

STABLE PAIR INVARIANTS OF SURFACES AND SEIBERG-WITTEN INVARIANTS

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ABSTRACT. The moduli space of stable pairs on a local surface $X = K_S$ is in general non-compact. The action of \mathbb{C}^* on the fibres of X induces an action on the moduli space and the stable pair invariants of X are defined by the virtual localization formula. We study the contribution to these invariants of stable pairs (scheme theoretically) supported in the zero section $S \subset X$. Sometimes there are no other contributions, e.g. when the curve class β is irreducible.

We relate these surface stable pair invariants to the Poincaré invariants of Dürr-Kabanov-Okonek. The latter are equal to the Seiberg-Witten invariants of S by work of Dürr-Kabanov-Okonek and Chang-Kiem. We give two applications of our result. (1) For irreducible curve classes the GW/PT correspondence for $X = K_S$ implies Taubes' GW/SW correspondence for S . (2) When $p_g(S) = 0$, the difference of surface stable pair invariants in class β and $K_S - \beta$ is a universal topological expression.

1. INTRODUCTION

In [23], R. Pandharipande and R. P. Thomas introduce stable pairs on projective 3-folds X and show their moduli space is a component of the moduli space of all complexes in the bounded derived category $D^b(X)$. Formally, a stable pair (F, s) on X consists of a pure dimension 1 sheaf F on X and a section $s \in H^0(F)$ with 0-dimensional cokernel. The moduli space of stable pairs has a perfect obstruction theory, which is symmetric in the case X is Calabi-Yau. The associated invariants are known as stable pair invariants and are closely related to the Donaldson-Thomas and Gromov-Witten invariants of X [3, 16, 17, 18, 21, 22, 23, 24, 31].

We consider the case where $X = K_S$ is the total space of the canonical bundle over a smooth projective surface S . Let $P_\chi(X, \beta)$ denote the moduli space of stable pairs (F, s) on X with class $\beta \in H_2(S)$ and $\chi(F) = \chi$. The space $P_\chi(X, \beta)$ carries a perfect obstruction theory but can be non-compact. Using the \mathbb{C}^* -action on the fibres of X gives an induced obstruction theory on $P_\chi(X, \beta)^{\mathbb{C}^*}$. The components of this fixed locus are compact. For any¹

¹We denote the \mathbb{C}^* -equivariant cohomology of X by $H_{\mathbb{C}^*}^*(X, \mathbb{Q})$. Endowing S with trivial \mathbb{C}^* -action, we then have $H_{\mathbb{C}^*}^*(X, \mathbb{Q}) \cong H_{\mathbb{C}^*}^*(S, \mathbb{Q})$.

$\sigma_1, \dots, \sigma_m \in H_{\mathbb{C}^*}^*(S, \mathbb{Q})$ the stable pair invariants of X are defined by the virtual localization formula of T. Graber and R. Pandharipande [12]:

$$(1) \quad P_{\chi, \beta}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) := \int_{[P_{\chi}(X, \beta)^{\mathbb{C}^*}]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \prod_{i=1}^m \tau_{\alpha_i}(\sigma_i).$$

Here $e(N^{\text{vir}})$ is the equivariant Euler class of the virtual normal bundle and $\tau_{\alpha}(\sigma)$ is the descendent insertion

$$(2) \quad \tau_{\alpha}(\sigma) := \pi_{P*}(\pi_X^*(\sigma) \cap \text{ch}_{\alpha+2}^{\mathbb{C}^*}(\mathbb{F})),$$

where $\alpha_i \geq 0$, \mathbb{F} is the universal sheaf on $P_{\chi}(X, \beta) \times X$, and $\text{ch}^{\mathbb{C}^*}$ denotes \mathbb{C}^* -equivariant Chern character. Note that these invariants are elements of $\mathbb{Q}[t, t^{-1}]$, where t is the equivariant parameter. In this paper we will only be concerned with *primary point insertions*

$$\tau_0(\text{pt}) := \pi_{P*}(\pi_X^*(\text{pt}) \cap \text{ch}_2^{\mathbb{C}^*}(\mathbb{F})),$$

where pt denotes the (Poincaré dual of) the point class in $H^4(S, \mathbb{Z})$.

The easiest component of $P_{\chi}(X, \beta)^{\mathbb{C}^*}$ consists of stable pairs which are scheme theoretically supported on the zero section $S \subset X$, i.e. $P_{\chi}(S, \beta)$. Denote the Hilbert scheme of effective divisors on S with class β by $H_{\beta} := \text{Hilb}_{\beta}(S)$ and the universal curve by $\mathcal{C} \rightarrow H_{\beta}$. Let n be determined by $\chi = 1 - h + n$, where h is the arithmetic genus of curves with class β

$$2h - 2 = \beta(\beta + k), \quad k := c_1(\mathcal{O}(K_S)) \in H^2(S, \mathbb{Z}).$$

Given a stable pair $[s : \mathcal{O}_S \rightarrow F]$ on S , the scheme theoretic support of F is a Gorenstein curve $C \subset S$. The cokernel Q of s gives rise to a 0-dimensional closed subscheme $Z \subset C$ via the surjection $\mathcal{O}_C \rightarrow \mathcal{E}xt^1(Q, \mathcal{O}_C)$ obtained by dualizing. This provides an isomorphism [25]

$$P_{\chi}(S, \beta) \cong \text{Hilb}^n(\mathcal{C}/H_{\beta}),$$

where $\text{Hilb}^n(\mathcal{C}/H_{\beta})$ is the relative Hilbert scheme of n points on the fibres of $\mathcal{C} \rightarrow H_{\beta}$. In this paper we only consider contributions to (1) of the “surface component” $P_{\chi}(S, \beta)$, i.e.

$$(3) \quad P_{\chi, \beta}(S, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) := \int_{[P_{\chi}(S, \beta)]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \prod_{i=1}^m \tau_{\alpha_i}(\sigma_i).$$

We group these invariants together into a generating function

$$Z_{\beta}^P(S, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) := \sum_{\chi \in \mathbb{Z}} P_{\chi, \beta}(S, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) q^{\chi}.$$

The following is our main theorem:

Theorem 1.1. *For any S, β , and $m := \frac{\beta(\beta-k)}{2}$*

$$Z_{\beta}^P(S, \tau_0(\text{pt})^m) = t^m P_S(\beta) (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2h-2},$$

where t is the equivariant parameter, $2h - 2 = \beta(\beta + k)$, and $P_S(\beta) \in \mathbb{Z}$ is the numerical part of the Poincaré invariant $P_S^+(\beta)$ of Dürr-Kabanov-Okonek.

In this theorem

$$P_S^+(\beta) \in \Lambda^* H^1(S, \mathbb{Z})^*$$

are the Poincaré invariants of S, β defined by M. Dürr, A. Kabanov, and Ch. Okonek [6]. These invariants are defined in terms of a natural virtual cycle on the Hilbert scheme of curves H_{β} . They define a corresponding invariant $P_S^-(\beta)$ in terms of a natural virtual cycle on $H_{k-\beta}$. We are only concerned with the numerical part (degree $b_1(S)$ in cohomology), which we denote by $P_S(\beta)$. Dürr-Kabanov-Okonek conjectured that $P_S^{\pm}(\beta)$ are equal to the *Seiberg-Witten invariants* of S, β . Up to a purely algebraic conjecture, they prove this using their wall-crossing and blow-up formula. This algebraic conjecture was subsequently proved by H.-l. Chang and Y.-H. Kiem via a beautiful application of cosection localization [4]. As a corollary of the ‘‘Poincaré/PT correspondence’’ of Theorem 1.1 and the (much deeper!) Poincaré/SW correspondence of [4, 6] we obtain:

Corollary 1.2. *In the notation of Theorem 1.1*

$$Z_{\beta}^P(S, \tau_0(\text{pt})^m) = t^m SW(\beta) (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2h-2},$$

where $SW(\beta) \in \mathbb{Z}$ is the Seiberg-Witten invariant of S, β .

We have two applications of Theorem 1.1 (and its Corollary 1.2). The first is to Gromov-Witten theory. For any g , let $\overline{M}'_{g,m}(X, \beta)$ be the moduli space of stable maps with possibly disconnected domain curve and no collapsed connected components. Its \mathbb{C}^* -fixed locus $\overline{M}'_{g,m}(S, \beta)$ has an induced perfect obstruction theory, which is the usual Gromov-Witten theory of S . The Gromov-Witten invariants of X are defined by virtual localization

$$R_{g,\beta}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) := \int_{[\overline{M}'_{g,m}(S,\beta)]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \prod_{i=1}^m \tau_{\alpha_i}(\sigma_i),$$

$$\tau_{\alpha_i}(\sigma_i) := \psi_i^{\alpha_i} \text{ev}_i^*(\sigma_i),$$

$$Z_{\beta}^{GW}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) := \sum_g R_{g,\beta}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)) u^{2g-2},$$

where ψ_i are the ψ -classes and ev_i the evaluation maps. From Theorem 1.1 (or rather Corollary 1.2) we will deduce:

Theorem 1.3. *Fix any S, β with β irreducible. Let $m := \frac{\beta(\beta-k)}{2}$ and $2h - 2 = \beta(\beta+k)$. The GW/PT correspondence² for $Z_\beta^{GW}(X, \tau_0(\text{pt})^m)$ and $Z_\beta^P(X, \tau_0(\text{pt})^m)$ is equivalent to the following equality*

$$Z_\beta^{GW}(X, \tau_0(\text{pt})^m) = t^m SW(\beta) (2 \sin(u/2))^{2h-2}.$$

In particular, setting $-q = e^{iu}$, the lowest order terms of $Z_\beta^{GW}(X, \tau_0(\text{pt})^m)$ and $Z_\beta^P(X, \tau_0(\text{pt})^m)$ in u coincide if and only if

$$SW(\beta) = \int_{[\overline{M}_{h,m}(S,\beta)]^{\text{vir}}} \prod_{i=1}^m \tau_0(\text{pt}).$$

We have a similar result for any S, β with $-K_S$ nef and β sufficiently ample (Remark 2.3). This shows that the GW/PT correspondence implies (a very special case of) Taubes' GW/SW correspondence [29, 30].

