Two-Fluid Theory for Spin Superfluidity in Magnetic Insulators

B. Flebus,[1] S. A. Bender,[2] Y. Tserkovnyak,[2] and R. A. Duine[1]
[1]*Institute for Theoretical Physics and Center for Extreme Matter and Emergent Phenomena, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*
[2]*Department of Physics and Astronomy, University of California, Los Angeles, California 90095, USA*

(Received 2 November 2015; published 14 March 2016)

We investigate coupled spin and heat transport in easy-plane magnetic insulators. These materials display a continuous phase transition between normal and condensate states that is controlled by an external magnetic field. Using hydrodynamic equations supplemented by Gross-Pitaevski phenomenology and magnetoelectric circuit theory, we derive a two-fluid model to describe the dynamics of thermal and condensed magnons, and the appropriate boundary conditions in a hybrid normal-metal–magnetic-insulator–normal-metal heterostructure. We discuss how the emergent spin superfluidity can be experimentally probed via a spin Seebeck effect measurement.

DOI: 10.1103/PhysRevLett.116.117201

Introduction.—It has been many years since Kapitza first observed that helium, when cooled below a temperature of 2.17 K, displays properties attributable to a new quantum phase of matter [1], such as the ability to flow without dissipation through thin capillaries, the quantization of the vorticity, and a record thermal conductivity. These properties are well understood within the framework of the two-fluid model proposed independently by Tisza [2] and Landau [3], in which He II is described as a mixture of a normal fluid, which is viscous and carries all the entropy of the system, and a superfluid that flows without friction and carries no thermal energy.

Only a few years later, the two-fluid model successfully threw light upon the apparent absence of the usual thermoelectric effects, such as the Seebeck and the Peltier effects, in the superconducting state [4]. Indeed, in superconductors, all the conventional thermoelectric properties vanish due to the coexistence of the thermal quasiparticle current with a dissipationless supercurrent that counterflows with it. The analogy between the supercurrent of electric charge in superconductors and the mass superflow in helium stems from the underlying common origin of these phenomena, i.e., the spontaneous breaking of the $U(1)$ symmetry underlying Bose-Einstein condensation (BEC, of either atoms or Cooper pairs) and the associated macroscopic quantum coherence. Therefore, a superfluid phase can be described by a two-fluid model, in which the condensed and itinerant atoms are, loosely speaking, identified with the superfluid and normal components, respectively. This concept can be extended to a variety of systems exhibiting $U(1)$ symmetry breaking and thus the coexistence of a normal and a Bose-Einstein condensed fluids, such as excitons [5,6], polaritons [7,8], and magnons [9–11].

A growing interest has recently arisen in magnonic systems as promising setups for achieving room-temperature Bose-Einstein condensation, motivated in part by the experimental progress of Demokritov *et al.* [12] on parametrically pumped magnon condensates. More recently, a theoretical proposal for the realization of a BEC of magnons by means of direct spin current injection from an adjacent normal metal with strong spin-orbit coupling was put forward by Bender *et al.* [13]. Unlike BEC of real particles, BEC of quasiparticles and, in particular, quasiequilibrium magnons, does not require low temperatures, since the high densities of magnons needed for the condensate to form can be produced via external pumping or by tuning the magnetic field, which is facilitated by their small effective mass (corresponding to strong exchange). In this Letter, we focus on a ferromagnetic insulator with easy-plane magnetic anisotropy as a simple model system that displays a transition between normal and BEC phases and exhibits superfluid behavior. The magnet is sandwiched between two metallic reservoirs that act like thermal baths, set at two different temperatures, and that may provide spin accumulation via the spin Hall effect (as illustrated in Fig. 1). The temperature difference applied across the ferromagnet induces a spin current into normal metals, which can be measured as an inverse spin Hall voltage and is dubbed the spin Seebeck effect [14]. By sweeping the magnetic field in the z direction, the system can be tuned to a state where the (xy) easy-plane rotational symmetry is spontaneously broken, and which, as a result, supports collective spin currents. We show that the spin Seebeck effect is then diminished, as a result of counterflow between condensate and thermal spin currents. As a practical utility, our results may provide novel routes to control thermal spin currents.

