-frame.*

Proof. (Sketch) Let A be a clopen subset of X. It is sufficient to show that for each $x \in A$, either x is a strongly maximal point or there exists a strongly maximal point $y \in A$ such that xRy. If $x \notin R^{-1}(A)$, then x is a strongly maximal point. Suppose that $x \in R^{-1}(A)$. Since A is clopen, there exists $a \in B$ such that $A = \varphi(a)$. Therefore, $x \in R^{-1}(\varphi(a))$. As $R^{-1}(\varphi(a)) = \varphi(\delta a)$, we obtain $x \in \varphi(\delta a)$. But $\delta a = \delta(a - \delta a)$. Thus, $x \in \varphi(\delta(a - \delta a)) = R^{-1}(\varphi(a - \delta a))$. This implies that there exists $y \in \varphi(a - \delta a)$ such that xRy. Now as $y \in \varphi(a - \delta a) = \varphi(a) - R^{-1}(\varphi(a))$, y must be a strongly maximal point of $\varphi(a) = A$. ⊣

It follows that if $\mathfrak{A} = \langle B, \delta\rangle$ is a **GL**-algebra, then $\mathfrak{X} = \langle X, R\rangle$ is a descriptive **GL**-frame and $\varphi : \langle B, \delta\rangle \to \langle \mathrm{Cp}(X), R^{-1}\rangle$ is an isomorphism of **GL**-algebras. Thus, each **GL**-algebra can be represented as the algebra of clopen subsets of the corresponding descriptive **GL**-frame. In particular, if $\mathfrak{A}$ is countable, then $\mathfrak{X}$ is second-countable.

As in the case of boolean algebras and Stone spaces, this representation extends to a dual equivalence of the appropriate categories, however we will not address this here and refer the interested reader to [1, 8].

This representation of **GL**-algebras extends in an obvious way to **GLP**-algebras and **GLB**-algebras.

Definition 2.10. We call a tuple $\mathfrak{X} = \langle X, \{R_n\}_{n\in\omega}\rangle$ a *descriptive* **GLP***-frame* if

(i) $\langle X, R_n\rangle$ is a descriptive **GL**-frame for each $n \in \omega$;

(ii) $R_n \subseteq R_m$ for each $m < n$;

(iii) xR_my and xR_nz imply zR_my for each $m < n$.

In particular, a triple $\mathfrak{X} = \langle X, R_0, R_1\rangle$ is a *descriptive* **GLB***-frame* if both $\langle X, R_0\rangle$ and $\langle X, R_1\rangle$ are descriptive **GL**-frames, $R_1 \subseteq R_0$, and xR_0y and xR_1z imply zR_0y.

Let $\mathfrak{A} = \langle B, \{\delta_n\}_{n\in\omega}\rangle$ be a **GLP**-algebra, X the Stone space of B, and xR_ny iff $a \in y$ implies $\delta_n a \in x$ for each $n \in \omega$ and $a \in B$.

Lemma 2.11. *If $\mathfrak{A} = \langle B, \{\delta_n\}_{n\in\omega}\rangle$ is a* **GLP***-algebra, then $\mathfrak{X} = \langle X, \{R_n\}_{n\in\omega}\rangle$ is a descriptive* **GLP***-frame. Moreover, $\varphi : \langle B, \{\delta_n\}_{n\in\omega}\rangle \to \langle \mathrm{Cp}(X), \{R_n^{-1}\}_{n\in\omega}\rangle$ is an isomorphism of* **GLP***-algebras.*

Proof. In view of the representation of **GL**-algebras, all we have to verify is that $R_n \subseteq R_m$ and xR_my and xR_nz imply zR_my for each $m < n$. Let xR_ny and $a \in y$. Then $\delta_n a \in x$. Since $\delta_n a \le \delta_m a$, also $\delta_m a \in x$. Therefore, xR_my, and so $R_n \subseteq R_m$. Now let xR_my and xR_nz. Suppose that $a \in y$. Since xR_my, we have $\delta_m a \in x$. If $\delta_m a \notin z$, then $-\delta_m a \in z$. As xR_nz, we have $\delta_n(-\delta_m a) \in x$. But $\delta_m a \in x$ and $\delta_m a \le -\delta_n(-\delta_m a)$ imply $-\delta_n(-\delta_m a) \in x$, a contradiction. Thus, $\delta_m a \in z$, and so zR_my. ⊣

In particular, Lemma 2.11 implies that if $\mathfrak{A} = \langle B, \delta_0, \delta_1\rangle$ is a **GLB**-algebra, then $\mathfrak{X} = \langle X, R_0, R_1\rangle$ is a descriptive **GLB**-frame, and $\varphi : \langle B, \delta_0, \delta_1\rangle \to \langle \mathrm{Cp}(X), R_0^{-1}, R_1^{-1}\rangle$ is an isomorphism of **GLB**-algebras.

2.3 Topological semantics

Our main interest in this paper is in topological semantics. Ordinarily, when modal logics are interpreted topologically, modal diamond is read as topological closure. However, as we already pointed out in the introduction, this only works if the logic in question contains the reflection axiom, since each set is a subset of its closure. For logics that do not contain the reflection axiom, of which **GL**, **GLB**, and **GLP** are all examples, $\Diamond$ can instead be interpreted as the derived set operator.

