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Priorities and Preferences: A Study Based on Contraction

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Abstract

Preference models lie at the core of the formalization for several related notions, such as non-monotonic reasoning, obligations, goals, beliefs, etc. Recently, the interest in integrating dynamic operators in the logics of belief, preference and obligation has gained momentum. This integration sheds light on similarities among several change operations traditionally studied independently of each other. While a prolific approach, important operations, such as the well-known contraction of beliefs or derogation of norms studied in the AGM tradition, have not received proper attention in this framework. In this work, we study codifications of contraction operations, stemming from the work on iterated belief change, in the logic of preferences, by means of both semantically defined operations and their counterpart in syntactical priority structures.

1 Introduction

Preference models lie at the core of the formalization of several related notions, such as non-monotonic reasoning, obligations, goals, beliefs, etc. Logics for these concepts are now well-established in the literature. Particularly, dynamic preference logic introduced in (Girard 2008) has shown great flexibility to easily encode these notions, as well as actions representing change policies in an agent's mental state. Therefore, this logic has the potential to become a common framework to study several related notions in philosophy and artificial intelligence.

Recently, while investigating the theory for extrinsic and intrinsic preferences in the sense of (von Wright 1966), (Liu 2011) showed that agent's preferences can be equivalently represented by means of priorities, in the sense of justifications for preferences. The author argues that a theory that embraces both perspectives in modelling preferences is of both technical and philosophical interest, since it may represent both the agent explicit preferences and their justification. Liu calls such a theory that embraces the 'two levels' of preferences, the two-level perspective on preferences.

We agree with Liu that the two-level perspective is an advantageous approach, when compared to a reductionist one, i.e. one that aims to reduce one notion to the other. In this work, however, we are interested in a much more pragmatical consequence on the connection between intrinsic and extrinsic preferences. By relating justification structures, usually described by syntactical constructions as priority bases, and possible worlds models, Liu's two-level perspective provides an ideal framework for a computational theory of preferences. In that theory, we can combine the well studied model-theory by means of preference models with the computation-friendly justification structures.

While changes in mental attitudes have been a well studied topic in the literature, e.g. (Alchourrón, Gärdenfors, and Makinson 1985; Hansson 1995; Herzig et al. 2011), the integration of such operations within the logics of beliefs, obligations, etc. is a somewhat recent development. To our knowledge, the work of (Segerberg 1999) is the first to propose the integration of dynamic logic-like operations within the logic of Beliefs and Knowledge to represent the doxastic changes as studied in AGM tradition (Alchourrón, Gärdenfors, and Makinson 1985).

This shift from extra-logical characterization of changes in the agents attitudes to their integration within the representation language has important consequences for the expressive power of the language. It allows, for example, the study of dynamic phenomena not representable in axiomatic approach of the AGM framework. This is the case, for example, of the well-known Moore sentences in epistemology.

Recently, inspired by Van Benthem and the Dutch School on the "dynamic turn" in logic (Van Benthem 1996), several dynamic logics for information change and dynamics of mental attitudes have been proposed (Van Benthem 2007; Baltag and Smets 2008; Van Benthem, Girard, and Roy 2009; Liu 2011; Van Benthem, Pacuit, and Roy 2011). While different change operations have been proposed in these frameworks, these operations are mostly concerned with different policies to revise or update one's mental state. Other important operations such as contractions, however, have received far less attention in this blooming literature.

It is unquestionable that revision and update policies are at the core of both theoretic and operational concerns about changes in mental attitudes. We believe, however, that the retraction of information is also a central point in the study of dynamics of mental attitudes, as corroborated by the results on the relation between the operations of contraction and revision (Alchourrón, Gärdenfors, and Makinson 1985).

In this work, we propose a semantic codification of contraction, in the framework of (Girard 2008) of a dynamic logic of preferences and provide a syntactical counterpart for contraction in the framework of priority graphs of (Liu 2011). We do this by introducing dynamic modalities in the logic of (Girard 2008), representing three different contraction operations in the literature of Iterated Belief Revision (Ramachandran, Nayak, and Orgun 2012). Through the study of contraction in this logic, we show that not all operations on preference models definable in Propositional Dynamic Logic (PDL) can be characterized by syntactic operations on Liu's priority graphs. This is a negative answer to the question posed by (Liu 2011) corresponding the feasibility of Liu's two-level perspective for the dynamics of preferences - a somewhat surprising result, considering other important change operations proposed in the literature have been proved to be syntactically representable in this framework.

2 Related Work

The study of changes in mental attitudes has grown as in important field in both philosophy and artificial intelligence in the past decades. Perhaps, one of the most important contributions in this area is the seminal paper (Alchourrón, Gärdenfors, and Makinson 1985) that introduced the AGM paradigm for belief revision. The AGM initial work focused on defining the requirements for rational changes in the agents beliefs, which the authors claim to encode the notion of minimality, based on three different belief altering operations named expansion, contraction, and revision.

While a semantic characterization of AGM belief change has been established by (Grove 1988), the AGM approach and the vast literature based on it rely mainly on extralogical characterization of change operations by means. The first attempt to integrate belief change operation within a logic for Belief and Knowledge that we are aware of is the work of (Segerberg 1999). The author defines the dynamic doxastic logic (DDL) as a doxastic logic augmented with modalities representing the actions of revision and contraction in the agents mental state, based on Grove's semantics for these operations. This is a significant change regarding the expressibility of the theory of belief change, allowing the codification of phenomena such as Moore Sentences which cannot be expressed in the AGM framework. These new phenomena have posed challenges to the area, such as defying the consistency of some well-known AGM postulates (Alchourrón, Gärdenfors, and Makinson 1985) in this general setting.

Inspired by the connection between belief revision policies and transformations in priority structures, presented by (Rott 2009), Van Benthem embeds some belief change operations in the setting of dynamic epistemic logic (Van Benthem 2007). (Baltag and Smets 2008) consolidates this connection by providing a semantic codification of different epistemic and doxastic attitudes, such as Safe Belief, Knowledge, Conditional Belief, etc. and providing axiomatization for both the static and dynamic parts of this language. Finally, (Baltag, Fiutek, and Smets 2014) shows that unlimited DDL is expressively equivalent to dynamic epistemic logic, and thus, just another formalism to express the same phenomena.

Studying the logic of preferences, (Girard 2008) and (Van Benthem 2009) generalize the results of (Baltag and Smets 2008), presenting a logic for preferences and order, which was used to encode several different notions, such as Conditional Preferences, Beliefs, Obligations, Contrary-to-Duty reasoning etc. These works extend the dynamic epistemic logic approach to investigate change in several other mental attitudes defining changes in preference models by means of propositional dynamic logic (PDL) programs. The work of (Liu 2011), on the other hand, provides the connection between extrinsic and intrinsic preferences, in the sense of (von Wright 1966), showing that syntactical representations of priorities are equivalent to the semantic encoding of preferences of an agent by means of preference models. The author further shows that several change operations over possible-worlds models, as defined in the dynamic epistemic logic tradition, can be equivalently represented as syntactic transformations on these priority structures.

In this work, we extend the research in dynamic operators for logics of preferences and order by means of both syntactic and semantic representations. Differently than the related work, our analysis, however, will focus on the operation of contraction, which has been largely neglected in the area and was shown to provide problems in its codification - especially on a syntactical level (Girard 2008; Liu 2011).

3 Preference Logic

Preference logic is a modal logic complete for the class of transitive and reflexive frames. It has been applied to model a plethora of phenomena in deontic logic (Van Benthem, Grossi, and Liu 2014), logics of preference (Boutilier 1994b; Lang, Van Der Torre, and Weydert 2003), logics of belief (Boutilier 1994a; Baltag and Smets 2008), etc.