Model and hydrodynamic equations.—We consider the following model Hamiltonian for an easy-plane magnetic insulator subjected to a field B oriented along the z axis:

$$\mathcal{H} = \int d^3r\left(-\frac{A}{2s}\hat{\mathbf{s}}\cdot\nabla^2\hat{\mathbf{s}} + B\hat{s}_z + \frac{K}{2s}\hat{s}_z^2\right), \qquad (1)$$

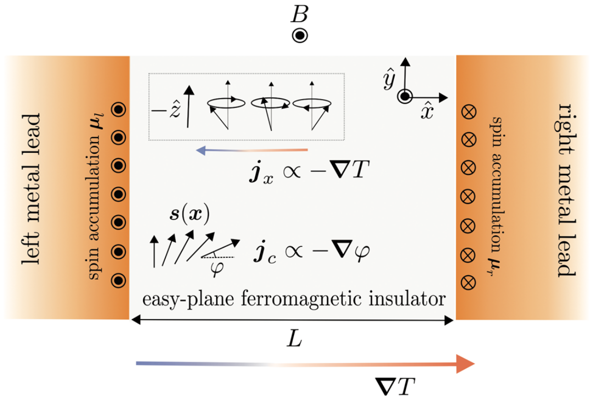

FIG. 1. Normal-metal–easy-plane insulator–normal-metal hybrid heterostructure. The state of the equilibrium magnetization, which is determined by the interplay between the magnetic field B and the anisotropy energy K, can be perturbed by magnon transport driven by temperature gradient $\boldsymbol{\nabla}T$ and spin accumulations $\boldsymbol{\mu}_{l,r} = \mu_{l,r}\hat{\mathbf{z}}$ sustained by the metal leads. At low magnetic fields, the spin Seebeck current (polarized along the z axis) $\mathbf{j}_x$ induced by the temperature gradient $\boldsymbol{\nabla}T$ coexists with a superfluid spin counterflow $\mathbf{j}_c$, as discussed in the text.

where $\hat{\mathbf{s}}$ is the spin density operator (in units of $\hbar$), A the exchange stiffness, $K > 0$ the constant governing the strength of the local easy-plane anisotropy, and s the saturation spin density. Performing the Holstein-Primakoff transformation [15], $\hat{s}_z = \hat{\Phi}^\dagger\hat{\Phi} - s$ and $\hat{s}_- = \sqrt{2s - \hat{\Phi}^\dagger\hat{\Phi}}\hat{\Phi}$, it is straightforward to recast the Heisenberg dynamics of $\hat{\mathbf{s}}$ as a superfluid coupled to a normal cloud (see, e.g., Ref. [16]). By, furthermore, including phenomenologically the Gilbert damping constant α, the corresponding Gross-Pitaevksi equation (following the Popov approximation [17]) reads as

$$(i - \alpha)\hbar\partial_t\Phi = (\hbar\Omega + Kn_c/s - iR)\Phi - A\nabla^2\Phi. \quad (2)$$

Here, $\Phi \equiv \langle\hat{\Phi}\rangle = \sqrt{n_c}e^{-i\varphi}$ is the superfluid order parameter, with φ being the precessional angle of the magnetization density in the xy plane and n_c (n_x) condensed (normal) magnon density. In particular, $s_z = n_c + n_x - s$. We are assuming small deviations from the ground state (in the absence of anisotropy), so that $n_c + n_x \ll s$, throughout. $\hbar\Omega \equiv B - K(1 - 2n_x/s)$ is the normal-phase magnon gap, and the collisional term R describes the coupling to the finite-temperature normal cloud [18], which is defined by $\hat{\phi} \equiv \hat{\Phi} - \Phi$, with $\langle\hat{\phi}^\dagger\hat{\phi}\rangle$ being the normal cloud density n_x. At zero temperature (and thus $R \to 0$), Eq. (2) recasts the Landau-Lifshitz-Gilbert equation [19] for small-angle dynamics of the spin density around the $-\mathbf{z}$ direction (see Fig. 2). It is, furthermore, illuminating to rewrite Eq. (2) as the superfluid hydrodynamic equations

$$\dot{n}_c + \boldsymbol{\nabla}\cdot\mathbf{j}_c = -\Gamma_{cx} - 2\alpha\omega n_c, \quad (3a)$$

$$\hbar(\omega - \Omega) - K\frac{n_c}{s} = A\left[(\boldsymbol{\nabla}\varphi)^2 - \frac{\nabla^2\sqrt{n_c}}{\sqrt{n_c}}\right], \quad (3b)$$