Definition 2.12. Let X be a topological space and $A \subseteq X$. We recall that $x \in X$ is a *limit point* of A if for each neighborhood U of x we have $A \cap (U - \{x\}) \neq \varnothing$. Let $d(A)$ denote the set of limit points of A. As usual, we call $d(A)$ the *derived set* of A. Obviously, the topological closure of A can then be defined as $\mathrm{cl}(A) = A \cup d(A)$ and topological interior as $\mathrm{int}(A) = A \cap t(A)$, where $t(A) := -d(-A)$.

Interpreting $\Diamond$ as a derived set operator provides an adequate semantics for **GL**. Let X be a topological space and let $v : \mathrm{Var} \to 2^X$ be a valuation. We extend v to the set of all formulas by setting

- $v(\varphi \vee \psi) = v(\varphi) \cup v(\psi)$, $v(\neg\varphi) = -v(\varphi)$, $v(\top) = X$, $v(\bot) = \varnothing$,
- $v(\Diamond\varphi) = d(v(\varphi))$, $v(\Box\varphi) = t(v(\varphi))$.

We will also write $X, x \overset{\mathrm{top}}{\vDash}_v \varphi$ for $x \in v(\varphi)$. When the valuation v is given from the context this can also be written as $X, x \overset{\mathrm{top}}{\vDash} \varphi$.

Definition 2.13. A formula φ is *valid in* X (denoted $X \overset{\mathrm{top}}{\vDash} \varphi$) if $\forall v, v(\varphi) = X$. The *logic of* X is the set of all formulas valid in X. If $\mathcal{C}$ is a class of spaces, the *logic of* $\mathcal{C}$ is the set of formulas valid in all members $X \in \mathcal{C}$.

Given a topological space X, we recall that $x \in X$ is an *isolated point* of X if $\{x\}$ is an open subset of X. Note that the set of isolated points of a subspace Y of X coincides with $Y - d(Y)$. We call X a *scattered space* if each nonempty subspace of X has an isolated point.

Theorem 2.14 ([13])**.** *A topological space X is scattered iff $X \overset{top}{\vDash}$* **GL***; moreover,* **GL** *is the logic the class of all scattered spaces.*

Typical examples of scattered spaces are ordinals (in the interval topology). Theorem 2.14 can be improved by showing that **GL** is the logic of all ordinals. In fact, **GL** is the logic of any ordinal $\alpha \geq \omega^\omega$:

Theorem 2.15 ([2, 10]). **GL** *is the logic of the class of all ordinals. In fact,* **GL** *is the logic of any ordinal* $\alpha \geq \omega^\omega$. *In particular,* **GL** *is the logic of* ω^ω.[2]

For the case of the polymodal language of **GLP** we consider *polytopological spaces*; that is, sets X equipped with a family of topologies $\{\tau_n\}_{n\in\omega}$. As our immediate task, we would like to understand which polytopological spaces satisfy all the axioms of **GLP**.

Let $\mathcal{X} = \langle X, \{\tau_n\}_{n\in\omega}\rangle$ be a polytopological space. Let, for each $n \in \omega$, d_n denote the derived set operator and t_n its dual with respect to τ_n. Theorem 2.14 tells us that each τ_n should be a scattered topology. Now we give necessary and sufficient conditions for axioms (iv) and (v) to be valid in $\mathcal{X}$.

Proposition 2.16. *Let* $\mathcal{X} = \langle X, \{\tau_n\}_{n\in\omega}\rangle$ *be a polytopological space and* $m < n$.

1. *For each* $A \subseteq X$ *we have* $d_m(A)$ *is* τ_n*-open iff* $d_m(A) \subseteq t_n(d_m(A))$.
2. $\tau_m \subseteq \tau_n$ *iff* $d_n(A) \subseteq d_m(A)$ *for each* $A \subseteq X$.

Proof. (1) We have:

$$\begin{array}{lll} d_m(A) \text{ is } \tau_n\text{-open} & \text{iff} & d_m(A) = \mathrm{int}_n(d_m(A)) \\ & \text{iff} & d_m(A) = d_m(A) \cap t_n(d_m(A)) \\ & \text{iff} & d_m(A) \subseteq t_n(d_m(A)). \end{array}$$

(2) Let $\tau_m \subseteq \tau_n$. Suppose that $A \subseteq X$, $x \in d_n(A)$, and U is a τ_m-open neighborhood of x. Then U is also a τ_n-open neighborhood of x, and so $A \cap (U - \{x\}) \neq \varnothing$, which implies that $x \in d_m(A)$. Conversely, let $\tau_m \not\subseteq \tau_n$. Then there exists $U \in \tau_m$ such that $U \notin \tau_n$. Since $U \notin \tau_n$, there exists $x \in U$ such that for each τ_n-open neighborhood V of x we have $V \cap -U \neq \varnothing$. Therefore, $U \cap d_n(-U) \neq \varnothing$ and yet $U \cap d_m(-U) = \varnothing$. Thus, $d_n(-U) \not\subseteq d_m(-U)$. ⊣

Theorem 2.14 and Proposition 2.16 suggest the following definition of a **GLP**-space.

Definition 2.17. Let $\mathcal{X} = \langle X, \{\tau_n\}_{n\in\omega}\rangle$ be a polytopological space. We call $\mathcal{X}$ a **GLP**-*space* if

(i) Each τ_n is a scattered topology;

(ii) $\tau_n \subseteq \tau_{n+1}$;

(iii) $d_n(A)$ is τ_{n+1}-open for each $A \subseteq X$.