Dynamic preference logic, introduced in (Girard 2008; Van Benthem, Girard, and Roy 2009), is the result of "dynamifying" preference logic, i.e. extending it with dynamic modalities - usually represented by programs in propositional dynamic logic (PDL). This logic is interesting for its expressibility, allowing the study of dynamic phenomena of attitudes such as Beliefs, Obligations, Preferences etc. For that, dynamic preference logic has the potential to become a common framework to study different but correlated notions in the areas of epistemology, deontic logic and agency theory.

In this section, we will define the language of preference logic and will further introduce different actions in the logic to represent contraction operations, such as studied in Iterated Belief Change literature (Ramachandran, Nayak, and Orgun 2012).

Definition 1. *Let P be a finite set of atomic variables. We define the language $\mathscr{L}_{\leq}(P)$ by the following grammar (where $p \in P$):*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid A\varphi \mid [\leq]\varphi \mid [<]\varphi$$

If the set P is clear, we will often refer to the language $\mathscr{L}(P)$ simply as $\mathscr{L}$.

Definition 2. *A preference model is a tuple $M = \langle W, \leq, v\rangle$ where W is a set of possible worlds, $\leq$ is a well-preorder over W, i.e. a reflexive, transitive and well-founded*[1] *relation over W, and v a valuation function.*

A preference model, informally, represents an ordering of the possible states of affairs according to the preferences of some agent, i.e. how such an agent would prefer the actual world to be. As such, we say that a state of affairs w is at least as preferred as w' if $w \leq w'$.

The interpretation of the formulas is defined as usual. We will only present the interpretations for the modalities, since the semantics of the propositional connectives is clear. The A modality is an universal modality satisfied iff all worlds in the model satisfy its argument. The $[\leq]$ modality is a box modality on the accessibility order $\leq$. The $[<]$ modality is the strict variant of $[\leq]$. They are interpreted as:

$$\begin{array}{lll} M,w \vDash A\varphi & \text{iff} & \forall w' \in W. M,w' \vDash \varphi \\ M,w \vDash [\leq]\varphi & \text{iff} & \forall w' \in W. w' \leq w \Rightarrow M,w' \vDash \varphi \\ M,w \vDash [<]\varphi & \text{iff} & \forall w' \in W. w' < w \Rightarrow M,w' \vDash \varphi \end{array}$$

As usual, given a model M and a formula φ, we will use the notation $[\![\varphi]\!]_M$ to denote the set of all the worlds in M satisfying φ. When it is clear to which model we are referring to, we will denote the same set by $[\![\varphi]\!]$. Also, given a set of worlds $[\![\varphi]\!]$ and a (pre-)order $\leq$, we will denote the minimal elements of $[\![\varphi]\!]$, according to $\leq$, by the notation $Min_{\leq}[\![\varphi]\!]$. This corresponds to the notion of 'best worlds satisfying φ' in the model.

An axiomatization for the logic is provided by the axiom schemata depicted in Figure1 based on the work of (Van Eijck 2008; Van Benthem, Girard, and Roy 2009; Liu 2011; Girard and Rott 2014).

Notice that (Girard and Rott 2014) proposed a similar axiomatization for a logic with modalities $[\leq], [<]$ and an additional epistemic modality $[\sim]$, were $a \sim b$ iff $a \leq b$ or $b \leq a$. The authors showed that some contraction operations can only be consistently defined over well-founded models and conjectured that the use of Löb Axiom in their axiomatization would suffice to guarantee well-foundedness but no completeness proof was presented.

In what follows, we give the completeness proof by showing that any mono-modal model for which the accessibility relation $\leq$ satisfies reflexivity and transitivity and its strict part $<$ satisfies Löb Axiom is well-founded. The proof is fairly easy and we use an equivalent formulation of Löb Axiom with possibilities instead of necessities:

$$\mathbf{L}': \quad \langle<\rangle\varphi \rightarrow \langle<\rangle(\varphi \wedge \neg\langle<\rangle\varphi)$$

Lemma 3. *Let $M = \langle W, \leq, v\rangle$ be a reflexive and transitive model and $\varphi \in \mathscr{L}_{\leq}(P)$, s.t. $[\![\varphi]\!]_M \neq \emptyset$. Then, for any $w \in W$: $M,w \vDash \langle<\rangle\varphi \rightarrow \langle<\rangle(\varphi \wedge \neg\langle<\rangle\varphi)$ iff $Min_{\leq}[\![\varphi]\!]_M \neq \emptyset$.*

Proof. $\Rightarrow$:
Since $[\![\varphi]\!] \neq \emptyset$, take $w \in [\![\varphi]\!]$. Either w is minimal and thus $Min_{\leq}[\![\varphi]\!] \neq \emptyset$ or $M,w \vDash \langle<\rangle\varphi$. Since, by our hypothesis w satisfies Löb Axiom for φ, then $M,w \vDash \langle<\rangle(\varphi \wedge \neg\langle<\rangle\varphi)$, thus there is a world $w' \in W$ s.t. $w' \in [\![\varphi]\!]$ and w' is minimal in this set.
$\Leftarrow$:
Take some world $w \in W$. The interesting case here is when $M,w \vDash \langle<\rangle\varphi$ holds, since Löb Axiom would hold vacuously otherwise. Suppose $M,w \vDash \langle<\rangle\varphi$, then there is some $w' \in W$, s.t. $w' < w$ and $M,w' \vDash \varphi$. Take $w'' \in Min_{\leq}[\![\varphi]\!] \neq \emptyset$. By minimality, $w'' \leq w'$ and $M,w'' \vDash \varphi \wedge \neg\langle<\rangle\varphi$. By transitivity $w'' < w$, thus $M,w \vDash \langle<\rangle(\varphi \wedge \neg\langle<\rangle\varphi)$. □

Corollary 4. *The logic depicted in Figure 1, together with the rules of modus ponens and necessitation for all three modalities, is sound and complete for the class frames with a well-founded pre-order relation $\leq$.*

$$\begin{array}{ll} \mathbf{K}_{\leq}: & [\leq](\varphi \rightarrow \psi) \rightarrow ([\leq]\varphi \rightarrow [\leq]\psi) \\ \mathbf{T}_{\leq}: & [\leq]\varphi \rightarrow \varphi \\ \mathbf{4}_{\leq}: & [\leq]\varphi \rightarrow [\leq][\leq]\varphi \\ & \\ \mathbf{K}_{<}: & [<](\varphi \rightarrow \psi) \rightarrow ([<]\varphi \rightarrow [<]\psi) \\ \mathbf{W}_{<}: & [<]([<]\varphi \rightarrow \varphi) \rightarrow [<]\varphi \\ <\leq_1: & [\leq]\varphi \rightarrow [<]\varphi \\ <\leq_2: & [<]\varphi \rightarrow [<][\leq]\varphi \\ <\leq_3: & [<]\varphi \rightarrow [\leq][<]\varphi \\ & \\ \mathbf{K}_A: & A(\varphi \rightarrow \psi) \rightarrow (A\varphi \rightarrow A\psi) \\ \mathbf{T}_A: & A\varphi \rightarrow \varphi \\ \mathbf{4}_A: & A\varphi \rightarrow AA\varphi \\ \mathbf{B}_A: & \varphi \rightarrow A\neg A\neg\varphi \\ A\leq: & A\varphi \rightarrow [\leq]\varphi \end{array}$$

Figure 1: Axiomatization of Preference Logic

Since the concept of best worlds satisfying a given formula φ will be of great use in this work, we define a formula encompassing this exact concept.

Definition 5. *We define the formula $\mu\varphi \leftrightarrow \varphi \wedge \neg\langle<\rangle\varphi$, that is satisfied by exactly the most preferred worlds satisfying φ, i.e. $[\![\mu\varphi]\!]_M = Min_{\leq}[\![\varphi]\!]_M$.*

Notice that, since our logic is complete for well-founded preference models, the formula $\mu\varphi$ is unsatisfiable iff so is φ. Now that the language and semantics of the logic have been established, we proceed to extend it with dynamic operations for representing different contraction operators from the literature.