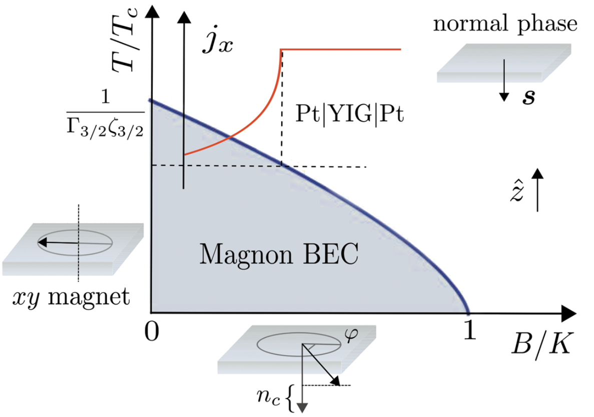

FIG. 2. Equilibrium phase diagram. The condensate phase boundary is at $T/T_c = (1 - B/K)^{2/3}/\Gamma_{3/2}\zeta_{3/2}$, where $T_c \equiv As^{2/3}$ estimates the Curie temperature. In the normal phase, the net spin density $\mathbf{s}$ is oriented along the (negative) z axis; the condensate spontaneously breaks U(1) symmetry around the z axis, as manifested by a static canting of the magnetization, whose deviation from its normal-state equilibrium value along the z axis is parametrized by the condensate density n_c. In the absence of an applied field B, the ferromagnet is a planar xy magnet. The reduction of the spin Seebeck current j_x (red curve) as the magnetic field B decreases below the transition point, at a fixed T, is a direct and observable signature of superfluidity.

where $\omega = \dot{\varphi}$ is the condensate frequency and $\mathbf{j}_c = n_c\mathbf{v}_c$ the condensate spin current (polarized out of the easy plane, i.e., in the z direction), where $\mathbf{v}_c = -\hbar\boldsymbol{\nabla}\varphi/m$ and $m \equiv \hbar^2/2A$ is the kinetic magnon mass. $\Gamma_{cx} = 2n_cR/\hbar$ is the collision term describing equilibration between the condensate and the thermal cloud, defined as $\Gamma_{cx} = 2\eta(\omega - \mu/\hbar)n_c$ [17], with η parametrizing the rate of the thermal cloud-condensate scattering [20]. Chemical potential μ and temperature T parametrize the Bose-Einstein distribution of the thermal cloud.

The equilibrium phase diagram of the easy-plane condensate is shown in Fig. 2, which is obtained by a mean-field self-consistency analysis for $n_c \geq 0$ coupled to the thermal cloud [20]. In the following, we will be interested in the linear response of magnons to a temperature gradient. Linearizing with respect to small nonequilibrium variables —ω, $\mathbf{v}_c$, and $\delta n_c \equiv n_c - n_c^{(0)}$ for the condensate and μ and $\delta T \equiv T - T^{(0)}$ for the cloud—Eqs. (3) become

$$\delta\dot{n}_c + n_c\boldsymbol{\nabla}\cdot\mathbf{v}_c = 2\eta(\mu/\hbar - \omega)n_c - 2\alpha\omega n_c, \quad (4a)$$

$$\hbar\omega = K\frac{\delta n_c + 2\delta n_x}{s} - A\frac{\nabla^2\delta n_c}{2n_c}. \quad (4b)$$

Here, $\delta n_x \equiv n_x - n_x^{(0)}$ can be expanded in terms of μ and δT (disregarding its subleading dependence on δn_c). The superscript (0), which was dropped in Eqs. (4) without danger of ambiguity, denotes the corresponding equilibrium values in the absence of the thermal flux.