In particular, a bitopological space $\langle X, \tau_0, \tau_1\rangle$ is a **GLB**-*space* if both τ_0 and τ_1 are scattered topologies, $\tau_0 \subseteq \tau_1$, and $d_0(A)$ is τ_1-open for each $A \subseteq X$.

Note that, because of condition (ii), condition (i) can be weakened to the requirement that only τ_0 be scattered. From Theorem 2.14 and Proposition 2.16 we directly obtain:

[2]For a simplified proof of this result we refer to [9].

Theorem 2.18. *A polytopological space $\mathcal{X} = \langle X, \{\tau_n\}_{n\in\omega}\rangle$ is a* **GLP***-space iff $\mathcal{X} \overset{top}{\vDash}$* **GLP***, and a bitopological space $\langle X, \tau_0, \tau_1\rangle$ is a* **GLB***-space iff $\mathcal{X} \overset{top}{\vDash}$* **GLB***.*

An obvious question is whether **GLP** (resp. **GLB**) is complete with respect to this semantics. But first we should be able to give examples of **GLP**-spaces (resp. **GLB**-spaces). Note that conditions (i) and (ii) are natural topological conditions and are easy to satisfy. On the other hand, condition (iii) is rather strong and somewhat unusual. Nevertheless, we will see shortly how to satisfy it.

Of course, if $\langle X, \tau_0\rangle$ is a scattered space and τ_1 is a discrete topology on X, then $\langle X, \tau_0, \tau_1\rangle$ is trivially a **GLB**-space. The first example of a **GLB**-space with two non-discrete topologies was given by Leo Esakia (private communication).

Example 2.19 (Esakia space)**.** Let α be an ordinal. Let τ_0 consist of all $<$-downsets and let τ_1 be the interval topology. It is easy to verify that both τ_0 and τ_1 are scattered topologies and that $\tau_0 \subset \tau_1$. Let $A \subseteq \alpha$. To see that $d_0(A)$ is τ_1-open observe that $d_0(A) = \{x \in \alpha : x > \min(A)\}$, which is clearly τ_1-open. Thus, $\langle \alpha, \tau_0, \tau_1\rangle$ is a **GLB**-space.

On the other hand, the next lemma shows that in order to define a third non-discrete topology on α, the ordinal should be very large. Recall that a topological space X is *first-countable* if every point $x \in X$ has a countable basis of open neighborhoods.

Proposition 2.20. *For any* **GLB***-space $\langle X, \tau_0, \tau_1\rangle$, if τ_0 is Hausdorff and first-countable, then τ_1 is discrete.*

Proof. It is easy to see that if $\langle X, \tau_0\rangle$ is first-countable and Hausdorff, then every point $a \in X$ is a (unique) limit of a countable sequence of points $A = \{a_n\}_{n\in\omega}$. Hence, there is a set $A \subseteq X$ such that $d_0(A) = \{a\}$. By condition (iii), this means that $\{a\}$ is τ_1-open. ⊣

Going back to $\langle \alpha, \tau_0, \tau_1\rangle$, observe that $\langle \alpha, \tau_1\rangle$ is always Hausdorff, and that $\langle \alpha, \tau_1\rangle$ is first-countable iff $\alpha \leq \omega_1$. Therefore, in order for us to be able to define a non-discrete τ_2 on α, the ordinal should be at least $\omega_1 + 1$. This is, in fact, sufficient as the following example shows.

Example 2.21 (club topology)**.** Recall that *cofinality* $\mathrm{cf}(\alpha)$ of a limit ordinal α is the least order type of an unbounded subset of α. If α is not a limit ordinal, we set $\mathrm{cf}(\alpha) = 0$. A set $A \subseteq \alpha$ is called a *club in* α if it is τ_1-closed (in the interval topology on α) and unbounded in α.

Define a topology τ_2 on α as follows: a set U is τ_2-open if, for each $\beta \in U$, either $\mathrm{cf}(\beta) \leq \omega$ or there is a club C in β such that $C \subseteq U$.

If $\mathrm{cf}(\beta) > \omega$, the intersection of countably many clubs in β is again a club. Hence, it is easy to check that τ_2 is indeed a topology. The filter of neighborhoods of β in τ_2 (restricted to β) coincides with the so-called *club filter* on β — a well-known concept in set theory (see [23]). Therefore, we call this topology the *club topology.*

Proposition 2.22. *$\langle\alpha, \tau_1, \tau_2\rangle$ is a* **GLB**-*space. In fact, the club topology τ_2 is the coarsest topology τ such that $\langle\alpha, \tau_1, \tau\rangle$ is a* **GLB**-*space.*

Proof. To verify condition (iii) notice that a set of the form $d_1(A) \cap \beta$ is a club in any $\beta \in d_1(A)$. Hence, $d_1(A)$ is τ_2-open. The other conditions are obvious.

On the other hand, assume $\langle\alpha, \tau_1, \tau\rangle$ is a **GLB**-space. We show that every τ_2-open neighborhood of any $\beta \in \alpha$ contains a τ-open neighborhood. If $\mathrm{cf}(\beta) \leq \omega$ then either β is isolated already in τ_1 (in the case β is not a limit ordinal), or β is a unique limit of an increasing ω-sequence A of ordinals. Then $\{\beta\} = d_1(A)$ and hence β is isolated in τ. If $\mathrm{cf}(\beta) > \omega$ and C is a club in β, then $d_1(C) \subseteq C \cup \{\beta\}$ is a τ-open neighborhood of β. ⊣

We are mainly interested in topological completeness of **GLP** and **GLB**. Note that no Esakia space can be an exact model of **GLB**. Looking at $\langle\alpha, \tau_0, \tau_1\rangle$, observe that τ_0 consists of the $<$-downsets of α. Since α is a linear order, the linearity axiom $[0]([0]^+p \to q) \vee [0]([0]^+q \to p)$ is valid in $\langle\alpha, \tau_0, \tau_1\rangle$, where $[0]^+\varphi$ is an abbreviation of $\varphi \wedge [0]\varphi$.

