

Monodromy of A -hypergeometric functions

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Abstract. Using Mellin–Barnes integrals we give a method to compute elements of the monodromy group of an A -hypergeometric system of differential equations. The method works under the assumption that the A -hypergeometric system has a basis of solutions consisting of Mellin–Barnes integrals. Hopefully these elements generate the full monodromy group, but this has only been verified in some special cases.

1. Introduction

At the end of the 1980s, Gel'fand, Kapranov and Zelevinsky (GKZ) defined in [11, 14, 15] a general class of hypergeometric functions, encompassing the classical one-variable hypergeometric functions, the multi-variable Appell and Lauricella functions and Horn's functions. They are called A -hypergeometric functions and they provide a beautiful and elegant basis of a theory of hypergeometric functions in several variables. For an introduction to the subject we refer the reader to [7, 34] or the book [30] by Saito, Sturmfels and Takayama. We briefly recall some main facts. Let $A \subset \mathbb{Z}^r$ be a finite set such that

- (i) the \mathbb{Z} -span of A is \mathbb{Z}^r ,
- (ii) there exists a linear form h such that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in A$.

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}^r$. Denote $A = \{\mathbf{a}_1, \dots, \mathbf{a}_N\}$ (with $N > r$). Writing the vectors \mathbf{a}_i in column form we get the so-called A -matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & & & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rN} \end{pmatrix}.$$

We introduce the complex variables v_1, \dots, v_N . For $i = 1, 2, \dots, r$ consider the first order differential operators

$$Z_i = a_{i1}v_1\partial_1 + a_{i2}v_2\partial_2 + \cdots + a_{iN}v_N\partial_N$$

where $\partial_j = \frac{\partial}{\partial v_j}$ for all j .

Let

$$L = \{(l_1, \dots, l_N) \in \mathbb{Z}^N : l_1\mathbf{a}_1 + l_2\mathbf{a}_2 + \cdots + l_N\mathbf{a}_N = \mathbf{0}\}$$

be the lattice of integer relations between the elements of A . For every $\mathbf{l} \in L$ we define the

so-called box-operator

$$\square_{\mathbf{l}} = \prod_{l_i > 0} \partial_i^{l_i} - \prod_{l_i < 0} \partial_i^{-l_i}.$$

The system of differential equations

$$\begin{aligned} (Z_i - \alpha_i)\Phi &= 0, \quad i = 1, \dots, r, \\ \square_{\mathbf{l}}\Phi &= 0, \quad \mathbf{l} \in L, \end{aligned}$$

is known as the system of *A -hypergeometric differential equations* and we denote it by $H_A(\boldsymbol{\alpha})$. It turns out that the solution space of $H_A(\boldsymbol{\alpha})$ is finite dimensional and in general the dimension is equal to the volume of the convex hull $Q(A)$ of A . In order to be more precise we have to introduce $C(A)$, the cone generated by the $\mathbb{R}_{\geq 0}$ -linear combinations of $\mathbf{a}_1, \dots, \mathbf{a}_N$.

Definition 1.1. We say that an A -hypergeometric system is *non-resonant* if the boundary of $C(A)$ has empty intersection with the shifted lattice $\boldsymbol{\alpha} + \mathbb{Z}^r$.

We have the following theorem.

Theorem 1.2 (GKZ, Adolphson). *The solution space of $H_A(\boldsymbol{\alpha})$ is finite dimensional. If, in addition the system is non-resonant, then the rank of $H_A(\boldsymbol{\alpha})$ equals the volume of the convex hull $Q(A)$ of the points of A . The volume is normalized so that a minimal $(r-1)$ -simplex with integer vertices in $h(\mathbf{x}) = 1$ has volume 1.*

Theorem 1.2 is proven in [15] (corrected in [13]) and [1, Corollary 5.20]).

The condition of the existence of a form h such that $h(\mathbf{a}_i) = 1$ for all i amounts to regularity of $H_A(\boldsymbol{\alpha})$, classically known as non-confluence. The condition of non-resonance ensures that our system is irreducible. This means that the corresponding D-module is irreducible.

Theorem 1.3 (GKZ). *Suppose the system $H_A(\boldsymbol{\alpha})$ is non-resonant. Then the system is irreducible.*

For a proof, see for example [12, Theorem 2.11], [32] or [6] for a slightly more elementary proof. Non-resonance also implies that A -hypergeometric systems whose parameter vectors are the same modulo \mathbb{Z}^r have isomorphic monodromy, see [6, Theorem 2.1], which is actually a theorem due to B. Dwork.

Among the many papers written on A -hypergeometric equations there are very few papers dealing with the monodromy group of these systems in general. We give a brief overview of several results in special cases, where the parameters are assumed generic. In the case of one-variable hypergeometric functions there is the paper [8] by Beukers and Heckman, which give a characterisation of monodromy groups as complex reflection groups. There is also a classical method to compute monodromy with respect to an explicit basis of functions using so-called Mellin–Barnes integrals, see Smith [33]. In the special case of the fourth order equation with symplectic monodromy there is a detailed calculation in [9]. A recent paper by Mimachi [26] uses computation of twisted cycle intersection.

For the two-variable Appell system F_1 and Lauricella's F_D , monodromy follows from the work by Picard [29], Terada [36] and Deligne–Mostow [10]. In Sasaki's paper [31] we

find explicit monodromy generators for Appell F_1 . They all use the fact that Lauricella functions of F_D -type can be written as one-dimensional twisted period integrals and monodromy is a representation of the pure braid group on $n + 1$ strands (where n is the number of variables).

The two-variable Appell F_2 has been considered explicitly by Kato [20]. The Appell F_3 system has the same A -set as Appell F_2 , and therefore gives nothing new. Finally, the Appell system F_4 has been considered completely explicitly by Takano [35] and later Kaneko [19]. In Haraoka–Ueno [18] we find some rigidity considerations on the monodromy of F_4 . In the paper [25] by Matsumoto and Yoshida, the authors provide generators for the monodromy of Lauricella F_A (and hence also F_B). Very recently there appeared the preprint [16] on the preprint archive where generators for the monodromy of Lauricella F_C are computed.

Finally, the complete monodromy of the Aomoto–Gel’fand system $E(3, 6)$ has been determined by Matsumoto, Sasaki, Takayama and Yoshida in [23] and further properties in [24]. See also Yoshida’s book [37].

In essentially all of the above studies the monodromy is computed by studying the behaviour of Euler integrals for hypergeometric functions under analytic continuation and corresponding deformation of the contours of integration. For this, knowledge of the fundamental group of the complement of the singular locus of the system of equations is required. It is the purpose of the present paper to avoid these geometric difficulties as long as possible and compute monodromy groups of A -hypergeometric systems by methods which are combinatorial in nature. We do this by starting with local monodromy groups which arise from series expansions of solutions of $H_A(\alpha)$. It is well known that such local expansions correspond to regular triangulations of A . This is a discovery by Gel’fand, Kapranov and Zelevinsky that we shall explain in Section 2. The local monodromy groups have to be glued together to build a global monodromy group. This glue is provided by multidimensional Mellin–Barnes integrals as defined in Section 3. Such integrals were defined around 1908 by Barnes with the exact purpose to study monodromy. In her PhD thesis from 2009, Lisa Nilsson [27] introduced (non-confluent) A -hypergeometric functions in terms of Mellin–Barnes integrals and initiated their study. The shape of these integrals is different from what is commonly used, since they do not give single hypergeometric functions, but combinations of these. It is these integrals we use in this paper.

Unfortunately, Mellin–Barnes integrals do not always provide a basis of solutions. But if they do (Assumption 4.1), the construction of the global group generated by the local contributions is completely combinatorial.

Let us make this more precise. First of all we restrict to totally non-resonant systems.

Definition 1.4. The system $H_A(\alpha)$ is *totally non-resonant* if the shifted lattice $\alpha + \mathbb{Z}^r$ has empty intersection with any hyperplane spanned by $r - 1$ independent elements of A .

Note that this is stronger than just non-resonance where only the faces of the cone spanned by the elements of A are involved. Total non-resonance implies T -non-resonance for every triangulation T in the terminology of Gel’fand, Kapranov and Zelevinsky. In particular this ensures that the local series solution expansions will not contain logarithms. So local monodromy representations act by characters. We prefer to leave the case of logarithmic local solutions for a later occasion. The main result of this paper is the following.

Theorem 1.5. *Let $H_A(\alpha)$ be a totally non-resonant system of rank D . Choose a parameter vector $\delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}^N$ such that $\delta_1 \mathbf{a}_1 + \dots + \delta_N \mathbf{a}_N = \alpha$. Let $d = N - r$.*

Suppose that around some point $\mathbf{v} \in (\mathbb{C}^\times)^N$ we have a Mellin–Barnes basis M_1, \dots, M_D of solutions (Assumption 4.1). Then, for every regular triangulation of A , the algorithm described in Section 5 produces d matrices of size $D \times D$, which are elements of the monodromy group with respect to the basis of functions $\mathbf{v}^{-\delta} M_1, \dots, \mathbf{v}^{-\delta} M_D$. Here, \mathbf{v}^δ denotes $v_1^{\delta_1} \cdots v_N^{\delta_N}$.

Remark 1.6. The reason we used $\mathbf{v}^{-\delta} M_i$ instead of M_i is that the former are more related to the classical hypergeometric functions, as explained in Section 2. The number d is the number of variables that occur in the classical counterpart of an A -hypergeometric function, see Section 2.

Remark 1.7. This theorem requires a basis of Mellin–Barnes solutions which, unfortunately, does not always exist. In many classical cases such a basis exists. For example, two variable Appell, Horn and higher Lauricella F_A, F_B, F_D . But, on the other hand, in the case of Lauricella F_C and many other Aomoto systems such a basis does not seem to exist.

Remark 1.8. Another weak point of the algorithm is that, if successful, it is not clear if the computed matrices generate the full monodromy group. In several cases, such as the one variable case and Appell F_1, F_2 , we do get the full group.