The above condensate equations are complemented by hydrodynamic equations for the thermal cloud, which can be easily constructed within the Boltzmann transport theory [20]:

$$\delta\dot{n}_x + \nabla\cdot\mathbf{j}_x = 2\eta(\omega - \mu/\hbar)n_c - g_{n\mu}\mu - g_{nT}(T - T_p), \tag{5a}$$

$$\delta\dot{u} + \nabla\cdot\mathbf{j}_q = -g_{uT}(T - T_p) - g_{u\mu}\mu. \tag{5b}$$

Here, u is the energy density of the thermal cloud, T_p is the phonon temperature, and the g coefficients parametrize relaxation of magnons by the (phononic) environment. [Note that a contribution to the energy rate equation (5b) from the condensate-cloud scattering is missing as it is quadratic in the nonequilibrium bias: $\delta\dot{u}|_{cx} \propto \hbar\omega(\hbar\omega - \mu)$.] The linear response spin $\mathbf{j}_x$ and heat $\mathbf{j}_q$, current densities, furthermore, can be expanded as

$$\mathbf{j}_x = -\sigma\nabla\mu - \varsigma\nabla T, \qquad \mathbf{j}_q = -\kappa\nabla T - \rho\nabla\mu, \tag{6}$$

where σ, κ, ς, and ρ are, respectively, the bulk spin and heat conductivities and the intrinsic spin Seebeck and Peltier coefficients.

Boundary conditions.—The spin and heat flow across the sample must be determined consistently with the boundary conditions defined at the $F|N$ interfaces at $x = 0, L$. Accounting for interfacial static spin-transfer and spin-pumping torques, the linearized z component of the condensate spin current density injected from the left reservoir with a nonequilibrium spin accumulation $\boldsymbol{\mu}_l = \mu_l\mathbf{z}$ is given by [25]

$$j_c|_{x=0} = n_c g_l^{\uparrow\downarrow}(\mu_l - \hbar\omega)/2\pi\hbar s, \tag{7}$$

where $g_l^{\uparrow\downarrow}$ is the real part of the (dimensionless) spin mixing conductance (per unit area). The thermal spin and heat currents flowing across the left interface are given by

$$j_x|_{x=0} = G(\mu_l - \mu)|_{x=0} + S(T_l - T)|_{x=0}, \tag{8a}$$

$$j_q|_{x=0} = K(T_l - T)|_{x=0} + \Pi(\mu_l - \mu)|_{x=0}. \tag{8b}$$

Here, T_l is the electron temperature and G, K, S, and Π are the interfacial magnon spin and thermal conductances and spin Seebeck and Peltier coefficients, respectively.

The boundary conditions, Eqs. (7) and (8) along with the analogous expressions for the right interface, together with the two-fluid hydrodynamic relations, Eqs. (4) and (5), constitute a complete set of linearized equations from which we can yield solutions for all the dynamical variables. We will now solve this problem in a steady state (i.e., $\delta\dot{n}_c = \delta\dot{n}_x = \delta\dot{u} = 0$ and $\omega = \text{const}$), when the normal-metal reservoirs are thermally biased: $T_l = T - \Delta T/2$ and $T_r = T + \Delta T/2$. We will suppose, for simplicity, that the phononic heat transport and thermal profile are only weakly disturbed by the magnons, so that $T_p = T + \Delta T(x/L - 1/2)$, where we, furthermore, neglected interfacial Kapitza resistances.

Results.—Let us investigate the flow of magnonic spin and heat across a mirror-symmetric $N|F|N$ structure driven by a small temperature bias ΔT. We will consider two limiting cases: the magnet is sandwiched (i) between two heavy metals acting as good spin sinks (as may be exemplified by Pt|YIG|Pt), in which case $\mu_{l,r} = 0$, or (ii) between two light metals being perfectly poor spin sinks (possibly approximated by Cu|YIG|Cu), in which case spin accumulations build in each lead to block the total spin current across the interfaces, $j_c + j_x \to 0$ at $x \to 0, L$.

Since the spin-preserving relaxation of magnon distribution towards the phonon temperature, as parametrized by g_{uT} in Eq. (5b), does not rely on relativistic spin-orbit interactions, we may expect it to be an efficient process at high temperatures (stemming, e.g., from the modulation of exchange coupling by lattice vibrations). The corresponding length scale, which is governed by the inelastic magnon-phonon scattering, $\lambda_u \equiv \sqrt{\kappa/g_{uT}}$, can therefore be taken to be shorter than other relevant length scales, which are associated with relativistic physics (i.e., λ_n and λ_{cx} defined below). In this regime, we can set $T \to T_p$, which decouples the spin transport from heat dynamics, resulting in the steady state, in the following diffusion equation for magnons:

$$\partial_x^2\mu - (\mu - \hbar\omega)/\lambda_{cx}^2 - \mu/\lambda_n^2 = 0, \tag{9}$$

which is solved by

$$\mu = (\lambda_m/\lambda_{cx})^2\hbar\omega + c_l e^{-x/\lambda_m} + c_r e^{(x-L)/\lambda_m}. \tag{10}$$