4 Introducing Contraction in Preference Logic

In this section, we will investigate the addition of different contraction operations to preference logic by means of transformations in the preference relation of a model. This operation will be further defined as a PDL program and, from that, we provide an axiomatization for the resulting logic. We will

[1]The relation $\leq$ is said well-founded if any non-empty subset set of W has minimal elements.

investigate in this work three different contraction operations from the literature in Iterated Belief Contraction: Natural Contraction, Moderate Contraction and Lexicographic Contraction. These operations are three of the best known operators in the literature, as well as the ones satisfying some important properties, such as the (generalized) Levi Identity and the Principled Factored Insertion (Ramachandran, Nayak, and Orgun 2012).

Natural Contraction Natural contraction is a dual operation for the natural revision proposed by (Boutilier 1993), a well-known iterated revision operation in the literature. Given a plausibility relation $\leq$ over possible worlds, its natural contraction by a sentence φ, represented by $\leq_{\varphi}^{-}$, is defined in (Ramachandran, Nayak, and Orgun 2012) by the following axioms:

NC1: If $w_1 \in Min_{\leq}W$ or $w_1 \in Min_{\leq}[\![\neg\varphi]\!]$, then $w_1 \leq_{\varphi}^{-} w_2$ for any $w_2 \in W$.

NC2: If $w_1, w_2 \notin Min_{\leq}W$ and $w_1, w_2 \notin Min_{\leq}[\![\neg\varphi]\!]$ then $w_1 \leq_{\varphi}^{-} w_2$ if and only if $w_1 \leq w_2$.

NC1 says that the best worlds that do not satisfy φ are promoted to the most plausible, i.e. best, worlds after the contraction by φ and NC2 states that the plausibility of all other worlds in the model is unaffected by natural contraction. We now define this operation as a transformation on preference models. The following example shows how the natural contraction changes a preference relation.

Example 6. *Contracting the preference relation $p \wedge q < p \wedge \neg q < \neg p \wedge q < \neg p \wedge \neg q$ by the sentence p results in the preference $p \wedge q \sim \neg p \wedge q < p \wedge \neg q < \neg p \wedge \neg q$, where $a \sim b$ stands for $a \leq b \wedge b \leq a$.*

Definition 7. *Let $M = \langle W, \leq, v\rangle$ be a preference model and φ a formula of $\mathscr{L}$. We say the model $M -^{N} \varphi = \langle W, \leq', v\rangle$ is natural contraction of M by φ, where:*

$$w \leq' w' \text{ iff } \begin{cases} w \in Min_{\leq}W \text{ or} \\ w \in Min_{\leq}[\![\neg\varphi]\!]_M \text{ or} \\ w \leq w' \text{ and } w \notin Min_{\leq}[\![\neg\varphi]\!]_M \end{cases}$$

It is clear that the class of reflexive, transitive and well-founded frames is closed under the above transformation, i.e. natural contraction preserves both preference models. The reflexivity is derived by the fact that the order is preserved within the sets $[\![\varphi]\!]_M$ and $[\![\neg\varphi]\!]_M$. Transitivity follows by the fact only the minimal worlds in $[\![\varphi]\!]_M$ have their positions in the preference relation altered and they become minimal. Well-foundedness follows from the fact that no infinite descending chain is created.

We can now introduce new modalities $[-^{N}\varphi]$ in the logic representing the contraction of the model by φ.

Definition 8. *Let $M = \langle W, \leq, v\rangle$ be a preference model, $w \in W$ and φ a formula of $\mathscr{L}$*

$$M, w \vDash [-^{N}\varphi]\psi \quad \text{iff} \quad M -^{N} \varphi, w \vDash \psi$$

In order to provide an axiomatization for preference logic augmented with a natural contraction operation, we will encode the semantic transformation defined above in a PDL program.

Natural contraction can be equivalently defined by the following PDL program:

$$\leq := (?\mu\top;\leq) \cup (?\mu\neg\varphi;\top) \cup (\leq;?\neg\mu\neg\varphi) \qquad (1)$$

Fact 9. *Let $M = \langle W, \leq, v\rangle$ be a preference model. The relation $\leq_{\varphi}^{-}$ is a natural contraction of $\leq$ by a propositional sentence φ, i.e. satisfies axioms NC1 and NC2, iff $\leq_{\varphi}^{-}$ is the result of computing the PDL program (1) over the relation $\leq$.*

Proof. $(\Rightarrow)$: Take $w, w' \in W$, s.t. $w \leq_{\varphi}^{-} w'$. We have three cases to consider: (i) $w \in [\![\mu\top]\!]$ (ii)$w \in [\![\mu\neg\varphi]\!]$ and (iii) $w, w' \notin [\![\mu\top]\!] \cup [\![\mu\neg\varphi]\!]$.

(i) Since $w \in [\![\mu\top]\!]$ implies that $w \in Min_{\leq}W$, then for any w', $w \leq w'$. As such, by the term $(?\mu\top;\leq)$ of (1), we have that $\langle w, w'\rangle$ is in the relation resulting of applying (1) to $\leq$.

(ii) By the term $(?\mu\neg\varphi;\top)$ of (1), we have that $\langle w, w'\rangle$ is in the relation resulting of applying (1) to $\leq$.

(iii) By NC2, $w \leq w'$. Thus, since $w' \notin [\![\neg\varphi]\!]$, by the term $(\leq;?\neg\mu\neg\varphi)$, we have that $\langle w, w'\rangle$ is in the relation resulting of applying (1) to $\leq$.

$(\Leftarrow)$: Similar to the case above. Take the relation $\leq'$ as the result of applying (1) to $\leq$ and two worlds w, w' s.t. $w \leq' w'$. The terms $(?\mu\top;\leq)$ and $(?\mu\neg\varphi;\top)$ imply NC1, while the term $(\leq;?\neg\mu\neg\varphi)$ implies NC2. ☐

From this PDL representation we can easily provide an axiomatization for the logic extended with the $[-^{N}.]$ modality, by traditional reduction techniques for PDL. Particularly, we will need the following result due to (Van Benthem and Liu 2007).

Fact 10. *Every relation-changing operation that is definable in PDL without iteration has a complete set of reduction axioms in dynamic epistemic logic.*

The reduction axioms are obtained by rewriting a formula $[-^{N}\varphi][\leq]\psi$ as to push the dynamic modality, i.e. $[-^{N}\varphi]$, by exploring the program structure. Let, thus the operation σ be defined by a PDL program $\pi(R)$ using only tests, i.e. $?\varphi$ programs, composition, union and the relation R, the following equivalence is valid.

$$[\sigma][R]\varphi \leftrightarrow [\pi(R)][\sigma]\varphi$$

In the following, we use this equivalence to provide the reduction axioms for the preference logic augmented with natural contraction.