The groups we shall calculate are determined with respect to a basis of solutions in Mellin–Barnes integral form. In the case of one variable ${}_{n+1}F_n$ they turn out to coincide with the matrices found in [8] (see Section 6.2). If one wishes to calculate monodromy matrices with respect to an explicit basis of local series expansions, one has to find an explicit calculation of a Mellin–Barnes integral as a linear combination of these series solutions. This is a tedious task which we like to carry out in a forthcoming paper. We remark that such a calculation has been carried out in the so-called confluent case (i.e. A does not lie in translated hyperplane) by Zhdanov and Tsikh [38].

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2. Gamma series solutions

The starting data of our computation will not be the set A and parameter vector α , but rather a dual version as follows. Let $d = N - r$. This will be the number of variables in the classical counterpart of the A -hypergeometric system (number of essential variables, e.g. $d = 2$ in the Appell cases). Choose a \mathbb{Z} -basis for the lattice L , which has rank d , and write the basis elements as rows of a $d \times N$ matrix B . In the literature the transpose of B is often called a *Gale dual* of A , we simply call B a *B -matrix*. The matrix B has the property that it has maximal rank d , the \mathbb{Z} -span of the columns is \mathbb{Z}^d and $A \cdot B^t$ is the zero matrix. We denote the columns of B by \mathbf{b}_j , $j = 1, \dots, N$. Then the lattice L is parameterised by the N -tuple $(\mathbf{b}_1 \cdot \mathbf{s}, \dots, \mathbf{b}_N \cdot \mathbf{s})$ with $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d$ as parameters.

In our computations we take a B -matrix as starting data and instead of the parameters α we choose $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_N) \in \mathbb{R}^N$ such that $\gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N = \alpha$. Notice that there is some ambiguity in the choice of $\boldsymbol{\gamma}$ which we will fix later. The reason we take B and $\boldsymbol{\gamma}$ as

starting data is that they are easily read off from the classical power series expansions. We give some examples of this.

Consider the system $H_A(\alpha)$ and a formal solution

$$\Phi_{\boldsymbol{\gamma}} = \sum_{\mathbf{l} \in L} \frac{v_1^{l_1 + \gamma_1} \cdots v_N^{l_N + \gamma_N}}{\Gamma(l_1 + \gamma_1 + 1) \cdots \Gamma(l_N + \gamma_N + 1)}$$

where $\boldsymbol{\gamma}$ is chosen such that $\alpha = \gamma_1 \mathbf{a}_1 + \cdots + \gamma_N \mathbf{a}_N$. This expansion was introduced in [15]. It is a formal Laurent series multiplied by generally non-integral powers of the variables v_i . It may not converge anywhere. We call such a series a formal Gamma series. A more concrete way to write this is to use the parametrization $(\mathbf{b}_1 \cdot \mathbf{s}, \dots, \mathbf{b}_N \cdot \mathbf{s})$ of L with $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{Z}^d$. We get

$$\Phi_{\boldsymbol{\gamma}} = \sum_{\mathbf{s} \in \mathbb{Z}^d} \frac{v_1^{\mathbf{b}_1 \cdot \mathbf{s} + \gamma_1} \cdots v_N^{\mathbf{b}_N \cdot \mathbf{s} + \gamma_N}}{\Gamma(\mathbf{b}_1 \cdot \mathbf{s} + \gamma_1 + 1) \cdots \Gamma(\mathbf{b}_N \cdot \mathbf{s} + \gamma_N + 1)}.$$

We have a freedom of choice in $\boldsymbol{\gamma}$ by shifts over $L \otimes \mathbb{R}$. We shall use this freedom to produce Gamma series which have a domain of convergence, in which case we simply call them Gamma series. Choose a subset $I \subset \{1, 2, \dots, N\}$ with $|I| = d = N - r$ such that \mathbf{b}_i with $i \in I$ are linearly independent. It is known that $|\det(\mathbf{b}_i)_{i \in I}| = |\det(\mathbf{a}_j)_{j \notin I}|$ and we denote this quantity by Δ_I . Choose $\boldsymbol{\gamma}$ such that $\gamma_i \in \mathbb{Z}$ for all $i \in I$. There are precisely Δ_I such choices for $\boldsymbol{\gamma}$ which are distinct modulo L . The series Φ now reads

$$\Phi_{\boldsymbol{\gamma}} = \sum_{\mathbf{s} \in \mathbb{Z}^d} \prod_{i \in I} \frac{v_i^{\mathbf{b}_i \cdot \mathbf{s} + \gamma_i}}{\Gamma(\mathbf{b}_i \cdot \mathbf{s} + \gamma_i + 1)} \times \prod_{j \notin I} \frac{v_j^{\mathbf{b}_j \cdot \mathbf{s} + \gamma_j}}{\Gamma(\mathbf{b}_j \cdot \mathbf{s} + \gamma_j + 1)}.$$

Since $\gamma_i \in \mathbb{Z}$ for all $i \in I$, this summation extends over all \mathbf{s} with $\mathbf{b}_i \cdot \mathbf{s} + \gamma_i \geq 0$ for all $i \in I$. The other terms vanish because $1/\Gamma(\gamma_i + \mathbf{b}_i \cdot \mathbf{s} + 1) = 0$ whenever $\gamma_i + \mathbf{b}_i \cdot \mathbf{s}$ is a negative integer for some $i \in I$. Hence $\Phi_{\boldsymbol{\gamma}}$ is now a power series with a region of convergence. We can see this more clearly by making a proper change of variables and consider $\mathbf{v}^{-\boldsymbol{\delta}} \Phi_{\boldsymbol{\gamma}}$ instead of $\Phi_{\boldsymbol{\gamma}}$. Recall that $\boldsymbol{\delta} = (\delta_1, \dots, \delta_N)$ is the fixed vector such that $\alpha = \delta_1 \mathbf{a}_1 + \cdots + \delta_N \mathbf{a}_N$, introduced in Theorem 1.5. Then $\boldsymbol{\gamma} - \boldsymbol{\delta} \in L \otimes \mathbb{R}$. Determine $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d) \in \mathbb{R}^d$ such that

$$\gamma_i - \delta_i = \mathbf{b}_i \cdot \boldsymbol{\sigma}$$

and consider $\mathbf{v}^{-\boldsymbol{\delta}} \Phi_{\boldsymbol{\gamma}}$, which is a infinite series with terms of the form $c_{\mathbf{s}} v_1^{\mathbf{b}_1 \cdot (\mathbf{s} + \boldsymbol{\sigma})} \cdots v_N^{\mathbf{b}_N \cdot (\mathbf{s} + \boldsymbol{\sigma})}$. Rewrite these terms, by collecting with respect to s_1, \dots, s_d , in the form $c_{\mathbf{s}} x_1^{s_1 + \sigma_1} \cdots x_d^{s_d + \sigma_d}$ where the x_i are monomials in v_1, \dots, v_N . Thus we get

$$\mathbf{v}^{-\boldsymbol{\delta}} \Phi_{\boldsymbol{\gamma}} = \sum_{\mathbf{s} \in \mathbb{Z}^d} \frac{x_1^{s_1 + \sigma_1} \cdots x_d^{s_d + \sigma_d}}{\prod_{i=1}^N \Gamma(\mathbf{b}_i \cdot \mathbf{s} + \gamma_i + 1)}$$

where the summation is over all \mathbf{s} such that for all $i \in I$, $\gamma_i + \mathbf{b}_i \cdot \mathbf{s} \geq 0$. We call this a classical hypergeometric series. It will satisfy a system of partial differential equations in x_1, \dots, x_d but they are generally much more complicated than the A -hypergeometric system.

It is not hard to see that a set of series $\Phi_{\boldsymbol{\gamma}}$ with $\boldsymbol{\gamma}$ -values which are distinct modulo $L \otimes \mathbb{R}$ is linearly independent over \mathbb{C} .

There is one important assumption we need for our construction to work. Namely the guarantee that none of the arguments $\gamma_j + \mathbf{b}_j \cdot \mathbf{s}$ is a negative integer when $j \notin I$. Otherwise

too many terms may vanish and in the worst case we may even end up with a trivial solution. Notice that

$$\alpha = \sum_{j=1}^N \gamma_j \mathbf{a}_j \equiv \sum_{j \notin I} \gamma_j \mathbf{a}_j \pmod{\mathbb{Z}^r}$$

So if $\gamma_j \in \mathbb{Z}$ for some $j \notin I$, the point α lies modulo \mathbb{Z}^r in a space spanned by the $r - 1$ remaining vectors \mathbf{a}_i . Under the assumption of total non-resonance this situation cannot occur, so from now on we assume that $H_A(\alpha)$ is totally non-resonant.

We denote the set of all sets I such that $\Delta_I = |\det(\mathbf{b}_i)_{i \in I}| \neq 0$ by \mathcal{J} . When $t = \Delta_I > 1$, we must take t copies of I in this list. To each $I \in \mathcal{J}$ there corresponds a choice of $\boldsymbol{\gamma}$ and we see to it that all these choices are distinct modulo $L \otimes \mathbb{R}$. So to an index set I which occurs t times there correspond t choices of $\boldsymbol{\gamma}$ that are distinct modulo $L \otimes \mathbb{R}$. The corresponding powerseries solutions are denoted by Φ_I .

Choose $I \in \mathcal{J}$ and $\rho_1, \dots, \rho_N \in \mathbb{R}$ such that $\rho_1 \mathbf{b}_1 \cdot \mathbf{s} + \dots + \rho_N \mathbf{b}_N \cdot \mathbf{s} > 0$ for any non-zero $\mathbf{s} \in \mathbb{Z}^d$ with $\mathbf{b}_i \cdot \mathbf{s} \geq 0$ for all $i \in I$. For example $\rho_i = 1$ if $i \in I$ and $\rho_i = 0$ if $i \notin I$. Then, according to Theorem [7, Proposition 16.2] or [34, Sections 3.3 and 3.4], the series $\Phi_{\boldsymbol{\gamma}}$ converges for all v_1, \dots, v_N with the property that for all i , $|v_i| = t^{\rho_i}$ and $t \in \mathbb{R}_{>0}$ sufficiently small. We call such an N -tuple ρ_1, \dots, ρ_N a *convergence direction* of Φ_I .