Here, $\lambda_m^{-2} \equiv \lambda_n^{-2} + \lambda_{cx}^{-2}$, $\lambda_n \equiv \sqrt{\sigma/g_{n\mu}}$ is the thermal magnon diffusion length, and $\lambda_{cx} \equiv \sqrt{\hbar\sigma/2\eta n_c}$ is the condensate-cloud equilibration length (where n_c is the condensate equilibrium density according to the phase diagram in Fig. 2). The boundary conditions are given by

$$j_x(0) = G_* c_l - \varsigma\Delta T/L = G[\mu_l - \mu(0)], \tag{11a}$$

$$j_x(L) = -G_* c_r - \varsigma\Delta T/L = G[\mu(L) - \mu_r], \tag{11b}$$

for the cloud (supposing $L \gg \lambda_m$), where $\mu(0, L) = (\lambda_m/\lambda_{cx})^2\hbar\omega + c_{l,r}$, $G_* \equiv \sigma/\lambda_m$, and

$$v_c(0) = g^{\uparrow\downarrow}(\mu_l - \hbar\omega)/2\pi\hbar s, \tag{12a}$$

$$v_c(L) = g^{\uparrow\downarrow}(\hbar\omega - \mu_r)/2\pi\hbar s, \tag{12b}$$

for the condensate. The reservoir spin accumulations are $\mu_l = \mu_r = 0$ in the good spin sink case and are found according to $n_c v_c + j_x = 0$ (at both interfaces) for the poor spin sinks. Integrating the steady-state version of Eq. (4a),

$$\partial_x v_c = 2\eta(\mu/\hbar - \omega) - 2\alpha\omega, \tag{13}$$

we get for $\Delta v_c \equiv v_c(L) - v_c(0)$,

$$\Delta v_c = \frac{2\eta\lambda_m(c_l + c_r)}{\hbar} - [2\alpha + 2\eta(\lambda_m/\lambda_n)^2]\omega L. \tag{14}$$

In the simpler, good spin sink case (where the spin Seebeck physics is manifested through the total spin currents injected into the metal reservoirs), we thus have 5 linear equations, (11), (12), and (14), for 5 unknowns: $c_{l,r}$, $\hbar\omega$, and v_c at $x = 0, L$. For poor spin sinks (where the spin Seebeck physics is manifested through the spin accumulations induced in the metal reservoirs), we have two additional unknowns, $\mu_{l,r}$, and two more equations (for the vanishing total spin current at the interfaces). Note that the differential equation (4b) for δn_c decouples in the linearized treatment. Adding and subtracting Eqs. (11), and substituting the difference of Eqs. (12) into Eq. (14) leads to

$$\begin{aligned}(G+G_*)c_- - \varsigma\Delta T/L - G\mu_- &= 0,\\ (G+G_*)c_+ + G(\lambda_m/\lambda_{cx})^2\hbar\omega - G\mu_+ &= 0,\\ \frac{2\eta\lambda_m c_+}{\hbar} - \left[\alpha + \eta\left(\frac{\lambda_m}{\lambda_n}\right)^2 + \frac{g^{\uparrow\downarrow}}{2\pi s L}\left(1 - \frac{\mu_+}{\hbar\omega}\right)\right]\omega L &= 0,\end{aligned} \tag{15}$$

where $c_\pm \equiv (c_l \pm c_r)/2$ and $\mu_\pm \equiv (\mu_l \pm \mu_r)/2$.

In the good spin sink case, $\mu_\pm = 0$, the last two equations above lead immediately to $\omega = 0$ and $c_+ = 0$. The remaining equation gives

$$c_l = \frac{\varsigma\Delta T/L}{G + G_*} = -c_r. \tag{16}$$

The spin currents at the two interfaces (which turn out to be purely thermal and equivalent) are thus given by

$$j_x = -\frac{\varsigma\Delta T/L}{1 + G_*/G}, \tag{17}$$

and vanish when either $\lambda_{cx} \to 0$ (strong condensate-cloud interaction regime, where $\lambda_m \to \lambda_{cx}$) or $\lambda_n \to 0$ (strong magnon damping regime, where $\lambda_m \to \lambda_n$), since $G_* \propto 1/\lambda_m \to \infty$. As, by decreasing field B, we go deeper into the condensate phase at a fixed T, and n_c is monotonically increasing, λ_{cx} decreases and thus the magnitude of j_x is reduced (see Fig. 2, where we took into account the dependence of λ_m on B but ignored the dependence of other quantities on B, which is valid as long as $T \gg K$ [20]). j_x is largest at the transition point to the normal state and is given by Eq. (17) with $\lambda_m \to \lambda_n$. Note that although the superfluid velocity v_c vanishes at both interfaces, it is nonzero inside the ferromagnet (at distances beyond λ_m from the interfaces), according to Eq. (13),

