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UNIFICATION IN INTERMEDIATE LOGICS

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Abstract. This paper contains a proof–theoretic account of unification in intermediate logics. It is shown that many existing results can be extended to fragments that at least contain implication and conjunction. For such fragments, the connection between valuations and most general unifiers is clarified, and it is shown how from the closure of a formula under the Visser rules a proof of the formula under a projective unifier can be obtained. This implies that in the logics considered, for the n-unification type to be finitary it suffices that the m-th Visser rule is admissible for a sufficiently large m. At the end of the paper it is shown how these results imply several well-known results from the literature.

§1. Introduction.

Unification theory is concerned with the problem whether two given terms can be identified via a substitution against a certain background theory of equality. In this paper the background theories are fragments of propositional intermediate logics that contain at least implication and conjunction. Thus the terms are formulas, and equality is logical equivalence in the logic. In this setting unification becomes the study of substitutions under which a formula becomes provable in a logic, in which case the substitutions are called the *unifiers* of the formula. This paper presents a proof-theoretic treatment of unification in these logics. It originates from the unpublished PhD-thesis [32] by the second author, while the first author obtained new proofs of the theorems in [32] and thereby strengthened and simplified the results. In this introduction we will explain what these results are and discuss related work. We start by explaining what unification types and admissible rules are.

1.1. Unifiers. In intermediate logics, any consistent formula is classically satisfied under some valuation (assigning 0 or 1 to the atoms). This valuation corresponds to a unifier of the formula, by taking $\top$ for 1 and $\bot$ for 0. Thus the existence of a unifier is equivalent to the consistency of the formula. Finding a maximal unifier for a formula is less trivial. A substitution is a *maximal unifier* (*mu*) of a formula if among the unifiers of the formula it is maximal in the following ordering:

$$\tau \leqslant \sigma \equiv_{def} \exists \pi (\tau =_{\mathsf{L}} \pi\sigma),$$

and it is a *most general unifier* (*mgu*) if it is also unique modulo $=_{\mathsf{L}}$. Here $=_{\mathsf{L}}$ is the equivalence relation on substitutions associated with the logic: $\sigma =_{\mathsf{L}} \tau$ if and only

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if $\sigma(p) \leftrightarrow \tau(p)$ is derivable for all atoms p. If $\tau \leqslant \sigma$ we say that τ is *less general* than σ. Mgus generate all unifiers of a formula, which is the reason that they play an important role in unification theory. The mgus that are important in the setting of logics are projective unifiers. σ is a *projective unifier* (*pu*) of a formula A if it is a unifier of A and A implies that σ is the identity:

$$\forall p: \; A \vdash \sigma(p) \leftrightarrow p.$$

In this case A is called *projective*. Projective unifiers were first introduced by Wroǹski under the name *transparent unifiers* [37], and they form one of the key notions in unification theory in logic.

In classical propositional logic every consistent formula has a mgu, but in non-classical logics this no longer is the case. In intuitionistic logic $p \vee \neg p$ is an example of such a formula: the two valuations that classically satisfy this formula correspond to substitutions $\sigma_1(p) = \top$ and $\sigma_2(p) = \bot$, that cannot both be less general than another unifier. For if τ is a unifier of $p \vee \neg p$, the logic derives either τp or $\neg \tau p$ by the disjunction property. Therefore either $\sigma_1 \leqslant \tau$ or $\sigma_2 \leqslant \tau$ holds, but not both. Hence $p \vee \neg p$ has no mgu.

1.2. Unification types. The phenomenon that certain unification problems do not have mgus gave rise to the definition of unification types, which classify theories according to the existence or non-existence of mgus and mus. Although this definition applies to all theories, here we restrict ourselves to logics. A *complete* set of unifiers for a formula is a set of unifiers such that every unifier of the formula is less general than a unifier in the set. A formula is *unifiable* if it has at least one unifier. A logic has *unitary* n*-unification type* if every unifiable formula of size (number of symbols) at most n has a mgu and *finitary* n*-unification type* if every unifiable formula of size at most n has a finite complete set of mus. It has *unitary unification type* if it has unitary n-unification type for all n, and similarly for the finitary unification type. The other two types, infinitary (every unifiable formula has a (in)finite complete set of mus) and nullary (some unifiable formula does not have mus) will not be discussed any further in this paper.

1.3. Admissible rules. As it turns out, many modal and intermediate logics have finitary unification, and this paper provides a proof-theoretic proof of this fact for intermediate logics. We use a notion that is closely related to unification types, that of admissible rules. A multi-conclusion rule Γ/Δ for finite sets of formulas Γ and Δ is *admissible*, written $\Gamma \mathrel{|\!\!\sim}_{\mathsf{L}} \Delta$, if and only if

$$\forall \sigma \; (\forall A \in \Gamma \vdash_{\mathsf{L}} \sigma A \;\Rightarrow\; \exists A \in \Delta \vdash_{\mathsf{L}} \sigma A).$$

Thus a rule is admissible if it can be added to the logic without leading to new theorems, and the set of admissible rules form the largest class of inferences allowed to obtain the theorems of a logic and nothing more. It may be useful to know the admissible rules of a logic for various reasons. They may shorten proofs, as is the case for the cut rule in many Gentzen calculi. Or they may express properties of the logic that are intuitive yet invisible in standard axiomatizations, such as the disjunction property in the case of intuitionistic logic. The description of the admissible rules of a logic is usually given via a basis, which is a set of rules that axiomatize the admissible rules of the logic, see [18] for further details.

1.4. Projective approximations. In recent years, these two notions, unification and admissible rules, have been shown to be intimately connected, starting from the pioneering work of Ghilardi. In [10] Ghilardi proves that IPC has finitary unification by showing that every formula has a finite *projective approximation* Π_A, which is a finite set of projective formulas such that $\bigvee \Pi_A \vdash A \mathrel{|\!\sim}_{\mathsf{IPC}} \Pi_A$. Note that this indeed implies that A has a finite complete set of maximal unifiers, namely the projective unifiers of the formulas in its projective approximation. The proofs in this paper that certain intermediate logics and fragments thereof have finitary or unitary unification follow the same pattern. By showing the stronger $\bigvee \Pi_A \vdash_{\mathsf{L}} A \vdash_{\mathsf{L}}^{\mathcal{R}} \Pi_A$, for some particular set of admissible rules $\mathcal{R}$, we simultaneously prove that $\mathcal{R}$ is a basis for the admissible rules of L. Here $\vdash_{\mathsf{L}}^{\mathcal{R}}$ denotes derivability in L extended by the rules $\mathcal{R}$. In this way results about unification are closely tied to results about admissible rules.

1.5. Results. What is new in this paper is the way in which it is proved that the formulas in the projective approximation are projective. This is related to the way the projective formulas are obtained. First we show, in the style of [21], that for every formula A there is a set of formulas Π_A, the irreducible projective approximation of A, such that $\bigvee \Pi_A \vdash_{\mathsf{L}} A \vdash_{\mathsf{L}}^{\overline{\mathsf{V}}} \Pi_A$ and every formula in Π_A is closed under the Visser rules $\overline{\mathsf{V}}$. The Visser rules will be introduced in Section 6 and closure under a rule means that if all formulas in the antecedent are derivable, then so is at least one of the formulas in the conclusion. Then we show that the formulas in Π_A are projective by showing that they are strongly satisfiable (to be defined below) and that, Theorem 5.5, strong satisfiability implies projectivity. The combination of these results imply that if the Visser rules are admissible, they form a basis and the logic has finitary or unitary unification.

The use of strong satisfiability also clarifies the connection between classical valuations and projective unifiers that is already present in Ghilardi's work [12]. In IPC, the condition of strong satisfiability is also necessary for projectivity, and therefore it can be viewed as an analogue of Ghilardi's semantical characterization of projective formulas.

Most theorems in this paper apply to any fragment of any intermediate logic that contains implication and conjunction. In this way some of the known results on unification and admissibility in intermediate logics are generalized to such fragments. But rather than this slight generalization, which is not very surprising or particularly useful, we think the merit of this approach is its generality, which stems from the fact that the proofs are purely syntactic and do not presuppose completeness with respect to a well-behaved class of Kripke models. It also provides short proofs of various other results on unification and admissible rules. That it is applicable also to other logics is shown in [20], where the method is applied to transitive modal logics.

