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**History**

# The how and why of constructions in classical geometry

**What can we learn today from the classical tradition of geometrical constructions? Viktor Blåsjö looks at construction methods through the ages, and points to lessons for teachers as well as philosophers.**

Mathematics was for many thousands of years something you drew in the sand. It was active, hands-on. We can see this for instance in Rafael's fresco *The School of Athens* (Figure 1). Among the many philosophers and scientists shown, one and only one is actually doing something with an instrument: Euclid, who is busy drawing figures with his compass. Likewise, Archimedes was killed by invading soldiers while tracing figures in the sand, as ancient sources report (e.g., Livy, *Ab Urbe Condita*, XXV.31) and as depicted in Figure 2. Figure 3 again shows geometry drawn in Greek sand. As the inscription explains, it is based on a story about shipwrecked men finding hope in these signs of intelligent life on the beach where they washed ashore.

This do-it-yourself element of ancient mathematics makes it a lot of fun and also well-suited to modern pedagogical

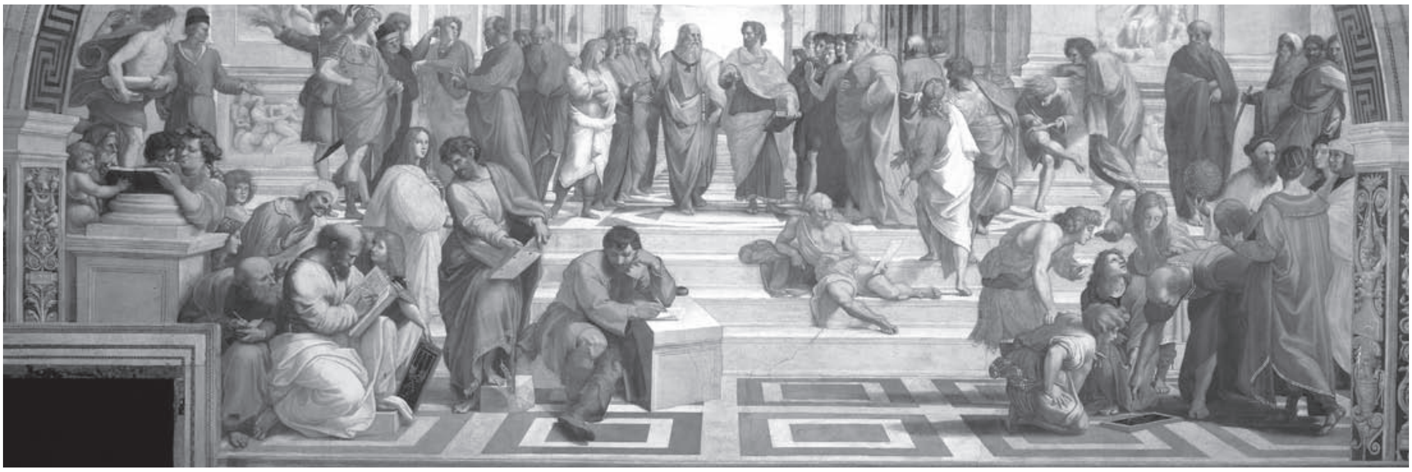

**Figure 1** Rafael's *School of Athens*. Early 16th century, the Vatican.

**Figure 2** The death of Archimedes in an 18th-century engraving. From Giovanni Maria Mazzuchelli, *Notizie Isotoriche e Critiche Intorno alla Vita, alle Invenzioni, ed agli Scritti di Archimede Siracusano*.

inclinations. But at the time the purpose of this kind of mathematics was not pedagogical, for engaging students. Rather, it was cutting-edge research pursued by the very best mathematicians of their time, and, as we shall see, these mathematicians had excellent and sophisticated reasons for insisting on drawing everything in the sand.

### The significance of the ruler and compass

The simplest embodiment of the tradition of which I speak is the ruler and compass of Euclidean geometry. In classical times, these simple tools were seen as embodying the very essence of mathematical meaning and method, a fact that is easily obscured by how deceptively simple and natural they are. Indeed, one could argue that the ruler and the compass are as old as geometry itself. According to ancient sources [15, p. 52], geometry began, as its name suggests, as a form of 'earth-measurement'. This was necessitated by the yearly overflowing of the Nile in Egypt: the flooding made the banks of the river fertile in an otherwise desert land, but it also wiped away boundaries between plots, so a geometer, or 'earth-measurer', had to be called upon to redraw a fair division of the precious farmable land. In fact, Egyptian geometers were not called 'earth-measurers' as in Greek but rather 'rope-stretchers'. The rope was their basic tool, as Figure 4 illustrates. There are two basic ways of generating a curve using a rope: if you pull both ends you get a straight line, and if you hold one end fixed and move the other while holding the rope fully stretched you get a circle. In this way the ruler and the compass can be seen as arising immediately from the most basic motivation for geometry in practical necessity.


**Figure 3** Frontispiece from *Apollonii Pergaei Conicorum*, Oxford, 1710.

Illustration: Linda Hall Library of Science, Engineering & Technology

But in Greek times, around the fifth century BC, these simple tools took on a much more sophisticated theoretical importance (see for example the historical introduction in [8]). Consider for instance $\sqrt{7}$. Today we think of $\sqrt{7}$ as a number, but this is highly problematic since it cannot be written as a fraction of integers or a finite or repeating decimal expansion. What, then, is this mysterious number called $\sqrt{7}$, which no one has ever seen actually written down in exact numerical form? Does it even exist at all?