As far as the **GLB**-space $\langle\alpha, \tau_1, \tau_2\rangle$ is concerned, the situation is more complicated. We know that it is consistent with ZFC that **GLB** is incomplete with respect to this space. This follows from a result of Blass [10] who analyzed the question of completeness of **GL** with respect to the club topology τ_2.[3] In particular, he has shown that it is consistent with ZFC that **GL** is incomplete with respect to τ_2 on any ordinal. He has also shown that, under the assumption $V = L$, **GL** *is* complete with respect to the space $\langle\aleph_\omega, \tau_2\rangle$. We conjecture that this result can be extended to a completeness result for **GLB** with respect to $\langle\aleph_\omega, \tau_1, \tau_2\rangle$.

In the next section we will be able to prove topological completeness of **GLB** while standing firmly on the basis of ZFC. However, the question of topological (in)completeness of any fragment of **GLP** with more than two modalities remains open. At the least, our method of proving completeness of **GLB** does not immediately generalize to three or more modalities.

While the full **GLP**, so far, eludes completeness, we note that the letterless fragment $\mathbf{GLP}_0$ allows for a simple topological treatment. Namely, $\mathbf{GLP}_0$ is sound and complete with respect to a natural polytopological space defined on the ordinal ϵ_0. This space, however, is not a **GLP**-space (see [17, 18]).

3 Topological Completeness of GLB

In this section we work in the language with two modalities [0] and [1]. Before we prove our main result, we need a few auxiliary notions.

[3] Blass did not introduce the topology explicitly, but formulated an equivalent semantics in terms of the club filter.

3.1 The logic J

Our proof of topological completeness will make use of a subsystem of **GLB** introduced in [5] and denoted **J**. This logic is defined by weakening axiom (iv) of **GLB** to the following axioms (vi) and (vii) both of which are theorems of **GLB**:

(vi) $[0]\varphi \to [1][0]\varphi$;

(vii) $[0]\varphi \to [0][1]\varphi$.

J is the logic of a simple class of frames, which is established by standard methods ([5, Theorem 1]).

Lemma 3.1. **J** *is sound and complete with respect to the class of (finite) frames* $\langle W, R_1, R_2 \rangle$ *such that, for all* $x, y, z \in W$,

1. R_0 *and* R_1 *are transitive and dually well-founded;*
2. *If* xR_1y, *then* xR_0z *iff* yR_0z;
3. xR_0y *and* yR_1z *imply* xR_0z.

If we let $\overline{R_1}$ denote the reflexive, symmetric, transitive closure of R_1, then we call each $\overline{R_1}$ equivalence class a *1-sheet*. By (2), all points in a 1-sheet are R_0 incomparable. But R_0 defines a natural ordering on 1-sheets in the following sense: if α and β are 1-sheets, then $\alpha R_0 \beta$, iff $\exists x \in \alpha, \exists y \in \beta, xR_0y$. By standard techniques, one can improve on Lemma 3.1 to show that **J** is complete for such frames, in which each 1-sheet is a tree under R_1, and if $\alpha R_0 \beta$ then xR_0y for all $x \in \alpha$, $y \in \beta$ (see [5, Theorem 2 and Corollary 3.3]). Thus, models of **J** can be seen as R_0-orders (and even tree-like orders), in which the nodes are 1-sheets that are themselves R_1-trees. We call such frames *tree-like J-frames*.

As shown in [5], **GLB** is reducible to **J** in the following sense. Let

$$M(\varphi) := \bigwedge_{i<s} ([0]\varphi_i \to [1]\varphi_i),$$

where $[0]\varphi_i$, $i < s$, are all subformulas of φ of the form $[0]\psi$. Also, let

$$M^+(\varphi) := M(\varphi) \wedge [0]M(\varphi) \wedge [1]M(\varphi).$$

Proposition 3.2 ([5]). **GLB** $\vdash \varphi$ *iff* **J** $\vdash M^+(\varphi) \to \varphi$.

This proposition generalizes straightforwardly to the case of **GLP**. In fact, we obtain another proof of this proposition, for the case of **GLB**, as a byproduct of the topological completeness proof below.[4]

[4]It is worth noting that Ignatiev's proof of *arithmetical* completeness of **GLP** establishes a similar reduction of **GLP** to a different frame complete subsystem of **GLP**.

3.2 Some notions related to partial orderings

Let $\langle X, \prec\rangle$ be a dually well-founded strict partial ordering. We consider bitopological spaces of the form $\langle X, \tau_0, \tau_1\rangle$, where τ_0 is the upset topology on $\langle X, \prec\rangle$ and τ_1 is generated by all semi-open intervals of the form

$$[a, b) := \{x \in X : a \preceq x \prec b\}$$

for $a \prec b$, and

$$[a, \infty) := \{x \in X : a \preceq x\}.$$

Notice that if $\langle X, \prec\rangle$ is a strict linear ordering, then τ_1 is the usual interval topology on X, and thus $\langle X, \tau_0, \tau_1\rangle$ is the Esakia space of the ordinal dual to $\langle X, \prec\rangle$.