1.6. Related work. We saw that classical logic has unitary unification type and any intermediate logic with the disjunction property does not. Ghilardi [10] proved that intuitionistic propositional logic has finitary unification, and he and Wroński [37] proved that De Morgan Logic and Gödel–Dummett logic have unitary unification, respectively. Intermediate logics of nullary type are also known to exist [13].

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Whether there is an intermediate logic with infinitary unification type is not known. Dzik located logics with unitary unification in the lattice of intermediate logics by showing that all extensions of De Morgan logic have nullary or unitary unification, and that for the latter the converse holds too [6]. There exist modal logics for which the unification problem is undecidable [36], but whether there exists an intermediate logic with this property is unknown. Rybakov showed that for intuitionistic and De Morgan logic with parameters the unification type is finitary [35].

For fragments the situation is as follows. Prucnal [29] proved that the implication fragment of IPC has unitary unification, and the same holds for the implication-conjunction and the implication-conjunction-negation fragment of any intermediate logic [25]. The implication-negation fragment, however, has finitary instead of unitary unification, as was shown by Cintula and Metcalfe [4], who in the same paper also provide a basis for the admissible rules of that fragment.

The paper is built-up as follows. Section 2 contains preliminaries and Section 3 explains the idea behind the proof of the main Theorem 6.7. Section 4 contains the technicalities needed to prove Theorem 5.5, which is done in Section 5. In Section 6 the relation between the Visser rules and strong satisfiability is established, which in Section 7 is used to obtain results about projective approximations, which imply results on unification in Section 8.

§2. **Preliminaries.** Let $\mathcal{L}$ be a language for propositional logic with propositional variables, or atoms, $\mathcal{P} = \{p_1, p_2, \dots\}$. p, q, r, s denote arbitrary elements of $\mathcal{P} \cup \{\bot\}$, and in case that $\bot$ is not part of the language, of $\mathcal{P}$. In this paper L can be any fragment of any propositional intermediate logic that contains the $\{\wedge, \to, \top\}$-fragment of intuitionistic logic IPC. $\top$ is not strictly necessary, but it will be convenient to have a separate symbol for truth available. Note that every logic is a fragment of itself. $\vdash_{\mathsf{L}}$ stands for derivability in logic L and we sometimes omit the "L".

We use $\Gamma, \Pi, \Delta, \Sigma$ to denote finite sets of formulas. *Sequents* are expressions $\Gamma \Rightarrow \Delta$. In the case that $\bot$ and negation do not belong to the language, we require that Δ is not empty, and in the case that disjunction does not belong to the language, we require that $|\Delta| \leq 1$. S, R range over sequents. Given a subset $\mathcal{L}'$ of $\mathcal{L}$, a formula, sequent or set of sequents is *in* $\mathcal{L}'$ if all symbols in it belong to $\mathcal{L}'$.

Atomic implications are implications of the form $p \to q$, where $p \neq \bot$. Note that $\neg p$ is an atomic implication. $\mathcal{L}_n$ is $\mathcal{L}$ minus the atoms p_i for $i > n$. Given a set of sequents $\mathcal{G}$, $\mathcal{L}_{\mathcal{G}}$ denotes $\mathcal{L}$ minus the atoms that do not occur in $\mathcal{G}$ and $n_{\mathcal{G}}$ is the number of different atoms that occur in $\mathcal{G}$. For a sequent S, $i(S)$ denotes the number $(n_{\{S\}} + 2m)^2$, where m is the number of implications in S, and similarly for formulas.

A sequent $(\Gamma \Rightarrow \Delta)$ is *irreducible* if Δ is empty or consists of atoms, and Γ is empty or consists of atoms and atomic implications. $\mathcal{S}_{\mathcal{G}}$ is the set of irreducible sequents in $\mathcal{L}_{\mathcal{G}}$. $\mathrm{var}(A)$ is the set of atoms that occur in A and similarly for sequents and sets of sequents. In general, we will use $\mathcal{S}$ and $\mathcal{T}$ for arbitrary finite sets of sequents and $\mathcal{G}$ and $\mathcal{H}$ for finite sets of irreducible sequents.

We need the following notation, where v stands for variable, i for implication, a for antecedent, and c for conclusion:

$$
\begin{array}{ll}
(\Gamma \Rightarrow \Delta)^a \equiv_{def} \Gamma & (\Gamma \Rightarrow \Delta)^c \equiv_{def} \Delta \\
\Gamma^v \equiv_{def} \{p \in \mathcal{P} \mid p \in \Gamma\} \quad \Gamma^i \equiv_{def} \Gamma \backslash \Gamma^v & \Gamma^i_\Delta \equiv_{def} \{(p \to q) \in \Gamma \mid p \in \Delta\} \\
\Gamma^a \equiv_{def} \{p \mid \exists q\,(p \to q) \in \Gamma\} & \Gamma^c \equiv_{def} \{q \mid \exists p\,(p \to q) \in \Gamma\} \\
\Pi_{\mathcal{S}} \equiv_{def} \bigcup\{S^{ai} \mid S \in \mathcal{S}\} & \Sigma^{\mathcal{G}}_{\mathcal{S}} \equiv_{def} \{p \mid \mathcal{G} \vdash \Pi_{\mathcal{S}} \Rightarrow p\}.
\end{array}
$$

For $k \in \{a, c\}$ and $l \in \{a, c, i, v\}$, S^{kl} abbreviates $(S^k)^l$.

Sequents are interpreted in the usual way: $I(S) = (\bigwedge S^a \to \bigvee S^c)$, where an empty conjunction is $\top$ and an empty disjunction $\bot$. For notational convenience we sometimes write S for $I(S)$, for example in $\vdash S$, which thus should be read as $\vdash I(S)$. Sets of sequents are interpreted as conjunctions:

$$I(\mathcal{G}) \equiv_{def} \bigwedge_{S \in \mathcal{G}} I(S),$$

interpreting empty disjunctions by $\bot$ and empty conjunctions by $\top$. For sets of irreducible sequents $\mathcal{G}$ we sometimes denote $I(\mathcal{G})$ by the noncalligraphic version of the name of the set. For example, $G \equiv_{def} I(\mathcal{G})$ and $G_i \equiv_{def} I(\mathcal{G}_i)$. When we speak of the (in)consistency of $\mathcal{G}$, we mean the (in)consistency of G.

We use σ and τ to denote substitutions, which are maps from propositional formulas to propositional formulas that commute with the connectives. ι is the identity substitution. As usual, $\tau\Gamma = \{\tau A \mid A \in \Gamma\}$ and $\tau S = (\tau S^a \Rightarrow \tau S^c)$. Throughout the paper substitutions are assumed to have finite domains, denoted by $dom(\cdot)$. We use the following notation for substitutions with the same domain:

$$\sigma \leftrightarrow \tau \equiv_{def} \bigwedge_{p \in dom(\sigma)} \sigma(p) \leftrightarrow \tau(p).$$

Observe that

$$\vdash \sigma \leftrightarrow \tau \text{ implies } \vdash \sigma A \leftrightarrow \tau A.$$

Given two finite sets of sequents $\mathcal{T}$ and $\mathcal{T}'$, $\mathcal{S}$ is *closed under* the multi-conclusion rule $\mathcal{T}/\mathcal{T}'$ if whenever $I(\mathcal{S})$ derives $I(\mathcal{T})$ it derives $I(S)$ for at least one sequent S in $\mathcal{T}'$. The rule is *admissible* if for all substitutions σ, $\mathcal{S}$ is closed under the rule $\sigma\mathcal{T}/\sigma\mathcal{T}'$. In the single-conclusion case, which means the case that $|\mathcal{T}'| \leq 1$, $\mathcal{T}/\mathcal{T}'$ is *derivable* if $I(\mathcal{S})$ derives $I(\mathcal{T}) \to I(\mathcal{T}')$. The same notions apply to logics by considering a logic L as the set of its theorems (thus by taking for $\mathcal{S}$ the set $\{(\Rightarrow A) \mid \vdash_{\mathsf{L}} A\}$). In this case we write $\mathcal{T} \mathrel{|\!\sim}_{\mathsf{L}} \mathcal{T}'$ if the rule is admissible, and $\mathcal{T} \vdash_{\mathsf{L}} \mathcal{T}'$ if the rule is derivable.