Proposition 11. *Preference logic with natural contraction is (soundly and completely) axiomatized by the axioms below added to the axioms provided in Figure 1 for preference logic and the rules of modus ponens and necessitation for all modalities.*

1. $\vDash [-^{N}\varphi]p \leftrightarrow p$
2. $\vDash [-^{N}\varphi]\neg\psi \leftrightarrow \neg[-^{N}\varphi]\psi$
3. $\vDash [-^{N}\varphi](\xi \wedge \psi) \leftrightarrow [-^{N}\varphi]\xi \wedge [-^{N}\varphi]\psi$
4. $\vDash [-^{N}\varphi]A\psi \leftrightarrow A[-^{N}\varphi]\psi$
5. $\vDash [-^{N}\varphi][\leq]\psi \leftrightarrow (\mu\top \rightarrow [\leq][-^{N}\varphi]\psi) \wedge (\mu\neg\varphi \rightarrow A([-^{N}\varphi]\psi)) \wedge [\leq](\neg\mu\neg\varphi \rightarrow [-^{N}\varphi]\psi)$

6. $\models\ [-^N\varphi][<]\psi \ \leftrightarrow\ (\mu\top \to [<][-^N\varphi]\psi)\wedge (\mu\neg\varphi \to A([-^N\varphi]\psi))\wedge [<](\neg\mu\neg\varphi \to [-^N\varphi]\psi)$

Sketch of the Proof. Axioms 1-4 are the usual PDL non-modal formulas. Axioms 5 and 6 above represent the description of the change in accessibility relations $\leq$ and $<$ after the application of the PDL program described in (1). Thus, in Axiom 5, we have that

$$[-^N\varphi][\leq]\psi \leftrightarrow [(?\mu\top;\leq)\cup(?\mu\neg\varphi;\top)\cup(\leq;?\neg\mu\neg\varphi)][-^N\varphi]\psi$$

and, through application of usual PDL axioms $[\pi\cup\sigma]\varphi \leftrightarrow [\pi]\varphi\wedge[\sigma]\varphi$, $[\pi;\sigma]\varphi \leftrightarrow [\pi][\sigma]\varphi$ and $[?\varphi]\psi \leftrightarrow \varphi\to\psi$, we obtain the described equivalence. □

Moderate Contraction As done for natural contraction, we will extend the language of preference logic to include a new modality $[-^M\varphi]$ representing the operation of contracting a formula φ of the model by means of moderate contraction.

Moderate contraction of a plausibility relation $\leq$ over possible worlds by a sentence φ, represented by $\leq^-_\varphi$, is defined in (Ramachandran, Nayak, and Orgun 2012) by the following axioms:

MC1: If $w_1 \models \varphi$ and $w_2 \models \varphi$ then $w_1 \leq^-_\varphi w_2$ if and only if $w_1 \leq w_2$.

MC2: If $w_1 \models \neg\varphi$ and $w_2 \models \neg\varphi$ then $w_1 \leq^-_\varphi w_2$ if and only if $w_1 \leq w_2$.

MC3: If $w_2 \models \varphi$, $w_1 \notin Min_{\leq}W$ and $w_1 \models \neg\varphi$ then $w_1 <^-_\varphi w_2$.

MC4: If $w_1 \in Min_{\leq}W$ or $w_1 \in Min_{\leq}[\![\neg\varphi]\!]$ then $w_1 \leq^-_\varphi w_2$, for any w_2.

MC1 and MC2 require that within the sets $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$ the order is preserved; MC3 requires that the worlds not satisfying φ are more plausible than any world satisfying φ that is not minimal in the original plausibility relation. Finally MC4 says that after the contraction the minimal elements of the resulting plausibility relation are exactly the original minimal elements plus the minimal elements in $[\![\neg\varphi]\!]$. As before we define moderate contraction as a transformation on models. The following example illustrates how moderate contraction changes a preference relation.

Example 12. *Contracting the preference relation $p\wedge q < p\wedge\neg q < \neg p\wedge q < \neg p\wedge\neg q$ by the sentence p results in the preference $p\wedge q \sim \neg p\wedge q < \neg p\wedge\neg q < p\wedge\neg q$.*

Definition 13. *Let $M = \langle W,\leq,v\rangle$ be a preference model and φ a formula of $\mathcal{L}$. We say the model $M -^M \varphi = \langle W,\leq',v\rangle$ is the moderate contraction of M by φ, where:*

$$w \leq' w' \text{ iff } \begin{cases} w \leq w' \text{ and } w,w' \in [\![\varphi]\!]_M \text{ or } w,w' \in [\![\neg\varphi]\!]_M \text{ or} \\ w \in [\![\neg\varphi]\!]_M, w' \in [\![\varphi]\!]_M \text{ and } w' \notin Min_{\leq}W \text{ or} \\ w \in Min_{\leq}W \text{ or} \\ w \in Min_{\leq}[\![\neg\varphi]\!]_M \end{cases}$$

As before, it is easy to see that the above transformation preserves preference models.

We can now introduce in the language of preference logic, the modality $[-^M\varphi]$, as done for natural contraction.

Definition 14. *Let $M = \langle W,\leq,v\rangle$ be a preference model, $w\in W$ and φ a formula of $\mathcal{L}$*

$$M,w \models [-^M\varphi]\psi \qquad \textit{iff} \qquad M -^M \varphi, w \models \psi$$

As before, we encode this transformation in a PDL program in order to provide an axiomatization for the augmented logic.

This operation can be equivalently represented by the following PDL program:

$$\leq := \leq_\varphi \cup(?\neg\varphi;\top;?\varphi\wedge\neg\mu\top)\cup(?\mu\top;\leq)\cup(?\mu\neg\varphi;\top)(2)$$

where

$$\leq_\varphi = (?\varphi;\leq;?\varphi)\cup(?\neg\varphi;\leq;?\neg\varphi)$$

Fact 15. *Let $M = \langle W,\leq,v\rangle$ be a preference model. The relation $\leq^-_\varphi$ is a natural contraction of $\leq$ by a propositional sentence φ, i.e. satisfies axioms MC1, MC2, MC3 and MC4, iff $\leq^-_\varphi$ is the result of computing the PDL program (2) over the relation $\leq$.*

The above program states that the preference order $\leq$ after the contraction is composed of the pairs $\langle w,w'\rangle$ s.t. $w,w' \in [\![\varphi]\!]$ or $w,w' \in [\![\neg\varphi]\!]$ and $w \leq w'$, or $w \in [\![\neg\varphi]\!]$ and $w' \in [\![\varphi \wedge \neg\mu\top]\!]$, i.e. w' is a non-minimal world satisfying φ, or $w \in [\![?\mu\top]\!]$ and $w \leq w'$, or $w \in [\![?\mu\neg\varphi]\!]$, i.e. w is a minimal world in $[\![\neg\varphi]\!]$.

As for the case of Natural Contraction (see Proposition 11), with the above PDL program, we can obtain an axiomatization for the preference logic augmented with a moderate contraction modality. For space concerns, we will omit the resulting axiomatization, pointing out that only axioms 5 and 6 are changed - as before, they can be obtained by the PDL reduction $[R := def(R)][R]\varphi \leftrightarrow [def(R)][R := def(R)]\varphi$, where R is either the relation $\leq$ or $<$, and $def(R)$ the PDL program (2).

Lexicographic Contraction Lexicographic contraction was introduced by (Nayak et al. 2006), as a product of the generalization of the well-known Harper identity (Alchourrón, Gärdenfors, and Makinson 1985). Despite the desirable properties it satisfies, such as the already mentioned generalized Harper and Levi identities and Principled Factored Insertion (Ramachandran, Nayak, and Orgun 2012), this operation presents difficulties in characterization as will be evident in Proposition 23 below. This is because the operation is defined by means of complete chains of worlds in a model - which encodes a great deal of information about the preference relation.

Lexicographic contraction of a plausibility relation $\leq$ over possible worlds by a sentence φ, represented by $\leq^-_\varphi$, is defined in (Ramachandran, Nayak, and Orgun 2012) by the following axioms:

LC1: If $w_1 \models \varphi$ and $w_2 \models \varphi$ then $w_1 \leq^-_\varphi w_2$ if and only if $w_1 \leq w_2$.

LC2: If $w_1 \models \neg\varphi$ and $w_2 \models \neg\varphi$ then $w_1 \leq^-_\varphi w_2$ if and only if $w_1 \leq w_2$.

LC3: Let ξ be a member of $\{\varphi,\neg\varphi\}$ and $\overline{\xi}$ the other. If $w_1 \models \xi$ and $w_2 \models \overline{\xi}$, then $w_1 \leq^-_\varphi w_2$ iff the length of a complete chain of worlds in $[\![\xi]\!]$ which ends in w_1 is less than or equal to the length of a complete chain of worlds in $[\![\bar{\xi}]\!]$ which ends in w_2.