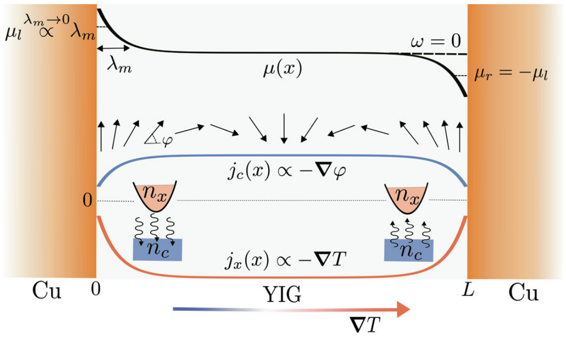

FIG. 3. In the presence of a temperature gradient ΔT, the magnon chemical potential $\mu(x)$ deviates near the interfaces from its zero bulk value in the ferromagnet (YIG). This is accompanied by the electronic spin accumulation buildup in adjacent metals (Cu, treated as a poor spin sink). The spin accumulation μ_l at the left interface exerts a torque on the magnetic order parameter, twisting it in the opposite direction with respect to the one induced by $\mu_r = -\mu_l$ at the right interface. In the mirror-symmetric case, the precession frequency ω vanishes. The condensate j_c and thermal j_x contributions to the spin currents are plotted for $\lambda_n = \lambda_{cx}$.

$$v_c = \frac{2\eta\lambda_m c_l}{\hbar} = \frac{2\eta\lambda_m}{\hbar}\frac{\varsigma\Delta T/L}{G + G_*}. \tag{18}$$

Already in this simple case we encounter the *conveyor-belt* physics, as the superfluid spin current $n_c v_c$ in the bulk counteracts the diffusive thermal flux $-\varsigma\Delta T/L$ and reduces the net spin Seebeck effect as measured at interfaces.

In the opposite limit of the poor spin sinks, we still find $\omega = 0$ and $c_+ = 0$, so that $\mu_+ = 0$, while

$$\mu_l = \frac{\varsigma\Delta T/L}{G_* + (1 + G_*/G)g^{\uparrow\downarrow}n_c/2\pi\hbar s} = -\mu_r. \tag{19}$$

This spin accumulation vanishes when either $\lambda_{cx} \to 0$ or $\lambda_n \to 0$ and decreases with decreasing field B, displaying an analogous behavior to the one of the spin current at the interfaces in the good spin sink case (see Fig. 2). While the total current now vanishes at the interfaces, j_x and v_c are both nonzero in the ferromagnet (see Fig. 3).

Discussion and conclusions.—In this work, we constructed a hydrodynamic theory which describes the interactions between thermal and condensed magnons in an easy-plane magnetic insulator in the presence of a thermal gradient. We predicted that spin superfluidity can be induced by sweeping the external magnetic field and experimentally probed via the spin Seebeck effect. We estimate that for YIG this drop should be observable [20]. Although we have explicitly considered a ferromagnetic insulator, we anticipate, according to Refs. [26] and [27], qualitatively similar behavior also for antiferromagnets. Future works should more systematically address the magnon-phonon relaxation mechanisms and study the role of magnons in the net heat transport. Nonlinear response, in the contexts of dynamic instabilities [16] and pinning by parasitic in-plane anisotropies [28], and higher-order manifestations of the microscopic irreversibility [29] of the coupled spin and heat transport will be addressed elsewhere.

The authors thank R. E. Troncoso and S. Takei for helpful discussions. This work is supported by the Stichting voor Fundamenteel Onderzoek der Materie (FOM) and is part of the D-ITP consortium, a program of the Netherlands Organisation for Scientific Research (NWO) that is funded by the Dutch Ministry of Education, Culture and Science (OCW), and in part by the NSF-funded MRSEC under Grant No. DMR-1420451 and US DOE-BES under Award No. DE-SC0012190.