§3. Proof idea.

The main technical part of the paper is the proof of Theorem 6.7, which is divided in two parts, the proof of Theorem 5.5 and the proof of Lemma 6.6. In this section we briefly explain these proofs in general, the next three sections provide the technical details. The proofs make use of a connection between valuations and substitutions that goes back to Prucnal [29, 31] and are crucial in the work of Ghilardi [10] and Rozière [32, 33].

Given a set of atoms I and a formula A we define the valuation v_I and substitution σ_I as follows:

$$v_I(p) \equiv_{def} \begin{cases} 1 & \text{if } p \in I \\ 0 & \text{if } p \notin I \end{cases} \qquad \sigma_I^A(p) \equiv_{def} \begin{cases} A \to p & \text{if } p \in I \\ A \wedge p & \text{if } p \notin I. \end{cases}$$

For sequents, $v_I(S)$ is short for $v_I(I(S))$ and σ_I^S for $\sigma_I^{I(S)}$. It is not difficult to see that in classical logic $v_I(A) = 1$ implies $\vdash_{\mathsf{CPC}} \sigma_I^A A$. This no longer holds in IPC, a counter example is provided below. There is, however, a weaker form of the statement that does hold. For a certain composition σ_A of substitutions of the form σ_I^A and a certain notion of satisfiablity called strong satisfiability, it is shown in Theorem 5.5 that if A is strongly satisfiable, then σ_A is a unifier for A. As σ_A is a composition of σ_I^A's, it is a projective unifier of A, thus showing that strong satisfiability implies projectivity. Lemma 6.6 shows that closure under the Visser rules (to be defined below) is a necessary and sufficient condition for strong satisfiability. Together with Theorem 5.5 it therefore proves Theorem 6.7.

The proof of Lemma 6.6 is fairly straightforward and will be discussed in Section 6, but the proof of Theorem 5.5 needs some clarification, which we provide in the remainder of this section.

Instead of formulas it is convenient to work with a set of irreducible sequents $\mathcal{G}$ and corresponding formula $G = I(\mathcal{G})$. As mentioned above, $v_I(S) = 1$ does not imply $\vdash_{\mathsf{IPC}} \sigma_I^S(S)$, $S = (p \Rightarrow q, r)$ and $I = \{q\}$ being a counter example. But it is not hard to see that $v_I(S) = 1$ does imply $\vdash_{\mathsf{IPC}} \sigma_I^S(S)$ in case S is an irreducible sequent and S^c contains at most one formula. So (the formulas corresponding to) these sequents are projective anyway. For the remaining sequents, those for which S^c contains more than one atom, we consider them relative to a set of irreducible sequents $\mathcal{S} \subseteq \mathcal{G}$ in which they are contained. The notion of satisfiability that we need is that of strong satisfiability with respect to $\mathcal{S}$, meaning that $v_I(S^{av}, S^{ai}_{\Sigma^{\mathcal{G}}_{\mathcal{S}}} \Rightarrow S^c \cap \Sigma^{\mathcal{G}}_{\mathcal{S}})$, denoted by $\overline{v}_I(S \mid \mathcal{S})$, equals 1.

The aim is to prove for $S_1 \in \mathcal{S}$ and a composition $\sigma = \sigma_m \dots \sigma_1$ of substitutions of the form σ_I^G that $\vdash \sigma S_1$ given that S_1 is strongly satisfiable with respect to $\mathcal{G}$. Let us denote $\sigma_m \dots \sigma_i$ by $\overline{\sigma}_i$, which means that $\overline{\sigma}_1 = \sigma$. Note that in order to prove $\vdash \sigma S_1$, one has to show that $\vdash \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1^c$. For this it suffices to show that for some $i_2 \geq 1$ and for all $S_2 \in \mathcal{G}$:

$$\vdash \overline{\sigma}_1 S_1^a \Rightarrow I(\overline{\sigma}_{i_2} S_2). \tag{1}$$

This would namely imply that $\vdash \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_{i_2} I(\mathcal{G})$, which means $\vdash \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_{i_2} G$. And as the σ_j are such that $\vdash G \to \sigma_{i_2-1} \dots \sigma_1 G$, an application of $\overline{\sigma}_{i_2}$ gives $\vdash \overline{\sigma}_{i_2} G \Rightarrow \overline{\sigma}_1 G$. Thus $\vdash \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 G$, which implies $\vdash \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1$, as $S_1 \in \mathcal{G}$. And thus $\vdash \overline{\sigma}_1 S_1^a \Rightarrow \overline{\sigma}_1 S_1^c$.

Repeating this argument shows that to prove (1) it suffices to show that for some $i_3 \geq i_2$ and for all $S_3 \in \mathcal{G}$:

$$\vdash \overline{\sigma}_1 S_1^a, \overline{\sigma}_{i_2} S_2^a \Rightarrow I(\overline{\sigma}_{i_3} S_3). \tag{2}$$

Continuing this argument, one sees that in order to prove $\vdash \sigma G$ it suffices to show that for all possible sequences $S_1, \dots, S_m$ of sequents from $\mathcal{G}$ and all numbers $1 \leq i_2 \leq i_3 \leq \dots \leq i_m$ there is a $j \geq i_m$ such that for all $S \in \mathcal{G}$:

$$\vdash \overline{\sigma}_1 S_1^a, \overline{\sigma}_{i_2} S_2^a, \dots, \overline{\sigma}_{i_m} S_{i_m}^a \Rightarrow I(\overline{\sigma}_j S). \tag{3}$$

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As one can see (and will be proved below), if $\sigma_j = \sigma_I^G$ for an I such that $v_I(S^{av}, S^{ai}_{\Sigma^{\mathcal{G}}_{\mathcal{S}}} \Rightarrow S^c \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}) = 1$, where $S \in \mathcal{S} = \{S_1, \ldots, S_m\}$, then (3) holds. This is the core idea behind the proof of Theorem 5.5 stating that strong satisfiability implies projectivity. This completes the informal explanation, and we continue with the technical details.

§**4. Substitutions and valuations.** The discussion above serves as a motivation for the notions introduced in this section. In this and the next section we consider an arbitrary finite set $\mathcal{G}$ of *irreducible* sequents, and corresponding formula $G = I(\mathcal{G})$, and assume the atoms that occur in $\mathcal{G}$ to be $\{p_1, \ldots, p_{n_{\mathcal{G}}}\}$. Most definitions are relative to $\mathcal{G}$ but for simplicity we do not always indicate this in our notation. $\mathcal{G}^\star$ consists of those sequents $S \in \mathcal{G}$ such that $(S^{av} \cap S^{cv})$ is empty. Note that all sequents in $\mathcal{G}$ that are not in $\mathcal{G}^\star$ are derivable.

We fix an arbitrary enumeration $J_1, \ldots, J_{2^{n_{\mathcal{G}}}}$ of all subsets of $\{p_1, \ldots, p_{n_{\mathcal{G}}}\}$, and let I range over subsets of $\{p_1, \ldots, p_{n_{\mathcal{G}}}\}$. Given a set I the valuation v_I has been defined above, and σ_I denotes σ_I^G, also defined at the beginning of Section 3. We extend it to a valuation for sequents S relative to a set of sequents $\mathcal{S}$: S is *strongly satisfiable* with respect to $\mathcal{S}$ if

$$\overline{v}_I(S \mid \mathcal{S}) \equiv_{def} v_I(S^{av}, S^{ai}_{\Sigma^{\mathcal{G}}_{\mathcal{S}}} \Rightarrow S^c \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}) = 1.$$

If $S^{ai}_{\Sigma^{\mathcal{G}}_{\mathcal{S}}}$ is empty, the right side is read as $v_I(S^{av} \Rightarrow S^c \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}) = 1$, similarly for S^{av} and $S^c \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}$. The valuations are extended to sets of sequents in the usual way: $\overline{v}_I(\mathcal{S}' \mid \mathcal{S}) = 1$ if and only if $\overline{v}_I(S \mid \mathcal{S}) = 1$ for all $S \in \mathcal{S}'$. We write $\overline{v}_I(\mathcal{S})$ for $\overline{v}_I(\mathcal{S} \mid \mathcal{S})$. $\mathcal{G}$ is *strongly satisfiable* if for all $\mathcal{S} \subseteq \mathcal{G}^\star$ there is an I such that $\overline{v}_I(\mathcal{S}) = 1$.

