# Duality symmetries in supergravity

New structures and deformations

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# Duality symmetries in supergravity

## New structures and deformations

Dualiteit symmetrieën in supergravitatie

Nieuwe structuren en deformaties

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof.dr. G.J. van der Zwaan, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op vrijdag 31 maart 2017 des middags te 4.15 uur.

door

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# Publications

The present work is based on:

1. F. Ciceri, B. de Wit, and O. Varela, "IIB Supergravity and the $E_{6(6)}$ covariant vector-tensor hierarchy," *JHEP* **1504** (2015) 094, 1412.8297.

2. F. Ciceri, A. Guarino, and G. Inverso, "The exceptional story of massive IIA supergravity," *JHEP* **1608** (2016) 154, 1604.08602.

3. D. Butter, F. Ciceri, B. de Wit, and B. Sahoo, "All N=4 Conformal Supergravities," *Phys. Rev. Lett.* **118** (2017) no. 8 081602, 1609.09083.

4. D. Butter, F. Ciceri, B. de Wit, and B. Sahoo, "N=4 Conformal Supergravity: the complete actions," in preparation.

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5. F. Ciceri and B. Sahoo, "Towards the full N=4 conformal supergravity action," *JHEP* **1601** (2016) 059, 1510.04999.

6. F. Ciceri, G. Dibitetto, A. Guarino, G. Inverso, and J. J. Fernandez-Melgarejo, "Double Field Theory at SL(2) angles," (2016), 1612.05230.

# Contents

# Chapter 1

# Introduction and Overview

Seeking unification of the laws of physics has so far proven to be the correct approach to gain a deeper understanding of our universe. Besides providing a more simple and elegant formulation of pre-existing laws, the hope for a unified framework is that it will also provide unsuspected new insights. A prime example is the unification of the laws describing the behaviour of electric and magnetic fields (Gauss's laws, Ampere's law and Faraday's law) into the Lorentz-covariant Maxwell's equations [1]. These equations then served as a major source of inspiration for Einstein's theory of special relativity [2]. This is only one in many examples, as the 20th century is punctuated by theoretical breakthroughs which stem from the identifications of seemingly unrelated theories with limits of a more fundamental framework.

Nowadays, after having successfully assembled many pieces of the puzzle, we are left with two reliable theoretical frameworks which have resisted various unification attempts. They are both illustrated in Einstein's equation [3],

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^4} T_{\mu\nu} \,, \tag{1.1}$$

where the left-hand side describes the dynamics of spacetime, *i.e.* gravity, while the right-hand side accounts for the presence of matter. The latter consists of various particles whose behaviour and interactions with the strong, weak and electromagnetic forces are accurately described by the Standard Model using the framework of quantum field theory. The composite and quantum nature of matter, referred to as the 'wood' by Einstein, sharply contrasts with the unique and geometric description of spacetime, the 'marble'. It is therefore not unreasonable to expect that the unification of both realms should involve a framework that provides a consistent (and predictive) quantisation of gravity and allows for a geometric interpretation of matter and gauge symmetries. In fact, this last idea was already explored in the early works of Kaluza [4] and Klein [5] which showed that four-dimensional gravity coupled to an Abelian gauge field and a scalar can be regarded as pure gravity in $4+1$ dimensions, with the fifth dimension being 'curled up' (or compactified) on a circle. In this example, the 'matter' fields simply descend from components of the five-dimensional metric with legs along the circle.

Supergravity [6] is an attempt, based on supersymmetry, to consistently unify both sides of Einstein's equation in the framework of quantum field theory. The key ingredient there is the invariance of the theory under supersymmetry transformations. Roughly speaking, these transformations take bosons into fermions and vice versa, such that the theory contains an equal number of bosonic and fermionic degrees of freedom. Importantly, in supergravity, supersymmetry is a local symmetry, and the commutator of two such transformations yields a general coordinate transformation. Invariance under local supersymmetry therefore automatically implies the presence of dynamical degrees of freedom associated to gravity. Nowadays, supergravity has flourished into a multitude of theories formulated in various dimensions and with different amount of supersymmetry. From the Kaluza-Klein point of view discussed previously, the rich matter content of some of the lower-dimensional supergravities can be seen as resulting from compactifications of the few simple higher-dimensional theories.

A major issue faced by supergravity theories is the appearance of divergences that are inherent to the perturbative approach in quantum field theory. Already in [7, 8], it was that realised that a perturbative quantisation of gravity leads to non-renormalizable ultraviolet divergences from the two-loops order onwards, and therefore nullifies the predictive power of the theory. Supergravity suffers from the same issues, although supersymmetry tends to soften the ultraviolet divergences. In four dimensions, the hope for a ultraviolet finite theory at all orders still remains and is carried by the maximally supersymmetric supergravity [9–11], especially in view of its similarities with the finite maximally supersymmetric Yang-Mills theory [12, 13]. So far, elaborate techniques have shown that the four-graviton amplitude remains finite up to four loops [14], while a divergence is expected to appear at the seven loop order, which currently remains out of reach. Regardless of the outcome of these loop computations, the finiteness issue is usually evaded by arguing that (super)gravity emerges as the low-energy effective limit of a ultraviolet finite theory, which must therefore rely on a radically different description of the physical degrees of freedom. This theory, whose detailed features only become apparent at the Planck scale ($10^{19}$ GeV), is superstring theory (for a comprehensive exposition, see for instance the textbooks [15, 16]).

In superstring theory, the fundamental constituents are tiny one-dimensional objects: closed or open strings of the order of the Planck length ($10^{-33}$ cm). These strings vibrate, interact and propagate in a ten-dimensional space, sweeping out a two-dimensional surface known as the world-sheet. The energy (or mass) spectrum of each string is discrete as a result of the boundary conditions that are imposed on their vibration. In this framework, the various elementary particles are characterised by specific modes of the string. In particular, the spectrum of massless modes of the closed string contains the degrees of freedom of a spin-2 particle corresponding to the graviton. This is in contrast with supergravity, where particles are described as point-like excitations of fields defined over the spacetime manifold. In fact, it is the 'stretching' of the point-like interactions into extended vertices for strings that is at the root of the (perturbative) finiteness of string theory. In addition to massless modes, the string spectrum contains an infinity of massive modes. In the low-energy limit, the massive modes are ignored and as a result, the one-dimensional profile of the string becomes irrelevant and can be approximated by a point. In this case, the classical dynamics of the massless modes becomes precisely encoded in the field equations of a ten-dimensional supergravity theory.

In total, there exist five different versions of superstring theory, which differ from one another by certain properties of the strings considered. They are known as type I, type IIA, type IIB and Heterotic (with gauge groups SO(32) and $\mathrm{E}_8 \times \mathrm{E}_8$) superstring theory. In the low-energy limit, each of these theories is effectively described by one of the five ten-dimensional supergravities that carry the same name. Furthermore, upon compactifications on different backgrounds, such as torii of unequal radii, different superstring theories have been shown to actually describe the same underlying quantum theory. The transformations that map such equivalent backgrounds define discrete groups, which are commonly referred to as duality symmetry groups [17]. These are believed to be manifestations of a large set of equivalence relations, or dualities, that relate all the superstring theories in ten dimensions [18]. Based on these insights, it was conjectured that the five superstring theories actually correspond to different realisations of a single and more profound theory, the so-called M-theory [19]. Despite many efforts, not much is known today about M-theory besides the fact that its low-energy limit should be described by the unique eleven-dimensional supergravity.

In this work, we exclusively focus on supergravity theories, and in particular, on their continuous rigid duality symmetries that appear in $D \leq 9$ dimensions and which can be understood as remnants of their discrete counterparts in superstring theory [17]. The aim is to improve our understanding of the unifying abilities and/or the higher-dimensional origin of these duality symmetries. In the long term, this will hopefully allow us to gain insights into the structure and properties of M-theory.

The outline of this introductory chapter is as follows. In sections 1.1 and 1.2, we review some aspects of supergravity theories which will be necessary for the derivation of the original results presented in the next chapters. In section 1.3, we give a short summary of these results and of our motivations.

## 1.1 Maximal supergravity

Supergravity theories can be formulated in $D \leq 11$ spacetime dimensions. These field theories are, by construction, invariant under general coordinate transformations as well as under local supersymmetry transformations. The latter are generated by supercharges which arrange into an irreducible Lorentz spinor. Extended supergravities result from considering $N > 1$ independent supersymmetries, and therefore $N$ sets of supercharges. As a result, all supergravity theories contain at least a graviton and a set of $N$ gravitini, *i.e.* the supersymmetry (fermionic) gauge fields. The maximal number of real supercharges for which supergravity theories can be constructed is 32. For instance, in $D = 4$, Majorana spinors have 4 real components and therefore supergravity theories only exist up to $N = 8$. Beyond this limit, supersymmetry requires the presence of higher-spin fields and a number of associated issues arise. In this chapter, we present a few selected features of supergravity theories. Detailed reviews of the subject can for example be found in [20, 21].

In this work we exclusively consider supergravity theories in $D \geq 3$ dimensions and with 32 real supercharges. The field contents of these so-called maximal supergravities are strongly constrained by the large amount of symmetries. In particular, the latter do not allow for the presence of matter supermultiplet. The various fields transforming into each other under supersymmetry transformations organise into the gravity supermultiplet. In addition to the metric and $N$ gravitini, the gravity supermultiplet typically comprises antisymmetric tensor fields (or $p$-forms, with $p \geq 1$) associated to Abelian gauge symmetries, and for $D \leq 10$, spin-1/2 fermions and scalars. In that sense, maximal supergravities stand as a true unification of gravity and matter.

The simplest, and arguably most elegant maximal supergravity is the unique N=1 eleven-dimensional theory constructed by Cremmer, Julia and Scherk [22]. Its bosonic Lagrangian simply reads

$$\begin{aligned}\mathcal{L}_{D=11} = & -\frac{1}{2}\, E\, R - \frac{1}{48}\, E\, F_{MNPQ}\, F^{MNPQ} \\ & - \frac{1}{(12)^3\sqrt{2}}\, i\, \varepsilon^{MNPQRSTUVWX} F_{MNPQ}\, F_{RSTU}\, C_{VWX}\,, \end{aligned} \tag{1.2}$$

where $R$ is the Ricci scalar and $E$ is the determinant of the elfbein $E_M{}^A$. Space-time and tangent space indices[1] are denoted by $M, N, \ldots$ and $A, B, \ldots$, respectively. The three-form potential $C_{MNP}$ possesses the following Abelian tensor gauge transformations

$$\delta C_{MNP} = 3\, \partial_{[M} \Xi_{NP]}\,, \tag{1.3}$$

which naturally leave the field strength $F_{MNPQ} = 4\, \partial_{[M} C_{NPQ]}$ invariant. The 128 bosonic on-shell degrees of freedom (44 from $E_M{}^A$ and 84 from $C_{MNP}$) appearing in (1.2) are consistently matched by the 128 fermionic on-shell degrees of freedom contained in a Majorana vector-spinor, the single gravitino of the theory.

In D=10, the situation is different due to the presence of Majorana-Weyl spinors which contain 16 real components. There exist two maximal $D = 10$ supergravities: the chiral $N = (2, 0)$ type IIB theory [23–25] in which the two gravitino are of the same chirality, and the non-chiral $N = (1, 1)$ type IIA theory [26]. They correspond to the low-energy limits of type IIB and type IIA superstring theory.

Type IIB and type IIA supergravity are respectively the pivots of chapter 2 and 3, and we refer to those chapters for a more detailed presentation. Nevertheless, it is already worth mentioning some of their characteristic features that will turn out to be particularly relevant for our purposes. The IIB theory is invariant under a rigid $\mathrm{SU}(1,1) \cong \mathrm{SL}(2)$ symmetry and its field content includes two real scalars parametrising an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space. On the other hand, the IIA theory can be obtained from a circle reduction of the eleven-dimensional theory, and therefore admits a rigid $\mathbb{R}^+$ symmetry associated to the rescaling of the circle (see next subsection). Finally, there also exists a massive deformation of the IIA

[1] The factor of $i$ in front of the Levi-Civita tensor density, $\varepsilon^{M\cdots X}$, accounts for the Pauli-Källén conventions that we adopt throughout this work, except in chapter 3.

theory, known as massive type IIA [27], for which no eleven-dimensional origin is known.

In each of the lower dimensions (*i.e.* for $D < 10$), there exists a unique,[2] non-chiral, maximal supergravity. As discussed below, these supergravities can be obtained from the dimensional reduction of either eleven-dimensional supergravity or type IIB supergravity. This is a consequence of the so-called T-duality which implies that type IIA and type IIB superstring theory compactified on torii (of different radii) correspond to two different formulations of the same theory.

### 1.1.1 Dimensional reduction and duality symmetries

The field content and the symmetries of maximal supergravities in $3 \leq D \leq 9$ are more complex than that of the ten- and eleven-dimensional theories. In this subsection, we focus exclusively on the ungauged theories, in which all scalar fields are neutral under the local Abelian symmetries. This is in contrast with gauged supergravities which will be discussed in section 1.1.2.

On top of the various local Abelian symmetries associated to $p$-forms (with $p \geq 1$), ungauged maximal supergravities develop increasingly large rigid symmetries, $G$, as the dimension decreases. These so-called *duality symmetries* leave the vielbein invariant but act non-lineary on the scalars that parametrize the coset spaces $G/K$, where $K$ is the maximal compact subgroup of $G$. Both $G$ and $K$ are listed in Table 1.1 for various dimensions. Note that, until chapter 3, we ignore the scaling symmetry (also known as trombone symmetry) of the field equations of all (super)gravity theories. When the duals of the scalars and the $p$-forms are introduced on-shell, each complete set of $p$-forms transforms linearly under $G$ according to the representations presented in Table 1.1. In the Lagrangians however, only physical degrees of freedom appear. In particular, due to the self-duality of $(D/2-1)$-forms in even dimensions, one is forced to select half of them by choosing a so-called symplectic frame. For instance, only 28 vectors enter the Lagrangian of $D = 4$, $N = 8$ supergravity [9–11]. For this reason, the duality symmetry is only realised at the level of the field equations in even dimensions. In general, the bosonic Lagrangians involve the $p$-forms[3] listed in blue and only half of those listed in red in Table 1.1. A detailed description of the fermionic sectors

[2] In $D = 6$, the existence of Weyl spinors allows again for a chiral $(4, 0)$ [28,29] and a non-chiral $(2, 2)$ theory. The former however however does not contain a graviton and will be ignored in what follows.

[3] Except in $D = 3$ where the standard ungauged Lagrangian only contains scalars.

of maximal supergravities is not necessary at this point but we note that all the fermions transform in linear representations under local $K$ transformations. The latter are also known as R-symmetry transformations. For instance, the gravitini of $N = 8$ supergravity in $D = 4$ and $D = 5$ transform in the $\mathbf{8}$'s of SU(8) and USp(8), respectively.

| | $G$ | $K$ | $\mathcal{R}_v$ $(p = 1)$ | $p = 2$ | $p = 3$ | $p = 4$ | $p = 5$ |
|---|---|---|---|---|---|---|---|
| $D = 9$ | $\mathbb{R}^+ \times \mathrm{SL}(2)$ | SO(2) | $\mathbf{1}_{-4} + \mathbf{2}_{+3}$ | $\mathbf{2}_{-1}$ | $\mathbf{1}_{+2}$ | $\mathbf{1}_{+2}$ | $\mathbf{2}_{-1}$ |
| $D = 8$ | $\mathrm{SL}(3) \times \mathrm{SL}(2)$ | $\mathrm{SO}(3) \times \mathrm{SO}(2)$ | $(\overline{\mathbf{3}}, \mathbf{2})$ | $(\mathbf{3}, \mathbf{1})$ | $(\mathbf{1}, \mathbf{2})$ | $(\overline{\mathbf{3}}, \mathbf{1})$ | $(\mathbf{3}, \mathbf{2})$ |
| $D = 7$ | SL(5) | SO(5) | $\overline{\mathbf{10}}$ | $\mathbf{5}$ | $\overline{\mathbf{5}}$ | $\mathbf{10}$ | $\mathbf{24}$ |
| $D = 6$ | SO(5,5) | $\mathrm{SO}(5) \times \mathrm{SO}(5)$ | $\mathbf{16}_c$ | $\mathbf{10}$ | $\mathbf{16}_s$ | $\mathbf{45}$ | |
| $D = 5$ | $\mathrm{E}_{6(6)}$ | USp(8) | $\overline{\mathbf{27}}$ | $\mathbf{27}$ | $\mathbf{78}$ | | |
| $D = 4$ | $\mathrm{E}_{7(7)}$ | SU(8) | $\mathbf{56}$ | $\mathbf{133}$ | | | |
| $D = 3$ | $\mathrm{E}_{8(8)}$ | SO(16) | $\mathbf{248}$ | | | | |

Table 1.1: Duality symmetry group $G$ and its maximal compact subgroup $K$ in $D$ dimensions [30]. The representations of $G$ carried by the various $p$-forms ($p < 6$) are indicated. The one-forms, which will often be loosely referred to as vectors, transform in the $\mathcal{R}_v$ representation of $G$. Note that this table also includes those $p$-forms which can be introduced on-shell by dualization. In contrast, only the blue and half of the red $p$-forms enter the standard ungauged Lagrangians as propagating degrees of freedom.

As mentioned previously, all the maximal (ungauged) supergravities of Table 1.1 can be obtained by dimensional reductions of the type IIB or eleven-dimensional supergravity (1.2). In these reductions, which are based on the early ideas of Kaluza and Klein (see [31] for a review in the context of supergravity), one assumes that the $(D+n)$-dimensional spacetime manifold factorizes into $\mathcal{M}_D \times T^n$, where $\mathcal{M}_D$ denotes an arbitrary $D$-dimensional spacetime, and subsequently shrinks the $n$-torus $T^n$ to a point. Due to the periodicity of the fields in the coordinates parameterizing $T^n$, they can be expanded on a Fourier basis. The massless modes are $\mathrm{U}(1)^n$ singlets[4] while the massive modes carry a charge proportional to the Fourier mode $m \in \mathbb{N}^\star$. Because of these charge assignments, one can consistently truncate the infinite tower of massive modes without running the risk of inadvertently throwing away massless modes. In the end, this truncation simply amounts to declaring that all the fields only depend on the $D$ coordinates parametrizing $\mathcal{M}_D$. From a low-energy perspective, the truncation becomes meaningful when the characteristic size of $T^n$ is set to vanish. In this case the massive modes,

[4]Here we mean the $\mathrm{U}(1)^n$ transformations that act as shifts of the compactified coordinates.

whose masses are inversely proportional to the radii of the torus, indeed become irrelevant in the framework of supergravity. Finally we emphasise that for compactifications on more general manifolds, the consistency of the truncation of the massive modes is a notoriously difficult problem to study. We come back to this issue later on.

Upon dimensional reduction, the various bosonic fields of the eleven-dimensional theory appearing in (1.2) can be shown to decompose into the field content of the lower-dimensional theories of Table 1.1. As an example, one obtains $15+6$ five-dimensional vectors from the eleven-dimensional three-form and metric, respectively. The three-form also contributes to 6 five-dimensional two-forms which, after dualisation, provide the remaining vectors to complete the $\overline{\mathbf{27}}$ representation of $\mathrm{E}_{6(6)}$[5]. The same result can be obtained by considering the reduction of type IIB supergravity along the same reasoning.

The emergence of exceptional duality symmetries upon dimensional reduction is a characteristic feature of supergravity theories. In comparison, the dimensional reduction of pure gravity only yields a theory which admits a rigid $\mathbb{R}^+\times\mathrm{SL}(n)$ symmetry related to the geometry of the $n$-torus. In order to clarify this point, let us split the coordinates of a $(D+n)$-dimensional spacetime as $x^M\to(x^\mu,y^m)$, with $\mu=0,\ldots,D-1$ and $m=1,\ldots,n$, and similarly for the tangent space indices $A\to(\alpha,a)$. The $(D+n)$-bein can always be rotated into the following upper triangular form

$$E_M{}^A=\begin{pmatrix}\Delta^{\frac{1}{(2-D+n)}}\,e_\mu{}^\alpha & B_\mu{}^m\,e_m{}^a\\ & \\ 0 & e_m{}^a\end{pmatrix}\,,\quad\text{with}\quad\det(e_m{}^a)=\Delta\,,\tag{1.4}$$

by gauge fixing the higher-dimensional Lorentz symmetry to $\mathrm{SO}(D-1,1)\times\mathrm{SO}(n)$. As explained previously, since we want to perform here a dimensional reduction, the $D$-bein $e_\mu{}^\alpha(x)$, the $n$ Kaluza-Klein vectors $B_\mu{}^m(x)$ and the $n^2$ scalars $e_m{}^a(x)$, are chosen independent of the $n$ compactified coordinates.[6] The residual $\mathrm{SO}(n)$ symmetry preserved by the Ansatz (1.4) acts as

$$\delta e_m{}^a=e_m{}^b\,h_b{}^a\,,\ \text{with}\ h\in\mathfrak{so}(n)\,,\tag{1.5}$$

[5]The notation $\mathrm{E}_{n(n)}$ refers to the split form of the exceptional Lie groups. In this form, the associated Lie algebra is real and contains the maximal number of non-compact generators.

[6]Note that in chapter 2, we consider such a Kaluza-Klein decomposition but we do *not* perform a reduction, *i.e.* we keep an arbitrary dependence on the full set of $D+n$ coordinates.

and can be used to reduce the number of physical scalars carried by ${e_m}^a$ to $n(n+1)/2$. The diffeomorphism parameters $\xi^M(x,y)$ split into $\xi^\mu(x)$, which generate the usual diffeomorphims in $D$ dimensions, and $\xi^m$, which induce two kinds of internal symmetries in the reduced theory. Because of the restricted coordinate dependence of the fields, the latter can be decomposed without loss of generality as

$$\xi^m(x,y) = \tilde{\xi}^m(x) + {\Lambda^m}_n\, y^n + \lambda\, y^m\,, \tag{1.6}$$

where $\tilde{\xi}^m(x)$ are arbitrary functions, and where ${\Lambda^m}_n$ is a real constant traceless matrix and $\lambda$ is a real number. It can be shown that the first term generates Abelian gauge transformations for the Kaluza-Klein vectors

$$\delta {B_\mu}^m = \partial_\mu\, \tilde{\xi}^m\,, \tag{1.7}$$

and leaves the scalars and ${e_\mu}^\alpha$ invariant. The second term in (1.6) generates rigid $\mathrm{SL}(n)$ transformations

$$\delta {B_\mu}^m = -{\Lambda^m}_n {B_\mu}^n\,, \quad \delta {e_m}^a = {\Lambda^n}_m {e_n}^a\,. \tag{1.8}$$

Finally, the third term in (1.6) corresponds to a rescaling of the internal torus. When appropriately combined with the on-shell scaling symmetry (the aforementioned trombone symmetry) of the $(D+n)$-dimensional theory, it induces two scaling symmetries in the reduced theory. One of them is again a trombone symmetry. The other one extends the rigid symmetry group of the reduced theory to $\mathbb{R}^+\times\mathrm{SL}(n)$. From (1.5) and (1.8), it can be shown that the $n(n+1)/2$ physical scalars parametrise an $\mathbb{R}^+\times\mathrm{SL}(n)/\mathrm{SO}(n)$ coset space. The reduced theory also admits a local $\mathrm{U}(1)^n$ symmetry associated to the Kaluza-Klein vectors (1.7).

Let us now reconsider the same reduction but with the bosonic sector of eleven-dimensional supergravity (1.2) as a starting point. The pure gravity picture presented above is now modified by the presence of the three-form $C_{MNP}$ in the gravity supermultiplet. For $n\geq 3$, it decomposes into various $D$-dimensional scalars and $p$-forms

$$C_{MNP}(x) = \{C_{\mu\nu\rho}(x), C_{\mu\nu m}(x), C_{\mu mn}(x), C_{mnp}(x)\}\,, \tag{1.9}$$

which, according to the reduction procedure, do not depend on the compactified coordinates. These fields transform linearly under the rigid $\mathrm{SL}(n)$ transformations induced by the internal diffeomorphisms (1.8). Moreover, the Abelian tensor gauge transformations (1.3) associated to the three-form in the higher-dimensional theory contribute to additional symmetries in the reduced theory. As with the internal diffeomorphisms, those internal tensor gauge parameters which are linear in the torus coordinates, *i.e.* $\Xi_{mn} = \rho_{mnp}\, y^p$ with constant $\rho_{mnp}$, induce the following rigid transformations on the $D$-dimensional scalars

$$\delta C_{mnp} = 3\, \rho_{[mnp]}\,. \tag{1.10}$$

It turns out that this additional rigid shift symmetry plays a crucial role in extending the $\mathbb{R}^+ \times \mathrm{SL}(n)$ symmetry of the reduced theory into the full duality symmetry group $G$ (see Table 1.1). In general, a symmetry enhancement occurs whenever new scalars arise from the dimensional reduction.[7] When $n \geq 6$, this also includes the scalars that are obtained by dualising the $(D-2)$-forms. A detailed investigation of the emergence of duality symmetries upon dimensional reduction can be found in [32, 33].

In essence, when reducing on torii of increasing dimensionality, the spacetime symmetries of the higher-dimensional theories gradually get transmuted into increasingly large rigid duality symmetries. When reaching the threshold of $D \leq 2$ dimensions, the symmetry groups become infinite-dimensional. In $D = 2$, ungauged maximal supergravity has been constructed and admits a rigid $\mathrm{E}_{9(9)}$ symmetry [34–36]. In $D = 1$ and $D = 0$, models have been developed based on the Kac-Moody algebras $\mathfrak{e}_{10}$ (see for instance [37–39]) and $\mathfrak{e}_{11}$ (see [40] and references therein), respectively. Although they provide exciting perspectives beyond the standard supergravity framework, they will not be discussed in this work.

### 1.1.2 Gauged supergravity

A systematic approach to the construction and classification of gauged maximal supergravities in $3 \leq D \leq 9$ is provided by the embedding tensor formalism [41–47]. In this subsection, we give a brief overview of this formalism, in view of the fact that the same structures underly the exceptional field theories discussed in chapter 3. A detailed presentation of gauged maximal supergravities can for instance be found in the reviews [30, 48].

[7] For $n = 2$, the $11D$ three-form does not decompose into 9D scalars and, in this case, the rigid symmetry of the reduced theory is then simply $\mathrm{R}^+ \times \mathrm{SL}(2)$. This symmetry can also be understood from the dimensional reduction of type IIB supergravity. In this case, the $\mathrm{R}^+$ is induced by the circle reduction while the rigid SL(2) symmetry is already present in ten dimensions.

### The embedding tensor

Consider as a starting point one of the ungauged maximal supergravities discussed in the previous subsection. There, the Abelian vector fields $A_\mu{}^{\mathcal{M}}$ transform in the $\mathcal{R}_v$ representation (see Table 1.1) of the duality symmetry group $G$. Such transformations, with constant parameters $\kappa^\alpha$, take the form

$$\delta_\kappa A_\mu{}^{\mathcal{M}} = -\kappa^\alpha \, [t_\alpha]_{\mathcal{N}}{}^{\mathcal{M}} A_\mu{}^{\mathcal{N}} \, , \tag{1.11}$$

where $t_\alpha$ denote the generators of $G$. In gauged supergravity, a subgroup of this rigid duality symmetry is promoted to a local invariance.[8] To this end, some of the vectors appearing in (1.11) are used as gauge connections for the chosen gauge group. The gauged supergravity Lagrangian and the symmetry variations of the fields are entirely specified by an embedding tensor $\Theta_{\mathcal{M}}{}^\alpha$, where the index $\alpha$ labels the adjoint representations of $G$. As its name suggests, this tensor encodes the embedding of the gauge group into $G$. In particular, the generators of the gauge group, $X_{\mathcal{M}}$, can be decomposed as

$$X_{\mathcal{M}} \equiv \Theta_{\mathcal{M}}{}^\alpha \, t_\alpha \; . \tag{1.12}$$

Following the conventional gauging procedure in field theory, the partial derivatives are then converted into covariant derivatives, $D_\mu = \partial_\mu - g \, A_\mu{}^{\mathcal{M}} X_{\mathcal{M}}$, where $g$ denotes the gauge coupling constant. For example, when acting on a covariant field $U^{\mathcal{M}}$ in the $\mathcal{R}_v$ representation of $G$, the covariant derivative reads

$$D_\mu U^{\mathcal{M}} = \partial_\mu U^{\mathcal{M}} + g \, A_\mu{}^{\mathcal{N}} X_{\mathcal{N}\mathcal{P}}{}^{\mathcal{M}} U^{\mathcal{P}} \, , \tag{1.13}$$

where the objects $X_{\mathcal{N}\mathcal{M}}{}^{\mathcal{P}} = [X_{\mathcal{N}}]_{\mathcal{M}}{}^{\mathcal{P}}$ play the role of the structure constants of the gauge group. In general, they are not purely antisymmetric in their lower indices. In the maximal theories considered here, $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ captures the same information as the embedding tensor $\Theta_{\mathcal{M}}{}^\alpha$.

Despite its name, the embedding tensor should be taken as a fixed object, which therefore explicitly breaks the rigid symmetry $G$ in order to gauge a subgroup. Its variations under diffeomorphisms, gauge transformations and supersymmetry all vanish by definition. However, it is often convenient to regard $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ as a spurious object, which is allowed to transform under $G$ together with the fields of the theory, thus allowing for a formally $G$-covariant formulation of gauged

[8] Here we exclude the gauging of the $\mathbb{R}^+$ trombone symmetry of maximal supergravities for which there is no Lagrangian formulation [49,50].

maximal supergravity. A set of quadratic and linear constraints must be imposed on the embedding tensor for consistency of the gauged theory. The quadratic constraint takes the following form

$$\begin{aligned} 2\,X_{[\mathcal{M}\mathcal{P}}{}^{\mathcal{R}}X_{\mathcal{N}]\mathcal{R}}{}^{\mathcal{Q}} + X_{[\mathcal{M}\mathcal{N}]}{}^{\mathcal{R}}X_{\mathcal{R}\mathcal{P}}{}^{\mathcal{Q}} &= 0\,, \\ X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{R}}X_{\mathcal{R}} &= 0\,, \end{aligned} \tag{1.14}$$

and ensure both the closure of the gauge algebra and the Jacobi identity (when this identity is evaluated in the vector representation $\mathcal{R}_v$). Using the quadratic constraint, it is easy to verify that the gauge covariance of the derivative (1.13) requires the gauge connections to transform as

$$\delta_\Lambda A_\mu{}^{\mathcal{M}} \equiv \partial_\mu \Lambda^{\mathcal{M}} - g A_\mu{}^{\mathcal{N}} \Lambda^{\mathcal{P}} X_{\mathcal{P}\mathcal{N}}{}^{\mathcal{M}} + \dots \tag{1.15}$$

under a gauge transformation with parameter $\Lambda^{\mathcal{M}}(x)$. Actually, this definition is valid up to terms of the form $A_\mu{}^{\mathcal{N}}\Lambda^{\mathcal{P}}X_{(\mathcal{P}\mathcal{N})}{}^{\mathcal{M}}$, which do not affect the gauge covariance of (1.13) as a consequence of the symmetric part of the quadratic constraint (1.14). It is therefore convenient to chose

$$\delta_\Lambda A_\mu{}^{\mathcal{M}} = \partial_\mu \Lambda^{\mathcal{M}} + g A_\mu{}^{\mathcal{P}} \Lambda^{\mathcal{N}} X_{\mathcal{P}\mathcal{N}}{}^{\mathcal{M}} = D_\mu \Lambda^{\mathcal{M}}\,. \tag{1.16}$$

The linear (or representation) constraint is required by supersymmetry. At the bosonic level, it also ensures that the number of physical degrees of freedom remains unaffected by the chosen gauging (cf. discussion below). In practice, the linear constraint restricts the embedding tensor to specific representations, $\mathcal{R}_X$, of $G$ contained in the tensor product of $\overline{\mathcal{R}_v}$ and the adjoint representation

$$\Theta \in \mathcal{R}_X \;\subset\; \overline{\mathcal{R}_v} \otimes \mathrm{adj}_G\,. \tag{1.17}$$

We list these representations in Table 1.2 for gauged supergravities in $3 \geq D \geq 9$.

### The tensor hierarchy

Prior to the discovery of the embedding tensor formalism, constructions of gauged supergravities were hampered by the necessity to first adapt the field content of the ungauged Lagrangian depending on the desired gauging. In particular, those vectors that were not used as gauge connections but which were still charged under the gauge symmetry had to be dualised before proceeding with the gauging [51–53].

| | $D=9$ | $D=8$ | $D=7$ | $D=6$ | $D=5$ | $D=4$ | $D=3$ |
|---|---|---|---|---|---|---|---|
| $G$ | $\mathrm{SL}(2)\times\mathbb{R}^+$ | $\mathrm{SL}(3)\times\mathrm{SL}(2)$ | $\mathrm{SL}(5)$ | $\mathrm{SO}(5,5)$ | $\mathrm{E}_{6(6)}$ | $\mathrm{E}_{7(7)}$ | $\mathrm{E}_{8(8)}$ |
| $\overline{\mathcal{R}_v}$ | $\mathbf{2}_{-3}+\mathbf{1}_{+4}$ | $(\mathbf{3},\mathbf{2})$ | $\mathbf{10}$ | $\mathbf{16}_s$ | $\mathbf{27}$ | $\mathbf{56}$ | $\mathbf{248}$ |
| $\mathrm{adj}_G$ | $\mathbf{3}_0+\mathbf{1}_0$ | $(\mathbf{8},\mathbf{1})+(\mathbf{1},\mathbf{3})$ | $\mathbf{24}$ | $\mathbf{45}$ | $\mathbf{78}$ | $\mathbf{133}$ | $\mathbf{248}$ |
| $\mathcal{R}_X$ | $\mathbf{2}_{-3}+\mathbf{3}_4$ | $(\mathbf{3},\mathbf{2})+(\overline{\mathbf{6}},\mathbf{2})$ | $\mathbf{15}+\overline{\mathbf{40}}$ | $\mathbf{144}_s$ | $\overline{\mathbf{351}}$ | $\mathbf{912}$ | $\mathbf{1}+\mathbf{3875}$ |
| $p^{\mathrm{Max}}$ | 4 | 4 | 3 | 3 | 2 | 2 | 1 |

Table 1.2: Relevant representations of the duality group $G$ for the embedding tensor formalism of $D$-dimensional gauged maximal supergravity. The embedding tensor is restricted to $\mathcal{R}_X$ by the linear constraint. The value of $p^{\mathrm{Max}}$ indicates the highest rank of the $p$-forms appearing in the Lagrangian.

The embedding tensor formalism allows to bypass this difficulty by relying on a hierarchy of $p$-forms[9] (those listed in Table 1.1) which also carry their own Abelian tensor gauge transformations [55–57]. When a gauging is turned on, the various gauge transformations of the theory become entangled in a way which is dictated by the corresponding embedding tensor. This enlarged set of gauge symmetries ensures that, for all consistent gaugings, the number of propagating degrees of freedom remains the same as in the ungauged theory. The distribution of these degrees of freedom over the various $p$-forms is automatically organised by the embedding tensor and therefore, the issue mentioned in the beginning of the paragraph is circumvented in this formalism.

Let us now shortly comment on the mechanism at origin of the tensor hierarchy. The spark that triggers this proliferation of $p$-forms occurs when initially trying to construct the field strengths $F_{\mu\nu}{}^{\mathcal{M}}$ associated to the gauge fields $A_\mu{}^{\mathcal{M}}$. More precisely, the naive expression

$$F_{\mu\nu}{}^{\mathcal{M}} = 2\,\partial_{[\mu}A_{\nu]}{}^{\mathcal{M}} - g\,X_{[\mathcal{N}\mathcal{P}]}{}^{\mathcal{M}}A_\mu{}^{\mathcal{N}}A_\nu{}^{\mathcal{P}}\,, \tag{1.18}$$

fails to tranform covariantly under the gauge transformations (1.16) due to terms proportional to $X_{(\mathcal{N}\mathcal{P})}{}^{\mathcal{M}}$. The lack of covariance of these field strengths can be absorbed in the gauge transformations of a set of two-forms. The latter are introduced through a redefinition of the expression (1.18) in the form of Stückelberg couplings (which depend on the embedding tensor). The same issues remerge

[9]The first form of such a hierarchy appeared in the peculiar $D=3$ case. There, the absence of vectors in the standard ungauged formulation (c.f. Table 1.1) originally left the introduction of gaugings as an open problem. This was resolved in [41, 54] where gauged Lagrangians were formulated by introducing 248 vectors (that reduce to the on-shell duals of the scalars in the ungauged limit) via a Chern-Simons term.

when attempting to construct the field strengths associated with the two-forms. Again, one is forced to introduce a set of three-forms in order to guarantee the gauge covariance. This process unfolds repeatedly and leads to a hierarchy of $p$-forms. At some point, however, this iterative procedure becomes irrelevant for the purpose of writing down a gauge invariant Lagrangian. Indeed, the $p$-forms that are present in the Lagrangian appear projected by certain components of the embedding tensor such that, as a result of the linear and quadratic constraints, they form a gauge invariant subset. In the embedding tensor formalism, the gauged Lagrangians thus only depend on those $p$-forms of rank $p \leq p^{\mathrm{Max}}$, where the value of $p^{\mathrm{Max}}$ is indicated in Table 1.2 for various dimensions.

#### Consistent truncations

As explained in section 1.1.1, ungauged supergravities can systematically be obtained as dimensional reductions of $D = 11$ (or equivalently $D = 10$, type IIB) supergravity on flat torii of appropriate dimensions. The higher-dimensional origins of gauged supergravities are usually more puzzling, and for some gaugings an uplift to eleven- or ten-dimensional supergravities is simply not possible. In general, the gaugings are inherited from the isometry group of the curved space on which the higher-dimensional theory is compactified, and/or from the presence of background fluxes. In these compactifications, the internal coordinates dependence of the higher-dimensional fields are captured by geometric quantities characterising the curved internal space, such as Killing vectors or tensors.

Assuming such ansätze for the fields, the challenging task is then to prove that the higher-dimensional theory can be consistently truncated to the lower-dimensional gauged supergravity. A consistent truncation means that the ansätze provide an embedding of the whole field configuration space of the lower-dimensional theory into that of the higher-dimensional theory. As a byproduct, this implies that any solution of the lower-dimensional theory can be embedded into a solution of the higher-dimensional one. This formally requires the dependence on the compactified coordinates to systematically factorise out of the field equations of the higher-dimensional theory after substitution of the ansätze, leaving the field equations of the lower-dimensional theory. In practice, due to the non-linearity of the field equations of supergravity theories, it is extremely difficult to verify explicitly such a requirement. For most of the successful cases known today, the consistency of the truncation was proven on the basis of group-theoretic arguments (such as in the trivial cases of reductions on torii discussed earlier). This includes the Scherk-Schwarz (SS) reductions [58] in which the dependence on the internal coordinates is carried by group-valued matrix, sometimes referred to as 'twist matrix', that must satisfy stringent conditions to ensure consistency of the truncation.

More involved truncations, such as those based on sphere compactifications, cannot be approached as standard SS reductions (as introduced in [58]). Their consistency requires highly non-trivial conditions. Nevertheless, in some cases, the consistency was established using more elaborate techniques which will be discussed later in this work.

## 1.2 Conformal (super)gravity

In this section, we shift our perspective from the duality symmetries presented above and focus on the space-time symmetries of supergravity. The theories discussed so far are by construction invariant under local super-Poincaré transformations. Requiring invariance under the larger group of superconformal transformations leads to conformal supergravity. As Poincaré supergravities, the conformal theories exist in various dimensions and with different amount of supersymmetry. In general, the higher degree of symmetry of these theories tends to organise and stiffens their structures. In this section and chapter 4, we restrict ourselves to four spacetime dimensions where superconformal theories exist up to $N = 4$ [59]. The same structures that underly conformal supergravities are also reflected in conformal gravity, which we present in this section. The supersymmetric extensions are technically much more involved but rely on the same mechanisms.

Conformal gravity is constructed as the gauge theory of the conformal group. In four dimensions, the conformal algebra is $\mathfrak{so}(4,2)$ and contains, in addition to the Poincare subalgebra, the generators of the conformal boosts $K^a$ and of the dilatations $D$. To each of these generators is associated a gauge field and a gauge parameter:

$$\begin{array}{lllll} \text{generators :} & P^a & M^{[ab]} & D & K^a \\ \text{gauge fields :} & e_\mu^a & \omega_\mu{}^{ab} & b_\mu & f_\mu{}^a \\ \text{parameters :} & \xi^a & \varepsilon^{ab} & \Lambda_D & \Lambda_K{}^a \end{array}$$

Here, $\mu = 0, \ldots, 3$ is a spacetime index. The indices $a, b, \ldots = 0, \ldots, 3$ label the coordinates of a flat manifold with Minkowski signature which is, at this point, not identified with the tangent space of the spacetime manifold. Indeed, the conformal symmetry is treated so far as an internal symmetry. In particular, the transformations rules of the gauge fields are dictated by the conformal algebra

$$
\begin{aligned}
\delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \Lambda_D e_\mu{}^a + \varepsilon^{ab} e_{\mu b}\,,\\
\delta \omega_\mu{}^{ab} &= \mathcal{D}_\mu\, \varepsilon^{ab} + 2\,\Lambda_K{}^{[a} e_\mu{}^{b]} + 2\,\xi^{[a} f_\mu{}^{b]}\,,\\
\delta b_\mu &= \partial_\mu \Lambda_D + \Lambda_K{}^a e_{\mu a} - \xi^a f_{\mu a}\,,\\
\delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_K{}^a + \Lambda_D f_\mu{}^a + \varepsilon^{ab} f_{\mu b}\,,
\end{aligned}
\tag{1.19}
$$

where $\mathcal{D}_\mu$ denotes the covariant derivative with respect to Lorentz transformations and dilatations. Under general coordinates transformations all the fields simply transform as one-forms, irrespective of their internal indices.

The covariant curvatures associated to the various symmetries can be constructed using the transformations (1.19). Here, we list the two that will be relevant for our purposes

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a &= 2\,\partial_{[\mu} e_{\nu]}{}^a + 2\,b_{[\mu} e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab} e_{\nu]b}\,,\\
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4\,e_{[\mu}{}^{[a} f_{\nu]}{}^{b]}\,.
\end{aligned}
\tag{1.20}
$$

In order to promote the internal transformations (1.19) to transformations of the spacetime manifold, a set of covariant algebraic constraints is imposed.[10] In particular, after identifying the gauge field $e_\mu{}^a$ with the (invertible) vierbein, it is easy to verify that, by imposing $R(P)_{\mu\nu}{}^a = 0$, the variation of vierbein under a $P$ transformations reduces to a (covariant) general coordinate transformation. As by a product, this constraint determines entirely the gauge field $\omega_\mu{}^{ab}$

$$
\omega_\mu{}^{ab}(e,b) = -2\,e^{\nu[a}\partial_{[\mu} e_{\nu]}{}^{b} - e^{\nu[a} e^{b]\sigma} e_{\mu c}\partial_\sigma e_\nu{}^c - 2\,e_\mu{}^{[a} e^{b]\nu} b_\nu\,. \tag{1.21}
$$

This is the analogue of Cartan's first structure equation which determines the spin connection in terms of the vierbein in general relativity. Note however that in (1.21), the spin connection $\omega_\mu{}^{ab}$ also depends on the dilatational gauge field $b_\mu$. A second curvature constraint is imposed, $e^\nu{}_b R(M)_{\mu\nu}{}^{ab} = 0$, and can be solved for the K-gauge field $f_\mu{}^a$

$$
f_\mu{}^a = \frac{1}{2} R(\omega,e)_\mu{}^a - \frac{1}{12} R(\omega,e)\, e_\mu{}^a\,. \tag{1.22}
$$

[10] In conformal supergravity, these constraints also ensure that the symmetry algebra is of the supergravity type, *i.e.* that the commutator of two SUSY transformations is a general coordinate transformation (see (B.7)).

Here we used the Ricci tensor $R(\omega,e)_\mu{}^a = R(\omega)_{\mu b}{}^{ab}$ and its corresponding Ricci scalar $R(\omega,e)$, where $R(\omega)_{\mu b}{}^{ab}$ is the Riemann tensor associated to the spin connection (1.21). Its expression is given by the second equation in (1.20) without the last term. The only independent fields of the theory are therefore $e_\mu{}^a$ and $b_\mu$. Finally, if one sets[11] $b_\mu = 0$, then $R(\omega)_{\mu\nu}{}^{ab}$, $R(\omega,e)_\mu{}^a$ and $R(\omega,e)$ reduce to the Riemann tensor, the Ricci tensor and the Ricci scalar of general relativity, respectively.

From the transformation rules (1.19), it can be shown that the curvature $R(M)_{\mu\nu}{}^{ab}$ is invariant under conformal boosts due to the constraint $R(P)_{\mu\nu}{}^a = 0$. Moreover, observe that $b_\mu$ is the only independent field transforming under conformal boosts. This means that, in the expression (1.20), the various terms that contain $b_\mu$ in fact cancel out. Therefore, the curvature $R(M)_{\mu\nu}{}^{ab}$ is simply equal to the Weyl tensor, which we henceforth denote as $C_{\mu\nu}{}^{ab}$. The unique conformally invariant Lagrangian is given by

$$\mathcal{L}_{CG} = e\, C_{\mu\nu}{}^{ab}\, C^{\mu\nu}{}_{ab}\,. \tag{1.23}$$

The Weyl tensor is invariant under dilatations while the determinant of the vierbein, $e$, carries a Weyl weight $w=-4$, such that

$$\delta_{\Lambda_D} e = -4\Lambda_D e\,. \tag{1.24}$$

With this information, it is trivial to verify the invariance of the Lagrangian (1.23) under all conformal transformations.

The setup of conformal gravity is primarily useful as a gauge equivalent formulation of Poincaré gravity. To illustrate this point, let us consider a real scalar field $\phi$ in a background of conformal gravity. The scalar is chosen inert under conformal boosts and carries a Weyl weight $w$, such that under dilatations

$$\delta_{\Lambda_D}\phi = w\Lambda_D\phi\,. \tag{1.25}$$

We introduce the derivative $D_\mu$ which is covariant with respect to all conformal transformations. The following expression can then be derived

$$D_\mu D^a\phi = (\partial_\mu - (w+1)\, b_\mu)D^a\phi - \omega_\mu{}^a{}_b D^b\phi + w\, f_\mu{}^a\phi\,, \tag{1.26}$$

[11] This is possible due the specific form of the variation of $b_\mu$ under conformal boosts (1.19), and because the vierbein is invertible.

where the appearance of the K-gauge field is tied to the presence of $b_\mu$ in $D_\mu\phi$. After some algebra, and using (1.19), (1.26) leads to

$$\delta_{\Lambda_D,\Lambda_K} D_a D^a\phi = \Lambda_D(2+w)D_aD^a\phi + 2\,\Lambda_K{}^a(1-w)D_a\phi\,. \tag{1.27}$$

With this result and (1.24), it is straightforward to check that, after fixing $w=1$, the following Lagrangian

$$\mathcal{L} = -e\,\phi D_a D^a\phi\,. \tag{1.28}$$

is invariant under all conformal transformations. In particular, invariance under conformal boosts implies that the various terms that contain $b_\mu$ cancel out in (1.28). The subsequent computation is therefore simplified by simply ignoring the field $b_\mu$. Upon using (1.26) and (1.22), the Lagrangian reduces to

$$\mathcal{L} = e\,\partial_a\phi\,\partial^a\phi - \frac{1}{6}eR(e)\,\phi^2\,, \tag{1.29}$$

where we also dropped a total derivative. Here, the Ricci scalar is that of Einstein's gravity since, from the start, $b_\mu$ was absent from the Lagrangian. Finally, when gauge fixing the dilatations by setting $\phi$ to a (dimensionful) constant, for instance $\phi = \sqrt{3}/\kappa$, the Lagrangian (1.29) reduces to the Einstein-Hilbert Lagrangian

$$\mathcal{L} = -e\,\frac{R(e)}{2\kappa^2}\,, \tag{1.30}$$

where $\kappa^2 = 8\pi G_N c^{-4}$ denotes Einstein's constant. This standard example illustrates the gauge equivalence between Einstein's gravity, which is invariant under local Poincaré transformations, and a scalar field $\phi$, known as a compensator field, coupled to conformal gravity. In general, a compensator is introduced in a theory (here Einstein's gravity), in order to compensate for a lack of gauge invariance (here conformal invariance). The gauge equivalence between the two formulations of the theory (with or without the compensator) is reflected through the counting of physical degrees of freedom. In our context, the $4+1$ degrees of freedom carried by $b_\mu$ and $\phi$ are canceled by the $4+1$ gauge degrees of freedom associated with the conformal boosts and the dilatations. After further substracting the $4+6$ gauge degrees of freedom associated to the Poincaré symmetries from the 16 degrees of freedom carried by $e_\mu{}^a$, only the 6 physical degrees of freedom of Einstein's gravity are left. As a separated remark, let us also point out that the same mechanism underlies the tensor hierarchy of the embedding tensor formalism for gauged supergravities (see section 1.1.2). In this case, the (non-abelian) gauge invariance of the Lagrangian is ensured by introducing a hierarchy of $p$-forms that act as compensators.

The picture presented so far is naturally generalised to the cases of four-dimensional conformal supergravities. There, the gauge symmetries are associated to a superconformal algebra which, in addition to the conformal generators listed in (1.19), contains an R-symmetry generator as well as two different kinds of supersymmetry generators, denoted by $Q^i$ and $S^i$ (where $i = 1, \ldots, N$). The former, known in this context as Q-supersymmetry generators, are the usual supersymmetry generators whose corresponding gauge fields are the $N$ gravitini. The presence of the additional supercharges $S^i$ is an exclusive feature of superconformal algebras. They generate the so-called S-supersymmetry transformations. In contrast with Poincaré supergravity, these extra symmetries impose additional restrictions on the multiplets, and as a result conformal supergravities only exist up to $N = 4$ in four spacetime dimensions (see chapter 4). For $N > 1$, the various independent superconformal gauge fields (which include those presented in (1.19)) must be completed by a set of 'matter' fields to form an irreducible off-shell representation of the superconformal algebra. These gauge and matter fields constitute the so-called Weyl supermultiplet. The associated invariant Lagrangian is the superconformal completion of the conformal gravity Lagrangian (1.23). In analogy with the non-supersymmetric case presented above, various matter supermultiplets can be coupled to conformal supergravity as compensators in order to obtain a gauge equivalent formulation of Poincare supergravity. This approach has proven very useful in order to gain insights on the structure of the Poincare theories [60].

We now close the discussion on conformal supergravity until chapter 4. In the following, we revert to the framework of Poincaré supergravity.

## 1.3 Content of this work

This section serves as a summary of the main results and motivations of this thesis, and at the same time, provides a more global perspective on the rather technical problems that will be addressed.

As discussed in section 1.1, maximal supergravity theories in various dimensions are known to possess intruiging duality symmetries (cf. Table 1.1) which can optionally be broken by non-abelian gauge interactions. Many of these theories can be described as truncations from eleven-dimensional supergravity or from ten-dimensional IIB supergravity in the context of dimensional compactification on an internal manifold of appropriate dimensionality. Already at an early stage this raised the question whether the higher-dimensional supergravities might somehow reflect the exceptional duality symmetries that are present in their lower-dimensional 'descendants'. This question has a long history and is also relevant for proving the existence of consistent truncations, implying that any solution of the lower-dimensional maximal supergravity can be uplifted to a higher-dimensional one.

An early attempt to answer this question was based on a reformulation of the full eleven-dimensional supergravity obtained by performing a suitable Kaluza-Klein decomposition to four dimensions while retaining the full dependence on the seven internal coordinates [61]. The key element here was to ensure that the resulting theory was invariant under the four-dimensional R-symmetry group SU(8). This symmetry was locally realized with respect to all the eleven coordinates, and it was introduced by a gauge equivalent re-assembling of the original Spin(10, 1) tangent space. The resulting supersymmetry transformation rules then took a form that was almost identical to the four-dimensional ones, which do indeed exhibit the typical characteristics of the $\mathrm{E}_{7(7)}$ dualities, but now with fields that still depend on all eleven spacetime coordinates. Eventually this set-up made it possible to establish the consistency of the $S^7$ truncation, meaning that the whole field configuration of four-dimensional SO(8) gauged supergravity can be uplifted as a submanifold in the full eleven-dimensional theory by specifying the dependence of the fields on the seven internal coordinates [62, 63] .

Recently this approach was substantially extended by including the supersymmetry transformations of dual fields, which opened a new window to accessing the $\mathrm{E}_{7(7)}$ duality properties of the full eleven-dimensional supergravity [64–67]. Given these recent insights, it is a natural question whether similar structures can be derived for IIB supergravity in the context of a $5 + 5$ split of the coordinates. In chapter 2, we confirm that this is indeed the case and we present a detailed analysis to support this claim. Qualitatively the results turn out to be rather similar to the case of eleven-dimensional supergravity, but many new features arise. In our case the tangent space is re-assembled such that the theory is manifestly invariant under local USp(8) transformations. This group contains the USp(4) subgroup of the $10D$ tangent space group and the explicit U(1) R-symmetry of IIB supergravity as subgroups. Obviously, in this case, the $\mathrm{SU}(1,1) \cong \mathrm{SL}(2)$ subgroup of $\mathrm{E}_{6(6)}$ is manifestly realized from the start. Another interesting aspect is that the five-dimensional gauged supergravity theories, when described in terms of the embedding tensor formalism, involve 27 vector and 27 two-rank tensor fields (cf. Tables 1.1 and 1.2) which constitute the beginning of an intricate $E_{6(6)}$ vector-tensor hierarchy [55,56]. As explained in chapter 2, these features are also present when one retains the dependence on the extra internal coordinates for IIB supergravity, so that this vector-tensor hierarchy does emerge in a ten-dimensional context. This is in line with more recent approaches, such as $E_{6(6)}$ Exceptional Field Theory (discussed below), which incorporates both eleven-dimensional and ten-dimensional IIB supergravity and where the vector-tensor hierarchy also plays a key role [68]. Irrespective of these features, the analysis presented in chapter 2 adresses a number of subtle technical complications that are absent in the corresponding analysis of eleven-dimensional supergravity. Many of those are caused by the fact that the field representation of IIB supergravity is more reducible than that of the eleven-dimensional one. Finally, our reformulation of the IIB theory exhibits the characteristic features of five-dimensional gauged supergravity, and therefore provides a convenient framework in which to study consistent truncations to five dimensions. In particular, we close chapter 2 by working out a partial proof of the consistency of the $S^5$ truncation to SO(6) gauged supergravity, and compare the results with those obtained from other approaches.

While it is clearly significant that the approach initiated in [61] can also be applied successfully to IIB supergravity, we should point out the existence of a wide variety of alternative approaches. These approaches are also aimed at understanding and/or exploiting the duality symmetries of the lower dimensional theories, and involve substantial extensions of the conventional supergravity framework. In particular, they make use of extended geometrical structures and, therefore, contrast with the work presented in chapter 2 which is exclusively based on the standard formulation of type IIB supergravity (as constructed in [23–25]). One such approach is based on generalized geometry [69, 70], and was later further extended to Exceptional Generalised Geometry (EGG) [71–74]. In the latter, one considers an extension of the internal tangent space of eleven-dimensional supergravity (or massless type II supergravities) in the context of specific dimensional split. This enables the unification of the internal diffeomorphims and the internal gauge transformations into the so-called generalised diffeomorphisms, which act on the fields by means of a generalised Lie derivative. In this approach, the purely internal components of the various fields are gathered into a generalised metric while the external components are either set to zero, or frozen to a background value. Also, the fields are taken to only depend on the internal coordinates. Nevertheless, the manifest exceptional duality structure of the (internal) generalised tangent space allows, for instance, to study consistent truncations using a generalisation of the conventional Scherk-Schwarz mechanism in which the twist matrices take values in the duality group. In particular, it was realised that the truncation of type IIB supergravity on $S^5$ can be cast as generalised Scherk-Schwarz reduction. In this way, the complete truncation ansätze for the scalars were provided in [75].

Building up on the idea that an exceptional generalised geometry underlies maximal supergravity [71], another step was envisaged in [76] and later concretized in [77], where the seven-dimensional internal space of eleven-dimensional supergravity (in the context of a 4+7 split) was extended to a 56-dimensional space in order to allow for the $\mathrm{E}_{7(7)}$ duality symmetry to act linearly on the coordinates. This idea was then adapted, in the case of a $5+6$ split, to also include the external (five-dimensional) spacetime, thus providing the first $\mathrm{E}_{6(6)}$-covariant formulation of the full (bosonic) eleven-dimensional supergravity [68,78]. In this so-called $\mathrm{E}_{6(6)}$ Exceptional Field Theory (EFT), all the fields depend arbitrarily on $5+27$ coordinates. The field equations of this theory can be encoded in an $\mathrm{E}_{6(6)}$-invariant action which is unambiguously constructed by requiring gauge invariance under external and internal generalised diffeomorphisms. Consistency of the theory however requires to impose a so-called section constraint which restricts the fields and the gauge parameters to depend on at most eleven coordinates, and therefore ultimately breaks the $\mathrm{E}_{6(6)}$ invariance. More precisely, the section constraint admits two independent maximal solutions, and upon choosing one solution or the other, $\mathrm{E}_{6(6)}$ EFT reduces to eleven-dimensional supergravity (or equivalently massless type IIA supergravity) or to type IIB supergravity in a $5+6$ or $5+5$ dimensional split, respectively [68,79]. The internal generalised diffeomorphisms then account for the internal diffeomorphisms and the internal gauge transformations of the corresponding supergravity theory.

Since then, EFT's have been constructed based on the various duality groups $G$ of maximal supergravities presented in Table 1.1 [68,80–85]. For simplicity, let us henceforth denote by $\mathrm{E}_{n(n)}$ the duality group in $D = 11 - n$ dimensions.[12] The construction of the $\mathrm{E}_{n(n)}$ EFT relies on the same structures and principles as those discussed above for the $\mathrm{E}_{6(6)}$ case. In particular, all the fields depend on external spacetime coordinates $x^\mu$ with $\mu = 0, \ldots, D-1$, as well as on an extended set of internal coordinates $y^{\mathcal{M}}$ with $\mathcal{M} = 1, \ldots, \dim \mathcal{R}_v$ that transform in the representation $\mathcal{R}_v$ of $\mathrm{E}_{n(n)}$ given in Table 1.1. As long as one does not commit to a specific solution of the section constraint, the theory can be regarded as being formally invariant under rigid $\mathrm{E}_{n(n)}$ transformations. The embedding of the massless type II and eleven-dimensional supergravities is systematically recovered by choosing

[12] Such that for instance $\mathrm{E}_{4(4)} \equiv \mathrm{SL}(5)$.

the appropriate solution of the section constraint. The generalised Lie derivative and other structures in EFT then reproduce the EGG associated with the internal space of the corresponding supergravity theory in an appropriate dimensional split. An alternative (trivial) option is to solve the section constraint by choosing the fields to be independent of all the internal coordinates $y^{\mathcal{M}}$. The $\mathrm{E}_{n(n)}$ EFT then reduces to ungauged supergravity in $D$ dimensions, and, only in this case, the rigid $\mathrm{E}_{n(n)}$ symmetry is preserved. A more detailed presentation of EFT's will be given in chapter 3. Their applications range from the study of consistent truncations [86–88] to loop computations of higher-derivative corrections to the M-theory effective action [89]. Finally, supersymmetric completions of the $\mathrm{E}_{6(6)}$, $\mathrm{E}_{7(7)}$ and $\mathrm{E}_{8(8)}$ EFT have been worked out [90–92], but in this work, we will only be concerned with bosonic EFT's.

While the embedding of the eleven-dimensional and massless type II supergravities into EFT is well understood, the one of massive IIA supergravity [27] remained until recently an open problem. In fact, a puzzle arises when facing this issue. On the one hand, being a fully consistent ten-dimensional maximal supergravity in its own right, massive type IIA should possess an associated EGG capturing its degrees of freedom and local symmetries in the same fashion as for the massless type II theories. It is therefore natural to expect that such a generalised geometry would descend from EFT after choosing some specific solution of the section constraint. On the other hand, solutions of the section constraint in EFT have been classified and are known to exclusively correspond to the massless type II and eleven-dimensional supergravities [89, 93–96]. It thus suggests that some violation of the section constraint could be needed in order to reproduce massive type IIA supergravity. In this case, however, there would be no direct relation with an EGG in ten dimensions.