The second application of Theorem 1.1 is a universal formula for the difference of stable pair invariants in class β and $k - \beta$. Instead of the stable pair invariants (3), one can define *reduced* stable pair invariants of X in class β

$$P_{\chi,\beta}^{\text{red}}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)).$$

These originate from stable pair theory on $P_\chi(X, \beta)$ by removing a trivial part of rank $p_g(S) := h^{0,2}(S)$ from the obstruction bundle. The reduced invariants coincide with the usual invariants when $p_g(S) = 0$. Reduced stable pair invariants have been studied by many people: see [14] and references therein. Consider the surface part of these invariants for any number of point insertions³

$$P_{\chi,\beta}^{\text{red}}(S, \tau_0(\text{pt})^m).$$

We recall the definition in Appendix A, where we give a formula for the reduced virtual cycle (Proposition A.1). This formula is not used in the main body of this text, but is of independent interest. It extends a formula from [15, Appendix], which was derived under the following condition

$$(4) \quad H^2(L) = 0 \text{ for all line bundles } L \text{ with } c_1(L) = \beta.$$

When Condition (4) is satisfied it is shown in [15] that $P_{\chi,\beta}^{\text{red}}(S, \tau_0(\text{pt})^m)$ is given by a universal⁴ function in $\beta^2, \beta.c_1(S), c_1(S)^2, c_2(S)$, and certain invariants of the ring structure of $H^*(S, \mathbb{Z})$. The precise statement is recalled in Theorem A.2 of Appendix A. It is natural to ask whether universality holds for *all* invariants $P_{\chi,\beta}^{\text{red}}(S, \tau_0(\text{pt})^m), P_{\chi,\beta}(S, \tau_0(\text{pt})^m)$. We show that this is *not* the case

²I.e. [23, Conj. 3.3] but for X a non-compact Calabi-Yau 3-fold. See also [19, Sect. 1.4].

³So m need not be $\frac{\beta(\beta-k)}{2}$ as in Theorems 1.1 and 1.3.

⁴This universality result is used in the recent proof of the Katz-Klemm-Vafa conjecture for all curve classes by R. Pandharipande and R. P. Thomas [26].

(Remark A.3). The reason is as follows. Theorem 1.1 relates $P_{\chi,\beta}(S, \tau_0(\text{pt})^m)$ to Poincaré invariants. Using examples of [6] we observe that Poincaré invariants do not satisfy universality (Examples B.1, B.3, B.5 of Appendix B).

Despite failure of universality there is an interesting “duality” for surfaces with $p_g(S) = 0$. If β or $k - \beta$ satisfies Condition (4), then one of

$$P_{\chi,\beta}(S, \tau_0(\text{pt})^m), P_{\chi,k-\beta}(S, \tau_0(\text{pt})^m)$$

is given by a universal expression and the other is zero. These cases are covered by [15]. The new case is when neither β nor $k - \beta$ satisfies (4). Then universality can fail for the individual invariants $P_{\chi,\beta}(S, \tau_0(\text{pt})^m)$, $P_{\chi,k-\beta}(S, \tau_0(\text{pt})^m)$ (Examples B.1, B.3, B.5 of Appendix B), but their *difference* satisfies a nice duality formula. Combining Theorem 1.1 and the wall-crossing formula of Dürr-Kabanov-Okonek will lead to the following theorem:

Theorem 1.4. *Fix S, β such that $p_g(S) = 0$ and neither β nor $k - \beta$ satisfies Condition (4). If $\beta(\beta - k) < 0$, then*

$$P_{\chi,\beta}(S, \tau_0(\text{pt})^m) = P_{\chi,k-\beta}(S, \tau_0(\text{pt})^m) = 0.$$

If $\beta(\beta - k) \geq 0$, then $\beta(\beta - k) = 0$, $q(S) := h^{0,1}(S) = 1$, and⁵

$$P_{\chi,\beta}(S, \tau_0(\text{pt})^m) = P_{\chi,k-\beta}(S, \tau_0(\text{pt})^m) = 0, \text{ for } m > 0$$

$$\frac{Z_{\beta}^P(S)}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2\beta^2}} - \frac{Z_{k-\beta}^P(S)}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})^{2(k-\beta)^2}} = \frac{1}{2}[\beta] - \frac{1}{2}[k - \beta], \text{ for } m = 0.$$

Examples of S, β with $p_g(S) = 0$, $\beta(\beta - k) \geq 0$, and neither β nor $k - \beta$ satisfying Condition (4) are given in Remark B.7 of Appendix B. Such surfaces are necessarily elliptic fibrations or blow-ups thereof. The results of this paper make heavy use of the work of Dürr-Kabanov-Okonek [6]. For the purposes of readability, we take the opportunity to survey part of their work along the way.

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⁵The fact that $\beta(\beta - k) \geq 0$ implies $\beta(\beta - k) = 0$ and $q(S) = 1$ is a non-trivial result of Dürr-Kabanov-Okonek [6]. This fact and its proof are recalled in Section 3 (Proposition 3.1). The number $[\gamma] \in \mathbb{Z}$ for any $\gamma \in H^2(S, \mathbb{Z})$ on a surface with $q(S) = 1$ is defined as follows. The class γ determines an element $\int_S \gamma \wedge \cdot \in \Lambda^2 H^1(S, \mathbb{Z})^*$. Since $q(S) = 1$ we have a canonical isomorphism $\Lambda^2 H^1(S, \mathbb{Z})^* \cong \mathbb{Z}$ induced by choosing an integral basis of $H^1(S, \mathbb{Z}) \subset H^1(S, \mathbb{R})$ compatible with the orientation coming from the complex structure. The integer obtained in this way is denoted by $[\gamma]$.

2. POINCARÉ/PT CORRESPONDENCE

In this section we give a formula for the virtual cycle $[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{vir}}$ (Proposition 2.1). We then exploit the “product structure” of this formula to prove Main Theorem 1.1, Corollary 1.2, and Theorem 1.3.

2.1. Virtual cycle. Let $\mathcal{C} \subset H_\beta \times S \rightarrow H_\beta$ be the universal curve over the Hilbert scheme $H_\beta = \mathrm{Hilb}_\beta(S)$ of effective divisors in class β . Recall from the introduction that $\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \cong P_\chi(S, \beta)$ is a component of the \mathbb{C}^* -fixed locus of the full 3-fold stable pair space $P_\chi(X, \beta)$. Also recall that $\chi = 1 - h + n$, where h is the genus of curves in class β . We start with the natural embedding

$$\iota : \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta,$$

where $S^{[n]}$ is the Hilbert scheme of n points on S . A point (Z, C) lies in $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ if and only if

$$s_C|_Z = 0 \in H^0(\mathcal{O}_Z(C)),$$

where s_C is the section cutting out $C \subset S$. The family version of this goes as follows. Let $\mathcal{Z} \subset S^{[n]} \times S$ be the universal subscheme and let

$$\pi : S^{[n]} \times S \times H_\beta \rightarrow S^{[n]} \times H_\beta$$

denote projection. Then

$$(5) \quad \mathcal{O}(\mathcal{C})^{[n]} := \pi_*(\mathcal{O}(S^{[n]} \times \mathcal{C})|_{\mathcal{Z} \times H_\beta})$$

is a rank n vector bundle on $S^{[n]} \times H_\beta$. It has a tautological section σ with zero locus $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$. This provides $\mathrm{Hilb}^n(\mathcal{C}/H_\beta)$ with a *relative* perfect obstruction theory over H_β . This construction does not provide an absolute perfect obstruction theory because H_β can be singular. The notation (5) is chosen for the following reason. Consider projections

$$\begin{array}{ccc} & \mathcal{Z} & \\ p \swarrow & & \searrow q \\ S & & S^{[n]} \end{array}$$

Then for any line bundle L on S

$$L^{[n]} := q_* p^* L$$

is a rank n vector bundle on $S^{[n]}$ known as a tautological bundle (e.g. see [7]). It is not hard to see from the definitions that for any point $p = [C] \in H_\beta$

$$(6) \quad \mathcal{O}(\mathcal{C})^{[n]} \Big|_{S^{[n]} \times \{p\}} \cong \mathcal{O}(C)^{[n]}.$$

Dürr-Kabanov-Okonek [6] constructed a natural perfect obstruction theory on H_β of the form

$$(R\pi_*\mathcal{O}_{\mathcal{C}}(\mathcal{C}))^\vee \rightarrow \mathbb{L}_{H_\beta}.$$

In [14, Appendix] this perfect obstruction theory on H_β and the relative perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$ are combined to construct an *absolute* perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$. See diagram (89) of [14, Appendix] for details. We denote the corresponding virtual cycles on H_β and $\text{Hilb}^n(\mathcal{C}/H_\beta)$ by $[H_\beta]^{\text{vir}}$ and $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{vir}}$. It is shown in [14, Appendix], that $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{vir}}$ coincides with the virtual cycle induced by \mathbb{C}^* -localization of stable pair theory on $X = K_S$ to the component $\text{Hilb}^n(\mathcal{C}/H_\beta)$ of the \mathbb{C}^* -fixed locus. Although H_β can be singular, we still have the following:

Proposition 2.1. *For any S, β*

$$\iota_*[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{vir}} = (S^{[n]} \times [H_\beta]^{\text{vir}}).c_n(\mathcal{O}(\mathcal{C})^{[n]})$$

and its virtual dimension is $v = \frac{\beta(\beta-k)}{2} + n$.

For the proof of this proposition, we need the following lemma.