We use the following abbreviations in this and the next section:

$$g = 2^{n_{\mathcal{G}}} \quad \sigma_G \equiv_{def} (\sigma_{J_g} \ldots \sigma_{J_1})^{(|\mathcal{G}|+1)}.$$

Thus σ_G is the concatenation of $g(|\mathcal{G}|+1)$ substitutions. The i-th substitution in σ_G (reading from right to left) is denoted by σ_i and for $i < j$, $\sigma_j \ldots \sigma_i$ is denoted by $\sigma_{j,i}$. We denote $\sigma_{g(|\mathcal{G}|+1),i} = \sigma_{g(|\mathcal{G}|+1)} \ldots \sigma_i$ by $\overline{\sigma}_i$. For example, $\sigma_2 = \sigma_{g+2} = \cdots = \sigma_{g|\mathcal{G}|+1} = \sigma_{J_2}$, $\overline{\sigma}_1 = \sigma_G$, and $\overline{\sigma}_{g+1} = \overline{\sigma}^{|\mathcal{G}|}$. Note that for $i < j$, $\overline{\sigma}_j$ is the tail of $\overline{\sigma}_i$. We denote by I_i the set J_j such that $\sigma_i = \sigma_{J_j}$. For valuations we define: $v_i \equiv_{def} v_{I_i}$.

Lemma 4.1. *For all* $i < j$: $\vdash G \to (\iota \leftrightarrow \sigma_i \leftrightarrow \overline{\sigma}_{j,i})$ *and* $\vdash \overline{\sigma}_j G \to \overline{\sigma}_i G$.

Proof. The first equivalence in the first statement is clear. The second equivalence follows from this and the fact that $\vdash (B \leftrightarrow C) \to (A[B/p] \leftrightarrow A[C/p])$ for any atom p.

The first statement implies that $\vdash G \to \sigma_{j-1,i} G$, which implies $\vdash \overline{\sigma}_j G \to \overline{\sigma}_i G$. ⊣

Define

$$F(i_1, \ldots, i_j, S_1, \ldots, S_j, A) \equiv_{def} \overline{\sigma}_{i_1} S_1^a, \overline{\sigma}_{i_2} S_2^a, \ldots \overline{\sigma}_{i_j} S_j^a \Rightarrow A. \tag{4}$$

Proposition 4.2. *For all* $\mathcal{S} = \{S_1, \ldots, S_j\} \subseteq \mathcal{G}$ *and all* $1 \leq i_1, \ldots, i_j \leq g(|\mathcal{G}|+1)$, *if* $\overline{v}_I(\mathcal{S}) = 1$, *then for all* $S \in \mathcal{S}$: $\vdash F(i_1, \ldots, i_j, S_1, \ldots, S_j, \sigma_I S)$.

Proof. First the case that $S^{av} \backslash I$ or $S^c \cap \Sigma^{\mathcal{G}}_{\mathcal{S}}$ is nonempty. In case $p \in S^{av} \backslash I$, $\sigma_I(p) = G \wedge p$, and thus $\sigma_I S^a$ derives G, and therefore it derives S^c and $\sigma_I S^c$ by Lemma 4.1. Thus proving that $\sigma_I S$ is derivable, even without the assumptions.

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In case $p \in S^c \cap \Sigma_S^{\mathcal{G}}$, $\sigma_I(p) = G \to p$, and $\vdash G \wedge \bigwedge_h S_h^a \to p$. Hence also $F(i_1, \ldots, i_j, S_1, \ldots, S_j, \sigma_I p)$, which gives $\vdash F(i_1, \ldots, i_j, S_1, \ldots, S_j, \sigma_I S)$ as $p \in S^c$.

Finally the remaining case: there is an implication $(p \to q) \in S^a$ with $p \in \Sigma_S^{\mathcal{G}}$, $p \in I$ and $q \notin I$. Thus $\sigma_I((p \to q)) = (G \to p) \to G \wedge q$. As $p \in \Sigma_S^{\mathcal{G}}$, $G \vdash \bigwedge_h S_h^a \to p$, and therefore Lemma 4.1 implies $\vdash \bigwedge_h \overline{\sigma}_{i_h} S_h^a \to (G \to p)$. Hence $\vdash \bigwedge_h \overline{\sigma}_{i_h} S_h^a \wedge \sigma_I S^a \to G \wedge q$. The fact that $S \in \mathcal{G}$ finally leads to $\vdash \bigwedge_h \overline{\sigma}_{i_h} S_h^a \wedge \sigma_I S^a \to \sigma_I S^c$, which is what we had to show. ⊣

§5. **Unifiers.** In this section we show that every strongly satisfiable set of sequents $\mathcal{G}$ has σ_G as a projective unifier. We need some terminology to be able to prove this theorem by backwards induction. A sequence of m numbers followed by m sequents $i_1, \ldots, i_m, S_1, \ldots, S_m$ is *appropriate* if $m \leq |\mathcal{G}|$,

$$1 = i_1 \leq g < i_2 \leq 2g \leq \cdots < i_m \leq mg,$$

and the sequents are distinct and belong to $\mathcal{G}$. It is *G-sufficient* if for all numbers j such that $mg < j \leq (m+1)g$ and $\overline{v}_j(\{S_1, \ldots, S_m\}) = 1$, the formula $F(i_1, \ldots, i_m, S_1, \ldots, S_m, \overline{\sigma}_j G)$ is derivable, where F is defined in (4).

LEMMA 5.1. *If $\mathcal{G}$ is strongly satisfiable, then for every appropriate sequence $i_1, \ldots, i_m, S_1, \ldots, S_m$ and any number $k > 0$ there exists a number h such that $kg < h \leq (k+1)g$ and $\overline{v}_h(\{S_1, \ldots, S_m\}) = 1$.*

PROOF. As $\mathcal{G}$ is strongly satisfiable, there is a $j \leq g$ such that $\overline{v}_j(\{S_1, \ldots, S_m\})$ equals 1. Since $v_j = v_{kg+j}$, the lemma follows. ⊣

LEMMA 5.2. *If $\mathcal{G}$ is strongly satisfiable then for all $m \leq |\mathcal{G}|$: if all appropriate sequences of length $2m$ are G-sufficient, then so are all appropriate sequences of length $2m - 2$.*

PROOF. Consider an appropriate $i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}$ and let j be such that $(m-1)g < j \leq mg$ and $\overline{v}_j(\{S_1, \ldots, S_{m-1}\}) = 1$. We have to show that for all $S \in \mathcal{G}$:

$$\vdash F(i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}, \overline{\sigma}_j S). \tag{5}$$

If $S \in \{S_1, \ldots, S_{m-1}\}$, then (5) follows from Proposition 4.2. If, on the other hand, $S \notin \{S_1, \ldots, S_{m-1}\}$, then $i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S$ is an appropriate sequence of length $2m$. By Lemma 5.1 there exists a number h such that $mg < h \leq (m+1)g$ and $\overline{v}_h(\{S_1, \ldots, S_{m-1}, S\}) = 1$. Therefore by G-sufficiency

$$\vdash F(i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S, \overline{\sigma}_h G).$$

Since $\vdash \overline{\sigma}_h G \to \overline{\sigma}_j G$ and $S \in \mathcal{G}$, this implies that

$$\vdash F(i_1, \ldots, i_{m-1}, j, S_1, \ldots, S_{m-1}, S, \overline{\sigma}_j S).$$

Hence $\vdash F(i_1, \ldots, i_{m-1}, S_1, \ldots, S_{m-1}, \overline{\sigma}_j S)$, which is what we had to show. ⊣

LEMMA 5.3. *If $S \in \mathcal{G}$ and $1, S$ is G-sufficient, then $\vdash \overline{\sigma} S$.*