In chapter 3, we provide a solution to this puzzle and, in doing so, we unveil an extension of the EFT framework which is based on consistent deformations of the generalised Lie derivative. In analogy with the embedding tensor formalism in gauged supergravity (see 1.1.2), the deformation parameter is $\mathrm{E}_{n(n)}$ Lie algebra-valued and must satisfy set of quadratic and linear constraints in order to guarantee the consistency of the deformed theory. In addition to the original section constraint of EFT, the presence of the deformation imposes further constraints on the coordinate dependence of the fields and gauge parameters. In particular, we prove that for a specific deformation, these constraints allow for a ten-dimensional solution which corresponds to massive type IIA supergravity. This is illustrated for the SL(5) EFT where we show explicitly that, for a suitable deformation and choice of physical coordinates, the deformed generalised diffeomorphisms reproduce the internal diffeomorphisms and gauge transformations of massive type IIA supergravity in a $7+3$ split. Later on we focus exclusively on the $\mathrm{E}_{7(7)}$ case, and we show that invariance under deformed generalised diffeomorphisms requires modifications of the tensor hierarchy and of the action. In particular, when the fields are simply chosen to be independent of the 56 internal coordinates, the theory reduces to the bosonic sector of $D=4$ gauged maximal supergravity and the deformation parameter serves as the embedding tensor.

The last part of this work is rather disconnected from the problematics discussed above and concerns the conformal supergravity framework introduced in section 1.2. More precisely, the question addressed in chapter 4 is that of the construction of an invariant action for $N=4$ conformal supergravity in four dimensions. In this case, the field representation and the full non-linear superconformal transformation rules have been derived long ago in [97]. One of the characteristic feature of the $N=4$ theory is the presence of scalar fields that parametrize an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset. When coupled to compensating matter multiplets, the theory is gauge-equivalent to half-maximal Poincaré supergravity where the $\mathrm{SU}(1,1)$ factor is part of rigid duality symmetry group [98]. Irrespective of this property, the construction of an $N=4$ invariant action for pure conformal supergravity is hampered by the absence of a chiral superspace and, so far, only partial results had been obtained [99, 100]. Moreover, it was suggested already in [101] that a large class of actions could exist. In chapter 4, we confirm this expectation by deriving the most general $N=4$ conformal supergravity action, which turns out to depend on a holomorphic function of the scalar fields. Our approach relies on the construction of a density formula which is expressed in terms of composite fields and provides a superconformally invariant skeleton for the full action. Ultimately, we explicitly present the purely bosonic term and postpone the presentation of the fermionic sector to an upcoming paper [102].

## Notations and conventions

To each chapter corresponds an independent set of notations and conventions. They are defined explicitly when necessary. Appendix A and B share the notations of chapter 2 and 4, respectively.

# Chapter 2

# $\mathbf{E_{6(6)}}$ structure of type IIB supergravity

In this chapter, we present a reformulation of type IIB supergravity in which some of the $E_{6(6)}$ duality features of five-dimensional maximal supergravity are reflected. Ultimately, this allows us to partially write down and study the ansätze for the truncation to SO(6) gauged supergravity. This chapter is directly based on [103], with the exception of section 2.6 which also takes into account the more recent results of [79, 86].

## 2.1 Introduction

As announced in section 1.3, it is possible to reformulate type IIB supergravity upon splitting the coordinates into 5 space-time and 5 internal coordinates, while retaining the full dependence on the two sets of coordinates. To ensure that the theory takes the form of a five-dimensional theory with fields that depend in addition on the five internal coordinates, one adopts a gauge-equivalent version of the tangent space such that the tangent space group will be restricted to the product group $SO(4,1) \times SO(5)$. Subsequently one combines the group SO(5) associated with the internal five-dimensional tangent space with the manifest local U(1) R-symmetry group of IIB supergravity. The crucial step is then to extend this product group to USp(8), which is the R-symmetry group for five-dimensional maximal supergravity. Hence we envisage

$$\begin{aligned}\mathrm{Spin}(9,1)\times\mathrm{U}(1) &\longrightarrow \mathrm{Spin}(4,1)\times\big[\mathrm{USp}(4)\times\mathrm{U}(1)\big]\\ &\longrightarrow \mathrm{Spin}(4,1)\times\mathrm{USp}(8)\,,\end{aligned}\tag{2.1}$$

where we now refer to the universal covering groups which are relevant for the fermions. Initially only the $\mathrm{USp}(4)\times\mathrm{U}(1)$ subgroup is realized as a local invariance that involves all ten coordinates. In order to realize the full local $\mathrm{USp}(8)$ invariance, it suffices to introduce a compensating $\mathrm{USp}(8)/[\mathrm{USp}(4)\times\mathrm{U}(1)]$ phase factor.

The ensuing analysis will be more subtle than for the $4+7$ split of $11D$ supergravity [61]. Indeed, the latter contains a single fermion field corresponding to the gravitino that decomposes directly into $4D$ gravitini transforming in the $\mathbf{8}$ representation and $4D$ spin-$1/2$ fermions transforming in the $\mathbf{48}+\mathbf{8}$ representation of $\mathrm{Spin}(7)$. As was first demonstrated in [104], these fields can be reassembled upon extending the group $\mathrm{Spin}(7)$ to the four-dimensional chiral $\mathrm{SU}(8)$ R-symmetry group, so that the gravitini transform in the $\mathbf{8}\oplus\overline{\mathbf{8}}$ representation and the spin-$1/2$ fields in the $\mathbf{56}\oplus\overline{\mathbf{56}}$ representation of $\mathrm{SU}(8)$. The IIB fermion representation, on the other hand, is already reducible in ten dimensions and consists of a complex gravitino and a complex dilatino field. The $\mathrm{USp}(4)$ tangent-space group can in principle be generalized for each of these fields to $\mathrm{SU}(4)\cong\mathrm{SO}(6)$. Furthermore, the fermions of the IIB theory transform under a locally realized $\mathrm{U}(1)$. Therefore, the R-symmetry group of the $5D$ fermions is extended from $\mathrm{SU}(4)$ to $\mathrm{SU}(4)\times\mathrm{U}(1)$. For the gravitini this group can be directly extended to the expected $\mathrm{USp}(8)$ R-symmetry group, under which the gravitini will transform in the $\mathbf{8}$ representation. However, for the spin-$1/2$ fermions one must combine the gravitino associated with the internal space, comprising 40 symplectic Majorana spinors, with the dilatino, comprising 8 such spinors, into an irreducible $\mathbf{48}$ representation of the group $\mathrm{USp}(8)$. It is clear that assembling the different IIB fermions into a single irreducible spinor that transforms covariantly under $\mathrm{USp}(8)$ is a subtle task.

Therefore our strategy is to read off the expression for the $\mathrm{Usp}(8)$ covariant spinor fields from the supersymmetry transformations of the bosonic fields. This requires us to first identify the correct $5D$ vector and two-rank tensor fields and their supersymmetry transformations, subject to the vector-tensor hierarchy that is known from the embedding tensor formulation of $5D$ maximal supergravity [42]. Unlike in the reformulation of eleven-dimensional supergravity one must also include the tensor fields in the analysis, because in five dimensions the dynamic degrees of freedom for generic gaugings are always carried by a mixture of vector and tensor fields (see 1.1.2). Hence the vector-tensor hierarchy plays a key role here at a much earlier stage of the analysis and it is not sufficient to rely exclusively on a proper preparation of the target space as indicated in (2.1). In the end, five of the tensor fields are still unaccounted for, but even without these missing tensors we have sufficient information to determine the generalized vielbeine, the USp(8) covariant spinor fields, and the supersymmetry transformations of the generalized vielbeine. Using the vector-tensor hierarchy as a guide, one can incorporate the missing five tensor fields which turn out to transform in a representation that coincides precisely with that of a descendant of the $10D$ dual graviton [105–107]. Hence the dual graviton emerges in the form of tensor fields, unlike in the eleven-dimensional situation [67] where the dual graviton resides in the vector sector. We present a basis for the vector and tensor fields which is manifestly in agreement with the $\mathrm{E}_{6(6)}$ assignments known from the $5D$ theory, which involves the invariant three-rank symmetric tensor of that group.

In spite of many subtle differences, the gross features of the present analysis are in agreement with those of 11-dimensional supergravity, implying that the approach that has been adopted is sufficiently robust to be applied to more complicated situations. The supersymmetry transformations of the fields are covariant under local USp(8) transformations and, as such, provide a convenient basis to study other detailed questions, such as consistent truncations to five-dimensions. Here, we focus on the truncation to the SO(6) gauging of maximal five-dimensional supergravity. We present a partial set of truncation ansätze which we compare with results obtained from different approaches.

This chapter is organized as follows. In section 2.2 the relevant properties of IIB supergravity are summarized and the conventions are defined. Subsequently, in section 2.3, the Kaluza-Klein decompositions are carried out to ensure that the fields transform covariantly from the viewpoint of the $5D$ space-time. Also, the conversion to $5D$ spinors and gamma matrices is discussed as well as the proper definitions of the $5D$ vector and tensor fields that emerge directly from the $10D$ bosonic fields. As it turns out, further redefinitions on the vector and tensor fields are required such that they transform under supersymmetry in a way that is consistent with the vector-tensor hierarchy of the $5D$ theory. In section 2.4, we introduce the $5D$ vector and tensor fields associated with the dual six-form potentials of the IIB theory. Again their proper identification is based on covariance in the $5D$ space-time and on the vector-tensor hierarchy. As it turns out there are only 22 tensor fields at this stage. It is then demonstrated how the missing fields can emerge from a component of the $10D$ dual graviton. This enables one to obtain the symmetric $\mathrm{E}_{6(6)}$ tensor that appears in the transformation rules of the tensor fields. At this point the supersymmetry transformations of the bosonic vector and tensor fields clearly resemble the transformation rules encountered in the pure $5D$ theory as presented in [42], including those related to the vector-tensor hierarchy. By direct comparison between the supersymmetry transformations of the vector fields arising from ten dimensions and the five-dimensional ones, explicit expressions for the generalized vielbeine are derived in section 2.5. In addition the USp(8) covariant definitions of the spinor fields are obtained, as well as the supersymmetry transformations of the generalized vielbeine. A similar strategy is then applied to the tensor fields, which leads to a corresponding set of generalized vielbeine. Upon adopting suitable normalizations of the vector and tensor fields one can show that this new set of vielbeine constitutes the inverse of the generalized vielbeine determined in the vector sector. Finally, in section 2.6 the question of the consistent truncation to SO(6) gauged maximal $5D$ supergravity is adressed. The results presented in this chapter relies on Appendix A, which deals with the definition and decomposition of gamma matrices as well as with the spinor and R-symmetry representations associated with the various groups emerging upon decomposing the tangent-space into two separate $5D$ subspaces.

## 2.2 IIB supergravity

Here we summarize the relevant results for IIB supergravity in ten spacetime dimensions [23–25]. We use $M, N, \ldots$ as ten-dimensional spacetime indices and $A, B, \ldots$ as tangent space indices.[1] The theory is described in terms of a zehnbein $E_M{}^A$, a gravitino field $\psi_M$, a spinor field $\lambda$, a complex three-rank tensor field strength, $G_{MNP}$, a five-rank field strength $F_{MNPQR}$ subject to a duality constraint, a complex vector $P_M$ and a U(1) gauge field $Q_M$. The fermions are complex and have opposite chirality,

$$\breve{\Gamma}_{11}\psi_M = \psi_M\,, \qquad \breve{\Gamma}_{11}\lambda = -\lambda\,, \tag{2.2}$$

where $\breve{\Gamma}_{11} = \mathrm{i}\breve{\Gamma}_1\breve{\Gamma}_2\cdots\breve{\Gamma}_{10}$, with $\breve{\Gamma}_A$ denoting the ten-dimensional gamma matrices. The fermions transform under the local U(1) transformations according to

$$\psi_M \to \mathrm{e}^{\mathrm{i}\Lambda/2}\,\psi_M\,, \qquad \lambda \to \mathrm{e}^{3\mathrm{i}\Lambda/2}\,\lambda\,. \tag{2.3}$$

[1] They should not be confused with the Usp(8) indices introduced in section 2.5.

The zehnbein $E_M{}^A$ and the field strength $F_{MNPQR}$ are invariant under these transformations, unlike the other quantities, which transform as follows,

$$G_{MNP} \to \mathrm{e}^{\mathrm{i}\Lambda}\, G_{MNP}\,, \qquad P_M \to \mathrm{e}^{2\mathrm{i}\Lambda}\, P_M\,, \qquad Q_M \to Q_M + \partial_M \Lambda\,. \tag{2.4}$$

The vectors $P_M$ and $Q_M$ satisfy the Maurer-Cartan equations associated with the coset space $\mathrm{SU}(1,1)/\mathrm{U}(1)$, which is parametrized by the scalar fields of the theory,

$$\partial_{[M} Q_{N]} = -\mathrm{i} P_{[M}\, \bar{P}_{N]}\,, \qquad \mathcal{D}_{[M} P_{N]} = 0\,. \tag{2.5}$$

In this section the derivative $\mathcal{D}_M$ is covariant with respect to local Lorentz and local $\mathrm{U}(1)$ transformations.

The coset representative can be expressed in terms of an $\mathrm{SU}(1,1)$ doublet $\phi^\alpha$, $(\alpha = 1, 2)$, transforming under $\mathrm{U}(1)$ as

$$\phi^\alpha \to \mathrm{e}^{\mathrm{i}\Lambda}\, \phi^\alpha\,, \tag{2.6}$$

and subject to the $\mathrm{SU}(1,1)$ invariant constraint,

$$|\phi^1|^2 - |\phi^2|^2 = 1\,. \tag{2.7}$$

In what follows we use the convenient notation $\phi_\alpha \equiv \eta_{\alpha\beta}(\phi^\beta)^*$, with $\eta_{\alpha\beta} = \mathrm{diag}(+1, -1)$, so that the above constraint reads $\phi_\alpha \phi^\alpha = 1$. In this convention the vector fields take the following form,

$$\begin{aligned} Q_M =& -\mathrm{i}\phi_\alpha\, \partial_M \phi^\alpha\,, \\ P_M =& \,\varepsilon_{\alpha\beta}\, \phi^\alpha\, \mathcal{D}_M \phi^\beta\,, \\ \bar{P}_M =& -\varepsilon^{\alpha\beta}\, \phi_\alpha\, \mathcal{D}_M \phi_\beta\,, \end{aligned} \tag{2.8}$$

where the Levi-Civita symbol is normalized by $\varepsilon_{12} = \varepsilon^{12} = 1$. Note that $\eta_{\alpha\beta}\varepsilon^{\beta\gamma}\eta_{\gamma\delta} = -\varepsilon_{\alpha\delta}$. We note the following useful identities,

$$\phi_\alpha\, \mathcal{D}_M \phi^\alpha = 0\,, \qquad \phi_\alpha P_M = \varepsilon_{\alpha\beta}\, \mathcal{D}_M \phi^\beta\,, \qquad \phi^\alpha \bar{P}_M = -\varepsilon^{\alpha\beta}\, \mathcal{D}_M \phi_\beta\,. \tag{2.9}$$

Let us now turn to the tensor field strengths. The theory contains two tensor fields $A^\alpha{}_{MN}$ transforming under $\mathrm{SU}(1,1) \cong \mathrm{SL}(2)$. Here we use a pseudoreal basis with $A^\alpha{}_{MN} = \varepsilon^{\alpha\beta}(A_{MN})_\beta$, where the convention for lowering and raising of indices is the same as for $\phi^\alpha$. Their field strengths are defined as follows,

$$
\begin{aligned}
3\,\partial_{[M}A^\alpha{}_{NP]} &= \phi^\alpha\,\bar G_{MNP} + \varepsilon^{\alpha\beta}\phi_\beta\,G_{MNP}\,,\\
G_{MNP} &= -\,3\,\varepsilon_{\alpha\beta}\,\phi^\alpha\,\partial_{[M}A^\beta{}_{NP]}\,,\\
\bar G_{MNP} &= 3\,\phi_\alpha\,\partial_{[M}A^\alpha{}_{NP]}\,.
\end{aligned}
\tag{2.10}
$$

The tensor fields are subject to rigid $\mathrm{SU}(1,1)$ transformations, just as the scalar fields $\phi^\alpha$, and to tensor gauge transformations. The latter read

$$
\delta A^\alpha{}_{MN} = 2\,\partial_{[M}\Xi^\alpha{}_{N]}\,.
\tag{2.11}
$$

Furthermore we have a 4-rank antisymmetric gauge potential $A_{MNPQ}$, which transforms under two types of gauge transformations

$$
\delta A_{MNPQ} = 4\,\partial_{[M}\Lambda_{NPQ]} + \tfrac{3}{4}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\Xi^\alpha{}_{[M}\,\partial_N A^\beta{}_{PQ]}\,.
\tag{2.12}
$$

The corresponding 5-rank field strength is defined by

$$
F_{MNPQR} = 5\,\partial_{[M}A_{NPQR]} - \tfrac{15}{8}\mathrm{i}\varepsilon_{\alpha\beta}\,A^\alpha{}_{[MN}\,\partial_P A^\beta{}_{QR]}\,.
\tag{2.13}
$$

The 3- and 5-rank field strengths satisfy the following Bianchi identities,

$$
\begin{aligned}
\mathcal{D}_{[M}G_{NPQ]} &= P_{[M}\,\bar G_{NPQ]}\,,\\
\partial_{[M}F_{NPQRS]} &= -\,\tfrac{5}{12}\mathrm{i}\,G_{[MNP}\,\bar G_{QRS]}\,.
\end{aligned}
\tag{2.14}
$$

In addition there is a constraint on the 5-rank field strengh which involves the dual field strength,

$$
\begin{aligned}
\tfrac{1}{120}\mathrm{i}\,\varepsilon_{ABCDEFGHIJ}\,F^{FGHIJ} &= F_{ABCDE} - \tfrac{1}{8}\mathrm{i}\,\bar\psi_M\breve\Gamma^{[M}\breve\Gamma_{ABCDE}\breve\Gamma^{N]}\psi_N\\
&\quad + \tfrac{1}{16}\mathrm{i}\,\bar\lambda\,\breve\Gamma_{ABCDE}\,\lambda\,.
\end{aligned}
\tag{2.15}
$$

From the chirality of the fermion fields it follows that the fermionic bilinears in (2.15) are anti-selfdual, which is obviously required because otherwise (2.15) would decompose into two independent constraints that would overconstrain the system. Originally (2.15) was derived in superspace [25]. Suppressing the fermionic terms would imply that the bosonic field strength should be self-dual. Note that the constraint (2.15) is supersymmetric and it must transform into the fermionic field equations. Upon combining it with the Bianchi identity (2.14), one obtains the field equations for $A_{MNPQ}$.

Let us now turn to the fermions $\psi_M$ and $\lambda$. The supersymmetry transformations for the spinor fields are as follows,

$$\begin{aligned} \delta\psi_M =& \mathcal{D}_M\epsilon - \tfrac{1}{480}\mathrm{i}F_{NPQRS}\,\breve{\Gamma}^{NPQRS}\,\breve{\Gamma}_M\epsilon - \tfrac{1}{96}G_{NPQ}\big(\breve{\Gamma}_M\,\breve{\Gamma}^{NPQ} + 2\,\breve{\Gamma}^{NPQ}\,\breve{\Gamma}_M\big)\epsilon^{\mathrm{c}}\,, \\ \delta\lambda =& -P_M\,\breve{\Gamma}^M\epsilon^{\mathrm{c}} - \tfrac{1}{24}G_{MNP}\breve{\Gamma}^{MNP}\epsilon\,, \end{aligned} \tag{2.16}$$

where the quantities $\breve{\Gamma}^{MN\cdots}$ denote anti-symmetrized products of $10D$ gamma matrices, and $\mathcal{D}_M\epsilon$ contains the spin-connection field ${\omega_M}^{AB}$ and the U(1) connection $Q_M$,

$$\mathcal{D}_M\epsilon = \Big(\partial_M - \tfrac{1}{4}{\omega_M}^{AB}\breve{\Gamma}_{AB} - \tfrac{1}{2}\mathrm{i}Q_M\Big)\epsilon\,. \tag{2.17}$$

Here $\epsilon$ is the space-time spinor parameter of supersymmetry. In (2.16) we have introduced the Majorana conjugate of a $10D$ spinor $\psi$, which is defined by

$$\psi^{\mathrm{c}} = \breve{C}_\pm^{-1}\bar{\psi}^{\mathrm{T}}\,, \qquad \psi = \breve{C}_\pm^{-1}\bar{\psi}^{\mathrm{cT}}\,. \tag{2.18}$$

Here $\breve{C}_\pm$ denotes the charge conjugation matrix in ten spacetime dimensions which can be either symmetric or anti-symmetric. The gamma matrix conventions are discussed in detail in appendix A.1, but for the convenience of the reader we note

$$\breve{C}_\pm\,\breve{\Gamma}_A\breve{C}_\pm^{-1} = \pm{\breve{\Gamma}_A}^{\mathrm{T}}\,, \quad {\breve{C}_\pm}^{\mathrm{T}} = \pm\breve{C}_\pm\,, \quad {\breve{C}_\pm}^{\dagger} = \breve{C}_\pm^{-1}\,. \tag{2.19}$$

We also note the following equation for spinor bilinears with strings of gamma matrices,

$$\bar{\chi}\,\Gamma_{A_1}\cdots\Gamma_{A_n}\,\psi = -(\pm)^{n+1}\bar{\psi}^{\mathrm{c}}\,\Gamma_{A_n}\cdots\Gamma_{A_1}\,\chi^{\mathrm{c}}\,. \tag{2.20}$$

In type IIB supergravity we have chiral spinors comprising 16 complex components. One can show that $\psi$ and $\psi^{\mathrm{c}}$ have the same chirality (see appendix A.1 for details) and since the spinors are complex (so that $\psi^{\mathrm{c}} \neq \psi$) one can adopt a pseudo-real representation by combining $\psi$ and $\psi^{\mathrm{c}}$ into a 32-component chiral spinor $\Psi = (\psi, \psi^{\mathrm{c}})$, subject to

$$\Psi = \sigma_1\,\breve{C}_\pm^{-1}\bar{\Psi}^{\mathrm{T}}\,, \tag{2.21}$$

where $\sigma_1$ denotes the standard $2\times 2$ Pauli spin matrix. We will also need the supersymmetry transformations for the bosons,

$$\begin{aligned} \delta{E_M}^A =& \tfrac{1}{2}(\bar{\epsilon}\,\breve{\Gamma}^A\psi_M + \bar{\epsilon}^{\mathrm{c}}\,\breve{\Gamma}^A\psi_M^{\mathrm{c}})\,, \\ \delta\phi^\alpha =& \tfrac{1}{2}\varepsilon^{\alpha\beta}\phi_\beta\,\bar{\epsilon}^{\mathrm{c}}\lambda\,, \\ \delta A^\alpha{}_{MN} =& -\tfrac{1}{2}\phi^\alpha\big(\bar{\lambda}\,\breve{\Gamma}_{MN}\epsilon - 4\,\bar{\epsilon}\,\breve{\Gamma}_{[M}{\psi_{N]}}^{\mathrm{c}}\big) + \tfrac{1}{2}\varepsilon^{\alpha\beta}\phi_\beta\,\big(\bar{\epsilon}\,\breve{\Gamma}_{MN}\lambda + 4\,\bar{\psi}_{[M}{}^{\mathrm{c}}\,\breve{\Gamma}_{N]}\epsilon\big)\,, \\ \delta A_{MNPQ} =& \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\,\breve{\Gamma}_{[MNP}\psi_{Q]} + \tfrac{1}{2}\mathrm{i}\bar{\psi}_{[M}\breve{\Gamma}_{NPQ]}\epsilon + \tfrac{3}{8}\mathrm{i}\,\varepsilon_{\alpha\beta}A^\alpha{}_{[MN}\,\delta A^\beta{}_{PQ]}\,. \end{aligned} \tag{2.22}$$

The above transformation rules (2.16) and (2.22) have been derived by imposing the supersymmetry algebra,

$$\big[\delta(\epsilon_1)\,,\delta(\epsilon_2)\big] = \xi^M D_M + \delta_\Xi\big(\Xi^\alpha{}_{MN}\big) + \delta_\Lambda(\Lambda_{MNP}) + \cdots\,, \tag{2.23}$$

where

$$\begin{aligned}
\xi^M &= \tfrac12\bar\epsilon_2\,\breve\Gamma^M\epsilon_1 + \tfrac12\bar\epsilon_2{}^{\mathrm c}\,\breve\Gamma^M\epsilon_1{}^{\mathrm c}\,,\\
\Xi^\alpha{}_M &= -\,\phi^\alpha\,\bar\epsilon_2\breve\Gamma_M\epsilon_1{}^{\mathrm c} - \varepsilon^{\alpha\beta}\phi_\beta\,\bar\epsilon_2{}^{\mathrm c}\breve\Gamma_M\epsilon_1\,,\\
\Lambda_{MNP} &= \tfrac18\mathrm{i}\big(\bar\epsilon_1\breve\Gamma_{MNP}\epsilon_2 - \bar\epsilon_2\breve\Gamma_{MNP}\epsilon_1\big)\\
&\quad + \tfrac{3}{16}\mathrm{i}\big(\varepsilon_{\alpha\beta}\phi^\alpha A^\beta_{[MN}\,\bar\epsilon_2\breve\Gamma_{P]}\epsilon_1{}^{\mathrm c} + \phi_\alpha A^\alpha_{[MN}\,\bar\epsilon_2{}^{\mathrm c}\breve\Gamma_{P]}\epsilon_1\big)\,,
\end{aligned} \tag{2.24}$$

and where $\xi^M D_M$ denotes a fully covariantized spacetime diffeomorphism.

For completeness, we also present the supersymmetry transformation rules for the Majorana conjugate spinors,

$$\begin{aligned}
\delta\psi_M{}^{\mathrm c} &= \mathcal{D}_M\epsilon^{\mathrm c} + \tfrac{1}{480}\mathrm{i}F_{NPQRS}\,\breve\Gamma^{NPQRS}\,\breve\Gamma_M\epsilon^{\mathrm c} - \tfrac{1}{96}\bar G_{NPQ}\big(\breve\Gamma_M\,\breve\Gamma^{NPQ} + 2\,\breve\Gamma^{NPQ}\,\breve\Gamma_M\big)\epsilon\,,\\
\delta\lambda^{\mathrm c} &= \pm\,\bar P_M\breve\Gamma^M\epsilon \pm \tfrac{1}{24}\bar G_{MNP}\,\breve\Gamma^{MNP}\epsilon^{\mathrm c}\,.
\end{aligned} \tag{2.25}$$

To understand the various field equations it is convenient to first consider the following $10D$ Lagrangian of IIB supergravity up to terms of fourth-order in the fermion fields, ignoring for the moment the constraint (2.15),

$$\begin{aligned}
\mathcal{L} = &-\tfrac12 E\,R - E\,\bar\psi_M\breve\Gamma^{MNP}\mathcal{D}_N\psi_P - \tfrac12 E\,\bar\lambda\breve{\not{\mathcal{D}}}\lambda - E\,|P_M|^2 - \tfrac{1}{24}E\,|G_{MNP}|^2\\
&- \tfrac{1}{60}E\,(F_{MNPQR})^2 + \tfrac{1}{384}\,\varepsilon^{MNPQRSTUVW}\,\varepsilon_{\alpha\beta}\,\partial_M A_{NPQR}\,A_{ST}{}^\alpha\,\partial_U A_{VW}{}^\beta\\
&- \tfrac12 E\,\big[\bar\psi_M{}^{\mathrm c}\,\breve\Gamma^N\breve\Gamma^M\lambda\,\bar P_N + \bar\lambda\,\breve\Gamma^M\breve\Gamma^N\psi_M{}^{\mathrm c}\,P_N\big]\\
&+ \tfrac{1}{240}\mathrm{i}E\,\bar\psi_M\breve\Gamma^{[M}\breve\Gamma^{ABCDE}\breve\Gamma^{N]}\psi_N\,F_{ABCDE}\\
&+ \tfrac{1}{48}E\,\big[\bar\psi_M\,\breve\Gamma^{[M}\breve\Gamma^{ABC}\,\breve\Gamma^{N]}\psi_N{}^{\mathrm c}\,G_{ABC} + \bar\psi_M{}^{\mathrm c}\,\breve\Gamma^{[M}\breve\Gamma^{ABC}\,\breve\Gamma^{N]}\psi_N\,\bar G_{ABC}\big]\\
&+ \tfrac{1}{48}E\,\big[\bar\psi_M\,\breve\Gamma^{ABC}\,\breve\Gamma^M\,\lambda\,\bar G_{ABC} - \bar\lambda\,\breve\Gamma^M\,\breve\Gamma^{ABC}\psi_M\,G^{ABC}\big]\\
&- \tfrac{1}{480}\mathrm{i}E\,\bar\lambda\,\breve\Gamma^{ABCDE}\,\lambda\,F_{ABCDE} + \cdots\,.
\end{aligned} \tag{2.26}$$

We have refrained from imposing the supersymmetric constraint (2.15) so that it makes sense to include a term proportional to $(F_{MNPQR})^2$, and furthermore we have included a Chern-Simons term that is invariant under tensor gauge transformations up to a total derivative. It is then straightforward to show that the field equation for the 4-form field that follows from this Lagrangian is consistent with the constraint (2.15) upon using the second Bianchi identity (3.94). Here we should remind the reader that there are extensive discussions in the literature about manifestly covariant Lagrangians that imply self-duality constraints for tensor fields (see, for instance, [108], where also the Chern-Simons terms is presented, and references cited therein). An example is the Lagrangian of $\mathrm{E}_{7(7)}$ exceptional field theory which will be discussed in chapter 3. However, these features are not relevant for the purpose of the present chapter. We also recall that the field equations are already encoded in the supersymmetry transformations, as supersymmetry is only realized on-shell, so that one can determine most terms in (2.26) by imposing super-covariance of the field equations, just as was done in [24]. Our results are also consistent with [25] where an on-shell superspace treatment of IIB supergravity was presented.

For further convenience we list some of the field equations,

$$
\begin{aligned}
&\mathcal{D}^M P_M + \tfrac{1}{24} G_{MNP}\, G^{MNP} \;= 0\,,\\
&\mathcal{D}^M G_{MNP} + P^M\, \bar{G}_{MNP} - \tfrac{2}{3}\mathrm{i} F_{NPQRS}\, G^{QRS} \;= 0\,,\\
&R_{MN} + 2\, P_{(M}\, \bar{P}_{N)} + \tfrac{1}{4}\big(\bar{G}_{PQ(M}\, G^{PQ}{}_{N)} - \tfrac{1}{12} g_{MN}\, |G_{PQR}|^2\big) + \tfrac{1}{6} F_M{}^{PQRS}\, F_{NPQRS} = 0\,,\\
&\breve{\Gamma}^M \hat{D}_M \lambda + \tfrac{1}{240}\mathrm{i}\breve{\Gamma}^{NPQRS} \lambda\, F_{NPQRS} \;= 0\,,\\
&\breve{\Gamma}^{MNP}\, \widehat{D_N \psi_P} \mp \tfrac{1}{2}\breve{\Gamma}^Q \breve{\Gamma}^M \lambda^{\mathrm{c}} P_Q - \tfrac{1}{48}\breve{\Gamma}^{QRS}\breve{\Gamma}^M \lambda\, \bar{G}_{QRS} \;= 0\,,
\end{aligned}
\tag{2.27}
$$

where $\hat{D}_M\lambda$ denotes the supercovariant derivative of the spinor $\lambda$ and $\widehat{D_{[M}\psi_{N]}}$ the supercovariant curl of the gravitino. Here we suppressed higher-order fermion terms.

However, in section 2.4, we will need the field equations for the two-form fields including the terms quadratic in the fermions. They follow directly from the Lagrangian (2.26) and can be written as follows,

$$
\partial_{[M} F_{NPQRSTU]\,\alpha} = 0\,, \tag{2.28}
$$

where the 7-rank anti-symmetric tensors $F_{MNPQRST\,\alpha}$ are equal to

$$
\begin{aligned}
F_{\alpha\, MNPQRST} =& -\tfrac{1}{7}\mathrm{i}\, E\, \varepsilon_{MNPQRSTUVW} \left(\varepsilon_{\alpha\gamma}\, \phi^\gamma \phi_\beta + \varepsilon_{\beta\gamma}\, \phi^\gamma \phi_\alpha\right) \partial^U A^{VW\,\beta}\\
&- 120\mathrm{i}\, \varepsilon_{\alpha\beta}\, A_{[MN}{}^{\beta} \big[\partial_P A_{QRST]} - \tfrac{1}{8}\mathrm{i}\varepsilon_{\gamma\delta}\, A_{PQ}{}^{\gamma}\, \partial_R A_{ST}{}^{\delta}\big]\\
&+ \tfrac{1}{7}\varepsilon_{\alpha\beta}\phi^\beta \left[\bar{\psi}_U\, \breve{\Gamma}^{[U}\breve{\Gamma}_{MNPQRST}\, \breve{\Gamma}^{V]}\psi_V{}^{\mathrm{c}} + \bar{\lambda}\, \breve{\Gamma}^U\, \breve{\Gamma}_{MNPQRST}\psi_U\right]\\
&+ \tfrac{1}{7}\phi_\alpha \left[\bar{\psi}_U{}^{\mathrm{c}}\, \breve{\Gamma}^{[U}\breve{\Gamma}_{MNPQRST}\, \breve{\Gamma}^{V]}\psi_V - \bar{\psi}_U\, \breve{\Gamma}_{MNPQRST}\breve{\Gamma}^U\, \lambda\right].
\end{aligned}
\tag{2.29}
$$

Note that the normalization of this tensor is arbitrary but the phase is dictated by the fact that its pseudo-reality condition is in line with that of the other pseudo-real fields.

## 2.3 Kaluza-Klein decompositions and additional field redefinitions

The strategy in this chapter is to describe IIB supergravity as a field theory in a five-dimensional space-time, while still retaining the dependence on the five additional coordinates that describe an internal space. Hence the $10D$ coordinates are decomposed according to $x^M \to (x^\mu, y^m)$, where $x^\mu$ are regarded as the five space-time coordinates and $y^m$ as the five coordinates of the internal manifold. Eventually, in a given background, the fields may be decomposed in terms of a complete basis of functions of the internal coordinates. As discussed in 1.1.1, for the $T^5$ background this is rather straightforward. On the other hand, the spectrum of the tower of Kaluza-Klein supermultiplets for $S^5$ has been studied in [109, 110]. However, at this stage we will not be assuming any particular space-time background and neither will we be truncating the theory in any way. We are only reformulating the theory in a form that emphasizes the five-dimensional spacetime.

A crucial ingredient in this reformulation is provided by a change of the tangent-space group, which we have already indicated in (2.1). First we impose a gauge choice, reducing the $10D$ local Lorentz group to the product group $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$, whose universal covering group equals $\mathrm{Spin}(4,1) \times \mathrm{USp}(4)$. The fermions then transform according to the product representation of this group, so that from a five-dimensional space-time perspective we are dealing with four complex $\mathrm{Spin}(4,1)$ spinors, each carrying four components. The fermions are subject to the extra local $\mathrm{U}(1)$ transformations (2.3), and the product group $\mathrm{USp}(4) \times \mathrm{U}(1)$ must be contained in the $5D$ R-symmetry group. Obviously we have to convert the $10D$ gamma matrices to those appropriate for five space-time dimensions, equiped with two sets of mutually commuting gamma matrices, one associated with space-time and the other one with the internal space. In due course we will also have to recombine the spin-1/2 fermion fields into an irreducible representation of the group $\mathrm{USp}(8)$, which is the R-symmetry group for eight symplectic Majorana supercharges in a $5D$ space-time. This last redefinition will be considered in section 2.5.

The next step is to redefine the fields such that they transform covariantly under the $5D$ space-time diffeomorphisms. These Kaluza-Klein decompositions were systematically discussed in the context of the $T^7$ reduction of $11D$ supergravity to $4D$ supergravity [104]. Furthermore, we will find that the vector and tensor fields require additional redefinitions beyond the Kaluza-Klein ones in order to generate transformations that reflect the vector-tensor hierarchy [42].

The standard Kaluza-Klein decompositions start with the vielbein field and its inverse, which we write in triangular form by exploiting the $10D$ local Lorentz transformations,

$$E_M{}^A = \begin{pmatrix} \Delta^{-1/3} e_\mu{}^\alpha & B_\mu{}^m\, e_m{}^a \\ \\ 0 & e_m{}^a \end{pmatrix} , \; E_A{}^M = \begin{pmatrix} \Delta^{1/3} e_\alpha{}^\mu & -\Delta^{1/3} e_\alpha{}^\nu B_\nu{}^m \\ \\ 0 & e_a{}^m \end{pmatrix} . \tag{2.30}$$

Here we used tangent-space indices $\alpha, \beta, \ldots$ associated with the $5D$ space-time and $a, b, \ldots$ associated with the $5D$ internal space.[2] The scalar factor $\Delta$ is defined by,

$$\Delta = \frac{\det[e_m{}^a(x,y)]}{\det[\mathring{e}_m{}^a(y)]} , \tag{2.31}$$

where $\mathring{e}_m{}^a$ is some reference frame for the internal space parametrized by the coordinates $y^m$. The rescaling of the fünfbein is such that the gravitational coupling constants in $10D$ and $5D$ are related by $\kappa^{-2}|_{10D} = \kappa^{-2}|_{5D} \int \mathrm{d}^5y \, \det[\mathring{e}_m{}^a]$, so that we are in the $5D$ Einstein frame.

An important feature of the gauge choice made in (2.30) is that it must be preserved under supersymmetry. This requires to add to the $10D$ supersymmetry transformations a uniform field-dependent Lorentz transformation with a parameter equal to

$$\epsilon^{\alpha a} = -\epsilon^{a\alpha} = -\tfrac{1}{2} e_a{}^m \big( \bar\epsilon\, \check\Gamma^\alpha \psi_m + \bar\epsilon^{\mathrm{c}}\, \check\Gamma^\alpha \psi_m{}^{\mathrm{c}} \big) , \tag{2.32}$$

where $\psi_a = e_a{}^m\, \psi_m$. The supersymmetry transformation of $e_a{}^m$ is not affected by the compensating Lorentz transformations. This implies

$$\delta\Delta = \tfrac{1}{2} \Delta \big( \bar\epsilon\, \check\Gamma^a \psi_a + \bar\epsilon^{\mathrm{c}}\, \check\Gamma^a \psi_a{}^{\mathrm{c}} \big) . \tag{2.33}$$

One can now determine the supersymmetry variation of the fünfbein $e_\mu{}^\alpha$, taking into account the compensating Lorentz transformation (2.32) and the effect of the

[2] Note that we are also using indices $\alpha, \beta \ldots$ for the $\mathrm{SU}(1,1)$ indices on the scalar doublet and the tensor fields. This should not cause any confusion.

factor $\Delta$. Insisting on the fact that $e_\mu{}^\alpha$ transforms into the $5D$ gravitino field in the same way as before, one then derives a modified gravitino field,

$$\psi_\mu{}^{\mathrm{KK}} \equiv \Delta^{1/6}\big[\psi_\mu - B_\mu{}^m\,\psi_m\big] + \tfrac{1}{3}\Delta^{-1/6}e_\mu{}^\alpha\,\breve\Gamma_\alpha\breve\Gamma^a\psi_a\,, \tag{2.34}$$

and likewise for $\psi_\mu{}^{\mathrm{c}}$. This field transforms covariantly under $5D$ space-time diffeomorphisms by virtue of the presence of the field $B_\mu{}^m$. Accordingly we also perform field-dependent scale transformations on the supersymmetry parameter, the gravitino components $\psi_a$ and the dilatino,

$$\epsilon^{\mathrm{KK}} = \Delta^{1/6}\epsilon\,,\qquad \psi_a{}^{\mathrm{KK}} = \Delta^{-1/6}\,e_a{}^m\,\psi_m\,,\qquad \lambda^{\mathrm{KK}} = \Delta^{-1/6}\lambda\,. \tag{2.35}$$

Subsequently we must convert to different gamma matrices that decompose into two commuting Clifford algebras corresponding to the 5-dimensional space-time and the 5-dimensional internal space, which must both commute with $\breve\Gamma_{11}$ so that they will be consistent with the $10D$ chirality restriction on the original spinors. As mentioned previously every $10D$ spinor decomposes into four complex $\mathrm{Spin}(4,1)$ spinors. The gamma matrix conversion is discussed in detail in appendix A.1 and the results can be summarized as follows. The $32\times 32$ gamma matrices $\breve\Gamma_A$ can be written as

$$\breve\Gamma_\alpha = -\mathrm{i}\big(\hat\gamma_\alpha\,\tilde\Gamma\big)\,,\qquad \breve\Gamma_{a+5} = -\mathrm{i}\big(\hat\Gamma_a\,\tilde\gamma\big)\,, \tag{2.36}$$

where $\breve\Gamma_{11} = \mathrm{i}\tilde\gamma\tilde\Gamma$ with $\tilde\gamma$ and $\tilde\Gamma$ mutually anti-commuting hermitian matrices that square to $\mathbb{1}_{32}$. The tangent space indices in the $5+5$ split were already defined below (2.30).[3] Both $\hat\gamma^\alpha$ and $\hat\Gamma^a$ anti-commute with $\tilde\gamma$ and $\tilde\Gamma$ (and therefore commute with $\breve\Gamma_{11}$ as insisted on before). They generate two commuting five-dimensional Clifford algebras. Furthermore, we will insist on the Majorana condition $\hat C^{-1}\bar\psi^{\mathrm{T}} = \psi^{\mathrm{c}}$ for all the $5D$ spinor fields, where $\hat C$ is defined in terms of the $10D$ charge conjugation matrix in (A.13). For the gravitino fields and the supersymmetry parameters this leads to the following relations between $10D$ and $5D$ fields,

$$\psi\big|^{\mathrm{KK}}_{10D} = \psi\big|_{5D}\,,\qquad \psi^{\mathrm{c}}\big|^{\mathrm{KK}}_{10D} = \psi^{\mathrm{c}}\big|_{5D}\,,\qquad \bar\psi\big|^{\mathrm{KK}}_{10D} = -\mathrm{i}\bar\psi\big|_{5D}\tilde\Gamma\,, \tag{2.37}$$

where $\psi$ denotes either $\psi_M$ or $\epsilon$.

For the dilatino field $\lambda$ the situation is somewhat different in view of the fact that we wish to change its chirality by absorbing the matrix $\tilde\Gamma$. This conversion is of course no longer consistent with $10D$ Lorentz invariance, but it is convenient to

[3] We employed Pauli-Källén conventions where $x^\alpha$ equals $\mathrm{i}x^0$ for $\alpha=1$, so that all gamma matrices are hermitian.

define all the spinor fields with the same (positive) chirality.

$$\lambda\big|_{10D}^{\mathrm{KK}} = \tilde{\Gamma}\,\lambda\big|_{5D}\,, \qquad \lambda^{\mathrm{c}}\big|_{10D}^{\mathrm{KK}} = \mp\tilde{\Gamma}\,\lambda^{\mathrm{c}}\big|_{5D}\,, \qquad \bar{\lambda}\big|_{10D}^{\mathrm{KK}} = \mathrm{i}\,\bar{\lambda}\big|_{5D}\,, \tag{2.38}$$

Once these modifications have been performed, one can simply restrict oneself to the 16-dimensional subspace corresponding to the eigenspace of $\breve{\Gamma}_{11}$ with eigenvalue $+1$. After this one drops the carets on $\gamma_\alpha$ and $\Gamma_a$ and thus obtains a description in term of 16-component complex spinors, with two mutually commuting sets of gamma matrices $\gamma_\alpha$ and $\Gamma_a$. Note that this is consistent with using the charge conjugation matrix $\hat{C}$, which was introduced as a 32-dimensional matrix but which commutes with the chirality operator (i.e. charge-conjugated fields carry the same chirality). With these conversions the relation (2.34) for the $10D$ gravitino field $\psi_M$ with $M=\mu$ in terms of the $5D$ fields reads

$$\Delta^{1/6}\psi_\mu = \psi_\mu{}^{\mathrm{KK}} - \tfrac{1}{3}\mathrm{i}\gamma_\mu\Gamma^m\psi_m{}^{\mathrm{KK}} + \Delta^{1/3}B_\mu{}^m\,\psi_m{}^{\mathrm{KK}}\,. \tag{2.39}$$

Observe that here and henceforth $\gamma_\mu \equiv e_\mu{}^\alpha\,\gamma_\alpha$ and $\Gamma_m \equiv e_m{}^a\,\Gamma_a$, where the vielbein fields $e_\mu{}^\alpha$ and $e_m{}^a$ are defined in (2.30).

In this way one finds the following transformation rules for the $5D$ fields emerging from $E_M{}^A$ as defined in (2.34) and (2.35),

$$\begin{aligned}
\delta e_\mu{}^\alpha &= \tfrac{1}{2}\big[\bar{\epsilon}\,\gamma^\alpha\psi_\mu + \bar{\epsilon}^{\mathrm{c}}\gamma^\alpha\psi_\mu{}^{\mathrm{c}}\big]\,,\\
\delta B_\mu{}^m &= \tfrac{1}{2}\Delta^{-1/3}e_a{}^m\big[\mathrm{i}\big(\bar{\epsilon}\,\Gamma^a\psi_\mu + \bar{\epsilon}^{\mathrm{c}}\,\Gamma^a\psi_\mu{}^{\mathrm{c}}\big)\\
&\qquad\qquad + \bar{\epsilon}\,\gamma_\mu(\delta^a{}_b + \tfrac{1}{3}\Gamma^a\Gamma_b)\psi^b + \bar{\epsilon}^{\mathrm{c}}\,\gamma_\mu(\delta^a{}_b + \tfrac{1}{3}\Gamma^a\Gamma_b)\psi^{b\mathrm{c}}\big]\,,\\
\delta e_m{}^a &= \tfrac{1}{2}\mathrm{i}\big[\bar{\epsilon}\,\Gamma^a\psi_m + \bar{\epsilon}^{\mathrm{c}}\,\Gamma^a\psi_m{}^{\mathrm{c}}\big]\,,
\end{aligned} \tag{2.40}$$

up to an infinitesimal $5D$ local Lorentz transformation with a parameter proportional to $\Gamma^m\psi_m$. Since we will be suppressing terms of higher orders in the spinor fields, these transformations will not play a role when evaluating the fermion transformation rules later in this section. Here and in the following we are exclusively considering the $5D$ fields, so that we have dropped the additional labels.

We also evaluate the supersymmetry variations of the scalars and the dilatini,

$$\delta\phi^\alpha = -\tfrac{1}{2}\mathrm{i}\,\varepsilon^{\alpha\beta}\phi_\beta\,\bar{\epsilon}^{\mathrm{c}}\lambda\,, \tag{2.41}$$

$$\begin{aligned}
\delta\lambda &= \Delta^{-1/3}\big[-\mathrm{i}P_\alpha\gamma^\alpha + P_a\Gamma^a\big]\epsilon^{\mathrm{c}}\\
&\quad + \tfrac{1}{24}\Delta^{-1/3}\big[G_{abc}\,\Gamma^{abc} - 3\mathrm{i}\,G_{ab\alpha}\,\Gamma^{ab}\gamma^\alpha + 3\,G_{a\alpha\beta}\,\Gamma^a\gamma^{\alpha\beta} - \mathrm{i}\,G_{\alpha\beta\gamma}\,\gamma^{\alpha\beta\gamma}\big]\epsilon\,,
\end{aligned}$$

$$
\begin{aligned}
\delta\lambda^{\rm c} = {}& \Delta^{-1/3}\big[ -\mathrm{i}\bar{P}_\alpha\gamma^\alpha + \bar{P}_a\Gamma^a\big]\epsilon \\
& + \tfrac{1}{24}\Delta^{-1/3}\big[\bar{G}_{abc}\,\Gamma^{abc} - 3\mathrm{i}\,\bar{G}_{ab\alpha}\,\Gamma^{ab}\gamma^\alpha + 3\,\bar{G}_{a\alpha\beta}\,\Gamma^a\gamma^{\alpha\beta} - \mathrm{i}\,\bar{G}_{\alpha\beta\gamma}\,\gamma^{\alpha\beta\gamma}\big]\epsilon^{\rm c}\,,
\end{aligned}
$$

where the tensors $P$ and $G$ refer to the components of $P_A$ and $G_{ABC}$, which are defined with $10D$ tangent-space indices.

Subsequently we derive the expressions for the supersymmetry variation of the gravitino fields up to terms of higher order in the fermion fields, which will now also involve the components of the field strength $F_{ABCDE}$ and the spin-connection fields written with $10D$ tangent-space indices. We first list the gravitino fields that carry a $5D$ space-time vector index,

$$
\begin{aligned}
\delta\psi_\mu = {}& \big[\partial_\mu - \tfrac{1}{6}\partial_\mu\ln\Delta - \Delta^{-1/3}e_\mu{}^\alpha\big(\tfrac{1}{4}\omega_\alpha{}^{\beta\gamma}\,\gamma_{\beta\gamma} + \tfrac{1}{2}\mathrm{i}\omega_\alpha{}^{\beta a}\,\Gamma_a\gamma_\beta + \tfrac{1}{4}\omega_\alpha{}^{ab}\,\Gamma_{ab} + \tfrac{1}{2}\mathrm{i}Q_\alpha\big)\big]\epsilon \\
& - B_\mu{}^m\big[\partial_m - \tfrac{1}{6}\partial_m\ln\Delta\big]\epsilon \\
& - \tfrac{1}{240}\mathrm{i}\Delta^{-1/3}\,\varepsilon^{abcde}\big[\mathrm{i}\,F_{abcde} - 5F_{\beta abcd}\gamma^\beta\Gamma_e - 5\mathrm{i}\,F_{\beta\gamma abc}\gamma^{\beta\gamma}\Gamma_{de}\big]\gamma_\mu\epsilon \\
& - \tfrac{1}{96}\Delta^{-1/3}\big[ -\mathrm{i}G_{bcd}\,\Gamma^{bcd}\gamma_\mu + 3G_{bc\alpha}\,\Gamma^{bc}\big(\gamma_\mu\gamma^\alpha + 2\,\gamma^\alpha\gamma_\mu\big) \\
& \qquad + 3\mathrm{i}G_{b\alpha\beta}\,\Gamma^b\big(\gamma_\mu\gamma^{\alpha\beta} - 2\,\gamma^{\alpha\beta}\gamma_\mu\big) + G_{\alpha\beta\gamma}\big(\gamma_\mu\gamma^{\alpha\beta\gamma} + 2\gamma^{\alpha\beta\gamma}\gamma_\mu\big)\big]\,\epsilon^{\rm c} \\
& + \tfrac{1}{3}\mathrm{i}\Delta^{-1/6}\gamma_\mu\,\Gamma^a\delta\psi_a\,, \\
\delta\psi_\mu{}^{\rm c} = {}& \big[\partial_\mu - \tfrac{1}{6}\partial_\mu\ln\Delta - \Delta^{-1/3}e_\mu{}^\alpha\big(\tfrac{1}{4}\omega_\alpha{}^{\beta\gamma}\,\gamma_{\beta\gamma} + \tfrac{1}{2}\mathrm{i}\omega_\alpha{}^{\beta a}\,\Gamma_a\gamma_\beta + \tfrac{1}{4}\omega_\alpha{}^{ab}\,\Gamma_{ab} - \tfrac{1}{2}\mathrm{i}Q_\alpha\big)\big]\epsilon^{\rm c} \\
& - B_\mu{}^m\big[\partial_m - \tfrac{1}{6}\partial_m\ln\Delta\big]\epsilon^{\rm c} \\
& + \tfrac{1}{240}\mathrm{i}\Delta^{-1/3}\varepsilon^{abcde}\big[\mathrm{i}\,F_{abcde} - 5F_{\beta abcd}\gamma^\beta\Gamma_e - 5\mathrm{i}\,F_{\beta\gamma abc}\gamma^{\beta\gamma}\Gamma_{de}\big]\gamma_\mu\epsilon^{\rm c} \\
& - \tfrac{1}{96}\Delta^{-1/3}\big[ -\mathrm{i}\bar{G}_{bcd}\,\Gamma^{bcd}\gamma_\mu + 3\,\bar{G}_{bc\alpha}\,\Gamma^{bc}\big(\gamma_\mu\gamma^\alpha + 2\,\gamma^\alpha\gamma_\mu\big) \\
& \qquad + 3\mathrm{i}\bar{G}_{b\alpha\beta}\,\Gamma^b\big(\gamma_\mu\gamma^{\alpha\beta} - 2\,\gamma^{\alpha\beta}\gamma_\mu\big) + \bar{G}_{\alpha\beta\gamma}\big(\gamma_\mu\gamma^{\alpha\beta\gamma} + 2\gamma^{\alpha\beta\gamma}\gamma_\mu\big)\big]\,\epsilon \\
& + \tfrac{1}{3}\mathrm{i}\Delta^{-1/6}\gamma_\mu\,\Gamma^a\delta\psi_a{}^{\rm c}\,.
\end{aligned} \tag{2.42}
$$

where we made use of the self-duality condition on the field strength (2.15) and the gamma matrices defined in appendix A.1, and in particular of (A.17), to simplify the terms involving the various components of the field strength $F_{ABCDE}$.

The transformation rules for the gravitini that carry a vector index of the internal $5D$ space are given by

$$
\begin{aligned}
\delta\psi_a = {}& \Delta^{-1/3}e_a{}^m\big[\partial_m - \tfrac{1}{4}\omega_m{}^{\alpha\beta}\,\gamma_{\alpha\beta} - \tfrac{1}{2}\mathrm{i}\omega_m{}^{\alpha a}\,\Gamma_a\gamma_\alpha - \tfrac{1}{4}\omega_m{}^{ab}\,\Gamma_{ab} - \tfrac{1}{2}\mathrm{i}Q_m - \tfrac{1}{6}\partial_m\ln\Delta\big]\epsilon \\
& + \tfrac{1}{240}\mathrm{i}\Delta^{-1/3}\varepsilon^{bcdef}\big[F_{bcdef} + 5\mathrm{i}\,F_{\alpha bcde}\gamma^\alpha\Gamma_f - 5\,F_{\alpha\beta bcd}\gamma^{\alpha\beta}\Gamma_{ef}\big]\Gamma_a\epsilon \\
& - \tfrac{1}{96}\Delta^{-1/3}\big[G_{bcd}\,\big(\Gamma_a\Gamma^{bcd} + 2\,\Gamma^{bcd}\Gamma_a\big) - 3\mathrm{i}G_{bc\alpha}\,\gamma^\alpha\big(\Gamma_a\Gamma^{bc} - 2\,\Gamma^{bc}\Gamma_a\big) \\
& \qquad + 3G_{b\alpha\beta}\,\gamma^{\alpha\beta}\big(\Gamma_a\Gamma^b + 2\,\Gamma^b\Gamma_a\big) + \mathrm{i}G_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\Gamma_a\big]\,\epsilon^{\rm c}\,,
\end{aligned}
$$

$$\begin{aligned}\delta\psi_a{}^{\mathrm{c}} = {}& \Delta^{-1/3} e_a{}^m \big[\partial_m - \tfrac{1}{4}\omega_m{}^{\alpha\beta}\,\gamma_{\alpha\beta} - \tfrac{1}{2}\mathrm{i}\omega_m{}^{\alpha a}\,\Gamma_a\gamma_\alpha - \tfrac{1}{4}\omega_m{}^{ab}\,\Gamma_{ab} + \tfrac{1}{2}\mathrm{i}Q_m - \tfrac{1}{6}\partial_m \ln\Delta\big]\epsilon^{\mathrm{c}} \\ & - \tfrac{1}{240}\mathrm{i}\Delta^{-1/3}\varepsilon^{bcdef}\big[F_{bcdef} + 5\mathrm{i}\,F_{\alpha bcde}\gamma^\alpha\Gamma_f - 5\,F_{\alpha\beta bcd}\gamma^{\alpha\beta}\Gamma^{ef}\big]\Gamma_a\epsilon^{\mathrm{c}} \\ & - \tfrac{1}{96}\Delta^{-1/3}\big[\bar{G}_{bcd}\,\big(\Gamma_a\Gamma^{bcd} + 2\,\Gamma^{bcd}\Gamma_a\big) - 3\mathrm{i}\bar{G}_{bc\alpha}\,\gamma^\alpha\big(\Gamma_a\Gamma^{bc} - 2\,\Gamma^{bc}\Gamma_a\big) \\ & \qquad + 3\bar{G}_{b\alpha\beta}\,\gamma^{\alpha\beta}\big(\Gamma_a\Gamma^b + 2\,\Gamma^b\Gamma_a\big) + \mathrm{i}\bar{G}_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\Gamma_a\big]\,\epsilon\,. \end{aligned} \tag{2.43}$$

The next topic concerns the 2-rank tensor fields $A^\alpha{}_{MN}$, which decompose into twenty scalars $A^\alpha{}_{mn}$, ten $5D$ vectors $A^\alpha{}_{\mu m}$ and two $5D$ 2-rank tensors $A^\alpha{}_{\mu\nu}$. Their consistent Kaluza-Klein definitions are as follows,

$$\begin{aligned} A^\alpha{}_{mn}{}^{\mathrm{KK}} &= A^\alpha{}_{mn}\,, \\ A^\alpha{}_{\mu m}{}^{\mathrm{KK}} &= A^\alpha{}_{\mu m} - B_\mu{}^p\,A^\alpha{}_{pm}\,, \\ A^\alpha{}_{\mu\nu}{}^{\mathrm{KK}} &= A^\alpha{}_{\mu\nu} + 2\,B_{[\mu}{}^p\,A^\alpha{}_{\nu]p} + B_\mu{}^p\,B_\nu{}^q A^\alpha{}_{pq}\,. \end{aligned} \tag{2.44}$$

Their supersymmetry variations take the form,

$$\begin{aligned} \delta A^\alpha{}_{mn} = {}& -\tfrac{1}{2}\mathrm{i}\,\phi^\alpha\big[\bar{\epsilon}^{\mathrm{c}}\,\Gamma_{mn}\lambda^{\mathrm{c}} - 4\,\bar{\epsilon}\,\Gamma_{[m}\psi_{n]}{}^{\mathrm{c}}\big] - \tfrac{1}{2}\mathrm{i}\,\varepsilon^{\alpha\beta}\phi_\beta\big[\bar{\epsilon}\,\Gamma_{mn}\lambda - 4\,\bar{\epsilon}^{\mathrm{c}}\,\Gamma_{[m}\psi_{n]}\big]\,, \\ \delta A^\alpha{}_{\mu m} = {}& -\tfrac{1}{2}\Delta^{-1/3}\phi^\alpha\big[2\mathrm{i}\,\bar{\epsilon}\,\Gamma_m\psi_\mu{}^{\mathrm{c}} - 2\,\bar{\epsilon}\,\gamma_\mu(\delta_m{}^n - \tfrac{1}{3}\Gamma_m\Gamma^n)\psi_n{}^{\mathrm{c}} + \bar{\epsilon}^{\mathrm{c}}\,\Gamma_m\gamma_\mu\lambda^{\mathrm{c}}\big] \\ & -\tfrac{1}{2}\Delta^{-1/3}\varepsilon^{\alpha\beta}\phi_\beta\big[2\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\,\Gamma_m\psi_\mu - 2\,\bar{\epsilon}^{\mathrm{c}}\gamma_\mu(\delta_m{}^n - \tfrac{1}{3}\Gamma_m\Gamma^n)\psi_n + \bar{\epsilon}\,\Gamma_m\gamma_\mu\lambda\big] \\ & -\delta B_\mu{}^p\,A^\alpha{}_{pm}\,, \\ \delta A^\alpha{}_{\mu\nu} = {}& -\tfrac{1}{2}\Delta^{-2/3}\phi^\alpha\big[-4\,\bar{\epsilon}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \tfrac{4}{3}\mathrm{i}\bar{\epsilon}\,\gamma_{\mu\nu}\Gamma^m\psi_m{}^{\mathrm{c}} + \mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big] \\ & -\tfrac{1}{2}\Delta^{-2/3}\varepsilon^{\alpha\beta}\phi_\beta\big[-4\,\bar{\epsilon}^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \tfrac{4}{3}\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^m\psi_m + \mathrm{i}\,\bar{\epsilon}\,\gamma_{\mu\nu}\lambda\big] \\ & + 2\,\delta B_{[\mu}{}^p\,A^\alpha{}_{\nu]p}\,, \end{aligned} \tag{2.45}$$

where we have suppressed the KK-label on both sides of the equations.