If we take the ruler and the compass as the basis of mathematical ontology this problem disappears at once. By the simple construction of Figure 5 the elusive $\sqrt{7}$ is readily caught on paper in the form of a line segment as humble as any other. Furthermore the construction is as precise as it is simple. I carried out this construction with my students the other day. We did it back-of-an-envelope style, with some basic compasses I bought at a toy store. We made no special efforts to ensure accuracy. Yet when we measured our $\sqrt{7}$ segment and averaged our values we were off by less than $0.0001$ from the true value.

But there is much more at stake here than the problem of incommensurability. Zeuthen famously advocated the thesis of "the geometrical construction as 'existence proof' in ancient geometry", i.e., that "the construction … served to ensure the existence of that which were to be constructed" [19, p. 223]. Phenomena such as the fact that $\sqrt{7}$ "does not exist" in the world of rational numbers could very well have suggested that concerns of this type are not as paranoid as they may seem at first sight. And legitimate existence doubts are by no means confined to intuitively obvious matters: the existence of the five regular polyhedra, for example, is far from obvious by any standard, until one sees constructions of them such as those with which Euclid crowned the *Elements* around 300 BC.

But the matter goes deeper still. Even existence questions aside, constructions are in the Greek tradition the very source of meaning in mathematics. It is a warrant guaranteeing that every mathematical proposition, no matter how subtle, has a definite 'cash value', as it were, i.e., that it has theory-independent, jargon-free, empirical content. Constructions, then, serve the purpose of grounding geometry in a concrete, pre-theoretical reality that is accessible and indisputable even to outsiders.

This means in particular that any proposition is in principle checkable without any understanding of its proof, since it can ultimately be boiled down to a construction recipe and an empirically checkable assertion about the resulting figure. That is to say, theorems in the Euclidean tradition are of the form "if you perform

Illustration: www.metmuseum.org

**Figure 4** Egyptian geometers, or 'rope-stretchers', delineating a field by means of a stretched rope. From the tomb of Menna, Egypt, c. 14th century BC.

such-and-such operations, this will result", e.g., if you draw a triangle and add up its angles they will make two right angles. By thus speaking about measurements and relations in figures whose constructions have been specified, theorems in the Euclidean tradition imply a recipe for checking them empirically in as many instances as desired. This has many potential uses, from convincing sceptic outsiders to aiding explorative research. It also makes it possible to display expertise without revealing one's methods — a common practice in mathematics as late as the seventeenth century, where constructions published without proofs are commonplace. These kinds of advantages of construction-based mathematics are quite incompatible with the emphasis in modern mathematics on grand 'systemic' theorems such as Rolle's Theorem in analysis, Cayley's Theorem in group theory, and so on. These modern kinds of theorems are not of the constructive, Euclidean type, whose very formulation implies a verification procedure.

Another way of putting it is in terms of the positivist paradigm that has dominated much of empirical science in modern times. The positivist principle is that science should only speak of that which is observable or measurable; it should not engage in speculation about qualities and sympathies and whatnot that do not

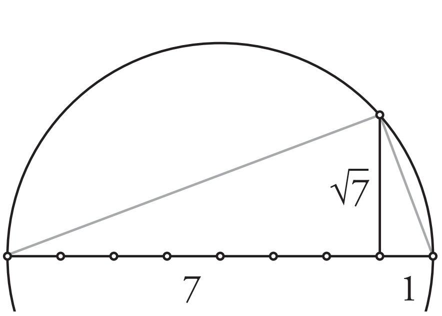


**Figure 5** Ruler-and-compass construction of $\sqrt{7}$. A line segment of length $7+1$ is used as the diameter of a circle. A perpendicular line is erected at the point between the two subsegments. The height of the perpendicular is $\sqrt{7}$, as is easily seen by similar triangles.

translate into a recipe for concrete measurement, no matter how convincing a story one might be able to spin in such 'metaphysical' terms. Greek geometry lives by this principle too. It speaks of nothing it cannot exhibit in the most tangible, concrete form right before our eyes. The definition of $\sqrt{7}$ as a number such that $\sqrt{7}\times\sqrt{7}=7$ could certainly be accused of being 'metaphysical' and therefore anti-scientific in the positivist sense. But once it has been concretely exhibited by a ruler-and-compass construction there is no longer any room for such a critique.

This foundational role of constructions was arguably the key characteristic that separated geometry from other scientific and philosophical theories in Greek times. For instance, the Greeks (following a program laid out by Plato in the fourth century BC) attempted to account for planetary motions using combinations of circles. This science was virtually a branch of mathematics. In particular, it was based on axiomatic-deductive reasoning, with its axioms even being supposedly 'obvious' assumptions such as that heavenly motions must be composed of circles since this is the most 'perfect' shape. But the one fundamental respect in which this science differed from geometry is that it was not constructive. It spoke of preexisting phenomena and tried to fit mathematical constructs to them, unlike geometry which built up all the objects of its theory from scratch using ruler and compass. Much the same can be said of many other branches of ancient science, such as the theory that all bodies are composed of four elements (earth, water, air, fire).

The importance of this distinction became all the more crucial when the scientific theories in question were refuted and abandoned in the sixteenth and seventeenth centuries. By the end of the seventeenth century no one believed anymore in the Aristotelian theory of the elements, or that the earth was in the center of the solar system, or that planetary motion consists of combinations of circular motions, despite these theories having been considered virtually the pinnacle of human understanding for thousands of years. Geometry, however, fared differently; it passed the test of time with flying colours. Not a single theorem of ancient mathematics needed to be revised.