Lemma 3.3. *$\langle X, \tau_0, \tau_1\rangle$ is a* **GLB***-space.*

Proof. Clearly $\tau_0 \subseteq \tau_1$. We show that sets of the form $d_0(A)$ are τ_1-open for any $A \subseteq X$. Let $\max(A)$ denote the set of maximal points of A. Since $\langle X, \prec\rangle$ is dually well-founded, $d_0(A)$ consists of all points below $\max(A)$; that is,

$$d_0(A) = \bigcup_{a \prec b \in \max(A)} [a, b).$$

Hence, $d_0(A)$ is a union of τ_1-open sets. ⊣

We call such **GLB**-spaces *general Esakia spaces.*

Next, we recall a few standard operations on strict partial orderings.[5] The *disjoint union* of the orderings X and Y is denoted $X \sqcup Y$. The *sum* of X and Y is denoted $X + Y$; that is, the ordering is obtained by putting Y on top of X. In particular, when X is a singleton $\{a\}$, $\{a\} + Y$ denotes the result of adding a new node at the bottom of Y.

A more general operation of *ordered sum* of a family $\{\mathcal{A}_i : i \in I\}$ of orderings $\mathcal{A}_i = \langle A_i, \prec_i\rangle$, where $\langle I, \prec\rangle$ is a strict partially ordered index set, is the ordering $\langle Y, \prec_Y\rangle$ such that $Y = \bigsqcup_{i\in I} A_i$. For $x, y \in Y$, we declare $x \prec_Y y$ iff either $x, y \in A_i$ and $x \prec_i y$ for some $i \in I$; or $x \in A_i$ and $y \in A_j$ for some $i \prec j$. We denote this ordering by $\sum_{i\in I} \mathcal{A}_i$. In particular, if $\langle I, \prec\rangle$ is the ordering $\langle \omega, >\rangle$ and all $\mathcal{A}_i$ are isomorphic to the same ordering $\mathcal{A}$, the ordering $\sum_{i\in I} \mathcal{A}$ consists of countably many copies of $\mathcal{A}$ ordered by ω^* and is denoted $\mathcal{A} \cdot \omega^*$.

3.3 Topological completeness theorem

Theorem 3.4 (Main Theorem)**.** **GLB** *is complete w.r.t. the class of general Esakia spaces.*

[5] The notations we use are dual to those given in [5], but they are more in line with the standard usage.

Proof. Assume **GLB** $\nvdash \varphi$. Consider a finite tree-like J-model $\mathcal{A}$ such that $\mathcal{A} \nvDash M^+(\varphi) \to \varphi$. We denote by Greek letters $\alpha, \beta, \dots$ the elements of $\mathcal{A}$.

Following [5], we associate with $\mathcal{A}$ a strict partial ordering called the *topological blow-up of* $\mathcal{A}$. First, we associate with each 1-sheet $\mathcal{S}$ of $\mathcal{A}$ a strict partial ordering $\mathcal{S}^\omega$ by induction on the R_1-depth of $\mathcal{S}$. Second, we consider the set $\mathbf{S}(\mathcal{A})$ of all 1-sheets of $\mathcal{A}$ ordered by R_0 and take the ordered sum of orders $\mathcal{S}^\omega$ with respect to this index set. This idea is expressed by the following two formal definitions.

Definition 3.5.

- If $\mathcal{A}_\alpha = \langle A_\alpha, R_1 \rangle$ is a tree with the root α, define a strict partial ordering $\mathcal{A}_\alpha^\omega$ by induction on the depth of α:

$$\mathcal{A}_\alpha^\omega := \{\alpha\} + \big(\bigsqcup_{i=1}^{n} \mathcal{A}_{\alpha_i}^\omega\big) \cdot \omega^*,$$

  where α_i are all the R_1-children of α. $\mathcal{A}_\alpha^\omega := \{\alpha\}$ if $\mathcal{A}_\alpha$ is the singleton $\{\alpha\}$.

- $\mathfrak{B}_\omega(\mathcal{A}) := \sum_{\mathcal{S} \in \mathbf{S}(\mathcal{A})} \mathcal{S}^\omega$.

The ordering $\mathfrak{B}_\omega(\mathcal{A})$ is called the *topological blow-up of* $\mathcal{A}$ and will define the general Esakia space we seek. The order relation on $\mathfrak{B}_\omega(\mathcal{A})$ will be denoted $\prec$; τ_0 and τ_1 are the topologies of the associated general Esakia space; d_0 and d_1 are the corresponding derived set operators.

It is worth noting that the blow-up construction here is much simpler than the one in [5] for two main reasons. Firstly, we only deal with the case of two modalities which avoids the iterative process involved in [5] and the complicated limit construction. Secondly, the type of the resulting structure is simpler (it is just a strict partial order) and, in addition, it needs fewer new points. The latter seems to be a helpful feature of the topological semantics we consider compared to relational semantics.

Next, we make a couple observations about the defined structures. Firstly, there is a natural embedding of $\mathcal{A}_\beta^\omega$ as an upset into $\mathcal{A}_\alpha^\omega$ whenever $\alpha \prec \beta$. This is easy to verify by induction on R_1-depth of α. Secondly, a natural *projection map* $\pi_\alpha : \mathcal{A}_\alpha^\omega \to \mathcal{A}_\alpha$ is defined inductively as follows: if $x \in \mathcal{A}_{\alpha_i}^\omega$, then $\pi_\alpha(x) := \pi_{\alpha_i}(x)$; otherwise, $\pi_\alpha(x) := \alpha$. This extends to a map $\pi : \mathfrak{B}_\omega(\mathcal{A}) \to \mathcal{A}$ in the obvious way.