Let

$$\mathbf{b}_I = \left\{ \sum_{i \in I} \lambda_i \mathbf{b}_i : \lambda_i > 0 \right\}$$

be the interior of the cone spanned by the \mathbf{b}_i , $i \in I$. The condition for (ρ_1, \dots, ρ_N) to be a convergence direction of Φ_I comes down to

$$\boldsymbol{\rho} := \rho_1 \mathbf{b}_1 + \dots + \rho_N \mathbf{b}_N \in \mathbf{b}_I.$$

By a slight abuse of language we call the vector $\boldsymbol{\rho}$ also a convergence direction.

Conversely, fix an element $\boldsymbol{\rho}$ which does not lie on any boundary of \mathbf{b}_I for any $I \in \mathcal{J}$. Define $\mathcal{J}_{\boldsymbol{\rho}} = \{I : \boldsymbol{\rho} \in \mathbf{b}_I\}$. Then, by the theory of Gel'fand, Kapranov and Zelevinsky the powerseries Φ_I with $I \in \mathcal{J}_{\boldsymbol{\rho}}$ form a basis of solutions with a common open region of convergence. We call such a set a basis of local solutions of $H_A(\alpha)$. It also follows from the theory that to the set $\mathcal{J}_{\boldsymbol{\rho}}$ there corresponds a regular triangulations of the set A given by $\{I^c : I \in \mathcal{J}_{\boldsymbol{\rho}}\}$.

The intersections of the simplicial cones \mathbf{b}_I define a subdivision of \mathbb{R}^d into open convex polyhedral cones whose closure of the union is \mathbb{R}^d . This is a polyhedral fan which is called the *secondary fan*. The open cones in the secondary fan are in one-to-one correspondence with the bases of local series solutions. As an example let us take the system Appell F_2 . The standard Appell F_2 -series reads

$$F_2(\alpha, \beta, \beta', \gamma, \gamma', x, y) = \sum_{m, n \geq 0} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{m! n! (\gamma)_m (\gamma')_n} x^m y^n.$$

We hope no confusion arise with the existing notations $\alpha, \boldsymbol{\gamma}$. Using the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

we see that the series is proportional to

$$\sum_{m, n \geq 0} \frac{x^m y^n}{\Gamma(-\alpha - m - n + 1) \Gamma(-\beta - m + 1) \Gamma(-\beta' - n + 1) \Gamma(\gamma + m) \Gamma(\gamma' + n) \Gamma(m + 1) \Gamma(n + 1)}.$$

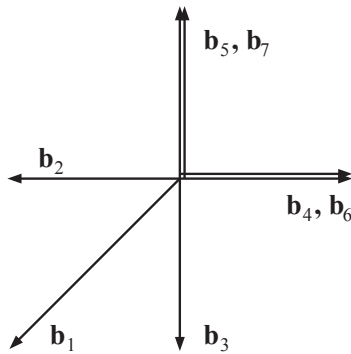
The basisvectors $(-1, -1, 0, 1, 0, 1, 0)$ and $(-1, 0, -1, 0, 1, 0, 1)$ of L are given to us naturally because these are the coefficient vectors of m and n respectively in the Γ -factors of the expansion just given. This follows from the shape of the canonical solution $\Phi_{\boldsymbol{\gamma}}$. So our B -matrix reads

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

A parameter vector $\boldsymbol{\gamma}$ can also be read off from the Γ -expansion, namely

$$(-\alpha, -\beta, -\beta', \gamma - 1, \gamma' - 1, 0, 0).$$

We choose this vector as our standard $\boldsymbol{\delta}$. We also see that the variables x, y actually come from $v_1^{-1}v_2^{-1}v_4v_6$ and $v_1^{-1}v_3^{-1}v_5v_7$. The column vectors of B are depicted here.



For example, consider the vector $(-0.5, 1)$ in this picture. We see that it is contained in the positive cones of the following pairs: $\{\mathbf{b}_2, \mathbf{b}_5\}$, $\{\mathbf{b}_2, \mathbf{b}_7\}$, $\{\mathbf{b}_1, \mathbf{b}_5\}$, $\{\mathbf{b}_1, \mathbf{b}_7\}$. Taking the complementary sets of indices of each pair we get

$$\{1, 3, 4, 6, 7\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}.$$

These form the index sets of the simplices of a triangulation of the set A . Take the alternative parameter vector

$$\boldsymbol{\gamma} = (\gamma' + \beta - \alpha - 1, 0, \gamma' - 1 - \beta', \gamma - \beta - 1, 0, -\beta, 1 - \gamma'),$$

which differs from the original choice by an element of $L \otimes \mathbb{R}$. The coordinates on positions 2, 5 are made zero, corresponding to the choice $\mathbf{b}_2, \mathbf{b}_5$. The formal solution for this new parameter vector, multiplied by $\mathbf{v}^{-\boldsymbol{\delta}}$, reads

$$\mathbf{v}^{-\boldsymbol{\delta}} \sum_{m,n} \frac{v_1^{\gamma' - \beta - \alpha - 1 - m - n} v_2^{-m} v_3^{\gamma' - \beta' - 1 - n} v_4^{\gamma - \beta - 1 + m} v_5^n v_6^{-\beta + m} v_7^{1 - \gamma' + n}}{\Gamma(\gamma' + \beta - \alpha - m - n) \Gamma(-m + 1) \Gamma(\gamma' - \beta' - n) \Gamma(\gamma - \beta + m) \Gamma(n + 1) \Gamma(-\beta + m) \Gamma(2 - \gamma' + n)}.$$

Since $1/\Gamma(n + 1) = 0$ when $n < 0$ and $1/\Gamma(-m + 1) = 0$ when $m > 0$, we see that our summation runs over $m \leq 0$ and $n \geq 0$. Replace m by $-m$ to get

$$\sum_{m,n \geq 0} \frac{v_1^{\gamma' + \beta - 1} v_2^\beta v_3^{\gamma' - 1} v_4^{-\beta} v_5^{1 - \gamma'} v_6^{-\beta} v_7^{1 - \gamma'} (v_1 v_2 v_4^{-1} v_6^{-1})^m (v_1^{-1} v_3^{-1} v_5 v_7)^n}{\Gamma(\gamma' + \beta - \alpha + m - n) \Gamma(m + 1) \Gamma(\gamma' - \beta' - n) \Gamma(\gamma - \beta - m) \Gamma(n + 1) \Gamma(-\beta - m) \Gamma(2 - \gamma' + n)}.$$

Note that $v_1 v_2 v_4^{-1} v_6^{-1} = x^{-1}$ and $v_1^{-1} v_3^{-1} v_5 v_7 = y$, so our Gamma series can be written as $x^\beta y^{1 - \gamma'}$ times an ordinary powerseries in x^{-1} and y . In the same way we can construct three other Gamma series and thus obtain a basis of local solutions of our system.

3. Mellin–Barnes integrals

3.1. Definition and properties. Let notations be as in the previous sections. Consider the parametrization of $L \otimes \mathbb{R}$ by the N -tuple $(\mathbf{b}_1 \cdot \mathbf{s}, \dots, \mathbf{b}_N \cdot \mathbf{s})$ with $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{R}^d$ as parameters. Choose $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_d) \in \mathbb{R}^d$.

We now complexify the parameters s_j and consider the integral

$$(MB) \quad M(v_1, \dots, v_N) = \int_{\boldsymbol{\sigma} + i\mathbb{R}^d} \prod_{j=1}^N \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s}) v_j^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s}} d\mathbf{s}$$

where $d\mathbf{s} = ds_1 \wedge \dots \wedge ds_d$ and the integration takes place over $-\infty < \text{Im}(s_j) < \infty$ and $\text{Re}(s_j) = \sigma_j$ for $j = 1, \dots, d$. This is an example of a so-called *Mellin–Barnes integral*. It will be crucial in the determination of the monodromy of A -hypergeometric systems.

We prove the following theorem.

Theorem 3.2. *Assume that $\gamma_j < -\mathbf{b}_j \cdot \boldsymbol{\sigma}$ for $j = 1, 2, \dots, N$. Then the Mellin–Barnes integral $M(v_1, \dots, v_N)$ satisfies the set of A -hypergeometric equations $H_A(\boldsymbol{\alpha})$.*

This will be done under the assumption that the Mellin–Barnes integral converges absolutely. We come to the matter of convergence in the next section.

Proof. The Mellin–Barnes integral clearly has the property

$$M(\mathbf{t}^{\mathbf{a}_1} v_1, \dots, \mathbf{t}^{\mathbf{a}_N} v_N) = \mathbf{t}^{\boldsymbol{\alpha}} M(v_1, \dots, v_N) \quad \text{for all } \mathbf{t} \in (\mathbb{C}^\times)^N.$$

So $M(\mathbf{v})$ satisfies the hypergeometric homogeneity equations.

Now let $\lambda \in L$ and put $\lambda = \lambda_+ - \lambda_-$ where λ_\pm have non-negative coefficients and disjoint support. Define $|\lambda| = \sum_{i=1}^N |\lambda_i|$. Then

$$\begin{aligned} \square_\lambda M(v_1, \dots, v_N) &= (-1)^{|\lambda|/2} \int_{\boldsymbol{\sigma} + i\mathbb{R}^d} \prod_{j=1}^N \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s} + \lambda_{+,j}) v_j^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s} - \lambda_{+,j}} d\mathbf{s} \\ &\quad - (-1)^{|\lambda|/2} \int_{\boldsymbol{\sigma} + i\mathbb{R}^d} \prod_{j=1}^N \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s} + \lambda_{-,j}) v_j^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s} - \lambda_{-,j}} d\mathbf{s}. \end{aligned}$$

Choose \mathbf{s}_λ such that $\mathbf{b}_j \cdot \mathbf{s}_\lambda = \lambda_j$ for $j = 1, \dots, N$. Then the second integral is actually integration over $\mathbf{s}_\lambda + i\mathbb{R}^d$ of the integrand of the first integral.