Lemma 2.2. *Let $\pi : M \rightarrow B$ be a flat morphism of \mathbb{C} -schemes of finite type with B projective. Let $E^\bullet \rightarrow \mathbb{L}_B, F^\bullet \rightarrow \mathbb{L}_M$ be perfect obstruction theories. Suppose that there exists a smooth projective variety A and a rank r vector bundle V on $A \times B$ with regular⁶ section s such that $M = s^{-1}(0) \subset A \times B$ and $\pi : M \rightarrow B$ commutes with projection $\pi_B : A \times B \rightarrow B$. This induces a canonical relative perfect obstruction theory $G^\bullet \rightarrow \mathbb{L}_{M/B}$ of the form $G^\bullet = \{V^*|_M \rightarrow \pi_A^*(\Omega_A)|_M\}$. Suppose there exists an exact triangle*

$$(7) \quad \pi^*E^\bullet \longrightarrow F^\bullet \longrightarrow G^\bullet.$$

Denote inclusion by $\iota : M \hookrightarrow A \times B$. Then

$$(8) \quad \iota_*[M]^{\text{vir}} = (A \times [B]^{\text{vir}}).c_r(V).$$

Proof. The content of the lemma is formula (8). For any perfect obstruction theory $F^\bullet \rightarrow \mathbb{L}_M$ with M projective, the following formula holds [28, Thm. 4.6] (see also [27])

$$(9) \quad [M]^{\text{vir}} = \left\{ s_\bullet(F^{\bullet\vee}) c_F(M) \right\}_v.$$

Here $s_\bullet(\cdot)$ is the total Segre class, v is the virtual dimension of M , and $c_F(M)$ is Fulton's canonical class which is defined as follows. Take any embedding $M \subset \mathcal{A}$ into a smooth variety \mathcal{A} , then

$$c_F(M) := c_\bullet(T_{\mathcal{A}}|_M) s_\bullet(C_{M/\mathcal{A}}),$$

⁶As defined in [9, B.3.4].

where $C_{M/A}$ is the normal cone of $M \subset \mathcal{A}$. This definition is independent of choice of embedding [9, Ex. 4.2.6]. Take an embedding $B \subset C$ into a smooth variety and consider

$$M \subset A \times B \subset A \times C =: \mathcal{A}.$$

By (7) we have

$$s_{\bullet}(F^{\vee}) = \pi^*(s_{\bullet}(E^{\vee})) \frac{c_{\bullet}(V|_M)}{\pi_A^*(c_{\bullet}(T_A))|_M}.$$

Since $M \subset A \times B$ is cut out by a regular section of V , we have

$$C_{M/A \times B} \cong N_{M/A \times B} \cong V|_M.$$

Consider the following short exact sequence of cones

$$N_{M/A \times B} \longrightarrow C_{M/A \times C} \longrightarrow C_{A \times B/A \times C}|_M.$$

We deduce

$$c_F(M) = \pi_A^*(c_{\bullet}(T_A))|_M \pi^*(c_{\bullet}(T_C|_B)) \frac{\pi^* s_{\bullet}(C_{B/C})}{c_{\bullet}(V|_M)}.$$

Formula (9) therefore implies

$$[M]^{\text{vir}} = \left\{ \pi^* \left(s_{\bullet}(E^{\vee}) c_{\bullet}(T_C|_B) s_{\bullet}(C_{B/C}) \right) \right\}_v = \pi^*[B]^{\text{vir}},$$

where the second equality follows from applying (9) to $E^{\bullet} \rightarrow \mathbb{L}_B$. The projection formula gives

$$\iota_*[M]^{\text{vir}} = (A \times [B]^{\text{vir}}) \cdot \iota_*[M].$$

Since $M \subset A \times B$ is cut out by a regular section of V , we have $\iota_*[M] = c_r(V)$ [9, Prop. 14.1] and the proposition is proved. \square

Proof of Proposition 2.1. Diagram (89) of [14, Appendix] provides the required exact triangle. It is left to show $\text{Hilb}^n(\mathcal{C}/H_{\beta}) \rightarrow H_{\beta}$ is flat and the tautological section σ of $\mathcal{O}(\mathcal{C})^{[n]}$ is regular. The fibre of the morphism $\text{Hilb}^n(\mathcal{C}/H_{\beta}) \rightarrow H_{\beta}$ over $C \in H_{\beta}$ is $C^{[n]}$, i.e. the Hilbert scheme of n points on the effective divisor C . The scheme $C^{[n]}$ is cut out by a tautological section of $L^{[n]}$ where $L := \mathcal{O}(C)$. Moreover, $C^{[n]} \subset S^{[n]}$ has codimension n (see [14, Footnote 18], which uses [1, 13]). Therefore $\sigma|_{S^{[n]} \times \{C\}}$ is regular for all $C \in H_{\beta}$. From this one can deduce that $\text{Hilb}^n(\mathcal{C}/H_{\beta}) \rightarrow H_{\beta}$ is flat and σ is regular. \square

2.2. Relation to Poincaré invariants. In Section 1 we introduced the stable pair invariants (1)

$$P_{\chi,\beta}(X, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m))$$

and the contribution to these invariants of the component $P_\chi(S, \beta) \cong \text{Hilb}^n(\mathcal{C}/H_\beta)$ of the \mathbb{C}^* -fixed locus

$$P_{\chi,\beta}(S, \tau_{\alpha_1}(\sigma_1) \cdots \tau_{\alpha_m}(\sigma_m)).$$

We only consider the case of primary point insertions

$$P_{\chi,\beta}(S, \tau_0(\text{pt})^m) = \int_{[P_\chi(S,\beta)]^{\text{vir}}} \frac{1}{e(N^{\text{vir}})} \tau_0(\text{pt})^m.$$

In the case $n = 0$, $\text{Hilb}^n(\mathcal{C}/H_\beta) \cong H_\beta$ and $[H_\beta]^{\text{vir}}$ was introduced many years ago by Dürr-Kabanov-Okonek [6, Def. 3.1]. They used this virtual cycle to define *Poincaré invariants*. We recall their definition. Consider the two Abel-Jacobi maps

$$\begin{aligned} \text{AJ}^+ : H_\beta &\rightarrow \text{Pic}^\beta(S), \\ \text{AJ}^- : H_{k-\beta} &\rightarrow \text{Pic}^{k-\beta}(S) \cong \text{Pic}^\beta(S), \end{aligned}$$

where $\text{Pic}^{k-\beta}(S) \cong \text{Pic}^\beta(S)$, $L \mapsto L^* \otimes K_S$. Then the Poincaré invariants are

$$(10) \quad P_S^+(\beta) := \text{AJ}_*^+ \left(\sum_i c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{\text{pt}\}})^i \cap [H_\beta]^{\text{vir}} \right),$$

$$P_S^-(\beta) := (-1)^{\chi(\mathcal{O}_S) + \frac{\beta(\beta-k)}{2}} \text{AJ}_*^- \left(\sum_i (-1)^i c_1(\mathcal{O}(\mathcal{C})|_{H_{k-\beta} \times \{\text{pt}\}})^i \cap [H_{k-\beta}]^{\text{vir}} \right).$$

In the first line, \mathcal{C} denotes the universal curve over H_β and in the second line, the universal curve over $H_{k-\beta}$. Note that⁷ $P_S^\pm(\beta) \in \Lambda^* H^1(S, \mathbb{Z})^*$. We write the (numerical) degree $2q(S)$ part of $P_S^+(\beta) \in \Lambda^* H^1(S, \mathbb{Z})^*$ by

$$P_S(\beta) \in \mathbb{Z}.$$

The product structure of the virtual cycle of Proposition 2.1 leads to Main Theorem 1.1 of the introduction:

Proof of Theorem 1.1. We want to calculate the invariant

$$(11) \quad P_{\chi,\beta}(S, \tau_0(\text{pt})^m) := \frac{1}{e(N^{\text{vir}})} \tau_0(\text{pt})^m \cap [\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{vir}},$$

⁷From the construction the Poincaré invariants take values in homology $H_*(\text{Pic}^\beta(S)) \cong \Lambda^* H^1(S, \mathbb{Z})$. We use Poincaré duality so the invariants take values in cohomology $\Lambda^* H^1(S, \mathbb{Z})^*$.

where $\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \cong P_\chi(S, \beta)$, and χ and n are related by $\chi = 1 - h + n$ (Section 1). Let $\varpi : \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \rightarrow H_\beta$ denote projection, then we claim

$$(12) \quad \tau_0(\mathrm{pt}) = \varpi^* c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{\mathrm{pt}\}}).$$

The proof can be found in [15, Proof Cor. 4.2], but we quickly reproduce it here. Consider the Cartesian diagram

$$\begin{array}{ccc} S & \xleftarrow{\pi_S} & \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \times S & \xrightarrow{\pi_P} & \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \\ & & \downarrow & & \downarrow \\ & & H_\beta \times S & \longrightarrow & H_\beta. \end{array}$$

By the definition (2), $\tau_0(\mathrm{pt}) = \pi_{P*}(\pi_S^*[\mathrm{pt}] \cdot c_1(\mathbb{F}))$, where \mathbb{F} is the universal sheaf on $\mathrm{Hilb}^n(\mathcal{C}/H_\beta) \times S$. Hence (12) follows from the fact that $c_1(\mathbb{F})$ is the pull-back of $c_1(\mathcal{O}(\mathcal{C}))$ from $H_\beta \times S$ and going around the Cartesian diagram.