Subsequently we consider the 4-rank tensor $A_{MNPQ}$ which decomposes into five $5D$ scalars $A_{mnpq}$, ten $5D$ vectors $A_{\mu mnp}$, ten $5D$ 2-rank tensors $A_{\mu\nu mn}$, five $5D$ 3-rank tensors $A_{\mu\nu\rho p}$ and one $5D$ 4-rank tensor $A_{\mu\nu\rho\sigma}$. Their consistent definition is

$$\begin{aligned} A_{mnpq}{}^{\mathrm{KK}} &= A_{mnpq}\,, \\ A_{\mu mnp}{}^{\mathrm{KK}} &= A_{\mu mnp} - B_\mu{}^q\,A_{qmnp}\,, \\ A_{\mu\nu mn}{}^{\mathrm{KK}} &= A_{\mu\nu mn} + 2\,B_{[\mu}{}^q\,A_{\nu]qmn} + B_\mu{}^p\,B_\nu{}^q\,A_{pqmn}\,, \\ A_{\mu\nu\rho m}{}^{\mathrm{KK}} &= A_{\mu\nu\rho m} + 3\,B_{[\mu}{}^p\,A_{\nu\rho]mp} + 3\,B_{[\mu}{}^p\,B_\nu{}^q\,A_{\rho]mpq} - B_\mu{}^p\,B_\nu{}^q\,B_\rho{}^r\,A_{pqrm}\,, \\ A_{\mu\nu\rho\sigma}{}^{\mathrm{KK}} &= A_{\mu\nu\rho\sigma} + 4\,B_{[\mu}{}^p\,A_{\nu\rho\sigma]p} + 6\,B_{[\mu}{}^p\,B_\nu{}^q\,A_{\rho\sigma]pq} + 4\,B_{[\mu}{}^p\,B_\nu{}^q\,B_\rho{}^r\,A_{\sigma]pqr} \\ &\qquad + B_\mu{}^p\,B_\nu{}^q\,B_\rho{}^r\,B_\sigma{}^s\,A_{pqrs}\,. \end{aligned} \tag{2.46}$$

The supersymmetry variations for these fields then take the following form,

$$
\begin{aligned}
\delta A_{mnpq} =& -\tfrac{1}{2}\bar\epsilon\,\Gamma_{[mnp}\psi_{q]} + \tfrac{1}{2}\bar\epsilon^{\rm c}\Gamma_{[mnp}\psi_{q]}^{\rm c} + \tfrac{3}{8}\mathrm{i}\,\varepsilon_{\alpha\beta}A^{\alpha}{}_{[mn}\,\delta A^{\beta}{}_{pq]}\,,\\
\delta A_{\mu mnp} =& \tfrac{1}{8}\Delta^{-1/3}\big[\bar\epsilon\,\Gamma_{mnp}\psi_\mu + 3\mathrm{i}\,\bar\epsilon\gamma_\mu\Gamma_{[mn}(\delta_{p]}^q - \tfrac{1}{9}\Gamma_{p]}\Gamma^q)\psi_q\big]\\
&+ \tfrac{1}{8}\Delta^{-1/3}\big[-\bar\epsilon^{\rm c}\Gamma_{mnp}\psi_\mu^{\rm c} - 3\mathrm{i}\,\bar\epsilon^{\rm c}\gamma_\mu\Gamma_{[mn}(\delta_{p]}^q - \tfrac{1}{9}\Gamma_{p]}\Gamma^q)\psi_q^{\rm c}\big]\\
&+ \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\big[A^{\alpha}{}_{\mu[m}\,\delta A^{\beta}{}_{np]} - \delta A^{\alpha}{}_{\mu[m}\,A^{\beta}{}_{np]} - \delta B_\mu{}^q A^{\alpha}{}_{q[m}\,A^{\beta}{}_{np]}\big]\\
&- \delta B_\mu{}^q\,A_{qmnp}\,,\\
\delta A_{\mu\nu mn} =& \tfrac{1}{4}\Delta^{-2/3}\big[\mathrm{i}\bar\epsilon\,\Gamma_{mn}\gamma_{[\mu}\psi_{\nu]} - \bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[m}(\delta_{n]}^p - \tfrac{1}{3}\Gamma_{n]}\Gamma^p)\psi_p\big]\\
&+ \tfrac{1}{4}\Delta^{-2/3}\big[-\mathrm{i}\bar\epsilon^{\rm c}\,\Gamma_{mn}\,\gamma_{[\mu}\psi_{\nu]}^{\rm c} + \bar\epsilon^{\rm c}\gamma_{\mu\nu}\Gamma_{[m}(\delta_{n]}^p - \tfrac{1}{3}\Gamma_{n]}\Gamma^p)\psi_p^{\rm c}\big]\\
&+ \tfrac{1}{16}\mathrm{i}\varepsilon_{\alpha\beta}\big[A^{\alpha}{}_{\mu\nu}\,\delta A^{\beta}{}_{mn} + A^{\alpha}{}_{mn}\,\delta A^{\beta}{}_{\mu\nu} - 4\,A^{\alpha}{}_{[\mu[m}\,\delta A^{\beta}{}_{\nu]n]}\big]\\
&+ \tfrac{1}{8}\mathrm{i}\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^p\big[A^{\alpha}{}_{\nu]p}\,A^{\beta}{}_{mn} - 2\,A^{\alpha}{}_{\nu][m}\,A^{\beta}{}_{n]p}\big]\\
&+ 2\,\delta B_{[\mu}{}^p\,A_{\nu]pmn}\,,\\
\delta A_{\mu\nu\rho m} =& \tfrac{1}{8}\Delta^{-1}\big[3\,\bar\epsilon\,\Gamma_m\gamma_{[\mu\nu}\psi_{\rho]} + \mathrm{i}\bar\epsilon\,\gamma_{\mu\nu\rho}(\delta_m{}^p - \Gamma_m\Gamma^p)\psi_p\big]\\
&+ \tfrac{1}{8}\Delta^{-1}\big[-3\,\bar\epsilon^{\rm c}\,\Gamma_m\gamma_{[\mu\nu}\psi_{\rho]}^{\rm c} - \mathrm{i}\bar\epsilon^{\rm c}\gamma_{\mu\nu\rho}(\delta_m{}^p - \Gamma_m\Gamma^p)\psi_p^{\rm c}\big]\\
&+ \tfrac{3}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\big[A^{\alpha}{}_{[\mu\nu}\,\delta A^{\beta}{}_{\rho]m} - \delta A^{\alpha}{}_{[\mu\nu}\,A^{\beta}{}_{\rho]m}\big]\\
&+ \tfrac{3}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^p\,\big[A^{\alpha}{}_{\nu\rho]}\,A^{\beta}_{pm} + 2A^{\alpha}{}_{\nu m}\,A^{\beta}{}_{\rho]p}\big]\\
&+ 3\,\delta B_{[\mu}{}^p\,A_{\nu\rho]mp}\,,\\
\delta A_{\mu\nu\rho\sigma} =& \tfrac{1}{2}\Delta^{-4/3}\big[\mathrm{i}\bar\epsilon\,\gamma_{[\mu\nu\rho}\psi_{\sigma]} + \tfrac{1}{3}\bar\epsilon\,\gamma_{\mu\nu\rho\sigma}\Gamma^p\psi_p\big]\\
&+ \tfrac{1}{2}\Delta^{-4/3}\big[-\mathrm{i}\bar\epsilon^{\rm c}\gamma_{[\mu\nu\rho}\psi_{\sigma]}^{\rm c} - \tfrac{1}{3}\bar\epsilon^{\rm c}\gamma_{\mu\nu\rho\sigma}\Gamma^p\psi_p^{\rm c}\big]\\
&+ \tfrac{3}{8}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{[\mu\nu}\,\delta A^{\beta}{}_{\rho\sigma]} - \tfrac{3}{4}\mathrm{i}\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^p\,A^{\alpha}{}_{\nu\rho}\,A^{\beta}{}_{\sigma]p}\\
&+ 4\,\delta B_{[\mu}{}^p\,A_{\nu\rho\sigma]p}\,,
\end{aligned}
\tag{2.47}
$$

where again we suppressed the KK-label on both sides of these equations.

Let us review the various fields that we have obtained so far and compare them with the fields that are generically contained in maximal $5D$ supergravity. First of all we have the fünfbein field $e_\mu{}^\alpha$ and the eight independent gravitini fields consisting of the fields $(\psi_\mu, \psi_\mu{}^{\rm c})$. Furthermore there are 48 spin-1/2 fields consisting of $(\psi_a, \psi_a{}^{\rm c})$, and $(\lambda, \lambda^{\rm c})$.

Then there are 42 scalar fields, consisting of $e_m{}^a$, $\phi^\alpha$, $A^\alpha{}_{mn}$ and $A_{mnpq}$. The field $e_m{}^a$ corresponds to 15 scalars and the fields $\phi^\alpha$ to 2 scalars upon subtracting the degrees of freedom associated with tangent space transformations of the internal space and local U(1) transformations. The fields $A^\alpha{}_{mn}$ and $A_{mnpq}$ describe 20 and

5 scalars, respectively. The total number of scalars is thus equal to the dimension of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space that parametrizes the scalars in $5D$ maximal supergravity.

To appreciate the systematics of the vector and tensor fields we introduce the following (re)definitions. The 25 vector fields that we have obtained at this stage will be denoted by

$$\begin{aligned}
C_\mu{}^m &= B_\mu{}^m\,,\\
C_\mu{}^\alpha{}_m &= A^\alpha{}_{\mu m}{}^{\mathrm{KK}}\,,\\
C_{\mu\,mnp} &= A_{\mu mnp}{}^{\mathrm{KK}} - \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^\alpha{}_{\mu[m}{}^{\mathrm{KK}}\,A^\beta{}_{np]}\,,
\end{aligned} \tag{2.48}$$

where the extra term in the definition of $C_{\mu\,mnp}$ has been included such that its supersymmetry variation will not contain any vector field. This is line with the supersymmetry variations of the vectors in $5D$ maximal supergravity. Observe also that in the above result we have suppressed the KK-label for the scalar field $A^\beta{}_{np}$; henceforth we will do this consistently for both $A^\beta{}_{np}$ and $A_{mnpq}$. The fields $C_\mu{}^m$ and $C_{\mu\,mnp}$ can be combined into the 15-dimensional anti-symmetric representation of SL(6). The remaining vector fields $C_\mu{}^\alpha{}_m$ transform as five doublets under $\mathrm{SU}(1,1)\cong\mathrm{SL}(2)$. As compared to the vector fields of $5D$ maximal supergravity, we should expect six such doublets. As we will show in the next section, the extra doublet will emerge from a dual tensor field, $A^\alpha{}_{MNPQRS}$, which leads to the fields $A^\alpha{}_{\mu mnpqr}$. In view of the self-duality constraint (2.15), we do not expect any tensor fields dual to $A_{MNPQ}$.

The 25 vector fields (2.48) transform as follows,

$$\begin{aligned}
\delta C_\mu{}^m =\ & \tfrac12\Delta^{-1/3}e_a{}^m\big[\mathrm{i}\big(\bar\epsilon\,\Gamma^a\psi_\mu+\bar\epsilon^{\mathrm{c}}\,\Gamma^a\psi_\mu{}^{\mathrm{c}}\big)\\
&\qquad +\bar\epsilon\,\gamma_\mu(\delta^a{}_b+\tfrac13\Gamma^a\Gamma_b)\psi^b+\bar\epsilon^{\mathrm{c}}\,\gamma_\mu(\delta^a{}_b+\tfrac13\Gamma^a\Gamma_b)\psi^{b\mathrm{c}}\big]\,,\\
\delta C_\mu{}^\alpha{}_m =\ & -\tfrac12\Delta^{-1/3}\phi^\alpha\big[2\mathrm{i}\,\bar\epsilon\,\Gamma_m\psi_\mu{}^{\mathrm{c}}-2\,\bar\epsilon\,\gamma_\mu(\delta_m{}^n-\tfrac13\Gamma_m\Gamma^n)\psi_n{}^{\mathrm{c}}+\bar\epsilon^{\mathrm{c}}\,\Gamma_m\gamma_\mu\lambda^{\mathrm{c}}\big]\\
&-\tfrac12\Delta^{-1/3}\varepsilon^{\alpha\beta}\phi_\beta\big[2\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\Gamma_m\psi_\mu-2\,\bar\epsilon^{\mathrm{c}}\gamma_\mu(\delta_m{}^n-\tfrac13\Gamma_m\Gamma^n)\psi_n+\bar\epsilon\,\Gamma_m\gamma_\mu\lambda\big]\\
&+\tfrac12\mathrm{i}\,\Delta^{-1/3}A^\alpha{}_{mp}\big[\bar\epsilon\,\Gamma^p\psi_\mu+\bar\epsilon^{\mathrm{c}}\,\Gamma^p\psi_\mu{}^{\mathrm{c}}\big]\\
&+\tfrac12\Delta^{-1/3}A^\alpha{}_{mp}\big[\bar\epsilon\,\gamma_\mu(e_a{}^p+\tfrac13\Gamma^p\Gamma_a)\psi^a+\bar\epsilon^{\mathrm{c}}\,\gamma_\mu(e_a{}^p+\tfrac13\Gamma^p\Gamma_a)\psi^{a\mathrm{c}}\big]\,,
\end{aligned}$$

$$\begin{aligned}
\delta C_{\mu\,mnp} =\ &\\
&\tfrac18\Delta^{-1/3}\big[\bar\epsilon\,\Gamma_{mnp}\psi_\mu+3\mathrm{i}\,\bar\epsilon\gamma_\mu\Gamma_{[mn}(\delta_{p]}{}^q-\tfrac19\Gamma_{p]}\Gamma^q)\psi_q\big]\\
&+\tfrac18\Delta^{-1/3}\big[-\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnp}\psi_\mu^{\mathrm{c}}-3\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_\mu\Gamma_{[mn}(\delta_{p]}{}^q-\tfrac19\Gamma_{p]}\Gamma^q)\psi_q^{\mathrm{c}}\big]\\
&-\tfrac{3}{16}\mathrm{i}\,\Delta^{-1/3}\varepsilon_{\alpha\beta}\,A^\alpha{}_{[mn}\phi^\beta\big[2\mathrm{i}\,\bar\epsilon\,\Gamma_{p]}\psi_\mu^{\mathrm{c}}-2\,\bar\epsilon\,\gamma_\mu(\delta_{p]}{}^n-\tfrac13\Gamma_{p]}\Gamma^n)\psi_n{}^{\mathrm{c}}+\bar\epsilon^{\mathrm{c}}\,\Gamma_{p]}\gamma_\mu\lambda^{\mathrm{c}}\big]
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{3}{16}\mathrm{i}\,\Delta^{-1/3}\,A^{\alpha}{}_{[mn}\,\phi_{\alpha}\big[2\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\,\Gamma_{p]}\psi_{\mu}-2\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu}(\delta_{p]}{}^{n}-\tfrac{1}{3}\Gamma_{p]}\Gamma^{n})\psi_{n}+\bar{\epsilon}\,\Gamma_{p]}\gamma_{\mu}\lambda\big]\\
&+\tfrac{1}{2}\mathrm{i}\,\Delta^{-1/3}\big[A_{mnpq}+\tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{[mn}\,A^{\beta}{}_{p]q}\big]\big[\bar{\epsilon}\,\Gamma^{q}\psi_{\mu}+\bar{\epsilon}^{\mathrm{c}}\,\Gamma^{q}\psi_{\mu}{}^{\mathrm{c}}\big]\\
&+\tfrac{1}{2}\Delta^{-1/3}\big[A_{mnpq}+\tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{[mn}\,A^{\beta}{}_{p]q}\big]\\
&\qquad\qquad\times\big[\bar{\epsilon}\,\gamma_{\mu}(e_{b}{}^{q}+\tfrac{1}{3}\Gamma^{q}\Gamma_{b})\psi^{b}+\bar{\epsilon}^{\mathrm{c}}\,\gamma_{\mu}(e_{b}{}^{q}+\tfrac{1}{3}\Gamma^{q}\Gamma_{b})\psi^{b\mathrm{c}}\big]\,. \qquad (2.49)
\end{aligned}
$$

Furthermore we have identified 12 two-rank tensor fields, which we define by

$$
\begin{aligned}
C_{\mu\nu}{}^{\alpha}&=A^{\alpha}{}_{\mu\nu}{}^{\mathrm{KK}}-C_{[\mu}{}^{p}\,C_{\nu]}{}^{\alpha}{}_{p}\,,\\
C_{\mu\nu\,mn}&=A_{\mu\nu mn}{}^{\mathrm{KK}}-\tfrac{1}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{\mu\nu}{}^{\mathrm{KK}}\,A^{\beta}{}_{mn}-C_{[\mu}{}^{p}\,C_{\nu]pmn}\,. \qquad (2.50)
\end{aligned}
$$

The supersymmetry transformations of these tensors are expressed by

$$
\begin{aligned}
&\delta C_{\mu\nu}{}^{\alpha}+C_{[\mu}{}^{p}\,\delta C_{\nu]}{}^{\alpha}{}_{p}+C_{[\mu}{}^{\alpha}{}_{p}\,\delta C_{\nu]}{}^{p}\\
&=-\tfrac{1}{2}\Delta^{-2/3}\phi^{\alpha}\big[-4\,\bar{\epsilon}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\tfrac{4}{3}\mathrm{i}\bar{\epsilon}\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}}+\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big]\\
&\quad-\tfrac{1}{2}\Delta^{-2/3}\varepsilon^{\alpha\beta}\phi_{\beta}\big[-4\,\bar{\epsilon}^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]}+\tfrac{4}{3}\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m}+\mathrm{i}\,\bar{\epsilon}\,\gamma_{\mu\nu}\lambda\big]\,,\\
&\delta C_{\mu\nu\,mn}+C_{[\mu}{}^{p}\,\delta C_{\nu]pmn}+C_{[\mu\,pmn}\,\delta C_{\nu]}{}^{p}+\tfrac{1}{4}\mathrm{i}\varepsilon_{\alpha\beta}\,C_{[\mu}{}^{\alpha}{}_{[m}\,\delta C_{\nu]}{}^{\beta}{}_{n]}\\
&=\tfrac{1}{4}\Delta^{-2/3}\big[\mathrm{i}\bar{\epsilon}\,\Gamma_{mn}\gamma_{[\mu}\psi_{\nu]}-\bar{\epsilon}\,\gamma_{\mu\nu}\Gamma_{[m}(\delta^{p}_{n]}-\tfrac{1}{3}\Gamma_{n]}\Gamma^{p})\psi_{p}\big]\\
&\quad+\tfrac{1}{4}\Delta^{-2/3}\big[-\mathrm{i}\bar{\epsilon}^{\mathrm{c}}\,\Gamma_{mn}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\Gamma_{[m}(\delta_{n]}{}^{p}-\tfrac{1}{3}\Gamma_{n]}\Gamma^{p})\psi_{p}{}^{\mathrm{c}}\big]\\
&\quad-\tfrac{1}{16}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{mn}\,\phi^{\beta}\big[-4\,\bar{\epsilon}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\tfrac{4}{3}\mathrm{i}\bar{\epsilon}\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}}+\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big]\\
&\quad+\tfrac{1}{16}\mathrm{i}\,\Delta^{-2/3}\,A^{\alpha}{}_{mn}\,\phi_{\alpha}\big[-4\,\bar{\epsilon}^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]}+\tfrac{4}{3}\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m}+\mathrm{i}\,\bar{\epsilon}\,\gamma_{\mu\nu}\lambda\big]\,. \qquad (2.51)
\end{aligned}
$$

These transformation rules are in line with what is known from the vector-tensor hierarchy that appears in the context of the embedding tensor formalism [42, 56]. We have actually verified that also the variation of the 3-rank $5D$ tensor fields, $A_{\mu\nu\rho m}{}^{\mathrm{KK}}$ listed in (2.46) will exhibit the same structure upon introducing a suitable modification. Since we will not be considering tensors of rank higer than two, we refrain from giving further details.

At this point the number of 2-rank $5D$ tensor fields is less than the 27 fields that one expects on the basis of $5D$ maximal supergravity in the context of the embedding tensor formalism. Ten extra tensors $A^{\alpha}{}_{\mu\nu mnpq}$ will be provided by the dual field, $A^{\alpha}{}_{MNPQRS}$, which will bring the total of tensors to 22. The dual vectors and tensors are evaluated in the next section.

## 2.4 Dual fields and the vector-tensor hierarchy

In (2.28) we presented the field equation for the tensor fields $A^{\alpha}{}_{MN}$ written as a Bianchi identity of the 7-rank field strength $F_{\alpha\,MNPQRST}$ defined in (2.29). The field equation thus implies that this field strength can be written in terms of a dual six-form field $A_{\alpha\,MNPQRS}$ according to

$$F_{\alpha\,MNPQRST} = 6\,\partial_{[M}A_{\alpha\,NPQRST]}\,. \tag{2.52}$$

It is not possible to derive an expression for $A_{\alpha\,MNPQRS}$ in closed form, but it is possible to determine how this field transforms under supersymmetry. Obviously, the Bianchi identity (2.28) should transform under supersymmetry into fermionic equations which are of at most first order in derivatives. Therefore one expects that $F_{\alpha\,MNPQRST}$ transforms into fermionic field equations and into terms that carry explicit space-time derivatives such that they can be identified as the result of the supersymmetry variation of the dual six-form. Because the field equations are supercovariant all the contributions of the variation of the six-form can be identified from the terms that are proportional to the derivative of the supersymmetry parameters. The consistency of this approach can easily be verified and it leads to the following result,

$$\begin{aligned}\delta A_{\alpha\,MNPQRS} =&\,\varepsilon_{\alpha\beta}\phi^{\beta}\big(\tfrac{1}{6}\bar{\lambda}\,\breve{\Gamma}_{MNPQRS}\epsilon + 2\,\bar{\epsilon}\,\breve{\Gamma}_{[MNPQR}\psi_{S]}{}^{\mathrm{c}}\big)\\ &-\phi_{\alpha}\big(\tfrac{1}{6}\bar{\epsilon}\,\breve{\Gamma}_{MNPQRS}\lambda - 2\,\bar{\psi}_{[M}{}^{\mathrm{c}}\,\breve{\Gamma}_{NPQRS]}\epsilon\big)\\ &-20\mathrm{i}\,\varepsilon_{\alpha\beta}A^{\beta}{}_{[MN}\,\big(\delta A_{PQRS]} - \tfrac{1}{8}\mathrm{i}\,\varepsilon_{\gamma\delta}A^{\gamma}{}_{PQ}\,\delta A^{\delta}{}_{RS]}\big)\,.\end{aligned} \tag{2.53}$$

In particular we note the dual fields $A^{\alpha}{}_{\mu mnpqr}$ and $A^{\alpha}{}_{\mu\nu mnpq}$, which constitute two $5D$ vector fields and twelve $5D$ tensor fields transforming under $\mathrm{SU}(1,1)$. We first consider the transformation rule of the vector field $A^{\alpha}{}_{\mu mnpqr}$, which takes the following form,

$$\begin{aligned}&\delta A_{\alpha\,\mu mnpqr} =\\ &\quad-\tfrac{1}{3}\mathrm{i}\,\Delta^{-1/3}\varepsilon_{\alpha\beta}\phi^{\beta}\big[\bar{\epsilon}\,\Gamma_{mnpqr}\psi_{\mu}{}^{\mathrm{c}} + 5\mathrm{i}\,\bar{\epsilon}\,\gamma_{\mu}(\Gamma_{[mnpq}\delta_{r]}{}^{s} - \tfrac{1}{15}\Gamma_{r]}\Gamma^{s})\psi_{s}^{\mathrm{c}} + \tfrac{1}{2}\mathrm{i}\bar{\epsilon}^{\mathrm{c}}\,\gamma_{\mu}\Gamma_{mnpqr}\lambda^{\mathrm{c}}\big]\\ &\quad-\tfrac{1}{3}\mathrm{i}\,\Delta^{-1/3}\phi_{\alpha}\big[\bar{\epsilon}^{\mathrm{c}}\,\Gamma_{mnpqr}\psi_{\mu} + 5\mathrm{i}\,\bar{\epsilon}^{\mathrm{c}}\,\gamma_{\mu}\Gamma_{[mnpq}(\delta_{r]}^{s} - \tfrac{1}{15}\Gamma_{r]}\Gamma^{s})\psi_{s} + \tfrac{1}{2}\mathrm{i}\bar{\epsilon}\,\gamma_{\mu}\Gamma_{mnpqr}\lambda\big]\\ &\quad-\tfrac{20}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\big(A^{\beta}{}_{\mu[m}\,\delta A_{npqr]} - 2\,A^{\beta}{}_{[mn}\,\delta A_{pqr]\mu}\big)\\ &\quad-\tfrac{5}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big(2\,A^{(\beta}{}_{\mu[m}A^{\gamma)}{}_{np}\,\delta A^{\delta}{}_{qr]} - A^{\beta}{}_{[mn}A^{\gamma}{}_{pq}\,\delta A^{\delta}{}_{r]\mu}\big)\\ &\quad+\tfrac{40}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\;\delta B_{\mu}{}^{s}A^{\beta}{}_{[mn}\big(A_{pqr]s} - \tfrac{1}{16}\mathrm{i}\varepsilon_{\gamma\delta}A^{\gamma}_{pq}\,A^{\delta}_{r]s}\big)\,,\end{aligned} \tag{2.54}$$

where on the right-hand side all the fields have been subject to Kaluza-Klein redefinitions. The field $A_{\alpha\,\mu mnpqr}$ already transforms consistently as a vector in the $5D$ spacetime because tensors antisymmetric in more than five internal-space indices must vanish. The consistency of the above result is confirmed by the fact that no terms are generated proportional to the Kaluza-Klein vector field $B_\mu{}^m$, simply because the corresponding terms are fully anti-symmetric in six internal-space indices and therefore vanish.

However, from the perspective of the vector-tensor hierarchy further redefinitions are required, as the supersymmetry variations should not contain any vector fields, but at most variations of vector fields. A preliminary analysis suggests to add modifications that are quadratic and cubic terms in the four- and two-form fields but here we have to make sure that also the modification itself transforms consistently as a vector in the $5D$ space-time. This leads us to the following redefinition,

$$C_{\mu\,\alpha mnpqr} = A_{\alpha\,\mu mnpqr} + \tfrac{20}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,C_\mu{}^\beta{}_{[m}\,A_{npqr]} - \tfrac{5}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,A^\beta{}_{[mn}\,C_\mu{}^\gamma{}_p\,A^\delta{}_{qr]}\,, \qquad (2.55)$$

where $C_\mu{}^\alpha{}_m$ is a proper vector field defined in (2.48). Under supersymmetry the field $C_{\mu\,\alpha mnpqr}$ transforms in the required way,

$$\begin{aligned}
\delta C_{\mu\,\alpha mnpqr} = \\
&- \tfrac{1}{3}\mathrm{i}\,\Delta^{-1/3}\varepsilon_{\alpha\beta}\phi^\beta\big[\bar\epsilon\,\Gamma_{mnpqr}\psi_\mu{}^{\mathrm{c}} + 5\mathrm{i}\,\bar\epsilon\,\gamma_\mu(\Gamma_{[mnpq}\delta_{r]}{}^s - \tfrac{1}{15}\Gamma_{r]}\Gamma^s)\psi_s{}^{\mathrm{c}} + \tfrac{1}{2}\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\gamma_\mu\Gamma_{mnpqr}\lambda^{\mathrm{c}}\big] \\
&- \tfrac{1}{3}\mathrm{i}\,\Delta^{-1/3}\phi_\alpha\big[\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnpqr}\psi_\mu + 5\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\gamma_\mu\Gamma_{[mnpq}(\delta_{r]}^s - \tfrac{1}{15}\Gamma_{r]}\Gamma^s)\psi_s + \tfrac{1}{2}\mathrm{i}\bar\epsilon\,\gamma_\mu\Gamma_{mnpqr}\lambda\big] \\
&+ \tfrac{20}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\big[\delta C_\mu{}^\beta{}_{[m}\,A_{npqr]} - 2\,\delta C_{\mu[mnp}\,A^\beta{}_{qr]} - 2\;\delta C_\mu{}^s A_{s[mnp}\,A^\beta{}_{qr]}\big] \\
&+ \tfrac{5}{2}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big[\delta C_\mu{}^\gamma{}_{[m}\,A^\delta{}_{np}\,A^\beta{}_{qr]} + \tfrac{1}{3}\delta C_\mu{}^s\,A^\gamma{}_{s[m}\,A^\beta{}_{np}\,A^\delta{}_{qr]}\big]\,,
\end{aligned} \qquad (2.56)$$

where, for conciseness, we refrained from substituting the explicit expressions for $C_\mu{}^m$, $\delta C^{\;\alpha}_{\mu\;m}$ and $\delta C_{\mu\,mnp}$ in the right-hand of the last equation.

Subsequently we consider the tensor field $A_{\alpha\,\mu\nu mnpq}$. To ensure that this field transforms as a proper $5D$ tensor one performs the standard Kaluza-Klein redefinition,

$$A_{\alpha\,\mu\nu mnpq}{}^{\mathrm{KK}} = A_{\alpha\,\mu\nu mnpq} + 2B_{[\mu}{}^r\,A_{\alpha\nu]mnpqr}\,. \qquad (2.57)$$

This modified tensor field transforms as

$$\begin{aligned}
\delta A_{\alpha\,\mu\nu mnpq} = \\
&- \tfrac{2}{3}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\phi^\beta\big[\mathrm{i}\bar\epsilon\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[mnp}(\delta_{q]}^r - \tfrac{1}{6}\Gamma_{q]}\Gamma^r)\psi_r{}^{\mathrm{c}} - \tfrac{1}{4}\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda^{\mathrm{c}}\big] \\
&- \tfrac{2}{3}\mathrm{i}\,\Delta^{-2/3}\phi_\alpha\big[\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]} - 2\,\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{[mnp}(\delta_{q]}^r - \tfrac{1}{6}\Gamma_{q]}\Gamma^r)\psi_r - \tfrac{1}{4}\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda\big]
\end{aligned}$$

$$
\begin{aligned}
&- \tfrac{4}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\big[A^{\beta}{}_{\mu\nu}\,\delta A_{mnpq} - 8\,A^{\beta}{}_{[\mu[m}\,\delta A_{\nu]npq]} + 6\,A^{\beta}{}_{[mn}\,\delta A_{pq]\mu\nu}\big] \\
&- \tfrac{1}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big(2\,A^{(\beta}{}_{\mu\nu}\,A^{\gamma)}{}_{[mn}\,\delta A^{\delta}{}_{pq]} + A^{\beta}{}_{[mn}\,A^{\gamma}{}_{pq]}\,\delta A^{\delta}{}_{\mu\nu}\big) \\
&+ \tfrac{2}{3}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big(A^{\beta}{}_{[\mu[m}\,A^{\gamma}{}_{\nu]n}\,\delta A^{\delta}{}_{pq]} + 2\,A^{(\beta}{}_{[\mu[m}\,A^{\gamma)}{}_{np}\,\delta A^{\delta}{}_{\nu]q]}\big) \\
&+ \tfrac{16}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^{r}\,\big(2A^{\beta}{}_{\nu][m}\,A_{npq]r} + 3\,A_{\nu]r[mn}\,A^{\beta}{}_{pq]}\big) \\
&+ \tfrac{1}{3}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,\delta B_{[\mu}{}^{r}\,\big(4\,A^{(\beta}{}_{\nu][m}A^{\gamma)}{}_{np}\,A^{\delta}{}_{q]r} + A^{\delta}{}_{\nu]r}A^{\beta}{}_{[mn}\,A^{\gamma}{}_{pq]}\big) \\
&+ 2\,\delta B_{[\mu}{}^{r}\,A_{\alpha\nu]mnpqr}\,,
\end{aligned}
\tag{2.58}
$$

where we again dropped KK-label on both sides of the equation.

Again this result is not consistent with regard to the vector-tensor hierarchy so that further redefinitions of the tensor field are required. As it turns out, they take the following form,

$$
\begin{aligned}
C_{\mu\nu\,\alpha mnpq} = {}& A_{\mu\nu\,\alpha mnpq} + \tfrac{4}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{\mu\nu}\,A_{mnpq} \\
&- \tfrac{1}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,\big[A^{\gamma}{}_{\mu\nu}\,A^{\beta}{}_{[mn}\,A^{\delta}{}_{pq]} - 8\,C_{[\mu}{}^{\beta}{}_{[m}\,C_{\nu]}{}^{\gamma}{}_{n}\,A^{\delta}{}_{pq]}\big] \\
&- \tfrac{16}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,C_{[\mu}{}^{\beta}{}_{[m}\,C_{\nu]npq]} - C_{[\mu}{}^{r}C_{\nu]\,\alpha mnpqr}\,,
\end{aligned}
\tag{2.59}
$$

where on the the right-hand side the KK-labels have again been suppressed. The transformation rule of $C_{\mu\nu\,\alpha mnpq}$ takes the form

$$
\begin{aligned}
&\delta C_{\mu\nu\,\alpha mnpq} = \\
&- \tfrac{2}{3}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\phi^{\beta}\Big[\big[\mathrm{i}\bar\epsilon\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[mnp}(\delta_{q]}{}^{r} - \tfrac{1}{6}\Gamma_{q]}\Gamma^{r})\psi_{r}{}^{\mathrm{c}} - \tfrac{1}{4}\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda^{\mathrm{c}}\big] \\
&\qquad + A_{mnpq}\big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \tfrac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}} + \mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big]\Big] \\
&- \tfrac{2}{3}\mathrm{i}\,\Delta^{-2/3}\phi_{\alpha}\Big[\big[\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]} - 2\,\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{[mnp}(\delta_{q]}{}^{r} - \tfrac{1}{6}\Gamma_{q]}\Gamma^{r})\psi_{r} - \tfrac{1}{4}\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda\big] \\
&\qquad - A_{mnpq}\big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \tfrac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m} + \mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\big]\Big] \\
&- 2\mathrm{i}\,\Delta^{-2/3}\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{[mn}\Big[\big[\mathrm{i}\bar\epsilon\,\Gamma_{pq]}\gamma_{[\mu}\psi_{\nu]} - \bar\epsilon\,\gamma_{\mu\nu}\Gamma_{p}(\delta_{q]}{}^{r} - \tfrac{1}{3}\Gamma_{q]}\Gamma^{r})\psi_{r}\big] \\
&\qquad - \big[\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{pq]}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} - \bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma_{p}(\delta_{q]}{}^{r} - \tfrac{1}{3}\Gamma_{q]}\Gamma^{r})\psi_{r}{}^{\mathrm{c}}\big]\Big] \\
&- \mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\,A^{\beta}{}_{[mn}\Big[\varepsilon_{\gamma\delta}\,A^{\gamma}{}_{pq]}\,\phi^{\delta}\big[\mathrm{i}\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \tfrac{1}{3}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{r}\psi_{r}{}^{\mathrm{c}} + \tfrac{1}{4}\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\big] \\
&\qquad - A^{\gamma}{}_{pq]}\,\phi_{\gamma}\big[\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \tfrac{1}{3}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{r}\psi_{r} + \tfrac{1}{4}\bar\epsilon\,\gamma_{\mu\nu}\lambda\big]\Big] \\
&+ \tfrac{16}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\big[C_{[\mu}{}^{\beta}{}_{[m}\,\delta C_{\nu]npq]} + C_{[\mu[npq}\,\delta C_{\nu]}{}^{\beta}{}_{m]}\big] - C_{[\mu}{}^{r}\,\delta C_{\nu]\,\alpha mnpqr} - C_{[\mu\,\alpha mnpqr}\,\delta C_{\nu]}{}^{r}\,.
\end{aligned}
\tag{2.60}
$$

To conclude this section let us summarize the situation regarding the vector and tensor fields. We have identified precisely 27 vector fields, namely,

$$C_\mu{}^M = \big\{ C_\mu^{\;m} \,, \, C_{\mu\, mnp} \,, \, C_{\mu\; m}^{\;\alpha} \,, \, C_{\mu\, \alpha mnpqr} \big\} \,. \tag{2.61}$$

For the tensor fields the situation is somewhat different. First of all, we expect 27 tensor fields whereas previously we found only 22 fields. Secondly, we note that the tensor fields, which we will denote by $C_{\mu\nu\, Q}$, carry different indices. The vector-tensor hierarchy implies that there must be 5 additional tensor fields and furthermore requires the existence of a constant tensor $d_{Q,MN}$, symmetric in $(M,N)$, in order to obtain the characteristic term $d_{Q,MN}\, C_{[\mu}{}^M\, \delta C_{\nu]}{}^N$ in $\delta C_{\mu\nu\, Q}$. Assuming that the overall covariance of this expression must be preserved and that precisely five additional fields are needed, one deduces that these five fields can be precisely represented by new fields $C_{\mu\nu\, m;npqrs}$, where the array $[npqrs]$ is fully antisymmetric. Hence the decomposition of the 27 tensors takes the following form, in direct analogy with (2.61),

$$C_{\mu\nu\, Q} = \big\{ C_{\mu\nu\, m;npqrs} \,, \, C_{\mu\nu\, mn} \,, \, C_{\mu\nu\, \alpha mnpq} \,, \, C_{\mu\nu}{}^{\alpha} \big\} \,. \tag{2.62}$$

The new field $C_{\mu\nu\, m;npqrs}$ indeed has the representation that is expected from the dualization of $10D$ gravity [105, 106] (although this dualization can not be fully understood at the non-linear level in $10D$ [107]).

The systematics of the vector and tensor fields can be improved upon converting to dual representations by extracting the anti-symmetric tensors $\mathring{e}\, \varepsilon_{mnpqr}$ and/or $\varepsilon_{\alpha\beta}$. Note that the first tensor depends only on the reference background of the internal space, because of the definition $\mathring{e}(y) \equiv \det[\, \mathring{e}_m{}^a(y)]$, and not on the space-time coordinates $x^\mu$. Hence these conversions have no bearing on the supersymmetry transformations nor the vector and tensor gauge transformations. Now consider the following redefinitions for the vector fields,

$$\begin{aligned} C_\mu^{\;m} &= C_\mu^{\;m} \,, & C_{\mu\, mnp} &= \tfrac{1}{128}\sqrt{5}\, \mathring{e}\, \varepsilon_{mnpqr}\, C_\mu^{\;qr} \,, \\ C_{\mu\; m}^{\;\alpha} &= \mathrm{i}\, \varepsilon^{\alpha\beta}\, C_{\mu\, \beta m} \,, & C_{\mu\, \alpha mnpqr} &= -\tfrac{1}{6}\sqrt{5} \mathring{e}\, \varepsilon_{mnpqr}\, C_{\mu\, \alpha} \,. \end{aligned} \tag{2.63}$$

For the tensor fields the corresponding redefinitions are

$$\begin{aligned} C_{\mu\nu\, m;npqrs} &\propto \mathring{e}\, \varepsilon_{npqrs}\, C_{\mu\nu\, m} \,, & C_{\mu\nu\, mn} &= C_{\mu\nu\, mn} \,, \\ C_{\mu\nu\, \alpha mnpq} &= \tfrac{1}{6}\sqrt{5}\mathrm{i} \mathring{e}\, \varepsilon_{mnpqr}\, \varepsilon_{\alpha\beta}\, C_{\mu\nu}{}^{\beta r} \,, & C_{\mu\nu}{}^{\alpha} &= C_{\mu\nu}{}^{\alpha} \,. \end{aligned} \tag{2.64}$$

Now the vector and tensor fields can be written as $C_\mu{}^M$ and $C_{\mu\nu\,M}$, respectively, where the indices $M$ decompose according to ${}^M = \{{}^m\,,\,{}^{mn}\,,\,{}_{\alpha m}\,,\,{}_\alpha\}$ and ${}_M = \{{}_m\,,\,{}_{mn}\,,\,{}^{\alpha m}\,,\,{}^\alpha\}$, respectively. Here we observe that the normalization of the vector and tensor fields is at this point completely arbitrary. Nevertheless, by identifying the (upper) index $M$ on $C_\mu{}^M$ with the $\overline{\mathbf{27}}$ representation of $\mathrm{E}_{6(6)}$ and the (lower) index $M$ on $C_{\mu\nu M}$ as the $\mathbf{27}$ representation, the decompositions (2.63) and (2.64) correspond to the branchings

$$\begin{aligned}
\overline{\mathbf{27}} &\stackrel{\mathrm{SL}(2)\times\mathrm{SL}(6)}{\longrightarrow} (\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{6}) \stackrel{\mathrm{SL}(2)\times\mathrm{SO}(5)}{\longrightarrow} (\mathbf{1},\mathbf{5}) + (\mathbf{1},\mathbf{10}) + (\mathbf{2},\mathbf{5}) + (\mathbf{2},\mathbf{1})\,,\\
\mathbf{27} &\stackrel{\mathrm{SL}(2)\times\mathrm{SL}(6)}{\longrightarrow} (\mathbf{1},\mathbf{15}) + (\mathbf{2},\overline{\mathbf{6}}) \stackrel{\mathrm{SL}(2)\times\mathrm{SO}(5)}{\longrightarrow} (\mathbf{1},\mathbf{5}) + (\mathbf{1},\mathbf{10}) + (\mathbf{2},\mathbf{5}) + (\mathbf{2},\mathbf{1})\,.
\end{aligned}$$

At this point it makes sense to compare our results for variations of the tensor fields to the corresponding expressions known from maximal $5D$ supergravity [42]. In the latter case these variations are encoded in the symmetric three-rank $\mathrm{E}_{6(6)}$ invariant tensor $d_{MNP}$,

$$\delta C_{\mu\nu\,M} - 2\,d_{MNP}\,C_{[\mu}{}^N\,\delta C_{\nu]}{}^P\,. \tag{2.65}$$

Expressions such as these are characteristic for the vector-tensor hierarchy. Obviously the tensor $d_{MNP}$ decomposes into three $\mathrm{SL}(2)\times\mathrm{SO}(5)$ invariant components,

$$d_{MNP} \propto \begin{cases} d({}_{mn}|{}^{\alpha p}|{}^{\beta q}) &= \delta_{mn}{}^{pq}\,\varepsilon^{\alpha\beta}\,,\\[1ex] d({}_{mn}|{}_{pq}|{}_r) &= \mathring{e}\,\varepsilon_{mnpqr}\,,\\[1ex] d({}_m|{}^{\alpha n}|{}^\beta) &= \delta_m{}^n\,\varepsilon^{\alpha\beta}\,, \end{cases} \tag{2.66}$$

where normalization factors are not specified because they can be changed by rescaling the normalization of the vector and tensor fields. Nevertheless the fact that a single symmetric tensor $d_{MNP}$ must encode the variations above for all the fields does pose certain restrictions on the relative normalizations of vectors and tensor fields, especially because the product of the normalization of a tensor and its corresponding dual vector is constrained, just as in the maximal $5D$ theory [42]. We return to this issue in the next section, but note that this normalization condition has been incorporated when adopting the rescalings of the vector and tensor fields in (2.63) and (2.64), repectively. It then turns out that the expressions for the independent components of the combined variations (2.65) must be equivalent to

$$\begin{aligned}
&\delta C_{\mu\nu}{}^{\alpha m} - \tfrac{1}{8}\mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu\,\beta n}\,\delta C_{\nu]}{}^{mn} + C_{[\mu}{}^{mn}\,\delta C_{\nu]\,\beta n}\big] - \mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu}{}^{m}\,\delta C_{\nu]\beta} + C_{[\mu\beta}\,\delta C_{\nu]}{}^{m}\big]\,,\\
&\delta C_{\mu\nu}{}^{\alpha} + \mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu}{}^{m}\,\delta C_{\nu]\,\beta m} + C_{[\mu\,\beta m}\,\delta C_{\nu]}{}^{m}\big]\,,\\
&\delta C_{\mu\nu\,mn} + \tfrac{1}{128}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\big[C_{[\mu}{}^{p}\,\delta C_{\nu]}{}^{qr} + C_{[\mu}{}^{qr}\,\delta C_{\nu]}{}^{p}\big] - \tfrac{1}{4}\mathrm{i}\,\varepsilon^{\alpha\beta}\,C_{[\mu\alpha[m}\,\delta C_{\nu]\beta n]}\,,\\
&\delta C_{\mu\nu\,m} - \mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu\,\alpha m}\,\delta C_{\nu]\,\beta} - C_{[\mu\,\alpha}\,\delta C_{\nu]\,\beta m}\big] + \tfrac{1}{256}\sqrt{5}\mathring{e}\,\varepsilon_{mnpqr}\,C_{[\mu}{}^{np}\,\delta C_{\nu]}{}^{qr}\,, \qquad (2.67)
\end{aligned}$$

where the last line is not derived directly from the $10D$ supergravity as the tensor field $C_{\mu\nu\,m}$ is associated with the elusive dual graviton. Nevertheless it is remarkable that one can also derive the coefficients in the variation of $C_{\mu\nu\,m}$ by comparing to the $5D$ vector-tensor hierarchy.

## 2.5 Generalized vielbeine and USp(8) covariant spinors

The spinor fields $\psi_\mu$, $\psi_\mu{}^{\mathrm{c}}$, $\psi_a$, $\psi_a^{\mathrm{c}}$, $\lambda$ and $\lambda^{\mathrm{c}}$, which were defined in section 2.3, obviously transform under the $\mathrm{Spin}(4,1)\times\mathrm{USp}(4)$ subgroup of the $10D$ tangent space group $\mathrm{Spin}(9,1)$. Hence every $10D$ spinor consists of four complex $\mathrm{Spin}(4,1)$ spinors which rotate among each other under $\mathrm{USp}(4)$ transformations. In the following we will not consider the $\mathrm{Spin}(4,1)$ aspects but concentrate on the extension of the $\mathrm{USp}(4)$ transformations to the full automorphism group of the $5D$ space-time Clifford algebra. This R-symmetry group contains also the $\mathrm{U}(1)$ group of IIB supergravity (which can be regarded as the $10D$ R-symmetry group) and it can be further extended by realizing that the spinors can actually transform under $\mathrm{SU}(4)\cong\mathrm{SO}(6)$ (for instance, by regarding them as chiral spinors of $\mathrm{SO}(6)$). It is then convenient to introduce corresponding $\mathrm{SO}(6)$ gamma matrices as well, which requires to combine the spinors with their charge conjugates, i.e. $\psi_\mu$ with $\psi_\mu{}^{\mathrm{c}}$, and likewise, $\psi_a$ with $\psi_a^{\mathrm{c}}$, and $\lambda$ with $\lambda^{\mathrm{c}}$. This is described in detail in appendix A.2. The $\mathrm{SO}(6)$ gamma matrices will be denoted by $\mathbf{\Gamma}_{\hat a}$, with $\hat a = 1,\ldots,6$, and act on the eight-component pseudo-real spinors. We may then introduce the chirality operator $\mathbf{\Gamma}_7 \equiv \mathrm{i}\mathbf{\Gamma}_1\mathbf{\Gamma}_2\cdots\mathbf{\Gamma}_6$, which decomposes as $\mathbf{\Gamma}_7 = \mathbb{1}_4\otimes\sigma_3$, so that the $\mathrm{SO}(6)$ chirality of the charge conjugate fermions is opposite to the original ones. Here we are using a basis where the positive-chirality (negative-chirality) components carry positive (negative) $\mathrm{U}(1)$ charge. In this section and henceforth we will be using these 8-component spinor arrays whenever possible (labeled by indices $A = 1, \ldots, 8$) and they will simply be denoted by ${\psi_\mu}^A$, ${\psi_a}^A$ and $\lambda^A$. Each of these spinors are then $5D$ symplectic Majorana spinors, i.e.,

$$C^{-1}{\bar{\psi}_A}{}^{\mathrm{T}} = \Omega_{AB}\,\psi^B\,, \tag{2.68}$$

where $C$ is the charge conjugation matrix in five space-time dimensions and $\Omega$ is the anti-symmetric USp(8) invariant tensor.

The appearance of $\Omega$ indicates that the full R-symmetry group is equal to USp(8), as is to be expected for $5D$ spinors. Indeed, the gravitini ${\psi_\mu}^A$ transform consistently in the **8** representation of this extended R-symmetry group. However, the fields $\psi_a$ and $\lambda$ cannot possibly transform in the **8** representation, in view of the fact that the U(1) charges of the fields ${\psi_a}^A$ and $\lambda^A$ are equal to $\pm 1/2$ and $\pm 3/2$, respectively. Therefore those fields must transform in a different representation of the USp(8) group. In view of the values for the U(1) charges and the fact that ${\psi_a}^A$ and $\lambda^A$ define precisely 48 $5D$ symplectic Majorana spinors, these fields must combine into the **48** representation of the group USp(8). At this point we should recall that only the $\mathrm{USp}(4)\times\mathrm{U}(1)$ subgroup is realized as a local gauge invariance, as they originate from the symmetries of $10D$ IIB supergravity that were already realized as local ones. As we have stressed in the introduction, the full USp(8) R-symmetry group can be realized locally upon introducing a compensating phase factor belonging to $\mathrm{USp}(8)/[\mathrm{USp}(4)\times\mathrm{U}(1)]$. We will postpone the introduction of this phase factor till later, so that the present calculations will describe the results subject to a gauge condition that sets the compensating phase factor equal to unity. However, it is important to realize that the local transformations depend on both sets of coordinates, $x^\mu$ and $y^m$. This is the reason why we adopted the indices $A, B, \ldots$ for the spinors in this case, while in the maximal $5D$ supergravity, the spinors will carry indices $i, j, \ldots$ with local R-symmetry transformations that depend only on the space-time coordinates $x^\mu$. This issue will be important in section 2.6, when considering a truncation of $10D$ supergravity to $5D$,

In the previous section we have identified 27 vector fields ${C_\mu}^M$ as listed in (2.63), which transform under supersymmetry into the symplectic Majorana spinors ${\psi_\mu}^A$, ${\psi_a}^A$ and $\lambda^A$. As it turns out the supersymmetry variations of these fields can be written in the same way as the variations of the vector fields in $5D$ maximal supergravity [42],

$$\delta {C_\mu}^M = 2\left[\mathrm{i}\bar{\Omega}^{AC}\,\bar{\epsilon}_C\,{\psi_\mu}^B + \bar{\epsilon}_C\,\gamma_\mu\chi^{ABC}\right]{\mathcal{V}_{AB}}^M\,, \tag{2.69}$$

except that, as explained above, we changed the USp(8) indices from $i, j, \ldots$ to $A, B, \ldots$. Here $\Omega^{AB}$ is the symplectic USp(8) invariant tensor introduced aboved and the $\mathcal{V}_{AB}{}^{M}$ depend on the 42 scalar fields. All these fields depend on coordinates $x^\mu$ and $y^m$. In the pure $5D$ theory the corresponding quantities $\mathcal{V}_{ij}{}^{M}$ are defined in terms of the $E_{6(6)}/\mathrm{USp}(8)$ coset representative. The transformations (2.69) are consistent with the USp(8) R-symmetry group and the anti-symmetric traceless spinors $\chi^{ABC}$ are symplectic Majorana spinors, satisfying

$$C^{-1}\bar{\chi}_{ABC}{}^{\mathrm{T}} = \Omega_{AD}\,\Omega_{BE}\,\Omega_{CF}\,\chi^{DEF}\,, \tag{2.70}$$

in direct correspondence with the $5D$ theory [42]. Because of the anti-symmetry in $[ABC]$ and the condition $\Omega_{AB}\,\chi^{ABC} = 0$, this representation is irreducible. Hence the spinor $\chi^{ABC}$ should be linearly related to the spinors $\psi_a{}^A$ and $\lambda^A$. Indeed, as we demonstrate in appendix A.2 (c.f. (A.26)) the branching of the **8** and **48** USp(8) representations of the fermions with respect to the $\mathrm{SU}(4)\times\mathrm{U}(1)$ subgroup accounts precisely for the fermion fields $\psi_\mu{}^A$, $\psi_a{}^A$ and $\lambda^A$ including their U(1) charge assignments.

The supersymmetry transformation rules for the vector fields $C_\mu{}^M$ in terms of the spinors $\psi_\mu{}^A$, $\psi_a{}^A$, $\lambda^A$ based on IIB supergravity follow from (2.49) and (2.56) upon taking into account the redefinitions (2.63). By comparing these expressions to (2.69) we obtain explicit representations of the so-called *generalized vielbeine* $\mathcal{V}_{AB}{}^M$, which depend on all $10D$ coordinates. Furthermore we can deduce the explicit relation between the USp(8) covariant spinor field $\chi^{ABC}$ and the fields $\psi_a{}^A$ and $\lambda^A$. In the same fashion one can evaluate the supersymmetry transformations of the tensor fields, a topic that will be dealt with at the end of this section.

Matrices in spinor space can be decomposed into direct products of the $5D$ gamma matrices $\gamma^\mu$ and the SO(6) gamma matrices. The latter products can be conveniently decomposed into 28 anti-symmetric matrices $\Omega$, $\Omega\,\mathbf{\Gamma}_{\hat{a}}$, $\Omega\,\mathbf{\Gamma}_{\hat{a}}\mathbf{\Gamma}_7$ and $\Omega\,\mathbf{\Gamma}_{\hat{a}\hat{b}}\mathbf{\Gamma}_7$, and 36 symmetric matrices $\Omega\,\mathbf{\Gamma}_7$, $\Omega\,\mathbf{\Gamma}_{\hat{a}\hat{b}}$ and $\Omega\,\mathbf{\Gamma}_{\hat{a}\hat{b}\hat{c}}$. The latter are proportional to the anti-hermitian generators of USp(8) (note that the matrices $\mathbf{\Gamma}_{\hat{a}\hat{b}}$ are the generators of the group $\mathrm{SU}(4)\cong\mathrm{SO}(6)$). Before obtaining a representation of the generalized vielbeine $\mathcal{V}_{AB}{}^M$ we note that the USp(8) transformations of the spinors $\psi_\mu{}^A$ and $\epsilon^A$ have been defined in appendix A.2, and they imply that the bilinears $\Omega^{AC}\,\bar{\epsilon}_C\psi_\mu{}^B$ transform in the **27** representation of USp(8). Since the vector fields are not subject to the R-symmetry, it follows that the generalized vielbeine $\mathcal{V}_{AB}{}^M$ transform in the same representation, so that they can be expanded in the corresponding gamma matrix combinations,

$$\begin{aligned}\mathcal{V}_{AB}{}^{M} =& \mathcal{V}_a{}^M\big(\Omega\,\boldsymbol{\Gamma}^a\big)_{AB} + \mathcal{V}_6{}^M\big(\Omega\,\boldsymbol{\Gamma}^6\big)_{AB} + \tilde{\mathcal{V}}_a{}^M\big(\Omega\,\boldsymbol{\Gamma}^a\boldsymbol{\Gamma}_7\big)_{AB} + \tilde{\mathcal{V}}_6{}^M\big(\Omega\,\boldsymbol{\Gamma}^6\boldsymbol{\Gamma}_7\big)_{AB}\\ &+ \mathcal{V}_{ab}{}^M\big(\Omega\,\boldsymbol{\Gamma}^{ab}\boldsymbol{\Gamma}_7\big)_{AB} + 2\,\mathcal{V}_{a6}{}^M\big(\Omega\,\boldsymbol{\Gamma}^{a6}\boldsymbol{\Gamma}_7\big)_{AB}\,,\end{aligned} \tag{2.71}$$

which defines the branching of the **27** representation of USp(8) with respect to SO(5) (which directly follows via the branching with respect to SO(6)),

$$\mathbf{27} \stackrel{\mathrm{SO}(6)}{\longrightarrow} \mathbf{6}+\bar{\mathbf{6}}+\mathbf{15} \stackrel{\mathrm{SO}(5)}{\longrightarrow} \mathbf{1}+\mathbf{5}+\mathbf{1}+\mathbf{5}+\mathbf{10}+\mathbf{5}\,. \tag{2.72}$$

The generalized vielbeine can now be directly determined from the supersymmetry transformations of the vector fields, which leads to

$$\begin{aligned}
\mathcal{V}_{AB}{}^{m} =&\, -\tfrac{1}{4}\mathrm{i}\,\Delta^{-1/3}\,e_a{}^m\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}^{a6}\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\,, \qquad (2.73)\\
\mathcal{V}_{AB}{}^{mn} =&\, -\tfrac{4}{5}\sqrt{5}\mathrm{i}\,\Delta^{2/3}\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}^{mn}\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\\
&+\tfrac{4}{5}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}A^{\alpha}{}_{pq}\,\mathcal{V}_{AB\,\alpha r}\\
&+\tfrac{32}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\big[A_{pqrs}-\tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{pq}A^{\beta}{}_{rs}\big]\,\mathcal{V}_{AB}{}^{s}\,,\\
\mathcal{V}_{AB\,\alpha m} =&\, \tfrac{1}{4}\mathrm{i}\,\Delta^{-1/3}\,\big[(\phi_\alpha-\varepsilon_{\alpha\beta}\phi^\beta)\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_m\,\Phi\big)_{AB}+(\phi_\alpha+\varepsilon_{\alpha\beta}\phi^\beta)\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_m\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\big]\\
&+\mathrm{i}\,\varepsilon_{\alpha\beta}A^{\beta}{}_{mn}\,\mathcal{V}_{AB}{}^{n}\,,\\
\mathcal{V}_{AB\,\alpha} =&\, \tfrac{1}{10}\sqrt{5}\mathrm{i}\,\Delta^{2/3}\big[(\phi_\alpha-\varepsilon_{\alpha\beta}\phi^\beta)\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_6\,\Phi\big)_{AB}+(\phi_\alpha+\varepsilon_{\alpha\beta}\phi^\beta)\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_6\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\big]\\
&+\tfrac{1}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}A^{\beta}{}_{mn}\,\mathcal{V}_{AB}{}^{mn}\\
&-\tfrac{1}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\big[A_{mnpq}\,\mathcal{V}_{AB\alpha r}+2\mathrm{i}\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{mn}\,A_{pqrs}\,\mathcal{V}_{AB}{}^{s}\big]\\
&-\tfrac{1}{40}\sqrt{5}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\big[\,A^{\beta}{}_{mn}\,A^{\gamma}{}_{pq}\,\mathcal{V}_{AB\,\gamma r}-\tfrac{1}{3}\mathrm{i}\,\varepsilon_{\gamma\delta}\,A^{\gamma}{}_{sm}\,A^{\delta}{}_{np}\,A^{\beta}{}_{qr}\,\mathcal{V}_{AB}{}^{s}\big]\,.
\end{aligned}$$

In the above equations we have now included the compensating phase factors $\Phi^A{}_B$ that were discussed earlier, which enable the USp(8) R-symmetry group to be realized locally. The phase factors are simply generated by a redefinition of the fermion fields, as $\Phi \in \mathrm{USp}(8)$ is assumed to transform under the action of USp(8) from the right and under $\mathrm{USp}(4)\times\mathrm{U}(1)$ from the left, so that fermion fields $\Phi^\dagger\Psi$, where $\Psi$ denotes the original fields in a proper basis, transform indeed under this local group. Previously we have assumed the gauge condition $\Phi = \mathbb{1}$ which suffices to carry out most of the various calculations. In fact, we will continue to use this gauge condition in most of what follows. The phase factors can always be introduced later to elevate the R-symmetry group to a local invariance group. This is in direct analogy with what was done for $11D$ supergravity [61].