Because of the assumption $\gamma_j < -\mathbf{b}_j \cdot \boldsymbol{\sigma}$ we see that

$$-\gamma_j - t\mathbf{b}_j \cdot (\mathbf{s}_\lambda + \boldsymbol{\gamma}) + \lambda_{+,j} > 0 \quad \text{for all } t \in [0, 1] \text{ and } j = 1, \dots, N.$$

Hence the $(d + 1)$ -dimensional domain $\{t(\mathbf{s}_\lambda + \boldsymbol{\gamma}) + i\mathbb{R}^d : 0 \leq t \leq 1\}$ does not contain any poles of the integrand and a homotopy argument gives that the two integrals are equal and cancel. □

Not all systems $H_A(\boldsymbol{\alpha})$ allow a choice of $\boldsymbol{\gamma}$ satisfying $\gamma_j < -\mathbf{b}_j \cdot \boldsymbol{\sigma}$. However, under the assumption of non-resonance it is known that two hypergeometric systems $H_A(\boldsymbol{\alpha})$ and $H_A(\boldsymbol{\alpha}')$ have the same monodromy if $\boldsymbol{\alpha} - \boldsymbol{\alpha}' \in \mathbb{Z}^r$ (see [6, Theorem 2.1]). We call such systems *contiguous*, and since we are interested in monodromy, we might as well consider contiguous

systems. Thus, by shifting $\boldsymbol{\gamma}$ over an integer vector we can always replace an irreducible A -hypergeometric system by a contiguous one which does allow a choice of $\gamma_j < -\mathbf{b}_j \cdot \boldsymbol{\sigma}$ for all j . In concrete cases we can also play with the value of $\boldsymbol{\sigma}$. From now on we make this assumption, i.e. all our Mellin–Barnes integrals are solution of an A -hypergeometric system. Of course there is also the question whether or not $M(v_1, \dots, v_N)$ is a trivial function. By Proposition 4.1 we will find that it is non-trivial.

3.3. Convergence of the Mellin–Barnes integral. We find from [2] the following estimate. Suppose $s = a + bi$ with $a_1 < a < a_2$, $i = \sqrt{-1}$ and $|b| \rightarrow \infty$. Then

$$|\Gamma(a + bi)| = \sqrt{2\pi} |b|^{a-1/2} e^{-\pi|b|/2} [1 + O(1/|b|)].$$

Notice also that for any $v \in \mathbb{C}^\times$ we have $|v^{a+bi}| = |v|^a e^{-b \arg(v)}$. Write $s_j = \sigma_j + i\tau_j$ for $j = 1, \dots, N - r$. Let us denote $\theta_j = \arg(v_j)$ and $l_j(\boldsymbol{\tau}) = l_j(\tau_1, \dots, \tau_d)$. The integrand in the Mellin–Barnes integral can now be estimated by

$$\left| \prod_{j=1}^N \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s}) v_i^{\gamma_j + \mathbf{b}_j \cdot \mathbf{s}} \right| \leq c_1 \max_j |\tau_j|^{c_2} \exp\left(-\sum_{j=1}^N \pi \frac{|\mathbf{b}_j \cdot \boldsymbol{\tau}|}{2} - \theta_j \mathbf{b}_j \cdot \boldsymbol{\tau}\right)$$

where c_1, c_2 are positive numbers depending only on γ_j, v_j, σ_j . In order to ensure convergence of the integral we must have that

$$(C) \quad \sum_{j=1}^N \pi \frac{|\mathbf{b}_j \cdot \boldsymbol{\tau}|}{2} + \theta_j \mathbf{b}_j \cdot \boldsymbol{\tau} > 0$$

for every non-zero $\boldsymbol{\tau} \in \mathbb{R}^d$. We apply the following lemma.

Lemma 3.4. *Let $\mathbf{p}_1, \dots, \mathbf{p}_N \in \mathbb{R}^d$ be a set of N vectors of rank d , and let $\mathbf{q} \in \mathbb{R}^d$. Then the following statements are equivalent:*

(i) *For all non-zero $\mathbf{x} \in \mathbb{R}^d$,*

$$|\mathbf{q} \cdot \mathbf{x}| < \sum_{j=1}^N |\mathbf{p}_j \cdot \mathbf{x}|.$$

(ii) *There exist $\lambda_1, \dots, \lambda_N$ with $-1 < \lambda_j < 1$ such that*

$$\mathbf{q} = \lambda_1 \mathbf{p}_1 + \dots + \lambda_N \mathbf{p}_N.$$

Proof. First suppose that $\mathbf{q} = \lambda_1 \mathbf{p}_1 + \dots + \lambda_N \mathbf{p}_N$. Then, for all non-zero $\mathbf{x} \in \mathbb{R}^d$,

$$|\mathbf{q} \cdot \mathbf{x}| = \left| \sum_{j=1}^N \lambda_j \mathbf{p}_j \cdot \mathbf{x} \right| \leq \sum_{j=1}^N |\lambda_j| |\mathbf{p}_j \cdot \mathbf{x}| < \sum_{j=1}^N |\mathbf{p}_j \cdot \mathbf{x}|.$$

To show the converse statement consider the set

$$V = \left\{ \sum_{j=1}^N \lambda_j \mathbf{p}_j : -1 < \lambda_j < 1 \right\}.$$

This is a convex set. Suppose $\mathbf{q} \notin V$. Then there exists a linear form h such that $h(\mathbf{q}) > h(\mathbf{p})$ for all $\mathbf{p} \in V$. In other words, there exists a vector $\mathbf{x} \in \mathbb{R}^d$ such that $\mathbf{q} \cdot \mathbf{x} > \sum_{j=1}^N \lambda_j \mathbf{p}_j \cdot \mathbf{x}$

all $-1 < \lambda_j < 1$. In particular,

$$|\mathbf{q} \cdot \mathbf{x}| \geq \sum_{j=1}^N |\mathbf{p}_j \cdot \mathbf{x}|$$

contradicting our assumption. Hence $\mathbf{q} \in V$. □

Application of Lemma 3.4 with $\mathbf{q} = \sum_{j=1}^N \theta_j \mathbf{b}_j$ and $\mathbf{p}_j = \frac{\pi \mathbf{b}_j}{2}$ to inequality (C) yields the following criterion.

Corollary 3.5. *Let notations be as above. Then the Mellin–Barnes integral converges absolutely if there exist $\lambda_j \in (-1, 1)$ such that*

$$\sum_{j=1}^N \frac{\theta_j}{2\pi} \mathbf{b}_j = \frac{1}{4} \sum_{j=1}^N \lambda_j \mathbf{b}_j.$$

Let us define

$$Z_B = \left\{ \frac{1}{4} \sum_{j=1}^N \lambda_j \mathbf{b}_j : \lambda_j \in (-1, 1) \right\}.$$

This is a so-called *zonotope* in d -dimensional space. The convergence condition for the Mellin–Barnes integral now reads

$$\sum_{j=1}^N \frac{\theta_j}{2\pi} \mathbf{b}_j \in Z_B.$$

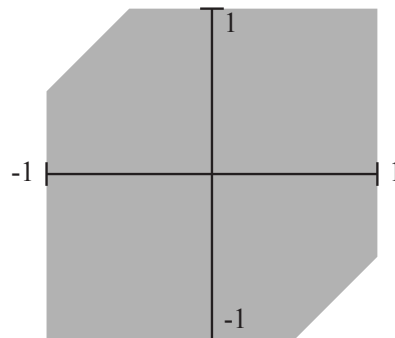
As an example let us again take the system Appell F_2 . Recall that our B -matrix reads

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

As before, a parameter vector $\boldsymbol{\gamma}$ can also be read off from the Γ -expansion, namely

$$\boldsymbol{\gamma} = (-\alpha, -\beta, -\beta', \gamma - 1, \gamma' - 1, 0, 0).$$

When $\alpha, \beta, \beta' > 0$ and $\gamma, \gamma' < 1$, we see that $\boldsymbol{\gamma}$ has negative components, except for the last two. By making a suitable choice for $\boldsymbol{\sigma} \in \mathbb{R}^2$ we can see to it that $\gamma_j < -\mathbf{b}_j \cdot \boldsymbol{\sigma}$ for all j . Thus the corresponding Mellin–Barnes integral is indeed a solution of the F_2 -system. The zonotope Z_B can be pictured as



and the convergence condition reads

$$\frac{1}{2\pi}(-\theta_1 - \theta_2 + \theta_4 + \theta_6, -\theta_1 - \theta_3 + \theta_5 + \theta_7) \in Z_B.$$

Note that the four points $(\pm 1/2, \pm 1/2)$ are contained in Z_B . They correspond to the arguments $\theta = 2\pi(0, 0, 0, 0, 0, \pm 1/2, \pm 1/2)$. These argument choices represent the same point in v_1, \dots, v_7 -space. Hence we have four Mellin–Barnes solutions of the system F_2 around one point. According to Proposition 3.6 these integrals are linearly independent, and hence form a basis of local solutions of the F_2 -system. We say that we have a *Mellin–Barnes basis* of solutions.

Proposition 3.6. *Let $\mathbf{v}_0 = (v_1^{(0)}, \dots, v_N^{(0)}) \in (\mathbb{C}^\times)^N$. Furthermore, let Θ be a finite set of N -tuples $\theta = (\theta_1, \dots, \theta_N)$ such that*

- $v_j^{(0)} = |v_j^{(0)}| \exp(i\theta_j)$,
- the sums $\frac{1}{2\pi} \sum_{j=1}^N \theta_j \mathbf{b}_j$ are distinct elements of Z_B .

To each $\theta \in \Theta$ denote the corresponding determination of the Mellin–Barnes integral in the neighbourhood of \mathbf{v}_0 by M_θ . Then the functions M_θ are linearly independent over \mathbb{C} .

The proof of this lemma depends on a d -dimensional version or if one wants, repeated application, of the following theorem.