As for the axioms MC1 and MC2, the axioms LC1 and LC2 require order preservation in $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$. LC3 requires that the plausibility relation will be computed by lexicographically unifying the equivalence classes in $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$, regarding the plausibility relation $\leq$.

Example 16. *Contracting the preference relation $p \wedge q < p \wedge \neg q < \neg p \wedge q < \neg p \wedge \neg q$ by the sentence p results in the preference $p \wedge q \sim \neg p \wedge q < p \wedge \neg q \sim \neg p \wedge \neg q$.*

Notice that to specify lexicographic contraction we need a way to represent that there is a chain of worlds in $[\![\varphi]\!]$ of length i ending in w. We will define a formula $dg_\varphi(i)$ below to represent this notion.

Definition 17. *Let $M = \langle W, \leq, v\rangle$ be a preference model and φ a formula of $\mathscr{L}$. We define the formula $dg_\varphi(i)$ as*

$$dg_\varphi(i) = \begin{cases} \varphi \wedge \neg\langle<\rangle\varphi & \text{if } i = 0 \\ \varphi \wedge \langle<\rangle\varphi & \text{if } i = 1 \\ \varphi \wedge \langle<\rangle dg_\varphi(i-1) & \text{if } i \geq 2 \end{cases}$$

It is easy to see that $dg_\varphi(i)$ encodes the notion of existence of a chain of worlds ending in w.

Lemma 18. *Let $M = \langle W, \leq, v\rangle$ be a preference model and φ a formula of $\mathscr{L}$ and $w \in W$. $M, w \vDash dg_\varphi(i), i > 0$ iff there is a chain of worlds of $w_1, w_2 \cdots, w_i$ such that, $w_j \in [\![\varphi]\!]$, for all j, and $w_i = w$.*

The lemma can be proved by a simple induction on the parameter i, noticing that $<$ is a transitive relation.

Particularly, if $\varphi = \top$ this formula encodes the notion of the degree of a world - similar to that of (Spohn 1988)[2].

With that, we can define lexicographic contraction as a model transformation.

Definition 19. *Let $M = \langle W, \leq, v\rangle$ be a preference model and φ a formula of $\mathscr{L}$. We say the model $M -^L \varphi = \langle W, \leq', v\rangle$ is the lexicographic contraction of M by φ, where:*

$$w \leq' w' \text{ iff } \begin{cases} w \leq w' & \text{if } w, w' \in [\![\varphi]\!] \\ & \text{ or } w, w' \notin [\![\varphi]\!] \\ w \in [\![dg_\varphi(i)]\!] \Rightarrow w' \in [\![dg_{\neg\varphi}(i)]\!] & \text{if } M, w \vDash \varphi \text{ and} \\ \text{for all } i \leq |W| & M, w \nvDash \varphi \\ w \in [\![dg_{\neg\varphi}(i)]\!] \Rightarrow w \in [\![dg_\varphi(i)]\!] & \text{if } M, w \vDash \neg\varphi \text{ and} \\ \text{for all } i \leq |W| & M, w \vDash \varphi \end{cases}$$

By similar arguments as for natural and moderate contraction, it is not difficult to see that lexicographic contraction preserves preference models. Then, we can now include the operation as a modality in the language of preference logic.

Definition 20. *Let $M = \langle W, \leq, v\rangle$ be a preference model, $w \in W$, and φ a formula of $\mathscr{L}$*

$$M, w \vDash [-^L\varphi]\psi \qquad \text{iff} \qquad M - \varphi, w \vDash \psi$$

[2]Notice that only the maximal i s.t. a world satisfies $M, w \vDash dg_\top(i)$ can be thought as the degree of world w as in the framework of Spohn's OCFs, since by transitivity $M, w \vDash dg_\top(j)$ for all $1 \leq j \leq i$.

It is not difficult to see that our preference logic satisfies the Finite Model Property, since $\leq$ is S4, $<$ is KW and $< \ \subseteq \ \leq$. It is also well known that for S4, any formula φ has a bound on the maximal size of a model that can refute φ. This bound, $\rho(\varphi)$, is dependent on the size of formula φ: $\rho(\varphi) = 2^{|\varphi|}$ (Halpern and Moses 1992). We will use this fact to provide the following PDL codification of Lexicographic Contraction:

$$\begin{aligned} \leq := \quad & (?\varphi;\leq;?\varphi) \cup (?\neg\varphi;\leq;?\neg\varphi) \cup \\ & \bigcup_{i=1}^{\rho(\varphi\wedge\psi)} \bigcup_{j=i}^{\rho(\varphi\wedge\psi)} \Big(((?\varphi;\overline{<};?\varphi)^i;\top;(?\neg\varphi;<;?\neg\varphi)^j) \cup \\ & \qquad ((?\neg\varphi;\overline{<};?\neg\varphi)^i;\top;(?\varphi;<;?\varphi)^j) \Big) \end{aligned} \tag{3}$$

Notice that the use of the converse relation $\overline{<}$ equivalent to $>$ in the formula above does not require greater expressibility of the logic, since it has been shown that this operator can be eliminated in PDL (De Giacomo 1996). Also, while lexicographic contraction does not explicitly requires well-foundedness of the model to be well defined, the Finite Model Property of S4 is necessary for the PDL codification. This is not a severe restriction regarding the logic, since the presented axiomatization for the static logic is complete for both well-founded pre-orders and finite pre-orders (Blackburn, van Benthem, and Wolter 2006).

While the inclusion of the inverse relation $\overline{<}$ adds no expressibility to the logic and can be eliminated, the inclusion of an inverse modality $[>]$ greatly simplifies the axiomatization of Lexicographic Contraction. This is due to the fact that a simple expression can encode the meaning of the term $(?\neg\varphi;\overline{<};?\neg\varphi)^i$ in the PDL expression above, much like the expression $dg_\varphi(i)$ defined before. We will, then, include a modality $[>]$ and define the inverse degree of a world.

Definition 21. *Let $M = \langle W, \leq, v\rangle$ be a preference model and φ a formula of $\mathscr{L}$. We define the formula $dg^>_\varphi(i)$ as*

$$dg^>_\varphi(i) = \begin{cases} \varphi \wedge \neg\langle>\rangle\varphi & \text{if } i = 0 \\ \varphi \wedge \langle>\rangle\varphi & \text{if } i = 1 \\ \varphi \wedge \langle>\rangle dg^>_\varphi(i-1) & \text{if } i \geq 2 \end{cases}$$

As before, this encoding is a correct description of descending chains in preference models.

Lemma 22. *Let $M = \langle W, \leq, v\rangle$ be a preference model and φ a formula of $\mathscr{L}$ and $w \in W$. $M, w \vDash dg^>_\varphi(i), i > 0$ iff there is a chain of worlds of $w_1, w_2 \cdots, w_i$ such that, $w_j \in [\![\varphi]\!]$ for all j and $w_1 = w$.*

With that, we provide the axiomatization for preference logic with Lexicographic Contraction.