Lemma 3.6.

1. *Assume* $x \in \mathcal{A}_\alpha^\omega$ *and* $\pi_\alpha(x) R_1 y$ *in* $\mathcal{A}_\alpha$. *Then there is a sequence* $(x_n)_{n\in\omega} \in \mathcal{A}_\alpha^\omega$ *such that* $x \in d_1(\{x_n : n \in \omega\})$ *and* $\pi_\alpha(x_n) = y$ *for all* $n \in \omega$.

2. *For all* $x, y \in \mathfrak{B}_\omega(\mathcal{A})$, *if* $\pi(x) R_1 y$, *then* $x \in d_1(\pi^{-1}(y))$.

Proof. (1) We argue by induction on the R_1-depth of α. If α has depth 0, the claim is trivial (no such x, y exist). Otherwise, $\mathcal{A}^\omega_\alpha = \{\alpha\} + \left(\bigsqcup_{i=1}^n \mathcal{A}^\omega_{\alpha_i}\right) \cdot \omega^*$.

If x belongs to some copy of $\mathcal{A}^\omega_{\alpha_i}$, we can select a sequence x_n in (the same copy of) $\mathcal{A}^\omega_{\alpha_i}$ by the induction hypothesis. We obviously have that $\pi_\alpha(x_n) = y$ by the definition of π_α. Also, $x \in d_1(\{x_n : n \in \omega\})$ in $\mathcal{A}^\omega_{\alpha_i}$. Since $\langle \mathcal{A}^\omega_{\alpha_i}, \tau_1\rangle$ is a subspace of $\langle \mathcal{A}^\omega_\alpha, \tau_1\rangle$ (any interval in one space is an interval in the other), we also have $x \in d_1(\{x_n : n \in \omega\})$ in $\mathcal{A}^\omega_\alpha$.

If x is the root of $\mathcal{A}^\omega_\alpha$, then $\pi_\alpha(x) = \alpha$. Suppose y is an immediate successor of α. Then $\mathcal{A}^\omega_\alpha$ contains a sequence of copies of $\mathcal{A}^\omega_y$ the roots of which converge to x. Otherwise, let β be the son of α such that $\beta \prec y$. Select an element $z \in \mathcal{A}^\omega_\beta$ such that $\pi_\beta(z) = y$. Let z_n be the element corresponding to z within the n-th copy of $\mathcal{A}^\omega_\beta$ above x. Then z_n's converge to x in $\mathcal{A}^\omega_\alpha$.

(2) If $\pi(x)R_1y$, then $\pi(x), y$ belong to the same 1-sheet $\mathcal{A}_\alpha$, $x \in \mathcal{A}^\omega_\alpha$ and $\pi = \pi_\alpha$ on $\mathcal{A}^\omega_\alpha$. Hence, one can apply (1) and obtain a sequence (x_n) in $\mathcal{A}^\omega_\alpha$ such that $x_n \in \pi^{-1}(y)$ and $x \in d_1(\{x_n : n \in \omega\})$. ⊣

Lemma 3.7. *For all $x, y \in \mathfrak{B}_\omega(\mathcal{A})$, if $x \in d_1(Y)$, then $\pi(x)R_1\pi(y)$ for infinitely many $y \in Y$.*

Proof. Let $\mathcal{A}_\alpha$ be the 1-sheet of $\pi(x)$. If $x \in d_1(Y)$, then Y is infinite because τ_1 is a T_1-topology; that is, each finite set is closed. Since $\mathcal{A}^\omega_\alpha$ is a semi-open interval in $\mathfrak{B}_\omega(\mathcal{A})$, there are infinitely many $y \in Y$ such that $y \in \mathcal{A}^\omega_\alpha$. Without loss of generality assume that this holds for all $y \in Y$ and that $x \notin Y$. We prove that $\pi(x)R_1\pi(y)$ for infinitely many $y \in Y$ by induction on the R_1-depth of α.

If the depth of α is 0, then $\mathcal{A}^\omega_\alpha = \{\alpha\}$. Therefore, all $y \in Y$ must coincide with α, contradicting that Y is infinite. Otherwise, $\mathcal{A}^\omega_\alpha = \{\alpha\} + \left(\bigsqcup_{i=1}^n \mathcal{A}^\omega_{\alpha_i}\right) \cdot \omega^*$.

Suppose x belongs to some copy of $\mathcal{A}_{\alpha_i}$. Since this copy is a semi-open interval in $\mathcal{A}^\omega_\alpha$, infinitely many $y \in Y$ are in this interval. By induction hypothesis, $\pi(x)R_1\pi(y)$ for infinitely many $y \in Y$.

If x is the root of $\mathcal{A}^\omega_\alpha$, then $\pi(x) = \pi_\alpha(x) = \alpha$. If $y \in Y$ then $y \neq x$ by assumption, and by the construction of $\mathcal{A}^\omega_\alpha$, $\pi(y) \neq \pi(x) = \alpha$. Since $\pi(y) \in \mathcal{A}_\alpha$ and α is the minimum of $\mathcal{A}_\alpha$, we have $\alpha R_1 \pi(y)$. ⊣

We define a valuation $v : \mathrm{Var} \to 2^{\mathfrak{B}_\omega(\mathcal{A})}$ by

$$x \in v(p) \iff \mathcal{A}, \pi(x) \vDash p.$$

Lemma 3.8. *For each subformula ψ of φ,*

$$\mathfrak{B}_\omega(\mathcal{A}), x \overset{top}{\vDash} \psi \iff \mathcal{A}, \pi(x) \vDash \psi.$$

Proof. By induction on the build-up of ψ. We only treat the cases of modalities. Let $X := \mathfrak{B}_\omega(\mathcal{A})$ and $v(\psi) := \{x \in X : X, x \overset{\text{top}}{\vDash} \psi\}$.