PROOF. By Lemma 5.1 there exists an $i \leq 2g$ such that $\overline{v}_i(\{S\}) = 1$. Hence $\vdash \overline{\sigma}_1 S^a \to \overline{\sigma}_i G$. Since $\vdash \overline{\sigma}_i G \to \overline{\sigma}_1 G$ by Lemma 4.1, this gives $\vdash \overline{\sigma}_1 S^a \to \overline{\sigma}_1 G$. As $S \in \mathcal{G}$, $\vdash \overline{\sigma}_1 S$ follows, that is, $\vdash \sigma_G S$. ⊣

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LEMMA 5.4. *Every appropriate sequence of length* $2|\mathcal{G}|$ *is* G*-sufficient.*

PROOF. Let $|\mathcal{G}| = m$ and consider an appropriate sequence $i_1, \dots, i_m, S_1, \dots, S_m$ and let j be such that $mg < j \leq (m+1)g$ and $\overline{v}_j(\{S_1, \dots, S_m\}) = 1$. Because $m = |\mathcal{G}|$ and the S_i are distinct, $\{S_1, \dots, S_m\} = \mathcal{G}$. Therefore Proposition 4.2 gives $\vdash F(i_1, \dots, i_m, S_1, \dots, S_m, \overline{\sigma}_j G)$. ⊣

THEOREM 5.5. *If* $\mathcal{G}$ *is strongly satisfiable, then* $\vdash \sigma_G G$.

PROOF. By Lemma 5.4 every appropriate sequence of length $2|\mathcal{G}|$ is G-sufficient. By repeated application of Lemma 5.2 it follows that $1, S$ is G-sufficient for every $S \in \mathcal{G}$. This implies $\vdash \sigma_G S$ by Lemma 5.3. ⊣

§6. Rules and satisfiability.

In this section we show that closure under the Visser rules, (in sequent notation)

$$\frac{\{\Gamma \Rightarrow \Delta\}}{\{\Gamma \Rightarrow A \mid A \in \Gamma^a \cup \Delta\}}\ \overline{\mathsf{V}} \qquad (\Gamma \text{ consists of implications only}),$$

implies strong satisfiability, thus proving by Theorem 5.5 that closure under the Visser rules implies projectivity. We make use of a property, being closed, that is equivalent to being closed under the Visser rules, Lemma 6.5, but easier to apply. Before giving the formal definition, we treat two examples indicating how closure conditions can imply satisisfiability. In both examples $\mathcal{G}$ is a consistent set of sequents, which means that it does not derive the empty sequent.

Given $S \in \mathcal{G}$, the only way in which $S = (\Gamma \Rightarrow \Delta)$ cannot be strongly satisfiable with respect to $\mathcal{G}$, which means $\overline{v}_I(S \mid \mathcal{G}) = v_I(S^{av}, S^{ai}_{\Sigma_\mathcal{G}} \Rightarrow S^v \cap \Sigma_\mathcal{G}) = 0$ for all I, is if Γ consists of implications such that $\Gamma^a \cap \Sigma_\mathcal{G} = \Delta \cap \Sigma_\mathcal{G} = \emptyset$. However, if $\mathcal{G}$ is closed under the Visser rules, it contains $(\Gamma \Rightarrow p)$ for at least one $p \in \Gamma^a \cup \Delta$. But then $p \in \Sigma_\mathcal{G}$, contradicting $\Gamma^a \cap \Sigma_\mathcal{G} = \Delta \cap \Sigma_\mathcal{G} = \emptyset$. This shows that for single $S \in \mathcal{G}$, $\mathcal{G}$ being closed under the Visser rules implies that S is strongly satisfiable with respect to $\mathcal{G}$.

If we consider more than one sequent, we need the notion of being closed, as illustrated by the following example. Consider sequents $S_1, S_2 \in \mathcal{G}$ such that $S_1 = (\Gamma, q \to r \Rightarrow q)$ and $S_2 = (q \Rightarrow \Delta)$ and Γ consists of implications not equal to $(s \to s)$ and $q \notin \Delta$. If $\{S_1, S_2\}$ is not strongly satisfiable with respect to $\mathcal{G}$, then $\Gamma^a \cup \Delta$ does not contain elements from $\Sigma_\mathcal{G}$. If $\mathcal{G}$ is closed under the Visser rules and as it derives $\Gamma, q \to r \Rightarrow \Delta$, it also derives $\Gamma \Rightarrow p$ for at least one $p \in \Gamma^a \cup \{q\} \cup \Delta$. This, however, does no lead to a contradiction as in the case above, as q can be taken for p. But as we will show in Lemma 6.5, closure under the Visser rules implies being closed, which implies that $\mathcal{G}$ derives $\Gamma \Rightarrow p$ for at least one $p \in \Gamma^a \cup \Delta$, which is a contradiction. Thus showing that closure under the Visser rules of $\mathcal{G}$ implies that $\{S_1, S_2\}$ is strongly satisfiable with respect to $\mathcal{G}$.

The seemingly stronger notion of being closed is defined as follows. We use the following notation for sets of formulas in which some implications are replaced by their antecedents:

$$\Gamma_I \equiv_{def} I \cup \Gamma \backslash \{(p \to q) \in \Gamma \mid p \in I\}.$$

$\mathcal{G}$ is *closed* if for all irreducible sequents $(\Gamma \Rightarrow \Delta)$ in $\mathcal{L}_{\mathcal{G}}$ such that Γ does not contain atoms, $\mathcal{G}$ is closed under the rule

$$\{\Gamma_J \Rightarrow \Delta \mid J \subseteq I\}/\{\Gamma \Rightarrow p \mid p \in (\Gamma^a \backslash I) \cup \Delta\}.$$

Note that the Visser rules are a special instance of the above rule, namely for $I = \emptyset$.

6.1. Resolution. To prove that being closed implies being strongly satisfiable, we argue by contradiction. Assuming that $\mathcal{G}$ is not strongly satisfiable, which means that $\overline{v}_I(\mathcal{G}) = 0$ for all $I \subseteq \Sigma_{\mathcal{G}}$, we use a resolution proof on the sequents $(S^{av}, S^{ai}_{\Sigma_{\mathcal{G}}} \Rightarrow S^v \cap \Sigma_{\mathcal{G}})$ for $S \in \mathcal{G}$, to conclude that $\mathcal{G}$ is not closed. The following lemmas about resolution proofs are needed to draw that conclusion.

Resolution proofs in the usual sense correspond to sequent derivations in which every sequent contains only atoms, and every inference is a cut. The only difference between resolution proofs and the Σ-resolution proofs we consider below is that in our case we use irreducible sequents. And although the only inference rules are cuts on atoms, the implications are present as additional information that will be used in the next lemmas.

A sequent S is *full in* Σ if $S^{aa} \cap \Sigma \subseteq S^c$. $\mathcal{F}^{\Sigma}_{\mathcal{G}}$ is the set of sequents that are full in Σ and that can be obtained from S by left implications in Σ:

$$\mathcal{F}^{\Sigma}_S \equiv_{def} \{S^a \backslash \Pi, \Pi^c \Rightarrow S^c, (S^a \backslash \Pi)^a \cap \Sigma \mid \Pi \subseteq S^{ai}_{\Sigma}\} \quad \mathcal{F}^{\Sigma}_{\mathcal{G}} \equiv_{def} \cup\{\mathcal{F}^{\Sigma}_S \mid S \in \mathcal{G}\}.$$

LEMMA 6.1. *$F^{\Sigma}_{\mathcal{G}} \dashv\vdash G$. For every $I \subseteq \Sigma$: $v_I(\mathcal{F}^{\Sigma}_S) = 1$ implies $v_I(S) = 1$.*

Given a set of atoms Σ, a Σ-*resolution proof of S from $\mathcal{G}$* is a finite binary tree labelled with sequents: the leafs are (labelled with) sequents in $\mathcal{F}^{\Sigma}_{\mathcal{G}}$, the root is S, and a sequent at an inner node is the result of a cut in Σ on the two sequents immediately above it. $\mathcal{C}_{\mathcal{R}}$ is the set of cut formulas that occur in $\mathcal{R}$. Thus $\mathcal{C}_{\mathcal{R}} \subseteq \Sigma$.