Proposition 23. *Preference logic with lexicographic contraction is (soundly and completely) axiomatized by the axioms below added to the axioms provided in Figure 1 for preference logic and the rules of modus ponens and necessitation for all modalities.*

1. $\vDash [>](\varphi \to \psi) \ \to \ ([>]\varphi \to [>]\psi)$
2. $\vDash \varphi \ \to \ [>]\langle<\rangle\varphi$
3. $\vDash \varphi \ \to \ [<]\langle>\rangle\varphi$

4. $\vDash [-^L\varphi]p \leftrightarrow p$
5. $\vDash [-^L\varphi]\neg\psi \leftrightarrow \neg[-^L\varphi]\psi$
6. $\vDash [-^L\varphi](\xi \wedge \psi) \leftrightarrow [-^L\varphi]\xi \wedge [-^L\varphi]\psi$
7. $\vDash [-^L\varphi]A\psi \leftrightarrow A[-^L\varphi]\psi$
8. $\vDash [-^L\varphi][\leq]\psi \quad \leftrightarrow$
   $(\varphi \to [\leq](\varphi \to [-^L\varphi]\psi))\wedge$
   $(\neg\varphi \to [\leq](\neg\varphi \to [-^L\varphi]\psi))\wedge$
   $\bigwedge_{i=1}^{\rho(\varphi\wedge\psi)} \bigwedge_{j=i}^{\rho(\varphi\wedge\psi)} dg^{>}_{\varphi}(i) \to A(dg_{\neg\varphi}(j) \to [-^L\varphi]\psi)\wedge$
   $\bigwedge_{i=1}^{\rho(\varphi\wedge\psi)} \bigwedge_{j=i}^{\rho(\varphi\wedge\psi)} dg^{>}_{\neg\varphi(i)} \to A(dg_{\varphi}(j) \to [-^L\varphi]\psi)$
9. $\vDash [-^L\varphi][<]\psi \quad \leftrightarrow$
   $(\varphi \to [<](\varphi \to [-^L\varphi]\psi))\wedge$
   $(\neg\varphi \to [<](\neg\varphi \to [-^L\varphi]\psi))\wedge$
   $\bigwedge_{i=1}^{\rho(\varphi\wedge\psi)} \bigwedge_{j=i}^{\rho(\varphi\wedge\psi)} dg^{>}_{\varphi}(i) \to A(dg_{\neg\varphi}(j) \to [-^L\varphi]\psi)\wedge$
   $\bigwedge_{i=1}^{\rho(\varphi\wedge\psi)} \bigvee_{j=i}^{\rho(\varphi\wedge\psi)} dg^{>}_{\neg\varphi(i)} \to A(dg_{\varphi}(j) \to [-^L\varphi]\psi)$

Sketch of the proof. Since we included the modality $[>]$ in the logic, the axiomatization of preference logic had to be extended with axioms 1-3, corresponding to the K axiom for the modality $[>]$ and the axioms expressing that the modalities $[>]$ and $[<]$ are the converse of one another, respectively. As before, the axioms 4-6 are the usual axioms for PDL programs over propositional formulas and axiom 7 for the universal modality. Axioms 10 and 11 are the reductions obtained by the PDL program in (3), using the formulas $dg_{\varphi}(i)$ and $dg^{>}_{\varphi}(i)$ to encode the programs $(?\varphi;<;?\varphi)^i$ and $(?\varphi;>;?\varphi)^i$, respectively. As Lemmas 18 and 22 show, this is a valid encoding, since the terms $(?\varphi;<;?\varphi)^i$ and $(?\varphi;>;?\varphi)^i$ encode the existence of ascending and descending chains reaching a given world. □

Now that we have a semantic encoding of the selected contraction operations, we turn to the search a suitable syntactical codification for them. For this enterprise, we will use the framework of priority graphs of (Liu 2011), which is a syntactical representation of preferences expressibly equivalent to preference models, concerning the logic presented in Section 3. We will investigate syntactical transformations in these structures aiming to characterise the previously studied contraction operations.

5 Changing preferences by shifting priorities

An interesting connection has been proposed by (Liu 2011) between preference models and the finite syntactic-based representations of priorities. The author uses the notion of a priority graph as a syntactic representation of preferences and connects its static and dynamic properties with that of the semantic models discussed earlier.

Definition 24. *(Van Benthem, Grossi, and Liu 2014) Let $\mathscr{L}(P)$ be the propositional language constructed over the set of propositional letters P, as usual. A P-graph is a tuple $\mathscr{G} = \langle\Phi,\prec\rangle$ where $\Phi \subset \mathscr{L}(P)$, is a finite set of propositional sentences and $\prec$ is a strict partial order on Φ.*

P-graphs encode an order relation between propositional sentences. Unsurprisingly, given a valuation function of propositions over a set of possible worlds, such an ordering can be lifted to an order over worlds. In fact, there are many ways of establishing such an ordering over worlds.

Definition 25. *(Liu 2011) Let $\mathscr{G} = \langle\Phi,\prec\rangle$ be a P-graph, W be a non-empty set of states or possible worlds, and $v : P \to 2^W$ be a valuation function. The betternenss relation $\leq_{\mathscr{G}} \subseteq W \times W$ is defined as follows: $w \leq_{\mathscr{G}} w'$ iff*

$$\forall\varphi \in \Phi : (w' \vDash \varphi \Rightarrow w \vDash \varphi) \vee (\exists\psi < \varphi : (w \vDash \psi \text{ and } w' \nvDash \psi))$$

Fact 26. *(Van Benthem, Grossi, and Liu 2014) Let $\mathscr{G} = \langle\Phi,\prec\rangle$ be a P-graph, W a non-empty set of states or possible worlds, and $v : P \to 2^W$ a valuation function. The relation $\leq_{\mathscr{G}}$, as defined above, is a pre-order whose strict part is well-founded.*

From the above definitions and fact, it is easy to see that from a priority graph we can construct a preference model for the logic $\mathscr{L}_{\leq}(P)$ over the propositional names of P.

Definition 27. *Let $\mathscr{G} = \langle\Phi,\prec\rangle$ a P-graph and $M = \langle W,\leq,v\rangle$ a preference model. We say M is induced by $\mathscr{G}$ iff $\leq \; = \; \leq_{\mathscr{G}}$.*

The induction of preference models from priority structures raises the question about the relations between these two structures. (Liu 2011) shows that any model with a reflexive and transitive accessibility relation has an equivalent priority graph.

Theorem 28. *(Liu 2011) Let $M = \langle W,R,v\rangle$ a mono-modal model. The following two statements are equivalent:*

1. *The relation R is reflexive and transitive;*
2. *There is a priority graph $\mathscr{G} = (\Phi,\prec)$ s.t. $\forall w,w' \in W.\ wRw'$ iff $w \leq_{\mathscr{G}} w'$.*

While a positive result relating preference models and priority graphs, Theorem 28 also delineates the limits of such a connection. Notice that for a (mono-modal) model M, it is a necessary condition for its accessibility relation R to be reflexive and transitive, i.e. R is a pre-order. This is only true, however, for the reasoning over static preferences, i.e. if we do not allow changes in preferences.

As it is well known from belief revision literature, e.g. (Hansson 1992), the dynamics of syntactic representations and of semantic modellings can differ vastly. It is also the case that many different model transformations can be defined, e.g. using PDL programs, that do not preserve either reflexivity or transitivity. A question that remains is if any PDL definable operation that maintains pre-orders is syntactically definable by an operation in P-graphs. We argue that this is not the case, analysing the operation of natural contraction.

A syntactical codification of natural contraction has been given by (Rott 2009). It amounts to change the prioritized belief base $\overrightarrow{h}$ to $\overrightarrow{h^{<\varphi}} \prec \overrightarrow{h^{\geq\varphi} \vee \neg\varphi}$[3]. This definition could be easily generalized for P-graphs. We claim, however, that this codification is not right, using the following example.

Example 29. *Taking the base $\overrightarrow{h} = p \prec q \prec r$ and its model $M = \langle 2^{\{p,q,r\}},\leq,v\rangle$ an induced model in the sense of Definition 27, with $w \in v(u)$ iff $u \in w$ for $u \in \{p,q,r\}$. Applying syntactic operation of Rott to contract q, the result would*

[3]Notice that Rott's belief bases are totally ordered P-graphs $\overrightarrow{h} = \langle\Phi,<\rangle$, $\overrightarrow{h^{<\varphi}} = \langle\{\psi \in h | \bigwedge\{\alpha \in h \mid \alpha < \psi\} \nvDash \varphi\}$ and $<\rangle$, $h \vee \alpha = \langle\{\psi \vee \alpha | \psi \in h\},<\rangle$.

be the base $\overrightarrow{h'} = p \vee \neg q \prec q \vee \neg q \prec r \vee \neg q \prec p$ in which the worlds $p \wedge \neg q \wedge r$ and $p \wedge \neg q \wedge \neg r$ are in the same preference cluster. Natural contraction, however, would require by order preservation in $[\![\neg q]\!]$ (Ramachandran, Nayak, and Orgun 2012) that $p \wedge \neg q \wedge r \prec p \wedge \neg q \wedge \neg r$.