The next task is to establish the relation between the USp(8) covariant spinors

$\chi^{ABC}$ and the spinors originating from $10D$, ${\psi_a}^A$ and $\lambda^A$. Comparing the terms proportional to these fields in the supersymmetry variations of the vector fields, one finds the following set of equations,

$$\begin{aligned}
{\psi_a}^A =&\, -\mathrm{i}\big[\chi^{ABC}\,\delta_a^b - \tfrac{1}{8}(\mathbf{\Gamma}_a\mathbf{\Gamma}^b)^A{}_D\,\chi^{DBC}\big] \\
&\times\big[\Omega\,\mathbf{\Gamma}_{b6}\mathbf{\Gamma}_7\big]_{BC}\,, \\
\big[(\mathbb{1}\pm\mathbf{\Gamma}_7)\mathbf{\Gamma}_{a6}\big]^A{}_D\,\lambda^D =&\, \pm\,\mathrm{i}\,\big[\Omega\,\mathbf{\Gamma}_a(\mathbb{1}\pm\mathbf{\Gamma}_7)\big]_{BC}\,\big(\mathbb{1}\pm\mathbf{\Gamma}_7\big)^A{}_D\,\chi^{DBC}\,, \\
\big(\mathbb{1}\pm\mathbf{\Gamma}_7\big)^A{}_D\,\lambda^D =&\, \pm\,\mathrm{i}\,\big(\Omega\,\mathbf{\Gamma}_6(\mathbb{1}\pm\mathbf{\Gamma}_7)\big)_{BC}\,\big(\mathbb{1}\pm\mathbf{\Gamma}_7\big)^A{}_D\,\chi^{DBC}\,, \\
\big[(\mathbf{\Gamma}_{[ab}(\delta_{c]}^d\mathbb{1}-\tfrac{1}{9}\mathbf{\Gamma}_{c]}\mathbf{\Gamma}^d)\mathbf{\Gamma}_7\big]^A{}_D\,{\psi_d}^D =&\, -\tfrac{1}{6}\varepsilon_{abcde}\big(\Omega\,\mathbf{\Gamma}^{de}\mathbf{\Gamma}_7\big)_{BC}\,\chi^{ABC}\,, \\
\big[(\Omega\,\mathbf{\Gamma}_a\big(\mathbb{1}\pm\mathbf{\Gamma}_7)\big]_{BC}\,\big(\mathbb{1}\mp\mathbf{\Gamma}_7\big)^A{}_D\,\chi^{DBC} =&\, \pm\,2\mathrm{i}\,\big[\big(\mathbb{1}\mp\mathbf{\Gamma}_7\big)\mathbf{\Gamma}_6\,\big(\delta_a^b\mathbb{1}-\tfrac{1}{3}\mathbf{\Gamma}_a\mathbf{\Gamma}^b\big)\big]^A{}_D\,{\psi_b}^D\,, \\
\big[(\mathbb{1}\pm\mathbf{\Gamma}_7)\mathbf{\Gamma}^a\big]^A{}_D\,{\psi_a}^D =&\, \mp\,\tfrac{3}{4}\mathrm{i}\,\big[\Omega\,\mathbf{\Gamma}_6(\mathbb{1}\mp\mathbf{\Gamma}_7)\big]_{BC}\,\big(\mathbb{1}\pm\mathbf{\Gamma}_7\big)^A{}_D\,\chi^{DBC}\,.
\end{aligned} \tag{2.74}$$

These are the relations that determine the (linear) relation between the spinors ${\psi_a}^A$ and $\lambda^A$ and the USp(8) covariant spinors $\chi^{ABC}$. Just as in $11D$ supergravity, where the expression for the $4D$ spin-1/2 given in [104] is only unique up to Fierz reordering, there are various different ways to express the solution for $\chi^{ABC}$. One solution follows by substituting the SO(6) covariant parametrization derived in appendix A.2 into (2.74), which then leads to (A.43). However, given that the ansatz for $\chi^{ABC}$ is not unique, one might wonder whether there exists an alternative version of this solution that may be even more concise. Indeed we find such a solution taking the form

$$\begin{aligned}
\chi^{ABC} =&\, -\tfrac{3}{8}\mathrm{i}\Big[\big(\mathbf{\Gamma}_6\,\bar\Omega\big)^{[AB}\,\big(\mathbf{\Gamma}_7\lambda\big)^{C]}+\big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar\Omega\big)^{[AB}\,\lambda^{C]}\Big] \\
&-\tfrac{3}{4}\mathrm{i}\,\big(\mathbf{\Gamma}^a\mathbf{\Gamma}_6\mathbf{\Gamma}_7\,\bar\Omega\big)^{[AB}\,{\psi_a}^{C]}-\tfrac{1}{4}\mathrm{i}\,\bar\Omega^{[AB}\,\big(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\mathbf{\Gamma}^a\psi_a\big)^{C]}\,.
\end{aligned} \tag{2.75}$$

which also satisfies (2.74). Its equivalence to (A.43) can be confirmed by showing that both solutions are related by Fierz reordering to a single expression that involves eight different structures. This result satisfies the reality condition (2.70) and vanishes upon contraction with $\Omega_{AB}$. Note also that the above expression should in principle have been contracted with three different phase factors $\Phi^\dagger$ as was discussed above. For clarity of the presentation we have set $\Phi=\mathbb{1}$.

Subsequently we derive a formula for the supersymmetry transformations of the generalized vielbeine ${\mathcal{V}_{AB}}^M$. For maximal $5D$ supergravity [42] there exists the following expression (with indices $i,j,\ldots$ replaced again by $A,B,\ldots$),

$$\begin{aligned}
\delta{\mathcal{V}_{AB}}^M =&\, -\mathrm{i}\big[4\,\Omega_{G[A}\,\bar\chi_{BCD]}\,\epsilon^G+3\;\Omega_{[AB}\,\bar\chi_{CD]G}\,\epsilon^{G}\big]\,\bar\Omega^{CE}\bar\Omega^{DF}\,{\mathcal{V}_{EF}}^M \\
=&\,\mathrm{i}\,\Omega_{AC}\Omega_{BD}\,\big[4\,\bar\Omega^{G[C}\,\bar\epsilon_G\,\chi^{DEF]}+3\,\bar\Omega^{[CD}\,\bar\epsilon_G\,\chi^{EF]G}\big]\,{\mathcal{V}_{EF}}^M\,.
\end{aligned} \tag{2.76}$$

This result is expected to be identical to the result that one obtains by calculating the variations of the generalized vielbeine (2.73) induced by the supersymmetry transformations of the scalar fields,

$$\begin{aligned}
\delta e_m{}^a &= \tfrac{1}{2} e_m{}^b\, \bar\epsilon\, \mathbf{\Gamma}^{a6} \mathbf{\Gamma}_7 \psi_b \,, \\
\delta\phi^\alpha &= -\tfrac{1}{4} \varepsilon^{\alpha\beta} \phi_\beta\, \bar\epsilon\, \mathbf{\Gamma}_6 (\mathbb{1} + \mathbf{\Gamma}_7)\lambda \,, \\
\delta\phi_\alpha &= -\tfrac{1}{4} \varepsilon_{\alpha\beta} \phi^\beta\, \bar\epsilon\, \mathbf{\Gamma}_6 (\mathbb{1} - \mathbf{\Gamma}_7)\lambda \,, \\
\delta A^\alpha{}_{mn} &= -\tfrac{1}{4} \mathrm{i}\, e_m{}^a\, e_n{}^b\, (\phi^\alpha + \varepsilon^{\alpha\beta}\phi_\beta)\, \bar\epsilon\, (\mathbf{\Gamma}_{ab}\lambda - 4\, \mathbf{\Gamma}_{[a}\psi_{b]}) \\
&\quad + \tfrac{1}{4} \mathrm{i}\, e_m{}^a\, e_n{}^b\, (\phi^\alpha - \varepsilon^{\alpha\beta}\phi_\beta)\, \bar\epsilon\, (\mathbf{\Gamma}_{ab}\mathbf{\Gamma}_7\lambda - 4\, \mathbf{\Gamma}_{[a}\mathbf{\Gamma}_7\psi_{b]}) \\
\delta A_{mnpq} &= -\tfrac{1}{2} \mathrm{i}\, e_m{}^a\, e_n{}^b\, e_p{}^c\, e_q{}^d\, \bar\epsilon\, \mathbf{\Gamma}_6 \mathbf{\Gamma}_{[abc}\psi_{d]} + \tfrac{3}{8} \mathrm{i} \varepsilon_{\alpha\beta} A^\alpha{}_{[mn}\, \delta A^\beta{}_{pq]} \,.
\end{aligned} \tag{2.77}$$

Based on the similar construction for $11D$ supergravity [61], we expect the supersymmetry transformations of the vielbeine induced by the variations (2.77) to coincide with (2.76) up to a uniform USp(8) transformation. By very laborious calculations it is possible to demonstrate that this expectation is correct so that (2.76) can be regarded as the supersymmetry transformation rule for the vielbeine. More precisely, the results induced by (2.77) take the form

$$\delta \mathcal{V}_{AB}{}^M = \delta \mathcal{V}_{AB}{}^M \big|_{(2.76)} - \Lambda^C{}_{[A}\, \mathcal{V}_{B]C}{}^M \tag{2.78}$$

where $\Lambda^A{}_B$ is the field-dependent infinitesimal USp(8) transformation given by

$$\begin{aligned}
\Lambda^A{}_B = &-\tfrac{1}{16} \bar\epsilon\, \mathbf{\Gamma}_7 [\mathbf{\Gamma}_{ab}\lambda + 4\, \mathbf{\Gamma}_{[a}\psi_{b]}]\, \left(\mathbf{\Gamma}^{ab6}\right)^A{}_B \\
&+ \tfrac{1}{48} \bar\epsilon\, \mathbf{\Gamma}_7 [\mathbf{\Gamma}_{abc6}\lambda + 2\, \mathbf{\Gamma}_{abcd6}\psi^d]\, \left(\mathbf{\Gamma}^{abc}\right)^A{}_B \\
&+ \tfrac{1}{4} \bar\epsilon\, \mathbf{\Gamma}_7 \mathbf{\Gamma}_{ac}\psi^c\, \left(\mathbf{\Gamma}^{a6}\right)^A{}_B + \tfrac{1}{4} \bar\epsilon\, \mathbf{\Gamma}_7 \mathbf{\Gamma}_{6[a}\psi_{b]}\, \left(\mathbf{\Gamma}^{ab}\right)^A{}_B \,.
\end{aligned} \tag{2.79}$$

We now proceed with the supersymmetry transformations of the tensor fields $C_{\mu\nu\, mn}$, $C_{\mu\nu}{}^{\alpha m}$ and $C_{\mu\nu}{}^\alpha$ that were defined in (2.62), following the same approach as for the vector fields. Their supersymmetry transformations follow upon substituting the results specified in (2.51) and (2.60). Subsequently we compare them to the five-dimensional transformation rules for the tensor fields [42] with the indices adjusted as before,

$$\begin{aligned}
&\delta C_{\mu\nu\, M} - 2\, d_{MNP}\, C_{[\mu}{}^N\, \delta C_{\nu]}{}^P \\
&\quad = \tfrac{4}{5}\sqrt{5}\, \mathcal{V}_M{}^{AB} \left[2\, \bar\psi_{[\mu A}\, \gamma_{\nu]} \epsilon^C\, \Omega_{BC} - \mathrm{i} \bar\chi_{ABC}\, \gamma_{\mu\nu} \epsilon^C\right] \\
&\quad = -\tfrac{4}{5}\sqrt{5}\, \mathcal{V}_M{}^{AB} \left[2\, \Omega_{AC}\, \bar\epsilon_B\, \gamma_{[\mu}\psi_{\nu]}{}^C + \mathrm{i}\Omega_{AD}\Omega_{BE}\, \bar\epsilon_C\, \gamma_{\mu\nu}\chi^{DEC}\right] .
\end{aligned} \tag{2.80}$$

In $5D$ maximal gauged supergravity the tensor fields constitute a **27** representation of $\mathrm{E}_{6(6)}$. From IIB supergravity we have initially identified only 22 different tensor fields. The missing five tensors $C_{\mu\nu\, m}$ have been identified as originating from a component of the $10D$ dual graviton. The second term on the left-hand side of (2.80) has already been specified in (2.67).

From the terms in (2.80) proportional to $\psi_\mu{}^C$ one can directly obtain the following expressions for the 22 components of $\mathcal{V}_M{}^{ij}$, by making use of the supersymmetry transformations of the corresponding tensors derived in the previous sections,

$$\begin{aligned}
\mathcal{V}_{mn}{}^{AB} &= -\tfrac{1}{32}\sqrt{5}\mathrm{i}\,\Delta^{-2/3} e_m{}^a e_n{}^b \big(\Phi^\dagger\,\boldsymbol{\Gamma}_{ab}\boldsymbol{\Gamma}_7\bar{\Omega}\,\bar{\Phi}\big)^{AB} + \tfrac{1}{8}\mathrm{i}\,\varepsilon_{\alpha\beta}\,A^\alpha{}_{mn}\,\mathcal{V}^{\beta\,AB}\,,\\
\mathcal{V}^{\alpha m\,AB} &= -\tfrac{1}{4}\mathrm{i}\Delta^{1/3}\,e_a{}^m\,\big[\big(\phi^\alpha - \varepsilon^{\alpha\beta}\phi^\beta\big)\,\big(\Phi^\dagger\,\boldsymbol{\Gamma}^e\bar{\Omega}\,\bar{\Phi}\big)^{AB}\\
&\qquad\qquad - \big(\phi^\alpha + \varepsilon^{\alpha\beta}\phi_\beta\big)\,\big(\Phi^\dagger\boldsymbol{\Gamma}^e\boldsymbol{\Gamma}_7\bar{\Omega}\,\bar{\Phi}\big)^{AB}\big]\\
&\quad + \tfrac{1}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\big[A_{npqr}\,\mathcal{V}^{\alpha\,AB} + \tfrac{3}{8}\mathrm{i}\,A^\alpha{}_{np}A^\beta{}_{qr}\,\varepsilon_{\beta\gamma}\mathcal{V}^{\gamma\,AB} - 6\,A^\alpha{}_{np}\mathcal{V}_{qr}{}^{AB}\big]\,,\\
\mathcal{V}^{\alpha AB} &= -\tfrac{1}{8}\sqrt{5}\mathrm{i}\,\Delta^{-2/3}\big[\big(\phi^\alpha - \varepsilon^{\alpha\beta}\phi_\beta\big)\,\big(\Phi^\dagger\,\boldsymbol{\Gamma}_6\bar{\Omega}\,\bar{\Phi}\big)^{AB}\\
&\qquad\qquad - \big(\phi^\alpha + \varepsilon^{\alpha\beta}\phi_\beta\big)\,\big(\Phi^\dagger\,\boldsymbol{\Gamma}_6\boldsymbol{\Gamma}_7\bar{\Omega}\,\bar{\Phi}\big)^{AB}\big]\,,
\end{aligned}\tag{2.81}$$

where we have again included the phase factors $\Phi$. Before discussing how to obtain the missing components of $\mathcal{V}_M{}^{AB}$ that are associated with the dual graviton, we first consider the contractions of the form $\mathcal{V}_M{}^{AB}\,\mathcal{V}_{AB}{}^N$ making use of the expressions (2.73) and (2.81). As it turns out the only non-zero contractions are given by

$$\begin{aligned}
\mathcal{V}_{mn}{}^{AB}\,\mathcal{V}_{AB}{}^{pq} &= 2\,\delta_{mn}{}^{pq}\,,\\
\mathcal{V}^{\alpha\,AB}\,\mathcal{V}_{AB\,\beta} &= \delta^\alpha{}_\beta\,,\\
\mathcal{V}^{\alpha m\,AB}\,\mathcal{V}_{AB\,\beta n} &= \delta^\alpha{}_\beta\,\delta^m{}_n\,,
\end{aligned}\tag{2.82}$$

suggesting that

$$\mathcal{V}_M{}^{AB}\,\mathcal{V}_{AB}{}^N = \delta_M{}^N\,.\tag{2.83}$$

This condition is actually identical to the one that holds in $5D$ maximal gauged supergravity. In the same spirit as in (2.67), we may assume that (2.83) holds in this case as well, and this then enables us to also determine the five missing components $\mathcal{V}_m{}^{AB}$,

$$\begin{aligned}
\mathcal{V}_m{}^{AB} &= -\tfrac{1}{2}\mathrm{i}\,\Delta^{1/3}e_m{}^a\big(\Phi^\dagger\boldsymbol{\Gamma}_{a6}\boldsymbol{\Gamma}_7\bar{\Omega}\bar{\Phi}\big)^{AB}\\
&\quad + \tfrac{16}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{npqrs}\big[A_{mqrs} - \tfrac{3}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}A^\alpha{}_{qr}A^\beta{}_{sm}\big]\mathcal{V}_{np}{}^{ij} - \mathrm{i}A^\alpha{}_{mn}\,\varepsilon_{\alpha\beta}\,\mathcal{V}^{\beta n\,AB}\\
&\quad + \tfrac{1}{15}\sqrt{5}\mathrm{i}\,\mathring{e}^{-1}\varepsilon^{npqrs}\,\varepsilon_{\alpha\beta}\big[A_{npqr}\,A^\beta{}_{sm} - \tfrac{1}{8}\mathrm{i}\,\varepsilon_{\gamma\delta}A^\beta{}_{np}A^\gamma{}_{qr}A^\delta{}_{sm}\big]\mathcal{V}^{\alpha\,AB}\,.
\end{aligned}\tag{2.84}$$

Note that the conditions (2.83) implies that also the supersymmetry transformations of the $\mathcal{V}_M{}^{AB}$ are determined and take the same form as the corresponding supersymmetry transformations in $5D$ maximal supergravity. Needless to say, the results obtained from the vector fields on the covariant spinors $\chi^{ABC}$ can be verified also from the perpective of the transformations of the tensor fields. The results turn out to be mutually consistent.

This completes the evaluation of the bosons and their supersymmetry transformations. We have succeeded in identifying these fields from IIB supergravity such that the results resemble as closely as possible the structure of the $5D$ maximal gauged supergravities [42] while retaining the full dependence on all ten coordinates. For the fields associated with the dual graviton, we obtained their supersymmetry transformations by requiring them to be consistent with the global structure exhibited for the other fields. In this way the results exhibit covariance with respect to the duality group $\mathrm{E}_{6(6)}$, although the IIB theory is not in any way invariant under this group. This is further confirmed by the fact that the following representation of the invariant tensor $d_{MNP}$ which was noted for maximal $5D$ supergravity [42],

$$d_{MNP} = \tfrac{2}{5}\sqrt{5}\,\mathcal{V}_M{}^{AB}\,\mathcal{V}_M{}^{CD}\,\mathcal{V}_M{}^{EF}\,\Omega_{BC}\,\Omega_{DE}\,\Omega_{FA}\,, \tag{2.85}$$

is also satisfied here, as this expression precisely reproduces the tensor $d_{MNP}$ as specified in (2.67).

We remind the reader that the generalized vielbeine are pseudo-real. This property is inherited form the (pseudo-)reality of $10D$ the tensors and the fermionic bilinears. Hence, taking complex conjugates of vielbeine that carry the $\mathrm{SU}(1,1)$ requires the contraction with a two-dimensional metric $\eta_{\alpha\beta} = \mathrm{diag}(+1,-1)$ in order to obtain a covariant quantity (see section 2.2).

Finally, in view of the above results for the variations of the bosonic fields, it is also of interest to consider the supersymmetry variations of the fermion fields $\psi_\mu{}^A$ and $\chi^{ABC}$, to verify whether they also take a $\mathrm{Usp}(8)$ covariant form. This analysis not only complements the previous results, but also allows for the proper identification of various bosonic $\mathrm{Usp}(8)$ tensors in terms of the $10D$ bosonic fields. Here, we refrain form presenting these results as they are not directly relevant for the purpose of the next section. Instead, we refer to [103] where an explicit analysis of the fermion transformation rules is provided, starting from the expressions (2.41), (2.42) and (2.43), and following the same strategy as was applied in the case of $11D$ supergravity [61].

## 2.6 On the consistent truncation to 5D SO(6) gauged supergravity

It is natural to expect that IIB supergravity compactified on $S^5$ will be related to $5D$ SO(6) gauged maximal supergravity [53], provided one can consistently remove the additional Kaluza-Klein states by imposing a consistent truncation. Such a question was first studied for $11D$ supergravity compactified on $S^7$, which was indeed shown to admit a consistent truncation to $4D$ maximal supergravity with an SO(8) gauge group [11, 62–64, 111, 112]. There, the strategy was to first reorganise the fields and their supersymmetry transformations in a form that was as close as possible to that of $4D$ maximal supergravity, while keeping the dependence on the full set of eleven coordinates. This is analogous to our reformulation of type IIB supergravity, which was presented in the previous sections, where the supersymmetry transformations take the form of those in $5D$ maximal supergravity, although all fields and symmetry parameters still depend on both the five space-time coordinates $x^\mu$ and the five internal coordinates $y^m$. In fact, one of the the original motivations for this work was to acquire further insight into a suitable truncation for IIB supergravity, especially in view of the subtleties of the IIB theory that are not present in $11D$ supergravity.

Before continuing, let us put aside supergravity for moment and consider the issue of consistent truncations in the context of Kaluza-Klein theory, where one compactifies general relativity in $d+4$ dimensions on some $d$-dimensional Euclidean space. In this case the $4D$ vector fields originate entirely from the higher-dimensional metric, and when the internal manifold has non-abelian isometries, they act as non-abelian gauge fields. A consistent truncation ansatz of the vector fields is generally given by the following expression of the $d+4$-dimensional metric[4]

$$g_{\mu n}(x,y)\, g^{nm}(x,y) = K(y)^m{}_{\mathcal{A}}\, V_\mu{}^{\mathcal{A}}(x)\,, \tag{2.86}$$

where $K(y)^m{}_A$ are the Killing vectors of the internal metric $\mathring{g}_{mn}(y)$ associated with the compactification manifold. The Killing vectors generate the isometries of this manifold and the isometry group (whose adjoint representation is labelled by $\mathcal{A}, \mathcal{B}, \ldots$) will coincide with the gauge group of the $4D$ massless vector fields $V_\mu{}^{\mathcal{A}}$ [113]. Indeed the structure constants of the gauge group, $f_{\mathcal{AB}}{}^{\mathcal{C}}$ emerge when

[4]For the purpose of this example, the spacetime and internal space indices are momentarily parametrised by $x^\mu$ and $y^m$ with $\mu, \nu = 0, \ldots, 4$ and $m, n = 1, \ldots, d$, respectively.

considering the effect of an isometry on a Killing vector, i.e.,

$$K(y)^n{}_{\mathcal{A}}\,\partial_n K(y)^m{}_{\mathcal{B}} - K(y)^n{}_{\mathcal{B}}\,\partial_n K(y)^m{}_{\mathcal{A}} = f_{\mathcal{AB}}{}^{\mathcal{C}}\,K(y)^m{}_{\mathcal{C}}\,. \tag{2.87}$$

Of course, one would also like to derive consistent truncation ansätze for the scalar fields contained in $g_{mn}(x,y)$ and the gravitons contained in $g_{\mu\nu}(x,y)$, but this is in general a non-trivial task. Nevertheless, for supergravity it is possible in certain cases to derive consistent truncation ansätze for all the fields.

Let us now return to truncations in the supergravity framework and review some of the recent developments. First of all, the analysis of the consistent truncation of $11D$ supergravity on $S^7$ has been further extended over the last few years, and now involves the dual gauge fields as well as various other quantites [114–116]. For the case of the truncation of type IIB supergravity on $S^5$, some of those extensions were implemented in [103]. Results on the truncation of IIB supergravity were also derived based on rather different approaches. For instance, a fairly general set of truncation ansätze appeared in [75] based on generalized geometry. Even more recently, the full consistent truncation of IIB supergravity has been worked out in detail from exceptional field theory [87]. Both generalized geometry and exceptional field theory are based on extensions of the conventional supergravity framework, and therefore, are in contrast with the more conservative approach followed in this chapter. In this last section, we will therefore describe how results on the consistent truncation can be derived based on the work of the preceding sections, and compare them with the literature. We should mention that these results were already partially presented [103], but at the time some subtleties were overlooked.

The first step of the analysis is to sweep out the full field configuration space of $5D$ maximal supergravity in the ten-dimensional field configuration space starting from the fully supersymmetric $\mathrm{AdS}_5\times S^5$ solution. This is done by writing the $10D$ fields as functions of the $5D$ fields, involving $y$-dependent functions, mostly constructed from the $S^5$ Killing spinors, in such a way that the $10D$ supersymmetry transformations remain consistent upon extracting these $y$-dependent factors. In the case at hand these eight independent, pseudo-real, Killing spinors, $\eta^A(y)$, satisfy

$$\big(\mathring{D}_m + \tfrac{1}{2}m_5\,\mathring{e}_m{}^a\,\mathbf{\Gamma}_{a6}\big)\eta(y) = 0\,. \tag{2.88}$$

Here $m_5$ denotes the inverse $S^5$ radius which is related to the background value of the field strength $F_{mnpqr}$ by $m_5 = \frac{1}{120}\,\mathring{e}\,\varepsilon^{mnpqr}\mathring{F}_{mnpqr}$. The background value of the associated potential is denoted by $\mathring{A}_{mnpq}(y)$. Furthermore $\mathring{D}_m$ equals the $S^5$

background covariant derivative and $\mathring{e}_m{}^a(y)$ is the globally defined fünfbein on $S^5$. The Killing spinor equation (2.88) is motivated by the fact that it characterizes the supersymmetry of the $\mathrm{AdS}_5 \times S^5$ solution of IIB supergravity. Note that the Killing spinors in this section will be commuting.

In view of what follows it is useful to first discuss these Killing spinors in some detail. Since (2.88) is a first-order differential equation it allows for eight independent solutions. However, in five Euclidean dimensions, the Clifford algebra associated with the SO(5) gamma matrices has an automorphism group equal to $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Consequently one can choose six independent spinors that are not related by the action of the automorphism group, so that the orbit that is then swept out under the action of the SU(2) automorphism group will yield the two remaining independent spinors. Bilinears constructed from the Killing spinors that involve only the original SO(5) gamma matrices will necessarily be invariant under the automorphism group and therefore the number of independent spinor bilinears of this type will constitute $6 \otimes 6$ independent bilinears which decompose into 15 anti-symmetric and 21 symmetric components. This argument, which incidentally also plays a role when analyzing the number of degrees of freedom of the generalized vielbeine in section 2.5, explains why the bilinears produce precisely 15 independent Killing vectors associated with the isometry group of $S^5$. More specifically it follows that

$$\mathring{e}_a{}^m\, \bar{\eta}_1(y)\, \mathbf{\Gamma}^{a6}\mathbf{\Gamma}_7\, \eta^2(y) = \sum_{[\hat{a}\hat{b}]} C^{\hat{a}\hat{b}}\, K^m{}_{\hat{a}\hat{b}}(y)\,, \tag{2.89}$$

where $\eta^{1,2}$ are two possible Killing spinors (with $\bar{\eta} \equiv \eta^\dagger$) and the indices $\hat{a}, \hat{b}, \ldots$ denote the components of the defining representation of the SO(6); in this background SO(6) corresponds to the isometry group of the sphere $S^5$. The fifteen Killing vectors are labeled with anti-symmetric pairs $[\hat{a}\hat{b}]$ which are raised and lowered using the SO(6) Cartan-Killing metric (and its inverse). The coefficients $C^{\hat{a}\hat{b}}$ are constant. To prove this relation one can write the gamma matrices in terms of the original SO(5) gamma matrices and/or one can prove directly that the left-hand side of (2.89) satisfies the Killing equation by virtue of (2.88).

Taking the derivative of the Killing vectors one finds additional tensors that are also anti-symmetric in $[\hat{a}\hat{b}]$,

$$\mathring{D}_m K_{n\,\hat{a}\hat{b}} = -m_5\, K_{mn\,\hat{a}\hat{b}}\,, \tag{2.90}$$

and which are known as a Killing tensors. They satisfy the equation

$$\mathring{D}_m K_{np\,\hat{a}\hat{b}} = 2\,m_5\,\mathring{g}_{m[n}\,K_{p]\,\hat{a}\hat{b}}\,. \tag{2.91}$$

and are orthogonal to the Killing vectors, i.e. $K_{mn\,\hat{a}\hat{b}}K^{p\,\hat{a}\hat{b}} = 0$. From the previous results one then derives

$$\mathring{e}_m{}^a\,\mathring{e}_n{}^b\,\bar{\eta}_1(y)\,\mathbf{\Gamma}_{ab}\mathbf{\Gamma}_7\,\eta^2(y) = \sum_{[\hat{a}\hat{b}]} C^{\hat{a}\hat{b}}\,K_{mn\,\hat{a}\hat{b}}(y)\,. \tag{2.92}$$

Note that the internal indices carried by objects characterising the $S^5$ geometry, such as the Killing vectors or tensors, are raised and lowered using the $S^5$ metric $\mathring{g}_{mn}(y)$.

After these observations we turn to the consistent truncation ansätze for the $10D$ fields. We start from eight independent Killing spinors, now labeled by indices $i, j, \ldots = 1, 2, \ldots, 8$, such that these spinors form an orthonormal basis in the USp(8) spinor space and are subject to a pseudo-reality condition,

$$\bar{\eta}^i(y)\,\eta_j(y) = \delta^i_j\,, \qquad \bar{\eta}^i{}_A = \bar{\Omega}^{ij}\,\Omega_{AB}\,\eta_j^B\,, \tag{2.93}$$

where $\bar{\Omega}^{ij}$ and $\Omega_{AB}$ are the symplectic matrices used before. The truncation for the fermions, the supersymmetry parameters and the space-time vielbein $e_\mu{}^\alpha$ are then assumed to take the form,[5]

$$\begin{aligned}
\psi_\mu{}^A(x,y) &= \psi_\mu{}^i(x)\,\eta_i^A(y)\,,\\
\epsilon^A(x,y) &= \epsilon^i(x)\,\eta_i^A(y)\,,\\
\chi^{ABC}(x,y) &= \chi^{ijk}(x)\,\eta_i^A(y)\,\eta_j^B(y)\,\eta_k^C(y)\,,\\
e_\mu{}^\alpha(x,y) &= e_\mu{}^\alpha(x)\,.
\end{aligned} \tag{2.94}$$

Observe that the supersymmetry transformations for $e_\mu{}^\alpha(x,y)$ are consistent under this truncation ansatz in the sense that $y$-dependence drops out of (2.40). However, for the other bosons the truncation ansatz is more subtle.

To derive the truncation ansätze for the remaining bosons one first considers their

[5] The phase factor $\Phi$ is only implicit in the formulae below, but it actually plays a crucial role to ensure that consistency is achieved (see e.g. [111]).

supersymmetry variations into the fermions, defined according to (2.94). For instance, consider (2.69), which will now take the form,

$$\delta C_\mu{}^M(x,y) = 2\left[\mathrm{i}\bar{\Omega}^{ik}\,\bar{\epsilon}_k(x)\,\psi_\mu^j(x) + \bar{\epsilon}_k(x)\,\gamma_\mu\chi^{ijk}(x)\right]\mathcal{V}_{ij}{}^M(x,y)\,, \tag{2.95}$$

where

$$\mathcal{V}_{ij}{}^M(x,y) = \eta_i^A(y)\,\eta_j^B(y)\,\mathcal{V}_{AB}{}^M(x,y)\,. \tag{2.96}$$

The consistency of the truncation now requires that the $y$-dependence of $C_\mu{}^M$ and $\mathcal{V}_{ij}{}^M$ to match.

Before deriving some of the additional truncation results, let us first compare the situation regarding the compactification on the torus $T^5$ and the sphere $S^5$. In the torus truncation all the fields $C_\mu{}^M$ will all appear and will be independent of the torus coordinates $y^m$. Consequently the generalized vielbeine $\mathcal{V}_{ij}{}^M$ will also be $y$-independent and they will be precisely equal to the corresponding quantities $U_{ij}{}^M(x)$ that are a representative of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space.[6] The tensor fields $C_{\mu\nu\,M}$ can be gauged away in the torus truncation where they carry no additional information and they are simply dual to the vector fields.

The situation for the $S^5$ compactification is different, as in this case the various 'physical' fields reside in both the $C_\mu{}^M$ and $C_{\mu\nu\,M}$ [53]. More precisely, there are fifteen vector fields transforming in the adjoint representation of the SO(6) subgroup of $\mathrm{E}_{6(6)}$ and twelve tensor fields transforming as a direct product of the vector representation of the same SO(6) subgroup and the doublet represention of the SU(1,1) subgroup of $\mathrm{E}_{6(6)}$. The remaining vector and tensor fields in the sphere truncation are the duals of these $15 \oplus 12$ fields. When writing the $5D$ Lagrangian in the embedding-tensor approach, these fields are contained in the Lagrangian but they can be gauged away. This decomposition in terms of the expected vector and tensor fields must be reflected in the truncation ansätze for the vectors and tensors.

It is important to realize that the fields $C_\mu{}^M$ and $C_{\mu\nu\,M}$ are gauge fields, which excludes field-dependent multiplicative redefinitions. Given that the $y$-dependence should be extracted in the form of the geometric quantities associated with the sphere $S^5$, an obvious truncation ansatz is to decompose the vector and tensor fields, $C_\mu{}^m$, $C_\mu{}^{mn}$, $C_{\mu\nu\,m}$ and $C_{\mu\nu\,mn}$ into the fifteen Killing vectors or tensors

[6] Here we deviate from the notation used in [42] where the $U_{ij}{}^M(x)$ are denoted also by $\mathcal{V}_{ij}{}^M$.

according to

$$C_\mu{}^m(x,y) = K^m{}_{\hat a\hat b}(y)\, A_\mu{}^{\hat a\hat b}(x)\,, \tag{2.97}$$

$$C_\mu{}^{mn}(x,y) = K^{mn}{}_{\hat a\hat b}(y)\, \tilde A_\mu{}^{\hat a\hat b}(x)\,, \tag{2.98}$$

$$C_{\mu\nu\, m}(x,y) = K_m{}^{\hat a\hat b}(y)\, B_{\mu\nu\,\hat a\hat b}(x)\,, \tag{2.99}$$

$$C_{\mu\nu\, mn}(x,y) = K_{mn}{}^{\hat a\hat b}(y)\, \tilde B_{\mu\nu\,\hat a\hat b}(x)\,. \tag{2.100}$$

There are no strong arguments why precisely these ansätze should be the correct ones, except for the first one, which is a standard Kaluza-Klein truncation ansatz (2.97). When continuing with the analysis of the other vectors and tensors, some of the ansätze (2.98), (2.99) and (2.100) will eventually lead to contradictions and will therefore require modifications. Hence for the moment we will just proceed on the basis of these ansätze and investigate whether it leads to a consistent proposal. Furthermore, as explained above, in $5D$ one has only fifteen vector and fifteen tensor fields in the $\mathrm{SU}(1,1)$ invariant sector, so that one must assume that $\tilde A_\mu{}^{\hat a\hat b}(x)$ must be proportional to $A_\mu{}^{\hat a\hat b}(x)$ (or equal to zero). A similar comment can be made for the tensor fields $B_{\mu\nu\,\hat a\hat b}(x)$ and $\tilde B_{\mu\nu\,\hat a\hat b}(x)$. Here, we will not deal with fixing the normalizations of all quantities when connecting to the $5D$ fields.

A similar decomposition must then apply to the generalized vielbeine $\mathcal{V}_{ij}{}^m$, $\mathcal{V}_{ij}{}^{mn}$, $\mathcal{V}_m{}^{ij}$ and $\mathcal{V}_{mn}{}^{ij}$, to ensure that the supersymmetry transformation (2.95) are consistent under the truncation (i.e. that the same $y$-dependence factorises on both side). Hence we write

$$\mathcal{V}_{ij}{}^m(x,y) = U_{ij}{}^{\hat a\hat b}(x)\, K^m{}_{\hat a\hat b}(y)\,, \tag{2.101}$$

$$\mathcal{V}_{ij}{}^{mn}(x,y) = U_{ij}{}^{\hat a\hat b}(x)\, K^{mn}{}_{\hat a\hat b}(y)\,, \tag{2.102}$$

$$\mathcal{V}_m{}^{ij}(x,y) = U_{\hat a\hat b}{}^{ij}(x)\, K_m{}^{\hat a\hat b}(y)\,, \tag{2.103}$$

$$\mathcal{V}_{mn}{}^{ij}(x,y) = U_{\hat a\hat b}{}^{ij}(x)\, K_{mn}{}^{\hat a\hat b}(y)\,, \tag{2.104}$$

where, as we have explained above, $U_{ij}{}^{\hat a\hat b}(x)$ and $U_{\hat a\hat b}{}^{ij}(x)$ are components of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset representative of the $5D$ maximal supergravity that contain the scalars. In particular, they satisfy,

$$U_{ij}{}^{\hat a\hat b}(x) U_{\hat c\hat d}{}^{ij}(x) = 2\,\delta_{\hat c\hat d}{}^{\hat a\hat b}\,. \tag{2.105}$$

Subsequently we consider the following identities that follow from direct calculation using the generalized vielbeine expressed in terms of the $10D$ quantities in

(2.73), and converting the USp(8) indices according to (2.96),

$$\begin{aligned}\bar{\mathcal{V}}^{ik\,m}\,\mathcal{V}_{kj}{}^{n} + \bar{\mathcal{V}}^{ik\,n}\,\mathcal{V}_{kj}{}^{m} &= -\,\tfrac{1}{4}\delta^{i}{}_{j}\,\bar{\mathcal{V}}^{kl\,m}\,\mathcal{V}_{kl}{}^{n}\,,\\ \bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,\mathcal{V}_{ij}{}^{m}\,\mathcal{V}_{kl}{}^{np} &= \tfrac{32}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{npqrs}\big[A_{qrst} + \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{qr}A^{\beta}{}_{st}\big]\,\bar{\mathcal{V}}^{ij\,m}\,\mathcal{V}_{ij}{}^{t}\,,\\ \bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,\mathcal{V}_{ij}{}^{m}\,\mathcal{V}_{kl}{}^{n} &= \tfrac{1}{2}\Delta^{-2/3}g^{mn}\,,\end{aligned} \tag{2.106}$$

where $g^{mn}(x,y)$ is the full (inverse) internal metric, which depends also on the space-time coordinates $x^\mu$ through its dependence on the $5D$ scalar fields, and $\Delta^2 = \det[g(x,y)]/\det[\mathring{g}(y)]$. It is therefore different from the $S^5$ metric $\mathring{g}_{mn}(y)$, unless the $5D$ scalar fields take their background value. We also remind the reader that the generalized vielbeine are pseudo-real so that complex conjugation give $\bar{\mathcal{V}}^{ij\,m} \equiv \big(\mathcal{V}_{ij}{}^{m}\big)^* = \bar{\Omega}^{ik}\,\bar{\Omega}^{jk}\,\mathcal{V}_{kl}{}^{m}$.

Let us first consider the implications from the first and third equations (2.106), which only involve the Killing vectors $K^m{}_{\hat a\hat b}(y)$,

$$\Delta^{-2/3}\,g^{mn}(x,y) = 2\,\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat a\hat b}(x)\,U_{kl}{}^{\hat c\hat d}(x)\;K^m{}_{\hat a\hat b}(y)\,K^n{}_{\hat c\hat d}(y)\,. \tag{2.107}$$

This result is rather generic and was first established in the case of the $11D$ supergravity compactified on $S^7$ [112]. For IIB supergravity compactified on $S^5$ the above result was established in [117] and is in agreement with the recent papers [75,87]. Note that the power of $\Delta$ depends on the dimensions of the space-time and the internal space. From this result it is clear that (2.106) and (2.107) imply that we must adopt the following normalization for the Killing vectors,

$$\sum_{[\hat a\hat b]} K^m{}_{\hat a\hat b}(y)\,K^{n\,\hat a\hat b}(y) = \tfrac{1}{2}\,\mathring{g}^{mn}(y)\,, \tag{2.108}$$

where we used that in the $S^5$ solution, in which all $5D$ scalar fields vanish and consequently $U_{ij}{}^{\hat a\hat b}(x) \equiv \tilde{U}_{ij}{}^{\hat a\hat b}$, we have

$$\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,\tilde{U}_{ij}{}^{\hat a\hat b}\,\tilde{U}_{kl\,\hat c\hat d} = \delta^{\hat a\hat b}_{\hat c\hat d}\,. \tag{2.109}$$

The next step is to study the consequences of the second equation (2.106). Substitution of the generalized vielbeine leads to the equation

$$\begin{aligned}\Delta^{-2/3}\big[A_{mnpq} + \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{[mn}A^{\beta}{}_{p]q}\big] = \tfrac{\sqrt{5}}{432}\,&\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat a\hat b}(x)\,U_{kl}{}^{\hat c\hat d}(x)\\ &\times\mathring{e}\,\varepsilon_{mnptu}\,K^r{}_{\hat a\hat b}(y)\,K^{tu}{}_{\hat c\hat d}(y)\,g_{qr}(x,y)\,.\end{aligned} \tag{2.110}$$

The derivation above follows the same approach as the one in the context of $11D$ supergravity [64], where it gave rise to the non-linear ansatz of the internal components of the $11D$ 3-rank tensor field.

However, at this point, we stumble upon a first inconsistency that forces us to reconsider one of the 'naive' truncation ansatz that we proposed earlier on. Indeed, a puzzle becomes apparent when considering (2.110) in the $S^5$ background solution (2.109). In that case the right-hand side vanishes, which indicates that the background four-form potential $\mathring{A}_{mnpq}(y)$ must vanish. However, this is impossible in view of the fact that its field strength $\mathring{e}\,\varepsilon^{mnpqr}\mathring{F}_{mnpqr} = 120\, m_5$ is non-zero. This was already observed in [103].

To avoid this inconsistency we must trace back our steps to (2.102) and modify the ansatz for $\mathcal{V}_{ij}{}^{mn}$ to

$$\mathcal{V}_{ij}{}^{mn}(x,y) = U_{ij}{}^{\hat a\hat b}(x)\, Z^{mn}{}_{\hat a\hat b}(y)\,, \tag{2.111}$$

where

$$Z^{mn}{}_{\hat a\hat b}(y) = K^{mn}{}_{\hat a\hat b}(y) + \frac{32}{15}\sqrt{5}\,\mathring{e}^{-1}\,\varepsilon^{mnpqr}\,\mathring{A}_{pqrs}(y)\,K^{s}{}_{\hat a\hat b}(y)\,, \tag{2.112}$$

such that (2.110) now takes a form that is consistent with the $S^5$ solution,

$$\begin{aligned}\Delta^{-2/3}\big[A_{mnpq} - \mathring{A}_{mnpq} + \tfrac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{[mn}A^{\beta}{}_{p]q}\big] = \tfrac{\sqrt{5}}{432}\,\bar\Omega^{ik}\,\bar\Omega^{jl}\,U_{ij}{}^{\hat a\hat b}(x)\,U_{kl}{}^{\hat c\hat d}(x)\\ \times\,\mathring{e}\,\varepsilon_{mnptu}\,K^{r}{}_{\hat a\hat b}(y)\,K^{tu}{}_{\hat c\hat d}(y)\,g_{qr}(x,y)\,.\end{aligned} \tag{2.113}$$

Furthermore, the modification in (2.111) implies that (2.98) should be modified in the same manner to preserve the consistency of the truncation:

$$C_{\mu}{}^{mn}(x,y) = Z^{mn}{}_{\hat a\hat b}(y)\,A_{\mu}{}^{\hat a\hat b}(x) \tag{2.114}$$

We observe that the modified ansatz (2.114) is in agreement with [87] (up to normalizations) and that the equation (2.113) agrees with a result presented in [75].

Subsequently we continue to the twelve vector and twelve tensor fields that transform under $\mathrm{SU}(1,1)$, namely $C_{\mu\,\alpha}$, $C_{\mu\,\alpha m}$, $C_{\mu\nu}{}^{\alpha}$ and $C_{\mu\nu}{}^{\alpha m}$, which should be decomposed into the twelve vector and the twelve tensor fields that one expects in the $5D$ gauged supergravity. However, in view of their number, it is not possible to expand these fields in terms of Killing vectors or tensors. Therefore we introduce the $\mathrm{SO}(6)$ vector fields $Y^{\hat a}(y)$ that satisfy $Y^{\hat a}(y)\,Y_{\hat a}(y) = 1$, whose parametrization in terms of the $y^m$ is based on the same $\mathrm{SO}(6)/\mathrm{SO}(5)$ coset representative as all other geometric quantities of $S^5$, such as the metric and the Killing vectors and tensors (see, e.g. [112]). In that case, one can parametrize the remaining vector and tensor fields in terms of the twelve expected $5D$ fields,

$$C_{\mu\,\alpha}(x,y) = Y^{\hat{a}}(y)\, A_{\mu\,\alpha\hat{a}}(x)\,, \tag{2.115}$$
$$C_{\mu\,\alpha m}(x,y) = \partial_m Y^{\hat{a}}(y)\, A_{\mu\,\alpha\hat{a}}(x)\,, \tag{2.116}$$
$$C_{\mu\nu}{}^{\alpha}(x,y) = Y_{\hat{a}}(y)\, B_{\mu\nu}{}^{\alpha\hat{a}}(x)\,, \tag{2.117}$$
$$C_{\mu\nu}{}^{\alpha m}(x,y) = \mathring{g}^{mn}\, \partial_n Y_{\hat{a}}(y)\, B_{\mu\nu}{}^{\alpha\hat{a}}(x)\,. \tag{2.118}$$

We emphasise that, as before, these truncation ansätze are a first approximation. Once again, we close our eyes for the moment, and proceed until some inconsistency forces us to modify these ansätze by the addition of extra terms that depend on $\mathring{A}_{mnpq}(y)$ (as in (2.114)). For consistency of the truncation, a similar decomposition must therefore apply to the generalized vielbeine $\mathcal{V}_{ij\,\alpha}$, $\mathcal{V}_{ij\,\alpha m}$, $\mathcal{V}^{\alpha\,ij}$ and $\mathcal{V}^{\alpha m\,ij}$ which appear in the variation of the above fields,

$$\mathcal{V}_{ij\,\alpha}(x,y) = U_{ij\,\alpha\hat{a}}(x)\, Y^{\hat{a}}(y)\,, \tag{2.119}$$
$$\mathcal{V}_{ij\,\alpha m}(x,y) = U_{ij\,\alpha\hat{a}}(x)\, \partial_m Y^{\hat{a}}(y)\,, \tag{2.120}$$
$$\mathcal{V}^{\alpha\,ij}(x,y) = U^{\alpha\hat{a}\,ij}(x)\, Y_{\hat{a}}(y)\,, \tag{2.121}$$
$$\mathcal{V}^{\alpha m\,ij}(x,y) = U^{\alpha\hat{a}\,ij}(x)\, \mathring{g}^{mn}\, \partial_n Y_{\hat{a}}(y)\,, \tag{2.122}$$

where $U_{ij\,\alpha\hat{a}}(x)$ and $U^{\alpha\hat{a}\,ij}(x)$ are again related to specific components of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space representative that appear in the $5D$ theory. They satisfy

$$U^{\alpha\hat{a}\,ij}(x)\, U_{ij\,\beta\hat{b}}(x) = \delta^{\alpha}{}_{\beta}\, \delta^{\hat{a}}{}_{\hat{b}}\,. \tag{2.123}$$

Finally, we consider the following identities that can be derived for the generalized vielbeine,

$$\begin{aligned} \bar{\Omega}^{ik}\, \bar{\Omega}^{jl}\, \mathcal{V}_{ij}{}^{m}\, \mathcal{V}_{kl\,\alpha n} &= \mathrm{i}\varepsilon_{\alpha\beta} A^{\beta}{}_{np}\, \bar{\mathcal{V}}^{ij\,m}\, \mathcal{V}_{ij}{}^{p}\,, \\ \varepsilon_{\alpha\gamma}\, \Omega_{ik}\, \Omega_{jl}\, \mathcal{V}^{\gamma ij}\, \mathcal{V}^{\beta\,kl} &= \tfrac{5}{4}\Delta^{-4/3}\big(\delta_{\alpha}{}^{\beta} - 2\,\phi_{\alpha}\phi^{\beta}\big)\,. \end{aligned} \tag{2.124}$$

From these identities we can obtain the following results upon substitution of the adequate truncation ansätze,

$$\begin{aligned} \Delta^{-2/3}\, A^{\alpha}{}_{mn} &= 2\mathrm{i}\, \varepsilon^{\alpha\beta}\, \bar{\Omega}^{ik}\, \bar{\Omega}^{jl}\, U_{ij}{}^{\hat{a}\hat{b}}(x)\, U_{kl\,\beta\hat{c}}(x)\, K^{p}{}_{\hat{a}\hat{b}}(y)\, g_{p[m}(x,y)\, \partial_{n]} Y^{\hat{c}}(y)\,, \\ \Delta^{-4/3}\big(\delta_{\alpha}{}^{\beta} - 2\,\phi_{\alpha}\phi^{\beta}\big) &= \tfrac{4}{5}\varepsilon_{\alpha\gamma}\, \Omega_{ik}\, \Omega_{jl}\, U^{\gamma\hat{a}\,ij}(x)\, U^{\beta\hat{b}\,kl}(x)\, Y_{\hat{a}}(y)\, Y_{\hat{b}}(y)\,. \end{aligned} \tag{2.125}$$

The first result has recently been derived based on generalized geometry [75], while the second one has been obtained long ago (under some mild assumptions) in [117] by using the same strategy as in this section. Note that no further modifications of the ansätze (2.116) and (2.117) (and hence of (2.120) and (2.121)) were necessary at this point. This is so far consistent with the findings of [87].

We end the analysis here but it is clear that we have only probed part of the possible identities that can be derived on the basis of this approach. The consistency of the identities presented in this section is only enough to confirm the ansätze (2.97), (2.114), (2.116) and (2.117) (and the associated ones for the generalised vielbeine). In principle, this program should be further carried out by establishing the mutual consistency of all possible equations quadratic in the generalized vielbeine. For some of the generalized vielbeine, this would reveal the need for modifications of their truncation ansätze along the lines of (2.111). Alternatively, the need for making such modifications would follow from requiring the various $S^5$ geometric quantities to transform covariantly under SO(6). In the end, the results should be compared with [87] where the complete truncation ansätze for all the fields (including the $5D$ three- and four-forms) have been worked out.

# Chapter 3

# Deformations of Exceptional Field Theory

This chapter deals with the duality covariant framework of exceptional field theory. We show that the generalised Lie derivative admits consistent deformations which, in particular, allow for the description of massive type IIA supergravity as a geometric solution of the section constraint. This chapter is directly based on [118].

## 3.1 Introduction

Exceptional field theory (EFT) provides a unified framework where to describe massless type II and eleven-dimensional supergravity [68, 80–85]. In EFT, covariance with respect to the $E_{n(n)}$ duality symmetry of maximal supergravity in $D = (11-n)$ dimensions is made manifest by adding extra coordinates to the ten- or eleven-dimensional spacetime. Consistency of the theory eventually requires to impose a section constraint which restricts all fields to depend, at most, on ten or eleven physical coordinates. After solving the section constraint, EFT reduces (locally in a coordinate patch) to an exceptional generalised geometry (EGG) formulation of massless type II or eleven-dimensional supergravities [73, 74]. In view of the unifying abilities of EFT, it is natural to ask whether there exist consistent modifications of the EFT framework that would also allow Roman's massive type IIA supergravity [27] to arise as a solution of the section constraint.

The Romans mass $m_{\mathrm{R}}$ has always manifested itself as a deformation parameter in any construction related to type IIA supergravity. This is the case, for instance, for the supersymmetric AdS vacua of [119–121]. When considering dual holographic models, the Romans mass translates into a deformation of the field theory in the form of a Chern-Simons term with level $k$ given by $k/(2\pi l_s) = m_{\mathrm{R}}$ [122–124]. A more recent example involves the consistent truncation of the massive IIA theory on the six-sphere [123, 125, 126]. In this case it was shown that, after truncation to four dimensions, the Romans mass appears as an electric-magnetic deformation parameter of the types constructed in [127, 128]. These results suggest that, in order to embed the massive IIA theory in EFT, one should investigate the possible deformations of the latter.

In this chapter, we show that EFT admits consistent deformations which still allow for ten- and/or eleven-dimensional solutions of the section constraint. For one of these deformations, there exists a purely geometric ten-dimensional solution which precisely corresponds to massive IIA supergravity, and thus defines as a byproduct the associated EGG. This new deformed EFT framework endows massive IIA supergravity with the same geometrical and group-theoretical tools that were so far exclusive to the massless theories.

$\mathrm{E}_{n(n)}$ EFT is based on an external spacetime and an internal extended space which are respectively parametrised by the coordinates $x^\mu$ and $y^{\mathcal{M}}$, where $\mu = 0, \ldots, D-1$, $\mathcal{M} = 1, \ldots, \dim \mathcal{R}_v$ and where the $\mathcal{R}_v$ representation of $\mathrm{E}_{n(n)}$ is reminded for convenience in Table 3.1. Internal generalised diffeomorphisms act on fields by means of a generalised Lie derivative $\mathbb{L}_\Lambda$. While all fields and parameters formally depend on the full set of coordinates $(x^\mu, y^{\mathcal{M}})$, the dependence on the internal coordinates is ultimately restricted to a physical subset by the section constraint

$$Y^{\mathcal{PQ}}{}_{\mathcal{MN}}\, \partial_{\mathcal{P}} \otimes \partial_{\mathcal{Q}} = 0\,, \tag{3.1}$$

where $\partial_{\mathcal{M}} \equiv \frac{\partial}{\partial y^{\mathcal{M}}}$ and $Y^{\mathcal{PQ}}{}_{\mathcal{MN}}$ is a specific $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ invariant tensor [94]. After choosing a maximal solution of this constraint, EFT effectively reduces to eleven-dimensional or type IIB supergravity in a $D+n$ or $D+(n-1)$ dimensional split, respectively. Such a split of the physical coordinates into the $D$-dimensional external spacetime and the $n$- or $(n-1)$-dimensional internal space explicitly breaks the Lorentz covariance of the eleven- or ten-dimensional theory but does not truncate any of its degrees of freedom. The generalised Lie derivative then encodes the ordinary internal diffeomorphisms and $p$-form gauge transformations of the physical theory in the corresponding dimensional split.

| $D$ | 9 | 8 | 7 | 6 | 5 | 4 |
|---|---|---|---|---|---|---|
| $\mathrm{E}_{n(n)}$ | $\mathrm{SL}(2)\times\mathbb{R}^+$ | $\mathrm{SL}(2)\times\mathrm{SL}(3)$ | $\mathrm{SL}(5)$ | $\mathrm{SO}(5,5)$ | $\mathrm{E}_{6(6)}$ | $\mathrm{E}_{7(7)}$ |
| $\mathcal{R}_v$ | $\mathbf{2}_3+\mathbf{1}_{-4}$ | $(\mathbf{2},\bar{\mathbf{3}})$ | $\overline{\mathbf{10}}$ | $\mathbf{16}_c$ | $\overline{\mathbf{27}}$ | $\mathbf{56}$ |
| $\mathcal{R}_X$ | $\mathbf{2}_{-3}+\mathbf{3}_4$ | $(\mathbf{2},\mathbf{3})+(\mathbf{2},\bar{\mathbf{6}})$ | $\mathbf{15}+\overline{\mathbf{40}}$ | $\mathbf{144}_c$ | $\overline{\mathbf{351}}$ | $\mathbf{912}$ |

Table 3.1: Relevant $\mathrm{E}_{n(n)}$ representations for the internal generalised coordinates $y^{\mathcal{M}}$ and for the $X$ deformation.

The construction of '$X$ deformed' exceptional field theories (XFT's) entirely relies on the following modification of the ordinary generalised Lie derivative

$$\widetilde{\mathbb{L}}_\Lambda = \mathbb{L}_\Lambda + \Lambda^{\mathcal{M}} X_{\mathcal{M}}\,, \tag{3.2}$$

where $X_{\mathcal{M}}$ is $\mathrm{E}_{n(n)}$ Lie algebra valued. In particular, it takes the form $(X_{\mathcal{M}})_N{}^P \equiv X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ when acting on a field in the $\mathcal{R}_v$ representation. Closure of the deformed generalised Lie derivative (3.2) and consistency of the EFT tensor hierarchy require $X$ to be restricted to the $\mathcal{R}_X$ representation of $\mathrm{E}_{n(n)}$ (see Table 3.1) and to satisfy a quadratic constraint

$$X_{\mathcal{M}\mathcal{P}}{}^{\mathcal{R}} X_{\mathcal{N}\mathcal{R}}{}^{\mathcal{Q}} - X_{\mathcal{N}\mathcal{P}}{}^{\mathcal{R}} X_{\mathcal{M}\mathcal{R}}{}^{\mathcal{Q}} + X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{R}} X_{\mathcal{R}\mathcal{P}}{}^{\mathcal{Q}} = 0\,. \tag{3.3}$$

in analogy with the constraints (1.17) and (1.14) appearing in gauged maximal supergravity [42–46,54,129]. Furthermore, an additional constraint involving both $X$ and $\partial_{\mathcal{M}}$ must be imposed

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\,\partial_{\mathcal{P}} \;=\; 0\,. \tag{3.4}$$

This so-called $X$-constraint can be interpreted as a compatibility condition between the $X$ deformation and the $y^{\mathcal{M}}$ dependence of the fields and gauge parameters. Together with the section constraint (3.1), these conditions guarantee the consistency of the internal generalised diffeomorphisms algebra and, ultimately, of the whole XFT.

For specific choices of $X$, (3.4) is still compatible with solutions of the section constraint (3.1) that preserve $n$ or $(n-1)$ internal coordinates. The resulting XFT's ultimately describe three types of eleven- and ten-dimensional maximal supergravities:

- 11-dimensional and massless type IIA supergravities with background fluxes.

- Type IIB supergravity with background fluxes.
- Massive type IIA supergravity with background fluxes.

The latter case is a genuine result of XFT. Indeed, the massive IIA supergravity, which cannot be described in EFT without violating the section constraint, now admits a geometric description using the XFT framework. The fluxes mentioned above simply correspond to constant background values for the internal components of certain field strengths in the corresponding supergravity theory.[1] It is therefore not surprising that the background fluxes can be reabsorbed in the dynamical fields of the theory without violating the section constraint. This is however not possible for the Romans mass, since there exists no corresponding 'zero-form field strength' in which it could be reabsorbed. As a result, the EGG of type IIA supergravity admits two inequivalent generalised Lie derivatives corresponding to the massless and massive theories, respectively[2].

The outline of the chapter is as follows. In section 3.2 we review the generic features of EFT and its generalised Lie derivative. Subsequently, we reproduce explicitly the internal gauge transformations of the massless IIA theory in a $7+3$ split by choosing the appropriate solution of the section constraint in SL(5) EFT. We then argue that a deformation of the generalised Lie derivative is needed in order to account for the modifications to the gauge transformations in the massive case. In section 3.3 we present the general properties of the $X$ deformation and show that it can accommodate the Romans mass parameter. We also classify the deformations of the SL(5) EFT that remain compatible with ten- or eleven-dimensional solutions of the section- and $X$-constraints. Finally, in section 3.4, we derive the tensor hierarchy and the bosonic action for the $\mathrm{E}_{7(7)}$ XFT.

## 3.2 Exceptional field theory and type IIA supergravity

As discussed previously, EFT's embed the eleven-dimensional and massless type II supergravities in a unified framework which captures their underlying exceptional generalised geometries, and where the exceptional duality symmetries of

[1] Here we insist that these background values are constant over the whole internal space. The dynamical part of all fields still carries an arbitrary dependence on the eleven or ten physical coordinates.

[2] A detailed construction of the exceptional generalised geometry for massive IIA supergravity can be found in [130].

the lower-dimensional theories are manifest. More concretely, the spacetime of eleven-dimensional supergravity is decomposed into a $D$ dimensional 'external' spacetime and an $n = 11 - D$ dimensional 'internal' space, without performing any truncation of degrees of freedom. Note that this procedure was described in detail in chapter 2 for the case of a $5 + 5$ split of type IIB supergravity. The internal diffeomorphisms are then extended to generalised diffeomorphisms which also account for the internal gauge transformations of the three- and dual six-form potentials (and of the dual graviton in $D = 4, 3$). A similar situation occurs for the $D+(n-1)$ dimensional split of the massless type II supergravities. The set of internal coordinates is then extended to $y^{\mathcal{M}}$, with $\mathcal{M} = 1, \ldots, \dim \mathcal{R}_v$, such that the coordinates transform linearly under $\mathrm{E}_{n(n)}$ and can be regarded as conjugate to the internal momenta and half-BPS charges of the theory [89, 131]. A section constraint must be imposed for consistency, and restricts the coordinate dependence of all fields and gauge parameters to a subset of the internal coordinates. As long as one does not commit to a specific solution of this constraint, EFT can be regarded as being (formally) invariant under rigid $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ transformations. The embedding of the original ten- and eleven-dimensional supergravities is recovered by choosing the appropriate solution of the section constraint. The generalised Lie derivative and other structures in EFT then reproduce (locally) the exceptional generalised geometry associated with the corresponding supergravity theory where the rigid $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ symmetry is broken. This symmetry only subsists if one solves the section constraint by choosing all the fields to be independent of $y^{\mathcal{M}}$. In this case, EFT reduces to ungauged maximal supergravity in $D$ dimensions and the $\mathbb{R}^+$ factor corresponds to the trombone symmetry (cf. 1.1.1).