LEMMA 6.2. *If $v_I(\mathcal{G}) = 0$ for all $I \subseteq \Sigma$, then there exists a Σ-resolution proof from $\mathcal{G}$ of a sequent S such that $S^{av} \cup (S^c \cap \Sigma) = \emptyset$.*

PROOF. Let $\overline{\Sigma}$ be the complement of Σ in $\mathcal{P}$. Consider

$$\mathcal{H} = \{(S^{av}, S^{ai} \backslash S^{ai}_{\overline{\Sigma}} \Rightarrow S^c \cap \Sigma) \mid S \in \mathcal{G}\}.$$

Suppose $v_I(\mathcal{G}) = 0$ for all $I \subseteq \Sigma$. This implies that $\mathcal{H}$ is classically inconsistent, as all positive atoms in the sequents in $\mathcal{H}$ belong to Σ. From Lemma 6.1 it follows that $\mathcal{F}^{\mathcal{P}}_{\mathcal{H}}$ is classically inconsistent. By the completeness of resolution refutations for classical logic, there exists a $\mathcal{P}$-resolution proof from $\mathcal{F}^{\mathcal{P}}_{\mathcal{H}}$ of a sequent that does not contain atoms. Since all cut formulas belong to Σ, this proof corresponds to an Σ-resolution proof from $\mathcal{G}$ of a sequent S for which $S^{av} \cup (S^c \cap \Sigma) = \emptyset$. ⊣

LEMMA 6.3. *If $\mathcal{R}$ is a Σ-resolution proof of S, then*

$$(S^{aa} \cap \Sigma) \backslash \mathcal{C}_{\mathcal{R}} \subseteq S^c.$$

PROOF. With induction to the number of cuts in $\mathcal{R}$. If $\mathcal{R}$ does not contain cuts, then $S \in \mathcal{F}^{\Sigma}_{\mathcal{G}}$, which clearly implies the statement. Suppose that $\mathcal{R}$ contains cuts, that $S = (\Gamma, \Pi \Rightarrow \Delta, \Lambda)$, and that the lowest cut is:

$$\frac{\Gamma \Rightarrow p, \Delta \quad p, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

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Consider $q \in (S^{aa} \cap \Sigma)\backslash\mathcal{C}_{\mathcal{R}}$. Since $p \in \mathcal{C}_{\mathcal{R}}$, $q \neq p$. Note that $q \in \Gamma^a$ or $q \in \Pi^a$. In both cases the induction hypothesis implies that $q \in \Delta \cup \Lambda$, which is what we had to show. ⊣

LEMMA 6.4. *For all Σ-resolution proofs $\mathcal{R}$ of $S \in \mathcal{G}$, and all $I \subseteq \mathcal{C}_{\mathcal{R}}$:*

$$G \vdash (S^a_I \Rightarrow S^c) \text{ or } I \cap S^c \neq \emptyset.$$

PROOF. With induction to the number of cuts in $\mathcal{R}$. If there are no cuts, the lemma follows immediately. Suppose there are cuts in $\mathcal{R}$ and consider the lowest cut

$$\frac{\Gamma \Rightarrow p, \Delta \quad p, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$$

Let S be the conclusion, and let $\mathcal{R}'$ and $\mathcal{R}''$ be resolution proofs of respectively the left and the right premise. Consider $I \subseteq \mathcal{C}_{\mathcal{R}}$ and let $J = I \cap \Gamma^a$ and $H = I \cap \Pi^a$ and assume that $I \cap (\Delta \cup \Lambda)$ is empty, otherwise we are done immediately. If there exists a $q \in H\backslash\mathcal{C}_{\mathcal{R}''}$, then $q \in \Lambda$ by Lemma 6.3, and thus $S^c \cap I$ is not empty, which contradicts our assumptions. Therefore $H \subseteq \mathcal{C}_{\mathcal{R}''}$. Thus G derives $(p, \Pi_H \Rightarrow \Lambda)$ by the induction hypothesis.

If $p \in I$, this implies that $\Pi_H \cup \{p\} \subseteq \Pi_I$, and thus G derives $(\Gamma_I \Pi_I \Rightarrow \Delta, \Lambda)$. If, on the other hand, $p \notin I$, then $p \notin J$. Thus $J \subseteq \mathcal{C}_{\mathcal{R}'}$. Hence G derives $(\Gamma_J \Rightarrow p, \Delta)$ by the induction hypothesis. Thus it derives $(\Gamma_J, \Pi_H \Rightarrow \Delta, \Lambda)$ and whence $(\Gamma_I, \Pi_I \Rightarrow \Delta, \Lambda)$. ⊣

LEMMA 6.5. *$\mathcal{G}$ is closed if and only if $\mathcal{G}$ is closed under the multi-conclusion Visser rules.*

PROOF. The direction from left to right follows from the observation at the beginning of this section. For the other direction, we first treat the case that disjunction is in the language. Consider I and $(\Gamma \Rightarrow \Delta)$. Let $\Gamma^{\langle H,\Delta\rangle}$ be the result of replacing every $p \in \Gamma^a \cap H$ by $(p \wedge \bigvee \Delta)$.

First we show that with induction to $|H|$ that for all $J, H \subseteq I$:

$$\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_J^{\langle H,\Delta\rangle} \leftrightarrow \Gamma_J.$$

Since the $\leftarrow$ part is trivial, it remains to show

$$\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_J^{\langle H,\Delta\rangle} \Rightarrow \Gamma_J. \tag{6}$$

If $H = \emptyset$, then $\Gamma_J^{\langle H,\Delta\rangle} = \Gamma_J$. Suppose $|H| > 0$, and consider $p_i \in H$, such that $(p_i \to q) \in \Gamma_J$ (if there is more than one implication in Γ with antecedent p_i, the argument remains the same). By the induction hypothesis

$$\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_{J\cup\{i\}}^{\langle H\backslash\{i\},\Delta\rangle} \Rightarrow \Gamma_{J\cup\{i\}}.$$

Since $\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_{J\cup\{i\}} \Rightarrow \Delta$, this implies that

$$\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_{J\cup\{i\}}^{\langle H\backslash\{i\},\Delta\rangle} \Rightarrow \Delta.$$

As $(p_i \wedge \bigvee \Delta \to q) \in \Gamma_J^{\langle H,\Delta\rangle}$, $(p_i \to q) \notin \Gamma_{J\cup\{i\}}$ and $p_i \in \Gamma_{J\cup\{i\}}$, this gives

$$\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_J^{\langle H,\Delta\rangle} \Rightarrow (p_i \to q).$$

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Thus $\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma_J^{\langle H,\Delta\rangle} \Rightarrow \Gamma_J^{\langle H\backslash\{i\},\Delta\rangle}$, and the induction hypothesis gives (6).

Thus we have shown (6), and in particular

$$\{(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} \vdash_{\mathsf{L}} \Gamma^{\langle I,\Delta\rangle} \leftrightarrow \Gamma.$$

The following observations now prove the lemma:

$$\begin{aligned}(\Gamma_J \Rightarrow \Delta) \mid J \subseteq I\} &\vdash \Gamma \Rightarrow \Delta\\ &\vdash \Gamma^{\langle I,\Delta\rangle} \Rightarrow \Delta\\ &\mathrel{|\!\sim} \{\Gamma^{\langle I,\Delta\rangle} \Rightarrow p \mid p \in (\Gamma^{\langle I,\Delta\rangle})^a \cup \Delta\}\\ &\vdash \{\Gamma \Rightarrow p \mid p \in (\Gamma^{\langle I,\Delta\rangle})^a \cup \Delta\}\\ &\vdash \{\Gamma \Rightarrow p \mid p \in \Gamma^a\backslash I \cup \Delta\}.\end{aligned}$$

The proof for the case that disjunction is not present is similar: we have to replace $(p \wedge \bigvee \Delta \to q)$ by $\bigwedge_{r\in\Delta}(p \wedge r \to q)$. ⊣

LEMMA 6.6. *If $\mathcal{G}$ is closed under $\overline{\mathsf{V}}$, it is strongly satisfiable.*

PROOF. Suppose that $\mathcal{G}$ is not strongly satisfiable. Observe that this implies in particular that for all $I \subseteq \Sigma_{\mathcal{G}}$, $v_I(\mathcal{G}) = 0$. By Lemma 6.2 there is a $\Sigma_{\mathcal{G}}$-resolution proof $\mathcal{R}$ of a sequent $S \in \mathcal{G}$ for which $S^{av} \cup (S^c \cap \Sigma_{\mathcal{G}}) = \emptyset$. Thus S^a consists of implications. Since $\mathcal{C}_{\mathcal{R}} \subseteq \Sigma_{\mathcal{G}}$, Lemma 6.4 implies that G derives $S^a_I \Rightarrow S^c$ for all $I \subseteq \mathcal{C}_{\mathcal{R}}$. If $\mathcal{G}$ would be closed, it would derive $(S^a \Rightarrow p)$ for some $p \in S^{aa}\backslash\mathcal{C}_{\mathcal{R}} \cup S^c$. Hence $p \in \Sigma_{\mathcal{G}}$. Thus $p \in S^c$ by Lemma 6.3, which contradicts that $S^c \cap \Sigma_{\mathcal{G}}$ is empty. Therefore $\mathcal{G}$ is not closed, and thus not closed under $\overline{\mathsf{V}}$ by Lemma 6.5. ⊣

Theorem 5.5 and Lemma 6.6 immediately give the following.