[1] P. Kapitza, Nature (London) **141**, 74 (1938).
[2] L. Tisza, Nature (London) **141**, 913 (1938).
[3] L. D. Landau, Phys. Rev. **60**, 356 (1941).
[4] V. L. Ginzburg, Zh. Eksp. Teor. Fiz. **14**, 134 (1944).
[5] T. Fukuzawa, E. E. Mendez, and J. M. Hong, Phys. Rev. Lett. **64**, 3066 (1990).
[6] J. P. Eisenstein and A. H. MacDonald, Nature (London) **432**, 691 (2004).
[7] Y. Yamamoto, Nature (London) **405**, 629 (2000).
[8] J. Kasprzak, M. Richard, S. Kundermann, A. Baas, P. Jeambrun, J. M. J. Keeling, F. M. Marchetti, M. H. Szymanska, R. Andre, J. L. Staehli, V. Savona, P. B. Littlewood, B. Deveaud, and Le Si Dang, Nature (London) **443**, 409 (2006).
[9] A. Oosawa, M. Ishii, and H. Tanaka, J. Phys. Condens. Matter **11**, 265 (1999).
[10] T. Nikuni, M. Oshikawa, A. Oosawa, and H. Tanaka, Phys. Rev. Lett. **84**, 5868 (2000).
[11] T. Giamarchi, C. Rüegg, and Tchernyshyvov, Nat. Phys. **4**, 198 (2008), and references therein.
[12] S. O. Demokritov, V. E. Demidov, O. Dzyapko, G. A. Melkov, A. A. Serga, B. Hillebrands, and A. N. Slavin, Nature (London) **443**, 430 (2006).
[13] S. A. Bender, R. A. Duine, and Y. Tserkovnyak, Phys. Rev. Lett. **108**, 246601 (2012).
[14] K. Uchida, H. Adachi, T. Ota, H. Nakayama, S. Maekawa, and E. Saitoh, Appl. Phys. Lett. **97**, 172505 (2010).
[15] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).
[16] S. A. Bender, R. A. Duine, A. Brataas, and Y. Tserkovnyak, Phys. Rev. B **90**, 094409 (2014).
[17] A. Griffin, T. Nikuni, and E. Zaremba, *Bose-Condensed Gases at Finite Temperatures* (Cambridge University Press, Cambridge, England, 2009).
[18] H. T. C. Stoof, J. Low Temp. Phys. **114**, 11 (1999).
[19] L. D. Landau and E. M. Lifshitz, *Statistical Physics*, Part 2, 3rd ed., Course of Theoretical Physics (Pergamon, Oxford, 1980), Vol. 9; T. L. Gilbert, IEEE Trans. Magn. **40**, 3443 (2004).
[20] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.116.117201 for the discussion of the equilibrium phase diagram of the magnet, a kinetic theory for the thermal magnon transport, and quantitative estimates of the superfluid features in the spin Seebeck effect in YIG-based heterostructures, which includes Refs. [21-24].
[21] S. A. Bender and Y. Tserkovnyak, Phys. Rev. B **93**, 064418 (2016).
[22] S. A. Bender and Y. Tserkovnyak, Phys. Rev. B **91**, 140402 (2015).
[23] C. Burrowes, B. Heinrich, B. Kardasz, E. A. Montoya, E. Girt, Y. Sun, Y.-Y. Song, and M. Wu, Appl. Phys. Lett. **100**, 092403 (2012).
[24] L. J. Cornelissen, J. Liu, R. A. Duine, J. Ben Youssef, and B. J. van Wees, Nat. Phys. **11**, 1022 (2015).
[25] Y. Tserkovnyak, A. Brataas, G. E. W. Bauer, and B. I. Halperin, Rev. Mod. Phys. **77**, 1375 (2005).
[26] B. I. Halperin and P. C. Hohenberg, Phys. Rev. **188**, 898 (1969).
[27] S. Takei, B. I. Halperin, A. Yacoby, and Y. Tserkovnyak, Phys. Rev. B **90**, 094408 (2014).
[28] E. B. Sonin, Adv. Phys. **59**, 181 (2010).
[29] C. Wang and D. E. Feldman, Phys. Rev. B **92**, 064406 (2015).