Exposing Piaget's Scheme: Empirical Evidence for the Ontogenesis of Coordination in Learning a Mathematical Concept

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Abstract: The combination of two methodological resources—natural-user interfaces (NUI) and multimodal learning analytics (MMLA)—is creating opportunities for educational researchers to empirically evaluate seminal models for the hypothetical emergence of concepts from situated sensorimotor activity. 76 participants (9-14 yo) solved tablet-based non-symbolic manipulation tasks designed to foster grounded meanings for the mathematical concept of proportional equivalence. Data gathered in task-based semi-structured clinical interviews included action logging, eye-gaze tracking, and videography. Successful task performance coincided with spontaneous appearance of stable dynamical gaze-path patterns soon followed by multimodal articulation of strategy. Significantly, gaze patterns included uncued non-salient screen locations. We present cumulative results to argue that these 'attentional anchors' mediated participants' problem solving. We interpret the findings as enabling us to revisit, support, refine, and elaborate on central claims of Piaget's theory of genetic epistemology and in particular his insistence on the role of situated motor-action coordination in the process of reflective abstraction.

Keywords: attentional anchor, coordination, eye-tracking, genetic epistemology, NUI, proportion

Background and objectives: Revitalizing LS interest in genetic epistemology as complementary to sociocultural models of conceptual development

The eminent cognitive-developmental psychologist Jean Piaget, who would be celebrating his 120th birthday this August 2016, has had a rocky career in the Learning Sciences. Despite a near-centennial stretch of prodigious, paradigm-changing academic oeuvre, despite the omnipresence of constructivist educational parlance in preK-12 STEM rhetoric, and despite his indirect yet formative and enduring mark on the design of commercial pedagogical products for discovery-based learning, Piaget's groundbreaking construct of a schema has received some bad press. The construct suffers, perhaps, via its too-convenient association with Piaget's oft-critiqued yet oft-misunderstood Stage Theory or the indefatigable attacks on the validity of his clinical methodologies. But whereas Piaget-bashing has generated many a dissertation and built entire research programs, his theoretical constructs and model of conceptual schemata rising from sensorimotor operatory schemes, we posit, have yet to find their match as explanantia for meaningful situated learning. At the very least, we concede, the waning of empirical Piagetian research is hampering our field's intellectual progress and increasingly vitiating its relevance to the changing terrain of educational media (Abrahamson, 2015; Abrahamson & Sánchez–Garcia, in press).

We are calling to renew Piagetian discourse specifically on mathematical learning, and more specifically, mathematics learning with state-of-the-art interactive media (Forman, 1988; Lindgren & Johnson-Glenberg, 2013; Marshall, Antle, van den Hoven, & Rogers, 2013; Moreno-Armella, Hegedus, & Kaput, 2008; Sarama & Clements, 2009). Even more specifically, we are looking for forms of empirical research in environments where both student and researcher respective activities avail of multimodality, with the student engaging in explorative activity that the researcher monitors, documents, measures, and analyzes, even in real time (Martin & Sherin, 2013; Schneider, Bumbacher, & Blikstein, 2015; Worsley & Blikstein, 2014). This brave new world of multimodality in design, instruction, and research demands theoretical infrastructure for thinking seriously, anew, about situated motor-action skill acquisition as it relates to conceptual development. In turn, we are thus also looking to draw on a century of progress in the somatic–kinesiological disciplines (Bernstein, 1996; Kelso & Engstrøm, 2006; Newell & Ranganathan, 2010; Thelen & Smith, 1994) as these bear on the action-to-concept learning process (Bamberger, 2013). In fact, we will argue that Piaget's genetic epistemology is key to populating learning-sciences discourse with this diversity of fresh, pertinent, and resonant perspectives. In a sense, we are stepping back to jump forward.

Up front, we wish to clarify that our call is to build on, rather than supplant, a research tradition of treating

Piagetian themes through qualitative analysis (Abrahamson, 2012; Dubinsky, 1991; Gray & Tall, 1994; Norton, 2008). In fact, it is precisely these types of investigations that we wish further to pursue by introducing new constructs and methodological techniques.