THEOREM 6.7. *If $\mathcal{G}$ is closed under $\overline{\mathsf{V}}$, then G is projective with projective unifier σ_G.*

IPC is the only intermediate logic with the disjunction property for which all multi-conclusion Visser rules are admissible [16]. In this logic the closure condition is a sufficient and necessary condition for strongly satisfiability.

PROPOSITION 6.8. *In* IPC, *$\mathcal{G}$ is closed under $\overline{\mathsf{V}}$ if and only if $\mathcal{G}$ is strongly satisfiable.*

PROOF. The direction from left to right is Lemma 6.6. The other direction uses that $\overline{\mathsf{V}}$ is admissible in IPC. Suppose that $\mathcal{G}$ is $\Sigma_{\mathcal{G}}$-satisfiable. Hence σ_G is a projective unifier of G by Theorem 5.5. Let S/S' be an instance of $\overline{\mathsf{V}}$ such that G derives S. Thus $\sigma_G S$ is derivable in IPC, and by the admissibility of $\overline{\mathsf{V}}$ in IPC, so is $\sigma_G S'$. Therefore G derives S'. ⊣

§7. Projective approximations. In the setting of formulas the multi-conclusion Visser rules are defined as follows:

$$\frac{\bigwedge\Gamma \to \bigvee\Delta}{\{\bigwedge\Gamma \to A\}_{A\in\Gamma^a\cup\Delta}}\ \overline{\mathsf{V}} \qquad (\Gamma \text{ implications only}).$$

Observe that we use the same symbol for the formula and sequent setting. This is justified by the fact that for every instance $S/S_1,\dots,S_m$ of the sequent version, $I(S)/I(S_1),\dots,I(S_m)$ is an instance of the formula version and vice versa.

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Recall that an irreducible formula is a formula of the form $I(S)$, where S is irreducible, and that $\mathcal{F}(p_1,\dots,p_n)$ and $\mathcal{S}(p_1,\dots,p_n)$ are the sets of formulas respectively sequents in which only atoms in $\{p_1,\dots,p_n\}$ occur.

Lemma 7.1. *For every n and every set of sequents $\mathcal{G}\subseteq\mathcal{S}(p_1,\dots,p_n)$, there exists a finite set of irreducible sequents $\mathcal{H}$ such that for every $\Delta\subseteq\mathcal{F}(p_1,\dots,p_n)$:*

1. $I(\mathcal{G})\mathrel{|\!\sim}\Delta$ *if and only if* $I(\mathcal{H})\mathrel{|\!\sim}\Delta$,
2. $I(\mathcal{G})\vdash^{\nabla}\Delta$ *if and only if* $I(\mathcal{H})\vdash^{\nabla}\Delta$,
3. $I(\mathcal{G})\vdash\bigwedge\sigma(\mathcal{H})$ *for some σ that is the identity on* $\mathcal{F}(p_1,\dots,p_n)$.

Proof. We follow the method of proof of a similar lemma in [4]. The *length* of a formula is the number of symbols occurring in it. Let $ml(\mathcal{G})$ be the multiset of the lengths of the formulas in the sequents in $\mathcal{G}$. We prove the lemma by induction on $ml(\mathcal{G})$, using the multiset ordering. At every step we construct a new set of sequents $\mathcal{G}'$ such that 1–3 hold and $ml(\mathcal{G}')<ml(\mathcal{G})$, until $\mathcal{G}'$ is irreducible. This will prove the lemma.

If $ml(\mathcal{G})\leq 1$, $\mathcal{G}$ consists of irreducible sequents, and we can take $\mathcal{G}$ for $\mathcal{G}'$. Therefore suppose $ml(\mathcal{G})>1$ and consider a formula A in a sequent $S\in\mathcal{G}$ that has length greater than 1. Thus A is not an atom. If $A=(B\wedge C)$ and $A\in S^a$, we replace S by $(S^a\backslash\{A\},B,C\Rightarrow S^c)$, and if $A\in S^c$ we replace S by $(S^a\Rightarrow S^c\backslash\{A\},B)$ and $(S^a\Rightarrow S^c\backslash\{A\},C)$. Similarly if A is a disjunction. For $\mathcal{H}'$ being the result of these replacements, 1–3 clearly hold.

Suppose $A=(B\to C)$. If $A\in S^c$ we choose a fresh atom p different from $p_1,\dots,p_n$ and replace S by $S_1=(S^a\Rightarrow S^c\backslash\{A\},p)$ and $S_2=(p,B\Rightarrow C)$. If $A\in S^a$, we choose fresh atoms p,q different from $p_1,\dots,p_n$ and replace S by $S_1=(S^a\backslash\{A\},p\to q\Rightarrow S^c)$, $S_2=(p\Rightarrow B)$, and $S_3=(C\Rightarrow q)$. In both cases call the result $\mathcal{G}'$ and note we have $I(S_1)\wedge I(S_2)\wedge I(S_3)\vdash I(S)$ and therefore $I(\mathcal{G}')\vdash I(\mathcal{G})$. Note that there is a substitution σ that is the identity on $p_1,\dots,p_n$ such that $I(\mathcal{G})\vdash I(\sigma\mathcal{G}')$. Namely, in the first case all such substitutions for which $\sigma(p)=B\to C$, and the second case all such substitutions for which $\sigma(p)=B$ and $\sigma(q)=C$. This implies 3.

The direction from left to right of 1. and 2. holds because $I(\mathcal{G}')\vdash I(\mathcal{G})$, as can be seen from the construction. For the other direction of 1., consider a unifier τ of $I(\mathcal{G})$. This can be extended to a unifier τ' of $I(\mathcal{G}')$ in the way explained in the previous paragraph. Thus $\vdash\tau'C$ for some $C\in\Delta$. As τ equals τ' on Δ, $\vdash\tau C$ follows, proving that $I(\mathcal{G})\mathrel{|\!\sim}\Delta$. To prove the direction from right to left of 2., assume that $I(\mathcal{G}')\vdash^{\nabla}\Delta$. For the substitution σ defined in the previous paragraph $I(\sigma\mathcal{G}')\vdash^{\nabla}\Delta$ holds by structurality and the fact that σ is the identity on Δ. As $I(\mathcal{G})\vdash I(\sigma\mathcal{G}')$, $I(\mathcal{G})\vdash^{\nabla}\Delta$ follows. ⊣

The following lemma has essentially been proved in [21].