In their latest formulations, EFT's have been constructed in any $D \geq 3$ [68, 80–85] following a prescription that mimics the embedding tensor formulation of maximal supergravities in the corresponding dimension [42–46, 54, 129]. In both the EFT's and the maximal supergravities, internal and spacetime symmetries completely specify the field content as well as its interactions in an elegant and unambiguous manner. In this section we introduce the basics of the EFT's which we will extensively use later on. We will focus on $D \geq 4$ throughout this chapter.

### 3.2.1 Generalised diffeomorphisms

EFT fields depend on spacetime coordinates $x^\mu$, $\mu = 0, \ldots, D - 1$, and extended internal coordinates $y^{\mathcal{M}}$. The fields and gauge parameters of the theory are arranged in objects that transform consistently under a set of internal generalised diffeomorphisms. On covariant objects generalised diffeomorphisms act with a generalised Lie derivative $\mathbb{L}_\Lambda$. For instance, the action of $\mathbb{L}_\Lambda$ on a $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ vector $U^{\mathcal{M}}$ of weight $\lambda(U) = \lambda_U$ reads[3]

$$\mathbb{L}_\Lambda U^{\mathcal{M}} = \Lambda^{\mathcal{N}}\partial_{\mathcal{N}}U^{\mathcal{M}} - U^{\mathcal{N}}\partial_{\mathcal{N}}\Lambda^{\mathcal{M}} + Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\,\partial_{\mathcal{N}}\Lambda^{\mathcal{P}}\,U^{\mathcal{Q}} + (\lambda_U - \omega)\,\partial_{\mathcal{P}}\Lambda^{\mathcal{P}}U^{\mathcal{M}}, \tag{3.5}$$

where $\Lambda^{\mathcal{M}}(x,y)$ is the gauge parameter, $Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}$ is a specific, constant $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ invariant tensor (so that $\delta_\Lambda Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} = \mathbb{L}_\Lambda Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}} = 0$), and $\omega = 1/(D-2)$. All parameters of generalised diffeomorphisms carry weight $\omega$.

Consistency of the generalised diffeomorphisms requires the algebra of the generalised Lie derivative to close, namely

$$[\mathbb{L}_\Lambda, \mathbb{L}_\Sigma]W^{\mathcal{M}} = \mathbb{L}_{[\Lambda,\Sigma]_{\mathrm{E}}}W^{\mathcal{M}}\,, \tag{3.6}$$

where the so-called E-bracket for parameters $\Lambda$ and $\Sigma$ is defined as

$$\begin{aligned}
[\Lambda,\Sigma]_{\mathrm{E}}^{\mathcal{M}} \equiv \frac{1}{2}\left(\mathbb{L}_\Lambda\Sigma^{\mathcal{M}} - \mathbb{L}_\Sigma\Lambda^{\mathcal{M}}\right) &= \Lambda^{\mathcal{P}}\partial_{\mathcal{P}}\Sigma^{\mathcal{M}} + \frac{1}{2}\,Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\,\partial_{\mathcal{N}}\Lambda^{\mathcal{P}}\,\Sigma^{\mathcal{Q}} \\
&\quad - (\Lambda \leftrightarrow \Sigma)\,.
\end{aligned} \tag{3.7}$$

The requirement (3.6) translates into a set of conditions [94] which severely restricts the dependence of the fields and parameters in the EFT on the generalised coordinates:

$$\begin{aligned}
&Y^{\mathcal{P}\mathcal{Q}}{}_{\mathcal{M}\mathcal{N}}\,\partial_{\mathcal{P}} \otimes \partial_{\mathcal{Q}} = 0\,,\\
&\left(Y^{\mathcal{M}(\mathcal{P}}{}_{\mathcal{T}\mathcal{Q}}\,Y^{\mathcal{T}|\mathcal{N})}{}_{\mathcal{R}\mathcal{S}} - Y^{\mathcal{M}(\mathcal{P}}{}_{\mathcal{R}\mathcal{S}}\,\delta^{\mathcal{N})}_{\mathcal{Q}}\right)\,(\partial_{\mathcal{P}}\partial_{\mathcal{N}}) = 0\,,\\
&\Big(Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{T}\mathcal{Q}}\,Y^{\mathcal{T}\mathcal{P}}{}_{[\mathcal{S}\mathcal{R}]} + 2\,Y^{\mathcal{M}\mathcal{N}}{}_{[\mathcal{R}|\mathcal{T}|}\,Y^{\mathcal{T}\mathcal{P}}{}_{\mathcal{S}]\mathcal{Q}}\\
&\qquad\qquad - Y^{\mathcal{M}\mathcal{N}}{}_{[\mathcal{R}\mathcal{S}]}\,\delta^{\mathcal{P}}_{\mathcal{Q}} - 2\,Y^{\mathcal{M}\mathcal{N}}{}_{[\mathcal{S}|\mathcal{Q}|}\,\delta^{\mathcal{P}}_{\mathcal{R}]}\Big)\;\partial_{(\mathcal{N}} \otimes \partial_{\mathcal{P})} = 0\,,\\
&\Big(Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{T}\mathcal{Q}}\,Y^{\mathcal{T}\mathcal{P}}{}_{(\mathcal{S}\mathcal{R})} + 2\,Y^{\mathcal{M}\mathcal{N}}{}_{(\mathcal{R}|\mathcal{T}|}\,Y^{\mathcal{T}\mathcal{P}}{}_{\mathcal{S})\mathcal{Q}}\\
&\qquad\qquad - Y^{\mathcal{M}\mathcal{N}}{}_{(\mathcal{R}\mathcal{S})}\,\delta^{\mathcal{P}}_{\mathcal{Q}} - 2\,Y^{\mathcal{M}\mathcal{N}}{}_{(\mathcal{S}|\mathcal{Q}|}\,\delta^{\mathcal{P}}_{\mathcal{R})}\Big)\;\partial_{[\mathcal{N}} \otimes \partial_{\mathcal{P}]} = 0\,.
\end{aligned} \tag{3.8}$$

The first condition in (3.8) is usually referred to as the section constraint. We will always impose that it holds on any combination of fields and/or parameters, including derivatives and products. As a result, the section constraint restricts all objects in the EFT to depend only on a subset of the internal coordinates. In

[3] The transformation rule for a covariant tensor $V_M$ is deduced by requiring that the contraction $S = U^{\mathcal{M}}V_{\mathcal{M}}$ transforms as a scalar density of weight $\lambda_U + \lambda_V$. The transformation rule for tensors follows immediately.

all the $\mathrm{E}_{n(n)}$ EFT's, the other equations in (3.8) are automatically satisfied if the section constraint is imposed [94].

The E-bracket in (3.7) fails to satisfy the Jacobi identity:

$$\big[[\Lambda,\Sigma]_{\mathrm{E}},\Gamma\big]_{\mathrm{E}}+\text{cycl.}=\frac{1}{3}\big\{[\Lambda,\Sigma]_{\mathrm{E}},\Gamma\big\}_{\mathrm{E}}+\text{cycl.}\,. \tag{3.9}$$

This fact plays a central role in the construction of EFT's, as it requires the introduction of a hierarchy of $p$-form fields and gauge transformations [68,78,80–82] similar to the one of gauged supergravities [55, 56], in order to guarantee the invariance of the field equations under generalised diffeomorphisms. For vectors of weight $\omega$, one finds that the symmetric bracket $\{\Lambda,\Sigma\}_{\mathrm{E}}$, reads

$$\big\{\Lambda,\Sigma\big\}_{\mathrm{E}}^{\mathcal{M}}\;=\;\frac{1}{2}\left(\mathbb{L}_{\Lambda}\Sigma^{\mathcal{M}}+\mathbb{L}_{\Sigma}\Lambda^{\mathcal{M}}\right)\;=\;\frac{1}{2}\,Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\left[\Sigma^{\mathcal{Q}}\partial_{\mathcal{N}}\Lambda^{\mathcal{P}}+\Lambda^{\mathcal{Q}}\partial_{\mathcal{N}}\Sigma^{\mathcal{P}}\right], \tag{3.10}$$

so that $\mathbb{L}_{\Lambda}\Sigma^{\mathcal{M}}=[\Lambda,\Sigma]_{\mathrm{E}}^{\mathcal{M}}+\big\{\Lambda,\Sigma\big\}_{\mathrm{E}}^{\mathcal{M}}$. Consistency of the tensor hierarchy in EFT then follows from the fact that, upon using the section constraint, $\{\Lambda,\Sigma\}_{\mathrm{E}}$ is a trivial gauge parameter, namely, that $\mathbb{L}_{\{\Lambda,\Sigma\}_{\mathrm{E}}}$ vanishes identically.

Covariance under internal generalised diffeomorphisms with parameters that depend on spacetime coordinates $x^{\mu}$ requires the introduction of appropriate covariant derivatives and associated connections [78]

$$\partial_{\mu}\;\to\;\mathcal{D}_{\mu}\equiv\partial_{\mu}-\mathbb{L}_{A_{\mu}}\,, \tag{3.11}$$

where $A_{\mu}{}^{\mathcal{M}}(x,y)$ are the vector fields of EFT. The requirement that $\mathcal{D}_{\mu}$ is covariant fixes the transformation properties of $A_{\mu}{}^{\mathcal{M}}$ up to the addition of trivial gauge parameters. It is customary to choose

$$\delta_{\Lambda}A_{\mu}{}^{\mathcal{M}}=\mathcal{D}_{\mu}\Lambda^{\mathcal{M}}=\partial_{\mu}\Lambda^{\mathcal{M}}-\mathbb{L}_{A_{\mu}}\Lambda^{\mathcal{M}}\,. \tag{3.12}$$

Making use of the fact that $\{\Lambda,A_{\mu}\}_{\mathrm{E}}^{\mathcal{M}}$ is a trivial parameter, we can also give a different expression for $\delta_{\Lambda}A_{\mu}{}^{\mathcal{M}}$ which will be convenient in the following section:

$$\delta_{\Lambda}A_{\mu}{}^{\mathcal{M}}=\partial_{\mu}\Lambda^{\mathcal{M}}+\mathbb{L}_{\Lambda}A_{\mu}{}^{\mathcal{M}}\,. \tag{3.13}$$

The difference between any two choices of $\delta_{\Lambda}A_{\mu}{}^{\mathcal{M}}$ is absorbed into the gauge transformations associated with the two-forms of the EFT tensor hierarchy. This is again analogous to the situation in gauged supergravity. The specifics of the tensor hierarchy in EFT depend on the dimension $D$, although a systematic treatment has been recently developed in [132]. We will discuss the $D = 4$ case thoroughly in section 3.4.

### 3.2.2 Massless IIA gauge transformations from EFT

In order to make contact with the eleven-dimensional and massless type II supergravities, it is necessary to pick a specific solution of the section constraint in (3.8). As a first step towards the implementation of the Romans mass as a deformation parameter in EFT, we first briefly exemplify how to recover the gauge transformations of ten-dimensional massless IIA supergravity from those of EFT.

Type IIA supergravity is the maximally supersymmetric non-chiral theory in ten dimensions. Its field content differs from the one of the type IIB theory discussed in chapter 2. In addition to the zehnbein, a dilaton, a Kalb-Ramond two-form $A_{MN}$ as well as two gravitini (and two dilatini) of different chirality, type IIA supergravity involves Ramond-Ramond (RR) $p$-form potentials of odd rank. The latter possess gauge transformations which are deformed by the Romans mass, $m_{\rm R}$, in the massive version of the IIA theory [27].

Let us then start by introducing the massless gauge transformations of the IIA ten-dimensional $p$-form potentials $A_M$ , $A_{MN}$ and $A_{MNP}$ . These are specified by gauge parameters $\lambda$ , $\Xi_M$ and $\theta_{NP} = -\theta_{PN}$ , where $M$, $N$, . . . are ten-dimensional spacetime indices, and take the form (we follow the conventions of [133])

$$\delta A_M = \partial_M \lambda\,, \quad \delta A_{MN} = 2\,\partial_{[M}\,\Xi_{N]}\,, \quad \delta A_{MNP} = 3\,\partial_{[M}\,\theta_{NP]} - 3\,A_{[MN}\,\partial_{P]}\lambda\,. \tag{3.14}$$

In this example, we consider a 7+3 dimensional split of the fields and parameters of the ten-dimensional type IIA supergravity. The $p$-forms of type IIA supergravity are decomposed in $7D$ scalars, $7D$ vectors and so on. As in chapter 3, appropriate Kaluza-Klein (KK) redefinitions are needed to ensure their covariance under the seven-dimensional external diffeomorphisms. Importantly, all fields and gauge parameters still depend on the ten-dimensional coordinates $x^M = (x^\mu\,, y^\alpha)$ with $\mu = 0, \ldots, 6$ and $\alpha = 1, 2, 3$ . The redefined $7D$ vectors originating from the ten-dimensional $p$-form potentials take the form

$$A^{\rm KK}_\mu = A_\mu - B_\mu{}^\delta\,A_\delta\,, \quad A^{\rm KK}_{\mu\beta} = A_{\mu\beta} - B_\mu{}^\delta\,A_{\delta\beta}\,, \quad A^{\rm KK}_{\mu\beta\gamma} = A_{\mu\beta\gamma} - B_\mu{}^\delta\,A_{\delta\beta\gamma}\,, \tag{3.15}$$

where $B_\mu{}^\alpha$ are the KK vector fields coming from the ten-dimensional metric. It is convenient to perform a second set of non-linear redefinitions[4]

$$C_\mu = A^{\rm KK}_\mu\,, \qquad C_{\mu\beta} = A^{\rm KK}_{\mu\beta} \qquad \text{and} \qquad C_{\mu\beta\gamma} = A^{\rm KK}_{\mu\beta\gamma} + A^{\rm KK}_\mu\, A_{\beta\gamma}\,. \tag{3.16}$$

After some algebra manipulations it can be shown that, under (3.14), these vectors transform as follows under internal diffeomorphisms with parameter $\xi^\alpha$ and internal gauge transformations with parameters $\lambda\,,\ \Xi_\alpha\,,\ \theta_{\alpha\beta}$ :

$$\begin{aligned}
\delta B_\mu{}^\alpha =& \,(\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\xi^\alpha + \xi^\delta\,\partial_\delta B_\mu{}^\alpha\,,\\
\delta C_\mu =& \,\xi^\delta\,\partial_\delta C_\mu + (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\lambda\,,\\
\delta C_{\mu\beta} =& \,\xi^\delta\,\partial_\delta C_{\mu\beta} + C_{\mu\delta}\,\partial_\beta \xi^\delta + (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\Xi_\beta + B_\mu{}^\delta\,\partial_\beta\,\Xi_\delta\,,\\
\delta C_{\mu\beta\gamma} =& \,\xi^\delta\,\partial_\delta C_{\mu\beta\gamma} + 2\,C_{\mu\delta[\gamma}\,\partial_{\beta]}\xi^\delta + (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\theta_{\beta\gamma} + 2\,B_\mu{}^\delta\,\partial_{[\beta|}\,\theta_{\delta|\gamma]}\\
&+ 2\,C_\mu\,\partial_{[\beta}\Xi_{\gamma]} - 2\,C_{\mu[\beta}\,\partial_{\gamma]}\lambda\,.
\end{aligned} \tag{3.17}$$

The $7+3$ dimensional split we have adopted to describe the massless IIA supergravity can be compared with the $D=7$ EFT, based on $\mathrm{E}_{4(4)} \equiv \mathrm{SL}(5)$ [84,93,95]. Analogous comparisons can be performed for other $D+(n-1)$ dimensional splits. The SL(5) EFT is characterised by generalised vectors $\Lambda^{\mathcal{M}}$ in the $\overline{\mathbf{10}}$ representation,
textiti.e. $\Lambda^{mn} = -\Lambda^{nm}$ , with $m = 1, ..., 5$ being a fundamental SL(5) index. The structure tensor of the SL(5) EFT is given by[5]

$$Y^{mn\,pq}{}_{rs\,tu} = \epsilon^{mnpqz}\,\epsilon_{rstuz}\,, \tag{3.18}$$

and the section constraint reduces to

$$\epsilon^{mnpqz}\,\partial_{mn} \otimes \partial_{pq} = 0\,. \tag{3.19}$$

There are two inequivalent solutions of (3.19) (up to SL(5) transformations [95]) corresponding to eleven-dimensional supergravity (which will be referred to as

[4]Similar redefinitions were discussed for the IIB case in chapter 2, and more generally, in [67,103,125].

[5]The entries in $Y^{mn\,pq}{}_{rs\,tu}$ are $0, \pm 1$ . Therefore, whenever an index pair $mn$ is contracted, a factor of $\frac{1}{2}$ must be explicitly included.

M-theory for conciseness) and type IIB supergravity:

$$\begin{aligned} &\text{M-theory:} && \partial_{\alpha 4} \neq 0\,,\ \partial_{45} \neq 0 \ \text{ and } \ \partial_{\alpha 5} = \partial_{\alpha\beta} = 0\,, \\ &\text{type IIB:} && \partial_{\alpha\beta} \neq 0 \ \text{ and } \ \partial_{\alpha 4} = \partial_{\alpha 5} = \partial_{45} = 0\,. \end{aligned} \tag{3.20}$$

The massless IIA case is obtained by further restricting to only three coordinates in the M-theory solution. We will set $\partial_{45} = 0$.

The SL(5) EFT contains $\overline{\mathbf{10}}$ vector fields $A_\mu{}^{\mathcal{M}} \equiv A_\mu{}^{mn}$ that transform under a generalised diffeomorphism as in (3.13). Using the massless IIA solution of the section constraint ($\partial_{\alpha 4} \neq 0$), we can identify the field content and gauge parameters of the supergravity theory with those of the EFT:

$$\begin{aligned} A_\mu{}^{mn} = (\, A_\mu{}^{\alpha 5}\,,\, A_\mu{}^{\alpha 4}\,,\, A_\mu{}^{\alpha\beta}\,,\, A_\mu{}^{45}\,) \ &= \ (\, \tfrac{1}{2}\, \epsilon^{\alpha\beta\gamma}\, C_{\mu\beta\gamma}\,,\, B_\mu{}^{\alpha}\,,\, \epsilon^{\alpha\beta\gamma}\, C_{\mu\gamma}\,,\, C_\mu\,)\,, \\ \Lambda^{mn} = (\, \Lambda^{\alpha 5}\,,\, \Lambda^{\alpha 4}\,,\, \Lambda^{\alpha\beta}\,,\, \Lambda^{45}\,) \ &= \ (\, \tfrac{1}{2}\, \epsilon^{\alpha\beta\gamma}\, \theta_{\beta\gamma}\,,\, \xi^{\alpha}\,,\, \epsilon^{\alpha\beta\gamma}\, \Xi_\gamma\,,\, \lambda\,)\,, \\ \partial_{mn} = (\, \partial_{\alpha 5}\,,\, \partial_{\alpha 4}\,,\, \partial_{\alpha\beta}\,,\, \partial_{45}\,) \ &= \ (\, 0\,,\, \partial_\alpha\,,\, 0\,,\, 0\,,\, 0\,)\,. \end{aligned} \tag{3.21}$$

After imposing the massless IIA solution of the section constraint, an explicit computation of the vector field transformations directly from (3.13) reproduces (3.17). A similar analysis can be repeated for the other types, such as the $7D$ scalars or the $7D$ two- and three-forms. However, the vector gauge transformations are enough for our purposes in the next section.

### 3.2.3 Massive IIA gauge transformations from a deformed Lie derivative

Let us now look at the gauge transformations of the ten-dimensional massive IIA theory. They are modifications of the massless expressions (3.14), and they read

$$\begin{aligned} \delta A_M &= \partial_M \lambda + m_{\mathrm{R}}\, \Xi_M\,, \\ \delta A_{MN} &= 2\, \partial_{[M}\, \Xi_{N]}\,, \\ \delta A_{MNP} &= 3\, \partial_{[M}\, \theta_{NP]} - 3\, A_{[MN}\, \partial_{P]} \lambda - m_{\mathrm{R}}\, A_{[MN}\, \Xi_{P]}\,. \end{aligned} \tag{3.22}$$

where $m_{\rm R}$ denotes the constant Romans mass. In the $7+3$ dimensional split, and after performing the field redefinitions (3.15) and (3.16), the internal diffeomorphisms and gauge transformations become

$$\begin{aligned}
\delta B_\mu{}^\alpha &= (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\xi^\alpha + \xi^\delta\,\partial_\delta B_\mu{}^\alpha\,,\\
\delta C_\mu &= \xi^\delta\,\partial_\delta C_\mu + (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\lambda - m_{\rm R}\,B_\mu{}^\delta\,\Xi_\delta\,,\\
\delta C_{\mu\beta} &= \xi^\delta\,\partial_\delta C_{\mu\beta} + C_{\mu\delta}\,\partial_\beta\xi^\delta + (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\Xi_\beta + B_\mu{}^\delta\,\partial_\beta\,\Xi_\delta\,,\\
\delta C_{\mu\beta\gamma} &= \xi^\delta\,\partial_\delta C_{\mu\beta\gamma} + 2\,C_{\mu\delta[\gamma}\,\partial_{\beta]}\xi^\delta + (\partial_\mu - B_\mu{}^\delta\,\partial_\delta)\,\theta_{\beta\gamma} + 2\,B_\mu{}^\delta\,\partial_{[\beta|}\,\theta_{\delta|\gamma]}\\
&\quad + 2\,C_\mu\,\partial_{[\beta}\Xi_{\gamma]} - 2\,C_{\mu[\beta}\,\partial_{\gamma]}\lambda - 2\,m_{\rm R}\,C_{\mu[\beta}\,\Xi_{\gamma]}\,.
\end{aligned} \tag{3.23}$$

Note that the extra terms in (3.23) compared to (3.17) do not contain internal derivatives. This poses an obstruction to recovering such variations from a standard EFT/generalised geometry Lie derivative like (3.5), whose terms always contain derivatives of either the gauge parameter or the field it acts on. However, the fact that massive IIA supergravity is a geometrically well-defined theory means that it should admit an exceptional generalised geometry description. This suggests that the solution to the above obstruction is to implement $m_{\rm R}$ as a deformation of $\mathbb{L}_\Lambda$, and therefore modify the notion of covariance in the exceptional generalised geometry associated with type IIA supergravity. The procedure that we follow in order to identify the correct deformation is the inverse of the one presented in the previous section. We still use the dictionary (3.21) for the SL(5) EFT, but we now repackage (3.23) into an expression

$$\delta_\Lambda A_\mu{}^{mn} = \partial_\mu \Lambda^{mn} + \widetilde{\mathbb{L}}_\Lambda A_\mu{}^{mn}\,, \tag{3.24}$$

where $\widetilde{\mathbb{L}}_\Lambda$ accounts for $m_{\rm R}$ and reduces to the standard EFT Lie derivative in the limit $m_{\rm R}\to 0$. We stress that vector fields transform faithfully under internal generalised diffeomorphisms, so that by covariance this procedure uniquely identifies the deformation induced by $m_{\rm R}$ on all the every other field. The resulting deformed Lie derivative reads

$$\widetilde{\mathbb{L}}_\Lambda A_\mu{}^{mn} = \mathbb{L}_\Lambda A_\mu{}^{mn} - X_{pq\,rs}{}^{mn}\,\Lambda^{pq}\,A_\mu{}^{rs}\,, \tag{3.25}$$

where the second term in the right-hand side is specified by an $X$ deformation of the form

$$X_{mn\,pq}{}^{rs} = 2\,X_{mn\,[p}{}^{[r}\,\delta^{s]}_{q]}\;, \tag{3.26}$$

with non-vanishing entries given by

$$X_{\alpha\beta\,\gamma}{}^{5} = -2\,m_{\rm R}\,\epsilon_{\alpha\beta\gamma}\;, \tag{3.27}$$

and where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol in three dimensions with $\epsilon_{123} = +1$. Note at this point that equations (3.26) and (3.27) correspond to the embedding tensor of the gauged maximal supergravity induced by a three-torus compactification of massive IIA supergravity[6].

Consistency requirements, like closure of $\widetilde{\mathbb{L}}$, will follow from consistency of the original massive IIA theory, at least as long as we restrict ourselves to the solution of the section constraint that corresponds to the type IIA theory. As we shall see, however, the new structures unveiled in this section can be immediately generalised to other dimensions as well as to generic $X$ deformations. Therefore we will discuss consistency of the deformed EFT's in a more general setting in the next section, and later come back to the case of the Romans mass.

## 3.3 Deformations of exceptional field theory

Motivated by the Romans mass deformation of the SL(5) EFT found in the previous section, we move on to investigate general deformations of EFT. Let us recall that we refer to the deformed version of EFT as XFT. In this section we will focus on the structure of generalised diffeomorphisms and discuss their closure and consistency conditions.

### 3.3.1 Deformed generalised Lie derivative

Motivated by our discussion of the internal gauge variations of massive IIA supergravity, we will now consider generic deformations of the exceptional generalised Lie derivative $\mathbb{L}_\Lambda$ in the $D$-dimensional EFT, by non-derivative terms specified by

[6]The reduced theory is a seven-dimensional gauged maximal supergravity with three vectors $A_\mu{}^{\alpha\beta}$ spanning an abelian $\mathbb{R}^3$ gauging specified by the three commuting generators $t^\gamma{}_5$.

a constant object $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$. As we will see, this object satisfies the same requirements as the embedding tensor of the $D$-dimensional gauged maximal supergravity: the quadratic constraint (1.14) is required for the closure of the generalised diffeomorphisms algebra and for the Jacobi identity, while the linear or representation constraint (see Table 3.1) is required for the consistency of the hierarchy of tensor fields. We exclude deformations of the trombone type from our discussion.

We thus start by introducing a deformed generalised Lie derivative which acts on $\mathrm{E}_{n(n)} \times \mathbb{R}^+$ vectors as

$$\widetilde{\mathbb{L}}_\Lambda U^{\mathcal{M}} = \mathbb{L}_\Lambda U^{\mathcal{M}} - X_{\mathcal{N}\mathcal{P}}{}^{\mathcal{M}}\, \Lambda^{\mathcal{N}}\, U^{\mathcal{P}}\,, \tag{3.28}$$

where the standard (undeformed) generalised Lie derivative $\mathbb{L}_\Lambda$ is defined in (3.5). A first consistency requirement is that $\widetilde{\mathbb{L}}_\Lambda$ is compatible with the rigid $\mathrm{E}_{n(n)}$ structure of the theory. Hence, $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ must decompose just as in (1.12). We can thus say that in general

$$\widetilde{\mathbb{L}}_\Lambda \equiv \mathbb{L}_\Lambda + \Lambda^{\mathcal{M}} X_{\mathcal{M}}\,, \tag{3.29}$$

where $X_{\mathcal{M}}$ is $\mathrm{E}_{n(n)}$ Lie algebra valued and acts in the appropriate representation.

Closure of generalised diffeomorphisms translates to the deformed version of (3.6):

$$\big[\widetilde{\mathbb{L}}_\Lambda, \widetilde{\mathbb{L}}_\Sigma\big] = \widetilde{\mathbb{L}}_{[\Lambda,\Sigma]_X}\,, \tag{3.30}$$

where the $X$-bracket $[\bullet\,,\bullet]_X$ takes the form

$$\big[\Lambda, \Sigma\big]^{\mathcal{M}}_X \equiv \frac{1}{2}\big(\widetilde{\mathbb{L}}_\Lambda \Sigma^{\mathcal{M}} - \widetilde{\mathbb{L}}_\Sigma \Lambda^{\mathcal{M}}\big) = \big[\Lambda, \Sigma\big]^{\mathcal{M}}_{\mathrm{E}} - X_{[\mathcal{P}\mathcal{Q}]}{}^{\mathcal{M}}\, \Lambda^{\mathcal{P}}\, \Sigma^{\mathcal{Q}}\,. \tag{3.31}$$

Requiring (3.30) induces a new set of consistency constraints. Since $\Lambda$ and $\Sigma$ are arbitrary parameters, these constraints can be disentangled based on the number of derivatives. The two-derivative ones do not contain $X_{\mathcal{M}}$ and therefore reduce to the original section constraint (3.8). An explicit computation yields

$$\begin{aligned}[][\widetilde{\mathbb{L}}_\Lambda, \widetilde{\mathbb{L}}_\Sigma] W^{\mathcal{M}} - \widetilde{\mathbb{L}}_{[\Lambda,\Sigma]_X} W^{\mathcal{M}} = {}& A^{\mathcal{M}}_{\mathcal{N}\mathcal{P}\mathcal{S}}\; \Lambda^{\mathcal{N}} \Sigma^{\mathcal{P}} W^{\mathcal{S}} + X_{[\mathcal{N}\mathcal{P}]}{}^{\mathcal{Q}}\; \Lambda^{\mathcal{N}} \Sigma^{\mathcal{P}} \partial_{\mathcal{Q}} W^{\mathcal{M}} \\ & + B^{\mathcal{M}\mathcal{Q}}_{\mathcal{N}\mathcal{R}\mathcal{S}}\; \big(\Lambda^{\mathcal{N}} \partial_{\mathcal{Q}} \Sigma^{\mathcal{R}} W^{\mathcal{S}} - \partial_{\mathcal{Q}} \Lambda^{\mathcal{R}} \Sigma^{\mathcal{N}} W^{\mathcal{S}}\big)\,, \end{aligned} \tag{3.32}$$

where, without loss of generality, we have already assumed (3.8) to hold. The right-hand side of (3.32) therefore defines $X$-dependent conditions. The $A$ and $B$

terms read

$$A^{\mathcal{M}}_{\mathcal{NPS}} = 2X_{[\mathcal{N}|\mathcal{Q}}{}^{\mathcal{M}}X_{\mathcal{P}]\mathcal{S}}{}^{\mathcal{Q}} - X_{\mathcal{QS}}{}^{\mathcal{M}}X_{[\mathcal{NP}]}{}^{\mathcal{Q}}\,, \tag{3.33}$$

$$\begin{aligned} B^{\mathcal{MQ}}_{\mathcal{NRS}} = {}& X_{(\mathcal{NR})}{}^{\mathcal{M}}\delta^{\mathcal{Q}}_{\mathcal{S}} - X_{\mathcal{NS}}{}^{\mathcal{Q}}\delta^{\mathcal{M}}_{\mathcal{R}} + Y^{\mathcal{MQ}}{}_{\mathcal{RP}}X_{\mathcal{NS}}{}^{\mathcal{P}} \\ & - Y^{\mathcal{PQ}}{}_{\mathcal{RS}}X_{\mathcal{NP}}{}^{\mathcal{M}} + Y^{\mathcal{MQ}}{}_{\mathcal{PS}}X_{[\mathcal{NR}]}{}^{\mathcal{P}} - \frac{1}{2}Y^{\mathcal{PQ}}{}_{\mathcal{RN}}X_{\mathcal{PS}}{}^{\mathcal{M}}\,. \end{aligned} \tag{3.34}$$

Note that the first line is the antisymmetric part of the quadratic constraint (1.14). Altogether, we have the requirements

$$A^{\mathcal{M}}_{\mathcal{NPS}} = 0\,, \quad X_{[\mathcal{NP}]}{}^{\mathcal{Q}}\,\partial_{\mathcal{Q}} = 0 \quad \text{and} \quad B^{\mathcal{MQ}}_{\mathcal{NRS}}\,\partial_{\mathcal{Q}} = 0\,. \tag{3.35}$$

The conditions above are not yet final. Just as for the E-bracket, the $X$-bracket fails to define a Lie algebra as the Jacobi identity does not hold. Instead, it yields a Jacobiator

$$\big[[\Lambda,\Sigma]_X,\Gamma\big]_X + \text{cycl.} = \frac{1}{3}\big\{[\Lambda,\Sigma]_X,\Gamma\big\}_X + \text{cycl.}\,, \tag{3.36}$$

where the $X$-modified symmetric bracket turns out to be

$$\{\Lambda,\Sigma\}^{\mathcal{M}}_X \equiv \frac{1}{2}(\widetilde{\mathbb{L}}_\Lambda\Sigma^{\mathcal{M}} + \widetilde{\mathbb{L}}_\Sigma\Lambda^{\mathcal{M}}) = \{\Lambda,\Sigma\}^{\mathcal{M}}_{\mathrm{E}} - X_{(PQ)}{}^{\mathcal{M}}\Lambda^{\mathcal{P}}\Sigma^{\mathcal{Q}}\,. \tag{3.37}$$

Consistency of the XFT requires that the Jacobiator again corresponds to a trivial gauge parameter, namely, $\widetilde{\mathbb{L}}_{\{\Lambda,\Sigma\}_X}$ vanishes. A direct computation shows that

$$\begin{aligned} \widetilde{\mathbb{L}}_{\{\Lambda,\Sigma\}_X}U^{\mathcal{M}} = {}& C^{\mathcal{MR}}_{\mathcal{SPQ}}\,(\Lambda^{\mathcal{Q}}\partial_{\mathcal{R}}\Sigma^{\mathcal{P}}U^{\mathcal{S}} + \partial_{\mathcal{R}}\Lambda^{\mathcal{P}}\Sigma^{\mathcal{Q}}U^{\mathcal{S}}) - X_{(\mathcal{PQ})}{}^{\mathcal{R}}\,\Lambda^{\mathcal{P}}\Sigma^{\mathcal{Q}}\,\partial_{\mathcal{R}}U^{\mathcal{M}} \\ & + X_{(\mathcal{PQ})}{}^{\mathcal{R}}\,X_{\mathcal{RS}}{}^{\mathcal{M}}\,\Lambda^{\mathcal{P}}\Sigma^{\mathcal{Q}}U^{\mathcal{S}}\,, \end{aligned} \tag{3.38}$$

with

$$C^{\mathcal{MR}}_{\mathcal{SPQ}} = X_{(\mathcal{PQ})}{}^{\mathcal{M}}\delta^{\mathcal{R}}_{\mathcal{S}} - Y^{\mathcal{MR}}{}_{\mathcal{TS}}\,X_{(\mathcal{PQ})}{}^{\mathcal{T}} - \frac{1}{2}\,Y^{\mathcal{TR}}{}_{\mathcal{PQ}}\,X_{\mathcal{TS}}{}^{\mathcal{M}}\,, \tag{3.39}$$

and where we have used the conditions (3.35) derived from (3.32). Therefore we must impose

$$X_{(\mathcal{PQ})}{}^{\mathcal{R}}\,X_{\mathcal{RS}}{}^{\mathcal{M}} = 0\,, \quad X_{(\mathcal{PQ})}{}^{\mathcal{R}}\,\partial_{\mathcal{R}} = 0 \quad \text{and} \quad C^{\mathcal{MR}}_{\mathcal{SPQ}}\,\partial_{\mathcal{R}} = 0\,. \tag{3.40}$$

The first equation in (3.40) combines with the first equation in (3.35) to produce the full set of quadratic constraints in (1.14). The middle equations in (3.35) and (3.40) combine into the $X$-constraint $X_{\mathcal{MN}}{}^{\mathcal{P}}\,\partial_{\mathcal{P}} = 0$. A careful analysis of the representation content of the remaining conditions (namely, the '$B$' and '$C$' terms) shows that they are entirely equivalent to the $X$-constraint. We thus arrive at the final set of consistency conditions for XFT:

$$Y^{\mathcal{M}\mathcal{N}}{}_{\mathcal{P}\mathcal{Q}}\,\partial_{\mathcal{M}}\otimes\partial_{\mathcal{N}}=0 \qquad (\text{ section constraint }), \tag{3.41}$$

$$X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}\,\partial_{\mathcal{P}}=0 \qquad (\ X\text{-constraint }), \tag{3.42}$$

and $X_{\mathcal{M}}$ must additionally satisfy the quadratic constraint in (1.14). The above conditions should be intended as acting on any field, parameter and combinations thereof. As a result, the new $X$-constraint (3.42) restricts the coordinate dependence to those coordinates left invariant by the $\mathrm{E}_{n(n)}$ elements generated by $X_{\mathcal{M}}$. Together with the linear and quadratic constraints on $X$, this is the only new condition required for consistency of XFT.

We should also emphasise that our notion of covariance under internal generalised diffeomorphisms is now given in terms of $\widetilde{\mathbb{L}}$, so that $\delta_\Lambda T=\widetilde{\mathbb{L}}_\Lambda T$ for any covariant tensor $T$. The deformation $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ by definition does not vary under any (internal or external) diffeomorphism and gauge transformations. Its generalised Lie derivative, however, does not necessarily vanish. Using the constraints above one can compute

$$\widetilde{\mathbb{L}}_\Lambda X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}=2\,\partial_{[\mathcal{M}}\,\Lambda^{\mathcal{R}}\,X_{|\mathcal{R}|\mathcal{N}]}{}^{\mathcal{P}}+Y^{\mathcal{P}\mathcal{Q}}{}_{\mathcal{R}\mathcal{N}}\,\partial_{\mathcal{Q}}\Lambda^{\mathcal{S}}\,X_{\mathcal{S}\mathcal{M}}{}^{\mathcal{R}}\ , \tag{3.43}$$

where we assigned the weight $\lambda(X)=-\omega$, as can be deduced by requiring that generalised Lie derivatives of tensors carry the same weights as the tensors themselves.

We close the section by stressing again that $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ must be restricted to the $\mathrm{E}_{n(n)}$ representations displayed in Table 3.1 for the consistency of the tensor hierarchy that will be presented in 3.4.

### 3.3.2 Section constraint and massive IIA supergravity

Equipped with the new generalised Lie derivative $\widetilde{\mathbb{L}}$, and the consistency conditions derived in the previous section, we now look at specific $X$ deformations to discuss their interpretation. We will come back to the construction of the full XFT action in section 3.4 where we discuss the $\mathrm{E}_{7(7)}$ case in detail. Starting from the M-theory solution of the section constraint (3.41), we now show that by turning on the $X$ deformation corresponding to the Romans mass $m_{\rm R}$, to which we refer as $X^{\rm R}$, a dependence of the fields and parameters on the M-theory coordinate is no longer allowed as a consequence of the $X$-constraint (3.42). The resulting XFT will then describe the massive IIA theory. The deformation $X^{\rm R}$ always corresponds to the embedding tensor obtained from reduction of massive IIA supergravity on a torus. We will also show that other ten-dimensional solutions of the section and $X$-constraints compatible with $X^{\rm R}$ exist and correspond to type II theories with background RR $p$-form fluxes T-dual to $m_{\rm R}$. For the $D=4$ case, we will also find an eleven-dimensional supergravity solution. Several of these solutions would be equivalent to each other in standard EFT, as they belong to the same $\mathrm{E}_{n(n)}$ orbit. However, the presence of $X^{\rm R}$ breaks $\mathrm{E}_{n(n)}$ to a subgroup which always contains at least an $\mathrm{SL}(n-1)$ factor, and solutions to the constraints must be classified in orbits of this subgroup only. We restrict our analysis to $D=9,7,4$ XFT but it should be clear that similar results can be obtain for the other $D\geq 4$ XFT's.

### $\mathbf{SL}(2)\times\mathbb{R}^+$ XFT

The EFT with $(D,n)=(9,2)$ features an $\mathrm{SL}(2)\times\mathbb{R}^+$ structure and has recently been constructed in [85]. The extended (internal) space has coordinates $y^{\mathcal{M}} = (y^\alpha\,,\,y^3)$ with $\alpha=1,2$ being a fundamental $\mathrm{SL}(2)$ index. The $\mathrm{SL}(2)\times\mathbb{R}^+$ invariant $Y$-tensor is given by

$$Y^{\alpha 3}{}_{\beta 3} = Y^{\alpha 3}{}_{3\beta} = Y^{3\alpha}{}_{\beta 3} = Y^{3\alpha}{}_{3\beta} = \delta^\alpha_\beta\ , \tag{3.44}$$

and the section constraint in (3.41) reduces to $\partial_\alpha\otimes\partial_3 = 0$. There are two inequivalent solutions corresponding to M-theory and type IIB supergravity

$$i)\ \partial_\alpha\neq 0\,,\ \partial_3 = 0\,,\ \text{(M-theory)}\quad \text{and}\quad ii)\ \partial_3\neq 0\,,\ \partial_\alpha=0\,,\ \text{(type IIB)}\,. \tag{3.45}$$

In the context of maximal $D=9$ supergravity [46,134,135], the Romans mass parameter induces an embedding tensor[7] whose only non-vanishing entry is $[X^{\rm R}]_{32}{}^1 = m_{\rm R}$. Taking it to be the $X$ deformation in XFT and substituting it into the $X$-constraint (3.42) yields

$$m_{\rm R}\,\partial_1 = 0\ . \tag{3.46}$$

[7]The corresponding gauging is simply a shift symmetry $\mathbb{R}$ generated by $t^2{}_1\in\mathrm{SL}(2)$ and spanned by the vector field $A_\mu{}^3$ [134,136].

As a result, any dependence on the M-theory coordinate $y^1$ is prohibited by the $X$-constraint (3.46), thus reflecting the fact that massive IIA cannot be embedded into M-theory. Furthermore, after imposing (3.46), the section constraint (3.41) reduces to $\partial_2 \otimes \partial_3 = 0$ and admits a type IIA and a type IIB solution, namely,

$$i)\ \partial_2 \neq 0\,,\ \partial_3 = 0\,,\ (\text{type IIA}) \quad \text{and} \quad ii)\ \partial_3 \neq 0\,,\ \partial_2 = 0\,,\ (\text{type IIB})\,. \tag{3.47}$$

In the IIA solution, the $X^{\mathrm{R}}$ deformation is identified with the Romans mass. In the IIB solution, the same $X^{\mathrm{R}}$ deformation corresponds to turning on a RR background flux $F_{(1)}$ (
textiti.e. associated with the axion) along the $y^3$ coordinate. The two solutions are related by a T-duality transformation

$$i)\ \text{massive IIA} \ \xrightarrow{T}\ ii)\ \text{IIB with } F_{(1)}\ , \tag{3.48}$$

exchanging $y^2 \leftrightarrow y^3$. Note finally that in (3.47), the IIB solution is single out by identifying the coordinate that does not transform under SL(2). As a consequence, this solution also inherits the SL(2) symmetry.

### SL(5) XFT

The EFT with $(D,n) = (7,4)$ possesses an SL(5) structure and has already been discussed in section 3.2.2. The SL(5) invariant $Y$-tensor and the section constraint, as well as its solutions, can be found in (3.18), (3.19) and (3.20). The consistent $X^{\mathrm{R}}$ deformation induced by the Romans mass was presented in section 3.2.3, resulting in eqs (3.26) and (3.27). In this case, the $X$-constraint (3.42) reads

$$m_{\mathrm{R}}\, \partial_{\alpha 5} = m_{\mathrm{R}}\, \partial_{45} = 0\ . \tag{3.49}$$

Note again that any dependence on the M-theory coordinate $y^{45}$ as well as on the (brane) coordinates' $y^{\alpha 5}$ is killed by the $X$-constraint (3.49). Substituting (3.49) in the section constraint (3.41) leads to $\epsilon^{\alpha\beta\gamma}\partial_{\alpha 4} \otimes \partial_{\beta\gamma} = 0$, which admits the 'natural' type IIA and IIB solutions

$$\begin{aligned} &i) && \partial_{\alpha 4} \neq 0\,,\ \partial_{\alpha\beta} = 0\,, && (\text{type IIA}) \\ &iv) && \partial_{\alpha\beta} \neq 0\,,\ \partial_{\alpha 4} = 0\,, && (\text{type IIB})\,, \end{aligned} \tag{3.50}$$

as well as two more solutions (with $\alpha \neq \beta \neq \gamma$)

$$\begin{aligned} &ii) \quad \partial_{\alpha 4}\,,\, \partial_{\beta 4}\,,\, \partial_{\alpha\beta} \neq 0\,, \quad \partial_{\gamma 4} = \partial_{\beta\gamma} = \partial_{\gamma\alpha} = 0 \quad \text{(type IIB)}\,, \\ &iii) \quad \partial_{\alpha 4}\,,\, \partial_{\alpha\beta}\,,\, \partial_{\gamma\alpha} \neq 0\,, \quad \partial_{\beta 4} = \partial_{\gamma 4} = \partial_{\beta\gamma} = 0 \quad \text{(type IIA)}\,, \end{aligned} \tag{3.51}$$

In the IIA solution $i$), the $X^{\rm R}$ deformation is identified with the Romans mass. In the IIB solution $ii$), it corresponds to a RR background flux $F_{(1)}$ along the single coordinate $y^{\alpha\beta}$. In the IIA solution $iii$), it is mapped back to a RR background flux $F_{(2)}$ along the two coordinates $(y^{\alpha\beta}, y^{\gamma\alpha})$. Finally, in the IIB solution $iv$), the $X$ deformation corresponds to a RR background flux $F_{(3)}$. The four solutions are connected via a chain of T-duality transformations

$$\begin{aligned} &i)\text{ massive IIA} \xrightarrow{T_\gamma} ii)\text{ IIB with } F_{(1)} \\ &\qquad \xrightarrow{T_\beta} iii)\text{ IIA with } F_{(2)} \xrightarrow{T_\alpha} iv)\text{ IIB with } F_{(3)}\,, \end{aligned} \tag{3.52}$$

where $T_\gamma$ exchanges $y^{\gamma 4} \leftrightarrow y^{\alpha\beta}$, with $\alpha \neq \beta \neq \gamma$.

### $\mathbf{E}_{7(7)}$ XFT

The EFT with $(D, n) = (4, 7)$ exhibits an $\mathrm{E}_{7(7)}$ structure and the coordinates $y^{\mathcal{M}}$ of the extended space transform in the **56** fundamental representation. The $\mathrm{E}_{7(7)}$ invariant $Y$-tensor reads [94]

$$Y^{\mathcal{MN}}{}_{\mathcal{PQ}} = -12\,[t_\alpha]^{\mathcal{MN}}[t^\alpha]_{\mathcal{PQ}} - \frac{1}{2}\,\Omega^{\mathcal{MN}}\,\Omega_{\mathcal{PQ}}\ , \tag{3.53}$$

where $[t_\alpha]_{\mathcal{M}}{}^{\mathcal{N}}$ are the $\mathrm{E}_{7(7)}$ generators. $\mathrm{E}_{7(7)}$ fundamental indices are raised and lowered using the Sp(56)-invariant (and thus $\mathrm{E}_{7(7)}$-invariant) antisymmetric tensor $\Omega_{\mathcal{MN}}$.[8] It will prove convenient to move to an SL(8)-covariant description of the theory where one has the $\mathrm{E}_{7(7)} \supset \mathrm{SL}(8)$ branching $\mathbf{56} \to \overline{\mathbf{28}} + \mathbf{28}$. For instance, the coordinates $y^{\mathcal{M}} = (y^{AB}\,,\, y_{AB})$ are expressed in terms of an antisymmetric pair $AB$ of SL(8) fundamental indices $A, B = 1, ..., 8$. The section constraint (3.41)

[8] We use the NW-SE conventions of [80], such that $[t_\alpha]^{\mathcal{MN}} = \Omega^{\mathcal{MP}}[t_\alpha]_P{}^{\mathcal{N}}$, $[t_\alpha]_{\mathcal{MN}} = [t_\alpha]_{\mathcal{M}}{}^{\mathcal{P}}\,\Omega_{\mathcal{PN}}$ and $\Omega^{\mathcal{MP}}\Omega_{\mathcal{NP}} = \delta^{\mathcal{M}}_{\mathcal{N}}$.

reads

$$
\begin{aligned}
\partial_{[AB} \otimes \partial_{CD]} - \frac{1}{4!}\, \epsilon_{ABCDEFGH}\, \partial^{EF} \otimes \partial^{GH} &= 0\,, \\
\partial_{AC} \otimes \partial^{BC} + \partial^{BC} \otimes \partial_{AC} - \frac{1}{8}\, \delta_A^B \left(\partial_{CD} \otimes \partial^{CD} + \partial^{CD} \otimes \partial_{CD}\right) &= 0\,, \\
\partial_{CD} \otimes \partial^{CD} - \partial^{CD} \otimes \partial_{CD} &= 0\,.
\end{aligned}
\tag{3.54}
$$

Branching now the SL(8) index with respect to $\mathrm{SL}(7) \subset \mathrm{SL}(8)$, namely $A = (I, 8)$ with $I = 1, ..., 7$, two solutions of (3.54) were identified in [80] (see also the discussion in section 3.2 of [137]) which are the ordinary M-theory and type IIB solutions[9]. These are the only maximal solutions up to $\mathrm{E}_{7(7)}$ transformations [89, 96] and involve a non-trivial dependence on the extended space of the form

$$
i)\ \partial_{I8} \neq 0\,,\ \text{(M-theory)} \qquad \text{and} \qquad ii)\ \partial_{\alpha 8} \neq 0\,,\ \partial^{\hat{\alpha}7} \neq 0\,,\ \text{(type IIB)}\,, \tag{3.55}
$$

where we have further split $I = (m, 7)$ and $m = (\alpha, \hat{\alpha})$ with $m = 1, ..., 6$, $\alpha = 1, 2, 3$ and $\hat{\alpha} = 4, 5, 6$.

In the context of a reduction to maximal $D = 4$ supergravity [44], the Romans mass induces a consistent embedding tensor of the form[10]

$$
[X^{\mathrm{R}}]^{AB}{}_{CD}{}^{EF} = -[X^{\mathrm{R}}]^{AB\,EF}{}_{CD} = -8\, \delta^{[A}_{[C} \xi^{B][E} \delta^{F]}_{D]}\,, \tag{3.56}
$$

with $\xi^{AB} = m_{\mathrm{R}}\, \delta^A_8\, \delta^B_8$. Taking now (3.56) to be the $X$ deformation in XFT results in an $X$-constraint (3.42) of the form

$$
m_{\mathrm{R}}\, \partial_{I8} = m_{\mathrm{R}}\, \partial^{IJ} = 0\,, \tag{3.57}
$$

which removes any dependence on the M-theory coordinate $y^{78}$ as well as on the coordinates $y^{m8}$ and $y_{IJ}$. Substituting (3.57) in the section constraint (3.41) leads to two conditions $\partial_{[IJ} \otimes \partial_{KL]} = 0$ and $\partial_{IJ} \otimes \partial^{J8} + \partial^{J8} \otimes \partial_{IJ} = 0$. Various type IIA/IIB solutions are recovered with a non-trivial dependence on the internal

[9]For the ordinary type IIB solution, we pick the one that is obtained by acting with three T-dualities upon the 'natural' IIA solution (which follows from the M-theory one after imposing $\partial_{78} = 0$).

[10]The induced gauging in four dimensions is an abelian $\mathbb{R}^7$ symmetry associated to the generators $t^I{}_8 \in \mathrm{SL}(8)$ and it is spanned by the vector fields $A_{\mu\, I8}$ [126].

extended space of the form

$$
\begin{array}{llll}
i) & \partial_{m7} \neq 0 & & (\text{type IIA}) \,, \\
ii) & \partial_{17}, ..., \partial_{57} \neq 0 \,, & \partial^{68} \neq 0 & (\text{type IIB}) \,, \\
iii) & \partial_{17}, ..., \partial_{47} \neq 0 \,, & \partial^{58} \,, \partial^{68} \neq 0 & (\text{type IIA}) \,, \\
iv) & \partial_{17}, ..., \partial_{37} \neq 0 \,, & \partial^{48}, ...\partial^{68} \neq 0 & (\text{type IIB}) \,, \\
v) & \partial_{17}, ..., \partial_{27} \neq 0 \,, & \partial^{38}, ...\partial^{68} \neq 0 & (\text{type IIA}) \,, \\
vi) & \partial_{17} \neq 0 \,, & \partial^{28}, ...\partial^{68} \neq 0 & (\text{type IIB}) \,, \\
vii) & & \partial^{m8} \neq 0 & (\text{type IIA}) \,.
\end{array}
\tag{3.58}
$$

Note that $vii)$ is actually a type IIA solution that is embeddable into a dual M-theory solution characterised by

$$
viii) \quad \partial^{I8} \neq 0 \,, \qquad\qquad (\text{dual M-theory}) \,. \tag{3.59}
$$

The different cases in (3.58) are related by a chain of T-duality transformations, in complete analogy with the analysis of the other XFT's. Starting from $i)$, where the $X^{\text{R}}$ deformation is identified with the Romans mass, one finds

$$
\begin{array}{llllll}
i) \text{ massive IIA} & \xrightarrow{T_{67}} & ii) \text{ IIB with } F_{(1)} & \xrightarrow{T_{57}} & iii) \text{ IIA with } F_{(2)} & \xrightarrow{T_{47}} \\
iv) \text{ IIB with } F_{(3)} & \xrightarrow{T_{37}} & v) \text{ IIA with } F_{(4)} & \xrightarrow{T_{27}} & vi) \text{ IIB with } F_{(5)} & \xrightarrow{T_{17}} \\
vii) \text{ IIA with } F_{(6)} & \rightsquigarrow & \multicolumn{4}{l}{viii) \text{ dual M-theory with } F_{(7)} = \star_{\text{11D}} F_{(4)} \,,}
\end{array}
\tag{3.60}
$$

where $\star$ denotes the Hodge dual in $11D$ and where the chain of T-duality transformations $T_{m7}$, with $m = 1, ..., 6$, exchanges the internal coordinates $y^{m7} \leftrightarrow y_{m8}$. The original Romans mass parameter gets consistently mapped into different RR $p$-form fluxes upon T-dualities. In the dual M-theory case, obtained by an oxidation of type IIA with $F_{(6)}$ (the solution $vii)$), the $X^{\text{R}}$ deformation corresponds to the Freund–Rubin (FR) parameter [138].[11]

[11]Note the difference with the SL(5) XFT discussed before for which a dual M-theory interpretation of the Romans mass was not possible. The reason is that, in the $D = 7$ case, the Freund-Rubin parameter in M-theory can only reduce to a Neveu–Schwarz–Neveu–Schwarz (NSNS) background flux $H_{(3)}$ in type IIA. The latter is not related to the Romans mass via duality transformations.

### 3.3.3 Extension to other background fluxes

In the previous section we have seen how, starting from a type IIA solution of the section constraint with a non-vanishing Romans mass $m_{\mathrm{R}} \neq 0$, other type II (or M-theory) background fluxes are obtained upon choosing T-dual solutions (with an extra oxidation). In these dual descriptions, the mass parameter $m_{\mathrm{R}}$ gets consistently mapped into other types of flux parameters which are still compatible with the quadratic constraints (1.14), the section constraint (3.41) and the $X$-constraint (3.42) in XFT. It is therefore natural to wonder whether different types of fluxes can coexist in $X$ for a fixed solution of the section constraint. This is what we investigate in this section where, for simplicity, we again restrict ourselves to the SL(5) XFT. We choose a representative M-theory, type IIA and type IIB solution of the section constraint, and classify all the $X$ deformations that solve the $X$-constraint without having to impose further restrictions on the coordinate dependence.