To contextualize and substantiate this call, we present and discuss empirical data collected during the implementation of experimental educational interventions, in which young study participants were engaged in technologically enabled embodied-interaction activities designed to foster presymbolic proportional reasoning. We will argue for the unique and pivotal traction of Piaget's thesis on our research by way of explaining the critical role that his constructs of sensorimotor scheme and reflective abstraction served in making sense of our data. Namely, the Piagetian perspective enabled us to posit the significance of nuanced changes in children's sensorimotor activity for their conceptual ontogenesis as well as the implications of these findings for both theory and practice of mathematics education. Please note: This proceedings paper is a précis for a full-length peer-reviewed journal article (Abrahamson et al., in press), where we are able to provide far more information on theory, methods, and results.

Theoretical framework

Piaget (1896-1980) was fascinated by children's opportunities for personal development through engaging in the social enactment of cultural practice, such as moral development through game play. However he viewed culture, along with its social agents, practices, norms, and material artifacts, moreso as the setting and playing field of ontogenesis than as its very fabric and constitution.

In the latter 20th century, a slew of monographs inspired by the cultural–historical psychology of Lev Vygotsky (1896-1934) impressed upon our intellectual community a set of views not readily perceived as concordant with Piaget's epistemological theory. Instead of foregrounding the child's piecemeal construction of cognitive structures, these views underscored the critical role of sociocultural activity structures as shaping individual disciplinary enskilment, such as mathematical competence. These alternative views include the theorization of: (a) artifact appropriation and contextual adaptation as the *sine qua non* of mediated maturation into communal techno-scientific practice, including visualizations, orientations, and discourse (Newman, Griffin, & Cole, 1989; Saxe, 2012; Wertsch, 1979); (b) learning as legitimate peripheral participation in the social co-enactment of purposeful cultural practice (Lave & Wenger, 1991; Rogoff, 1990); and (c) discourse as the vehicle and substance of knowing (Sfard, 2010).

Scholars holding constructivist views of cognition have retaliated that, notwithstanding, meaning must be grounded in tacit, presymbolic sensorimotor routines and innate/early cognitive capacity (Allen & Bickhard, 2013; Denison & Xu, 2014; Harnad, 1990) and concepts are built painstakingly by coordinating multiple personal and situated resources for *ad hoc* productive engagement (Case & Okamoto, 1996; Noss & Hoyles, 1996; Smith, diSessa, & Roschelle, 1993). In the fray of this grand altercation some scholars are looking to forge dialogue between these die-hard entrenched camps (Abrahamson, 2012; Cole & Wertsch, 1996; diSessa, Levin, & Brown, 2015). By and large, though, the field is at a stalemate, with each faction chiding the other, "Show me!"

We have something new to show that might jostle the field out of its stalemate. We reasoned that if only we could demonstrate empirically student behaviors that are better accounted for by constructivist than sociocultural theory, then we would be in a better position to argue for a dialectical view of mathematical learning—a view of learning as action-based ontogenesis in facilitated settings—settings that we regard as culturally–historically evolved instrumented fields of promoted action (Abrahamson & Trninic, 2015). Moreover, by way of reemphasizing the critical role of sensorimotor activity in conceptual development, we could justify an introduction, into learning-sciences discourse, of evolving models of teaching and learning imported from disciplines focused on motor-action skill development and methodology (Abrahamson & Sánchez–Garcia, in press).

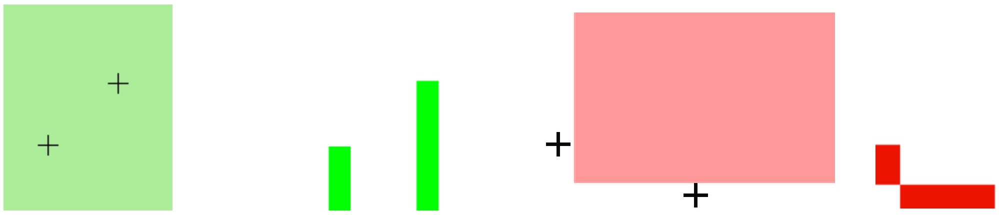
Our renewed interest in Piaget's theory of learning, and in particular his model of reflective abstraction, emerged from unexpected quarters. Namely, philosophers of radical enactive cognition (Chemero, 2009; Hutto & Myin, 2013), who reject exclusively representationalist epistemologies, have been seeking corroboration via partnerships with social scientists engaged in the empiricism of skill acquisition. For example, Hutto and Sánchez-García (2015), respectively a philosopher of cognition and a sociologist of sport, collaborated in articulating a radical-enactivist interpretation of athletic performance. In particular they developed the construct of an attentional anchor, which then became central to our own work (Abrahamson & Sánchez–Garcia, in press), as we now explain.