In order to calculate $e(N^{\mathrm{vir}})$, we use a formula for the \mathbb{C}^* -equivariant K -theory class of N^{vir} from [15]. Consider the projections

$$\begin{array}{ccc} & S^{[n]} \times H_\beta & \\ p_1 \swarrow & & \searrow p_2 \\ S^{[n]} & & H_\beta. \end{array}$$

Then [15, Eqn. (12)] reads

$$(13) \quad [N^{\mathrm{vir}}] = [(\mathcal{O}(\mathcal{C})^{[n]})^* - p_1^* \Omega_{S^{[n]}} - p_2^*(R\pi_* \mathcal{O}_C(\mathcal{C}))^\vee] \Big|_{\mathrm{Hilb}^n(\mathcal{C}/H_\beta)} \otimes \mathfrak{t},$$

where \mathfrak{t} is the irreducible representation of \mathbb{C}^* of weight 1. Recall from (5) that $\mathcal{O}(\mathcal{C})^{[n]}$ is a vector bundle on $S^{[n]} \times H_\beta$, $R\pi_* \mathcal{O}_C(\mathcal{C})$ is a complex on H_β , and π denotes projection $H_\beta \times S \rightarrow H_\beta$. By pushing forward along the inclusion $\iota : \mathrm{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta$ and using (11), (12), (13), we see that $P_{\chi, \beta}(S, \tau_0(\mathrm{pt})^m)$ equals

$$\frac{e(p_1^* \Omega_{S^{[n]}} \otimes \mathfrak{t}) \cdot e(p_2^*(R\pi_* \mathcal{O}_C(\mathcal{C}))^\vee \otimes \mathfrak{t})}{e((\mathcal{O}(\mathcal{C})^{[n]})^* \otimes \mathfrak{t})} \cdot \varpi^* c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{\mathrm{pt}\}})^m \cap \iota_* [\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{vir}}.$$

Next we want to use the formula for $\iota_* [\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{vir}}$ from Proposition 2.1. Recall from the assumptions of the theorem that $m := \frac{\beta(\beta-k)}{2}$. Since the virtual dimension of $[H_\beta]^{\mathrm{vir}}$ is also $\frac{\beta(\beta-k)}{2}$, the cycle

$$c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{\mathrm{pt}\}})^m \cap [H_\beta]^{\mathrm{vir}}$$

is 0-dimensional and can be written as $\sum_i \mu_i p_i$, where μ_i are integers and $p_i = [C_i] \in H_\beta$ are points. Then

$$P_S(\beta) = \sum_i \mu_i$$

by definition of the Poincaré invariants (10). Therefore $P_{\chi, \beta}(S, \tau_0(\text{pt})^m)$ equals

$$\sum_i \mu_i \int_{S^{[n]}} \frac{e(p_1^* \Omega_{S^{[n]}} \otimes \mathfrak{t}) \cdot e(p_2^* (R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C}))^\vee \otimes \mathfrak{t})}{e((\mathcal{O}(\mathcal{C})^{[n]})^* \otimes \mathfrak{t})} c_n(\mathcal{O}(\mathcal{C})^{[n]}) \Big|_{S^{[n]} \times \{p_i\}}.$$

In order to go from equivariant Euler classes to Chern classes we use the following formula (e.g. [15, Eqn. (16)]). For any complex E of rank r

$$(14) \quad e(E \otimes \mathfrak{t}) = t^r c_{-1/t}(E^\vee),$$

where $c_x(E) = 1 + c_1(E)x + c_2(E)x^2 + \dots$ is the total Chern class and $t := c_1(\mathfrak{t})$ is the equivariant parameter. Define $L_i := \mathcal{O}_S(C_i)$, where $p_i = [C_i] \in H_\beta$ was introduced earlier in the proof. Then (6) implies

$$(15) \quad \mathcal{O}(\mathcal{C})^{[n]} \Big|_{S^{[n]} \times \{p_i\}} \cong L_i^{[n]}.$$

Similarly

$$(16) \quad p_2^* R\pi_* \mathcal{O}_{\mathcal{C}}(\mathcal{C}) \Big|_{S^{[n]} \times \{p_i\}} \cong R\Gamma(\mathcal{O}_{C_i}(C_i)) \otimes \mathcal{O}.$$

Using (14), (15), (16) shows that $P_{\chi, \beta}(S, \tau_0(\text{pt})^m)$ equals

$$\begin{aligned} & \sum_i \mu_i \int_{S^{[n]}} \frac{t^{2n} c_{-1/t}(T_{S^{[n]}}) \cdot t^{1-h+\beta^2} c_{-1/t}(R\Gamma(\mathcal{O}_{C_i}(C_i)) \otimes \mathcal{O})}{t^n c_{-1/t}(L_i^{[n]})} c_n(L_i^{[n]}) \\ &= \sum_i \mu_i \int_{S^{[n]}} t^{n+m} \left(\frac{-1}{t} \right)^n \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L_i^{[n]})} c_n(L_i^{[n]}) \\ (17) \quad &= (-1)^n t^m \sum_i \mu_i \int_{S^{[n]}} \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L_i^{[n]})} c_n(L_i^{[n]}), \end{aligned}$$

where the second equality uses $m := \frac{\beta(\beta-k)}{2}$ and the factor $(-1/t)^n$ arises from the fact that $c_n(L_i^{[n]})$ has degree n and $S^{[n]}$ has dimension $2n$.

By [7], for each n there exists a universal polynomial $P_n(x_1, x_2, x_3, x_4)$ such that for all i we have

$$P_n(c_1(L_i)^2, c_1(L_i) \cdot k, k^2, c_2(S)) = \int_{S^{[n]}} c_n(L_i^{[n]}) \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L_i^{[n]})}.$$

Since $c_1(L_i) = \beta$ for all i , all these integrals are the same. Using $P_S(\beta) = \sum_i \mu_i$, formula (17) becomes

$$(18) \quad P_{X,\beta}(S, \tau_0(\text{pt})^m) = (-1)^n t^m P_S(\beta) \int_{S^{[n]}} c_n(L^{[n]}) \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L^{[n]}},$$

where $L := L_i$ for arbitrary choice of i .

For *any* S, L , the integral in (18) is given by $P_n(c_1(L)^2, c_1(L) \cdot \mathbf{k}, \mathbf{k}^2, c_2(S))$. For any S, L with the additional property that L is globally generated, we can compute the integral in (18). If L is globally generated, we can write $L = \mathcal{O}(C)$ for a smooth curve $C \subset S$. Then the Hilbert scheme $C^{[n]}$ of n points on C is cut out smoothly and transversally by a tautological section of $L^{[n]}$. Hence

$$\int_{S^{[n]}} c_n(L^{[n]}) \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L^{[n]}} = \int_{C^{[n]}} c_n(T_{C^{[n]}}) = e(C^{[n]}).$$

These Euler characteristics are given by the well-known expression

$$\sum_{n=0}^{\infty} e(C^{[n]}) q^n = (1 - q)^{2g-2},$$

where g is the genus of C . We conclude that

$$(19) \quad P_n(c_1(L)^2, c_1(L) \cdot \mathbf{k}, \mathbf{k}^2, c_2(S)) = \int_{S^{[n]}} c_n(L^{[n]}) \frac{c_\bullet(T_{S^{[n]}})}{c_\bullet(L^{[n]}} = (-1)^n \binom{2g-2}{n},$$

where $2g-2 = c_1(L)^2 + c_1(L) \cdot \mathbf{k}$.

Since (19) holds for any S, L with L globally generated and P_n is polynomial, it holds for *any* S, L . The theorem follows by combining (18) and (19). \square

2.3. Application to Seiberg-Witten invariants. Dürr-Kabanov-Okonek conjectured that Poincaré invariants (10) are equal to Seiberg-Witten invariants from 4-manifold theory [6, Conj. 5.3]. Using a wall-crossing formula and blow-up formula for $P_S^\pm(\beta)$, they reduced their conjecture to a purely algebraic statement about $H_{\mathbf{k}}$, which was proved by Chang-Kiem [4]. By these (non-trivial!) results, we can write the degree $2q(S)$ part of $P_S^+(\beta)$ as

$$P_S(\beta) = SW(\beta) \in \mathbb{Z},$$

where $SW(\beta)$ are the original Seiberg-Witten invariant of S, β (see [32, 20]). Combining the Poincaré/PT correspondence of Theorem 1.1 with the (much deeper!) Poincaré/SW correspondence of [6, 4] gives Corollary 1.2. An application of this corollary is that for S, β with β *irreducible* and $m = \frac{\beta(\beta-\mathbf{k})}{2}$ point insertions the GW/PT correspondence encodes (a very special case of) Taubes' GW/SW correspondence [29, 30]. This is the content of Theorem 1.3 of the introduction:

Proof of Theorem 1.3. Since β is irreducible, $P_\chi(X, \beta)^{\mathbb{C}^*} \cong P_\chi(S, \beta)$ for all χ . Hence $P_\beta(X, \tau_0(\text{pt})^m) = P_\beta(S, \tau_0(\text{pt})^m)$ and the result follows from Theorem 1.1. Note that the equivariant parameter t of the leading term of both generating functions match by [14, Lem. 3.3]. \square

Remark 2.3. The following is a variation on Theorem 1.3. Fix any S, β such that $-K_S$ is nef and β is sufficiently ample⁸. Assume the GW/PT correspondence⁹ holds for $Z_\beta^{GW}(X, \tau_0(\text{pt})^m)$, $Z_\beta^P(X, \tau_0(\text{pt})^m)$. Also assume that the BPS spectrum of X is finite¹⁰. Then

$$Z_\beta^{GW}(X, \tau_0(\text{pt})^m) = t^m SW(\beta) (2 \sin(u/2))^{2h-2},$$

$$SW(\beta) = \int_{[\overline{M}'_{h,m}(S,\beta)]^{\text{vir}}} \prod_{i=1}^m \text{ev}_i^*[\text{pt}].$$

The proof goes as follows. Since $h \geq 1$ and the BPS spectrum is assumed finite, applying the coordinate transformation $-q = e^{iu}$ to $Z_\beta^{GW}(X, \tau_0(\text{pt})^m)$ gives a Laurent *polynomial* in q . Moreover, it is symmetric under $q \leftrightarrow q^{-1}$, so of the form

$$(20) \quad a_b q^{-b} + a_{b-1} q^{-(b-1)} + \cdots + a_{b-1} q^{b-1} + a_b q^b,$$

for some $b \geq 0$. By [14, Prop. 5.1], we have $P_\chi(X, \beta)^{\mathbb{C}^*} \cong P_\chi(S, \beta)$ for all $\chi \leq h - 1$. Combining this with Theorem 1.1 and (20) gives the result. \circlearrowright

Remark 2.4. One can speculate that for *any* algebraic S, β , Taubes' GW/SW correspondence follows from the GW/PT correspondence. This requires dealing with other components of $P_\chi(X, \beta)^{\mathbb{C}^*}$. Conversely, one can try to derive cases of the GW/PT correspondence for $X = K_S$ from Taubes' GW/SW correspondence as is done in Theorem 1.3. These are interesting questions for future research. \circlearrowright

3. WALL-CROSSING AND DUALITY

In this section we study the stable pair invariants $P_{\chi,\beta}(S, \tau_0(\text{pt})^m)$ for any m and any surface S with $p_g(S) = 0$. The results of [15] (recalled in Theorem A.2 of Appendix A) suggest that these invariants are *always* given by universal functions in the topological numbers β^2 , $\beta \cdot c_1(S)$, $c_1(S)^2$, $c_2(S)$ and certain invariants of the ring structure of $H^1(S, \mathbb{Z})$. In Appendix B we show that this is *not* the case. The reason is that $P_{\chi,\beta}(S, \tau_0(\text{pt})^m)$ is related to a Poincaré

⁸I.e. β such that $h \geq 1$ and β is $(4h - 3)$ -very ample [15, Prop. 5.1].