1. Suppose $\mathcal{A}, \pi(x) \vDash \langle 1\rangle\psi$. Then there is a y such that $\pi(x)R_1y$ and $\mathcal{A}, y \vDash \psi$. Since $\pi(x)R_1y$, we have $x \in d_1(\pi^{-1}(y))$. By inductive hypothesis, $\pi^{-1}(y) \subseteq v(\psi)$, hence $x \in d_1(v(\psi))$ and $X, x \overset{\text{top}}{\vDash} \langle 1\rangle\psi$.

2. Suppose $X, x \overset{\text{top}}{\vDash} \langle 1\rangle\psi$. Then $x \in d_1(v(\psi))$. Setting $Y := v(\psi)$, by Lemma 3.7, there is a $y \in Y$ such that $\pi(x)R_1\pi(y)$. By inductive hypothesis, $\mathcal{A}, \pi(y) \vDash \psi$, hence $\mathcal{A}, \pi(x) \vDash \langle 1\rangle\psi$.

3. If $\mathcal{A}, \pi(x) \vDash \langle 0\rangle\psi$, then $\exists y(\pi(x)R_0 y \ \&\ \mathcal{A}, y \vDash \psi)$. Since π is a p-morphism, there is a $y' \succ x$ such that $\pi(y') = y$. This yields $X, y' \overset{\text{top}}{\vDash} \psi$ and $X, x \overset{\text{top}}{\vDash} \langle 0\rangle\psi$.

4. If $X, x \overset{\text{top}}{\vDash} \langle 0\rangle\psi$, then $\exists y\ (x \prec y \ \&\ X, y \overset{\text{top}}{\vDash} \psi)$. We have $\pi(x)R_0\pi(y)$ or $\pi(x)$,$\pi(y)$ belong to the same 1-sheet. In the first case we are done. In the second case, let α be the R_1-maximal point such that $x \in \mathcal{A}^\omega_\alpha$, and let z be the $\prec$-minimal point of $\mathcal{A}^\omega_\alpha$. Obviously $\pi(z) = \alpha$.

Notice that $\mathcal{A}^\omega_\beta$ is an upwards closed submodel of $\mathcal{A}_\alpha$ whenever $\alpha R_1 \beta$. Then, since $x \in \mathcal{A}^\omega_\alpha$, we must also have $y \in \mathcal{A}^\omega_\alpha$; hence $z \prec y$. Since $\pi(z) = \alpha$ and $z \prec y$, we have $\pi(z)R_1\pi(y)$. By inductive hypothesis, $\mathcal{A}, \pi(y) \vDash \psi$ and hence $\mathcal{A}, \pi(z) \vDash \langle 1\rangle\psi$. By the monotonicity axioms in $\mathcal{A}$ this yields $\mathcal{A}, \pi(z) \vDash \langle 0\rangle\psi$. Since $\pi(x)$ belongs to the same 1-sheet as $\alpha = \pi(z)$, we also have $\mathcal{A}, \pi(x) \vDash \langle 0\rangle\psi$. ⊣

Hence, we obtain $\mathfrak{B}_\omega(\mathcal{A}) \overset{\text{top}}{\nvDash} \varphi$, which proves the theorem. ⊣

Corollary 3.9. $\mathbf{GLB} \vdash \varphi$ *iff* $\mathbf{J} \vdash M^+(\varphi) \to \varphi$.

Proof. The non-trivial implication from left to right follows from the proof of topological completeness theorem. We have shown that if the conclusion is false, there must exist a **GLB**-space falsifying φ, hence $\mathbf{GLB} \nvdash \varphi$. ⊣

4 Discussion

We have established topological completeness results for two fragments of **GLP**: for the bimodal fragment **GLB**, and for the the letterless fragment $\mathbf{GLP}_0$ (see [18]). There are some questions that remain open, which we summarize below.

1. Is **GLP** topologically complete?
2. Is **GLB** complete with respect to the **GLB**-space $\langle\alpha, \tau_1, \tau_2\rangle$, for some ordinal α, under the assumption $V = L$? (Here, τ_1 is the interval topology and τ_2 is the club topology on α.)
3. There is a natural notion of an *ordinal* **GLP**-*space*. Consider a space of the form $\langle\alpha, \{\tau_n\}_{n\in\omega}\rangle$, where τ_1 is the interval topology on α, and τ_{n+1} is generated from τ_n and all sets of the form $d_n(A)$, for $A \subseteq \alpha$. Is **GLP** complete with respect to some ordinal **GLP**-space?