An *attentional anchor* (AA) is a phenomenological aspect of an agent's implicit or explicit goal-oriented interaction with the environment. AAs may be a specific object (real or imagined), area, or other pattern or behavior of the perceptual manifold that an agent detects, invokes, selects, and uses to enact the activity at hand. For example, a juggler struggling to coordinate the trajectories of many balls simultaneously flying through the air might imagine a tall rectangle rising in front of her and aim the balls to its vertices. Whether discovered or taught, the AA interpolates itself into the agent–environment relation to serve as an enabling task constraint—it becomes a new systemic element that hones and channels attention during perception–action couplings. The AA reduces operational complexity, rendering ergonomic and feasible otherwise overwhelming tasks. An agent acting on an AA experiences it as a "steering wheel" overlaid upon the perceptual field—the attentional anchor becomes the mediating proxy both for operating on the environment and interpreting feedback from the environment. Specifically, the AA brings forth to the agent new latent affordances by objectifying, specifying, and foregrounding the environment's task-oriented, invariant goal information structures (Chow et al., 2007; Kelso & Engstrøm, 2006; Newell & Ranganathan, 2010). As such, the AA thesis resonates well also with central tenets of Enactivism (Varela, Thompson, & Rosch, 1991).

We are intrigued and motivated by an apparent resemblance of the constructs *motor-action coordination* and *attentional anchor* from ecological-dynamics theory to those of *coordination* and *category* in Piaget's model of reflective abstraction. Reflective abstraction is the highest of three abstraction levels that Piaget distinguished. It concerns a learner's coordination of actions. Piaget viewed this type of abstraction as 'constructive' in the sense that new syntheses emerge that bring forth interaction regularities toward encapsulating and generalizing new capacity. In short, reflective abstraction is "the construction of mental objects and of mental actions on these objects" (Dubinsky, 1991, p. 101). It should come as no surprise that Piaget's work resonates with dynamical-systems theory, given his deep commitment to anti-representationalist, situated structuralism (Piaget, 1970; Turner, 1973). In the remainder of the paper we will discuss an empirical study, in which we have tracked what appear to be children's sensorimotor behaviors that mark a new coordination focused on an attentional anchor; a coordination leading to the reflective abstraction of a higher-order functional structure and its conscious articulation as a new phenomenal category. As we explain, this situated cognitive process is pivotal to a designed activity on proportional relations.

Methods

In total, 76 volunteering students from the Netherlands participated in two studies. Study 1 included 30 students of 5th/6th grade (mean age = 11[3] years; 13 male, 17 female) from five elementary schools. They all worked on the Parallel Bars activity (see Figure 1a). Study 2 included 46 students of 7th/8th grade (29 male, 17 female; mean age = 13[5]) from two prevocational schools. Of these, one group (26 students) only worked on "parallel tasks"—Parallel Pluses (Figure 1a) followed by Parallel Bars (Figure 1b); the other group (20 students) only worked on "orthogonal tasks"—Orthogonal Pluses (see Figure 1c) followed by Orthogonal Bars (see Figure 1d).



a. Parallel Pluses: user slides pluses each along its vertical axis to reposition them; full-screen color feedback

b. Parallel Bars: user drags bar tips each along its vertical axis to extend or shorten bars; color feedback on bars.

c. Orthogonal Pluses: user slides pluses along their vertical (left) and horizontal (right) axes; full-screen color feedback.

d. Orthogonal Bars: user drags bar tips along their vertical (left) and horizontal (right) axes to extend or shorten bars; color feedback on bars.

Figure 1. Sample screenshots from enacting four activity modules in the touch-screen tablet application. To make the screen green, participants had to manipulate either the positions of cursors (1a, 1c) or the extension of bars (1b, 1d) either along parallel (1a, 1b) or Cartesian axes (1c, 1d).