⁹The GW/PT correspondence has been proved in many cases [18, 19, 21, 22].

¹⁰I.e. after writing $Z_\beta^{GW}(X, \tau_0(\text{pt})^m)$ in BPS form [10, 11], [23, Eqn. (3.13)], we assume there are only finitely many nonzero $n_{g,\beta'}$.

invariant by Main Theorem 1.1 and it is easy to cook up surfaces S with $p_g(S) = 0$ whose Poincaré invariants are *not* given by universal functions (Examples B.1, B.3, B.5 of Appendix B).

However, Dürr-Kabanov-Okonek prove that when $p_g(S) = 0$ the difference of Poincaré invariants in class β and $\mathbf{k} - \beta$ satisfies a universal formula. Combining their formula with Main Theorem 1.1 gives an expression for the difference of $P_{\chi, \beta}(S, \tau_0(\text{pt})^m)$ and $P_{\chi, \mathbf{k} - \beta}(S, \tau_0(\text{pt})^m)$. This is Theorem 1.4 of the introduction and the second application of Main Theorem 1.1.

3.1. Dürr-Kabanov-Okonek's wall-crossing. We recall the wall-crossing formula for Poincaré invariants [6, Thm. 3.16]. Since we use this formula to establish Theorem 1.4, and for the sake of completeness, we recall Dürr-Kabanov-Okonek's interesting argument. Moreover, their results lead to a nice observation about the *reduced* virtual cycle for stable pairs, which is of independent interest (Proposition A.1 in Appendix A). The results and arguments presented in this section come *entirely* from their paper [6].

The following is contained in [6, Lem. 2.17] and its proof (see also [6, Cor. 3.15]).

Proposition 3.1 (Dürr-Kabanov-Okonek). *Let S be any surface. Suppose that β satisfies the following conditions:*

- (i) *For any effective $L \in \text{Pic}^\beta(S)$ with $c_1(L) = \beta$ we have $H^2(L) = 0$.
Note: this is automatic when $p_g(S) = 0$.*
- (ii) $\beta(\beta - \mathbf{k}) \geq 0$.
- (iii) H_β and $H_{\mathbf{k} - \beta}$ are both non-empty.

Then $\beta(\beta - \mathbf{k}) = 0$ and $\chi(\mathcal{O}_S) = 0$. Note: $\chi(\mathcal{O}_S) = 0$ is equivalent to $q(S) = 1$ when $p_g(S) = 0$.

Proof. The result follows by showing

$$\frac{\beta(\beta - \mathbf{k})}{2} + \chi(\mathcal{O}_S) = 0 \text{ and } \chi(\mathcal{O}_S) \geq 0.$$

Let $p : \text{Pic}^\beta(S) \times S \rightarrow \text{Pic}^\beta(S)$ denote projection and let \mathcal{P} be a choice of Poincaré bundle on $\text{Pic}^\beta(S) \times S$.

Condition (i) is equivalent to the statement that the images (Brill-Noether loci) of the two maps $H_{\mathbf{k} - \beta} \rightarrow \text{Pic}^{\mathbf{k} - \beta}(S) \cong \text{Pic}^\beta(S)$ and $H_\beta \rightarrow \text{Pic}^\beta(S)$ are disjoint. In other words, their complements U and V satisfy $\text{Pic}^\beta(S) = U \cup V$. Moreover, for any $L \in \text{Pic}^\beta(S)$, we have $H^2(L) = 0$ when $L \in U$ and $H^0(L) = 0$ when $L \in V$. In other words

$$R^2 p_* \mathcal{P}|_U = 0, \quad R^0 p_* \mathcal{P}|_V = 0.$$

This implies

$$\mathrm{rk} R p_* \mathcal{P} = \frac{\beta(\beta - \mathbf{k})}{2} + \chi(\mathcal{O}_S) \leq 0.$$

Since H_β and $H_{\mathbf{k}-\beta}$ are both non-empty (Condition (iii)), S cannot be rational because otherwise we get a section of K_S . Moreover, S cannot be ruled: for F the class of a fibre either $\beta.[F] < 0$ in which case $H_\beta = \emptyset$ or $\beta.[F] \geq 0$ in which case $(\mathbf{k} - \beta).[F] < 0$ so $H_{\mathbf{k}-\beta} = \emptyset$. Similarly, S cannot be the blow-up of a ruled surface. We conclude that the Kodaira dimension of S is ≥ 0 . Therefore $\chi(\mathcal{O}_S) \geq 0$ and

$$\frac{\beta(\beta - \mathbf{k})}{2} + \chi(\mathcal{O}_S) \geq 0. \quad \square$$

Theorem 3.2 (Dürr-Kabanov-Okonek). *Let S be a surface with $p_g(S) = 0$. Let \mathcal{P} be a choice of normalized Poincaré bundle on $\mathrm{Pic}^\beta(S)$, i.e. $\mathcal{P}|_{\mathrm{Pic}^\beta(S) \times \{\mathrm{pt}\}} \cong \mathcal{O}$. Denote projection by $p : \mathrm{Pic}^\beta(S) \times S \rightarrow \mathrm{Pic}^\beta(S)$. Then*

$$P_S^+(\beta) - P_S^-(\beta) = \sum_{i \geq 1 - \chi(\beta)} s_i(p! \mathcal{P}),$$

where $1 - \chi(\beta) = q(S) - \frac{\beta(\beta - \mathbf{k})}{2}$.

Proof. We first note that β satisfies Condition (4) of the introduction if and only if $H_{\mathbf{k}-\beta} = \emptyset$. Indeed if β satisfies Condition (4) we clearly have $H_{\mathbf{k}-\beta} = 0$. Conversely $H_{\mathbf{k}-\beta} = \mathbb{P}(R^2 p_* \mathcal{P})$ by [6, Lem. 2.15], so if $H_{\mathbf{k}-\beta} = \emptyset$ we have $R^2 p_* \mathcal{P} = 0$ and hence β satisfies Condition (4) by cohomology and base change. Similarly $\mathbf{k} - \beta$ satisfies Condition (4) if and only if $H_\beta = \emptyset$ (using $H_\beta = \mathbb{P}(R^2 p_* \mathcal{P}^*(K_S))$ [6, Lem. 2.15]).

The rest of the proof of [6] runs as follows. If $\frac{\beta(\beta - \mathbf{k})}{2} < 0$, then the virtual dimension of H_β and $H_{\mathbf{k}-\beta}$ are negative so the LHS is zero. Moreover the RHS is zero because of degree reasons ($\mathrm{Pic}^\beta(S)$ has dimension $q(S)$). For the remainder of the proof assume $\frac{\beta(\beta - \mathbf{k})}{2} \geq 0$.

Let \mathcal{P} be a choice of Poincaré bundle on $\mathrm{Pic}^\beta(S) \times S$ and let

$$p : \mathrm{Pic}^\beta(S) \times S \rightarrow \mathrm{Pic}^\beta(S)$$

denote projection. In Appendix A we describe a construction, which embeds H_β into a smooth ambient space in a natural way. For *sufficiently ample* divisor $A \subset S$ define $\gamma := [A] + \beta$ and let \mathcal{Q} be a choice of Poincaré bundle on $\mathrm{Pic}^\gamma(S) \times S$. Again we denote projection by $p : \mathrm{Pic}^\gamma(S) \times S \rightarrow \mathrm{Pic}^\gamma(S)$. By sufficient ampleness of A , the Abel-Jacobi map

$$\mathrm{AJ} : H_\gamma \longrightarrow \mathrm{Pic}^\gamma(S)$$

is a projective bundle and $H_\gamma \cong \mathbb{P}(p_*\mathcal{Q})$. Moreover we can embed $H_\beta \hookrightarrow H_\gamma$ by adding the divisor A . There exists a natural sheaf F on H_γ with tautological section cutting out $H_\beta \hookrightarrow H_\gamma$. Since $p_g(S) = 0$, the sheaf F is a vector bundle on a Zariski open neighbourhood of H_β . See Appendix A for the details. Let r be the rank of F and let $h := c_1(\mathcal{O}(1))$ on $\mathbb{P}(p_*\mathcal{Q})$. If $H_\beta \neq \emptyset$, then

$$\iota_*[H_\beta]^{\text{vir}} = c_r(F)$$

on H_γ (Proposition A.1 of Appendix A for $n = 0$) and

$$(21) \quad \text{AJ}_*(c_1(\mathcal{O}(\mathcal{C})|_{H_\beta \times \{\text{pt}\}})^i \cap [H_\beta]^{\text{vir}}) = \text{AJ}_*(c_r(F)h^i).$$

A similar formula holds for $[H_{\mathbf{k}-\beta}]^{\text{vir}}$ when $H_{\mathbf{k}-\beta} \neq \emptyset$. Moreover by [6, Prop. 2.18] (or [15, Lem. 4.3])

$$(22) \quad \text{AJ}_*(c_r(F)h^i) = s_{i-\chi(\beta)+1}(\tau_{\leq 1}p_!\mathcal{P}),$$

where $\chi(\beta)$ denotes the holomorphic Euler characteristic of β . Equation (22) also holds when $H_\beta = \emptyset$.