The structure of $X$ deformations in SL(5) XFT parallels that of maximal $D = 7$ supergravity [43]. In the latter, deformations (or gaugings) are described in terms of an embedding tensor that falls into the $\mathbf{15} + \overline{\mathbf{40}}$ representations of SL(5), and which is therefore characterised by $Y_{mn} = Y_{(mn)}$ and $Z^{mn,p} = Z^{[mn],p}$ with $m = 1, ..., 5$ and $Z^{[mn,p]} = 0$. In terms of these two objects, the $X$ deformation in XFT takes the form

$$X_{mn\,pq}{}^{rs} = 2\, X_{mn\,[p}{}^{[r}\, \delta^{s]}_{q]} \qquad \text{with} \qquad X_{mn\,p}{}^{r} = \delta^{r}_{[m}\, Y_{n]p} - 2\, \epsilon_{mnpst}\, Z^{st,r} \ . \tag{3.61}$$

#### Type IIA fluxes in SL(5) XFT

We start by selecting the type IIA solution of the section constraint in (3.50), in which the three internal coordinates are identified with $y^{\alpha 4}$ $(\alpha = 1, 2, 3)$, or equivalently with $\partial_{\alpha 4} \neq 0$. An explicit computation shows that the most general $X$ deformation compatible with this solution of the section constraint, as well as with the $X$-constraint, has (independent) non-vanishing components of the form

$$\begin{aligned} \tfrac{1}{4} Y_{\alpha 4} &= \tfrac{1}{2} \epsilon_{\alpha\beta\gamma}\, Z^{\beta\gamma,5} \equiv H_{\alpha} \,, & Y_{44} &\equiv \tfrac{1}{3!} \epsilon^{\alpha\beta\gamma}\, H_{\alpha\beta\gamma} \,, \\ Z^{5\alpha,5} &\equiv \tfrac{1}{2} \epsilon^{\alpha\beta\gamma} F_{\beta\gamma} \,, & Z^{45,5} &\equiv \tfrac{1}{2} m_{\mathrm{R}} \,, \end{aligned} \tag{3.62}$$

| SL(5) | $\mathbb{R}_1^+ \times \mathrm{SL}(4)$ | $\mathbb{R}_1^+ \times \mathbb{R}_2^+ \times \mathrm{SL}(3)$ |
|---|---|---|
| $\mathbf{10}\ (\partial_M)$ | $\mathbf{4}_{-\frac{3}{2}}\ (\partial_i) + \mathbf{6}_1$ | $\mathbf{1}_{(-\frac{3}{2},\frac{3}{2})} + \mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})}\ (\partial_\alpha) + \mathbf{3}_{(1,1)} + \bar{\mathbf{3}}_{(1,-1)}$ |
| $\mathbf{24}$ | $\bar{\mathbf{4}}_{-\frac{5}{2}}\ (C_{jkl}) + \mathbf{4}_{\frac{5}{2}} + (\mathbf{1}+\mathbf{15})_0$ | $\mathbf{1}_{(-\frac{5}{2},-\frac{3}{2})} + \bar{\mathbf{3}}_{(-\frac{5}{2},\frac{1}{2})}\ (A_{\beta\gamma}) + \mathbf{1}_{(\frac{5}{2},\frac{3}{2})} + \mathbf{3}_{(\frac{5}{2},-\frac{1}{2})}$ <br> $+\mathbf{1}_{(0,0)}\ (\phi) + \mathbf{8}_{(0,0)} + \mathbf{3}_{(0,-2)}\ (A_\beta) + \bar{\mathbf{3}}_{(0,2)} + \mathbf{1}_{(0,0)}$ |
| $\mathbf{15}\ (Y_{MN})$ | $\mathbf{1}_{-4}\ (\partial_{[i}C_{jkl]}) + \mathbf{4}_{-\frac{3}{2}} + \mathbf{10}_1$ | $\mathbf{1}_{(-4,0)}\ (\partial_{[\alpha}A_{\beta\gamma]}) + \mathbf{1}_{(-\frac{3}{2},\frac{3}{2})} + \mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})}\ (\partial_\alpha\phi)$ <br> $+\mathbf{1}_{(1,3)} + \mathbf{3}_{(1,1)} + \mathbf{6}_{(1,-1)}$ |
| $\overline{\mathbf{40}}\ (Z^{MN,P})$ | $\overline{\mathbf{20}}_{-\frac{3}{2}} + \mathbf{6}_1 + \overline{\mathbf{10}}_1 + \bar{\mathbf{4}}_{\frac{7}{2}}$ | $\mathbf{8}_{(-\frac{3}{2},\frac{3}{2})} + \bar{\mathbf{6}}_{(-\frac{3}{2},-\frac{1}{2})} + \bar{\mathbf{3}}_{(-\frac{3}{2},-\frac{5}{2})}\ (\partial_{[\alpha}A_{\beta]})$ <br> $+\mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})}\ (\partial_\alpha\phi) + \mathbf{3}_{(1,1)} + \bar{\mathbf{3}}_{(1,-1)}$ <br> $+\underbrace{\mathbf{1}_{(1,-3)}}_{m_{\mathrm{R}}} + \bar{\mathbf{3}}_{(1,-1)} + \bar{\mathbf{6}}_{(1,1)} + \mathbf{1}_{(\frac{7}{2},-\frac{3}{2})} + \bar{\mathbf{3}}_{(\frac{7}{2},\frac{1}{2})}$ |

Table 3.2: Relevant branchings for the embeddings of M-theory (middle column) and type IIA (right column) into SL(5) XFT. The internal derivatives ( $\subset \mathbf{10}$ ), gauge potentials and dilaton ( $\subset \mathbf{24}$ ) and gauge fluxes ( $\subset \mathbf{15}+\overline{\mathbf{40}}$ ) are highlighted both in the M-theory (blue) and the natural type IIA (red) solutions of the section constraint. The Romans mass parameter $m_{\mathrm{R}}$ is singled out. Note that only a linear combination of the two $\mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})} \subset \mathbf{15}\,,\ \overline{\mathbf{40}}$ is sourced by the dilaton flux $\partial_\alpha\phi$ so that there are 1 and 8 free real deformation parameters in M-theory and type IIA, respectively.

and therefore accounts for $3+1+3+1 = 8$ free real parameters. Using purely group-theoretical arguments, we establish in Table 3.2 a dictionary between type IIA fluxes and the deformations. The various components in (3.62) are then identified with the dilaton ( $H_\alpha$ ), NSNS three-form ( $H_{\alpha\beta\gamma}$ ) and RR two-form ( $F_{\alpha\beta}$ ) fluxes, as well as with the Romans mass parameter[12]

The $X$ deformation induced by (3.62) accounts for all the background gauge fluxes that can thread the three-dimensional internal space. However, this does not imply that all the parameters can be turned on simultaneously as they still have to obey the quadratic constraints (1.14). These take the form of

$$m_{\mathrm{R}}\, H_\alpha = 0 \qquad \text{and} \qquad \tfrac{1}{2}\,\epsilon^{\alpha\beta\gamma}\, H_\alpha\, F_{\beta\gamma} + \tfrac{1}{4!}\tfrac{1}{3!}\,\epsilon^{\alpha\beta\gamma}\, m_{\mathrm{R}}\, H_{\alpha\beta\gamma} = 0\ , \tag{3.63}$$

[12]The Romans mass can be dynamically generated in a non-geometric manner (not even locally geometric in the language of [139]) by allowing the RR one-form to have a non-trivial dependence on the type IIB coordinates $\tilde{y}_\alpha \equiv \frac{1}{2}\,\epsilon_{\alpha\beta\gamma}\, y^{\beta\gamma}$ associated with $\tilde{\partial}^\alpha \equiv \bar{\mathbf{3}}_{(1,-1)}$ (see Table 3.3). Using representation theory one finds

$$m_{\mathrm{R}} \equiv \mathbf{1}_{(1,-3)} = \bar{\mathbf{3}}_{(1,-1)} \otimes \mathbf{3}_{(0,-2)}\big|_{\mathbf{1}} \equiv \tilde{\partial}^\alpha A_\alpha\ .$$

As discussed in [140] in the context of double field theory, the dependence on $\tilde{y}_\alpha$ would violate the section constraint and, in order to recover massive IIA, one would have to explore the non-geometric side of the EFT's where the fields pick up a dependence on physical and dual coordinates at the same time.

| $\mathrm{SL}(5)$ | $\mathbb{R}^+_1 \times \mathbb{R}^+_2 \times \mathrm{SL}(3)$ |
|---|---|
| $\mathbf{10}\ (\partial_M)$ | $\mathbf{1}_{(-\frac{3}{2},\frac{3}{2})} + \mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})} + \mathbf{3}_{(1,1)} + \bar{\mathbf{3}}_{(1,-1)}\ \big(\tilde{\partial}^\alpha \equiv \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\partial_{\beta\gamma}\big)$ |
| $\mathbf{24}$ | $\mathbf{1}_{(-\frac{5}{2},-\frac{3}{2})}\ \big(C_0\big) + \bar{\mathbf{3}}_{(-\frac{5}{2},\frac{1}{2})} + \mathbf{1}_{(\frac{5}{2},\frac{3}{2})}\ \big(\gamma_0\big) + \mathbf{3}_{(\frac{5}{2},-\frac{1}{2})}\ \big(B^{\beta\gamma}\big)$ <br> $+\,\mathbf{1}_{(0,0)}\ \big(\phi\big) + \mathbf{8}_{(0,0)} + \mathbf{3}_{(0,-2)}\ \big(C^{\beta\gamma}\big) + \bar{\mathbf{3}}_{(0,2)} + \mathbf{1}_{(0,0)}$ |
| $\mathbf{15}\ (Y_{MN})$ | $\mathbf{1}_{(-4,0)} + \mathbf{1}_{(-\frac{3}{2},\frac{3}{2})} + \mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})} + \mathbf{1}_{(1,3)} + \mathbf{3}_{(1,1)} + \mathbf{6}_{(1,-1)}$ |
| $\overline{\mathbf{40}}\ (Z^{MN,P})$ | $\mathbf{8}_{(-\frac{3}{2},\frac{3}{2})} + \bar{\mathbf{6}}_{(-\frac{3}{2},-\frac{1}{2})} + \bar{\mathbf{3}}_{(-\frac{3}{2},-\frac{5}{2})}\ \big(\tilde{\partial}^\alpha C_0\big) + \mathbf{3}_{(-\frac{3}{2},-\frac{1}{2})}$ <br> $+\,\mathbf{3}_{(1,1)} + \bar{\mathbf{3}}_{(1,-1)}\ \big(\tilde{\partial}^\alpha\phi\big) + \mathbf{1}_{(1,-3)}\ \big(\tilde{\partial}^{[\alpha}C^{\beta\gamma]}\big) + \bar{\mathbf{3}}_{(1,-1)}\ \big(\tilde{\partial}^\alpha\phi\big) + \bar{\mathbf{6}}_{(1,1)}$ <br> $+\,\mathbf{1}_{(\frac{7}{2},-\frac{3}{2})}\ \big(\tilde{\partial}^{[\alpha}B^{\beta\gamma]}\big) + \bar{\mathbf{3}}_{(\frac{7}{2},\frac{1}{2})}\ \big(\tilde{\partial}^\alpha\gamma_0\big)$ |

Table 3.3: Relevant branchings for the embedding of type IIB into SL(5) XFT. The purely internal derivatives ( $\subset \mathbf{10}$ ), gauge potentials and scalars ( $\subset \mathbf{24}$ ) and gauge fluxes ( $\subset \overline{\mathbf{40}}$ ) are highlighted (red). Note that only a linear combination of the two $\bar{\mathbf{3}}_{(1,-1)} \subset \overline{\mathbf{40}}$ is sourced by the dilaton flux $\tilde{\partial}^\alpha\phi$ . The $\mathbb{R}^+_{\mathrm{S}} \in \mathrm{SL}(2)$ charge of the type IIB theory (S-duality) is given by $q_{\mathrm{S}} = q_{\mathbb{R}^+_1} + q_{\mathbb{R}^+_2}$ .

and correspond to the flux-induced tadpole cancellation conditions in absence of O8/D8 and O6/D6 sources, respectively. Solving the quadratic constraints (3.63) yields two families of $X$ deformations, and hence, of consistent XFT's. The first one is a six-parameter family of XFT's specified by the two conditions

$$a) \qquad \epsilon^{\alpha\beta\gamma}\, H_\alpha\, F_{\beta\gamma} = 0 \ , \qquad m_{\mathrm{R}} = 0 \ , \tag{3.64}$$

whereas the second one is a four-parameter family of XFT's specified by the four conditions

$$b) \qquad H_{\alpha\beta\gamma} = 0 \ , \qquad H_\alpha = 0 \ . \tag{3.65}$$

As a result, the dilaton flux $H_\alpha$ and the $H_{(3)}$ flux on the one hand, and the Romans mass parameter $m_{\mathrm{R}}$ on the other cannot be turned on simultaneously.

### M-theory fluxes in SL(5) XFT

The same analysis can be performed for the M-theory extension of the type IIA solution in (3.50). In this case, the four internal coordinates $y^i$ of the eleven-dimensional supergravity are identified with $y^{\alpha 4}$ and $y^{45}$ , the latter being the

M-theory coordinate. The most general $X$ deformation compatible with the $X$-constraint has a unique non-vanishing component given by

$$Y_{44} \equiv f_{\text{FR}} \ , \tag{3.66}$$

and is identified (see Table 3.2) with the Freund–Rubin (FR) parameter [138]. This parameter corresponds to a purely internal background value for the field strength of the three-form potential of eleven-dimensional supergravity (1.2). Hence, there is a one-parameter family of XFT's describing eleven-dimensional supergravity with an $f_{\text{FR}}$ flux.

### Type IIB fluxes in SL(5) XFT

Finally, let us consider the type IIB solution in (3.50), where the three internal coordinates are identified with $\tilde{y}_{\alpha} \equiv \frac{1}{2}\,\epsilon_{\alpha\beta\gamma}\,y^{\beta\gamma}$, or equivalently with $\tilde{\partial}^{\alpha} \neq 0$. The most general $X$ deformation satisfying the $X$-constraint is compatible with the SL(2) symmetry (S-duality) of the IIB theory and has (independent) non-vanishing components of the form

$$\begin{array}{ll}\text{SL(2)-doublet}: & Z^{45,5} \equiv \frac{1}{3!}\,\epsilon_{\alpha\beta\gamma}\,F^{\alpha\beta\gamma}\,, \quad Z^{45,4} \equiv \frac{1}{3!}\,\epsilon_{\alpha\beta\gamma}\,H^{\alpha\beta\gamma}\,, \\ \text{SL(2)-triplet}: & Z^{\alpha 5,5} \equiv F^{\alpha}\,, \;\; Z^{\alpha 5,4} = Z^{\alpha 4,5} \equiv H^{\alpha}\,, \;\; Z^{\alpha 4,4} \equiv \hat{F}^{\alpha}\,,\end{array} \tag{3.67}$$

accounting for $2 \times 1 + 3 \times 3 = 11$ free real parameters. Using the dictionary between type IIB fluxes and deformations in Table 3.3, one identifies the SL(2)-doublet of RR ( $F^{\alpha\beta\gamma}$ ) and NS-NS ( $H^{\alpha\beta\gamma}$ ) three-form fluxes[13] of the IIB theory. In addition, there is also an SL(2)-triplet of one-form deformations $(F^{\alpha}, H^{\alpha}, \hat{F}^{\alpha})$. The latter account for an internal dependence of the type IIB axion-dilaton and can be dualised into nine-form fluxes for the SL(2)-triplet of RR eight-form potentials of the IIB theory [142, 143].

The computation of the quadratic constraints in (1.14) for the type IIB fluxes in (3.67) produces the set of relations

$$\epsilon_{\alpha\beta\gamma}\,F^{\beta}\,H^{\gamma} = 0\,, \quad \epsilon_{\alpha\beta\gamma}\,F^{\beta}\,\hat{F}^{\gamma} = 0 \quad \text{and} \quad \epsilon_{\alpha\beta\gamma}\,\hat{F}^{\beta}\,H^{\gamma} = 0\ , \tag{3.68}$$

which corresponds to flux-induced tadpole cancellation conditions for an SL(2)-triplet of 7-branes (and related orientifold planes). Indeed such objects must be

[13] See also [141] for a discussion on generalised fluxes in SL(5) EFT.

absent in order to preserve maximal supersymmetry. Note that (3.68) is SL(2)-covariant and can be rewritten as $\mathcal{H}_{[A} \wedge \mathcal{H}_{B]} = 0$ with $A = 1, 2, 3$ labelling the adjoint representation and $\mathcal{H}_A = (F, H, \hat{F})$. Solving (3.68) yields a seven-parameter family of XFT's that describes such ten-dimensional type IIB with various fluxes along the internal directions.

We close this section by commenting on the number of deformation parameters allowed in other $D \geq 4$ XFT's. The most general $X$ deformation compatible with the section constraint (3.41) and the $X$-constraint (3.42) includes: *i*) the Freund–Rubin parameter in M-theory (only for $D = 7, 4$) *ii*) the Romans mass $m_{\mathrm{R}}$ (for any $D$) as well as the dilaton (for any $D$) and standard $p$-form gauge fluxes (when permitted by $D$) in type IIA *iii*) the SL(2)-triplet of one-form fluxes (for any $D$) as well as the standard internal $p$-form gauge fluxes (when permitted by $D$) in type IIB. In order for XFT to be consistent, the $X$ deformation must still satisfy the quadratic constraint (1.14). The latter can be seen as tadpole cancellation conditions that ensure the absence supersymmetry breaking sources. A last remark concerns the incompatibility of the $X$-constraint with the presence of a metric flux $\omega$ of the Scherk–Schwarz (SS) type [58]. Indeed, suppose that it was possible to introduce $\omega$ in XFT. After choosing a solution of the section and $X$-constraint, the resulting $X$-deformation would modify the action of the ordinary internal diffeomorphisms rather than that of the internal ($p$-form) gauge transformations. This would be incompatible with an arbitrary dependence of all fields on eleven or ten physical coordinates. It is therefore with no surprise that we find the $X$-constraint to actually exclude metric fluxes.

## 3.4 Dynamics of $\mathrm{E}_{7(7)}$ XFT

In this section we illustrate the generic features of the deformations introduced in section 3.3.1 by constructing explicitly the gauge invariant $\mathrm{E}_{7(7)}$ XFT. While the field content of the theory remains identical to the one of $\mathrm{E}_{7(7)}$ EFT, changes occur at the level of the tensor hierarchy and in the action due to the presence of the $X$ deformation. We refer to section 3.3.2 for a detailed discussion of the section constraint of the $\mathrm{E}_{7(7)}$ XFT. We present below some specifics of the deformed $\mathrm{E}_{7(7)}$ generalised diffeomorphisms, followed by the tensor hierarchy and the full bosonic action. The latter consistently reduces to the action of $D = 4$ gauged maximal supergravity when all fields are taken independent of the **56** exceptional coordinates $y^{\mathcal{M}}$, and to the one of the $\mathrm{E}_{7(7)}$ EFT when the $X$ deformation is turned off. Finally, when fixing $X$ to (3.56) and choosing an appropriate solution of the section and $X$-constraints (see (3.58)), one recovers the bosonic sector of massive type IIA supergravity in a $4+6$ dimensional split. The results of this section are in parallel with those of [80] to which we refer for an in-depth discussion of the $\mathrm{E}_{7(7)}$ EFT dynamics.

### 3.4.1 Modified Lie derivative and trivial parameters

The expression of the $\mathrm{E}_{7(7)}$ invariant $Y$-tensor was given in (3.53). Both the $\mathrm{E}_{7(7)}$ generators $[t_\alpha]_{\mathcal{M}}{}^{\mathcal{N}}$ and the symplectic form $\Omega_{\mathcal{M}\mathcal{N}}$ are invariant under the deformed generalised Lie derivative (3.29). For $\mathrm{E}_{7(7)}$ the distinguished weight to be introduced in (3.5) is $\omega = \frac{1}{2}$. The section constraint decomposes into two irreducible pieces in the $\mathbf{1}+\mathbf{133}$ irreps:

$$\Omega^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} = 0\,, \qquad [t_\alpha]^{\mathcal{M}\mathcal{N}}\partial_{M} \otimes \partial_{\mathcal{N}} = 0\,. \tag{3.69}$$

We will use the shorthand notation $(\mathbb{P}_{\mathbf{1}+\mathbf{133}})^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{M}} \otimes \partial_{\mathcal{N}} = 0$ to reflect these two constraints.

As explained previously, the $X$ deformation satisfies the same linear and quadratic constraints as the embedding tensor in gauged maximal supergravity [44]. The linear constraints in $D=4$ read

$$X_{\mathcal{N}\mathcal{M}}{}^{\mathcal{M}} = X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{M}} = 0\,, \qquad X_{(\mathcal{M}\mathcal{N}\mathcal{P})} = 0\,, \tag{3.70}$$

and restrict $X$ to belong to the $\mathbf{912}$ representation. Consequently, the quadratic constraint (1.14) can be rephrased as

$$\Omega^{\mathcal{M}\mathcal{N}}\, X_{\mathcal{M}} \otimes X_{\mathcal{N}} = 0\,. \tag{3.71}$$

The deformed EFT requires to impose the section- and the $X$-constraint (3.41). In $D=4$ the latter can be decomposed into the $\mathbf{133}+\mathbf{1539}$ irreps, corresponding to $X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{P}}\partial_{\mathcal{P}}$ and $X_{[\mathcal{M}\mathcal{N}]}{}^{\mathcal{P}}\partial_{\mathcal{P}}$ respectively. Using representation theory it is possible to find other equivalent ways to express these constraints. Two such expressions are particularly useful

$$\Omega^{\mathcal{M}\mathcal{N}}\Theta_{\mathcal{M}}{}^{\alpha}\partial_{\mathcal{N}} = 0\,, \qquad \Theta_{\mathcal{M}}{}^{(\alpha}[t^{\beta)}]^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{N}} = 0\,, \tag{3.72}$$

and correspond to the $\mathbf{133}$ and $\mathbf{1539}$, respectively.

The construction of the $E_{7(7)}$ XFT tensor hierarchy relies on the form of certain trivial parameters appearing in the symmetric $X$-bracket (3.37). Specifically, for two arbitrary generalised vectors of weight $\omega$ we have

$$\begin{aligned}\{U, V\}_X^{\mathcal{M}} = &- 6\,[t_\alpha]^{\mathcal{M}\mathcal{N}}[t^\alpha]_{\mathcal{P}\mathcal{Q}}\,\partial_{\mathcal{N}}\big[U^{\mathcal{P}}V^{\mathcal{Q}}\big] - U^{\mathcal{N}}V^{\mathcal{P}}X_{(\mathcal{N}\mathcal{P})}{}^{\mathcal{M}} \\ &- \frac{1}{4}\,\Omega^{\mathcal{M}\mathcal{N}}\,\Omega_{\mathcal{P}\mathcal{Q}}\big[V^{\mathcal{Q}}\,\partial_{\mathcal{N}}U^{\mathcal{P}} + U^{\mathcal{Q}}\,\partial_{\mathcal{N}}V^{\mathcal{P}}\big]\;.\end{aligned} \tag{3.73}$$

Both lines of (3.73) are trivial parameters provided all fields satisfy the section constraint and the symmetric part of the $X$-constraint (3.42). This ensures that the Jacobi identity for $\widetilde{\mathbb{L}}$ is satisfied. More generally, the following generic parameters do not generate generalised diffeomorphisms:

$$\Lambda^{\mathcal{M}} = [t^\alpha]^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{N}}\chi_\alpha + \frac{1}{6}\,Z^{\mathcal{M},\alpha}\,\chi_\alpha\,, \tag{3.74}$$

$$\Lambda^{\mathcal{M}} = \Omega^{\mathcal{M}\mathcal{N}}\chi_{\mathcal{N}}\;, \tag{3.75}$$

for arbitrary $\chi_\alpha$. The intertwining tensor $Z^{M,\alpha}$ is constructed from $X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}$ making use of the linear constraint:

$$Z^{\mathcal{M},\alpha} = -X_{\mathcal{P}\mathcal{Q}}{}^{\mathcal{M}}\,[t^\alpha]^{\mathcal{P}\mathcal{Q}} = -\frac{1}{2}\Omega^{\mathcal{M}\mathcal{N}}\Theta_{\mathcal{N}}{}^{\alpha}\;. \tag{3.76}$$

Similarly to the EFT case, $\chi_{\mathcal{M}}$ is covariantly constrained in the sense that it must itself satisfy the section constraints

$$\big(\mathbb{P}_{\mathbf{1}+\mathbf{133}}\big)^{\mathcal{M}\mathcal{N}}\chi_{\mathcal{M}}\partial_{\mathcal{N}} \;=\; 0 \;=\; \big(\mathbb{P}_{\mathbf{1}+\mathbf{133}}\big)^{\mathcal{M}\mathcal{N}}\chi_{\mathcal{M}}\chi_{\mathcal{N}}\,, \tag{3.77}$$

where $\mathbb{P}_{\mathbf{1}+\mathbf{133}}$ denotes the projector onto the $\mathbf{1}\oplus\mathbf{133}$ representation of the $\mathbf{56}\otimes\mathbf{56}$. In XFT, the field $\chi_{\mathcal{M}}$ is further covariantly constrained by

$$X_{(\mathcal{M}\mathcal{N})}{}^{\mathcal{P}}\chi_{\mathcal{P}} = 0\,, \tag{3.78}$$

or equivalently by $\Omega^{\mathcal{M}\mathcal{N}}\Theta_{\mathcal{M}}{}^{\alpha}\,\chi_N = 0$. The importance of the covariantly constrained parameters (3.75) will become apparent when constructing the tensor hierarchy.

### 3.4.2 Yang-Mills sector and tensor hierarchy

In analogy with EFT, we introduce an external derivative which is covariant with respect the deformed internal generalised diffeomorphisms

$$\mathcal{D}_\mu \equiv \partial_\mu - \widetilde{\mathbb{L}}_{A_\mu} \ . \tag{3.79}$$

Covariance determines the variation of $A_\mu{}^M$ to be

$$\delta_\Lambda A_\mu{}^{\mathcal{M}} = \mathcal{D}_\mu \Lambda^{\mathcal{M}} \simeq \partial_\mu \Lambda^{\mathcal{M}} + \widetilde{\mathbb{L}}_\Lambda A_\mu{}^{\mathcal{M}} \ , \tag{3.80}$$

where the equivalence holds up to the addition of a trivial gauge parameter that can be subsequently reabsorbed in other gauge transformations higher up in the tensor hierarchy. This is completely in line with the situation in EFT.

Following the construction of the tensor hierarchy in the original EFT's, we first define the field strength for the vector fields $A_\mu{}^{\mathcal{M}}$ as

$$F_{\mu\nu}{}^{\mathcal{M}} = 2\,\partial_{[\mu} A_{\nu]}{}^{\mathcal{M}} - \big[A_\mu, A_\nu\big]_X^{\mathcal{M}} \ . \tag{3.81}$$

Since the Jacobiator of the $X$-bracket does not vanish, the above expression does not transform covariantly under generalised diffeomorphisms. The procedure to restore gauge covariance is analogous to those of gauged supergravity and EFT. In fact, it turns out to be a superposition of the two cases. We define a modified field strength by introducing the two-form fields $B_{\mu\nu\,\alpha}$ and $B_{\mu\nu\,\mathcal{M}}$ in the form of the two trivial parameters (3.74) and (3.75)

$$\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = F_{\mu\nu}{}^{\mathcal{M}} - 12\,[t^\alpha]^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} B_{\mu\nu\,\alpha} - 2\,Z^{\mathcal{M},\alpha} B_{\mu\nu\,\alpha} - \frac{1}{2}\,\Omega^{\mathcal{M}\mathcal{N}} B_{\mu\nu\,\mathcal{N}} \ , \tag{3.82}$$

where $B_{\mu\nu\,\mathcal{N}}$ is a covariantly constrained field as in (3.77) and (3.78). Note that this construction only deviates from EFT by the term proportional to $Z^{\mathcal{M},\alpha}$, which is precisely the one needed to make contact with gauged supergravities when all the fields are taken to be $y^{\mathcal{M}}$-independent. It is easy to verify that, since $F_{\mu\nu}{}^{\mathcal{M}}$ only differs from $\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}$ by a trivial parameter, we have

$$\big[\mathcal{D}_\mu, \mathcal{D}_\nu\big] = -2\,\widetilde{\mathbb{L}}_{\partial_{[\mu} A_{\nu]}} + 2\,\widetilde{\mathbb{L}}_{A_{[\mu}} \widetilde{\mathbb{L}}_{A_{\nu]}} = -\widetilde{\mathbb{L}}_{F_{\mu\nu}} = -\widetilde{\mathbb{L}}_{\mathcal{F}_{\mu\nu}} \ . \tag{3.83}$$

Using the explicit expression for the symmetric $X$-bracket (3.73), the general variation of the modified field strength (3.82) now reads

$$\delta\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = 2\,\mathcal{D}_{[\mu}\delta A_{\nu]}{}^{\mathcal{M}} - 12\,[t^{\alpha}]^{\mathcal{MN}}\partial_{\mathcal{N}}\Delta B_{\mu\nu\,\alpha} \\ - 2\,Z^{\mathcal{M},\alpha}\Delta B_{\mu\nu\,\alpha} - \frac{1}{2}\,\Omega^{\mathcal{MN}}\,\Delta B_{\mu\nu\,\mathcal{N}}\,, \tag{3.84}$$

where, as in EFT, we have defined

$$\begin{aligned}\Delta B_{\mu\nu\,\alpha} &= \delta B_{\mu\nu\,\alpha} + [t_{\alpha}]_{\mathcal{NP}}A_{[\mu}{}^{\mathcal{N}}\delta A_{\nu]}{}^{\mathcal{P}}\ ,\\ \Delta B_{\mu\nu\,\mathcal{N}} &= \delta\mathcal{B}_{\mu\nu\,\mathcal{N}} + \Omega_{\mathcal{PQ}}\big[\mathcal{A}_{[\mu}{}^{\mathcal{Q}}\partial_{\mathcal{N}}\delta A_{\nu]}{}^{\mathcal{P}} + \partial_{\mathcal{N}}A_{[\mu}{}^{\mathcal{P}}\delta A_{\nu]}{}^{\mathcal{Q}}\big]\ .\end{aligned} \tag{3.85}$$

We define the vector gauge variations of the two-forms as follows[14]:

$$\begin{aligned}\Delta_{\Lambda}B_{\mu\nu\,\alpha} &= [t_{\alpha}]_{\mathcal{NP}}\Lambda^{\mathcal{N}}\mathcal{F}_{\mu\nu}{}^{\mathcal{P}}\,,\\ \Delta_{\Lambda}B_{\mu\nu\,\mathcal{N}} &= \Omega_{\mathcal{PQ}}\big[\Lambda^{\mathcal{Q}}\partial_{\mathcal{N}}\mathcal{F}_{\mu\nu}{}^{\mathcal{P}} + \mathcal{F}_{\mu\nu}{}^{\mathcal{Q}}\partial_{\mathcal{N}}\Lambda^{\mathcal{P}}\big]\,.\end{aligned} \tag{3.86}$$

Substituting the above variations back in (3.84) and making use of (3.73) and (3.83) yields

$$\delta_{\Lambda}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = \big[\mathcal{D}_{\mu},\mathcal{D}_{\nu}\big]\Lambda^{\mathcal{M}} + 2\big\{\Lambda,\mathcal{F}_{\mu\nu}\big\}_{X}^{\mathcal{M}} = \widetilde{\mathbb{L}}_{\Lambda}\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\,, \tag{3.87}$$

which shows that $\mathcal{F}_{\mu\nu}{}^{\mathcal{M}}$ transforms covariantly.

On top of the generalised diffeomorphisms (also referred to as vector gauge transformations in the following), the field strength (3.82) is invariant under tensor gauge transformations associated with the two-forms

$$\begin{aligned}\delta_{\Xi}A_{\mu}{}^{\mathcal{M}} &= 12[t^{\alpha}]^{\mathcal{MN}}\partial_{\mathcal{N}}\Xi_{\mu\,\alpha} + 2\,Z^{\mathcal{M},\alpha}\,\Xi_{\mu\,\alpha} + \frac{1}{2}\,\Omega^{\mathcal{MN}}\Xi_{\mu\,\mathcal{N}}\,,\\ \Delta_{\Xi}B_{\mu\nu\,\alpha} &= 2\,\mathcal{D}_{[\mu}\Xi_{\nu]\alpha}\,,\\ \Delta_{\Xi}B_{\mu\nu\,\mathcal{M}} &= 2\,\mathcal{D}_{[\mu}\Xi_{\nu]\mathcal{M}} + 48\,[t^{\alpha}]_{\mathcal{L}}{}^{\mathcal{K}}(\partial_{\mathcal{K}}\partial_{\mathcal{M}}A_{[\mu}{}^{\mathcal{L}})\,\Xi_{\nu]\alpha} + 4\,\Theta_{\mathcal{P}}{}^{\alpha}\,\partial_{\mathcal{M}}A_{[\mu}{}^{\mathcal{P}}\,\Xi_{\nu]\alpha}\,,\end{aligned} \tag{3.88}$$

where the tensor gauge parameters $\Xi_{\mu\,\alpha}$ and $\Xi_{\mu\,\mathcal{M}}$ carry weight 1 and $\frac{1}{2}$, respectively. For an arbitrary generalised vector $W_{\alpha}$ in the adjoint of $\mathrm{E}_{7(7)}$ with weight $\lambda'$, the deformed generalised Lie derivative acts as follows:

$$\widetilde{\mathbb{L}}_{\Lambda}W_{\alpha} = \Lambda^{\mathcal{R}}\partial_{\mathcal{R}}W_{\alpha} - 12\,f_{\gamma\alpha}{}^{\beta}[t^{\gamma}]_{\mathcal{L}}{}^{\mathcal{K}}\,\partial_{\mathcal{K}}\Lambda^{\mathcal{L}}W_{\beta} \\ + \lambda'\partial_{\mathcal{R}}\Lambda^{\mathcal{R}}W_{\alpha} - \Lambda^{\mathcal{N}}\Theta_{\mathcal{N}}{}^{\gamma}f_{\gamma\alpha}{}^{\beta}W_{\beta}\ , \tag{3.89}$$

[14]It will be convenient for compatibility with [80] to take $\delta_{\Lambda}A_{\mu}{}^{\mathcal{M}} = \mathcal{D}_{\mu}\Lambda^{\mathcal{M}}$ as the variation for the vector fields under generalised diffeomorphisms (cf. the discussion below (3.80)).

where we have used the definition (3.29) and the relation between the generators in the adjoint and the structure constants $[t_\gamma]_\alpha{}^\beta = -f_{\gamma\alpha}{}^\beta$. In order to verify the invariance of the field strength under tensor gauge transformations, it is necessary to study the following expression in terms of a covariant object $W_\alpha$ :

$$T^{\mathcal{M}} \equiv [t^\alpha]^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} W_\alpha + \frac{1}{6}\, Z^{\mathcal{M},\alpha}\, W_\alpha\ . \tag{3.90}$$

Under generalised diffeomorphisms, it transforms as

$$\begin{aligned}\delta_\Lambda T^{\mathcal{M}} = \widetilde{\mathbb{L}}_\Lambda T^{\mathcal{M}} + \Omega^{\mathcal{M}\mathcal{N}} \big( [t^\alpha]_{\mathcal{L}}{}^{\mathcal{K}} W_\alpha \partial_{\mathcal{N}} \partial_{\mathcal{K}} \Lambda^{\mathcal{K}} + \tfrac{1}{12}\, \Theta_{\mathcal{P}}{}^\alpha W_\alpha \partial_{\mathcal{N}} \Lambda^{\mathcal{P}} \big)\\ + (\lambda' - 1) [t^\alpha]^{\mathcal{M}\mathcal{N}} W_\alpha \partial_{\mathcal{N}} \partial_{\mathcal{K}} \Lambda^{\mathcal{K}}\ .\end{aligned} \tag{3.91}$$

where $T^{\mathcal{M}}$ carries weight $\lambda(T^{\mathcal{M}}) = (\lambda' - \frac{1}{2})$ . As in [80], in order to cancel the non-covariant terms in the first line, a compensating field $W_{\mathcal{M}}$ subject to the covariant constraints (3.77),(3.78) is introduced such that the combination

$$\hat{T}^{\mathcal{M}} \equiv T^{\mathcal{M}} + \frac{1}{24}\, \Omega^{\mathcal{M}\mathcal{N}}\, W_{\mathcal{N}}\ , \tag{3.92}$$

transforms covariantly with $\lambda(\hat{T}^{\mathcal{M}}) = \frac{1}{2}$ provided that $\lambda' = 1$. This is ensured only if the compensating field transforms under generalised diffeomorphisms as

$$\delta_\Lambda W_{\mathcal{M}} = \widetilde{\mathbb{L}}_\Lambda W_{\mathcal{M}} - 24\, [t^\alpha]_{\mathcal{L}}{}^{\mathcal{K}} W_\alpha \partial_{\mathcal{M}} \partial_{\mathcal{K}} \Lambda^{\mathcal{L}} - 2\, \Theta_{\mathcal{P}}{}^\alpha\, W_\alpha \partial_{\mathcal{M}} \Lambda^{\mathcal{P}}\ , \tag{3.93}$$

where $\lambda(W_M) = \frac{1}{2}$ . Note that (3.93) preserves the covariant constraints (3.77) and (3.78) by virtue of the section constraint (3.41) and the $X$-constraint (3.42). With the observation that structures of the form (3.92) transform covariantly, it becomes straightforward to verify the invariance of the field strength under both tensor gauge transformations.

The field strengths $H_{\mu\nu\rho\,\alpha}$ and $H_{\mu\nu\rho\,\mathcal{M}}$ associated to the two-forms are defined through the Bianchi identity

$$3\, \mathcal{D}_{[\mu} \mathcal{F}_{\nu\rho]}{}^{\mathcal{M}} = -12\, [t^\alpha]^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} H_{\mu\nu\rho\,\alpha} - 2\, Z^{\mathcal{M},\alpha} H_{\mu\nu\rho\,\alpha} - \frac{1}{2}\, \Omega^{\mathcal{M}\mathcal{N}}\, H_{\mu\nu\rho\,\mathcal{N}}\,, \tag{3.94}$$

up to terms that get projected out under $6\, [t^\alpha]^{\mathcal{M}\mathcal{N}} \partial_{\mathcal{N}} + Z^{\mathcal{M},\alpha}$ . The field strength $H_{\mu\nu\rho\,\mathcal{M}}$ is again covariantly constrained as in (3.77) and (3.78) and transforms according to (3.93) under generalised diffeomorphisms.

### 3.4.3 Bosonic action

In analogy with [80], the full dynamics of the theory can be encoded into a formally $E_{7(7)}$-invariant pseudo-action supplemented by a first-order duality equation[15] for the **56** gauge fields $A_\mu{}^{\mathcal{M}}$

$$\mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = -\tfrac{1}{2}\, e\, \varepsilon_{\mu\nu\rho\sigma}\, \Omega^{\mathcal{M}\mathcal{N}}\, \mathcal{M}_{\mathcal{N}\mathcal{K}} \mathcal{F}^{\rho\sigma\mathcal{K}}\ , \tag{3.95}$$

where $e$ denotes the determinant of the vierbein and $\mathcal{M}_{\mathcal{M}\mathcal{N}}$ is the scalar matrix parameterising the coset space $E_{7(7)}/SU(8)$. This ensures that only half of the vectors are independent.

The field equations can be conveniently derived by varying the following gauge invariant pseudo-action, and subsequently imposing (3.95):

$$\begin{aligned} S_{\text{XFT}} = \int d^4x\, d^{56}y\, e\, \big[\, \hat{R}(X) \,+\, \tfrac{1}{48}\, g^{\mu\nu}\, \mathcal{D}_\mu \mathcal{M}^{\mathcal{M}\mathcal{N}}\, \mathcal{D}_\nu \mathcal{M}_{\mathcal{M}\mathcal{N}} \\ -\, \tfrac{1}{8}\, \mathcal{M}_{\mathcal{M}\mathcal{N}}\, \mathcal{F}^{\mu\nu\mathcal{M}} \mathcal{F}_{\mu\nu}{}^{\mathcal{N}} + e^{-1}\, \mathcal{L}_{\text{top}}(X) - V_{\text{XFT}}(\mathcal{M}, g, X)\, \big]\, . \end{aligned} \tag{3.96}$$

For the purpose of this chapter we shall always assume that integration over the internal space is actually performed only on the physical coordinates after choosing a solution of the section constraint, so that rigid integration over the internal manifold is well defined. While the general form of the action matches the one of EFT, the differences with the latter lie in the expressions of the field strengths, the covariant derivatives and the scalar potential, $V_{\text{XFT}}$, which explicitly depend on the $X$ deformation. As in EFT, the XFT action is uniquely determined by requiring gauge invariance under the bosonic symmetries. More specifically, each term in (3.96) is invariant under internal generalised diffeomorphisms while the relative coefficients are fixed by external diffeomorphisms.

In what follows we discuss the invariance of the different terms under vector (*i.e.* generalised diffeomorphisms) and tensor gauge transformations. In the forthcoming computations we will consistently drop all the vector gauge transformations of scalar density of weight 1. Indeed, these take the form of boundary terms in the extended space.

[15]We remind that in this chapter, we do not adopt Pauli-Källén conventions for the external spacetime.

**The kinetic terms:** The first term in the action is the Einstein-Hilbert term. As in EFT, it is built from a modified Riemann tensor

$$\hat{R}_{\mu\nu}{}^{ab}(X) = R_{\mu\nu}{}^{ab}[\omega] + \mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\, e^{a\rho}\, \partial_{\mathcal{M}} e_{\rho}{}^{b} \,, \tag{3.97}$$

where the curvature of the four dimensional spin connection $\omega_\mu{}^{ab}$ reads

$$R_{\mu\nu}{}^{ab}[\omega] = 2\, \mathcal{D}_{[\mu}\omega_{\nu]}{}^{ab} - 2\, \omega_{[\mu}{}^{ac}\, \omega_{\nu]c}{}^{b} \,. \tag{3.98}$$

The second term in (3.97) has been added in order for the modified Riemann tensor to transform covariantly under the four dimensional local Lorentz transformations acting on the spin connection as $\delta_\lambda \omega_\mu{}^{ab} = -\mathcal{D}_\mu \lambda^{ab}$. The spin connection can in turn be expressed via Cartan's (covariantised) first structure equation in terms of the vierbein $e_\mu{}^a$ which is an $\mathrm{E}_{7(7)}$ scalar of weight $\frac{1}{2}$. Consequently, the spin connection and the Riemann tensor both carry weight 0. Furthermore, using the section constraint and the $X$-constraint, it is straightforward to show that the internal derivative of an $\mathrm{E}_{7(7)}$ scalar $S$ of weight $\lambda(S)$ transforms under vector gauge transformations as

$$\delta_\Lambda(\partial_{\mathcal{M}} S) = \widetilde{\mathbb{L}}_\Lambda(\partial_{\mathcal{M}} S) + \lambda(S)\, S\, \partial_{\mathcal{M}}\partial_{\mathcal{N}}\Lambda^{\mathcal{N}} \,, \quad \text{with} \quad \lambda(\partial_{\mathcal{M}} S) = \lambda(S) - \tfrac{1}{2} \,. \tag{3.99}$$

Hence, the modified Riemann tensor does not transform covariantly due to the second term in (3.97). The non-covariant part of the variation vanishes when contracted with vierbeine and therefore, the modified Ricci scalar $\hat{R}(X)$ is a scalar of weight -1. This proves the invariance of the XFT Einstein-Hilbert term under gauge transformations.

The second and third term in (3.96) are respectively the kinetic terms for the scalars and the vector fields. They only differ from the ones in EFT by the implicit presence of the $X$ deformation. The scalar matrix $\mathcal{M}_{\mathcal{M}\mathcal{N}}$ is a tensor of weight 0 while $\mathcal{F}_{\mu\nu}{}^{\mathcal{N}}$ carries weight $\frac{1}{2}$. Using (3.87) and $\delta_\Xi \mathcal{F}_{\mu\nu}{}^{\mathcal{M}} = 0$, it is clear that both terms are invariant under vector and tensor gauge transformations.

**The topological term:** Following [80], we present the topological term as a surface term in five spacetime dimensions

$$\begin{aligned} S_{\text{top}}(X) &= -\frac{1}{24} \int_{\Sigma^5} d^5x \int d^{56}y\, \varepsilon^{\mu\nu\rho\sigma\tau}\, \mathcal{F}_{\mu\nu}{}^{\mathcal{M}}\, \mathcal{D}_\rho \mathcal{F}_{\sigma\tau\,\mathcal{M}} \\ &\equiv \int_{\partial\Sigma^5} d^4x \int d^{56}y\, \mathcal{L}_{\text{top}}(X) \,, \end{aligned} \tag{3.100}$$

where once again the difference with EFT lies in the definition of the field strength and the covariant derivative. Although this term is manifestly gauge invariant, its general variation is needed to derive the field equations for the vectors and two-forms

$$\begin{aligned}\delta\mathcal{L}_{\rm top} = -\frac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}\big[\delta A_\mu{}^{\mathcal{M}}\,\mathcal{D}_\nu\mathcal{F}_{\rho\sigma\,\mathcal{M}} \\ + \mathcal{F}_{\mu\nu\,\mathcal{M}}\big(6\,[t^\alpha]^{\mathcal{M}\mathcal{N}}\partial_{\mathcal{N}}\Delta B_{\rho\sigma\,\alpha} + Z^{\mathcal{M},\alpha}\Delta B_{\rho\sigma\,\alpha} + \tfrac{1}{4}\,\Omega^{\mathcal{M}\mathcal{N}}\Delta B_{\rho\sigma\,\mathcal{N}}\big)\big]\,.\end{aligned} \tag{3.101}$$

This requires to use the Bianchi identity (3.94) and the fact that for any three vectors of weight $\frac{1}{2}$ the following identity holds

$$\Omega_{\mathcal{M}\mathcal{N}}\,U^{\mathcal{M}}\big\{V,W\big\}_X^{\mathcal{N}} + \text{cyclic} \,=\, 12\,[t_\alpha]_{(\mathcal{M}}{}^{\mathcal{Q}}[t^\alpha]_{\mathcal{N}\mathcal{P})}\,\partial_{\mathcal{Q}}(U^{\mathcal{M}}V^{\mathcal{N}}W^{\mathcal{P}})\,. \tag{3.102}$$

The $X$-dependent part of the left-hand side vanishes using (3.70), and hence the identity takes the same form as in EFT. Using these results one can explicitly verify that (3.101) vanishes for vector and tensor gauge transformations.

**The potential:** The scalar potential of XFT takes the following form:

$$V_{\rm XFT}(\mathcal{M},g,X) \,=\, V_{\rm EFT}(\mathcal{M},g) \,+\, V_{\rm SUGRA}(\mathcal{M},X) \,+\, V_{\rm cross}(\mathcal{M},X)\,, \tag{3.103}$$

where the scalar potential of EFT is independent of the $X$ deformation

$$\begin{aligned}V_{\rm EFT} = &-\frac{1}{48}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}\,\partial_{\mathcal{M}}\mathcal{M}^{\mathcal{K}\mathcal{L}}\,\partial_{\mathcal{N}}\mathcal{M}_{\mathcal{K}\mathcal{L}} + \frac{1}{2}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}\,\partial_{\mathcal{M}}\mathcal{M}^{\mathcal{K}\mathcal{L}}\,\partial_{\mathcal{L}}\mathcal{M}_{\mathcal{N}\mathcal{K}} \\ &-\frac{1}{2}\,g^{-1}\partial_{\mathcal{M}}g\,\partial_{\mathcal{N}}\mathcal{M}^{\mathcal{M}\mathcal{N}} - \frac{1}{4}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}\,g^{-1}\partial_{\mathcal{M}}g\,g^{-1}\partial_{\mathcal{N}}g \\ &-\frac{1}{4}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}\,\partial_{\mathcal{M}}g^{\mu\nu}\,\partial_{\mathcal{N}}g_{\mu\nu}\,,\end{aligned} \tag{3.104}$$

while the parts exclusive to XFT are

$$V_{\rm SUGRA} = \frac{1}{168}\,\big[\,X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}X_{\mathcal{Q}\mathcal{R}}{}^{\mathcal{S}}\mathcal{M}^{\mathcal{M}\mathcal{Q}}\mathcal{M}^{\mathcal{N}\mathcal{R}}\mathcal{M}_{\mathcal{P}\mathcal{S}} + 7X_{\mathcal{M}\mathcal{N}}{}^{\mathcal{P}}X_{\mathcal{Q}\mathcal{P}}{}^{\mathcal{N}}\mathcal{M}^{\mathcal{M}\mathcal{Q}}\,\big]\,, \tag{3.105}$$

and

$$V_{\rm cross} = \frac{1}{12}\,\mathcal{M}^{\mathcal{M}\mathcal{N}}\mathcal{M}^{\mathcal{K}\mathcal{L}}X_{\mathcal{M}\mathcal{K}}{}^{\mathcal{P}}\,\partial_{\mathcal{N}}\mathcal{M}_{\mathcal{P}\mathcal{L}}\,. \tag{3.106}$$

The full potential boils down to the one of EFT when the $X$ deformation is set to zero. Additionally, it precisely reduces to the potential of $D = 4$ gauged maximal supergravity (3.105) when the fields are taken to be $y^{\mathcal{M}}$-independent.[16] The term in (3.106) is a purely novel feature as it is absent in both EFT and gauged maximal supergravity.

We finally give a few guidelines on the construction of the XFT potential. The various terms and coefficients in (3.103) are uniquely determined by requiring invariance under vector gauge transformations up to boundary terms, while each term is manifestly invariant under tensor gauge transformations. Throughout the computation, one has to repeatedly make use of the section constraint, the linear (or representation) constraint and the $X$-constraint. The starting point is the variation of the EFT potential under vector gauge transformations which can easily be computed using (3.99) and

$$\begin{aligned}\delta_\Lambda(\partial_{\mathcal{M}}\mathcal{M}_{\mathcal{K}\mathcal{L}}) = \widetilde{\mathbb{L}}_\Lambda(\partial_{\mathcal{M}}\mathcal{M}_{\mathcal{K}\mathcal{L}}) + 2\,\mathcal{M}_{\mathcal{N}(\mathcal{K}}\,\partial_{\mathcal{L})}\partial_{\mathcal{M}}\Lambda^{\mathcal{N}} + \mathcal{M}_{\mathcal{K}\mathcal{L}}\,\partial_{\mathcal{M}}\partial_{\mathcal{N}}\Lambda^{\mathcal{N}} \\ - 2\,Y^{\mathcal{Q}\mathcal{R}}{}_{\mathcal{N}(\mathcal{K}}\mathcal{M}_{\mathcal{L})\mathcal{Q}}\,\partial_{\mathcal{M}}\partial_{\mathcal{R}}\,\Lambda^{\mathcal{N}} + 2\,X_{\mathcal{N}(\mathcal{K}}{}^{\mathcal{Q}}\mathcal{M}_{\mathcal{L})\mathcal{Q}}\,\partial_{\mathcal{M}}\Lambda^{\mathcal{N}}\,,\end{aligned} \tag{3.107}$$

where $\lambda(\partial_{\mathcal{M}}\mathcal{M}_{\mathcal{K}\mathcal{L}}) = \frac{1}{2}$. After the cancellations described in [80], the only non-covariant variations remaining are the ones depending (linearly) on the $X$ deformation. In order to cancel them, one needs to add counterterms to the potential which are of first order in the derivatives and the $X$. The only term[17] of this type which does not vanish by virtue of the various constraints is (3.106). At this stage of the computation, it is important to realise that both the $X$ and the combination $\mathcal{M}^{-1}X\mathcal{M}$ take value in the $\mathrm{E}_{7(7)}$ Lie algebra. Consequently, the adjoint projector satisfies

$$\begin{aligned}(\mathbb{P}_{\mathbf{133}})^{\mathcal{M}}{}_{\mathcal{N}}{}^{\mathcal{K}}{}_{\mathcal{L}}\,X_{\mathcal{P}\mathcal{K}}{}^{\mathcal{L}} = X_{\mathcal{P}\mathcal{N}}{}^{\mathcal{M}}\,, \\ (\mathbb{P}_{\mathbf{133}})^{\mathcal{M}}{}_{\mathcal{N}}{}^{\mathcal{K}}{}_{\mathcal{L}}\,\mathcal{M}^{\mathcal{L}\mathcal{P}}X_{\mathcal{Q}\mathcal{P}}{}^{\mathcal{R}}\mathcal{M}_{\mathcal{R}\mathcal{K}} = \mathcal{M}^{\mathcal{M}\mathcal{P}}X_{\mathcal{Q}\mathcal{P}}{}^{\mathcal{R}}\mathcal{M}_{\mathcal{R}\mathcal{N}}\,.\end{aligned} \tag{3.108}$$

The vector gauge transformation of (3.106) also yields additional non-covariant variations which are quadratic in $X$. These must be cancelled by extending further the potential with counterterms quadratic in $X$ and that do not contain derivatives. It again turns out that (3.105) are the only non-vanishing terms of this type.

Finally, as already mentioned in the EFT case [80], it should be possible to write down a true Lagrangian formulation of $\mathrm{E}_{7(7)}$ XFT by choosing a symplectic frame that selects 28 physical vector fields. In this case, all the formally $E_{7(7)}$-covariant

[16] The different normalisation of $V_{\mathrm{SUGRA}}$ with respect to [44] is due to the different normalisation of the Einstein–Hilbert term.

[17] Up to equivalent rewriting using the linear constraint for the $X$ deformation.

field equations can be derived by varying the Lagrangian. However, just as in $D = 4$ maximal supergravity [144], the Lagrangian itself is not $\mathrm{E}_{7(7)}$-invariant. Moreover, the kinetic terms for the physical vectors are not invariant under generalised diffeomorphisms, and their variations must cancel against that of a new topological term. Such true Lagrangian formulations have been discussed in the context of double field theory [145].

# Chapter 4

# $N = 4$ conformal supergravity

This final chapter is somewhat removed from the topics discussed in chapter 2 and 3. Here, we present the construction of all $N = 4$ conformal supergravity actions. We find that these are encoded in a function of the scalar fields that parametrize an SU(1,1)/U(1) coset space. When the function equals a constant, the theory is invariant under rigid continuous $\mathrm{SU}(1,1)$ transformations. For a short introduction to the symmetries and mechanisms underlying conformal supergravity, we point the reader to section 1.2, where a review of the construction of the $N = 0$ theory (i.e. conformal gravity) was given. This chapter is based on [146].

## 4.1 Introduction

Conformal supergravities in four dimensions are invariant under the local symmetries associated with the superconformal algebra $\mathfrak{su}(2,2|N)$. They are supersymmetric generalizations of the conformally invariant action of general relativity quadratic in the Weyl tensor (1.23). The latter corresponds to the traceless part of the Riemman tensor and can be conveniently expressed in terms of the Ricci tensor and Ricci scalar as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - 2\, g_{[\mu[\rho}\, R_{\sigma]\nu]} + \frac{1}{3}\, R\, g_{[\mu[\rho} g_{\sigma]\nu]}\,. \tag{4.1}$$

For $N = 1$ and 2 conformal supergravity, the transformation rules and the corresponding invariant Lagrangians are known [147, 148]. For $N = 4$ the full non-linear transformation rules of the fields, which constitute the off-shell Weyl supermultiplet, have been determined [97]. This is the largest possible conformal supergravity that can exist in four space-time dimensions [149], and until recently a complete action was not known. A unique feature of the $N = 4$ theory is the presence of dimensionless scalar fields that parametrize an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space. The U(1) factor is realized as a local symmetry with a composite connection, and acts chirally on the fermions. It is also worth pointing out that, similarly to the non-supersymmetric case (cf. section 1.2), pure $N = 4$ Poincaré supergravity is gauge equivalent to a system of six on-shell $N = 4$ abelian vector multiplets in a conformal supergravity background. The Poincaré theory is recovered after an appropriate gauge fixing of some of the superconformal symmetries [98]. From this perspective, the connection between the rigid $\mathrm{SU}(1,1)$ duality symmetry of the $N = 4$ Poincaré theory and the coset structure of the scalar sector of $N = 4$ conformal supergravity becomes apparent.

Already in the early works on conformal supergravity, the issue of 'non-minimal' couplings, where the terms quadratic in the Weyl tensor are multiplied by a function of the scalar fields, was raised. The possible existence of such Lagrangians was indeed suggested long ago in [101, 150]. Meanwhile indirect evidence came from string theory, where the threshold corrections in the effective action of IIA string compactifications on $\mathrm{K3} \times T^2$ revealed the presence of terms proportional to the square of the Weyl tensor multiplied by a modular function [151]. The same terms emerge in the semiclassical approximation of microscopic degeneracy formulae for dyonic BPS black holes [152–154]. Finally higher-derivative couplings derived for $N = 4$ Poincaré supergravity [155, 156] also exhibit non-trivial scalar interactions. These various results presciently led to expect that a large variety of $N = 4$ conformal supergravity actions were to exist.

In this chapter, we present the construction of all $N = 4$ conformal supergravity actions. These turn out to be encoded in a holomorphic function that is homogeneous of zeroth degree in scalar fields that parametrize the SU(1,1)/U(1) coset space [146]. The bosonic part of this class of actions will be presented explicitly while the fermionic terms will be reported in an upcoming paper [102]. When the function of the coset scalars is chosen to be constant, the bosonic action turns out to agree with a recent result derived by imposing supersymmetry on terms that are at most quadratic in the fermions [100]. In this case, the action is also invariant under continuous rigid $\mathrm{SU}(1,1)$ transformations. For a generic function, this symmetry is however broken.

The organisation of this chapter is as follows: in 4.2 and 4.3 we review the field content and the non-linear transformation rules of $N = 4$ conformal supergravity, respectively. In Section 4.4, we construct a so-called density formula which captures the form of the most general invariant action in terms of arbitrary functions of the Weyl multiplet fields. These functions, or composites, must obey a set of constraints to guarantee the invariance of the density formula under the (off-shell) $N = 4$ superconformal transformations. In Section 4.5, we construct explicit expressions for these functions in terms of the Weyl multiplet fields and show that they are left to depend on a holomorphic function of the coset scalars. The bosonic terms of the resulting invariant action are then explicitly presented. Finally in 4.6, we comment on the relation of the conformal theory with Poincaré supergravity and the potential applications of our result.

## 4.2 The $N = 4$ Weyl supermultiplet

$N = 4$ conformal supergravity [97] is built upon the gauging of the superconformal algebra $\mathfrak{su}(2,2|4)$. Its bosonic subalgebra[1] contains the generators of the conformal group $\mathrm{SU}(2,2)$ (1.19), and the generators of a chiral SU(4) R-symmetry. The fermionic generators consist of sixteen Q-supercharges and sixteen S-supercharges. In addition, the theory contains a local chiral U(1) symmetry which extends the R-symmetry group to $\mathrm{SU}(4) \times \mathrm{U}(1)$.

The fields of the theory describe 128+128 bosonic and fermionic off-shell degrees of freedom. Among them are gauge fields corresponding to the various local symmetries which include the vierbein $e_\mu{}^a$, SU(4) gauge fields $V_\mu{}^i{}_j$, a gauge field $b_\mu$ associated with dilatations as well as the gravitini $\psi_\mu{}^i$ (the gauge fields of Q-supersymmetry). In analogy with the $N = 0$ case presented in section 1.2, there is also a set of composite gauge fields, namely the spin connection $\omega_\mu{}^{ab}$, the gauge field $f_\mu{}^a$ associated with conformal boosts, the U(1) gauge field $a_\mu$ and the S-supersymmetry gauge fields $\phi_{\mu i}$. These do not count as independent degrees of freedom as they are expressed in terms of the other fields of the theory via a set of conventional constraints presented in Appendix B. All of the gauge fields are summarised in Table 4.1 along with their algebraic restrictions, their SU(4) representation, their Weyl weight $w$ under local dilatations and their U(1) chiral weight $c$. In this chapter, $\mu, \nu, \ldots$ and $a, b, \ldots$ denote four-dimensional spacetime and tangent space indices, respectively. The indices $i, j, \ldots$ label the fundamental

[1] The optional U(1) central charge is suppressed [59]. Note that it does not correspond to the one of the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space.

representation of SU(4). We adopt Pauli-Källen conventions which imply that all gamma-matrices are hermitian.

Table 4.1: Gauge fields of $N = 4$ conformal supergravity.
The gauge fields listed in red are composite fields.

| | Field | Symmetry (Generator) | Name/Restrictions | SU(4) | $w$ | $c$ |
|---|---|---|---|---|---|---|
| Bosons | $e_\mu{}^a$ | Translations (P) | vierbein | $\mathbf{1}$ | $-1$ | 0 |
| | $\omega_\mu{}^{ab}$ | Lorentz (M) | spin connection | $\mathbf{1}$ | 0 | 0 |
| | $b_\mu$ | Dilatation (D) | dilatational gauge field | $\mathbf{1}$ | 0 | 0 |
| | $V_\mu{}^i{}_j$ | SU(4) (V) | SU(4) gauge field | $\mathbf{15}$ | 0 | 0 |
| | | | $V_{\mu i}{}^j \equiv (V_\mu{}^i{}_j)^* = -V_\mu{}^j{}_i$ | | | |
| | | | $V_\mu{}^i{}_i = 0$ | | | |
| | $f_\mu{}^a$ | Conformal boosts (K) | K-gauge field | $\mathbf{1}$ | 1 | 0 |
| | $a_\mu$ | U(1) | U(1) gauge field | $\mathbf{1}$ | 0 | 0 |
| Fermions | $\phi_{\mu_i}$ | S-supersymmetry (S) | S-gauge field | $\overline{\mathbf{4}}$ | $\frac{1}{2}$ | $\frac{1}{2}$ |
| | | | $\gamma_5\, \phi_{\mu_i} = \phi_{\mu_i}$ | | | |
| | $\psi_\mu{}^i$ | Q-supersymmetry (Q) | gravitino; $\gamma_5\, \psi_\mu^i = \psi_\mu^i$ | $\mathbf{4}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ |

In addition to the gauge fields, the theory contains complex anti-selfdual tensor fields $T_{ab}{}^{ij}$, whose complex conjugates are the selfdual fields $T_{abij}$, complex scalars $E_{ij}$, pseudo-real scalars $D^{ij}{}_{kl}$ and two spin-1/2 fermions $\chi^{ij}{}_k$ and $\Lambda_i$. Finally there exists a doublet of complex scalars $\phi_\alpha$, which are invariant under dilatations and transform under rigid SU(1, 1) transformations ($\alpha = 1, 2$). They are subject to the SU(1, 1) invariant constraint

$$\phi^\alpha\, \phi_\alpha = 1\,, \quad \phi_1 \equiv (\phi^1)^*\,, \quad \phi_2 \equiv -(\phi^2)^*\,, \tag{4.2}$$

and as a result the fields $\phi^\alpha$ and $\phi_\alpha$ parametrise SU(1, 1) matrices. Because these fields are also subject to the local U(1) symmetry, they describe two physical degrees of freedom associated with an SU(1, 1)/U(1) coset space[2]. These various matter fields[3] are listed in Table 4.2 along with their various algebraic properties, and their representation assignments. Note that only the positive chirality fermions were presented in Table 4.1 and 4.2. The negative chirality fermions transform under the corresponding SU(4) conjugate representations and carry opposite chiral weight.

[2] Just as the scalars of type IIB supergravity which were presented in 2.2.

[3] The term "matter" here is used for lack of a better word. These fields are part of the Weyl supermultiplet and therefore have no relation with components of $N = 4$ matter multiplets.