It was only natural, therefore, to seek the distinguishing characteristic that set geometry apart from the other sciences. And thinkers such as Descartes, Leibniz, and Hobbes found the answer in the constructive character of geometry. It is in this light that we must understand for instance Hobbes's otherwise peculiar-sounding claim (in 1656) that political philosophy, rather than physics or astronomy, is the field of knowledge most susceptible to mathematical rigour:

> "Of arts, some are demonstrable, others indemonstrable; and demonstrable are those the construction of the subject whereof is in the power of the artist himself, who, in his demonstration, does no more but deduce the consequences of his own operation. The reason whereof is this, that the science of every subject is derived from a precognition of the causes, generation, and construction of the same; and consequently where the causes are known, there is place for demonstration, but not where the causes are to seek for. Geometry therefore is demonstrable, for the lines and figures from which we reason are drawn and described by ourselves; and civil philosophy is demonstrable, because we make the commonwealth ourselves." [10, pp. 183–184]

As bizarre as this may sound to modern ears, it makes perfect sense when we keep in mind the all-important role of constructions in classical geometry.

## Higher curves in Greek geometry

This strong emphasis on constructions was by no means a seventeenth-century afterthought. Rather, it was a key tenet of the Greek tradition all along. Beyond Euclidean ruler-and-compass geometry, three construction problems dominated in large part the development of Greek geometry: the duplication of the cube (finding a cube with twice the volume of a given cube), the quadrature of the circle (finding a square with area equal to that of a given circle), and the trisection of an angle (dividing an angle into three equal parts). (See for instance [17, IX].) And it is with good reason that these problems were seen as fundamental. They are very pure, prototypical problem — not to say picturesque embodiments — of key concepts of geometry: proportion, area, angle. The doubling of a plane figure, the area of a rectilinear figure, and the bisection of an angle are all fundamental results that the geometer constantly relies upon, and the three classical problems are arguably nothing but the most natural way of pushing the boundaries of these core elements of geometrical knowledge. The great majority of higher curves and constructions studied by the Greeks were pursued solely or largely because one or more of the classical construction problems can be solved with their aid.

For trisecting an angle, one of the Greek methods went as follows (see Figure 6). Consider a horizontal line segment $OA$. Raise the perpendicular above $A$ and let $B$ be any point on this line. We wish to trisect $\angle AOB$. Draw the horizontal through $B$ and find (somehow!) a point $E$ on this line such that when it is connected to $O$, the part $EC$ of it to the right of $AB$ is twice the length of $OB$. I say that $\angle AOC = \frac{1}{3}\angle AOB$, so we have trisected the angle, as desired. This is easy to show by introducing the midpoint $D$ of $EC$, and drawing the horizontal through this point. This line will bisect $BC$, and by comparing angles in the resulting triangles the result follows easily.

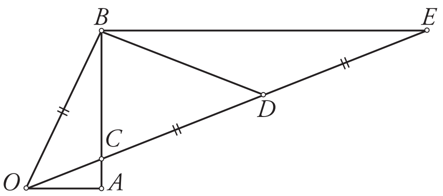


**Figure 6** Trisecting an angle $\angle AOB$ by finding an auxiliary point $E$.

**Figure 7** The defining property of the conchoid of Nicomedes.

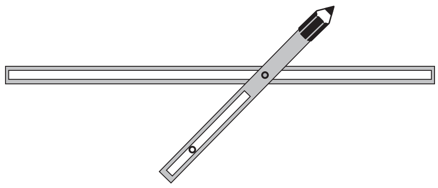

**Figure 8** Instrument for drawing the conchoid of Nicomedes.

But how exactly are we supposed to find the point $E$? This can in fact not be done by ruler and compass only. For this purpose Nicomedes invented a new curve and an instrument for drawing it. The curve is the conchoid of Figure 7, and the instrument must have been something like that shown in Figure 8. The point $E$ in the above construction can be found using this instrument, by setting it up using $O$ as the origin point, $AB$ as the axis, and $2OB$ as the protruding length. The intersection of the resulting conchoid with the horizontal through $B$ determines the desired point $E$.

This construction is strongly analogous to the ruler-and-compass construction of $\sqrt{7}$ above. In both cases, we would perhaps be inclined to think of the sought entity as an unproblematic numerical quantity. But in Greek geometry the task is to exhibit it concretely by carrying out definite construction steps using exact tools.

In terms of practical feasibility, however, the analogy rather breaks down. This in itself is quite illuminating and tells us a lot about Greek geometry. As we saw above, the $\sqrt{7}$ construction is theoretically profound as well as practically precise. So we could not know what relative importance Euclid and others attached to these two factors. But the conchoid construction is nowhere near as accurate in practice and surely not the best way to trisect an angle for any practical purpose. Thus it strongly suggests that the Greeks indeed attached great importance to the more philosophical reasons for insisting on constructions.

If you don't want to take my word for it you can build a conchoid instrument for yourself and use it to trisect an angle. I have done this with my students, and most of us were off by several degrees in our trisections. Fortunately for the construction enthusiasts among us, building Nicomedes's instrument is easier than it looks if you have access to a well-stocked hardware store. I found a tool called a 'templater' consisting of linked rulers (see Figure 9) which served the purpose very well. As a plane of construction I found it useful to use a large sheet of very thick paper. To mark points in a manner that can attach to and support rulers I used flat-headed nails piercing through the paper from underneath.