Lemma 7.2. *For every set of irreducible sequents $\mathcal{H}$ there exist sets of irreducible sequents $\mathcal{H}_1,\dots,\mathcal{H}_m$ such that the $I(\mathcal{H}_i)$ are projective and for all i:*

$$I(\mathcal{H}_i)\vdash I(\mathcal{H})\vdash^{\nabla}\{I(\mathcal{H}_1),\dots,I(\mathcal{H}_m)\}.$$

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Proof. Define the following (rewrite) relation on finite sets of finite sets of irreducible sequents in $\mathcal{L}_{\mathcal{H}}$, where X and Y range over such sets and Γ over sets of implications:

$$X \cup \{\mathcal{G} \cup \{\Gamma \Rightarrow \Delta\}\} \mapsto X \cup \{\mathcal{G} \cup \{\Gamma \Rightarrow \Delta, \Gamma \Rightarrow p\} \mid p \in \Gamma^a \cup \Delta\}.$$

Slightly ambiguous, we also use $\mapsto$ for the transitive closure of this relation. A set of sequents $\mathcal{G}$ is in $\mapsto$-*normal form* if there is no $\mathcal{H} \supset \mathcal{G}$ such that $\mathcal{G} \mapsto \mathcal{H}$. As the number of atoms in $\mathcal{H}$ is finite and all sequents involved are irreducible and contain no atoms than those in $\mathcal{H}$, there are $\mathcal{H}_1, \ldots, \mathcal{H}_n$ such that $\{\mathcal{H}\} \mapsto \{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ and the $\mathcal{H}_i$ are in $\mapsto$-normal form. Observe that the latter means that the $\mathcal{H}_i$ are closed under $\overline{\mathsf{V}}$, and thus that $I(\mathcal{H}_i)$ is projective by Theorem 6.7. It is easy to see that they satisfy the other properties in the lemma as well. ⊣

Combining the previous two lemmas gives the following theorem.

Theorem 7.3. *For every n and every set of sequents $\mathcal{G} \subseteq \mathcal{S}(p_1, \ldots, p_n)$, there exist sets of irreducible sequents $\mathcal{H}_1, \ldots, \mathcal{H}_m$ such that all $I(\mathcal{H}_i)$ are projective and for every $\Delta \subseteq \mathcal{F}(p_1, \ldots, p_n)$:*

1. *$I(\mathcal{G}) \vdash^{\overline{\mathsf{V}}} \Delta$ if and only if $I(\mathcal{H}_i) \vdash^{\overline{\mathsf{V}}} \Delta$ for all i.*
2. *$I(\mathcal{G}) \vdash^{\overline{\mathsf{V}}} \{I(\sigma\mathcal{H}_1), \ldots, I(\sigma\mathcal{H}_m)\}$ for some σ that is the identity on $\mathcal{F}(p_1, \ldots, p_n)$.*

Proof. Given $\mathcal{G}$, construct $\mathcal{H}$ and σ as in Lemma 7.1 and then sets of irreducible sequents $\mathcal{H}_1, \ldots, \mathcal{H}_m$ as in Lemma 7.2. It is easy to see that 1. holds. For 2., observe that by Lemma 7.2 and structurality, $I(\sigma\mathcal{H}) \vdash^{\overline{\mathsf{V}}} \{I(\sigma\Pi_1), \ldots, I(\sigma\Pi_m)\}$. As $I(\mathcal{G}) \vdash I(\sigma\mathcal{H})$, 2. follows. ⊣

Given a formula $A \in \mathcal{F}(p_1, \ldots, p_n)$, a set $\{B_1, \ldots, B_m\}$ of projective formulas is an *irreducible projective approximation* of A if and only if there are sets of irreducible sequents $\mathcal{H}_1, \ldots, \mathcal{H}_m$ such that $B_i = I(\mathcal{H}_i)$ and for all B in $\mathcal{F}(p_1, \ldots, p_n)$:

1. $A \mathrel{|\!\sim} B$ if and only if $B_i \vdash B$ for all i;
2. $A \mathrel{|\!\sim} \{\sigma B_1, \ldots, \sigma B_m\}$ for some σ that is the identity on $\mathcal{F}(p_1, \ldots, p_n)$.

Corollary 7.4. *For every formula A, if $\overline{\mathsf{V}}_{i(A)}$ is admissible, then A has an irreducible projective approximation.*

Proof. Consider a formula A and let n and m be the number of atoms and implications in A, respectively. Let $\mathcal{H}_1, \ldots, \mathcal{H}_n$ be as in Theorem 7.3, where $\mathcal{G} = \{(\Rightarrow A)\}$, and let $B_i = I(\mathcal{H}_i)$. Thus the B_i are projective. We prove that $\{B_1, \ldots, B_m\}$ is an irreducible projective approximation of A. From the constructions in Lemma 7.1 and Lemma 7.2 it can be seen that the sequents in the $\mathcal{H}_i$ contain at most $(n+2m)^2$ implications at the left: in Lemma 7.1 at most $2m$ new atoms will be introduced and as the sequents in $\mathcal{H}_i$ are irreducible, there are at most $(n+2m)^2$ different implications occurring in their antecedents. Recall that $i(A)$ is defined as $(n+2m)^2$. This implies that $\Gamma \vdash^{\mathsf{V}_{i(A)}} \{\bigwedge \sigma\Pi_1, \ldots, \bigwedge \sigma\Pi_m\}$. Therefore the admissibility of $\overline{\mathsf{V}}_{i(A)}$ implies the second requirement of projective approximations. The proof of the first requirement follows easily by observing that $C \mathrel{|\!\sim} B$ if and only if $C \vdash B$ for projective C and all B. ⊣

Corollary 7.5. *If $\overline{\mathsf{V}}_{i(A)}$ is admissible in L, then A has a finite complete set of unifiers.*

Proof. Given a formula A, let $B_1, \ldots, B_n$ be its irreducible projective approximation, which exists by Corollary 7.4. Thus there exists a σ that is the identity on $\mathcal{F}(p_1, \ldots, p_n)$ such that $A \mathrel{|\!\sim} \{\sigma B_1, \ldots, \sigma B_m\}$. Let σ_i' be the projective unifier of B_i and let σ_i be equal to σ_i' on the atoms in A and the identity everywhere else. We verify that $\{\sigma_1, \ldots, \sigma_n\}$ is a complete set of unifiers for A. Therefore suppose that $\vdash_{\mathsf{L}} \tau A$. Then for σ as in 2. of Theorem 7.3, $\tau A \vdash^{\overline{\mathsf{V}}} \{\tau\sigma B_1, \ldots, \tau\sigma B_m\}$. Thus $\vdash \tau\sigma B_i$ for at least one $i \leq n$ by the admissibility of $\overline{\mathsf{V}}$. Hence $\tau\sigma \leqslant \sigma_i'$. Thus $\tau \leqslant \sigma_i$. ⊣

§8. Unification types. In this section we apply the results of the previous section to obtain results on unification. The following theorem has been proved for the case $\mathsf{L} = \mathsf{IPC}$ in [10, 32], and for intermediate logics in [21]. In the latter paper the result for fragments is implicit.

Theorem 8.1. *If* $\overline{\mathsf{V}}_{9n^2}$ *is admissible, then the* n*-unification type of the logic is finitary.*

Proof. Immediate from Corollary 7.5, using that for formulas A of size at most n, $i(A) \leq 9n^2$ ⊣

IPC is the only intermediate logic with the disjunction property for which all multi-conclusion Visser rules are admissible [16]. Given this fact, the corollaries above immediately imply what has been proved by Mints, Ghilardi, and Rozière before.

Corollary 8.2 ([10, 26, 32]). *Any fragment of* IPC *that contains* $\mathsf{IPC}_{\wedge,\to}$ *has unitary or finitary unification. If it does not contain disjunction it has unitary unification.*

The fact that $\overline{\mathsf{V}}_{n+1}$ is admissible in the n-th Gabbay-De Jongh logic T_n [9, 16] implies the following.

Corollary 8.3 ([14]). *Any fragment of* T_n *that contains conjunction and implication has finitary* $\sqrt{n+1/9}$*–unification. If it does not contain disjunction it has unitary* $\sqrt{n+1/9}$*–unification under the same condition.*

Another consequence of Corollary 7.5 is the following result by Mints (for IPC) and Minari and Wroński (for all intermediate logics).

Corollary 8.4 ([25,26]). *The implication-conjunction(-negation) fragment of any intermediate logic has unitary unification.*

Using the fact that in Gödel–Dummett logic LC a disjunction $A \vee B$ is equivalent to $((A \to B) \to B) \wedge ((B \to A) \to A)$ we can extend the result by Minari and Wroński [25] that an intermediate logic has projective unification (every formula has a pu) if and only if it contains LC, to fragments.

Corollary 8.5. *Any fragment of* LC *that contains conjunction and implication has projective unification.*

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