Table 4.2: Matter fields of $N = 4$ conformal supergravity

| | Field | Properties | SU(4) | $w$ | $c$ |
|---|---|---|---|---|---|
| Bosons | $\phi_\alpha$ | $\phi^\alpha = \eta^{\alpha\beta}\,\phi_\alpha,\ \phi_\alpha\,\phi^\alpha = 1$ | $\mathbf{1}$ | 0 | $-1$ |
| | $E_{ij}$ | $E_{ij} = E_{ji}$ | $\overline{\mathbf{10}}$ | 1 | $-1$ |
| | $T_{ab}{}^{ij}$ | $\frac{1}{2}\varepsilon_{ab}{}^{cd}T_{cd}{}^{ij} = -T_{ab}{}^{ij}$ | $\mathbf{6}$ | 1 | $-1$ |
| | | $T_{ab}{}^{ij} = -T_{ab}{}^{ji}$ | | | |
| | $D^{ij}{}_{kl}$ | $D^{ij}{}_{kl} = \frac{1}{4}\varepsilon^{ijmn}\varepsilon_{klpq}D^{pq}{}_{mn}$ | $\mathbf{20'}$ | 2 | 0 |
| | | $D_{kl}{}^{ij} \equiv (D^{kl}{}_{ij})^* = D^{ij}{}_{kl}$ | | | |
| | | $D^{ij}{}_{kj} = 0$ | | | |
| Fermions | $\Lambda_i$ | $\gamma_5\Lambda_i = \Lambda_i$ | $\overline{\mathbf{4}}$ | $\frac{1}{2}$ | $-\frac{3}{2}$ |
| | $\chi^{ij}{}_k$ | $\gamma_5\chi^{ij}{}_k = \chi^{ij}{}_k;\ \chi^{ij}{}_k = -\chi^{ji}{}_k$ | $\mathbf{20}$ | $\frac{3}{2}$ | $-\frac{1}{2}$ |
| | | $\chi^{ij}{}_j = 0$ | | | |

In view of their central role in the construction of invariant actions, we present some further details regarding the scalar fields $\phi_\alpha$ and $\phi^\alpha$, which we refer to as the holomorphic and the anti-holomorphic fields, respectively. The holomorphic fields are invariant under S-supersymmetry and transform under Q-supersymmetry into the positive chirality spinors $\Lambda_i$

$$\delta_S\,\phi_\alpha = 0\,, \qquad \delta_Q\,\phi_\alpha = -\bar{\epsilon}^i\Lambda_i\,\varepsilon_{\alpha\beta}\,\phi^\beta\,. \tag{4.3}$$

where $\epsilon^i$ denotes the Q-supersymmetry parameter. The supersymmetry transformations of the remaining fields is postponed to the next section. The supercovariant constraint that determines the expression of the composite U(1) gauge field $a_\mu$ and the generalized supercovariant derivatives of the coset fields, $P_a$ and $\bar{P}_a$, are defined by

$$\begin{aligned} \phi^\alpha D_a\phi_\alpha &= -\tfrac{1}{4}\bar{\Lambda}^i\gamma_a\Lambda_i\,, \\ P_a = \phi^\alpha\,\varepsilon_{\alpha\beta}\,D_a\phi^\beta\,, &\quad \bar{P}_a = -\phi_\alpha\,\varepsilon^{\alpha\beta}\,D_a\phi_\beta\,, \end{aligned} \tag{4.4}$$

where $D_a$ denotes the derivative covariantised with respect to all the superconformal symmetries as well as the local U(1) symmetry. Observe that $P_a$ and $\bar{P}_a$ carry Weyl weight $w = 1$ and chiral weights $c = 2$ and $c = -2$, respectively. From these definitions one may derive the supercovariant extension of the Maurer-Cartan equations associated with the SU(1,1)/U(1) coset space,

$$
\begin{aligned}
F(a)_{ab} &= -\,2i\,P_{[a}\,\bar P_{b]} - \tfrac{1}{2}i\big(\bar\Lambda^i\gamma_{[a}D_{b]}\Lambda_i - \text{h.c.}\big)\,,\\
D_{[a}P_{b]} &= -\,\tfrac{1}{2}\bar\Lambda_i\gamma_{[a}\Lambda^i\,P_{b]} + \tfrac{1}{4}\bar\Lambda^i\,R(Q)_{ab\,i}\,,
\end{aligned}
\tag{4.5}
$$

where $F(a)_{ab}$ and $R(Q)_{ab\,i}$ denote the supercovariant U(1) and Q-supersymmetry curvatures, respectively. The superconformal curvatures and their transformation rules are presented in Appendix B. Note that the expressions (4.3) and (4.4), when combined with those for the anti-holomorphic fields, reflect the structure of the three left-invariant vector fields associated with the group SU(1,1),

$$
\begin{aligned}
\mathcal{D}^0 &= \phi^\alpha\,\frac{\partial}{\partial\phi^\alpha} - \phi_\alpha\,\frac{\partial}{\partial\phi_\alpha}\,,\\
\mathcal{D}^\dagger &= \phi_\alpha\,\varepsilon^{\alpha\beta}\,\frac{\partial}{\partial\phi^\beta}\,,\qquad \mathcal{D} = -\phi^\alpha\,\varepsilon_{\alpha\beta}\,\frac{\partial}{\partial\phi_\beta}\,,
\end{aligned}
\tag{4.6}
$$

which satisfy the commutation relations

$$
\big[\mathcal{D}^0,\mathcal{D}\big] = 2\,\mathcal{D}\,,\qquad \text{and}\qquad \big[\mathcal{D},\mathcal{D}^\dagger\big] = \mathcal{D}^0\,. \tag{4.7}
$$

Using these definitions, the Q-supersymmetry variation and the supercovariant derivative of arbitrary functions of the coset scalars $\mathcal{H}(\phi_\alpha,\phi^\beta)$ can be written as

$$
\begin{aligned}
\delta\mathcal{H} &= -\,\big[\bar\epsilon^i\Lambda_i\,\mathcal{D} + \bar\epsilon_i\Lambda^i\,\mathcal{D}^\dagger\big]\,\mathcal{H}\,,\\
D_a\mathcal{H} &= \big[\bar P_a\,\mathcal{D} + P_a\,\mathcal{D}^\dagger + \tfrac{1}{4}\bar\Lambda^i\gamma_a\Lambda_i\,\mathcal{D}^0\big]\,\mathcal{H}\,.
\end{aligned}
\tag{4.8}
$$

Of particular importance for the next sections are functions $\mathcal{H}(\phi_\alpha)$ that are homogeneous of zeroth degree in the holomorphic variables, so that $\mathcal{D}^\dagger\mathcal{H}(\phi_\alpha)=0$ and $\mathcal{D}^0\mathcal{H}(\phi_\alpha)=0$. Using the above commutation relations, it then follows that $\mathcal{D}^\dagger\,\mathcal{D}^n\mathcal{H}(\phi_\alpha)\propto\mathcal{D}^{n-1}\mathcal{H}(\phi_\alpha)$ for $n>1$, and vanishes for $n=1$ so that $\mathcal{D}\mathcal{H}(\phi_\alpha)$ is holomorphic while $\mathcal{D}^2\mathcal{H}$ is not.

## 4.3 Superconformal transformation rules

Let us now come back to the various fields presented in Tables 4.1 and 4.2. Their full non-linear superconformal transformation rules were derived in [97], and are such that the superconformal algebra closes off-shell on the independent fields which therefore constitute the $N=4$ Weyl supermultiplet. We refrain from presenting the bosonic symmetry transformations as they simply follow from the various properties of the fields given in the Tables 4.1 and 4.2 and from the usual conformal algebra (1.19). It should nevertheless be mentioned that all matter fields are invariant under conformal boosts. We then turn to the Q-supersymmetry variations of the gauge fields which read

$$
\begin{aligned}
\delta_Q\, e_\mu{}^a &= \bar\epsilon^i\gamma^a\psi_{\mu i} + \text{h.c.}\,,\\
\delta_Q\, \omega_\mu{}^{ab} &= -\tfrac12\,\bar\epsilon^i\gamma^{ab}\phi_{\mu i} + \bar\epsilon^i\gamma_\mu R(Q)^{ab}{}_i - 2\,T^{ab}{}_{ij}\,\bar\epsilon^i\psi_\mu{}^j + \text{h.c.}\,,\\
\delta_Q\, V_\mu{}^i{}_j &= \bar\epsilon^i\phi_{\mu j} + \bar\epsilon^k\gamma_\mu\chi^i{}_{kj} - \tfrac12\,\varepsilon_{jkmn}E^{ik}\,\bar\epsilon^m\psi_\mu{}^n - \tfrac16\,E^{ik}\bar\epsilon_j\gamma_\mu\Lambda_k\\
&\quad + \tfrac14\,\varepsilon^{iklm}\,T^{ab}{}_{lj}\,\bar\epsilon_k\gamma_{ab}\gamma_\mu\Lambda_m + \tfrac13\,\bar\epsilon^i\gamma_\mu \not{P}\Lambda_j\\
&\quad - \tfrac14\,\varepsilon^{iklp}\varepsilon_{jmnp}\,\bar\epsilon^m\gamma_a\psi_{\mu k}\,\bar\Lambda_l\gamma^a\Lambda^n - (\text{h.c.; traceless})\,,\\
\delta_Q\, b_\mu &= \tfrac12\,\bar\epsilon^i\phi_{\mu i} + \text{h.c.}\,,\\
\delta_Q\, f_\mu{}^a &= -\tfrac18\,e_{\mu b}\,\varepsilon^{abcd}\,\bar\epsilon_i R(S)_{cd}{}^i - \bar\epsilon_i\gamma_\mu D_b R(Q)^{ab\,i} - 2\,T_{\mu b}{}^{ij}\,\bar\epsilon_i R(Q)^{ab}{}_j\\
&\quad + \text{h.c.} + [\text{terms}\ \propto\psi_\mu]\,,\\
\delta_Q\, a_\mu &= \tfrac12\,i\bar\epsilon_i\gamma_\mu\bar{\not{P}}\Lambda^i + \tfrac14\,iE_{ij}\,\bar\Lambda^i\gamma_\mu\epsilon^j + \tfrac18\,i\,\varepsilon_{ijkl}\,T_{ab}{}^{kl}\,\bar\Lambda^i\gamma_\mu\gamma^{ab}\epsilon^j\\
&\quad - \tfrac14\,i(\bar\Lambda^i\gamma_a\Lambda_j - \delta^i_j\,\bar\Lambda^k\gamma_a\Lambda_k)\,\bar\epsilon_i\gamma^a\psi_\mu{}^j + \text{h.c.}\,,\\
\delta_Q\, \psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac12\,\gamma^{ab}T_{ab}{}^{ij}\gamma_\mu\epsilon_j + \varepsilon^{ijkl}\,\bar\psi_{\mu j}\epsilon_k\,\Lambda_l\,,\\
\delta_Q\, \phi_\mu{}^i &= -2 f_\mu{}^a\gamma_a\epsilon^i + \tfrac14\,T_{ab}{}^{ij}\,T^{cd}{}_{jk}\,\gamma_{cd}\gamma_\mu\gamma^{ab}\epsilon^k - \tfrac14\,\bar\epsilon_j R(Q)_{ab\,k}\,\gamma^{ab}\gamma_\mu\Lambda_l\varepsilon^{ijkl}\\
&\quad + \tfrac16\big[\gamma_\mu\gamma^{ab} - 3\,\gamma^{ab}\gamma_\mu\big]\big[R(V)_{ab}{}^i{}_j\,\epsilon^j - \tfrac12\,iF_{ab}\,\epsilon^i + \tfrac12\,D_aT_{cd}{}^{ij}\gamma^{cd}\gamma_b\epsilon_j\big]\\
&\quad + [\text{terms}\ \propto\psi_\mu\ \text{or}\ \phi_\mu]\,,
\end{aligned}
\tag{4.9}
$$

where the complex vectors $P_a$ were defined in (4.4) and where $\mathcal{D}_\mu$ is covariant with respect to the all the bosonic symmetries except the conformal boosts. For instance, we have

$$
\mathcal{D}_\mu\epsilon^i = \big[\partial_\mu - \tfrac14\omega_\mu{}^{ab}\gamma_{ab} + \tfrac12(b_\mu + ia_\mu)\big]\epsilon^i - V_\mu{}^i{}_j\,\epsilon^j\,,
\tag{4.10}
$$

Note here that, contrary to [97], the U(1) gauge field was chosen real. We also recall that the Levi-Civita symbol satisfies $\varepsilon_{\mu\nu\rho\sigma} = e^{-1}e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d\,\varepsilon_{abcd}$, and that $\varepsilon_{0123} = +1$. The variations of the matter fields under Q-supersymmetry are given by

$$
\begin{aligned}
\delta_Q\, \bar P_a &= -\bar\epsilon^i D_a\Lambda_i - \tfrac14\,\bar\Lambda_i\gamma^{bc}\,T_{bc}{}^{ij}\gamma_a\epsilon_i - \tfrac12\,\bar\epsilon^i\Lambda_i\,\bar\Lambda^j\gamma_a\Lambda_j\,,\\
\delta_Q\, \Lambda_i &= -2\,\bar{\not{P}}\epsilon_i + E_{ij}\epsilon^j + \tfrac12\,\varepsilon_{ijkl}\,T_{bc}{}^{kl}\gamma^{bc}\,\epsilon^j\,,\\
\delta_Q\, E_{ij} &= 2\,\bar\epsilon_{(i}\not{D}\Lambda_{j)} - 2\,\bar\epsilon^k\chi^{mn}{}_{(i}\,\varepsilon_{j)kmn} - \bar\Lambda_i\Lambda_j\,\bar\epsilon_k\Lambda^k + 2\,\bar\Lambda_k\Lambda_{(i}\,\bar\epsilon_{j)}\Lambda^k\,,\\
\delta_Q\, T_{ab}{}^{ij} &= 2\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]} + \tfrac12\,\bar\epsilon^k\gamma_{ab}\chi^{ij}{}_k + \tfrac14\,\varepsilon^{ijkl}\,\bar\epsilon_k\gamma^c\gamma_{ab}D_c\Lambda_l - \tfrac16\,E^{k[i}\,\bar\epsilon^{j]}\gamma_{ab}\Lambda_k
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{1}{3}\,\bar\epsilon^{[i}\gamma_{ab}\,\bar{\slashed{P}}\Lambda^{j]}\,,\\
\delta_Q\,\chi^{ij}{}_k =\ & D^{ij}{}_{kl}\,\epsilon^l - \gamma^{ab}R(V)_{ab}{}^{[i}{}_k\,\epsilon^{j]} - \tfrac{1}{2}\,\gamma^{ab}\slashed{D}T_{ab}{}^{ij}\epsilon_k - \tfrac{1}{2}\,\varepsilon^{ijlm}\,\slashed{D}E_{kl}\,\epsilon_m\\
&- \tfrac{1}{2}\,\varepsilon_{klmn}E^{l[i}\gamma^{ab}T_{ab}{}^{j]n}\epsilon^m + \tfrac{1}{2}\,E_{kl}\,E^{l[i}\,\epsilon^{j]} - \tfrac{1}{2}\,\varepsilon^{ijlm}\,\bar{\slashed{P}}\gamma_{ab}T^{ab}{}_{kl}\epsilon_m\\
&+ \tfrac{1}{4}\,\gamma^a\epsilon_n\big[2\,\varepsilon^{ijln}\bar\chi^m{}_{lk} - \varepsilon^{ijlm}\bar\chi^n{}_{lk}\big]\gamma_a\Lambda_m + \tfrac{1}{4}\,\epsilon^{[i}\big[2\,\bar\Lambda^{j]}\slashed{D}\Lambda_k + \bar\Lambda_k\slashed{D}\Lambda^{j]}\big]\\
&- \tfrac{1}{4}\,\gamma^{ab}\epsilon^{[i}\big[2\,\bar\Lambda^{j]}\gamma_aD_b\Lambda_k - \bar\Lambda_k\gamma_aD_b\Lambda^{j]}\big] - \tfrac{5}{12}\,\varepsilon^{ijlm}\Lambda_m\,\bar\epsilon_l\big[E_{kn}\Lambda^n + 2\,\slashed{P}\Lambda_k\big]\\
&+ \tfrac{1}{12}\,\varepsilon^{ijlm}\Lambda_m\,\bar\epsilon_k\,\big[E_{ln}\Lambda^n + 2\,\slashed{P}\Lambda_l\big] - \tfrac{1}{2}\,\gamma^{ab}T_{ab}{}^{ij}\gamma^c\epsilon_{[k}\,\bar\Lambda^l\gamma_c\Lambda_{l]}\\
&- \tfrac{1}{2}\,\gamma^{ab}T_{ab}{}^{l[i}\gamma^c\epsilon_{[k}\,\bar\Lambda^{j]}\gamma_c\Lambda_{l]} + \tfrac{1}{2}\,\epsilon^{[i}\bar\Lambda^{j]}\Lambda^m\,\bar\Lambda_k\Lambda_m - (\text{traces})\,,\\
\delta_Q D^{ij}{}_{kl} =\ & -4\bar\epsilon^{[i}\slashed{D}\chi^{j]}{}_{kl} + \varepsilon_{klmn}\,\bar\epsilon^{[i}\big[-2\,E^{j]p}\chi^{mn}{}_p + \tfrac{1}{2}\gamma^{ab}T_{ab}{}^{mn}\,\overset{\leftrightarrow}{\slashed{D}}\,\Lambda^{j]}\\
&\qquad + \tfrac{1}{3}E^{j]m}E^{np}\Lambda_p - \tfrac{2}{3}\bar{\slashed{P}}\Lambda^m E^{j]n} + \tfrac{1}{2}\gamma^{ab}T_{ab}{}^{mn}\Lambda_p\,\bar\Lambda^{j]}\Lambda^p\big]\\
&+ \bar\epsilon^{[i}\big[2\,\gamma^a\chi^m{}_{kl}\,\bar\Lambda^{j]}\gamma_a\Lambda_m + 2\,\bar{\slashed{P}}\,\gamma_{ab}T^{ab}{}_{kl}\,\Lambda^{j]} + \tfrac{2}{3}\Lambda_{[k}E_{l]m}\,\bar\Lambda^{j]}\Lambda^m\\
&\qquad + \tfrac{1}{6}\gamma^{ab}\slashed{P}\Lambda^{j]}\,\bar\Lambda_k\gamma_{ab}\Lambda_l\big] + \varepsilon^{ijmn}\,\bar\epsilon^p\,T^{ab}{}_{kl}\big[2\,T_{ab\,np}\Lambda_m + T_{ab\,mn}\Lambda_p\big]\\
&+ (\text{h.c.; traceless})\,.
\end{aligned}
\tag{4.11}
$$

We continue with the S-supersymmetry variations of all the fields

$$
\begin{aligned}
\delta_S\,e_\mu{}^a &= 0\,,\\
\delta_S\,\omega_\mu{}^{ab} &= \tfrac{1}{2}\bar\psi_\mu{}^i\gamma^{ab}\eta_i + \text{h.c.}\,,\\
\delta_S\,V_\mu{}^i{}_j &= -\big[\bar\psi_\mu{}^i\eta_j - \tfrac{1}{4}\delta^i_j\,\bar\psi_\mu{}^k\eta_k\big] - \text{h.c.}\,,\\
\delta_S\,b_\mu &= -\tfrac{1}{2}\bar\psi_\mu{}^i\,\eta_i + \text{h.c.}\,,\\
\delta_S\,f_\mu{}^a &= \tfrac{1}{2}\bar\eta_i\gamma^a\phi_\mu{}^i - \tfrac{1}{4}\bar\eta_iR(Q)_\mu{}^{a\,i} + \tfrac{1}{12}\bar\eta_i\gamma^{bc}T_{bc}{}^{ij}\gamma^a\psi_{\mu j} + \text{h.c.}\,,\\
\delta_S\,a_\mu &= 0\,,\\
\delta_S\,\psi_\mu{}^i &= -\gamma_\mu\eta^i\\
\delta_S\,\phi_\mu{}^i &= 2\,\mathcal{D}_\mu\eta^i - \tfrac{1}{6}\gamma_\mu\gamma^{ab}T_{ab}{}^{ij}\eta_j + \tfrac{1}{2}\varepsilon^{ijkl}\bar\eta_k\Lambda_l\psi_{\mu j}\,,\\
\delta_S\,\bar P_a &= -\tfrac{1}{2}\bar\eta^i\gamma_a\Lambda_i\,,\\
\delta_S\,\Lambda_i &= 0\,,\\
\delta_S\,E_{ij} &= 2\,\bar\eta_{(i}\Lambda_{j)}\,,\\
\delta_S\,T_{ab}{}^{ij} &= -\tfrac{1}{4}\varepsilon^{ijkl}\,\bar\eta_k\gamma_{ab}\Lambda_l\,,\\
\delta_S\,\chi^{ij}{}_k &= \tfrac{1}{2}\,T_{ab}\gamma^{ab}\eta_k + \tfrac{2}{3}\,\delta^{[i}_k\,T_{ab}{}^{j]l}\,\gamma^{ab}\eta_l - \tfrac{1}{2}\,\varepsilon^{ijlm}E_{kl}\,\eta_m - \tfrac{1}{4}\,\bar\Lambda_k\gamma^a\Lambda^{[i}\gamma_a\eta^{j]}\\
&\qquad + \tfrac{1}{12}\,\delta^{[i}_k\big[\bar\Lambda_l\gamma^a\Lambda^l\gamma_a\eta^{j]} - \bar\Lambda_l\gamma^a\Lambda^{j]}\,\gamma_a\eta^l\big]\,,\\
\delta_S\,D^{ij}{}_{kl} &= 0\,.
\end{aligned}
\tag{4.12}
$$

where $\eta^i$ denotes the S-supersymmetry parameter.

We close this section by commenting on the properties of the $N = 4$ conformal superspace. As a consequence of the Q-supersymmetry variation of the coset scalars (4.3), the Weyl multiplet defines an SU(1, 1) doublet of chiral superfields with lowest components $\phi_\alpha$ and with fermionic components of positive chirality [97]. They are in fact constrained superfields which explains the apparent discrepancy between their expected number of components and the 128+128 independent degrees of freedom of the Weyl multiplet. This doublet of superfields contains the gravitational degrees of freedom and therefore defines a curved $N = 4$ superspace. In contrast with $N = 2$, the twisted geometry of this superspace is such that it proscribes the existence of independent $N = 4$ chiral supermultiplets coupled to conformal supergravity (other than the Weyl multiplet itself). This is due to the presence of specific superspace torsion components proportional to the spin-1/2 field $\Lambda_i$ which do not preserve the chiral representation. These torsion components are reflected in the $N = 4$ superconformal algebra (see Appendix B) where the commutator of two Q-supersymmetry variations with parameters $\epsilon_1$ and $\epsilon_2$ yields, among other irrelevant transformations,

$$\left[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)\right] \longrightarrow \delta_Q(\epsilon_3^i = \varepsilon^{ijkl}\bar{\epsilon}_{1k}\,\epsilon_{2l}\,\Lambda_j)\,. \tag{4.13}$$

This means that for an arbitrary chiral superfield with lowest components $\{A, \Psi_i, \ldots\}$, such that $\delta_Q\, A = \bar{\epsilon}^i\Psi_i$, the relation (4.13) implies

$$\Psi_{[i}\Lambda_{j]} = 0\,, \tag{4.14}$$

and shows that any chiral superfield must be proportional to the Weyl multiplet. The absence of a chiral superspace formulation in $N = 4$ conformal supergravity undoubtedly makes the construction of invariants much more involved than in the $N = 2$ case.

## 4.4 A density formula

The construction of an invariant action has remained so far an open issue. Since the $N = 4$ superconformal algebra closes off-shell and the full transformation rules are known, one possible approach is to start from the Weyl tensor squared (1.23) (optionally multiplied by an arbitrary function of the coset scalars) and iteratively add terms in order for the total Q-supersymmetry variation to vanish. This supersymmetrization procedure is conceptually straightforward and, provided sufficient computational efforts are invested, it is guaranteed to produce a superconformal invariant. In practice however, such a computation rapidly becomes unmanageable due to the rich field content and the non-linearity of the transformation rules. This method was used in [100] to obtain an action up to terms which are at most quadratic in the fermions.

In this section, we will follow a somewhat different path and construct the most general invariant action by making use of the fact that any supersymmetric component Lagrangian can be written as the Hodge dual of a four-form built in terms of the vierbein, gravitini, and possibly other connections, multiplied by supercovariant coefficient functions that we will treat as composite fields. This approach is known as the superform method [157–159]. As explained previously, $N = 4$ chiral superspace does not exist and therefore we aim to construct such a density formula directly, assuming that only the vierbein and gravitini may appear explicitly within the four-form. Schematically we will thus consider a four-form decomposed into five types of forms, namely $\psi^4$, $e\,\psi^3$, $e^2\,\psi^2$, $e^3\,\psi$ and finally $e^4$. The Weyl weight of these forms ranges from $w = -2$ for the first one to $w = -4$ for the last one. This last form will be multiplied by a composite coefficient function with $w = 4$ that contains all the purely bosonic terms of the Lagrangian (as well as the purely supercovariant fermionic terms).

The overall structure of the Lagrangian is dictated by the transformation of the lowest-dimensional (i.e. with lowest Weyl weight) supercovariant composites. We thus start by considering the quartic gravitino forms, which we postulate to be of the following type,

$$
\begin{aligned}
\mathcal{L} = & -i\,\varepsilon^{\mu\nu\rho\sigma}\,\bar\psi_{\mu i}\,\psi_{\nu j}\;\bar\psi_{\rho}{}^{k}\,\psi_{\sigma}{}^{l}\;A^{ij}{}_{kl} \\
& - \tfrac{i}{4}\,\varepsilon^{\mu\nu\rho\sigma}\,\bar\psi_{\mu i}\,\psi_{\nu j}\;\bar\psi_{\rho k}\,\psi_{\sigma l}\,\varepsilon^{klrs}\,C^{ij}{}_{rs} \\
& - \tfrac{i}{4}\,\varepsilon^{\mu\nu\rho\sigma}\,\bar\psi_{\mu}{}^{i}\,\psi_{\nu}{}^{j}\;\bar\psi_{\rho}{}^{k}\,\psi_{\sigma}{}^{l}\,\varepsilon_{klrs}\,\bar C^{rs}{}_{ij} + \dots\,,
\end{aligned}
\tag{4.15}
$$

where the supercovariant composites $A^{ij}{}_{kl}$, $C^{ij}{}_{kl}$, and $\bar C^{ij}{}_{kl}$ are assumed to be S-supersymmetric, and are therefore also invariant under conformal boosts.[4] All three composites have $w = 2$ and are restricted to the $\mathbf{20'}$ representation of SU(4). The composite $A^{ij}{}_{kl}$ is chosen to be pseudo-real while the other two are complex conjugate to each other, i.e. $(C^{ij}{}_{kl})^* \equiv \bar C^{kl}{}_{ij}$. The Lagrangian is therefore real.[5] It turns out that the three terms in (4.15) are sufficient to generate the full class

[4] This is because the anti-commutator of two S-supersymmetry transformations is a conformal boost.

[5] We remind that the factors of $i$ account for the Pauli-Källén conventions we have adopted in this chapter.

of superconformal Lagrangians. It may be possible to include other quartic gravitino terms in different representations, but this modification will only give rise to additional total derivatives in the final Lagrangians. This subtle point is discussed in more detail in an upcoming paper [102].

The density formula (4.15) is completed by iteratively imposing Q-supersymmetry at each order in the gravitini. To elucidate this calculation we present its first step explicitly and evaluate the Q-supersymmetry variations that remain proportional to four gravitini. For this purpose, we note that in the Q-supersymmetry transformation of the gravitino fields (4.9), we may ignore the second term, but neither the third nor the first. The reason the first term is relevant is that, in writing the variation of the action in a manifestly supercovariant form, we must integrate the derivative by parts and reconstruct supercovariant quantities. This then leads to further gravitino terms in two ways. The first is when the derivative hits another gravitino, which must be converted into the supercovariant Q-supersymmetry curvature (B.2) by adding appropriate terms,

$$\mathcal{D}_{[\mu}\psi_{\nu]}{}^i = \tfrac{1}{2}R(Q)_{\mu\nu}{}^i - \tfrac{1}{4}\varepsilon^{ijkl}\bar\psi_{[\mu j}\psi_{\nu]k}\Lambda_l + \dots \,. \tag{4.16}$$

The second way is when the derivative hits a supercovariant composite, such as $C^{ij}{}_{kl}$, which we rewrite as

$$\mathcal{D}_\mu C^{ij}{}_{kl} = D_\mu C^{ij}{}_{kl} + \tfrac{1}{2}\big[\bar\psi_\mu{}^m\,\tilde C^{ij}{}_{kl\,,m} + \bar\psi_{\mu m}\,\tilde C^{ij}{}_{kl}{}^{,m}\big]\,. \tag{4.17}$$

Here we wrote the Q-supersymmetry transformations of the scalar composites as

$$\begin{aligned}\delta_Q\, C^{ij}{}_{kl} &= \bar\epsilon^m\,\tilde C^{ij}{}_{kl\,,m} + \bar\epsilon_m\,\tilde C^{ij}{}_{kl}{}^{,m}\,,\\ \delta_Q\, A^{ij}{}_{kl} &= \bar\epsilon^m\,\tilde A^{ij}{}_{kl\,,m} + \bar\epsilon_m\,\tilde A^{ij}{}_{kl}{}^{,m}\,,\end{aligned} \tag{4.18}$$

with supercovariant fermionic composites $\tilde C^{,m}$, $\tilde A^{,m}$ and $\tilde C_{,m}$, $\tilde A_{,m}$ in the $\overline{\mathbf{60}}+\overline{\mathbf{20}}$ and $\mathbf{60}+\mathbf{20}$, respectively. Naturally the transformations (4.18) also induce variations of (4.15) proportional to $\psi^4$ times $\tilde C$ and $\tilde A$. Finally there is yet another way to generate variations quartic in gravitini originating from four-forms of the type $e\,\psi^3$ in the Lagrangian, and induced by the transformation of the vierbein (4.9). Note that connections other than the gravitini will also be generated, for example from (4.16), but those turn out to cancel at the end.

Collecting all the resulting variations proportional to the various possible quartic gravitini four-forms and requiring them to vanish imposes the following constraints on the traceless parts of the fermionic composites in (4.18),

$$[\tilde{C}^{ij}{}_{kl\,,m}]_{\overline{\mathbf{60}}} = [2\Lambda_m\, \tilde{A}^{ij}{}_{kl}]_{\overline{\mathbf{60}}}\ , \quad [\tilde{C}^{ij}{}_{kl}{}^{,m}]_{\mathbf{60}} = 0\ ,$$
$$[\tilde{A}^{ij}{}_{kl\,,m}]_{\overline{\mathbf{60}}} = [\Lambda_m\, \bar{C}^{ij}{}_{kl}]_{\overline{\mathbf{60}}}\ , \tag{4.19}$$

along with their complex conjugates[6], where $[\bullet]_{\mathbf{r}}$ denotes projection onto the SU(4) representation $\mathbf{r}$. The remaining terms in the fermionic composites $\tilde{C}$ and $\tilde{A}$ lie in the $\mathbf{20}$ and $\overline{\mathbf{20}}$ representations and must be proportional to the fermionic composites multiplying the $e\,\psi^3$ in the Lagrangian. We find

$$\delta^{[i}_{m}\, \Upsilon_{kl}{}^{j]} + \delta^{[i}_{[k}\, \Upsilon_{l]m}{}^{j]} = [\tilde{C}^{ij}{}_{kl\,,m} + 2\,\Lambda_m\, A^{ij}{}_{kl}]_{\overline{\mathbf{20}}}\,,$$
$$\delta^{[i}_{m}\, \Psi_{kl}{}^{j]} + \delta^{[i}_{[k}\, \Psi_{l]m}{}^{j]} = [\tilde{A}^{ij}{}_{kl\,,m} + 3\,\Lambda_m\, \bar{C}^{ij}{}_{kl}]_{\overline{\mathbf{20}}}\,, \tag{4.20}$$

where the fermionic composites $\Upsilon_{ij}{}^{k}$ and $\Psi_{ij}{}^{k}$ appear in the density formula (4.15) as follows

$$\frac{i}{8}\varepsilon^{\mu\nu\rho\sigma} e_\sigma{}^a \bar{\psi}_{\mu i}\psi_{\nu j}\bar{\psi}_{\rho k}\gamma_a\, \Upsilon_{rs}{}^{k}\, \varepsilon^{ijrs} + \frac{i}{4}\varepsilon^{\mu\nu\rho\sigma} e_\sigma{}^a \bar{\psi}_\mu{}^i \psi_\nu{}^j \bar{\psi}_{\rho k}\gamma_a\, \Psi_{ij}{}^{k} + \text{h.c.} \tag{4.21}$$

Note that the component $[\tilde{C}^{ij}{}_{kl}{}^{,m}]_{\mathbf{20}}$ is left unconstrained by Q-supersymmetry as it appears projected out in the variations quartic in the gravitini that we have considered. The Q-supersymmetry variation (4.18) of the composite $C^{ij}{}_{kl}$ therefore involves an composite that lie in the $\mathbf{20}$ representation and which does not appear in the Lagrangian.

This procedure must be continued by considering the remaining Q-supersymmetry variations proportional to the four-forms $e\,\psi^3$, $e^2\,\psi^2$, etc., to determine the relations between all the supercovariant composites in the Lagrangian and their transformation rules. This calculation makes use of the Q-supersymmetry transformations of almost all the Weyl multiplet fields presented in section 4.3. The fact that no inconsistencies arise at this level is a first indication that our original assumptions regarding the $\psi^4$ four-forms are correct.

Finally, we must check that the derived transformation rules of the composites satisfy the same off-shell superconformal algebra as the Weyl multiplet. This is a straightforward but technically involved calculation for which we make extensive use of the computer algebra package *Cadabra* [160, 161]. In this process one also identifies the missing S-supersymmetry variations of the composites, which so far

[6] Note that since $A^{ij}{}_{kl}$ is pseudo-real, the fermionic composites $\tilde{A}^{ij}{}_{kl\,,m}$ and $\tilde{A}^{ij}{}_{kl}{}^{,m}$ are related by charge conjugation.

were only specified for $A^{ij}{}_{kl}$ and $C^{ij}{}_{kl}$. This then provides a complete density formula built upon (4.15) and the constraints (4.19), that is invariant under all local superconformal symmetries

$$\begin{aligned}\mathcal{L} = 6eF + 2e\Big(\bar\psi_{\mu i}\,\Omega^{\mu i} + \bar\psi_{\mu i}\gamma^\mu\,\Omega^i + \text{h.c}\Big) \\ + \frac{1}{8}e\,\bar\psi_{[\mu i}\gamma^\mu\psi_{\nu]}{}^j\,\mathcal{E}^{\nu i}{}_j - \frac{i}{8}\Big(e\,\bar\psi_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\,\mathcal{E}^{ij} + \frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\bar\psi_{\mu i}\psi_{\nu j}\,\mathcal{E}_{\rho\sigma}{}^{ij} + \text{h.c}\Big) \\ + \frac{i}{8}\,\varepsilon^{\mu\nu\rho\sigma}\Big(\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\gamma_\sigma\Upsilon_{rs}{}^k\,\varepsilon^{ijrs} + 2\,\bar\psi_\mu{}^i\psi_\nu{}^j\bar\psi_{\rho k}\gamma_\sigma\,\Psi_{ij}{}^k + \text{h.c}\Big) \\ - \frac{i}{4}\,\varepsilon^{\mu\nu\rho\sigma}\Big(\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\psi_{\sigma l}\,\varepsilon^{klrs}\,C^{ij}{}_{rs} + 2\,\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_\rho{}^k\psi_\sigma{}^l A^{ij}{}_{kl} + \text{h.c}\Big)\,. \qquad (4.22)\end{aligned}$$

Note, for instance, that the the four-form of the type $e^4$ corresponds to the first term. This can be seen by writing

$$eF = \tfrac{1}{4!}\,\varepsilon^{\mu\nu\rho\sigma}e_\mu{}^a e_\nu{}^b e_\rho{}^c e_\sigma{}^d\big(\varepsilon_{abcd}F\big)\,. \qquad (4.23)$$

Similar manipulations can be used to rewrite the second line and the bracket of the first line as four-forms of the type $e^2\psi^2$ and $e^3\psi$, respectively. The various supercovariant composites appearing in the density formula and their properties are listed in Table 4.3. In what follows, it will be useful to use the same 'tilded'

Table 4.3: Composites appearing in the density formula

| Type | Composite | Properties | SU(4) | $w$ | $c$ |
|---|---|---|---|---|---|
| Bosonic | $A^{ij}{}_{kl}$ | $(A^{ij}{}_{kl})^* = A^{kl}{}_{ij}$ | $\mathbf{20'}$ | 2 | 0 |
| | $C^{ij}{}_{kl}$ | $(C^{ij}{}_{kl})^* \equiv \bar C^{kl}{}_{ij}$ | $\mathbf{20'}$ | 2 | $-2$ |
| | $\mathcal{E}^{ij}$ | | $\mathbf{10}$ | 1 | $-1$ |
| | $\mathcal{E}_{ab}{}^{ij}$ | $\mathcal{E}_{(ab)}{}^{ij} = 0$ | $\mathbf{6}$ | 1 | $-1$ |
| | $\mathcal{E}_a{}^i{}_j$ | | $\mathbf{15}$ | 1 | 0 |
| | $F$ | $F = F^*$ | $\mathbf{1}$ | 0 | 0 |
| Fermionic | $\Upsilon_{ij}{}^k$ | $\gamma_5\Upsilon_{ij}{}^k = \Upsilon_{ij}{}^k$ | $\overline{\mathbf{20}}$ | $\frac{3}{2}$ | $-\frac{3}{2}$ |
| | $\Psi_{ij}{}^k$ | $\gamma_5\Psi_{ij}{}^k = \Psi_{ij}{}^k$ | $\overline{\mathbf{20}}$ | $\frac{3}{2}$ | $+\frac{1}{2}$ |
| | $\Omega^i$ | $\gamma_5\,\Omega^i = \Omega^i$ | $\mathbf{4}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |
| | $\Omega_a{}^i$ | $\gamma_5\,\Omega_a{}^i = -\Omega_a{}^i\,,\ \gamma^a\,\Omega_a{}^i = 0$ | $\mathbf{4}$ | $\frac{1}{2}$ | $-\frac{1}{2}$ |

notation as in (4.18) to denote the Q-supersymmetry variations of the composites. More precisely, for a bosonic composite $\mathcal{B}$ in an arbitrary representation of SU(4) and of the Lorentz group, we define

$$\delta_Q\,\mathcal{B} = \bar\epsilon^i\tilde{\mathcal{B}}_{,i} + \bar\epsilon_i\mathcal{B}^{,i}\,, \tag{4.24}$$

while the variation of an arbitrary fermionic composite $\mathcal{F}$ of positive chirality can be decomposed without loss of generality as

$$\delta_Q\,\mathcal{F} = \bar\epsilon^i\tilde{\mathcal{F}}_{,i} + \gamma^{ab}\bar\epsilon^i\tilde{\mathcal{F}}^-_{ab\,,i} + \gamma^a\bar\epsilon_i\tilde{\mathcal{F}}_a{}^{,i}\,, \tag{4.25}$$

where $\tilde{\mathcal{F}}^-_{ab\,,i}$ is an anti-selfdual tensor. The various composites appearing in the density formula (4.22) are related to $A^{ij}{}_{kl}$ and $C^{ij}{}_{kl}$ by Q-supersymmetry. Using the notations (4.24) and (4.25), they can be written as follows

$$\begin{aligned}
\Upsilon_{ij}{}^k &= \tilde C^{lk}{}_{ij\,,l} + 2\,\Lambda_l\,A^{lk}{}_{ij}\,,\\
\Psi_{ij}{}^k &= \tilde A^{lk}{}_{ij\,,l} + 3\,\Lambda_l\,\bar C^{lk}{}_{ij}\,,\\
\mathcal{E}^{ij} &= \tfrac{1}{3}\varepsilon^{lrs(i}\big(\tilde\Upsilon_{rs}{}^{j)}{}_{,l} - \bar\Lambda_l\Psi_{rs}{}^{j)}\big)\,,\\
\mathcal{E}_a{}^i{}_j &= i\tilde\Psi_{a\,kj}{}^{i,k} - 3\,i\bar\Lambda^k\gamma_a\Upsilon_{kj}{}^i\,,\\
\mathcal{E}_{ab}{}^{ij} &= -\tfrac{4}{5}\tilde\Psi^{+}_{ab}{}^{ij}{}_l{}^{,l} - \tfrac{12}{5}\,\varepsilon^{klrs}\,T_{ab\,kl}C^{ij}{}_{rs}\\
&\quad - \tfrac{3}{5}\varepsilon^{ijrs}\,\tilde\Upsilon^-_{ab\,rs}{}^k{}_{,k} - 4\,T_{ab}{}^{rs}A^{ij}{}_{rs} + \tfrac{1}{10}\,\varepsilon^{ijrs}\bar\Lambda_k\gamma_{ab}\Psi_{rs}{}^k\,,\\
\Omega_a{}^i &= \tfrac{i}{256}\,\gamma^{bc}\gamma_a\left(\Lambda_l\,\mathcal{E}_{bc\,jk}\,\varepsilon^{ijkl} - \tilde{\mathcal{E}}_{bc}{}^{ij}{}_{,j}\right),\\
\Omega^i &= -\tfrac{i}{40}\left(\gamma^{ab}\,T_{ab}{}^{jk}\Psi_{jk}{}^i + \tilde{\mathcal{E}}^{ij}{}_{,j}\right) - \tfrac{1}{640}\,\gamma^a\tilde{\mathcal{E}}_a{}^i{}_j{}^{,j}\,,\\
F &= \tfrac{i}{192}\left(T_{ab\,ij}\,\mathcal{E}^{abij} - T_{ab}{}^{ij}\mathcal{E}^{ab}{}_{ij}\right) - \tfrac{1}{12}\left(\tilde\Omega_i{}^{,i} + \tilde\Omega^i{}_{,i}\right),
\end{aligned} \tag{4.26}$$

Note that the first two expressions were already presented in (4.20). The full Q- and S-supersymmetry transformations of these composites are not particularly illuminating for our purposes and will be reported in an upcoming paper [102]. As observed above, the full transformations rules will involve additional composites which do not appear in the density formula (4.22). The complete set of composites however forms a multiplet on which the superconformal algebra closes off-shell.

Let us close this section by emphasizing that this density formula makes no assumptions about the specific dependence of the supercovariant composites on the Weyl multiplet fields. It only relies on the existence of composites $A^{ij}{}_{kl}$ and $C^j{}_{kl}$ satisfying the constraints (4.19).

## 4.5 The invariant action

In order to obtain an explicit expression of the $N = 4$ conformal supergravity action, we now have to express the composites $A^{ij}{}_{kl}$ and $C^{ij}{}_{kl}$ in terms of the supercovariant fields of the $N=4$ Weyl multiplet; they will then yield corresponding expressions for the remaining composites through their transformation rules (4.26). As it turns out there exist only four scalar S-supersymmetric expressions in the $\mathbf{20}'$ representation,

$$
\begin{aligned}
X_{(1)}{}^{ij}{}_{kl} &= D^{ij}{}_{kl} \ , \\
X_{(2)}{}^{ij}{}_{kl} &= \tfrac{1}{2} T_{ab}{}^{ij} T^{ab\,mn} \varepsilon_{klmn} - \tfrac{1}{4} \varepsilon^{ijmn} E_{mk} E_{nl} \\
&\quad + 2 \bar{\Lambda}_{[k} \chi^{ij}{}_{l]} - \text{traces} \ , \\
X_{(3)}{}^{ij}{}_{kl} &= -\tfrac{1}{4} T_{ab}{}^{ij} \bar{\Lambda}_k \gamma^{ab} \Lambda_l + \tfrac{1}{4} \varepsilon^{ijmn} E_{m[k} \bar{\Lambda}_{l]} \Lambda_n - \text{traces} \ , \\
X_{(4)}{}^{ij}{}_{kl} &= -\tfrac{1}{24} \varepsilon^{ijmn} \bar{\Lambda}_k \Lambda_m \bar{\Lambda}_n \Lambda_l \ ,
\end{aligned} \tag{4.27}
$$

each of which is homogeneous in the Weyl multiplet fields. The first one is pseudoreal and the others are complex. The composites $A^{ij}{}_{kl}$ and $C^{ij}{}_{kl}$ must then be written as linear combinations of $X_{(n)}{}^{ij}{}_{kl}$ and $\bar{X}_{(n)}{}^{ij}{}_{kl}$ multiplied by, apriori arbitrary, functions of the coset scalars with the appropriate U(1) weights. Hence, we have

$$
\begin{aligned}
C^{ij}{}_{kl} = {}& A_1^{-2}\, D^{ij}{}_{kl} & A^{ij}{}_{kl} = {}& B_1^0\, D^{ij}{}_{kl} \\
&+ W_2^0\, X_2{}^{ij}{}_{kl} + Z_2^{-4}\, \bar{X}_2{}^{ij}{}_{kl} & &+ C_2^{+2}\, X_2{}^{ij}{}_{kl} + \bar{C}_2^{-2}\, \bar{X}_2{}^{ij}{}_{kl} \\
&+ W_3^{+2}\, X_3{}^{ij}{}_{kl} + Z_3^{-6}\, \bar{X}_3{}^{ij}{}_{kl} & &+ C_3^{+4}\, X_3{}^{ij}{}_{kl} + \bar{C}_3^{-4}\, \bar{X}_3{}^{ij}{}_{kl} \\
&+ W_4^{+4}\, X_4{}^{ij}{}_{kl} + Z_4^{-8}\, \bar{X}_4{}^{ij}{}_{kl} & &+ C_4^{+6}\, X_4{}^{ij}{}_{kl} + \bar{C}_4^{-6}\, \bar{X}_4{}^{ij}{}_{kl} \ ,
\end{aligned} \tag{4.28}
$$

while the expression for $\bar{C}^{ij}{}_{kl}$ follows from complex conjugation. The various factors $A_1^{-2}, B_1^0, W_2^0, \ldots$ depend on the coset scalars and their U(1) weight is indicated explicitly as a superscript. From these expressions one determines the corresponding fermionic composites via (4.18), and subsequently imposes the constraints (4.19). The latter then turn into constraints on the factors that depend on the coset scalars in (4.28). In particular, the constraint $[\tilde{C}^{ij}{}_{kl}{}^{,m}]_{\mathbf{60}} = 0$ implies

$$
\begin{aligned}
\mathcal{D}^\dagger W_2^0 &= 0 \ , & \mathcal{D}^\dagger A_1^{-2} &= -Z_2^{-4} \ , \\
\mathcal{D}^\dagger W_3^{+2} &= 2 W_2^0 \ , & \mathcal{D}^\dagger Z_2^{-4} &= -Z_3^{-6} \ , \\
\mathcal{D}^\dagger W_4^{+4} &= 6 W_3^{+2} \ . & \mathcal{D}^\dagger Z_3^{-6} &= -Z_4^{-8} \ ,
\end{aligned} \tag{4.29}
$$

where the SU(1,1) derivative $\mathcal{D}^\dagger$ was defined in (4.6). It is already clear from the first constraint that $W_2^0$ must be at least be independent of the anti-holomorphic coset scalars $\phi^\alpha$. In fact, a careful analysis of all the constraints that follow from (4.19) leads to two linearly independent solutions for $A^{ij}{}_{kl}$ and $C^{ij}{}_{kl}$.

The first solution, in which $W_2^0 = 0$, can be shown to correspond to a Lagrangian that is a total derivative. We therefore ignore it here and refer to [102] where this will be demonstrated. The other solution is parametrised by an holomorphic function $\mathcal{H}(\phi_\alpha)$ that is homogeneous of zeroth degree in the coset scalars, such that

$$\mathcal{D}^\dagger\mathcal{H} = 0 = \mathcal{D}^0\mathcal{H}\,, \tag{4.30}$$

and by an associated complex potential $\mathcal{K}(\phi_\alpha, \phi^\beta)$ which carries a chiral weight $c = 0$, and which obeys

$$\mathcal{H} = \mathcal{D}^\dagger\mathcal{D}\,\mathcal{K}\,, \tag{4.31}$$

and its complex conjugate. Note that $\mathcal{K}$ can be seen as a Kähler potential for $\mathcal{H}$ since (4.31) is left invariant by shifts of the form $\mathcal{K} \to \mathcal{K} + \rho(\phi_\alpha) + \bar\rho(\phi^\alpha)$. The scalar factors in the expressions of the composites $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ (4.28) only depend on these two functions. They read

$$\begin{aligned}
A_1^{-2} &= -\tfrac{1}{2} i\,\mathcal{D}^\dagger\bar{\mathcal{K}}\,, & B_1^0 &= -\tfrac{1}{4}\,i(\mathcal{H} - \bar{\mathcal{H}})\,,\\
W_2^0 &= \tfrac{1}{2}\,i\mathcal{H}\,, & C_2^{+2} &= -\tfrac{1}{2}\,i\,\mathcal{D}\big(\mathcal{K} - \tfrac{1}{2}\mathcal{H}\big)\,,\\
W_3^{+2} &= -i\,\mathcal{D}\mathcal{K}\,, & C_3^{+4} &= i\,\mathcal{D}^2\big(\mathcal{K} - \tfrac{1}{4}\mathcal{H}\big)\,,\\
W_4^{+4} &= -\tfrac{1}{12}\,i\,\mathcal{D}^2\big(\mathcal{K} - \tfrac{1}{2}\mathcal{H}\big)\,, & C_4^{+6} &= -\tfrac{3}{2}\,i\,\mathcal{D}^3\big(\mathcal{K} - \tfrac{1}{6}\mathcal{H}\big)\,,\\
Z_2^{-4} &= -\mathcal{D}^\dagger A_1^{-2}\,, & &\\
Z_3^{-6} &= (\mathcal{D}^\dagger)^2 A_1^{-2}\,, & &\\
Z_4^{-8} &= -(\mathcal{D}^\dagger)^3 A_1^{-2}\,. & &
\end{aligned} \tag{4.32}$$

The function $\mathcal{H}$ is uniquely determined as it appears in $C^{ij}{}_{kl}$ as a distinctive term equal to $\tfrac{1}{2} i\, X_{(2)}{}^{ij}{}_{kl}\,\mathcal{H}$. The holomorphicity of $\mathcal{H}$ can then be seen as a direct consequence of the constraints (4.19), which also determine how derivatives of $\mathcal{H}$ and $\mathcal{K}$ appear within $C^{ij}{}_{kl}$ and all the other composites. The expression of the remaining composites in terms of the Weyl multiplets are then derived from the transformation rules (4.26).

Once the expressions of the various composites are substituted in the density formula (4.22), it is possible to show that all the terms that depend explicitly on the potential $\mathcal{K}$ can be removed by splitting off a total derivative, such that the final Lagrangian is written only in terms of the function $\mathcal{H}$. Once again, we refer to [102] for a detailed proof of this last point. This step concludes the construction of the class[7] of $N=4$ conformal supergravity actions which is characterized by the holomorphic function $\mathcal{H}$. Here we present the bosonic terms of the Lagrangians which are, up to total derivatives, fully captured by the first term of (4.22). They read

$$
\begin{aligned}
e^{-1}\mathcal{L} = \mathcal{H}\Big[ &\tfrac{1}{2}\, R(M)^{abcd}\, R(M)^{-}_{abcd} + R(V)^{ab\,i}{}_j\, R(V)^{-\,j}_{ab}{}_i + \tfrac{1}{8}\, D^{ij}{}_{kl}\, D^{kl}{}_{ij} + \tfrac{1}{4}\, E_{ij}\, D^2 E^{ij} \\
&- 4\, T_{ab}{}^{ij}\, D^a D_c\, T^{cb}{}_{ij} - \bar P^a D_a D_b P^b + P^2 \bar P^2 + \tfrac{1}{3}(P^a \bar P_a)^2 - \tfrac{1}{6}\, P^a \bar P_a\, E_{ij}\, E^{ij} \\
&- 8\, P_a\, \bar P^c\, T^{ab}{}_{ij}\, T_{bc}{}^{ij} - \tfrac{1}{16}\, E_{ij}\, E^{jk}\, E_{kl}\, E^{li} + \tfrac{1}{48}\,[E_{ij}\, E^{ij}]^2 \\
&+ T^{ab}{}_{ij}\, T_{ab\,kl}\, T^{cd\,ij}\, T_{cd}{}^{kl} - T^{ab}{}_{ij}\, T_{cd}{}^{jk}\, T_{ab\,kl}\, T^{cd\,li} \\
&- \tfrac{1}{2}\, E^{ij}\, T^{ab\,kl}\, R(V)_{ab}{}^m{}_i\, \varepsilon_{jklm} + \tfrac{1}{2}\, E_{ij}\, T^{ab}{}_{kl}\, R(V)_{ab}{}^i{}_m\, \varepsilon^{jklm} \\
&- \tfrac{1}{16}\, E_{ij} E_{kl}\, T^{ab}{}_{mn}\, T_{ab\,pq}\, \varepsilon^{ikmn}\, \varepsilon^{jlpq} - \tfrac{1}{16}\, E^{ij} E^{kl}\, T^{ab\,mn}\, T_{ab}{}^{pq}\, \varepsilon_{ikmn}\, \varepsilon_{jlpq} \\
&- 2\, T^{ab\,ij}\left(P_{[a} D_{c]} T_b{}^{c\,kl} + \tfrac{1}{6}\, P^c D_c T_{ab}{}^{kl} + \tfrac{1}{3}\, T_{ab}{}^{kl} D_c P^c\right) \varepsilon_{ijkl} \\
&- 2\, T^{ab}{}_{ij}\left(\bar P_{[a} D_{c]} T_b{}^c{}_{kl} - \tfrac{1}{2}\, \bar P^c D_c T_{ab\,kl}\right) \varepsilon^{ijkl}\Big] \\
+ \mathcal{D}\mathcal{H}\Big[ &\tfrac{1}{4}\, T_{ab}{}^{ij}\, T_{cd}{}^{kl} R(M)^{abcd}\, \varepsilon_{ijkl} + T^{ab\,ij}\, T_a{}^{c\,kl}\, R(V)_{bc}{}^m{}_k\, \varepsilon_{ijlm} \\
&- \tfrac{1}{8}\, D^{ij}{}_{kl}\big(T^{ab\,mn}\, T_{ab}{}^{kl}\, \varepsilon_{ijmn} - \tfrac{1}{2}\, E_{im}\, E_{jn} \varepsilon^{klmn}\big) + E_{ij}\, T^{ab\,ik}\, R(V)_{ab}{}^j{}_k \\
&- \tfrac{1}{24}\, E_{ij}\, E^{ij}\, T^{ab\,kl}\, T_{ab}{}^{mn}\, \varepsilon_{klmn} - \tfrac{1}{6}\, E^{ij}\, T_{ab}{}^{kl}\, T^{ac\,mn}\, T^b{}_c{}^{pq}\, \varepsilon_{iklm}\, \varepsilon_{jpqn}\Big] \\
+ \mathcal{D}^2\mathcal{H}\Big[ &\tfrac{1}{384}\, E_{ij}\, E_{kl}\, E_{mn}\, E_{pq}\, \varepsilon^{ikmp}\, \varepsilon^{jlnq} + \tfrac{1}{6}\, E_{ij}\, T_{ab}{}^{ik}\, T^{ac\,jl}\, T^b{}_c{}^{mn}\, \varepsilon_{klmn} \\
&- \tfrac{1}{8}\, E_{ij}\, E_{kl}\, T_{ab}{}^{ik}\, T^{ab\,jl} + \tfrac{1}{32}\, T^{ab\,ij}\, T^{cd\,pq}\, T_{ab}{}^{mn}\, T_{cd}{}^{kl}\, \varepsilon_{ijkl}\, \varepsilon_{mnpq} \\
&- \tfrac{1}{64}\, T^{ab\,ij}\, T^{cd\,pq}\, T_{ab}{}^{kl}\, T_{cd}{}^{mn}\, \varepsilon_{ijkl}\, \varepsilon_{mnpq}\Big] \\
+ 2\,\mathcal{H}\, &e_a{}^\mu f_\mu{}^c\, \eta_{cb}\Big[P^a\, \bar P^b - P^d\, \bar P_d\, \eta^{ab}\Big] + \text{h.c.}
\end{aligned}
\tag{4.33}
$$

where $R(M)^{-}_{abcd}$ and $R(V)^{-\,i}_{ab}{}_j$ denote the anti-selfdual part of the supercovariant curvatures (see (B.6)). When suppressing the fermionic terms, they become equal to the Weyl tensor (4.1) and the SU(4) field strengths. The fermonic terms of this class of Lagrangians will be reported in [102]. Let us emphasise again here that the function $\mathcal{H}$ is only restricted by the conditions (4.30). Nonetheless, these

[7]As already claimed previously, this is in fact the most general class of $N=4$ conformal supergravity actions. An argument based on the uniqueness of the supercurrent that couples (exclusively) to $N=4$ conformal supergravity will be presented in an upcoming paper [102].

conditions imply that any non-constant function is entirely expressed in terms of the coset coordinate $S = \phi_1/\phi_2$, and therefore breaks the rigid SU(1, 1) symmetry.

In deriving the result (4.33), we have introduced additional total derivative terms in order to bring the formula into a concise form at the cost of generating terms that explicitly depend on the conformal boost gauge field $f_\mu{}^a$. Indeed, the presence of this gauge field can at first seem disorienting since they were originally excluded from the density formula. Finally, a strong check of the Lagrangian (4.33) is provided by the fact that it agrees (up to a total derivative) with the bosonic Lagrangian derived in [100] when the holomorphic function is equal to a real constant. In this case, it is clear that the Lagrangian is also invariant under rigid SU(1, 1) transformations.

## 4.6 Potential applications

In the future, it would be interesting to consider the relevance of our results in the Poincare limit. As mentioned previously, the $N = 4$ Poincaré theory can be described as a system of six abelian $N = 4$ vector multiplets coupled to conformal supergravity. In this case, the field equations for the vector multiplets are invariant under the rigid SU(1, 1) transformations[8] associated to the coset scalars [98]. It is natural to wonder whether deformations of the type considered in this chapter, *i.e.* that break the SU(1, 1) symmetry, could also be implemented in this context. Since the vector multiplet is on-shell, its field equations are encoded in the superconformal transformation rules of its components. A rather simple analysis reveals that closure of the superconformal algebra actually requires the SU(1, 1) symmetry and thus proscribes the implementation of deformations. After eliminating some of the matter fields and choosing the Poincaré gauge, the associated action reduces to that of $N = 4$ Poincaré supergravity and the SU(1, 1) subsists as a duality symmetry of the field equations.

The results of this chapter could be directly incorporated into Poincaré supergravity as four-derivative couplings following the construction carried out originally in [98], by including the superconformal Lagrangian (4.33) before proceeding to the standard gauge choices. This could be interesting in view of the recent results concerning the finiteness of the $N = 4$ Poincaré theory.

[8] In this context, they also correspond to electric/magnetic duality transformations.

On one hand, sophisticated loop computations have shown that the Poincaré theory remains ultraviolet finite up to three loops [162]. The divergences that arise at four loops are associated to terms which are not invariant under the U(1) subgroup of the SU(1, 1) duality group. On the other hand, our class of higher-derivative conformal invariants (4.33), after a careful gauge fixing of the conformal symmetries and the elimination of auxiliary fields, should capture the one-loop effective Poincaré action. Due to the holomorphic function discussed in section 4.5, the effective action obtained in this way will generically break the SU(1, 1) invariance. This would be line with the presence of the U(1) anomaly [163].

Another possible application of our results concerns the calculation of subleading corrections to $N=4$ supersymmetric black hole entropy that are known to originate from precisely the class of Lagrangians (4.33). In principle this can be done by generalizing the analysis of [164], which can also be utilized for localization along the lines of [165, 166]. Both these approaches have so far only been considered in a truncated $N = 2$ setting, and therefore it should be interesting to understand these results in the context of a manifestly $N=4$ supersymmetric formulation.

# Appendix A

# Decomposition of type IIB gamma matrices et fermions

In chapter 2, we consider a $5+5$ dimensional split of type IIB supergravity. The $10D$ tangent space is decomposed accordingly into a direct product of two five-dimensional spaces, one corresponding to a five-dimensional space-time and one corresponding to a five-dimensional internal space. Since we are dealing with spinor fields, it is then important to identify the gamma matrices appropriate to this product space in terms of the original $10D$ gamma matrices. This identification is worked out in section A.1. Based on these results, we present in section A.2 a detailed derivation of the Usp(8) covariant expressions of the $5D$ spinors fields (discussed in section 2.5) in terms of the type IIB fermion fields.