If β satisfies Condition (4) (i.e. $H_{\mathbf{k}-\beta} = \emptyset$), then F is a vector bundle on H_γ , and $s_i(p_!\mathcal{P}) = s_i(\tau_{\leq 1}p_!\mathcal{P})$. The formula follows from (22) and (21). If $\mathbf{k} - \beta$ satisfies Condition (4) (i.e. $H_\beta = \emptyset$), then the formula follows similarly using Serre duality $Rp_*\mathcal{P}^*(K_S) \cong (Rp_*\mathcal{P}[2])^\vee$.

We are left with the case where neither β nor $\mathbf{k} - \beta$ satisfies Condition (4), i.e. H_β and $H_{\mathbf{k}-\beta}$ are both non-empty. The wall-crossing formula is equivalent to

$$\sum_{i \geq 1-\chi(\beta)} \left(s_i(\tau_{\leq 1}p_!\mathcal{P}) + (-1)^i s_i(\tau_{\leq 1}p_!\mathcal{P}^*(K_S)) \right) = \sum_{i \geq 1-\chi(\beta)} s_i(p_!\mathcal{P}).$$

By Proposition 3.1, $\beta(\beta - \mathbf{k}) \geq 0$ and $H_\beta, H_{\mathbf{k}-\beta}$ are both non-empty implies $\chi(\mathcal{O}_S) = 0$ and $\beta(\beta - \mathbf{k}) = 0$. Since $p_g(S) = 0$, we have $q(S) = 1$. Since $s_1(\tau_{\leq 1}p_!\mathcal{P}) = c_1(R^1p_*\mathcal{P}) - c_1(p_*\mathcal{P})$, it suffices to show

$$s_1(\tau_{\leq 1}p_!\mathcal{P}^*(K_S)) = c_1(R^2p_*\mathcal{P}).$$

Take a locally free resolution $[E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2]$ of $Rp_*\mathcal{P}$. Then Serre duality $Rp_*\mathcal{P}^*(K_S) \cong (Rp_*\mathcal{P}[2])^\vee$ implies

$$\begin{aligned} s_1(\tau_{\leq 1}p_!\mathcal{P}^*(K_S)) &= c_1(\ker(d^{0*})) - c_1(E^{2*}) = c_1(E^2) + c_1((\text{coker } d^0)^*), \\ c_1(R^2p_*\mathcal{P}) &= c_1(E^2) - c_1(\text{im } d^1) = c_1(E^2) + c_1((E^1/\ker d^1)^*). \end{aligned}$$

In the proof of Proposition 3.1 we saw that $R^1p_*\mathcal{P}$ is torsion. Dualizing the short exact sequence

$$0 \rightarrow R^1p_*\mathcal{P} \rightarrow \text{coker } d^0 \rightarrow E^1/\ker d^1 \rightarrow 0$$

shows $(\text{coker } d^0)^* \cong (E^1/\ker d^1)^*$. This proves the desired result. \square

Proof of Theorem 1.4. Fix S, β such that $p_g(S) = 0$ and neither β nor $k - \beta$ satisfies Condition (4) of the introduction. If $\beta(\beta - k) < 0$, the virtual dimensions of $[H_\beta]^{\text{vir}}$ and $[H_{k-\beta}]^{\text{vir}}$ are zero and we use Proposition 2.1. Assume $\beta(\beta - k) \geq 0$. By Proposition 3.1 this implies $q(S) = 1$ and $\beta(\beta - k) = 0$. By Proposition 2.1, the invariants are zero when point insertions are present ($m > 0$). In the case $m = 0$, Theorem 1.1 implies

$$\begin{aligned} Z_\beta^P(S) &= P_S^+(\beta) (q^{1/2} + q^{-1/2})^{2\beta^2}, \\ Z_{k-\beta}^P(S) &= P_S^+(k - \beta) (q^{1/2} + q^{-1/2})^{2(k-\beta)^2} = P_S^-(\beta) (q^{1/2} + q^{-1/2})^{2(k-\beta)^2}. \end{aligned}$$

The result follows from Dürr-Kabanov-Okonek's wall-crossing formula Theorem 3.2 and a Grothendieck-Riemann-Roch computation giving $s_1(p_! \mathcal{P}) = \frac{1}{2}[2\beta - k]$. \square

APPENDIX A. REDUCED STABLE PAIR INVARIANTS

Recall from Section 2.1 that the natural embedding

$$\text{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\beta$$

can be realized as the zero locus of a tautological section of the vector bundle $\mathcal{O}(\mathcal{C})^{[n]}$ on $S^{[n]} \times H_\beta$ (see (5)). As we discussed, this induces a relative perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$. We mentioned how the absolute perfect obstruction theory on H_β of Dürr-Kabanov-Okonek was used in [14] to construct an absolute perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$.

The Hilbert scheme H_β has *another* perfect obstruction theory also originally discovered by Dürr-Kabanov-Okonek [6]. This perfect obstruction theory comes from embedding H_β in a compact smooth ambient space as follows. Let A be a sufficiently ample divisor¹¹ and define $\gamma := [A] + \beta$. Then the Abel-Jacobi map makes $H_\gamma := \text{Hilb}_\gamma(S)$ into a projective bundle over the Picard variety $\text{Pic}^\gamma(S)$. In particular, H_γ is smooth. Consider the closed embedding

$$H_\beta \hookrightarrow H_\gamma, \quad C \mapsto A \cup C.$$

A point D lies in the image of this map if and only if it contains A , i.e.

$$s_D|_A = 0 \in H^0(\mathcal{O}_A(D)),$$

where s_D denotes the section cutting out $D \subset S$. The family version of this goes as follows. Let $\mathcal{D} \rightarrow H_\gamma$ be the universal curve and $\pi : H_\gamma \times S \rightarrow H_\gamma$ projection. Then the *sheaf*

$$(23) \quad F := \pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$$

¹¹It suffices to pick A such that $H^1(L(A)) = H^2(L(A)) = 0$ for all $L \in \text{Pic}^\beta(S)$.

has a tautological section with zero locus H_β . Suppose that β satisfies the following condition (Condition (i) of Proposition 3.1)

$$(24) \quad H^2(L) = 0 \text{ for all effective line bundles } L \text{ with } c_1(L) = \beta.$$

Note that this condition is weaker than Condition (4) of the introduction. Then $H^1(\mathcal{O}_A(A + C)) = 0$ for any $C \in H_\beta$. By semicontinuity and base change, $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is zero on a Zariski open neighbourhood of H_β . Hence F is a vector bundle on a Zariski open neighbourhood of H_β . This construction gives a perfect obstruction theory on H_β which we refer to as the *reduced* perfect obstruction theory (this terminology was not used by Dürri-Kabanov-Okonek). We denote the corresponding virtual cycle by $[H_\beta]^{\text{red}}$. The reduced perfect obstruction theory on H_β can be combined with the relative perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$ to give another absolute perfect obstruction theory on $\text{Hilb}^n(\mathcal{C}/H_\beta)$. This was carried out in [15, Appendix]. It turns out that the resulting virtual cycle $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{red}}$ coincides with the one coming from \mathbb{C}^* -localization of *reduced* stable pair theory of $X = K_S$ to the component $\text{Hilb}^n(\mathcal{C}/H_\beta)$ of the \mathbb{C}^* -fixed locus [15, Appendix]. Note that Condition (24) is automatic when $p_g(S) = 0$. In this case one can show that $[H_\beta]^{\text{red}} = [H_\beta]^{\text{vir}}$ and $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{red}} = [\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{vir}}$.

If β satisfies the stronger condition

$$H^2(L) = 0 \text{ for all line bundles } L \text{ with } c_1(L) = \beta,$$

i.e. Condition (4) of the introduction, then $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A}) = 0$ on H_γ and F is a vector bundle on H_γ . Denote the embedding

$$\text{Hilb}^n(\mathcal{C}/H_\beta) \hookrightarrow S^{[n]} \times H_\gamma$$

by ι . Similarly to (5), define

$$\mathcal{O}(\mathcal{D} - A)^{[n]} := \pi_* \left(\mathcal{O}(S^{[n]} \times \mathcal{D} - S^{[n]} \times A \times H_\gamma) \Big|_{\mathcal{Z} \times H_\gamma} \right),$$

where $\pi : S^{[n]} \times S \times H_\gamma \rightarrow S^{[n]} \times H_\gamma$ denotes projection. When Condition (4) holds one can compute the virtual cycle as follows [14, Thm. A.7]

$$(25) \quad \iota_* [\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{red}} = c_r(F) \cdot c_n(\mathcal{O}(\mathcal{D} - A)^{[n]}),$$

where $r := \chi(\beta(A)) - \chi(\beta)$. Here $\chi(\beta)$ is the holomorphic Euler characteristic of curves in H_β

$$2\chi(\beta) = \beta(\beta - k) + 2\chi(\mathcal{O}_S)$$

and $\chi(\beta(A))$ is defined similarly. The virtual dimension of $[\text{Hilb}^n(\mathcal{C}/H_\beta)]^{\text{red}}$ is

$$v = \frac{\beta(\beta - k)}{2} + p_g(S) + n,$$

which is $p_g(S)$ larger than the virtual dimension of $[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{vir}}$.

When only the weaker Condition (24) is satisfied we make the following somewhat surprising observation, which is more or less an immediate corollary of [6, Lem. 2.17].

Proposition A.1. *Fix S, β such that Condition (24) is satisfied, $H_\beta \neq \emptyset$, and $\beta(\beta - k) \geq 0$. Then F is a vector bundle on H_γ even though $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is in general non-zero. Consequently*

$$\iota_*[\mathrm{Hilb}^n(\mathcal{C}/H_\beta)]^{\mathrm{red}} = c_r(F) \cdot c_n(\mathcal{O}(\mathcal{D} - A))^{[n]}$$

and its virtual dimension is $v = \frac{\beta(\beta-k)}{2} + p_g(S) + n$.