I would argue that even conic sections were defined in terms of an instrument-construction along the above lines. Certainly the definition in terms of slicing a cone at first seems very different in character from the instrument-based constructions of circles or conchoids. But, in fact, one might argue that it really *is* a constructive definition *precisely* in the mould of Euclid. At


**Figure 9** A hardware-store tool useful for building classical curve-tracing instruments.

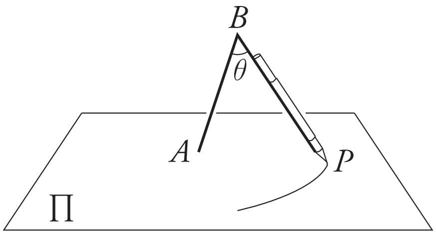


**Figure 10** Generalised compasses for drawing conic sections. The angle $\theta$ and the direction of the axis $AB$ are fixed. As the other leg rotates around the axis, the pen slides up and down in its cylinder, so as to always reach the plane $\Pi$. Figure from [16, p. 29], with altered notation.

least if one thinks of a cone as generated by the rotation of a line about an axis, for then, in a sense, a construction by 'generalised compasses' as in Figure 10 is really nothing but the physical manifestation of the literal meaning of the definition of a conic section. Such generalised compasses were described in the medieval Islamic commentary literature and could very well have been considered quite evident in Greek times. A tip for building these kinds of conic compasses today is to use a laser pointer in place of the pen, which removes the otherwise mechanically quite tricky issue of the pen needing to be able to slide freely up and down.

Very little is known about the early history of conic sections, but arguably the two main facts known about it are: (i) at an early stage conics were used for the duplication of the cube (since this amounts in modern terms to solving $x^3=2$, it can be accomplished by combining the hyperbola $xy=1$ with the parabola $y=x^2/2$) and other problems of this type; (ii) in the earliest records, cones were defined as line segments rotated about an axis and conic sections as the intersection of a cone with a plane *perpendicular* to its side. The perpendicularity restriction in (ii) at first appears very artificial and strange. It makes no sense in terms of the natural applications of conic section theory in astronomical gnomonics and perspective optics, nor does it make any theorems about conics easier to prove. This suggests that the study of conic sections was not originally an end in itself, but only a way of interpreting curves already necessitated elsewhere. The solution of (i) came first, and the notion of a conic section was concocted as a way of explicating the curves involved in this important construction.

In fact, in my view, the reason for the perpendicularity condition in (ii) lies in the conic compass approach of Figure 10. For suppose you want to set up this compass to trace for instance a parabola. How would you go about doing this, in such a way that you knew exactly what parabola you would get? The easiest way is to start with the pen arm $BP$ perpendicular to the ground plane $\Pi$, which corresponds precisely to the perpendicularity condition in (ii).

To explain the complete construction of the parabola, it is convenient for us to use modern algebraic notation and modern coordinates. Of course the ancient Greeks did not have such methods, but everything we shall do was well within their reach by other means. Start with $BP$ perpendicular to $\Pi$, and let this initial position of the pen point be denoted $P'$ to distinguish it from a general point on the traced curve. The next step is to choose the angle $\theta$. To get a parabola this angle needs to be $45^\circ$. If we take $BP'=1$ as our unit length, $AP'$ will be $1$ as well. Let us introduce a coordinate system with $P'=(0,0,0)$ as the origin, $A=(0,1,0)$ as the point determining the direction of the $y$-axis, and $B=(0,0,1)$. As we now let the pen arm $BP$ rotate about the axis $AB$, the pen point $P=(x,y,0)$ traces out a certain curve. We can find the equation for this curve by considering the inner product of $\overrightarrow{BA}=(0,1,0)-(0,0,1)=(0,1,-1)$ with $\overrightarrow{BP}=(x,y,0)-(0,0,1)=(x,y,-1)$. The inner product identity

$$\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3=|\vec{a}\,||\vec{b}\,|\cos\theta$$

in this case becomes

$$y+1=\sqrt{2}\sqrt{x^2+y^2+1}\frac{1}{\sqrt{2}}$$

which reduces to

$$y=\frac{1}{2}x^2.$$

If we want to trace the hyperbola $xy=1$ instead, we can define our coordinate system by $P'=(1,1,0)$, $B=(1,1,1)$, and $A=(2,2,0)$. Then $AP'=\sqrt{2}$ and $AB=\sqrt{3}$, so that $\cos(\theta)=1/\sqrt{3}$. In this case $\overrightarrow{BA}=(2,2,0)-(1,1,1)=(1,1,-1)$ and $\overrightarrow{BP}$ $(x,y,0)-(1,1,1)=(x-1,y-1,-1)$, and the inner product identity is

$$\begin{aligned}&x-1+y-1+1\\&\quad=\sqrt{3}\sqrt{(x-1)^2+(y-1)^2+1}\frac{1}{\sqrt{3}}\end{aligned}$$

which reduces to

$$xy=1.$$

By tracing both of these curves we have solved the problem of the duplication of the cube, for the $x$-coordinate of the intersection of $y=x^2/2$ and $xy=1$ is $\sqrt[3]{2}$, which is the side length needed for the cube of twice the volume of a unit cube.

Of course the Greeks did not have modern algebraic notation, vectors and inner products, so their proofs would have been more laborious. Nevertheless the fact remains that starting with the pen in a perpendicular position is very natural and convenient in this context. Moreover, the above demonstrations are very well suited for modern classroom use, not only to investigate this bit of history but to show the great power of the notion of inner product. In these examples, an otherwise very complicated geometrical problem is reduced to a line or two of simple algebra; it's a genuinely impressive application of the inner product that would go well in any vector geometry class.

In my view it is highly plausible that the Greeks solved the problem of the duplication of the cube by giving a conic-compass construction of precisely the above kind, and that this was how conic sections were first encountered by mathematicians. This is a new hypothesis regarding the origin of the study of conic sections. For previous attempts — less convincing ones, in my opinion — at explaining the origin of the study of conic sections and the perpendicularity condition in particular, see [18, chapter 21] and [14], or, for a brief overview of their views, [1] . For an overview of the early history of conic sections, see [9, I.I].