In fact, any such syntactical characterization will be flawed by the simple fact that the minimal elements of $[\![\neg\varphi]\!]$ will be dependent of the model to be transformed.

Fact 30. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ a P-graph and φ a propositional formula. There is no propositional formula μ_φ s.t. for every model $M = \langle W, \leq_{\mathscr{G}}, v \rangle$ induced by $\mathscr{G}$ and all $w \in W$, $w \vDash \mu_\varphi$ iff $w \in Min_{\leq_{\mathscr{G}}}[\![\varphi]\!]$.*

Sketch of the proof. It is easy to see that the above fact holds by observing that giving a model M having a chain of at least two worlds satisfying φ, we can construct a submodel M' - which is also a model induced by $\mathscr{G}$ - by removing all the minimal φ elements of M. Supposing there is a propositional formula μ_φ encoding the notion of 'minimal element satisfying φ', it is clear that the minimal elements satisfying φ in M' cannot satisfy μ_φ - since μ_φ is propositional formula and M' is submodel of M. It is also clear that M' has minimal elements satisfying φ, since M has a chain of at least two worlds satisfying φ. But this is a contradiction to the hypothesis that μ_φ encodes the notion of 'minimal element satisfying φ', thus we conclude it cannot exist a propositional formula encoding minimality. □

The fact and example above indicate that Natural contraction is inherently model-dependent and cannot be characterised by means of priority graphs.

Theorem 31. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph and φ a propositional formula. There is no P-graph $\mathscr{G}' = \langle \Phi', \prec' \rangle$ that for any two preference models M_1 and M_2 induced by $\mathscr{G}$, both $M_1 -_N \varphi$ and $M_2 -_N \varphi$ are preference models induced by $\mathscr{G}'$, if there is a chain of at least two worlds in M_1 satisfying φ.*

Proof. Let M_1 be a preference model induced by $\mathscr{G}$. Since $M_1 = \langle W_1, \leq_1, v_1 \rangle$ is a preference model and $[\![\varphi]\!]_{M_1} \neq \emptyset$, then there is minimal element in $[\![\varphi]\!]_{M_1}$. Lets call such element $w_1 \in Min_{\leq_1}[\![\varphi]\!]_{M_1}$. We can construct $M_2 = \langle W_2 = W_1 \setminus [\![\varphi]\!]_{M_1}, \leq_2, v_2 \rangle$ s.t. $\leq_2$ and v_2 are the restriction of $\leq_1$ and v_1 to W_2, respectively.

Now, suppose there is a P-graph $\mathscr{G}' = \langle \Phi', \prec' \rangle$, s.t. $M_1 -_N \varphi = \langle W_1, \leq_N, v \rangle$ is a preference model induced by $\mathscr{G}'$. By the definition of natural contraction, we have that $M_1 -_N \varphi$ is exactly like M_1, except that $Min_{\leq_N} W_1 = Min_{\leq_1} W_1 \cup Min_{\leq_1}[\![\varphi]\!]_{M_1}$. By definition of an induced preference model, it means that for every formula $\xi \in \Phi'$ and any world $w \in W_1$, $M_1, w \vDash \xi$ iff $M_1, w_1 \vDash \xi$.

Let's then look at the case of $w_2 \in Min_{\leq_2}[\![\varphi]\!]_{M_2}$. We know that w_2 exists since M_2 was created by removing the minimal elements of $[\![\varphi]\!]_{M_1}$ from the set W_1 and, by hypothesis, $[\![\varphi]\!]_{M_1}$ has a chain of at least two elements, thus not all elements of $[\![\varphi]\!]_{M_1}$ are minimal. Now, if M_2 is induced by $\mathscr{G}'$, we have that for every formula $\xi \in \Phi'$ and any world $w \in W_1$, $M_1, w \vDash \xi$ iff $M_1, w_2 \vDash \xi$. In particular, for every formula $\xi \in \Phi'$, $M_1, w_1 \vDash \xi$ iff $M_1, w_2 \vDash \xi$. From this, we can affirm that w_2 is a minimal element in $M_1 -_N \varphi$, and thus from M_1, which is a contradiction to the hypothesis that $w_1 <_1 w_2$. So M_2 cannot be induced by $\mathscr{G}'$. □

The theorem above is a definitive answer to the question posed by Liu (Liu 2011) about whether any PDL-definable transformation preserving preference models can be characterised by means of syntactic transformations in priority graphs. The root of the problem here lies in the fact that priority graphs are defined over propositional formulas only, and the notion of 'minimal element' is necessarily modal - as substantiated by Fact 30. Notice however that allowing modal formulas in priority graphs incurs in invalidating Theorem 28 since it allows for the construction of inconsistent graphs (Liu 2011).

A codification for moderate contraction has also been provided by (Rott 2009) and a similar result can be achieved for the impossibility of syntactic representation.

A codification for lexicographic contraction has not been yet proposed and, in fact, not an easy one to provide. Since lexicographic contraction rearranges the preference order in a model in such a way that the orders the sets $[\![\varphi]\!]$ and $[\![\neg\varphi]\!]$ are preserved and the change in the 'degree' of a world, characterized by $dg_\top(i)$, in the resulting model is dependent only on its position within either $[\![\varphi]\!]$ or $[\![\neg\varphi]\!]$, this operation can be syntactically characterized. To do that, however, we will need a further result about priority graphs.

Lemma 32. *(Liu 2011) Each priority graph has an equivalent graph whose propositions form a partition of the logical space.*

The above result means that any priority graph can be rewritten to an equivalent one in such a way that any world in a model satisfies exactly one of such nodes. To describe the transformation in the priority graph, we will need some auxiliary constructions to auxiliate the description of the full transformation.

The first one we present is the support of φ in a graph $\mathscr{G}$.

Definition 33. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph and φ a propositional formula. We define the support of φ in $\mathscr{G}$ as the graph $\mathscr{G}_\varphi = \langle \Phi_\varphi, \prec_\varphi \rangle$ where: $\Phi_\varphi = \{\xi \wedge \varphi \in \Phi \mid \xi \nvDash \neg\varphi\}$ and $\xi \wedge \varphi \prec_\varphi \xi' \wedge \varphi$ iff $\xi \prec \xi'$.*

Notice that the support of φ in a graph $\mathscr{G}$ contains all information about the chains of worlds satisfying φ for models induced by $\mathscr{G}$. In other words, the support of φ in $\mathscr{G}$ is a complete description of the order in $[\![\varphi]\!]$, for any model M induced by $\mathscr{G}$. This is what we show in the following result - which can be easily proved by an induction on n.

Lemma 34. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph whose propositions form a partition of the logical space, $M = \langle W, \leq, v \rangle$ a model induced by $\mathscr{G}$ and φ a propositional formula. For any $w \in W$, it holds that $M, w \vDash dg_\varphi(n)$ iff there is a sequence $\xi_1, \ldots, \xi_{n+1} \in G_\varphi$ s.t. $\xi_i < \xi_{i+1}$ and $M, w \vDash \xi_{n+1}$.*

The second definition we will need is that of depth of a formula in the graph.