## A.1 Decomposition of gamma matrices and spinors

We start from $32 \times 32$ hermitian gamma matrices $\breve{\Gamma}_A$, where $A = 1, 2, \ldots 10$, satisfying the Clifford algebra anti-commutation relation, $\{\breve{\Gamma}_A, \breve{\Gamma}_B\} = 2\,\delta_{AB}\,\mathbb{1}_{32}$, and proceed in a way that is independent of a specific representation for these gamma matrices. The hermitian chirality operator, $\breve{\Gamma}_{11}$, is defined by

$$\breve{\Gamma}_{11} = \mathrm{i}\breve{\Gamma}_1\,\breve{\Gamma}_2 \cdots \breve{\Gamma}_{10}\,, \tag{A.1}$$

and satisfies

$$\breve{\Gamma}_{11}{}^2 = \mathbb{1}_{32}\,, \qquad \{\breve{\Gamma}_A, \breve{\Gamma}_{11}\} = 0\,. \tag{A.2}$$

Moreover we note the identity,

$$\breve{\Gamma}^{ABCDE} = \tfrac{1}{120}\mathrm{i}\varepsilon^{ABCDEFGHIJ}\,\breve{\Gamma}_{FGHIJ}\,\breve{\Gamma}_{11}\,. \tag{A.3}$$

Let us now decompose the gamma matrices into two sets, $\breve{\Gamma}_\alpha$ with $\alpha = 1, 2, \ldots, 5$ and $\breve{\Gamma}_{a+5}$ with $a = 1, 2, \ldots, 5$.[1] Subsequently one introduces hermitian matrices associated with the two five-dimensional sectors,

$$\tilde{\gamma} = \breve{\Gamma}_1\,\breve{\Gamma}_2\,\breve{\Gamma}_3\,\breve{\Gamma}_4\,\breve{\Gamma}_5\,, \qquad \tilde{\Gamma} = \breve{\Gamma}_6\,\breve{\Gamma}_7\,\breve{\Gamma}_8\,\breve{\Gamma}_9\,\breve{\Gamma}_{10}\,. \tag{A.4}$$

which satisfy the following properties,

$$\tilde{\gamma}^2 = \mathbb{1}_{32}\,, \quad \tilde{\Gamma}^2 = \mathbb{1}_{32}\,, \quad \{\tilde{\gamma}, \tilde{\Gamma}\} = 0\,, \quad \breve{\Gamma}_{11} = \mathrm{i}\tilde{\gamma}\,\tilde{\Gamma}\,, \tag{A.5}$$

Subsequently one defines two sets of mutually commuting *hermitian* gamma matrices,

$$\hat{\gamma}_\alpha = \mathrm{i}\breve{\Gamma}_\alpha\,\tilde{\Gamma}\,, \qquad \hat{\Gamma}_a = \mathrm{i}\breve{\Gamma}_{a+5}\,\tilde{\gamma}\,, \tag{A.6}$$

so that $\{\hat{\gamma}_\alpha, \hat{\gamma}_\beta\} = 2\,\delta_{\alpha\beta}\mathbb{1}_{32}$, $\{\hat{\Gamma}_a, \hat{\Gamma}_b\} = 2\,\delta_{ab}\mathbb{1}_{32}$, and $[\hat{\gamma}_\alpha, \hat{\Gamma}_a] = 0$. The matrices $\hat{\gamma}_\alpha$ will refer to the five-dimensional space-time (to account for the signature one may write one of the five gamma matrices, say $\hat{\gamma}^1$ as $\mathrm{i}\hat{\gamma}^0$) and the matrices $\hat{\Gamma}_a$ to the five-dimensional internal space. The matrices $\hat{\gamma}_\alpha$ and $\hat{\Gamma}_a$ commute with $\breve{\Gamma}_{11}$, as one can easily verify from the above equations. It is important to note that

$$\begin{aligned}\hat{\gamma}_{[\alpha}\hat{\gamma}_\beta\hat{\gamma}_\gamma\hat{\gamma}_\delta\hat{\gamma}_{\tau]} &= \varepsilon_{\alpha\beta\gamma\delta\tau}\,\Gamma_{11}\,,\\ \hat{\Gamma}_{[a}\hat{\Gamma}_b\hat{\Gamma}_c\hat{\Gamma}_d\hat{\Gamma}_{e]} &= -\,\varepsilon_{abcde}\,\Gamma_{11}\,.\end{aligned} \tag{A.7}$$

where $\varepsilon_{12345} = +1$. Obviously, by choosing an explicit representation for the $10D$ gamma matrices, one obtains explicit expressions for the various matrices that we have defined above which will reflect their properties.

Let us now consider the charge conjugation matrix. In ten dimensions there exist two possible options for the charge conjugation matrix, denoted by $\breve{C}_\pm$, satisfying

$$\breve{C}_\pm\,\breve{\Gamma}_A\breve{C}_\pm^{-1} = \pm\breve{\Gamma}_A{}^{\mathrm{T}}\,, \quad \breve{C}_\pm{}^{\mathrm{T}} = \pm\breve{C}_\pm\,, \quad \breve{C}_\pm{}^{\dagger} = \breve{C}_\pm^{-1}\,, \tag{A.8}$$

[1] At this stage there is no difference between upper and lower indices, so that we are dealing with a positive Euclidean metric.

which lead to the following results,

$$\breve{C}_{\pm}\breve{\Gamma}_{11}\breve{C}_{\pm}^{-1} = -\breve{\Gamma}_{11}{}^{\mathrm{T}}\,, \quad \breve{C}_{\pm}\tilde{\gamma}\breve{C}_{\pm}^{-1} = \pm\tilde{\gamma}^{\mathrm{T}}\,, \quad \breve{C}_{\pm}\tilde{\Gamma}\breve{C}_{\pm}^{-1} = \pm\tilde{\Gamma}^{\mathrm{T}}\,. \tag{A.9}$$

From the first equation (A.9), it follows that $\breve{C}_{\pm}$ satisfy

$$(\breve{C}_{\pm}\breve{\Gamma}_{11})^{\mathrm{T}} = \breve{\Gamma}_{11}{}^{\mathrm{T}}\,\breve{C}_{\pm}{}^{\mathrm{T}} = \mp(\breve{C}_{\pm}\,\breve{\Gamma}_{11})\,, \tag{A.10}$$

so that the two options for the charge conjugation matrix can simply be related by multiplication with $\Gamma_{11}$. Furthermore we note that both $\breve{C}_{\pm}\tilde{\Gamma}$ and $\breve{C}_{\pm}\tilde{\gamma}$ are symmetric and unitary matrices. Up to a phase factor, these can act as the charge conjugation matrices in the $5D$ context, as is demonstrated by

$$(\breve{C}_{\pm}\tilde{\Gamma})\hat{\gamma}_{\alpha}(\breve{C}_{\pm}\tilde{\Gamma})^{-1} = \hat{\gamma}_{\alpha}{}^{\mathrm{T}}\,, \quad (\breve{C}_{\pm}\tilde{\Gamma})\hat{\Gamma}_{a}(\breve{C}_{\pm}\tilde{\Gamma})^{-1} = \hat{\Gamma}_{a}{}^{\mathrm{T}}\,. \tag{A.11}$$

Similar relations hold for $(\breve{C}_{\pm}\tilde{\gamma})$.

To appreciate the significance of this result, let us consider the definition of the Dirac conjugate in the $5D$ context, defined by $\psi^{\dagger}\mathrm{i}\hat{\gamma}^{0}$, where $\hat{\gamma}^{0}$ was related to $\hat{\gamma}^{1}$ as explained below (A.6). From these relations it follows straightforwardly that the $5D$ Dirac conjugate $\bar{\psi}\big|_{5D}$ is related to the $10D$ conjugate according to

$$\bar{\psi}\big|_{5D} = \mathrm{i}\bar{\psi}\big|_{10D}\,\tilde{\Gamma}\,. \tag{A.12}$$

Consequently, identifying the Majorana conjugate defined in (2.18) in the $10D$ context with the one in the $5D$ context, one concludes that the charge conjugation matrix in the $5D$ context equals

$$\hat{C} = \mathrm{i}\tilde{\Gamma}^{\mathrm{T}}\breve{C}_{\pm} = \pm\mathrm{i}\breve{C}_{\pm}\,\tilde{\Gamma}\,, \tag{A.13}$$

so that $\hat{C}^{-1}\big[\bar{\psi}|_{5\mathrm{D}}\big]^{\mathrm{T}} = \psi^{\mathrm{c}}$, and likewise $\psi^{\mathrm{T}} = \bar{\psi}^{\mathrm{c}}|_{5\mathrm{D}}\,\hat{C}^{-1}$. As a consequence the two commuting sets of $32\times 32$ gamma matrices, $\hat{\gamma}_{\alpha}$ and $\hat{\Gamma}_{a}$, satisfy the relations known from five dimensions,

$$\hat{C}\hat{\gamma}_{\alpha}\hat{C}^{-1} = \hat{\gamma}_{\alpha}{}^{\mathrm{T}}\,, \qquad \hat{C}\hat{\Gamma}_{a}\hat{C}^{-1} = \hat{\Gamma}_{a}{}^{\mathrm{T}}\,, \qquad \hat{C}^{\mathrm{T}} = \hat{C}\,, \qquad \hat{C}^{\dagger} = \hat{C}^{-1}\,. \tag{A.14}$$

This leads to the rearrangement formula,

$$\bar{\chi}\,\Gamma\psi = -\bar{\psi}^{\mathrm{c}}\,\hat{C}^{-1}\Gamma^{\mathrm{T}}\hat{C}\,\chi^{\mathrm{c}}\,, \tag{A.15}$$

where $\Gamma$ denotes any matrix in the spinor space, which in all cases of interest takes the form a product of gamma matrices $\hat{\Gamma}^a$ and $\hat{\gamma}_\alpha$. Observe that the new charge conjugation matrix is not anti-symmetric, as one might expect on the basis of a single irreducible $5D$ Clifford algebra representation. We return to this issue shortly.

In chapter 2 we discuss the type-IIB theory where the spinor fields are chiral and complex. Therefore the above formulae have to be projected on an eigenspace of $\Gamma_{11}$ and the effective $5D$ gamma matrices defined in (A.6) are consistent with the $10D$ chirality constraint on the spinor fields, because they are proportional to an even number of the original $10D$ gamma matrices. However, it is important to realize that IIB supergravity contains independent spinor fields of *opposite* chirality, namely $\psi_M$ and $\lambda$. This leads to a subtlety in view of (A.7), which indicates that different chirality spinors involve inequivalent gamma matrix representations in $5D$. However, one has to keep in mind that the chirality assignment can easily been changed in the $5D$ context by redefining the spinors by multiplication with one of the matrices (A.4).

Let us now assume that we are starting from $10D$ with fermion fields of *positive* chirality. Hence we can choose a Weyl basis where $\breve{\Gamma}_{11}$ is diagonal and make use of the fact that it commutes with the mutually commuting gamma matrices $\hat{\gamma}_\alpha$ and $\hat{\Gamma}_a$. Hence we write

$$\hat{\gamma}_\alpha = \sigma_3 \otimes \gamma_\alpha \otimes \mathbb{1}_4 \,, \qquad \hat{\Gamma}_a = \sigma_3 \otimes \mathbb{1}_4 \otimes \Gamma_a \,, \tag{A.16}$$

where $\breve{\Gamma}_{11} = \sigma_3 \otimes \mathbb{1}_{16}$ and $\gamma_\alpha$ and $\Gamma_a$ are $4 \times 4$ matrices. It then follows from (A.7) that they define irreducible representations of the respective Clifford algebras, as

$$\gamma_{[\alpha}\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_{\tau]} = \varepsilon_{\alpha\beta\gamma\delta\tau}\,\mathbb{1}_4 \,, \qquad \Gamma_{[a}\Gamma_b\Gamma_c\Gamma_d\Gamma_{e]} = -\varepsilon_{abcde}\,\mathbb{1}_4 \,. \tag{A.17}$$

The $10D$ chiral spinors thus transform under the direct product group $\mathrm{Spin}(1,4) \times \mathrm{USp}(4)$, whose generators are provided by the anti-symmetrized products of gamma matrices, $\gamma_{\alpha\beta}$ and $\Gamma_{ab}$, respectively. Correspondingly the charge conjugation matrix $\hat{C}$ can be written (adjusting possible phase factors) as the direct product of the two $5D$ *anti-symmetric* charge conjugation matrices,

$$\hat{C}_{(16)} = C \otimes \Omega_{(4)} \,, \tag{A.18}$$

where $C$ denotes the anti-symmetric charge conjugation matrix for a $5D$ space-time spinor and $\Omega_{(4)}$ is the symplectic matrix that is invariant under the $\mathrm{USp}(4)$

R-symmetry. In this case we may write (A.14) as

$$C\,\gamma_\alpha C^{-1} = {\gamma_\alpha}^{\mathrm{T}}\,, \qquad \Omega_{(4)}\,\Gamma_a\Omega_{(4)}^{-1} = {\Gamma_a}^{\mathrm{T}}\,. \tag{A.19}$$

However, the chiral spinors are complex which implies that the fields $(\psi, \psi^{\mathrm{c}})$, which constitute the 32-component spinor $\Psi$, can again be rearranged in a pseudo-real form as in (2.21). The doubling of field components enables one to realize the extension of the R-symmetry group from $\mathrm{USp}(4)\times\mathrm{U}(1)$ to $\mathrm{USp}(8)$. It then follows from (2.21) that the extended $\mathrm{USp}(8)$ invariant tensor must take the form

$$\Omega = \Omega_{(4)} \otimes \sigma_1\,. \tag{A.20}$$

Consequently, (2.21) and (A.18) imply the symplectic Majorana condition,

$$C^{-1}\bar{\Psi}^{\mathrm{T}} = \Omega\,\Psi\,, \tag{A.21}$$

where $\Omega$ is an $8\times 8$ anti-symmetric matrix. Both matrices $C$ and $\Omega$ are anti-symmetric and unitary.

We close this appendix with some additional definitions that will be useful in the next appendix A.2. First of all we write the anti-symmetric tensor $\Omega_{(4)}$ as $\Omega_{(4)IJ}$ and its complex conjugate as ${\bar{\Omega}_{(4)}}{}^{IJ}$, so that $\Omega_{(4)IJ}\,{\bar{\Omega}_{(4)}}{}^{JK} = -{\delta_I}^K$, where $I,J,K = 1,\ldots,4$. The gamma matrices $\Gamma_a$ are then written as ${\Gamma_a}^I{}_J$, so that

$${\Omega_{(4)}}^{\mathrm{T}} = -\Omega_{(4)}\,, \quad (\Omega_{(4)}\,\Gamma_a)^{\mathrm{T}} = -(\Omega_{(4)}\,\Gamma_a)\,, \quad (\Omega_{(4)}\,\Gamma_{ab})^{\mathrm{T}} = (\Omega_{(4)}\,\Gamma_{ab})\,, \tag{A.22}$$

with similar relations for $(\Gamma_a\,\bar{\Omega}_{(4)})^{IJ}$ and $(\Gamma_{ab}\,\bar{\Omega}_{(4)})^{IJ}$. The six matrices $\Omega_{(4)IJ}$ and $(\Omega_{(4)}\,\Gamma_a)_{IJ}$ form a complete set of $4\times 4$ anti-symmetric matrices, and the ten matrices $(\Omega_{(4)}\,\Gamma_{ab})_{IJ}$ a complete set of $4\times 4$ symmetric matrices This leads to the completeness relations

$$\begin{aligned} \Omega_{(4)IJ}\,{\bar{\Omega}_{(4)}}{}^{KL} + (\Omega_{(4)}\,\Gamma_a)_{IJ}\,(\Gamma^a\,\bar{\Omega}_{(4)})^{KL} &= 4\,\delta_{[I}{}^K\,\delta_{J]}{}^L\,,\\ (\Omega_{(4)}\,\Gamma_{ab})_{IJ}\,(\Gamma^{ab}\,\bar{\Omega}_{(4)})^{KL} &= 8\,\delta_{(I}{}^K\,\delta_{J)}{}^L\,. \end{aligned} \tag{A.23}$$

## A.2 The R-symmetry group and the fermion representations

In the previous section we considered a $10D$ chiral spinor and described its properties in the context of a product of a five-dimensional space-time and a five-dimensional internal space. The gamma matrices and the charge conjugation matrices were decomposed accordingly. The $10D$ spinors then transform under a subgroup of the original $\mathrm{Spin}(1,9)$ transformations consisting of the $\mathrm{Spin}(1,4)$ group associated with the $5D$ space-time and the group $\mathrm{USp}(4)$ associated with the internal space.

However, $\mathrm{USp}(4)$ is not the full automorphism group (or R-symmetry group) of the eight symplectic Majorana spinors. This group is actually equal to $\mathrm{USp}(8)$, which consists of the unitary transformations that leave the symplectic and unitary tensor $\Omega$, invariant. The generators of this group can be easily identified in terms of direct products of the $4\times4$ gamma matrices $\Gamma_a$, defined in (A.16), their anti-symmetrized products $\Gamma_{ab}$ and the unit matrix $\mathbb{1}_4$, and the $2\times2$ matrices $(\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3)$. As a result one derives all the 36 generators of the Lie algebra $\mathfrak{usp}(8) = \mathfrak{su}(8) \cap \mathfrak{sp}(8, \mathbb{R})$, by constructing the complete set of traceless and anti-hermitian matrices that preserve the symplectic form $\Omega$,

$$
\begin{aligned}
&T \equiv \mathrm{i}\mathbb{1}_4 \otimes \sigma_3, \qquad T_a \equiv \mathrm{i}\Gamma_a \otimes \sigma_3, \\
&T_{ab}{}^0 \equiv \Gamma_{ab} \otimes \mathbb{1}_2, \qquad T_{ab}{}^1 \equiv \Gamma_{ab} \otimes \sigma_1, \qquad T_{ab}{}^2 \equiv \Gamma_{ab} \otimes \sigma_2 \,.
\end{aligned}
\tag{A.24}
$$

It can be verified that these matrices close under commutation, and that the resulting structure constants are real, in agreement with $\mathfrak{usp}(8)$ being a real form. The $T_{ab}{}^0$ are the generators of the group $\mathrm{USp}(4) \cong \mathrm{SO}(5)$. When extended with the generators $T_a$ one obtains the group $\mathrm{SU}(4) \cong \mathrm{SO}(6)$ which obviously commutes with the generator $T$. As we will exhibit later, $T$ corresponds to the $\mathrm{SO}(6)$ chirality operator. The latter commutes with the $\mathrm{U}(1)$ transformations of the original $10D$ theory (see (2.3) and(2.4)). Clearly $\mathrm{SU}(4)\times\mathrm{U}(1)$ is a maximal subgroup of $\mathrm{USp}(8)$.

A chiral $10D$ spinor $\Psi$ can be decomposed into eight $5D$ symplectic Majorana spinors $\psi^A$, where $A = 1, \ldots, 8$. Note that from now on we employ indices $A, B, \ldots$ to label the symplectic Majorana spinors. The same indices were previously used in the $10D$ theory (in particular in section 2.2 and section A.1) to denote the $10D$ tangent-space components. This should not give rise to confusion in view of the fact that the $10D$ tangent space will no longer play a role in what follows.

In view of the direct-product structure indicated in (A.24) the indices $A$ can be written as index pairs $A = (I\alpha)$, where $I = 1,\ldots,4$ are USp(4) indices and $\alpha = +,-$. Here $\alpha = +$ ($\alpha = -$) indicates that we are dealing with a chiral (anti-chiral) SO(6) spinor with positive (negative) U(1) charge[2]. Based on this direct-product structure the eight $5D$ gravitini ${\psi_\mu}^A$ transform under the USp(8) R-symmetry group with generators that can be read off directly from (A.24). It is thus clear that that each of the ${\psi_\mu}^A$ decomposes into two components of opposite SO(6) chirality which therefore carry opposite values of the U(1) charge. This fact enables us to unambiguously identify the various chiral fermionic components on the basis of this charge. Furthermore we note that the symplectic Majorana constraint (A.21) relates fermion fields of opposite U(1) charges, which is consistent with the form of the symplectic matrix $\Omega$ defined in (A.20). For instance, for the gravitini we have

$$C^{-1}{\bar{\psi}_{\mu\,I+}}^{\mathrm{T}} = (\Omega_{(4)})_{IJ}\,{\psi_\mu}^{J-}\,, \tag{A.25}$$

where $C$ denotes the charge conjugation matrix associated with the five-dimensional space-time.

Let us now turn to the spin-1/2 fermions which originate from the fields $(\psi_a, {\psi_a}^{\mathrm{c}})$ and $\lambda, \lambda^{\mathrm{c}}$ and constitute 48 independent $5D$ symplectic Majorana spinors. From $5D$ maximal supergravity we know that these spinors can be written as a symplectic traceless, fully anti-symmetric three-rank USp(8) tensor $\chi^{ABC}$. This is consistent with the fact that the spin-1/2 fields carry U(1) charges $\pm 1/2$ and $\pm 3/2$. We intend to determine the (linear) relation between the components of $\chi^{ABC}$ and the fields ${\psi_a}^A$ and $\lambda^A$ by making use of the fact that these fields do all transform consistently under the action of the maximal subgroup $\mathrm{SU}(4)\times\mathrm{U}(1)$ of USp(8). To see how this works let us present the branching of ${\psi_\mu}^A$ and $\chi^{ABC}$ under the $\mathrm{SU}(4)\times\mathrm{U}(1)$ subgroup,

$$\begin{aligned}
\mathbf{8} &\stackrel{\mathrm{SU}(4)\times\mathrm{U}(1)}{\longrightarrow} \big(\mathbf{4},\tfrac{1}{2}\big)\oplus\big(\overline{\mathbf{4}},-\tfrac{1}{2}\big)\,,\\
\mathbf{48} &\stackrel{\mathrm{SU}(4)\times\mathrm{U}(1)}{\longrightarrow} \big(\overline{\mathbf{4}},\tfrac{3}{2}\big)\oplus\big(\mathbf{4},-\tfrac{3}{2}\big)\oplus\big(\mathbf{20},\tfrac{1}{2}\big)\oplus\big(\overline{\mathbf{20}},-\tfrac{1}{2}\big)\,.
\end{aligned} \tag{A.26}$$

The chiral representations on the right-hand side are now unambiguously identified by the corresponding U(1) charge, so that they must correspond to the fields $\psi_\mu$, ${\psi_\mu}^{\mathrm{c}}$, and $\lambda$, $\lambda^{\mathrm{c}}$, $\psi_a$ and ${\psi_a}^{\mathrm{c}}$, respectively.[3]

[2]We ignore the various redefinitions of the spinors that are considered in section 2.3. These redefinitions should be performed before making the decompositions described in this appendix, but their precise details are not relevant here.

[3]A vector-spinor in odd dimension $d$ can consistently transform under $\mathrm{SO}(d{+}1)$ by describing it as an irreducible chiral vector-spinor in $d+1$ dimensions.

After these general observations we determine the precise relationship between the various chiral components, which must carry positive U(1) charges and are thus related to the fields $\lambda$ and $\psi_a$. Writing the three indices of the symplectic Majorana fields $\chi^{ABC}$ in the direct-product representation introduced before, with $A = I\alpha$, $B = J\beta$ and $C = K\gamma$. In addition the fields should vanish upon contraction with the symplectic matrix $\Omega_{I\alpha\, J\beta}$ that was defined in (A.20). Since $\alpha, \beta, \gamma$ take only two possible index values, two of them must always be equal. Hence we may distinguish the fields $\chi^{I\pm\, J\pm\, K\pm}$, which must be fully anti-symmetric in the indices $I, J, K$, and thus correspond to $4 + 4$ symplectic Majorana fields, and the fields $\chi^{I\pm\, J\pm\, K\mp}$, which are anti-symmetric in the indices $I, J$, and thus define $24+24$ fields. The remaining fields $\chi^{I\alpha\, J\beta\, K\gamma}$ follow then from imposing the overall anti-symmetry. However, unlike the fields $\chi^{I\pm\, J\pm\, K\pm}$, the fields $\chi^{I\pm\, J\pm\, K\mp}$ are not manifestly traceless with respect to contractions with the symplectic matrix $\Omega$. This implies that one must impose the additional condition

$$\chi^{I\pm\, J\pm\, K\mp}\,(\Omega_{(4)})_{JK} = 0\,, \tag{A.27}$$

which reduces the number of independent spinors in this sector to $20 + 20$, as it should.

Let us first analyze the correspondence for the spinors $\chi^{ABC}$ with positive U(1) charge $+\frac{3}{2}$, which must be linearly related to the $10D$ spinor $\lambda$. The former must be given by $\chi^{I+\, J+\, K+}$, which must necessarily be fully anti-symmetric in USp(4) indices. From (A.22) one then concludes that $\chi^{I+\, J+\, K+}$ can be decomposed into two terms, namely $(\bar\Omega_{(4)})^{[IJ}\,(\lambda^{\rm c}\,)^{K]}$ and $(\Gamma^a\bar\Omega_{(4)})^{[IJ}(\Gamma_a\lambda^{\rm c}\,)^{K]}$. However, the first completeness relation (A.23) leads to

$$(\Gamma^a\bar\Omega_{(4)})^{IJ}(\Gamma_a\psi)^K = -4\,(\bar\Omega_{(4)})^{K[I}\,\psi^{J]} - (\bar\Omega_{(4)})^{IJ}\,\psi^K\,, \tag{A.28}$$

for an arbitrary USp(4) spinor $\psi$, so that the two terms are in fact related. Hence we may adopt the following ansatz,

$$\chi^{I+\, J+\, K+} = c_{3/2}\,(\bar\Omega_{(4)})^{[IJ}\,\lambda^{K]}\,, \tag{A.29}$$

where $c_{3/2}$ is a complex proportionality factor which is undetermined at this stage. The fields with charge $-\frac{3}{2}$ are then defined through the symplectic Majorana condition,

$$\chi^{I-\, J-\, K-} \equiv -\,(\bar\Omega_{(4)})^{IL}\,(\bar\Omega_{(4)})^{JM}\,(\bar\Omega_{(4)})^{KN}\,C^{-1}\,\bar\chi_{L+\, M+\, N+}{}^{\rm T}$$

$$= \bar{c}_{3/2}\, (\bar{\Omega}_{(4)})^{[IJ}\, (\lambda^{\rm c})^{K]}\,. \tag{A.30}$$

The relation between the spinors $\chi^{I+\,J+\,K-}$ and $\psi_a$ with U(1) charge $+\frac{1}{2}$ is more subtle. First consider the following ansatz,

$$\chi^{I+\,J+\,K-} = c_{1/2}\left[(\Gamma^a\bar{\Omega}_{(4)})^{IJ}\,(\hat{\psi}_a)^K - (\bar{\Omega}_{(4)})^{IJ}\,(\Gamma^a\,\hat{\psi}_a)^K\right], \tag{A.31}$$

where $\hat{\psi}_a = \psi_a + \alpha\,\Gamma_a\Gamma^b\psi_b$ with $\alpha$, so that we have now introduced two new parameters, $c_{1/2}$ and $\alpha$ whose values will be given later. The linear combination in (A.31) is chosen such that the USp(8) constraint (A.27) is satisfied. An alternative version of (A.31), which is the one that we will actually use, is

$$\begin{aligned}\chi^{I+\,J+\,K-} =\, & c_{1/2}\left[(\Gamma^a\bar{\Omega}_{(4)})^{IJ}\,(\psi_a)^K - (\bar{\Omega}_{(4)})^{IJ}\,(\Gamma^a\,\psi_a)^K\right] \\ & + c'_{1/2}\left[(\bar{\Omega}_{(4)})^{IJ}\,(\Gamma^a\psi_a)^K + \tfrac{2}{3}(\bar{\Omega}_{(4)})^{K[I}\,(\Gamma^a\psi_a)^{J]}\right],\end{aligned} \tag{A.32}$$

but also this expression can be rewritten by making use of the identity

$$(\Gamma^a\,\bar{\Omega}_{(4)})^{[IJ}\,(\psi_a)^{K]} = -(\bar{\Omega}_{(4)})^{[IJ}(\Gamma^a\,\psi_a)^{K]}\,. \tag{A.33}$$

As before we define the spinor components with U(1) charge $-\frac{1}{2}$ by

$$\begin{aligned}\chi^{I-\,J-\,K+} \equiv\, & -(\bar{\Omega}_{(4)})^{IL}\,(\bar{\Omega}_{(4)})^{JM}\,(\bar{\Omega}_{(4)})^{KN}\,C^{-1}\,\bar{\chi}_{L+\,M+\,N-}{}^{\rm T} \\ =\, & \bar{c}_{1/2}\left[(\Gamma^a\bar{\Omega}_{(4)})^{IJ}\,({\psi_a}^{\rm c})^K - (\bar{\Omega}_{(4)})^{IJ}\,(\Gamma^a\,{\psi_a}^{\rm c})^K\right] \\ & + \bar{c}'_{1/2}\left[(\bar{\Omega}_{(4)})^{IJ}\,(\Gamma^a{\psi_a}^{\rm c})^K + \tfrac{2}{3}(\bar{\Omega}_{(4)})^{K[I}\,(\Gamma^a{\psi_a}^{\rm c})^{J]}\right].\end{aligned} \tag{A.34}$$

Hence we have obtained the linear relation between $\chi^{ABC}$ and the original $10D$ spinors, depending on three unknown complex constants, $c_{3/2}$, $c_{1/2}$ $c'_{1/2}$. Their values are determined in section 2.5, as we will be discussing at the end of this appendix.

We will now merge the chiral and anti-chiral spinors with opposite U(1) charges into eight-component symplectic Majorana spinors. In that case it is convenient to introduce SO(6) gamma matrices and chiral projection operators. The $8\times 8$ gamma matrices $(\mathbf{\Gamma}_{\hat{a}})^A{}_B$, where $\hat{a} = 1,\ldots,6$, are defined in terms of direct products of $4\times 4$ and $2\times 2$ matrices, just as in (A.24),

$$\mathbf{\Gamma}_a \equiv \Gamma_a \otimes \sigma_1\,, \qquad \mathbf{\Gamma}_6 \equiv \mathbb{1}_4 \otimes \sigma_2\,. \tag{A.35}$$

These (hermitian) gamma matrices satisfy the Clifford property

$$\{\mathbf{\Gamma}_{\hat{a}}\ ,\mathbf{\Gamma}_{\hat{b}}\} = 2\,\delta_{\hat{a}\hat{b}}\,\mathbb{1}_8\,. \tag{A.36}$$

and satisfy the following charge-conjugation properties,

$$\Omega\,\mathbf{\Gamma}_{\hat{a}}\,\Omega^{-1} = {\mathbf{\Gamma}_{\hat{a}}}^{\mathrm{T}}\,,\ \text{with}\ \Omega^{\mathrm{T}} = -\Omega\,,\quad \Omega^{-1} = -\bar{\Omega}\,, \tag{A.37}$$

where the anti-symmetric charge conjugation matrix $\Omega_{AB}$ was defined in (A.20). The chirality operator $\mathbf{\Gamma}_7$ is obtained in the standard way,

$$\mathbf{\Gamma}_{[\hat{a}}\,\mathbf{\Gamma}_{\hat{b}}\cdots\mathbf{\Gamma}_{\hat{f}]} = -\mathrm{i}\varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}}\,\mathbf{\Gamma}_7\,,\quad \text{where}\ \mathbf{\Gamma}_7 = \mathbb{1}_4\otimes\sigma_3\,. \tag{A.38}$$

Observe that $\mathbf{\Gamma}_7$ is hermitian and behaves under charge conjugation as $\Omega\,\mathbf{\Gamma}_7\,\Omega^{-1} = -{\mathbf{\Gamma}_7}^{\mathrm{T}}$. Furthermore $\mathbf{\Gamma}_7$ coincides with the U(1) charge that was already present in the original $10D$ theory.

The gamma matrices $\mathbf{\Gamma}_{\hat{a}}$ and their multiple anti-symmetrized products define a complete basis for matrices in the 8-dimensional spinor space. They can conveniently decomposed into 28 anti-symmetric matrices $\Omega$, $\Omega\mathbf{\Gamma}_{\hat{a}}$, $\Omega\mathbf{\Gamma}_{\hat{a}}\mathbf{\Gamma}_7$ and $\Omega\mathbf{\Gamma}_{\hat{a}\hat{b}}\mathbf{\Gamma}_7$, and 36 symmetric matrices $\Omega\mathbf{\Gamma}_7$, $\Omega\mathbf{\Gamma}_{\hat{a}\hat{b}}$ and $\Omega\mathbf{\Gamma}_{\hat{a}\hat{b}\hat{c}}$. The latter are related to the anti-hermitian generators of USp(8) that were already defined in (A.24),

$$\begin{aligned} T &= \mathrm{i}\mathbf{\Gamma}_7\,, & T_a &= \mathbf{\Gamma}_{a6}\,, & {T_{ab}}^1 &= \tfrac{1}{6}\varepsilon_{abcde6}\,\mathbf{\Gamma}^{cde}\,, \\ & & {T_{ab}}^0 &= \mathbf{\Gamma}_{ab}\,, & {T_{ab}}^2 &= \mathbf{\Gamma}_{ab6}\,. \end{aligned} \tag{A.39}$$

We have now obtained a parametrization of the relation between the fields $\chi^{ABC}$ and the fields $\lambda$, $\lambda^{\mathrm{c}}$, $\psi_a$ and ${\psi_a}^{\mathrm{c}}$ originating from the $10D$ theory in terms of (anti-)chiral components. This relation is in accordance with the $\mathrm{SU}(4)\times\mathrm{U}(1)$ branching of the spinor fields presented in (A.26). The resulting expressions for given charges were given in (A.29), (A.30), (A.32), (A.34), which can be converted in terms of the SO(6) gamma matrices $\mathbf{\Gamma}_{\hat{a}}$. Since we have established this relation for chiral and anti-chiral components separately, it is convenient to introduce chiral projection operators

$$\mathbb{P}_\pm = \tfrac{1}{2}\big(\mathbb{1}\pm\mathbf{\Gamma}_7\big)\,. \tag{A.40}$$

The spinor $\chi^{ABC}$ is subsequently decomposed in tri-spinors with all possible chiralities,

$$\chi^{ABC} = \chi^{ABC}_{+++} + \chi^{ABC}_{---} + \chi^{ABC}_{++-} + \chi^{ABC}_{+-+} + \chi^{ABC}_{-++} + \chi^{ABC}_{--+} + \chi^{ABC}_{-+-} + \chi^{ABC}_{+--}\,. \tag{A.41}$$

For the spinors with U(1) charge equal to +3/2 and +1/2 we derive, respectively,

$$\chi^{ABC}{}_{(+++)} = \mathrm{i}\, c_{3/2}\, \mathbb{P}_+{}^A{}_D\, \mathbb{P}_+{}^B{}_E\, \mathbb{P}_+{}^C{}_F\, \big[\mathbf{\Gamma}_7\mathbf{\Gamma}_6\, \bar{\Omega}\big]^{[DE}\, \lambda^{F]}\,, \tag{A.42}$$

$$\begin{aligned}\chi^{ABC}{}_{(++-)} = {} & \mathrm{i}\, c_{1/2}\, \mathbb{P}_+{}^A{}_D\, \mathbb{P}_+{}^B{}_E\, \mathbb{P}_-{}^C{}_F \Big[\big[\mathbf{\Gamma}^a\, \bar{\Omega}\big]^{DE} \big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\, \psi_a\big)^F - \big[\mathbf{\Gamma}_7\mathbf{\Gamma}_6\, \bar{\Omega}\big]^{DE} \big(\mathbf{\Gamma}^a\, \psi_a\big)^F\Big] \\ & + \mathrm{i}\, c'_{1/2}\, \mathbb{P}_+{}^A{}_D\, \mathbb{P}_+{}^B{}_E\, \mathbb{P}_-{}^C{}_F \Big[\big[\mathbf{\Gamma}_7\mathbf{\Gamma}_6\, \bar{\Omega}\big]^{DE} \big(\mathbf{\Gamma}^a\, \psi_a\big)^F - \tfrac{2}{3}\, \bar{\Omega}^{F[D} \big[\mathbf{\Gamma}_7\mathbf{\Gamma}^{a6}\psi_a\big]^{E]}\Big]\,,\end{aligned}$$

where the spinors $\lambda$ and $\psi_a$ are now 8-component spinors consisting of $(\lambda, \lambda^{\mathrm{c}})$ and $(\psi_a, \psi_a{}^{\mathrm{c}})$. The labels $(+++)$ and $(++-)$ on the left-hand side indicate how the indices are contracted with the chiral projectors. Note that the combinations $(+-+)$ and $(-++)$ are related upon interchanging the indices $A, B, C$ correspondingly. The corresponding spinors with charges $-3/2$ and $-1/2$ read the same with $c_{3/2}$, $c_{1/2}$ and $c'_{1/2}$ replaced by their complex conjugates and with opposite projectors.

Confronting the above decompositons to the equations (2.74) uniquely determines the three constants to $c_{3/2} = -\frac{3}{4}$, $c_{1/2} = -\frac{1}{4}$ and $c'_{1/2} = -\frac{1}{2}$. The corresponding expression for $\chi^{ABC}$ equals

$$\begin{aligned}\chi^{ABC} = & -\tfrac{3}{8}\mathrm{i}\Big[\big(\mathbf{\Gamma}_6\, \bar{\Omega}\big)^{[AB} \big(\mathbf{\Gamma}_7\lambda\big)^{C]} + \big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\, \bar{\Omega}\big)^{[AB}\, \lambda^{C]}\Big] \\ & -\tfrac{3}{8}\mathrm{i}\Big[\big(\mathbf{\Gamma}^a\, \bar{\Omega}\big)^{[AB} \big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\psi_a\big)^{C]} - \big(\mathbf{\Gamma}_7\mathbf{\Gamma}^a\, \bar{\Omega}\big)^{[AB} \big(\mathbf{\Gamma}_6\psi_a\big)^{C]}\Big] \\ & -\tfrac{3}{8}\mathrm{i}\Big[\big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\, \bar{\Omega}\big)^{[AB} \big(\mathbf{\Gamma}^a\psi_a\big)^{C]} - \big(\mathbf{\Gamma}_6\, \bar{\Omega}\big)^{[AB} \big(\mathbf{\Gamma}_7\mathbf{\Gamma}^a\psi_a\big)^{C]}\Big] \\ & -\tfrac{1}{2}\mathrm{i}\, \bar{\Omega}^{[AB} \big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\mathbf{\Gamma}^a\psi_a\big)^{C]}\,.\end{aligned} \tag{A.43}$$

Here we shoud stress that this form of the solution is not unique as it can be rewritten by Fierz reordering. In section 2.5 we have presented an equivalent but shorter expression.

# Appendix B

# N=4 superconformal curvature constraints and algebra

In this appendix, we present some relevant material for the arguments and the computations of chapter 4. Most of the following expressions already appeared in [97] and we remind them here for the reader's convenience.

In analogy with the conformal gravity case discussed in 1.2, some of the $N = 4$ superconformal gauge fields (see Table 4.1) are composite fields. They are expressed in terms of the other fields of the theory via a set of supercovariant constraints, which we here choose to be

$$\begin{aligned} R(P)_{\mu\nu}{}^{a} &= 0\,, \\ e^{\nu}{}_{b}\, R(M)_{\mu\nu}{}^{ab} &= 0\,, \\ \gamma^{\mu}\, R(Q)_{\mu\nu}{}^{i} &= 0\,. \end{aligned} \tag{B.1}$$

All the curvatures can be derived by supercovariantizing the curl of the associated gauge connections. They read

$$\begin{aligned} R(P)_{\mu\nu}{}^{a} &= 2\,\mathcal{D}_{[\mu}e_{\nu]}{}^{a} - \bar{\psi}_{[\mu}{}^{i}\gamma^{a}\psi_{\nu]i}\,, \\ R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^{b} - 4\,e_{[\mu}{}^{[a}f_{\nu]}{}^{b]} \\ &\quad + \tfrac{1}{2}\big[\bar{\psi}_{[\mu}\gamma^{ab}\phi_{\nu]i} - 2\,\bar{\psi}_{[\mu i}\,\gamma_{\nu]}\,R(Q)^{ab\,i} + 2\,\bar{\psi}_{\mu i}\,\psi_{\nu j}\,T^{ab\,ij} + \text{h.c}\big]\,, \\ R(Q)_{\mu\nu}{}^{i} &= 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^{i} - \gamma_{[\mu}\phi_{\nu]}{}^{i} - \tfrac{1}{2}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]j}T_{ab}{}^{ij} + \tfrac{1}{2}\varepsilon^{ijkl}\bar{\psi}_{[\mu j}\,\psi_{\nu]k}\,\Lambda_{l}\,, \end{aligned} \tag{B.2}$$

where the derivative $\mathcal{D}_{\mu}$, defined in (4.10), is covariant with respect to all the bosonic symmetries except for the conformal boosts. Note that by setting the fermions to vanish, the curvatures (B.2) reduce to those presented in (1.20) for the $N = 0$ case. The first, second and third constraint in (B.1) respectively determines the field $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^a$ in terms of the independent fields of the Weyl supermultiplet. The U(1) gauge field $a_\mu$ is also a composite field and is expressed as the solution of the supercovariant constraint (4.4). Its associated curvature, $F(a)_{\mu\nu}$, was given in (4.5).

The remaining supercovariant curvatures, which appear in the Q-supersymmetry transformations of the fields (4.11) and the action (4.33), are given by

$$\begin{aligned} R(V)_{\mu\nu}{}^i{}_j &= 2\,\partial_{[\mu} V_{\nu]}{}^i{}_j - 2\,V_{[\mu}{}^i{}_k\,V_{\nu]}{}^k{}_j - \big[\delta_{Q|_{\psi_{[\mu}}} + \delta_{S|_{\phi_{[\mu}}}\big] V_{\nu]}{}^i{}_j\,, \\ R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{6}\gamma_{[\mu}\gamma^{ab}\phi_{\nu]j}T_{ab}{}^{ij} - \tfrac{1}{2}\varepsilon^{ijkl}\bar\phi_{[\mu k}\,\Lambda_l\,\psi_{\nu]j} - \delta_{Q|_{\psi_{[\mu}}}\phi_{\nu]}{}^i\,, \end{aligned} \tag{B.3}$$

where $\delta_{|_\bullet}$ denotes an infinitesimal transformation in which the parameter is replaced by $\bullet$ . Using (4.9) and (4.12), the explicit expressions of the curvatures (B.3) can be easily derived. Finally, the curvatures associated to the dilatations and the conformal boosts can be written as

$$\begin{aligned} R(D)_{\mu\nu} &= 0\,, \\ R(K)_{\mu\nu}{}^c &= e_\mu{}^a\,e_\mu{}^b\,D_d R(M)_{ab}{}^{dc}\,. \end{aligned} \tag{B.4}$$

These relations between the supercovariant curvatures follow from combining the constraints (B.1) with the SU(2, 2|4) Bianchi identities. Additional relations of this kind can be derived and were presented in [97]. Let us simply mention that the only non-vanishing independent curvatures are $R(M)_{\mu\nu}{}^{ab}$, $R(V)_{\mu\nu}{}^i{}_j$, $R(Q)^{-}_{\mu\nu}{}^i$ and $R(S)^{+}_{\mu\nu}{}^i$. Indeed, one can derive

$$\begin{aligned} R(Q)^{+}_{\mu\nu}{}^i &= 0\,, \\ R(S)^{-}_{\mu\nu}{}^i &= e_\mu{}^a\,e_\nu{}^b\,\not{D}R(Q)_{ab}{}^i\,. \end{aligned} \tag{B.5}$$

where the (anti)self-dual part of an arbitrary antisymmetric Lorentz tensor $L_{ab}$ is defined as

$$L^{\pm}_{ab} = \tfrac{1}{2}(L_{ab} \pm \tfrac{1}{2}\varepsilon_{abcd}L^{cd})\,.\,. \tag{B.6}$$

The Q- and S-supersymmetry variations of the curvatures can be derived using the variations of the fields (4.9) and (4.12) and the expressions of the curvatures given above. Part of them can once again be found in [97]. Their complete expressions are not particularly useful for the purpose of chapter 4 and therefore will only be presented in [102].

Finally, let us comment on the symmetry algebra of $N=4$ conformal supergravity. As mentioned previously, the theory is originally based on the $\mathfrak{su}(2,2|4)$ algebra [59]. Because of the curvature constraints (B.2) that must be imposed, some of the structure constants receive field-dependent modifications and, as such, characterise a so-called 'soft algebra'. Here, let us restrict ourselves to those commutation relations which differ from the original $\mathfrak{su}(2,2|4)$ relations. They correspond to the following commutators

$$\begin{aligned}\big[\delta_Q(\epsilon_1),\delta_Q(\epsilon_2)\big] =&\,\delta_{cgct}(\xi^\mu)+\delta_M(\varepsilon^{ab})+\delta_Q(\epsilon_3^i)+\delta_S(\eta^i)\\ &+\delta_{\mathrm{SU}(4)}(\Lambda^i{}_j)+\delta_{\mathrm{U}(1)}(\Lambda)+\delta_K(\Lambda_K^a)\,,\end{aligned}\tag{B.7}$$

$$\big[\delta_Q(\epsilon),\delta_S(\eta)\big] =\delta_D(\tilde\Lambda_D)+\delta_M(\tilde\varepsilon^{ab})+\delta_S(\tilde\eta^i)+\delta_{\mathrm{SU}(4)}(\tilde\Lambda^i{}_j)+\delta_K(\tilde\Lambda_K^a)\tag{B.8}$$

where $\delta_{cgct}$ denotes a covariant general coordinate transformation. The various (field-dependent) parameters read

$$\begin{aligned}\xi^\mu =&\,2\,\epsilon_1^i\gamma^\mu\,\bar\epsilon_{2i}+\text{h.c.}\,,\\
\varepsilon^{ab} =&\,4\,\bar\epsilon_1^i\epsilon_2^j\,T_{ab\,ij}+\text{h.c.}\,,\\
\epsilon_3^i =&\,\varepsilon^{ijkl}\bar\epsilon_{1k}\epsilon_{2l}\,\Lambda_j\\
\eta^i =&\,-2\,\bar\epsilon_1^k\,\epsilon_2^l\chi^i{}_{kl}-\tfrac12\big(\bar\epsilon_2{}^k\gamma^a\epsilon_{1j}+\text{h.c.}\big)\big(\gamma_a\chi^{ij}{}_k-\tfrac12\varepsilon^{ijlm}\gamma_{cd}\gamma_a\Lambda_l T^{cd}{}_{km}\big)\\
&-\tfrac1{12}\big(\bar\epsilon_2{}^i\gamma_a\epsilon_{1j}-\delta^i_j\,\bar\epsilon_2{}^l\gamma_a\epsilon_{1l}+\text{h.c.}\big)\gamma^a\big(E^{jk}\Lambda_k+2\bar{\not{P}}\Lambda^j\big)\\
&+\tfrac12\varepsilon^{ijkl}\bar\epsilon_{1k}\epsilon_{2l}\,\not{D}\Lambda_j+\tfrac23\bar\epsilon_2{}^{[i}\epsilon_1{}^{j]}\big(E_{jk}\Lambda^k+2\not{P}\Lambda_j\big)\,,\\
\Lambda^i{}_j =&\,E^{ik}\varepsilon_{klmj}\,\epsilon_2^l\epsilon_1^m+\tfrac12\big(\bar\epsilon_2^k\gamma_a+\text{h.c.}\big)\bar\Lambda^i\gamma_a\Lambda_k-\tfrac14\big(\bar\epsilon_2^k\gamma_a\epsilon_{1k}+\text{h.c.}\big)\bar\Lambda^i\gamma^a\Lambda_j\\
&-\tfrac14\big(\bar\epsilon_2 i^\gamma_a\epsilon_{1j}+\text{h.c.}\big)\bar\Lambda^k\gamma^a\Lambda_k-(\text{h.c.;traceless})\,,\\
\Lambda =&\,\tfrac12\big(\bar\epsilon_2^i\gamma_a\epsilon_{1j}+\text{h.c.}\big)\big(\bar\Lambda^j\gamma^a\Lambda_i-\delta_i^j\bar\Lambda^k\gamma^a\Lambda_k\big)\\
\Lambda_K^a =&\,\tfrac13\,\varepsilon^{abcd}\bar\epsilon_{2i}\gamma_b\epsilon_1{}^j\,R(V)_{cd}{}^i{}_j+\tfrac83\bar\epsilon_{2i}\epsilon_{1j}D_bT^{ab\,ij}-\tfrac16\,i\,\varepsilon^{abcd}\bar\epsilon_{2i}\gamma_b\epsilon_1^iF(a)_{cd}\\
&+\tfrac14\bar\epsilon_{2i}\gamma_{ab}\gamma_{cd}\epsilon_1{}^kT^{ab}{}_{jk}T^{cd\,ij}+\text{h.c.}\,,\end{aligned}\tag{B.9}$$

and

$$\begin{aligned}\tilde\Lambda_D =&\,-\bar\eta_i\epsilon^i+\text{h.c.}\,, &\tilde\varepsilon^{ab} =&\,-\bar\eta_i\gamma^{ab}\epsilon^i+\text{h.c.}\,,\\
\tilde\eta^i =&\,-\tfrac14\,\varepsilon^{ijkl}\bar\eta_k\gamma_a\epsilon_j\gamma^a\Lambda_l\,, &\tilde\Lambda_K^a =&\,\tfrac16\bar\eta_i\gamma^{bc}\gamma^a\epsilon_jT_{cb}{}^{ij}+\text{h.c.}\,,\\
\tilde\Lambda^i{}_j =&\,-2\bar\epsilon^i\eta_j-\text{h.c.;traceless}\,.\end{aligned}\tag{B.10}$$

# Samenvatting

Theoretisch natuurkundigen proberen wiskundige modellen te bedenken die de wereld om ons beschrijven. Dit theoretische werk had in de jaren 70' geleid tot twee belangrijke, maar incomplete, beschrijvingen. Deze beschrijvingen zijn gebaseerd op de algemene relativiteitstheorie van Einstein aan de ene kant, en kwantumveldentheorie aan de andere kant. De eerste beschrijft zwaartekracht en haar effect op macroscopische objecten zoals planeten en sterren. De tweede beschrijft natuurkunde op microscopische schaal zoals de wisselwerkingen van elementaire deeltjes onder drie fundamentele krachten: de sterke, zwakke, en elektromagnetische kracht. Het probleem schuilt in het feit dat deze succesvolle beschrijvingen niet op een eenduidige manier gecombineerd kunnen worden. De algemene relativiteitstheorie gaat niet samen met de regels van de kwantummechanica omdat de combinatie niet leidt tot consistente voorspellingen. Natuurkundigen geloven daarom dat er een meer fundamentele theorie van kwantumzwaartekracht moet zijn. Deze theorie zou consistent moeten zijn op alle schalen. Ondanks grote inspanningen in de laatste 50 jaar, en de vele kandidaattheorieën, is het bestaan en de beschrijving van zo'n theorie een open vraag. Bovendien is het testen van zo'n theorie problematisch omdat de zwaartekracht zo'n 40 ordes van grootte zwakker is dan de andere drie fundamentele krachten.

Een van de populaire kandidaten voor een theorie van kwantumzwaartekracht is supersnaartheorie. In dit raamwerk wordt alles in ons universum beschreven in enkel soort fundamentele objecten: vibrerende snaren met een typische lengte van $10^{33}$ cm (de Planck lengte). Deze snaren kunnen open of gesloten zijn en bewegen in een tien-dimensionale ruimte. Net zoals een snaar van een piano een oneindige hoeveelheid tonen kan creëren, kan een supersnaar op verschillende manieren vibreren om zo een oneindige toren van elementaire deeltjes te maken. Een van deze excitaties is het graviton dat correspondeert met de kwantumexcitatie van het zwaartekrachtveld. In supersnaartheorie is zwaartekracht daarom expliciet kwantummechanisch.

Door de jaren heen, zijn er vijf verschillende supersnaartheorieën ontdekt. Een tijdlang werd dit gezien als bewijs dat er geen unieke fundamentele natuurkundige theorie bestaat. Echter het werd langzamerhand duidelijk dat de vijf theorieën aan elkaar verbonden zijn via bepaalde relaties die ook wel dualiteiten worden genoemd. Het bleek dat de vijf theorieën allemaal realisaties zijn van een enkele onderliggende theorie in elf dimensies. Het ontrafelen van deze theorie, ook wel M-theorie genoemd, is het doel van menig onderzoeksprogramma.

Net zoals onze oren ongevoelig zijn voor noten boven een bepaalde frequentie, kunnen we alleen maar deeltjes waarnemen die corresponderen met lage frequenties van de supersnaren. In deze lage-energielimiet kunnen snaren benaderd worden door puntdeeltjes wiens gedrag en interacties beschreven worden door veldentheorieën genaamd superzwaarekracht. Deze theorieën zijn supersymmetrische extensies van algemene relativiteitstheorie. Supersymmetrie is een symmetrie die de twee meest algemene categorieën van deeltjes (de deeltjes verschillen alleen wat betreft spin-statistiek) relateert namelijk bosonen en fermionen. Superzwaartekrachttheorieën kunnen geformuleerd worden in elf dimensies and voor verschillende hoeveelheden supersymmetrie. Specifiek blijkt het dat er vijf tien-dimensionale en een elf-dimensionale superzwaartekrachten bestaan. Deze beschrijven de lage-energie limiet van de vijf supersnaartheorieën en M-theorie. Deze thesis gaat alleen over het lage-energie raamwerk van superzwaartekracht en in het bijzonder over theorieën met een maximale hoeveelheid supersymmetrie. Dit bevat ook de elf-dimensionale theorie en de zogenaamde type IIB en (massaloze en massieve) type IIA theorieën in tien dimensies.

Het bestaan van zo'n groot aantal dimensies is verontrustend, omdat het in tegenspraak is met ons leven in drie ruimtelijke dimensies. Deze paradox kan worden opgelost door de extra dimensies 'op te rollen' zodat ze onmeetbaar klein zijn in ons dagelijks leven. Dit is te vergelijken hoe een kabel van afstand een een-dimensionaal object lijkt terwijl een kever die rondom de kabel loopt ook de twee-dimensionale structuur waarneemt. In superzwaartekracht kunnen de deeltjes bewegen in de extra opgerolde dimensies en hun gedrag wordt daardoor beïnvloed. Het resultaat is dat de effectieve lage-dimensionale natuurkunde die wij waarnemen afhangt van de geometrie van de microscopische opgerolde ruimte.

In dit proefschrift bekijken we de zogenaamde dualiteit symmetrieën (en hun deformaties) die ontstaan in maximale superzwaartekrachttheorieën geformuleerd in minder dan tien dimensies. Veel van deze theorieën kunnen verkregen worden door compactificatie van elf- en/of tien-dimensionale superzwaartekracht. In sommige gevallen kunnen dezelfde gecompactificeerde theorieën verkregen worden door verschillende compactificatie van verscheidene hogerdimensionele voorouders en, in zo'n geval, kunnen de dualiteit symmetrieën begrepen worden als de lage-energie overblijfselen van de supersnaar dualiteiten. Het doel van dit werk is om de hogerdimensionale manifestaties en/of de unificerende eigenschappen te onderzoeken van dualiteit symmetrieën in superzwaartekracht. In de toekomst leidt dit hopelijk tot meer inzicht in de structuur van M-theorie.

De belangrijkste hoofdstukken van deze thesis zijn niet direct verbonden maar vormen meer een mozaïek van resultaten waarin dualiteit symmetrieën van maximale superzwaartekrachtstheorieën een central role spelen. In het eerste hoofdstuk introduceren we het vereiste achtergrondmateriaal. In het tweede hoofdstuk werken we een herformulering uit van de tien-dimensionale type IIB superzwaartekracht. Deze herformulering laat expliciet de karakteristieke eigenschappen zien van de dualiteit symmetrie van vijf-dimensionale maximale superzwaartekrachtstheorieën. Uiteindelijk, de herformulering maakt het mogelijk om de consistentie te onderzoeken van een specifieke truncatie van de IIB theorie naar vijf dimensies. Het derde hoofdstuk gaat over het raamwerk van exceptionele veldentheorieën dat afhangt van onfysische vrijheidsgraden om zo tegelijk elf- en massaloze tien-dimensionale superzwaartekrachtstheorieën te beschrijven op een dualiteitsymmetrische manier. We introduceren op een consistente manier deformaties van dit raamwerk en laten zien dat op de deze manier ook type IIA superzwaartekracht kan worden ingebed. De laatste hoofdstukken hebben tot slot betrekking op schaalinvariante superzwaartekracht. We identificeren en presenteren voor de eerste keer de meest algemene klasse van Lagrangianen voor de maximale theorie, beter bekend als $N = 4$ schaalinvariante superzwaartekracht. Deze klasse wordt gekarakteriseerd door een holomorfe functie die zich gedraagt als een deformatie van de dualiteit symmetrie van de theorie.

# Summary

Theoretical physicists aim to establish models, formulated in a mathematical language, that accurately predict the natural phenomena which can be observed around us. By the 1970's, theoretical research had resulted in two main incomplete descriptions of nature that are characterised by Einstein's general relativity and quantum (field) theory. The former accurately describes gravity and its effects on large-scales objects, such as planets and stars. The latter governs the physics on the microscopic scales, such as the interactions of elementary particles with the remaining three forces, namely the strong, weak and electromagnetic force. However, when these two theories are considered in each other's domain of validity, they lose any sensible predictive power. For instance, on the microscopic scales, general relativity does not comply with the rules of quantum mechanics and suffers from technical inconsistencies. This has led physicists to expect that they should in fact correspond to fragments of a unique, more fundamental theory of quantum gravity. The latter would then provide a unified description of all forces and matter that would remain consistent at all scales. Despite the many efforts that were invested during the past 50 years, and the various proposals that have been put forward, the existence and the details of such a unified theory remain today an open question. Furthermore, coming up with realistic experimental tests of such a theory would also constitute a real challenge in view of the fact that gravity is about $10^{40}$ times weaker than the other three forces.

Among the popular candidates for a theory of quantum gravity is superstring theory. In this framework, our whole universe is described in terms of a single type of objects: vibrating strings (of the order of the Planck length, $10^{-33}$ cm), that can be closed or open, and that propagate and interact in a ten-dimensional space. Similarly to a single piano string that can produce an infinity of notes depending on its vibration mode, the various energy modes (or harmonics) of the strings in superstring theory correspond to an infinity of different elementary particles. Among them is the so-called graviton which corresponds to a quanta

of excitation of the gravitational field. In superstring theory, gravity therefore appears as quantised from the start.

Along the years, a total of five different superstring theories were discovered. While once again, this seemed to contradict the idea that nature should be described by a unique theory, it was gradually realised that the five theories are in fact connected by a large group of equivalence relations, known as dualities, and should correspond to different realisations of a single underlying eleven-dimensional theory. Unraveling the structure and properties of this so-called M-theory, which remain to this day unknown, is the long-term goal of many ongoing research programme.

In analogy with our ears which remain insensitive to notes above a certain frequency, we are only able to perceive some of the particles that are associated with the low-energy modes of the strings. In this low-energy limit, because of the trivial vibrational profiles of the relevant modes, strings can be approximated by points whose behaviour and interactions are effectively described by field theories known as supergravities. The latter are supersymmetric extensions of general relativity. Supersymmetry, which is also a key ingredient in superstring theory, is a symmetry that relates the two most general categories of elementary particles (which only differ by their so-called spin-stastics), *i.e.* the bosons and the fermions. Supergravity theories can be formulated in up to eleven spacetime dimensions and with various amount supersymmetry. In particular, there exist five supergravity theories in ten dimensions and only one in eleven dimensions. They are known to describe the low-energy limits of the five superstring theories and the elusive M-theory, respectively. The present work is exclusively concerned with the low-energy framework of supergravity and, more specifically, with those theories which have a maximal amount of supersymmetries. These include the eleven-dimensional theory and the so-called type IIB and (massless and massive) type IIA theories in ten dimensions.

The possibility for such large numbers of spacetime dimensions might at first appear disconcerting, as it naturally conflicts with our daily perception of only three space dimensions. This mismatch is resolved by considering the extra space dimensions to be compactified, or curled up, to a size that is unobservable at our scale. The same idea for instance underlies the fact that, from far away, one discerns say an electric cable as a one-dimensional object, while a small bug that moves along and around the cable is able to experience its two-dimensional structure. In supergravity, the particles are allowed to move in these compactified dimensions, and are therefore sensitive to their shapes. As a result, the effective lower-dimensional physics that we observe depends on the chosen geometry of the compactified space.

In this work, we consider the so-called duality symmetries (and their deformations) that appear in maximal supergravities formulated in less than ten spacetime dimensions. Many of these supergravities can be obtained from various compactifications of eleven and/or ten-dimensional maximal supergravities. In particular, some of these lower-dimensional theories equivalently descend from different compactifications of multiple higher-dimensional ancestors and, in this case, their duality symmetries can be understood as the low-energy remnants of the superstring dualities. The aim of the present work is to investigate the higher-dimensional manifestations and/or the unifying abilities of the duality symmetries in supergravity. In the future, this might hopefully lead to some insights into the structure of M-theory.

The main chapters of this thesis are not directly connected, but rather constitute a mosaic of results in which the duality symmetries of maximal supergravities play a central role. In the first chapter, we introduce some of the necessary background material. In the second chapter, we work out a reformulation of the ten-dimensional type IIB supergravity which exhibits the characteristic features of the duality symmetry of five-dimensional maximal supergravities. This reformulation ultimately allows us to study the consistency of a specific truncation of the IIB theory to five dimensions. The third chapter is concerned with the framework of exceptional field theories, which relies on spurious (or unphysical) degrees of freedom in order to simultaneously describe the eleven- and massless ten-dimensional supergravities in a duality symmetric formalism. In particular, we introduce consistent deformations of this framework and show that those further allow for the embedding of massive type IIA supergravity. Finally, the last chapter deals with conformal supergravity in four dimensions. For the first time, we construct and partially present the most general class of Lagrangians for the maximal theory, which is known as $N = 4$ conformal supergravity. This class is characterised by a holomorphic function which, from a certain perspective, can be seen as a deformation of the duality symmetry of the theory.

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