**Greek tradition in the seventeenth century**

The importance of constructions in the Greek geometrical tradition was still keenly felt in the seventeenth century. In particular, it plays a crucial role in Descartes's *Géométrie* of 1637. In this work Descartes taught the world coordinate geometry and the identification of curves with equations. However, Descartes's take on these topics is radically different from the modern view in numerous respects. In particular, Descartes did *not* argue that the geometry of algebraic curves was a replacement for classical geometry, or a radically new approach to geometry. On the contrary, he argued at great length that it was in fact *subsumed* by classical geometry, and he would never have accepted it if it wasn't.

Descartes, accordingly, began by generalising the curve-tracing procedures of Euclid and then went on to show that the curves that could be generated in this way were precisely the algebraic curves (i.e., curves with polynomial equation of any degree), thereby establishing a pleasing harmony between classical construction-based geometry and the new methods of analytic geometry. Indeed, the historical record shows that Descartes's early geometrical research was devoted to curve-tracing procedures and instruments, and it was in this context that Descartes was gradually led to the idea of analytic and coordinate geometry. (See [5].)

By the time he published his *Géométrie*, Descartes had settled on his favoured curve-tracing method and become convinced that it encompassed all algebraic curves, and nothing else. Convincing his readers of this — and thereby justifying the new algebraic methods in terms of the standards of classical, construction-based geometry — is the dominant theme of the entire *Géométrie*. As he writes:

> "To treat all the curves I mean to introduce here [i.e., all algebraic curves], only one additional assumption [beyond ruler and compasses] is necessary, namely, that two or more lines can be moved, one by the other, determining by their intersection other curves. This seems to me in no way more difficult [than the classical constructions]." [7, p. 43]

An example of this curve-tracing procedure is shown in Figure 11. To find the equation for the curve traced by the intersection $C$, take $A$ as the origin of a coordinate system with $AB = y$ and $BC = x$, and introduce the notation $AK = t$, $LK = c$, $AG = a$ and $m = KL/NL$. Thus $t$ is variable while $c$, $a$ and $m$ are constants. In term of these quantities we can then express the equations of the lines $CNK$ and $GCL$, and then combine them so as to eliminate $t$, which gives the equation for the traced curve in terms of $x$, $y$, and constants.

We may ask ourselves: how can we use this method to generate for example the standard hyperbola $xy = 1$? This comes down to finding a suitable choice of constants in the equation we just derived. In fact, the choices $a = c = 1$ and $m = 0$ will do; this gives $xy = x + 1$ which is the sought curve except for a trivial vertical shift by one unit. Once you know the necessary constants, the instrument in question can quite easily be built using the 'templater' tool of Figure 9. I have done this with my students. Figure 12 shows the result. One sees that a quite small portion of the curve $xy = 1$ has been traced. I did this by placing a pen through the hole at the intersection of those two rulers. One can then move the pen without touching any other part of the setup. Because of the constraints of the rulers, the pen is restricted to move only along the sought curves.

Once a curve has been generated this way it in turn can be taken in place of the initial straight line $KNC$, and so on. An example used by Descartes himself is that replacing the line $KNC$ by a circle produces a conchoid (see Figure 13). Thus we have seen how Descartes's curve-tracing method quickly yields two of the key curves from the Greek tradition that were already crucial in our story above. This is no coincidence. Descartes was intimately familiar with the Greek tradition, and firmly devoted to preserving it. He did so with aplomb, and that was his proudest achievement in mathematics.

It goes without saying that the virtues Descartes saw in his construction procedures were theoretical in nature. His con-