The tasks were variations on the Mathematical Imagery Trainer for Proportion (Reinholz, Trninic, Howison, & Abrahamson, 2010), an interactive technological device designed for students first to develop new operatory schemes underlying mathematical concepts and then mathematize these schemes using standard frames of references (e.g., a grid, numerals). The task is implemented in a multi-touch tablet, with each hand (or each index finger) controlling one element on the screen, either a plus-shaped cursor (the "plus" task conditions) or the edge of a stretch/shrink rectangle (the "bars" task conditions). The task objective is to move these elements on the screen so as to achieve a specified goal state: keeping green either the whole screen ("plus") or elements thereof ("bars"). The software mediating between user-action input and screen-color output instantiates mathematical datum point (e.g., 10 and 20 cm, respectively, above the screen base) then calculates their quotient (e.g., 10/20). A match with a preset ratio (e.g., 1:2) makes for green, otherwise red (Figure 1). Thus in the case of a 1:2 ratio, users might move their index fingers along the screen constantly keeping the right-hand double as high as the left ("parallel" conditions, Figures 1a, 1b) or double as far from the origin ("orthogonal" conditions, Figs. 1c, 1d), or they might attend to other properties of the performance, such as the distance between their hands or their speeds (Abrahamson et al., 2014).

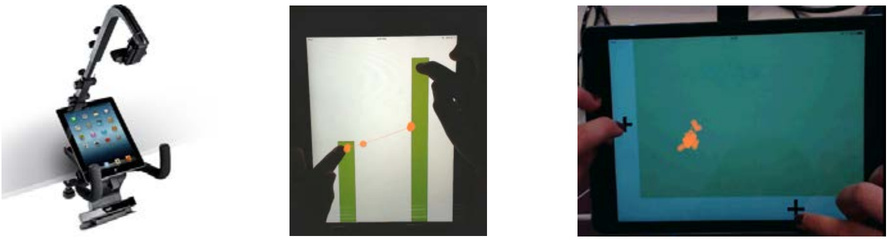


Figure 2. On left: Tobii Mobile Device Stand for X2. The stand is attached to the edge of a desk. The iPad is positioned in the center. The eye-tracker is placed on the stand base, with the camera on the top. Center: sample integrated eye-tracking and video data from the Parallel Bars condition. On right: Orthogonal Pluses data sample (left hand moves the plus up/down; right hand moves the plus righ/left).

The intervention and analysis followed principles of task-based semi-structured clinical interviews (e.g., Ginsburg, 1997). Our data set comprises videography (of student actions and multimodal student–tutor discourse), streaming logs of touchscreen activity, and eye-gaze tracking (see Figure 2). This complex data constellation was designed so as to serve us in developing a more detailed and comprehensive theoretical model for the spontaneous emergence of new sensorimotor coordinations grounding mathematical conceptions. We used visualization software that superimposes the eye-tracking paths onto the videography, so that we could see which particular locations on the screen were in the users' foveal vision as they were manipulating the virtual objects in dialogue with the researcher. Computational analyses of users' visual pathways on the screen fed into micro-analyses of their concurrent actions and multimodal utterance (Siegler, 2006). The analyses enabled us to discern general patterns in students' search for, and articulation of, effective bimanual manipulation strategies. We were particularly interested in implicating the emergence of attentional anchors that first support the bimanual motor-action then come forth into dyadic discourse as new mathematical objects and solution procedures. Furthermore, we evaluated whether Piaget's four phases of reflective abstraction—interiorization, coordination, encapsulation, and generalization—could be genuinely implicated and differentiated in the data as depictive markers parsing students' activity flow. For reliability, two researchers independently analyzed part of the data corpus, then shared their findings, and finally watched the videos repeatedly until reaching agreement over all their observations.

Results

Cumulative findings: A Piagetian analysis of learning proportion as reflective abstraction

Both within and across age and condition groups, students differed along several dimensions relevant to the study, including: (a) duration of time elapsed until discovery of a first effective interaction routine; (b) time to complete the whole task; and (c) pace of finger movement (fast or slow) at the initial exploration phase. Participants also differed in the incorrect rules they initially posited, their eye-gaze patterns accompanying successful hand coordination strategies, and their lines of reasoning toward effective solutions. These individual differences notwithstanding, the progress of *all* participants through the activity bore the pattern presented in Table 1. The discoveries students made en route to figuring out "green" interaction rules replicate our earlier findings (Reinholz et al., 2010). However adding eye-tracking visualization into the data manifold now enables us better to model the emergence of these discoveries from students' interactions and characterize the discoveries in terms of reflective abstraction phases.