Definition 35. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph and $\xi \in \Phi$ a propositional formula. We define the depth of ξ in $\mathscr{G}$, denoted by $d_{\mathscr{G}}(\xi) = k$, as the size of the longest chain $\xi_1, \ldots, \xi_k$ in $\mathscr{G}$ s.t. ξ_1 is minimal in $\mathscr{G}$ and $\xi_k = \xi$.*

With that we define the ranked disjunction of two graphs $\mathscr{G}$ and $\mathscr{G}'$.

Definition 36. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$, $\mathscr{G}' = \langle \Phi', \prec' \rangle$ be a P-Graphs. The disjunction of $\mathscr{G}$ and $\mathscr{G}'$ is the P-Graph $\mathscr{G} \vee \mathscr{G}' = \langle \Phi \vee \Phi', \prec \vee \prec' \rangle$ where:*

$$\Phi \vee \Phi' = \{\xi \vee \xi' \mid \xi \in \Phi \text{ and } \xi' \in \Phi'\}$$

and

$$\begin{aligned} \prec \vee \prec' \quad = \quad & \{\langle \xi_1 \vee \xi_1', \xi_2 \vee \xi_2' \rangle \in \Phi \vee \Phi' \mid \\ & d_{\mathscr{G}}(\xi_1) + d_{\mathscr{G}'}(\xi_1') < d_{\mathscr{G}}(\xi_2) + d_{\mathscr{G}'}(\xi_2')\} \end{aligned}$$

We will construct the lexicographic contraction of a graph $\mathscr{G}$ by a formula φ by conflating the order in the supports of the formulas φ and $\neg\varphi$ and preserving the status of the elements of the graph that neither support nor contradict φ. To do this, first we create a partially ordered (p.o.) set which corresponds to an intermediate graph in which the order relation is not strict, as in a P-Graph, and later we cluster the nodes in a same equivalence class to form a strict ordered graph, which will be the resulting P-graph.

Definition 37. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph whose propositions form a partition of the logical space and φ a propositional formula. The symmetric contraction of $\mathscr{G}$ by φ, is the p.o set $(\mathscr{G} \perp \varphi) = \langle \Phi \perp \varphi, \preceq' \rangle$ where:*

$$\Phi \perp \varphi = \Phi_\varphi \vee \Phi_{\neg\varphi}$$

$$\begin{aligned} \xi_1 \preceq_{\perp\varphi} \xi_2 \text{ iff } \quad & \xi_1 (\prec_\varphi \vee \prec_{\neg\varphi}) \xi_2 \text{ or} \\ & d_{\mathscr{G}_\varphi}(\xi_{\varphi_1}) + d_{\mathscr{G}_{\neg\varphi}}(\xi_{\neg\varphi_1}) = d_{\mathscr{G}_\varphi}(\xi_{\varphi_2}) + d_{\mathscr{G}_{\neg\varphi}}(\xi_{\neg\varphi_2}) \end{aligned}$$

with $\xi_i = \xi_{\varphi_i} \vee \xi_{\neg\varphi_i}$.

Finally, we construct the P-graph corresponding to the lexicographic contraction of a propositional formula φ from the graph $\mathscr{G}$ by joining all equivalent nodes in the symmetric contraction defined above, by means of disjunction. To make the definition more readable, we will use the notation $[\varphi]_\preceq = \{\xi \in \Phi \mid \xi \preceq \varphi \text{ and } \varphi \preceq \xi\}$ to denote the equivalence class of φ in the p.o. set $\langle \Phi, \preceq \rangle$.

Definition 38. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph whose propositions form a partition of the logical space and φ a propositional formula. We define the lexicographic contraction of φ from $\mathscr{G}$ as the P-Graph $\mathscr{G} -^L \varphi = \langle \Phi', \prec' \rangle$ s.t.*

$$\Phi' = \{\bigvee [\xi]_{\preceq_{\perp\varphi}} \mid \xi \in \Phi \perp \varphi\}$$

$$\bigvee [\xi]_{\preceq_{\perp\varphi}} \prec' \bigvee [\psi]_{\preceq_{\perp\varphi}} \qquad \text{iff} \qquad \xi \preceq_{\perp\varphi} \psi \text{ and } \psi \not\preceq_{\perp\varphi} \xi$$

To make it more concrete, we present the following example.

Example 39. *Lets take the graph $\mathscr{G}$ constituted by two nodes $A \prec B$. A model induced by such a graph is the model containing four worlds $A \wedge B < A \wedge \neg B < \neg A \wedge B < \neg A \wedge \neg B$. The lexicographic contraction of such model by the formula B, as defined in Section 4, would result in the model $A \wedge B \equiv A \wedge \neg B < \neg A \wedge B \equiv \neg A \wedge \neg B$. The computation of lexicographic contraction defined of B from the graph $\mathscr{G}$ above is depicted in the Figure 2*[4]*, where (a) represents graph $\mathscr{G}$; (b) the transformation of graph $\mathscr{G}$ into an equivalent graph $\mathscr{G}'$ partitioning the logical space; (c) the symmetric contraction of $\mathscr{G}'$ by formula B, and, (d) the resulting graph after lexicographic contraction.*

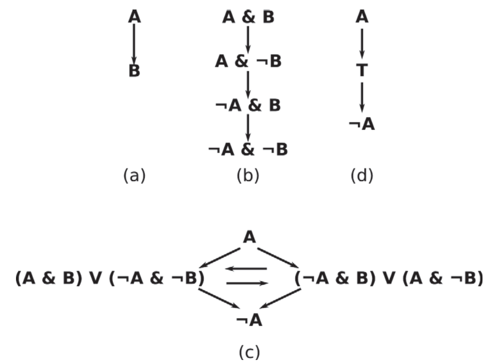

Figure 2: Contracting graph $\mathscr{G}$ by formula B from Example 39

We can prove harmony between the semantically defined lexicographic contraction on preference models and syntactic transformation in priority graphs. Due to space limitation we will not show the proof here, but the intuition behind it is that the support of φ in a graph $\mathscr{G}$ will represent the chains in $[\![\varphi]\!]$ in the induced model. Similarly for the support of $\neg\varphi$. Taking the disjunction of both subgraphs, we conflate the chains in a way that if a world belongs in a chain of size i in either $[\![\varphi]\!]$ or $[\![\neg\varphi]\!]$, it will be preferred to any world belonging in a chain of size $j > i$ in either $[\![\varphi]\!]$ or $[\![\neg\varphi]\!]$, as required by the definition of lexicographic contraction.

Proposition 40. *Let $\mathscr{G} = \langle \Phi, \prec \rangle$ be a P-Graph whose propositions form a partition of the logical space, $M = \langle W, \leq, v \rangle$ a model induced by $\mathscr{G}$ and φ a propositional formula. The model $M -^L \varphi$ is induced by the graph $\mathscr{G} -^L \varphi$.*

6 Conclusion

This work has explored some codifications of iterated contraction operators in the dynamic preference logic of (Van Benthem, Girard, and Roy 2009) using the syntatic representation of preference models by meas of Liu's priority structures (Liu 2011). We provided a proof of completeness for well-founded preference models, semantic codifications of contraction operations by transformation of preference models and investigated the harmony cases between these semantic transformations and syntactic manipulations on priority graphs.

While important cases of harmony between semantic operations in preference models and syntactic manipulations of priority graphs have been presented in the literature, we show that this is not the case for all PDL-definable operations that preserve pre-orders. We believe this is a very interesting result of our study, since this has been an open question posed in (Liu 2011). It remains to answer, however, if PDL programs in which the test operation $?\varphi$ can only be defined if φ is a propositional formula can always be represented by transformations of priority graphs. The intuition we obtain from the syntactic codification of lexicographic contraction is that this is indeed true.

[4]In the figure, an arrow starting in node a and ending in node b represents the relation $a \prec b$ of the graph.

Our work can, thus, be seen as a step in the direction of reducing the "semantical gap" in agent programming languages, by providing the syntactic characterization and proof of harmony for an important instance of a well-behaved operation: lexicographic contraction.

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