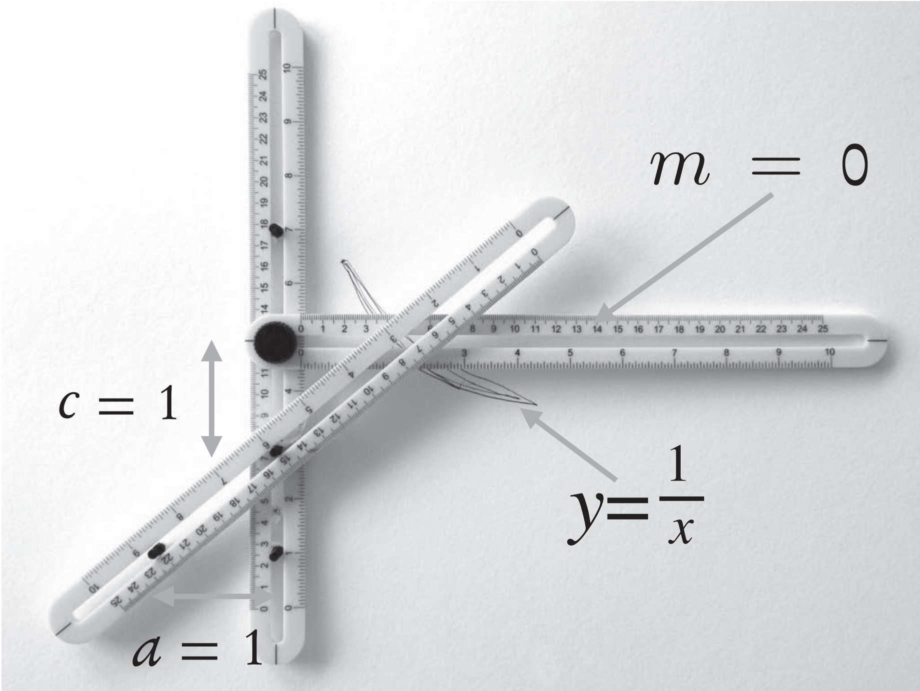


**Figure 12** A practical implementation of Descartes's method of Figure 11 in the case $xy = 1$.

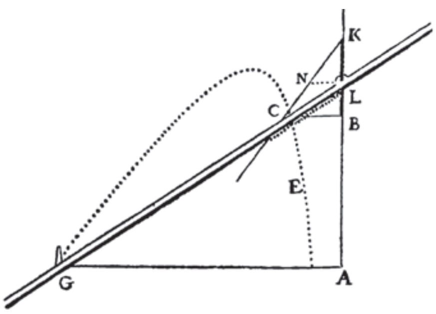


**Figure 11** Descartes's method for tracing a hyperbola. The triangle $KNL$ moves vertically along the axis $ABLK$. Attached to it at $L$ is a ruler, which is also constrained by the peg fixed at $G$. Therefore the ruler makes a mostly rotational motion as the triangle moves upwards. The intersection $C$ of the ruler and the extension of $KN$ defines the traced curve, in this case a hyperbola. (From [6, p. 321].)

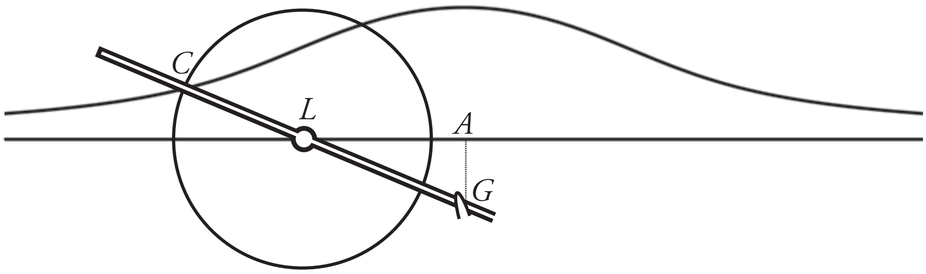


**Figure 13** Construction of the conchoid using Descartes's method of Figure 11 with a circle in place of the line $KNC$.

structions are obviously quite hopeless to apply in practice in any but the very simplest cases. The setup of Figure 12 is already crude to say the least, and it soon gets much worse when curves of higher degree are involved. Thus the following anecdote could very well have much truth in it:

> "[Descartes] was so learned that all learned men made visits to him, and many of them would desire him to show them ... his instruments ... He would drawe out a little drawer under his table, and show them a paire of Compasses with one of the legges broken: and then, for his ruler, he used a sheet of paper folded double." (Aubrey's *Brief Lives*, 1898 ed., vol. 1, p. 222, quoted from [13, p. 42].)

Nevertheless, as we stressed already in the case of Nicomedes's instrument above, even when practical feasibility goes out the window, constructions remain the theoretical cornerstone of mathematics. They are indeed what gives meaning to mathematical concepts. Without them, an equation such as $xy=1$ is nothing but empty, meaningless symbols.

From a modern point of view we might object that the equation $xy=1$, or its equivalent $y=1/x$, has a definite geometrical meaning without the peculiar tracing tools of Descartes. Namely: fill in various $x$-values, compute the corresponding $y$-values, and plot the corresponding points. In this way as many points of the curve as desired can be produced, making its geometrical meaning clear. The problem with this, in seventeenth-century eyes, is that it does not generate the curve as a whole, and therefore it might 'miss' the one point we are looking for. Christiaan Huygens expressed this well. For the present audience I may quote him in the original Dutch:

> "Doch soo en kan men niet seggen dat het beschrijven van een kromme linie door gevonden puncten geometrisch ofte volkomen sij, of dat sulcke beschreven linien konnen dienen tot geometrische constructie van eenighe problemata, dewijl hiertoe, nae mijn opinie, geen kromme linien en konnen dienen als die door eenigh instrument vervolgens beschreven konnen worden, gelijck den Cirkel door een passer; en de Conische Sectien, Conchoides en andere door de instrumenten daertoe geinventeert. Want de linien met de handt van punt tot punt getrocken alleenlijck de gesochte quantiteyt ten naesten bij konnen geven en dienvolgens niet naer de Geometrische perfectie. Want wat helpt het sooveel puncten te vinden als men wil, indien men dat eene punct dat gesocht werdt niet en vindt?" [11]

["One cannot say that the description of a curved line through found points is geometrical, that is to say complete, or that lines so described can serve as a geometrical construction for some problems, because for this, in my opinion, no curved lines can serve except those that can subsequently be described by some instrument, as the circle by a pair of compasses; and the conic sections, conchoids and others by the instruments invented thereto. For the lines drawn by hand from point to point can only give the sought quantity approximately and consequently not according to geometrical perfection. For what does it help to find as many points as one wishes, in case one does not find the one point that is sought?"]

Indeed this is what happens in the example above of using the intersection of $y=x^2/2$ and $xy=1$ to solve the duplication of the cube. We solve the problem by *finding* the $x$-coordinate of the point of intersection. If the curves were defined in terms of plugging in $x$-values this would clearly be circular reasoning.

Finding the equation for the traced curve in Descartes's construction in the above manner is certainly a good exercise in any course on analytic geometry. This is made all the more satisfying if it is followed by the physical tracing of the curve. And it will certainly be very healthy for students to be confronted with the excellent reasons seventeenth-century mathematicians had for preferring such methods of curve construction. Note well that Huygens's critique of pointwise curve constructions applies to the way graphing calculators plot curves: Descartes and Huygens would have been none to impressed by such gadgets as far as exact geometry is concerned; on grounds of theoretical rigour they had good reason to stick with their mechanical instruments instead.