Proof. Let $p : \mathrm{Pic}^\beta(S) \times S \rightarrow \mathrm{Pic}^\beta(S)$ be projection and let \mathcal{P} be a choice of Poincaré bundle on $\mathrm{Pic}^\beta(S) \times S$. Let

$$\mathbb{E} := [E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2]$$

be a resolution of $Rp_*\mathcal{P}$ by locally free sheaves. Then Dürr-Kabanov-Okonek found out that $\ker d^1$ is locally free (Claim). The reason for Claim is the following. If $H_{k-\beta} = \emptyset$, then $R^2p_*\mathcal{P} = 0$ because $H_{k-\beta} = \mathbb{P}(R^2p_*\mathcal{P})$ [6, Lem. 2.15]. In this case d^1 is surjective and $\ker d^1$ is locally free. Suppose $H_\beta, H_{k-\beta}$ are both non-empty. Then we saw in Proposition 3.1 and its proof (i.e. [6, Lem. 2.17] and its proof) that

$$\begin{aligned} R^2p_*\mathcal{P}|_U &= 0, \quad R^0p_*\mathcal{P}|_V = 0, \\ \mathrm{Pic}^\beta(S) &= U \cup V, \quad \mathrm{rk} Rp_*\mathcal{P} = 0, \end{aligned}$$

where U, V are the complements of the images of $H_{k-\beta} \rightarrow \mathrm{Pic}^{k-\beta}(S) \cong \mathrm{Pic}^\beta(S)$, $H_\beta \rightarrow \mathrm{Pic}^\beta(S)$. We see at once that $R^1p_*\mathcal{P}$ is torsion. Moreover $R^1p_*\mathcal{P}|_V$ is a subsheaf of $E^1/\mathrm{im} d^0|_V$. Also $E^1/\mathrm{im} d^0|_V$ is locally free because $(d^0)^*|_V$ is surjective. This implies $R^1p_*\mathcal{P}|_V$ is zero. Therefore $\ker d^1|_V = \mathrm{im} d^0|_V = E^0|_V$ is locally free. Since we already know $\ker d^1|_U$ is locally free (because $d^1|_U$ is surjective), this establishes Claim.

Back to the resolution \mathbb{E} of $Rp_*\mathcal{P}$. Take \mathbb{E} of the following form. Let $[E^1 \xrightarrow{d^1} E^2]$ be a resolution of $Rp_*\mathcal{P}_A(A)$ by locally free sheaves and set $E^0 := p_*\mathcal{P}(A)$. Note that $p_*\mathcal{P}(A)$ is locally free by choice of A . We define \mathbb{E} by the following diagram of exact triangles

$$\begin{array}{ccccc} \mathbb{E} & \dashrightarrow & E^0 & \dashrightarrow & [E^1 \xrightarrow{d^1} E^2] \\ \downarrow & & \parallel & & \downarrow \cong \\ Rp_*\mathcal{P} & \longrightarrow & p_*\mathcal{P}(A) & \longrightarrow & Rp_*\mathcal{P}_A(A). \end{array}$$

Here \mathcal{P}_A is short hand for $\mathcal{P}|_{H_\beta \times A}$. By Claim, $p_*\mathcal{P}_A(A) \cong \ker d^1$ is locally free. Next, let \mathcal{Q} be a choice of Poincaré bundle on $\text{Pic}^\gamma(S)$. The Abel-Jacobi map

$$\text{AJ} : H_\gamma = \mathbb{P}(p_*\mathcal{Q}) \rightarrow \text{Pic}^\gamma(S)$$

is a projective bundle with tautological bundle $\mathcal{O}(1)$. Note that $\mathcal{Q}(1) \cong \mathcal{O}(\mathcal{D})$ on $H_\gamma \times S$, therefore (23)

$$F = \pi_*\mathcal{O}(\mathcal{D}|_{H_\gamma \times A}) \cong \text{AJ}^*(p_*\mathcal{Q}_A)(1).$$

Since $\text{Pic}^\beta(S) \cong \text{Pic}^\gamma(S)$ sends $p_*\mathcal{P}_A(A)$ to $p_*\mathcal{Q}_A$ we indeed see that F is locally free. Finally the proposition states that $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is in general non-zero. This is proved in Remark A.3 below. \square

If S, β satisfies Condition (4), then the invariants $P_{\chi, \beta}^{\text{red}}(S, \tau_0(\text{pt})^m)$ are calculated in [15] in the following sense. Via wedging together and integrating over S , the classes $\beta, \mathbf{k} \in H^2(S, \mathbb{Z})$, and $1 \in H^4(S, \mathbb{Z})$ give elements

$$[\beta], [\mathbf{k}] \in \Lambda^2 H^1(S, \mathbb{Z})^*, \text{ and } [1] \in \Lambda^4 H^1(S, \mathbb{Z})^*.$$

Wedging together any combination produces an element

$$\Lambda^a[\beta] \wedge \Lambda^b[\mathbf{k}] \wedge \Lambda^c[1] \in \Lambda^{2q(S)} H^1(S, \mathbb{Z})^* \cong \mathbb{Z}, \text{ where } a + b + 2c = q(S).$$

Here the canonical isomorphism with \mathbb{Z} comes from choosing any integral basis of $H^1(S, \mathbb{Z}) \subset H^1(S, \mathbb{R})$ compatible with the orientation coming from the complex structure. We then have:

Theorem A.2. [15, Thm. 1.2] *Fixing q, p_g, m, n , there exists a universal function $F_{q, p_g, m, n}(\mathbf{x})$ with variables $\mathbf{x} := (x_1, x_2, x_3, x_4, \{x_{abc}\}_{a+b+2c=q}, t)$ such that for any S with $q(S) = q$, $p_g(S) = p_g$, and $\beta \in H^2(S, \mathbb{Z})$ satisfying Condition (4), $P_{\chi, \beta}^{\text{red}}(S, \tau_0(\text{pt})^m)$ is equal to*

$$F_{q, p_g, m, n}(\beta^2, \beta \cdot \mathbf{k}, \mathbf{k}^2, c_2(S), \{\Lambda^a[\beta] \wedge \Lambda^b[\mathbf{k}] \wedge \Lambda^c[1]\}_{a+b+2c=q}, t),$$

where $\chi = 1 - h + n$ and $2h - 2 = \beta(\beta + \mathbf{k})$ is the arithmetic genus of β .

Remark A.3. Suppose the setting is as in Proposition A.1. We now explain why $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ is in general non-zero. In Proposition A.1 we show that F is a vector bundle and the reduced virtual cycle is given by (25) when the weaker Condition (24) is satisfied. If $R^1\pi_*(\mathcal{O}(\mathcal{D})|_{H_\gamma \times A})$ were zero, then the invariants $P_{\chi, \beta}^{\text{red}}(S, \tau_0(\text{pt})^m)$ satisfy the same universal formula as Theorem A.2 by the calculation of [15]. However we show by explicit examples in Appendix B that some invariants $P_{\chi, \beta}^{\text{red}}(S, \tau_0(\text{pt})^m)$ do *not* satisfy universality (Examples B.1, B.3, B.5).

APPENDIX B. FAILURE OF UNIVERSALITY: EXAMPLES

In this appendix we show that Theorem A.2 does *not* hold for all stable pair invariants $P_{\chi,\beta}^{\text{red}}(S, \tau_0(\text{pt})^m)$, $P_{\chi,\beta}(S, \tau_0(\text{pt})^m)$. By Main Theorem 1.1 it suffices to prove $P_S(\beta)$ is not given by universal functions. We show this on elliptic surfaces with $p_g(S) = 0$ using calculations of Dürr-Kabanov-Okonek [6].

Let $\pi : S \rightarrow C$ be an elliptic fibration over a curve of genus g [2, Ch. V.6]. We are only interested in the case S, C are algebraic. The generic fibre F is a smooth elliptic curve and we denote by m_1F_1, \dots, m_rF_r the multiple fibres. The canonical divisor is given by

$$(26) \quad K_S = \pi^*D + \sum_{i=1}^r (m_i - 1)F_i,$$

for some divisor D of degree $2g - 2 + \chi(\mathcal{O}_S)$ on C [2, Cor. 12.3]. In this section, we will make frequent use of logarithmic transformations [2, Ch. V.13]. Given a generic point $x \in C$, a logarithmic transformation replaces the fibre F over x by a multiple mF , $m > 1$. The new elliptic fibration $\pi' : S' \rightarrow C$ has fibre mF over $x \in C$ and the restrictions $\pi^{-1}(C \setminus \{x\})$, $\pi'^{-1}(C \setminus \{x\})$ are biholomorphic as fibre bundles over $C \setminus \{x\}$. One should not think of a logarithmic transformation as a sort of birational transformation. The topology of S can change and S can even become non-algebraic [2, Ch. V.13].

Example B.1. Let $\mathbb{P}^1 \subset |\mathcal{O}(3)|$ be a generic pencil of cubics on \mathbb{P}^2 and let $S \rightarrow \mathbb{P}^1$ be the universal curve. This is a rational elliptic fibration so $q(S) = p_g(S) = 0$ and $K_S = -F$ (Equation (26)). We take $\beta = 6k$. Clearly $|6K_S| = \emptyset$ so $P_S(\beta) = 0$. Let S' be obtained from S by replacing one general fibre F by a double fibre $2F_1$ and another by a triple fibre $3F_2$. Then S' is one of the famous Dolgachev surfaces¹². The surface S' is known to be algebraic satisfying $q(S') = p_g(S') = 0$ and $K_{S'} = -F + F_1 + 2F_2$ (Equation (26)). In the Chow group, one has relations $2F_1 = 3F_2 = F$ so $k' = \frac{1}{6}[F]$ in $H^2(S', \mathbb{Q})$. Taking $\beta' = 6k'$, we see that $|6K_{S'}| = |F| \neq \emptyset$, whereas $|K_{S'} - 6K_{S'}| = |-5K_{S'}| = \emptyset$. Consequently, $P_{S'}(k' - \beta') = 0$. Dürr-Kabanov-Okonek's wall-crossing formula (Theorem 3.2) states $P_{S'}(\beta') - P_{S'}(k' - \beta') = 1$, so $P_{S'}(\beta') = 1$. Since the Chern numbers of S, β and S', β' are the same, this is a counter-example to universality. Note that this does not contradict Theorem A.2. Indeed

$$H^2(\mathcal{O}(6K_S)) = H^0(\mathcal{O}(-5K_S)) = H^0(\mathcal{O}(5F)) \neq 0,$$

so β' satisfies Condition (4) but β only satisfies the weaker Condition (24). \circ

¹²The surfaces S, S' provide homeomorphic compact simply connected 4-manifolds. S. K. Donaldson famously proved their C^∞ -structures are different [5]. One can also establish this by showing their Seiberg-Witten invariants are distinct (see [20]).