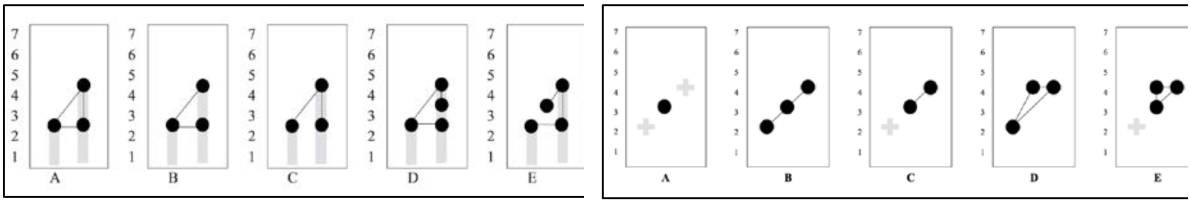
Table 1: Participants' cross-condition behavioral sequence follows Piaget's reflective abstraction phases

1. Interiorization: Exploring task environment	Students: (a) explored the task environment without any clearly discernable plan or strategy; (b) found greens haphazardly; (c) could not replicate green positions; (d) attempted strategies that did not bear out, e.g., moving fingers in equal pace; (e) realized there should be a spatial relation between

	the hands; (f) attempted to coordinate actions. Concurrently, eye gaze shifted between the moving fingertips.
2. Coordination: stable sensory patterns emerge concurrent with effective motor-action performance	In the course of attempting to develop an effective bimanual dynamical motor-action scheme for keeping green, gaze patterns emerged (see Figure 3) that: (a) followed tentative localized discovery of effective positions and constraints on action; (b) manifested as iterated rapid shifts among specific interface elements; (c) included at least one un-manipulated point; (d) settled on consistent, stable, and reoccurring forms; (e) coincided with significant improvement in overall performance; (d) coincided with more continuous as opposed to abrupt motor action; (f) enabled to reconstruct/replicate/repair previous green locations; and (g) preceded logical–mathematical reflective reasoning, discovery, or articulation of rules.
3. Encapsulation: Articulating sensorimotor patterns results in objectifying tacit elements, enhanced performance	Probed to articulate their strategy, students objectified an attentional anchor and then elaborated on it, forming new conjectures. Initially, though, their conjectures tended to belie their actions, such as speaking of a fixed distance between the moving fingers in the Parallel conditions, whereas in fact they had been changing the distance covariate with height. As they enacted their thoughts, however, they gradually came to appreciate the error, such as noticing that the distance in fact increases with height. After several replications they expressed their inference, such as saying, “No I was wrong.” At times, the experimenter guided this process by either challenging students or orienting them on critical features in the visual display. In turn, process of articulating and evaluating effective strategies resulted in better performance.
4. Generalization: From iterated, qualitative-process rule to explicit functional rule: articulating a latent mathematical relation as a constant property	Once students had validated an effective strategy, their actions were no longer explorative. Their gaze pattern intensified, e.g., more consistent and more rapid eye-gaze shifts along the triangular attentional anchor in Parallel Bars. Concurrently, their utterance included qualitative properties of objects and prospective actions. Introducing the grid precipitated a shift toward quantitative reasoning, e.g., “When they are lower they are one line apart, when they are in the middle they are more lines apart, and when the right hand is at the very top they are most apart.” Supplementing the numerals resulted in students unpacking the bimanual composite into ordered pairs, e.g., left at 1, right at 2; left at 2, right at 4; etc. Eventually they recognized a constant intra-pair quantitative (multiplicative) relation, e.g., “Oh wait it’s a half… I know it’s a half, the left is always half of the right.” They thus shifted from a scalar, inter-position process rule for iterated enactment (the higher you go, the bigger the distance) to an explicit intra-position functional rule with predictive power (wherever right is, left is half). That is, they articulated the notion of a constant ratio that underlies proportional equivalence.