Theorem 3.7 (Mellin inversion theorem). *Let $\phi(z)$ be a function on \mathbb{C} satisfying the following properties:*

- (a) ϕ is analytic in a vertical strip of the form $\alpha < x = \operatorname{Re}(z) < \beta$ where $\alpha, \beta \in \mathbb{R}$,
- (b) $\int_{-\infty}^{\infty} |\phi(x + iy)| dy$ converges for all $x \in (\alpha, \beta)$,
- (c) $\phi(z) \rightarrow 0$ uniformly as $|y| \rightarrow \infty$ in $\alpha + \epsilon < x < \beta - \epsilon$ for all $\epsilon > 0$.

Denote for all $t > 0$,

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} t^{-z} \phi(z) dz.$$

Then,

$$\phi(z) = \int_0^\infty t^{z-1} f(t) dt.$$

In particular, if $f(t) = 0$ for all $t > 0$, then $\phi(z) = 0$ for all z

For a proof of this theorem see [22, Appendix 4, pp. 341–342]. A treatment of the multi-dimensional case can also be found for example in [3].

Proof of Proposition 3.6. Suppose we have a non-trivial relation $\sum_{\theta \in \Theta} \lambda_\theta M_\theta = 0$. Let us use the notation $(\theta \cdot B)(\mathbf{s}) = \theta_1 \mathbf{b}_1 \cdot \mathbf{s} + \dots + \theta_N \mathbf{b}_N \cdot \mathbf{s}$. The relation can be written as

$$0 = \int_{\mathbf{s} \in \sigma + i\mathbb{R}^d} \left(\sum_{\theta \in \Theta} \lambda_\theta e^{(\theta \cdot B)(\mathbf{s})} \right) |v_1|^{\mathbf{b}_1 \cdot \mathbf{s}} \dots |v_N|^{\mathbf{b}_N \cdot \mathbf{s}} \prod_{j=1}^N \Gamma(-\gamma_j - \mathbf{b}_j \cdot \mathbf{s}) d\mathbf{s}$$

where we took out the factor $\prod_{i=1}^N v_i^{\gamma_i}$. Let us now write $x_j = |v_1|^{b_{1j}} \cdots |v_N|^{b_{Nj}}$ where b_{ij} are the entries of the B -matrix. Then $|v_1|^{\mathbf{b}_1 \cdot \mathbf{s}} \cdots |v_N|^{\mathbf{b}_N \cdot \mathbf{s}} = x_1^{s_1} \cdots x_d^{s_d}$. By repeated use of the Mellin Inversion Theorem 3.7 we conclude that the vanishing of the integral implies the identical vanishing of $\sum_{\theta \in \Theta} \lambda_{\theta} e^{(\theta \cdot B)(\mathbf{s})}$. Since the exponents in the exponentials are distinct linear forms in s_1, \dots, s_d , this implies that $\lambda_{\theta} = 0$ for all $\theta \in \Theta$. \square

Another proof of Proposition 3.6 can be found in [27, Lemma 5.5]. However, I hesitate somewhat about its completeness and decided to give the proof above.

Since we assume that $\gamma_j < -\mathbf{b}_j \cdot \boldsymbol{\sigma}$ for all j , all Mellin–Barnes integrals are solutions of the corresponding A -hypergeometric system. It would be very convenient if such a basis of solutions given by Mellin–Barnes integrals would always exist. It turns out that with the exception of Appell F_4 , see Section 7.3 all $d = 2$ systems Appell F_1, F_2, F_3 and Horn $G_1, G_2, G_3, H_1, \dots, H_7$ this is the case. A theoretical framework for a result like this may be provided in the PhD thesis of Lisa Nilsson [27] suggesting that there does indeed exist such a basis if the complement of the so-called coamoeba of the A -resultant is non-empty. In [27] this is elaborated for the case $d = 2$.

4. Monodromy computation

Let us now make the following assumption on our system $H_A(\boldsymbol{\alpha})$.

Assumption 4.1. There exists a point $\mathbf{v}_0 \in (\mathbb{C}^\times)^N$ and a basis of solutions of $H_A(\boldsymbol{\alpha})$ in a neighbourhood of \mathbf{v}_0 given by Mellin–Barnes integrals.

More precisely, with the notations of Proposition 3.6 there exist D choices of $\boldsymbol{\theta}$ such that all sums $\sum_{j=1}^N \theta_j \mathbf{b}_j$ are distinct.

An even more practical formulation of Assumption 4.1 is provided by the following proposition.

Proposition 4.2. *Let $H_A(\boldsymbol{\alpha})$ be a non-resonant system of rank D . Then Assumption 4.1 holds if and only if the zonotope Z_B contains D distinct points $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_D$ whose coordinates differ by integers. Remember that Z_B is an open set in \mathbb{R}^d .*

Proof. From the discussion above it follows that the existence of a Mellin–Barnes basis corresponds to the choice of D N -tuples $(\theta_1, \dots, \theta_N)$, representing argument choices of a given point. Hence the differences between these N -tuples have coordinates which are integer multiples of 2π . The sums $\frac{1}{2\pi} \sum_{j=1}^N \theta_j \mathbf{b}_j$ are distinct by Assumption 4.1, hence the D Mellin–Barnes basis elements correspond to D points $\boldsymbol{\tau}_j \in Z_B$ whose coordinates also differ by integers.

Suppose conversely we have D points $\boldsymbol{\tau}_k \in Z_B$ whose coordinates differ by integers. Since the \mathbb{Z} -span of the columns \mathbf{b}_j is \mathbb{Z}^d , we can find for every k integers n_{k1}, \dots, n_{kN} such that $\boldsymbol{\tau}_k - \boldsymbol{\tau}_1 = n_{k1} \mathbf{b}_1 + \cdots + n_{kN} \mathbf{b}_N$. So if $(\theta_1, \dots, \theta_N)$ is an argument choice for $\boldsymbol{\tau}_1$, then the N -tuples $(\theta_1 + 2\pi n_{j1}, \dots, \theta_N + 2\pi n_{jN})$ represent argument choices for $\boldsymbol{\tau}_k$ with $k = 1, 2, \dots, D$. \square

For later use, we consider the vector of arguments θ_k and the vector τ_k in Z_B as column vectors. Then, in matrix multiplication notation, $2\pi\tau_k = B\theta_k$ for $k = 1, 2, \dots, D$.

Denote the basis elements by M_1, \dots, M_D and the corresponding points in the zonotope Z_B by τ_1, \dots, τ_D . It is the goal of this section to compute the local monodromy groups with respect to the basis M_1, \dots, M_D . To this end we shall determine the transition matrices from M_1, \dots, M_D to each of the bases of local series expansions.

To each point τ_k there corresponds a (not necessarily unique) choice of arguments

$$\theta_k = (\text{Arg}(v_1), \dots, \text{Arg}(v_N)).$$

We assume that the arguments are chosen such that the differences $\theta_k - \theta_1$ have all their components equal to integer multiples of 2π (see Proposition 4.2). Let $\mathbf{v}_0 \in (\mathbb{C}^\times)^N$ be a point whose coordinates have arguments θ_1 . In particular we have a basis of Mellin–Barnes solutions around \mathbf{v}_0 . The Mellin–Barnes integral corresponding to the argument vector θ_k is denoted by M_k .

Fix a converge direction ρ . To each $I \in \mathcal{I}$ such that ρ is contained in

$$\mathbf{b}_I = \left\{ \sum_{j \in I} \lambda_j \mathbf{b}_j : \lambda_j > 0 \right\}$$

construct a converging Gamma series solution. Let us denote these series by f_1, \dots, f_D . For sufficiently small $t > 0$ the realm of convergence of these local series expansions contains an open neighbourhood of the torus $T: |v_j| = r_j := t^{\rho_j}$ for $j = 1, \dots, N$. Choose a point $\mathbf{v}'_0 \in T$ with the same argument values as \mathbf{v}_0 and let Π be a path from \mathbf{v}_0 to \mathbf{v}'_0 while keeping the arguments fixed. We remark that we do not meet any singular points of the system, since the Mellin–Barnes integrals converge around every point along Π . This is because the arguments do not change along Π . In a neighbourhood of \mathbf{v}'_0 we also have a Mellin–Barnes basis of solutions which are simply the analytic continuation of M_1, \dots, M_D along Π . Let us call them M_1, \dots, M_D again. For any N -tuple of integers $\mathbf{n} = (n_1, \dots, n_N)$ we consider the loop

$$c = c(\mathbf{n}) = c(n_1, \dots, n_N) : (e^{2\pi i n_1 t} v_1^{(0)'}, \dots, e^{2\pi i n_N t} v_N^{(0)'}), \quad t \in [0, 1],$$

where $\mathbf{v}'_0 = (v_1^{(0)'}, \dots, v_N^{(0)'})$. Note that after analytic continuation of the basis element M_1 along the path $c((\theta_k - \theta_1)/2\pi)$ we end up with the Mellin–Barnes solution M_k for every k . We denote this path by c_k .