In order to find more counter-examples to universality, we use the following result [6, Prop. 4.8] (see also R. Friedman and J. W. Morgan [8, Prop. 4.4]).

Proposition B.2 (Dürr-Kabanov-Okonek). *Suppose $\beta \in H^2(S, \mathbb{Z})$ satisfies $\beta^2 = \beta \cdot [F] = 0$. Then*

$$P_S(\beta) = \sum_{\substack{d[F] + \sum_i a_i [F_i] = \beta \\ d \geq 0, 0 \leq a_i < m_i}} (-1)^d \binom{2g - 2 + \chi(\mathcal{O}_S)}{d}.$$

Here we should recall the usual conventions on binomial coefficients. For each $b \geq 0$, define

$$\binom{a}{b} = \frac{1}{b!} a(a-1) \cdots (a-b+1).$$

In particular, $\binom{a}{b} = 1$ for $b = 0$, $\binom{a}{b} = 0$ for $0 \leq a < b$, and $\binom{-a}{b} = (-1)^b \binom{a+b-1}{b}$.

Example B.3. Let S be an hyper-elliptic surface and $\beta = d[F]$ for any $d \geq 0$. Note that $q(S) = 1$ and $p_g(S) = 0$. Proposition B.2 implies $P_S(\beta) = 0$ for $d > 0$ and $P_S(\beta) = 1$ for $d = 0$. Since $\beta^2 = \beta \cdot \mathbf{k} = \mathbf{k}^2 = c_2(S) = [\beta] = [\mathbf{k}] = 0$ for any $d \geq 0$, this also provides a counter-example to universality. Although K_S is a non-trivial torsion element of $A^1(S)$, its class $\mathbf{k} = 0 \in H^2(S, \mathbb{Q})$. The class $\beta = 0$ satisfies Condition (24) but not the stronger Condition (4) since $H^2(\mathcal{O}(K_S)) \neq 0$. Hence, there is no contradiction with Theorem A.2. \circ

Finally, we apply Proposition B.2 to a special class of logarithmic transformations discussed in [6, Sect. 4.2]. They will provide more interesting counter-examples to universality. We recall their construction. Fix an elliptic curve $F = \mathbb{C}/\Gamma$ with lattice $\Gamma = \langle 1, \omega \rangle \subset \mathbb{C}$. We apply logarithmic transformations to $\mathbb{P}^1 \times F \rightarrow \mathbb{P}^1$ as follows. Fix a point $t_1 \in \mathbb{P}^1$ and an m_1 -torsion point $\zeta_1 \in F$, $m_1 > 0$. The logarithmic transformation $L_{t_1}(m_1, \zeta_1)(\mathbb{P}^1 \times F)$ replaces the fibre over t_1 by $m_1 F_1$. Continuing in this fashion with other distinct points $t_2, \dots, t_r \in \mathbb{P}^1$, one obtains a smooth compact complex surface

$$S := L_{\underline{t}}(\underline{m}, \underline{\zeta})(\mathbb{P}^1 \times F),$$

which is an elliptic fibration over \mathbb{P}^1 . It has generic fibre F and multiple fibres $m_1 F_1, \dots, m_r F_r$. The following proposition [6, Sect. 4.2] summarizes the relevant geometry.

Proposition B.4 (Dürr-Kabanov-Okonek). *Suppose ζ_1, \dots, ζ_r are of the form $\zeta_i = \frac{u_i + v_i \omega}{m_i}$ for integers u_i, v_i satisfying $\gcd(m_i, u_i, v_i) = 1$.*

- (1) *The surface S is projective if and only if $\sum_{i=1}^r \zeta_i = 0$.*

(2) Suppose (1) is satisfied. Then $H^2(S, \mathbb{Z}) \cong \mathbb{Z} \oplus G$, where G is the free abelian group generated by $[F], [F_1], \dots, [F_r]$ modulo the relations

$$\begin{aligned} m_1[F_1] &= \dots = m_r[F_r] = [F], \\ u_1[F_1] + \dots + u_r[F_r] &= 0, \quad v_1[F_1] + \dots + v_r[F_r] = 0. \end{aligned}$$

(3) Suppose (1) is satisfied. Let $\Gamma' \subset \mathbb{C}$ be the lattice generated by the elements $1, \omega, \zeta_1, \dots, \zeta_r$ and consider the Albanese map $\text{Alb} : S \rightarrow \text{Alb}(S)$. Then there exists an isomorphism $\text{Alb}(S) \cong \mathbb{C}/\Gamma'$ such that the following diagram commutes

$$\begin{array}{ccc} F & \xrightarrow{\text{Alb}|_F} & \text{Alb}(S) \\ \parallel & & \downarrow \cong \\ \mathbb{C}/\Gamma & \longrightarrow & \mathbb{C}/\Gamma', \end{array}$$

where the bottom map is induced by $\Gamma \subset \Gamma'$.

The following example is used in [6, Ex. 4.14] to provide a surface S with $p_g(S) = 0$ and $P_S(\mathbf{k}) \neq 0$. We use it to give an interesting example where universality fails.

Example B.5. Take $\zeta_1 = \frac{1+\omega}{3}$, $\zeta_2 = \frac{1}{3}$, $\zeta_3 = \frac{1}{3}$, and $\zeta_4 = -\frac{3+\omega}{3}$. By Proposition B.4, S is projective, $[F] = 3[F_1]$, $[F_4] = [F_1]$, $[F_3] = 2[F_1] - [F_2]$, and

$$\begin{aligned} H^2(S, \mathbb{Z}) &\cong \mathbb{Z} \oplus \langle [F_1], [F_2] \mid 3[F_1] = 3[F_2] \rangle_{\mathbb{Z}} \\ &\cong \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

By Equation (26), $K_S = 2F_1$ and $\mathbf{k} = \frac{2}{3}[F] \in H^2(S, \mathbb{Q})$. We fix $\beta = n[F_1] + \epsilon[F_2]$ with $n \in \mathbb{Z}_{\geq 0}$ and $\epsilon = 0, 1, 2$. The surface S satisfies $q(S) = 1$ and $p_g(S) = 0$. Clearly, $\beta^2 = \beta \cdot \mathbf{k} = \mathbf{k}^2 = c_2(S) = 0$. Let E be the class of the fibre of $S \rightarrow \text{Alb}(S)$. Then Proposition B.4 (3) implies $E \cdot F = 9$. Hence $[\beta] = \beta \cdot E = 3(n + \epsilon)$. By Proposition B.2

$$P_S(\beta) = \sum_{\substack{d \geq 0, \\ a_i = 0, 1, 2}} (d+1). \quad (3d + a_1 + 2a_3 + a_4)[F_1] + (a_2 - a_3)[F_2] = n[F_1] + \epsilon[F_2]$$

For all $(n, \epsilon) \notin \{(0, 0), (1, 0), (2, 0), (0, 1)\}$, this is equal to

$$P_S(\beta) = [\beta] - 3.$$

For $(n, \epsilon) = (0, 0), (1, 0), (2, 0), (0, 1)$, we get the sporadic values

$$P_S(\beta) = 1, 2, 4, 1.$$

This gives another counter-example to universality. ◻

Remark B.6. One can consider reduced stable pair invariants with other insertion classes such as

$$(27) \quad P_{\chi, \beta}^{\text{red}}(S, \tau_0(\text{pt})^m \tau_0(\gamma_1) \dots \tau_0(\gamma_{2q(S)})),$$

where $\gamma_1, \dots, \gamma_{2q(S)} \in H_1(S)/\text{torsion}$ is an integral oriented basis [15, Sect. 3]. The H_1 -insertions cut H_β down to a linear system $|L| \subset H_\beta$. Fix any S, β with β satisfying Condition (24) but not necessarily the stronger Condition (4). Suppose $H_\beta \neq \emptyset$ and $\beta(\beta - k) \geq 0$. Using Proposition A.1, it is easy to see that [15, Sect. 3] continues to hold. Therefore (27) is given by a universal polynomial in $\beta^2, \beta.k, k^2, c_2(S)$ exactly as in [15, Thm. 1.1]. Note that this does not contradict Example B.1 where $|6K_S| = \emptyset$. \circlearrowright

Remark B.7. The conditions for the duality formula of Theorem 1.4 are: $p_g(S) = 0$, $H_\beta, H_{k-\beta}$ are both non-empty, and $\beta(\beta - k) \geq 0$. Proposition 3.1 implies $\beta(\beta - k) = 0$, $q(S) = 1$, and S is not a ruled surface or a blow-up of a ruled surface. Therefore S is hyper-elliptic, minimal properly elliptic, or a blow-up thereof. Conversely, any hyper-elliptic surface S or blow-up thereof with $\beta = k$ satisfies the conditions of Theorem 1.4. These examples are boring because $P_S^\pm(k) = 1$ (by Example B.3 and the blow-up formula [6, Thm. 3.12]). More exciting examples are provided by S as in Proposition B.4 and $\beta = k$. From Theorem 3.2 it follows that these surfaces generally have $H_k \neq \emptyset$. Blowing up these surfaces, one obtains examples with $H_k \neq \emptyset$ and $k^2 \neq 0$. \circlearrowright

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