Parallel Bars: Triangle A most prevalent

Parallel Pluses: Pattern B most prevalent



The top of Left Bar is “projected” to an un-manipulated point on Right Bar. Students dynamically calibrated the height of Left Bar relative to half the height of Right Bar (& vice versa).

The focal gaze point between the pluses is an un-manipulated location. It emerged and was constructed in relation to the two plus signs, constantly moving them and moved by them.

Orthogonal Pluses

Orthogonal Bars

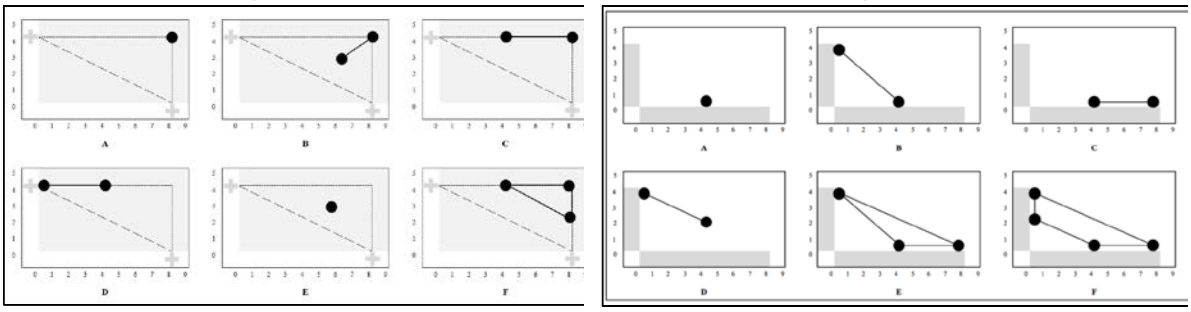


Figure 3. Schematic overview of the variety of emergent dynamical gaze patterns reveals attentional anchors.

In Figure 3, circles represent focal gaze points, lines are gaze paths. *Triangulated with the tablet action-logging and clinical data, we interpret these gaze patterns as evidence for ecologically coupled sensorimotor attentional anchors mediating effective enactment of problem solutions for the embodied-interaction task*. In our collaborative-analysis sessions, as we watched the superimposed gaze/video data, we have been compelled by the dynamical evolution of these forms, and in particular when we played these movies in fast motion: It is as if bits and pieces of a would-be instrument—a handle or steering wheel—assemble in the task environment as solution means; as actionable media "between" student and objective. These media, the attentional anchors, emerge via co-evolving dialectical process: attentional anchors are invented *for and by* the sensorimotor scheme that wields it as a means of accomplishing the situated task objective. As the interview advances, the attentional anchors ascend: from latent aspects of the task environment; to tacit, dynamical, ecologically coupling patterns; to bonafide articles of discourse and reasoning.

Summary: Beyond representations—appreciating Piaget as a non-cognitivist

The structure that the child constructs through goal-oriented engagement in the task environment is not a representation in the sense of some accessible mental content in her head. Rather, the structure is a cognitive construct, a tacit relation that emerges between the subject and the objective world through adaptive efforts toward equilibrating effective engagement. The structure functions as a dynamical systemic dialectic, by which are formed both the subject's schematized action routines of engagement with the world and, reciprocally, those worldly categories being engaged—aspects of the world toward which this schematized sensorimotor activity is oriented and transforming; categories by which the child is effecting aspects of the environment. In the particularities of the child's engagement with the MIT-P technological system, the emergent operatory schemes are correlational. For example, in the case of the Parallel Pluses the reciprocally emergent category upon which these schemes are operating is often the interval between the hands. The emergent correlational manipulation of the interval coordinates two operations upon it—transforming its elevation, transforming its size—so that the higher the interval is (or the farther it is along the screen), the bigger it should be so as to effect and maintain the desired worldly state (making and keeping the screen green). This covariational coordination is created through a process Piaget called reflective abstraction, that is, the construction of a higher-order operational structure—the organization of a new phenomenal invariance that breaks away from, yet contains and coordinates, existing routinized operations that hitherto had been sufficient for productive engagement with simpler categories yet hence prove insufficient. To iterate, this coordination is centered on the new category, the interval between the hands.