From the results of Blass (see our discussion of Problem 2 at the end of Section 2.3) we know that a positive answer to Problem 3 would require some set-theoretic assumptions outside ZFC. Some partial results in this direction have already been obtained. In particular, we know that the assumption that the third topology τ_3 of an ordinal **GLP**-space is nontrivial is equiconsistent with the existence of a weakly compact cardinal. In other words, non-discreteness of τ_3 (and similarly for further topologies τ_n) is a large cardinal assumption. We do not know the exact consistency strength of this assumption for $n > 3$. However, we know a reasonable sufficient condition for all τ_n to be non-discrete — the existence of the so-called Π^1_n-indescribable cardinals for each $n \in \omega$.[6] Therefore, it is hopeful to obtain completeness of **GLP** with respect to an ordinal **GLP**-space if we simultaneously assume things like $V = L$ and the existence of Π^1_n-indescribable cardinals. These results, in fact, show that there are deeper connections between the theory of ordinal **GLP**-spaces and parts of set theory dealing with infinitary combinatorics and stationary reflection.

References

[1] M. Abashidze. Some properties of Magari algebras. In *Studies in logic and semantics*, pages 111–127. "Metsniereba", Tbilisi, 1981.

[2] M. Abashidze. Ordinal completeness of the Gödel-Löb modal system (Russian). In *Intensional logics and the logical structure of theories*, pages 49–73. Metsniereba, Tbilisi, 1985.

[3] L.D. Beklemishev. Provability algebras and proof-theoretic ordinals, I. *Annals of Pure and Applied Logic*, 128:103–123, 2004.

[4] L.D. Beklemishev. Reflection principles and provability algebras in formal arithmetic. *Uspekhi Matematicheskikh Nauk*, 60(2):3–78, 2005. In Russian. English translation in: *Russian Mathematical Surveys*, 60(2): 197–268, 2005.

[5] L.D. Beklemishev. Kripke semantics for provability logic GLP. Logic Group Preprint Series 260, University of Utrecht, November 2007. `http://preprints.phil.uu.nl/lgps/`. To appear in *Annals of Pure and Applied Logic*, doi:10.1016/j.apal.2009.06.011.

[6] L.D. Beklemishev, J. Joosten, and M. Vervoort. A finitary treatment of the closed fragment of Japaridze's provability logic. *Journal of Logic and Computation*, 15(4):447–463, 2005.

[7] J. van Benthem and G. Bezhanishvili. Modal logics of space. In *Handbook of spatial logics*, pages 217–298. Springer, Dordrecht, 2007.

[8] C. Bernardi and P. d'Aquino. Topological duality for diagonalizable algebras. *Notre Dame Journal of Formal Logic*, 29(3):345–364, 1988.

[9] G. Bezhanishvili and P. J. Morandi. Classification of scattered and hereditarily irresolvable spaces. submitted, 2008.

[6] The first author thanks Philipp Schlicht for finding out this condition and for further advice on set theory involved here.

[10] A. Blass. Infinitary combinatorics and modal logic. *Journal of Symbolic Logic*, 55(2):761–778, 1990.

[11] G. Boolos. *The Logic of Provability.* Cambridge University Press, Cambridge, 1993.

[12] W. Buszkowski and T. Prucnal. Topological representation of co-diagonalizable algebras. In G. Wechsung, editor, *Frege Conference' 1984*, pages 63–65, Berlin, 1984. Akademie-Verlag.

[13] L. Esakia. Diagonal constructions, Löb's formula and Cantor's scattered space (Russian). In *Studies in logic and semantics*, pages 128–143. Metsniereba, Tbilisi, 1981.

[14] L. Esakia. Intuitionistic logic and modality via topology. *Annals of Pure and Applied Logic*, 127:155–170, 2003.

[15] M. Gerson. An extension of $S4$ complete for the neighbourhood semantics but incomplete for the relational semantics. *Studia Logica*, 34(4):333–342, 1975.

[16] M. Gerson. A neighbourhood frame for T with no equivalent relational frame. *Z. Math. Logik Grundlagen Math.*, 22(1):29–34, 1976.

[17] T.F. Icard, III. Models of the polymodal provability logic. M.Sc. Thesis, ILLC, University of Amsterdam, http://www.illc.uva.nl/Publications/ResearchReports/MoL-2008-06.text.pdf, 2008.

[18] T.F. Icard, III. A topological study of the closed fragment of GLP. *Journal of Logic and Computation*, 2009. Submitted.

[19] K.N. Ignatiev. The closed fragment of Dzhaparidze's polymodal logic and the logic of Σ_1-conservativity. ITLI Prepublication Series X–92–02, University of Amsterdam, 1992.

[20] K.N. Ignatiev. On strong provability predicates and the associated modal logics. *The Journal of Symbolic Logic*, 58:249–290, 1993.

[21] G.K. Japaridze. The modal logical means of investigation of provability. Thesis in Philosophy, in Russian, Moscow, 1986.

[22] G.K. Japaridze. The polymodal logic of provability. In *Intensional Logics and Logical Structure of Theories: Material from the fourth Soviet–Finnish Symposium on Logic, Telavi, May 20–24, 1985*, pages 16–48. Metsniereba, Tbilisi, 1988. In Russian.

[23] T. Jech. *Set Theory. The Third Millenium Edition.* Springer, 2002.

[24] R. Magari. The diagonalizable algebras (the algebraization of the theories which express Theor.:II). *Bollettino della Unione Matematica Italiana,* Serie 4, 12, 1975. Suppl. fasc. 3, 117–125.

[25] J. C. C. McKinsey and A. Tarski. The algebra of topology. *Annals of Mathematics*, 45:141–191, 1944.

[26] V. Shehtman. On neighbourhood semantics thirty years later. In S. Artemov et al., editor, *We Will Show Them! Essays in Honour of Dov Gabbay, v.2*, pages 663–692. King's College Publications, 2005.