Denote the choices of γ corresponding to f_1, \dots, f_D by $\gamma^{(1)}, \dots, \gamma^{(D)}$. We regard the latter as D row vectors. Then there exist scalars λ_k such that

$$M_1 = \lambda_1 f_1 + \dots + \lambda_D f_D$$

in a neighbourhood of \mathbf{v}'_0 . After continuation along c_j the integral M_1 changes into M_j for every j . Under the paths c_j the local expansions $f_k, k = 1, \dots, D$, are multiplied by scalars. The space spanned by M_1, \dots, M_D is D -dimensional. The space spanned by the images of $\lambda_1 f_1 + \dots + \lambda_D f_D$ under $c_j, j = 1, 2, \dots, D$, is at most equal to the number of non-zero λ_k . Hence we conclude that $\lambda_i \neq 0$ for all i . Let us renormalise the f_k such that

$$M_1 = f_1 + \dots + f_D.$$

Then after continuation along c_j we get

$$M_j = e^{i\gamma^{(1)}(\theta_j - \theta_1)} f_1 + \dots + e^{i\gamma^{(D)}(\theta_j - \theta_1)} f_D \quad \text{for every } j.$$

Define

$$X_\rho = \begin{pmatrix} 1 & \cdots & 1 \\ e^{i\boldsymbol{\gamma}^{(1)}(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)} & \cdots & e^{i\boldsymbol{\gamma}^{(D)}(\boldsymbol{\theta}_2 - \boldsymbol{\theta}_1)} \\ \vdots & & \vdots \\ e^{i\boldsymbol{\gamma}^{(1)}(\boldsymbol{\theta}_D - \boldsymbol{\theta}_1)} & \cdots & e^{i\boldsymbol{\gamma}^{(D)}(\boldsymbol{\theta}_D - \boldsymbol{\theta}_1)} \end{pmatrix}$$

where $\boldsymbol{\rho}$ is the convergence direction we started with. It depends only on the cone of the secondary fan in which $\boldsymbol{\rho}$ lies. Then

$$\begin{pmatrix} M_1 \\ \vdots \\ M_D \end{pmatrix} = X_\rho \begin{pmatrix} f_1 \\ \vdots \\ f_D \end{pmatrix},$$

hence X_ρ is the desired transition matrix. Let us now consider any closed path of the form $\Pi^{-1}c(\mathbf{n})\Pi$, $\mathbf{n} \in \mathbb{Z}^N$, beginning and ending in \mathbf{v}_0 . Continuation of M_1, \dots, M_D along Π is trivial since, as remarked before, the Mellin–Barnes integrals converge throughout. However these integrals do not converge anymore if we continue along the loop $c(\mathbf{n})$. For that we have to change to the local basis $f_k, k = 1, \dots, D$. Analytic continuation along $c(\mathbf{n})$ changes them into $e^{2\pi i \mathbf{n} \cdot \boldsymbol{\gamma}^{(k)}} f_k$ for $k = 1, \dots, D$. Express these solutions in terms of the M_k again and continue back along Π^{-1} .

Instead of taking the monodromy with respect to the M_k and f_k we consider it with respect to the functions $\mathbf{v}^{-\boldsymbol{\delta}} M_k$ and $\mathbf{v}^{-\boldsymbol{\delta}} f_k$, which correspond to the classical hypergeometric series as explained in Section 2. The functions $\mathbf{v}^{-\boldsymbol{\delta}} f_k$ change by a factor $e^{2\pi i \mathbf{n} \cdot (\boldsymbol{\gamma}^{(k)} - \boldsymbol{\delta})}$ when continued along $c(\mathbf{n})$. Moreover, if \mathbf{n} is a row of the matrix A , then $\mathbf{n} \cdot (\boldsymbol{\gamma}^{(k)} - \boldsymbol{\delta}) = \boldsymbol{\alpha} - \boldsymbol{\alpha} = \mathbf{0}$. Hence the local monodromy is trivial for any row vector \mathbf{n} of A and, a fortiori, for any \mathbf{n} which lies in the row span of A . So as a set of generators it suffices to choose generators of $\mathbb{Z}^N / \text{rowspan}(A)$.

Putting it all together we arrive at the following proposition.

Proposition 4.3. *Let notations be as above. For any vector $\mathbf{n} \in \mathbb{Z}^N$ let $\chi_\rho(\mathbf{n})$ be the diagonal $D \times D$ -matrix with entries $e^{2\pi i \mathbf{n} \cdot (\boldsymbol{\gamma}^{(k)} - \boldsymbol{\delta})}$, $k = 1, \dots, D$. It is the monodromy matrix for the path $c(\mathbf{n})$ with respect to $\mathbf{v}^{-\boldsymbol{\delta}} f_1, \dots, \mathbf{v}^{-\boldsymbol{\delta}} f_D$. For the monodromy matrix with respect to $\mathbf{v}^{-\boldsymbol{\delta}} M_1, \dots, \mathbf{v}^{-\boldsymbol{\delta}} M_D$ we get the conjugated matrix $X_\rho \chi_\rho(\mathbf{n}) X_\rho^{-1}$.*

The monodromy matrix is trivial if \mathbf{n} belongs to the rowspan of A .

Thus we see that all local monodromies can be written with respect to a fixed Mellin–Barnes basis.

5. An implementation

The considerations in the previous sections, together with some practical tricks, lead to an algorithm to compute monodromy matrices, which we describe in this section.

We start with a totally non-resonant hypergeometric system $H_A(\boldsymbol{\alpha})$ and we assume that there exists a Mellin–Barnes basis, which has rank D . The starting data are a $d \times N$ B -matrix B

and the fixed parameter vector $\delta = (\delta_1, \dots, \delta_N)$, introduced in the introduction, such that

$$\sum_{j=1}^N \delta_j \mathbf{a}_j = \alpha.$$

In general both B and a value of δ can easily be read off from an explicit series solution of a hypergeometric system. For example, from the expansion of Appell F_2 as on page 188.

We also determine a fixed set of loops which we use for the computation of all local monodromies. Let $I_0 \in \mathcal{I}$ be such that $|\det(\mathbf{b}_j)_{j \in I_0}| = 1$. For every $m = 1, \dots, d$ we define $\mathbf{n}_m \in \mathbb{Z}^N$ such that \mathbf{n}_j has support in I_0 and $B\mathbf{n}_m = \mathbf{e}_m$, the m -th standard basis vector in \mathbb{R}^d . We take the loops $c(\mathbf{n}_1), \dots, c(\mathbf{n}_m)$ for the local monodromies as considered in Proposition 4.3.

Step 1. Using the B -matrix we determine the zonotope Z_B and find D distinct points in it, whose coordinates differ by integers. Since we assumed the existence of a Mellin–Barnes basis, these points exist. Call the points τ_1, \dots, τ_D . From the proof of Proposition 4.2 we know that to each τ_k there exists a column vector of arguments $\theta_k \in \mathbb{R}^n$ such that $2\pi\tau_k = B\theta_k$. However, we do not compute these angle vectors.

Step 2. Construct the set \mathcal{I} of all subsets I of cardinality d of the columns $\{\mathbf{b}_1, \dots, \mathbf{b}_N\}$ with $\Delta_I = |\det_{j \in I}(\mathbf{b}_j)| \neq 0$. As a fine point, if $\Delta_I > 1$, we include Δ_I copies of I in \mathcal{I} . For each I there exists a parameter vector γ^I in the following way. Denote the rows of the B -matrix B by $\mathbf{l}_1, \dots, \mathbf{l}_d$, recall that this is a basis of the lattice L . In case $\Delta_I = 1$ we take the uniquely determined real numbers μ_1, \dots, μ_d such that $\delta + \mu_1\mathbf{l}_1 + \dots + \mu_d\mathbf{l}_d$ has j -th coordinate 0 for all $j \in I$ and call this sum γ^I . In case $\Delta_I > 1$ we make Δ_I choices for (μ_1, \dots, μ_d) , distinct modulo \mathbb{Z}^d , such that $\delta + \mu_1\mathbf{l}_1 + \dots + \mu_d\mathbf{l}_d$ has integer coordinates on the j -th position for all $j \in I$. In this way we get Δ_I different parameter vectors γ^I (in row form) for a given $I \in \mathcal{I}$. However, in the computation we only retain the (row vectors) $\mu^I = (\mu_1, \dots, \mu_d)$ for each I . They have the property that $\gamma^I - \delta = \mu^I B$ for all I .

Step 3. To every $I \in \mathcal{I}$ we associate a column vector X_I of length D given by

$$(5.1) \quad X_I = (1, \exp(2\pi i \mu^I (\tau_2 - \tau_1)), \dots, \exp(2\pi i \mu^I (\tau_D - \tau_1)))^t.$$

Since for each $j = 1, 2, \dots, D$ we have

$$2\pi \mu^I (\tau_j - \tau_1) = \mu^I B(\theta_j - \theta_1) = (\gamma^I - \delta)(\theta_j - \theta_1),$$

the j -th components of X_I differs by a factor $\exp(-i\delta(\theta_j - \theta_1))$ from the similar components in the columns of the transition matrices X_ρ . The only effect is that the transition matrices built out of our present X_I will give us the transition of the Mellin–Barnes basis to a renormalised local basis. This will have no effect on the monodromy computation.

Step 4. For every cone in the secondary fan (specified by a convergence direction ρ inside that cone) we determine the sets $I \in \mathcal{I}$ such that the cone or, equivalently, ρ lies in the positive real cone spanned by the vectors $\{\mathbf{b}_j\}_{j \in I}$. Call this set of sets \mathcal{J}_ρ . The theory of Gel'fand, Kapranov and Zelevinsky tells us that \mathcal{J}_ρ contains precisely D sets (when we count possible repetitions of a set with $\Delta_I > 1$). Let X_ρ be the $D \times D$ -matrix whose columns are the

vectors X_I with $I \in \mathcal{J}_\rho$. The matrices X_ρ are the transition matrices from the Mellin–Barnes basis to the local power series basis, all of whose elements contain ρ as a convergence direction.

Step 5. For every cone in the secondary fan (specified by a convergence direction ρ) we determine the characters of the local monodromies of the corresponding power series solutions. For this we use Proposition 4.3 with the loops $c(\mathbf{n}_1), \dots, c(\mathbf{n}_d)$. The character corresponding to \mathbf{n}_m and the function $\mathbf{v}^{-\delta} \Phi_I$ reads $\exp(2\pi i \mathbf{n}_m(\boldsymbol{\gamma}^I - \boldsymbol{\delta})) = \exp(2\pi i \mu_m^I)$.

Step 6. This is the final step in which we compute d monodromy matrices for every cone of the secondary fan. For a cone, specified by a convergence direction ρ , and a loop $c(\mathbf{n}_m)$ we construct the matrix X_ρ as in Step 4, and a diagonal matrix $\chi_{\rho,m}$ with entries $\exp(2\pi i \mu_m^I)$, $I \in \mathcal{J}_\rho$, as in Step 5. We see to it that both in X_ρ and $\chi_{\rho,m}$ we keep the same ordering of the set \mathcal{J}_ρ . Then construct the matrix $M_{\rho,m} = X_\rho \chi_{\rho,m} X_\rho^{-1}$.