## "A little boat will serve"

Huygens himself continued the construction tradition where Descartes had left off. According to Descartes, his curve-tracing could produce all algebraic curves, and

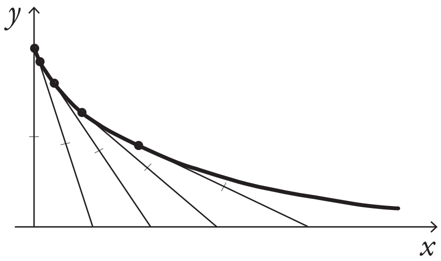


**Figure 14** The tractrix, i.e., the curve traced by a weight dragged along a horizontal surface by a string whose other end moves along a straight line.

nothing but algebraic curves. The next frontier, thus, was non-algebraic curves, i.e., graphs of functions that cannot be expressed by a polynomial equation. The logarithm function is arguably the most fundamental function of this type. So Huygens faced the problem of finding a curve-tracing method, analogous to those above, which could be used to find the logarithm of any number.

Huygens [12] found the answer in the tractrix (Figure 14). In the *physique de salon* of seventeenth-century Paris, a pocket watch on a chain was a popular way for gentlemen to trace this curve, as shown in Figure 15. A pocket watch is quite well-suited for the purpose since it is quite heavy and has a low center of mass, which prevents undue slippage or wobbling. Also, since its back is typically somewhat rounded it only has one point of contact with the table top surface, as the mathematical idealisation requires. By dipping the watch in paint or rubbing it with soot one can ensure that it leaves a trace of its path. This is all very replicable in a modern classroom. Huygens himself investigated such matters in great detail and decided in favour of a more ambitious method: having a small boat trace the tractrix in a tub of syrup (Figure 16).

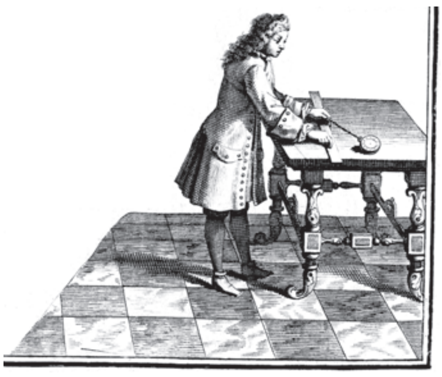

**Figure 15** Tracing the tractrix by means of a pocket watch. From Giovanni Poleni, *Epistolarum mathematicarum fasciculus*, 1729.

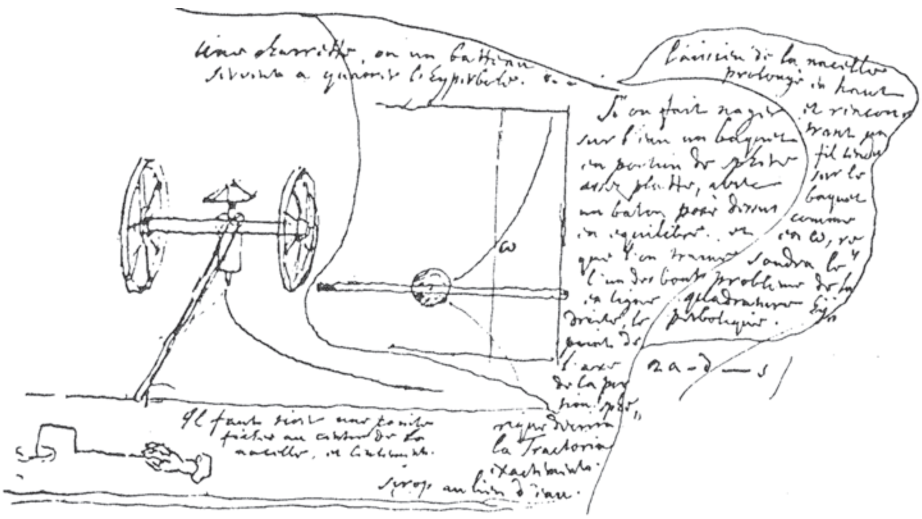

**Figure 16** Detail of a 1692 manuscript by Christiaan Huygens on the tractrix (reproduced from [4, p. 30]). The sentence in the top left corner reads: "Une charette, ou un batteau servira a quarrer l'hyperbole" ("a little cart or boat will serve to square the hyperbola"). "Squaring a hyperbola" means finding the area under a hyperbola such as $y=1/x$, so it is equivalent to computing logarithms, as Huygens was well aware. The bottom line reads: "sirop au lieu d'eau" ("syrup instead of water"). Syrup offers the necessary resistance and a boat leaves a clear trace in it.

The connection between the tractrix and the logarithm may be seen as follows. Let's say that the length of the string is $1$. At any point during the motion, we can consider it as the hypothenuse of a triangle with its other sides parallel to the axes. Thus the sides of this triangle are $1$ for the hypothenuse, $y$ for the height, and $\sqrt{1-y^2}$ for the last side by the Pythagorean Theorem. We can now find a differential equation for the tractrix by equating two different expressions for its slope: first the usual $dy/dx$ and then the slope expressed in terms of the above triangle. This gives

$$dx=-\frac{\sqrt{1-y^2}}{y}dy.$$

To solve this differential equation we make the substitution $u^2=1-y^2$, which gives

$$dx=\frac{u^2}{1-u^2}du.$$

We can split the right-hand-side expression into the partial fractions

$$\frac{u^2}{1-u^2}=\frac{1}{2}\Big(\frac{u}{1-u}-\frac{u}{1+u}\Big),$$

each of which we can integrate by making the substitutions $t_1=1-u$ and $t_2=1+u$ respectively, giving

$$x=\frac{1}{2}(-\log(t_1)+t_1+t_2-\log(t_2))+C.$$

Substituting back and simplifying, we get

$$x=\log\Big(\frac{1+\sqrt{1-y^2}}{y}\Big)-\sqrt{1-y^2}.$$

The desired configuration corresponds to $C=0$ since $x(1)=0$ and $x\to\infty$ as $y\to 0$. By $\log$ we mean the natural logarithm.