Looking at results from implementing the MIT-P system in an eye-tracking study, we have attempted to make sense of our data from this Piagetian perspective. In particular, we have been curious about shifts in students' visual attention toward the objects they are manipulating—shifts that co-occur with, or briefly anticipate, an apparent organization of new action patterns as well as the multimodal discursive articulation of these patterns into proto-mathematical propositions. Emblematic of these pattern shifts is that students will incorporate into their new routine a visual attention toward a location on the screen that is not a constituent part of the objects being manipulated. For example, they may stare at a point, a blank locus between two objects that they are manipulating—a point that apparently is strategic for constructing and applying the new coordination, such as the "higher-bigger" dynamical correlation discussed above. Whenever these new coordinations constitute schemes that we evaluate as proto-conceptual, such as schemes leading to proportional reasoning, it is very tempting to state that the children are re-inventing mathematical concepts within our designed fields of promoted action. That is, we seem to be witnessing the process of reflective abstraction, and this process is mediated by the children's participation in the discovery and enactment of a cultural practice of our design, a sensorimotor practice they are never shown but are steered toward.

We are thus offering an explication of mathematical learning as a Piagetian constructivist process embedded in a Vygotskian cultural–historical framework. In so doing, we are also endeavoring to redress a lacuna in Piaget's theoretical thesis, namely his little concern for sociocultural enframings of children's logico-mathematical ontogenesis. As Turner (1973) writes:

> Piaget's model of psychogenesis is formulated in an artificial sociological vacuum; he has never confronted the question of the socio-cultural components of the mind at the level of the basic structure of the psychogenetic process itself. (p. 364) [Piaget] has, in other words, not yet come to grips with the problem of the specific social and cultural mechanisms through which cultures and societies participate in and control the genetic development of the individual psyches of their members. (p. 369)

Conclusions and implications

When Piaget began publishing on cognitive developmental psychology a whole century ago, clinical interviews were cutting-edge scientific method. By detecting systematic patterns in children's action and utterance during interviews, as they attempted to respond to his questions and solve his puzzles, and building on a colossal battery of cross-sectional studies, Piaget put forth a cognitive theory of genetic epistemology. Central to this theory was a painstaking explanation for individuals' subjective construction of psychological objects—new phenomenal categories that come forth to enhance, mediate, and regulate effective worldly transactions. These new categories and attendant sensorimotor schemes coalesce as the child's cognitive adaptations—emergent interaction routines enabled yet constrained by innate cognitive architecture. That is, the mind constructs a new category and, whenever doing so, extends and tightens its grip on the world.

Though much water has since flowed under Geneva's Mont-Blanc bridge, the Learning Sciences have not advanced much in evaluating Piaget's central claims respecting the child's construction of new psychological objects as solutions to problems of sensorimotor interaction. To be sure replication, qualification, and elaboration have been offered aplenty, and yet abstraction itself—the construct and process—have not been validated via independent measures.

These are early days in our quest to witness the psychological construction of new objects as it occurs. And yet our findings to date embolden and impel us to submit that we are literally seeing reflective abstraction. Empirical data from our task-based interviews, and in particular children's eye-gaze patterns triangulated against their tablet action-logs and audio–video recordings, are aligning remarkably well with Piaget's constructivist model of cognitive development. What more, by seeing what the child is looking at and manipulating we now understand far better our own successes and failures as educational designers in guiding the children to mathematize these tacit constructions.

More broadly, we have demonstrated alignment between a core construct from Piaget's theory of genetic epistemology—reflective abstraction—and tenets of Enactivism, dynamical-systems theory, ecological psychology, and socio-kinesiology. We thus join Allen and Bickhard (2013) in challenging and encouraging our colleagues to revisit Piaget's seminal contributions; to see for themselves the emergence of conceptual categories; to understand what this might all mean in practice; and make that practice a reality.

The ICLS 2016 conference theme "directs our gaze to the commitment of the Learning Sciences to provide a more insightful understanding of how people learn." By directing our gaze to students' gaze, we hope to do just so.

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Acknowledgements

This work was funded by an Education and Learning grant, Utrecht University, January 2014.