Let F be the number of open cones in the secondary fan. Then we get dF monodromy matrices $M_{\rho,m}$ in this way. As remarked before, it is not clear if they generate the full monodromy group of our system.

6. Examples

6.1. Appell F_2 . Recall that a B -matrix is given by

$$B = \begin{pmatrix} -1 & -1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

and a parameter vector $\boldsymbol{\delta} = (-\alpha, -\beta, -\beta', \gamma - 1, \gamma' - 1, 0, 0)$. We trust that no confusion will arise with the existing notations $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Two powerseries solution expansions have already been given on pages 188 and 188. The set \mathcal{J} consists of fifteen elements, $\{1, 3\}, \{1, 4\}, \{1, 6\}, \{1, 5\}, \{1, 7\}, \{1, 2\}, \{2, 3\}, \{2, 5\}, \{2, 7\}, \{3, 4\}, \{3, 6\}, \{4, 5\}, \{4, 7\}, \{5, 6\}, \{6, 7\}$. Here is a table with the corresponding values of μ^J .

J	μ^J	J	μ^J	J	μ^J
1	$\{1, 3\} \quad -\alpha + \beta', -\beta'$	6	$\{1, 2\} \quad -\beta, -\alpha + \beta$	11	$\{3, 6\} \quad 0, -\beta'$
2	$\{1, 4\} \quad 1 - \gamma, -1 - \alpha + \gamma$	7	$\{2, 3\} \quad -\beta, -\beta'$	12	$\{4, 5\} \quad 1 - \gamma, 1 - \gamma'$
3	$\{1, 6\} \quad 0, -\alpha$	8	$\{2, 5\} \quad -\beta, 1 - \gamma'$	13	$\{4, 7\} \quad 1 - \gamma, 0$
4	$\{1, 5\} \quad -1 - \alpha + \gamma', 1 - \gamma'$	9	$\{2, 7\} \quad -\beta, 0$	14	$\{5, 6\} \quad 0, 1 - \gamma'$
5	$\{1, 7\} \quad -1 + \gamma', -\alpha, 0$	10	$\{3, 4\} \quad 1 - \gamma, -\beta'$	15	$\{6, 7\} \quad 0, 0$

As noted earlier, the zonotope Z_B contains the four points $(\pm 1/2, \pm 1/2)$. Define

$$\boldsymbol{\tau}_1 = (-1/2, -1/2)^t, \quad \boldsymbol{\tau}_2 = (1/2, -1/2)^t, \quad \boldsymbol{\tau}_3 = (-1/2, 1/2)^t, \quad \boldsymbol{\tau}_4 = (1/2, 1/2)^t.$$

Then the vectors X_J , as defined in Step 3 of our algorithm, read $1, e(\mu_1^J), e(\mu_2^J), e(\mu_1^J + \mu_2^J)$ where we use the notations $e(x) = e^{2\pi i x}$ and $a = e(\alpha)$, $b = e(\beta)$, $b' = e(\beta')$, $c = e(\gamma)$, $c' = e(\gamma')$. Here is the list of all X_J with the same ordering as in the previous table.

J					X_J					J					X_J																																
1	{1, 3}	1	b'/a	$1/b'$	$1/a$	9	{2, 7}	1	$1/b$	1	$1/b$	10	{3, 4}	1	$1/c$	$1/b'$	$1/cb'$	11	{3, 6}	1	1	$1/b'$	$1/b'$	12	{4, 5}	1	$1/c$	$1/c'$	$1/cc'$	13	{4, 7}	1	$1/c$	1	$1/c$	14	{5, 6}	1	1	$1/c'$	$1/c'$	15	{6, 7}	1	1	1	1
2	{1, 4}	1	$1/c$	c/a	$1/a$																																										
3	{1, 6}	1	1	$1/a$	$1/a$																																										
4	{1, 5}	1	c'/a	$1/c'$	$1/a$																																										
5	{1, 7}	1	$1/a$	1	$1/a$																																										
6	{1, 2}	1	$1/b$	b/a	$1/a$																																										
7	{2, 3}	1	$1/b$	$1/b'$	$1/bb'$																																										
8	{2, 5}	1	$1/b$	$1/c'$	$1/bc'$																																										

Table 1

To write down local monodromies we use Step 5 of our algorithm. The characters $e(\mu_1)$ are said to correspond to path I and the characters $e(\mu_2)$ correspond to path II. We do not need to write down a separate table for them since they are simply the second and third component of the vectors X_J .

As an example for the action of path I on a local basis we take the four local basis solutions with convergence direction $-0.5, 1$, as before. The transition matrix X_ρ consists of the vectors X_J with numbering 4, 5, 8, 9 in Table 1. The transition matrix reads

$$X_\rho = \begin{pmatrix} 1 & 1 & 1 & 1 \\ c'/a & 1/a & 1/b & 1/b \\ 1/c' & 1 & 1/c' & 1 \\ 1/a & 1/a & 1/bc' & 1/b \end{pmatrix}.$$

For the path I we get the monodromy matrix

$$X_\rho \begin{pmatrix} c'/a & 0 & 0 & 0 \\ 0 & 1/a & 0 & 0 \\ 0 & 0 & 1/b & 0 \\ 0 & 0 & 0 & 1/b \end{pmatrix} X_\rho^{-1}$$

which equals

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ (-1 + c')/ab & 1/b + 1/a + c'/a & c'/ab & -c'/a \\ 0 & 0 & 0 & 1 \\ -1/ab & 1/a & 0 & 1/b \end{pmatrix}$$

with respect to the Mellin–Barnes basis. For the path II we get

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/c' & 0 & 1 + 1/c' & 0 \\ 0 & -1/c' & 0 & 1 + 1/c' \end{pmatrix}.$$

The calculation so far has been carried out for the convergence direction $(-0.5, 1)$. In fact we get the same matrices for every convergence direction in the cone spanned by $\mathbf{b}_2, \mathbf{b}_5$ of the secondary fan. We can proceed in the same way with the other four cones. In each case we find two monodromy matrices. After removing duplicate matrices we end up with six monodromy matrices. They are given in the Appendix of this paper, together with a comparison of them with the five generators given by Kato in [20]. There it turns out that the group generated by our six generators is conjugate to the group computed in [20].

6.2. Clausen ${}_3F_2$. In this section we apply our method to the case of one variable

$${}_3F_2 \left(\begin{matrix} \alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2 \end{matrix} \middle| z \right)$$

which, up to a constant factor, is defined by the series

$$\sum_{n \geq 0} \frac{\Gamma(\alpha_1 + n)\Gamma(\alpha_2 + n)\Gamma(\alpha_3 + n)}{\Gamma(\beta_1 + n)\Gamma(\beta_2 + n)n!} z^n.$$

Using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z$ we see that the series is proportional to

$$\sum_n \frac{(-z)^n}{\Gamma(1-\alpha_1-n)\Gamma(1-\alpha_2-n)\Gamma(1-\alpha_3-n)\Gamma(\beta_1+n)\Gamma(\beta_2+n)\Gamma(1+n)}.$$

So the B -matrix is given by

$$B = (-1, -1, -1, 1, 1, 1)$$

and Z_B is simply the open interval $(-3/2, 3/2)$. In it we can take the three points $\tau_1 = -1$, $\tau_2 = 0$, $\tau_3 = 1$ and so we see that we have a Mellin–Barnes basis of solutions. For the set I_0 we take $\{6\}$ and

$$\delta = (-\alpha_1, -\alpha_2, \alpha_3, \beta_1, \beta_2, 0).$$

We consider the components modulo \mathbb{Z} . The set of columns of B has six subsets of cardinality 1 and the corresponding values of μ_1 are

$$\alpha_1, \alpha_2, \alpha_3, -\beta_1, -\beta_2, 0.$$

Letting $a_i = e(\alpha_i)$ and $b_j = e(\beta_j)$ we get for the vectors X_J ,

$$(1, a_1, a_1^2), \quad (1, a_2, a_2^2), \quad (1, a_3, a_3^2), \quad (1, b_1, b_1^2), \quad (1, b_2, b_2^2), \quad (1, 1, 1).$$

There is only one loop to consider for every local basis. Consider the convergence direction -1 . This lies in the positive cones spanned by $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ respectively. The transition matrix reads

$$X_\rho = \begin{pmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ a_1^2 & a_2^2 & a_3^2 \end{pmatrix}$$

and the diagonal character matrix

$$\chi_\rho = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}.$$

We get

$$X_\rho \chi_\rho X_\rho^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a_1 a_2 a_3 & -a_2 a_3 - a_2 a_1 - a_3 a_1 & a_1 + a_2 + a_3 \end{pmatrix}.$$

This is precisely the matrix representation for the monodromy matrix around $z = \infty$ for ${}_3F_2$ as given in [8]. We get a similar result for the monodromy matrix around $z = 0$ (with $b_1, b_2, 1$ instead of a_1, a_2, a_3)

7. Existence of Mellin–Barnes bases

In this section we show that certain families of hypergeometric equations satisfy Assumption 4.1, and some do not (the case of Lauricella F_C).

7.1. Lauricella F_A . The Lauricella system F_A in n variables is a system of rank 2^n . From the power series

$$F_A(a, \mathbf{b}, \mathbf{c}|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|} (\mathbf{b})_{\mathbf{m}}}{(\mathbf{c})_{\mathbf{m}} \mathbf{m}!} \mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \dots, x_n)$ we see that an $n \times (3n + 1)$ B -matrix is given by

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & & & & & \vdots & & & & & \vdots & & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

The B -zonotope is thus given by the points

$$\lambda(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + \sum_{i=1}^m \mu_i \mathbf{e}_i$$

where $|\lambda| < 1/4$ and $|\mu_i| < 3/4$ for $i = 1, \dots, n$. Let us choose $\epsilon > 0$ sufficiently small. Consider the 2^n points

$$\left(\frac{3}{4} - 2\epsilon\right)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) - k_1 \mathbf{e}_1 - \cdots - k_n \mathbf{e}_n$$

where $k_i \in \{0, 1\}$ for all i . Each such point equals

$$\left(\frac{1}{4} - \epsilon\right)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + \left(\frac{1}{2} - k_1 - \epsilon\right)\mathbf{e}_1 + \cdots + \left(\frac{1}{2} - k_n - \epsilon\right)\mathbf{e}_n$$

which is clearly contained in the B -zonotope.