This solution shows that the tractrix is related to logarithms. It does not reveal an easy way of finding the logarithm of some given number, but Huygens managed to extract such a recipe. Let's say that we seek $\log(1/Y)$. Huygens's construction goes as follows. Consider first the auxiliary triangle shown in Figure 17, where the length of the leg $a$ is chosen so that the hypothenuse equals this leg plus $Y$. We see that $(a+Y)^2=a^2+1^2$, so $a=\frac{1-Y^2}{2Y}$, which tells us how to find the $a$ (and thus construct the triangle) for the given $Y$. Next Huygens cuts off a portion of length $1$ of the hypothenuse, as in the second part of Figure 17. Here we find that $b^2+y^2=1^2$, so that $b=\sqrt{1-y^2}$, and $\frac{y}{1}=\frac{1}{a+Y}$, so that $y=\frac{2Y}{1+Y^2}$, which when substituted into the above solution formula for the tractrix gives

$$\begin{aligned} x&=\log\Big(\frac{1+\sqrt{1-y^2}}{y}\Big)-\sqrt{1-y^2}\\ &=\log\left(\frac{1+\sqrt{1-\Big(\frac{2Y}{1+Y^2}\Big)^2}}{\frac{2Y}{1+Y^2}}\right)-b\\ &=\log(1/Y)-b. \end{aligned}$$

Thus measuring $x+b$ gives $\log(1/Y)$, so we have found the sought logarithm as a concrete, measurable line segment. This construction fits very well in the long tradition outlined above. As ever, the sought quantity is determined by the intersections of curves generated by continuous motion. Nor was this the end of such attempts. Leibniz for instance devised a generalisation of the tractrix, involving a variable string length, which can be used to solve even more problems in the same spirit (see [2]). My recent dissertation [3] shows that these kinds of constructions were a key part of mainstream mathematics at this time and played an important role in the early development of the infinitesimal calculus.

Interestingly, the arc length of the tractrix from $y=1$ to any point $y=C$ is $\log(C)$. So in fact all the hassle with the auxiliary triangles in Figure 17 could have been avoided and instead the result could have been found by simply putting a measuring tape along the tractrix and directly reading off the answer. But that would have been unacceptable and in violation of tradition. Above, for instance, we found the trisection of an angle in a complicated fashion. If we could simply have put a measuring tape along a circular arc and marking it into three equal pieces the problem would obviously have been trivial; or, perhaps

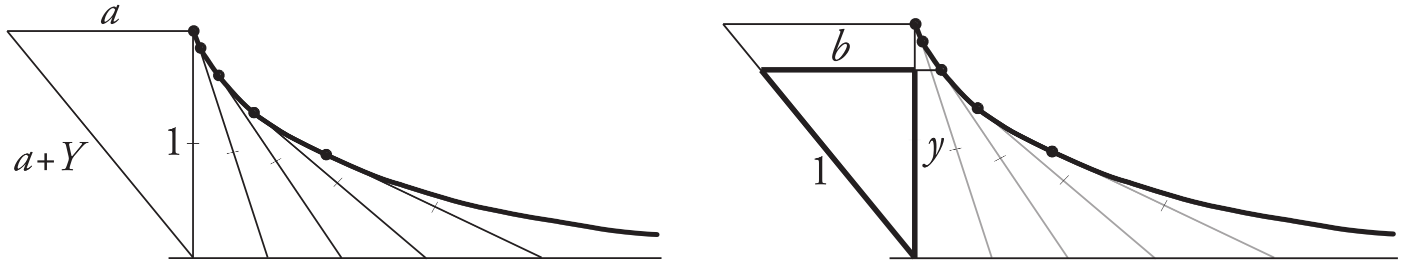


**Figure 17** Huygens's construction of logarithms from a tractrix.

better put, not a problem at all. The same goes for one of the other famous problems of antiquity: the quadrature of the circle. This too becomes elementary if one is allowed to simply measure the circumference of a circle with a measuring tape.

In all of these cases, the measuring tape solution is simple and very accurate for all practical purposes. Yet it was not accepted, and convoluted constructions based on intersections of curves were sought instead. This confirms once again that the construction paradigm was very much a theoretical obsession in the seventeenth century. But why insist on these standards, even theoretically? An official rationale can be given along the lines that tape-measure methods are not constructions in the proper sense; instead of straightforwardly producing what is sought they come dangerously close to simply assuming it to be done. But one wonders if the real reason is not a more opportunistic one: tape-measure methods could not sustain a mathematical research programme. Allowing such methods would solve too much: the quadrature of the circle and the multisection of an angle would collapse into trivialities at once, and where's the fun in that? These problems are supposed to be the great prizes of mathematics, not child's play. This in itself is reason enough to deem such methods to be beyond the rules of the game.

## Conclusion

In conclusion, we have seen that a consistent vision of mathematics as founded on constructions was a major guiding force in the development of mathematics from ancient to early modern times. How dismayed these classical mathematicians would be at the casual neglect of constructions in modern mathematics! Today we are happy to reason about entities such as the square root or logarithm of a given number, the third of a given angle, a cube of a given volume, or the graph of an algebraic function, without first asking ourselves how we could produce these things from first principles on the blank canvas of a Mediterranean beach using nothing but sticks and stones. By ancient standards we live in a state of blissful ignorance. We may yet learn a thing or two from our ancient friends by opening our eyes from this complacent slumber.

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