We remark that Lauricella F_B has an A -polytope which is essentially the same as that of F_A . Therefore the B -zonotopes are congruent and the same conclusion follows for F_B .

7.2. Lauricella F_D . The Lauricella system F_D in n variables is a system of rank $n + 1$. From the power series

$$F_D(a, \mathbf{b}, c|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|} (\mathbf{b})_{\mathbf{m}}}{(c)_{|\mathbf{m}|} \mathbf{m}!} \mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \dots, x_n)$ we deduce an $n \times (2n + 2)$ B -matrix

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & -1 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & \ddots & & & & & & & \ddots & & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Hence the B -zonotope consists of the points

$$\lambda_0(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + \lambda_1\mathbf{e}_1 + \cdots + \lambda_n\mathbf{e}_n$$

where $|\lambda_i| < 1/2$ for $i = 0, 1, \dots, n$. Choose $\epsilon > 0$ sufficiently small and consider the $n + 1$ points

$$-\epsilon(n\mathbf{e}_1 + (n - 1)\mathbf{e}_2 + \cdots + 2\mathbf{e}_{n-1} + \mathbf{e}_n) + \sum_{i=0}^k \mathbf{e}_i$$

for $k = 0, 1, 2, \dots, n$. Each such point can be rewritten as

$$\left(\frac{1}{2} - \left(n - k - \frac{1}{2}\right)\epsilon\right)(\mathbf{e}_1 + \cdots + \mathbf{e}_n) + \sum_{j=1}^n \left(\pm\frac{1}{2} - \left(k - j - \frac{1}{2}\right)\epsilon\right)\mathbf{e}_j$$

where $\pm 1/2$ is $1/2$ if $k > j + 1/2$ and $-1/2$ if $k < j + 1/2$. Hence they are contained in the B -zonotope and we have found a Mellin–Barnes basis for Lauricella F_D .

7.3. Lauricella F_C . The Lauricella system F_C in n variables is system of rank 2^n . From the powerseries

$$F_C(a, b, \mathbf{c}|\mathbf{x}) = \sum_{\mathbf{m} \geq 0} \frac{(a)_{|\mathbf{m}|}(\mathbf{b})_{|\mathbf{m}|}}{(c)_{\mathbf{m}}\mathbf{m}!} \mathbf{x}^{\mathbf{m}}$$

in $\mathbf{x} = (x_1, \dots, x_n)$ we deduce an $n \times (2n + 2)$ B -matrix

$$\begin{pmatrix} 1 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 0 & -1 & 0 & \cdots & 0 & 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & & & & & \ddots & & & & & & \ddots & & \vdots \\ 1 & 1 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Note that the B -zonotope is the same as for Lauricella F_D , but this time we have to find a Mellin–Barnes basis of 2^n solutions. Clearly this is impossible if $n > 1$.

7.4. Aomoto–Gel’fand system $E(3, 6)$. This system forms the subject of the second part of Yoshida’s book [37]. It is an Aomoto system which can be reinterpreted as an A -hypergeometric system. It has four essential variables ($d = 4$) and rank 6 system that corresponds to configurations of six points (or lines) in \mathbb{P}^2 . We start by giving the A -matrix of the system,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

and the parameters $(\alpha_1, \alpha_2, 2 - \alpha_4, 2 - \alpha_5, 2 - \alpha_6)$. The integrand of the Euler integral as defined in [5, p. 607] reads

$$\frac{s^{\alpha_1-1} t^{\alpha_2-1} t_1^{1-\alpha_4} t_2^{1-\alpha_5} t_3^{1-\alpha_6}}{1 - t_1(s + t + 1) - t_2(s + v_1 t + v_3) - t_3(s + v_2 t + v_4)} ds \wedge dt \wedge dt_1 \wedge dt_2 \wedge dt_3.$$

Perform the substitutions

$$t_1 \rightarrow \frac{t_1}{s + t + 1}, \quad t_2 \rightarrow \frac{t_2}{s + v_1 t + v_3} \quad \text{and} \quad t_3 \rightarrow \frac{t_3}{s + v_2 t + v_4}.$$

We obtain the integrand

$$\frac{t_1^{1-\alpha_4} t_2^{1-\alpha_5} t_3^{1-\alpha_6}}{1 - t_1 - t_2 - t_3} \prod_{i=1}^6 (L_i)^{\alpha_i-1} ds \wedge dt \wedge dt_1 \wedge dt_2 \wedge dt_3$$

where

$$\begin{aligned} L_1 &= s, & L_2 &= t, & L_3 &= 1, \\ L_4 &= s + t + 1, & L_5 &= s + v_1 t + v_3, & L_6 &= s + v_2 t + v_4 \end{aligned}$$

and $\alpha_3 = 3 - \alpha_1 - \alpha_2 - \alpha_4 - \alpha_5 - \alpha_6$. Integration with respect to t_1, t_2, t_3 leaves us with a 2-form which is the integrand given in [37, p. 221], but with v_i instead of x^i . Thus we see that our A -matrix corresponds to a Aomoto–Gel’fand system which is associated to configurations of six lines in \mathbb{P}^2 . The system is a four variable system of rank 6. It is irreducible if and only if none of the α_i is an integer, see [24, Proposition 2]. A possible B -matrix reads

$$B = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^t.$$

The set \mathcal{I} consists of 81 sets, and hence 81 distinct local solutions. The number of local solution bases is equal to the number of regular triangulations, which is 108. In a straightforward manner one can check that the B -zonotope Z_B contains the points

$$p, \quad p + (0, 0, 0, 1), \quad p + (1, 0, 0, 0), \quad p + (1, 0, 1, 1), \quad p + (1, 1, 0, 1), \quad p + (1, 1, 1, 1)$$

where $p = (-0.9, -0.4, -0.5, -0.7)$. Hence the system $E(3, 6)$ has a Mellin–Barnes basis of solutions. Computation of a set of elements for the monodromy group is now straightforward. We get 82 matrices, but have not made an attempt to check whether they generate the group computed in [23], which is generated by twenty elements.

8. Hermitean forms

In the cases where we carried out the algorithm given above, it turns out that whenever $\alpha \in \mathbb{R}^r$, and the system is totally non-resonant, there exists a unique (up to a constant factor) hermitean form which is invariant under the group elements. Subsequent studies lead us to the following conjecture.

Conjecture 8.1. Let $H_A(\alpha)$ be a non-resonant A -hypergeometric system with $\alpha \in \mathbb{R}^r$. Then there exists a non-trivial unique (up to scalars) Hermitean form, invariant under the monodromy group. More concretely, there exist a Hermitean $D \times D$ -matrix H such that $\bar{g}^t H g = H$ for all elements g of the monodromy group. Here D denotes the rank of $H_A(\alpha)$.

Moreover, when the system is totally non-resonant, the signature of H is determined by the signs of the numbers $\prod_{i \notin I} \sin(\pi \gamma_i^I)$ as I runs through the elements of \mathcal{J}_ρ for some convergence direction ρ .

A detailed calculation shows that the signatures thus obtained are in accordance with the results on $E(3, 6)$ in [24, Proposition 1] (except for a small printing error). Note that signature $(5, 1)$ does not occur. Similarly, calculations for Lauricella F_D give us results which are in accordance with Picard [29], Terada [36] and Deligne–Mostow [10]. We like to come back to this conjecture in a future paper. The most likely approach is that via intersection forms on twisted cycles as in [17, 21]. The twisted cycles should then come from twisted period integrals as described in [12] and [5, Section 7] (make sure to take $\mathbf{t}^{-\alpha}$ instead of the erroneous \mathbf{t}^α).

A. Appendix

We reproduce the six matrices obtained from the monodromy calculation of Appell F_2 :

$$\begin{aligned}
 G_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(1+c)/ab' & c/ab' & 1/b' + 1/a + c/a & -c/a \\ -1/ab' & 0 & 1/a & 1/b' \end{pmatrix}, \\
 G_2 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b/ab' - 1/c' & b/c' & 1 + b/a + 1/c' & b(-1 + 1/b' - 1/c') \\ -1/ab' & 0 & 1/a & 1/b' \end{pmatrix}, \\
 G_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1/c' & 0 & 1 + 1/c' & 0 \\ 0 & -1/c' & 0 & 1 + 1/c' \end{pmatrix}, \\
 G_4 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -b'/ab - 1/c & 1 + b'/a + 1/c & b'/c & b'(-1 + 1/b - 1/c) \\ 0 & 0 & 0 & 1 \\ -1/ab & 1/a & 0 & 1/b \end{pmatrix}, \\
 G_5 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1/c & 1 + 1/c & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/c & 1 + 1/c \end{pmatrix}
 \end{aligned}$$

and

$$G_6 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(1+c')/ab & 1/b + 1/a + c'/a & c'/ab & -c'/a \\ 0 & 0 & 0 & 1 \\ -1/ab & 1/a & 0 & 1/b \end{pmatrix}.$$

Now let g_1, g_2, g_3, g_4, g_5 be the monodromy matrices defined in [20, formulas (2.7)–(2.11)] where our symbols a, b, b', c, c' are Kato's symbols $e(a), e(b), e(b'), e(c), e(c')$. Define the conjugation matrix

$$S = \begin{pmatrix} -1 & c & c' & -cc' \\ -1 & 1 & c' & -c' \\ -1 & c & 1 & -c \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

Then the relations between the G_i and g_j are given by

$$\begin{aligned} G_1 &= S^{-1}g_2g_3g_5S, & G_2 &= S^{-1}g_2g_5S, & G_3 &= S^{-1}g_2S, \\ G_4 &= S^{-1}g_1g_4S, & G_5 &= S^{-1}g_1S, & G_6 &= S^{-1}g_1g_3g_4S. \end{aligned}$$

From these relations it follows that the group we computed and the group computed in [20] are conjugate.

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