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STUDENTS' THINKING MODES AND THE EMERGENCE OF SIGNS IN LEARNING LINEAR ALGEBRA

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To analyse the use of a dynamic geometry environment for linear algebra by two students, we combine a semiotic mediation approach with a lens of students' thinking modes. This way of approaching the data shows promise for a detailed understanding of the observed phenomena.

RESEARCH QUESTION AND THEORETICAL FRAMEWORK

In this paper, we address the research question, 'Does a combination of the semiotic mediation approach and students' thinking modes provide a better understanding of the learning of the notion of parameter using a dynamic geometry environment (DGE)?' To elaborate on this question, we first consider the notion of thinking modes and the theory of semiotic mediation as a theoretical framework, and then reanalyse certain of the data presented in Turgut (2015).

Thinking modes in learning linear algebra

Sierpinska (2000) distinguishes three thinking modes (TM) in linear algebra; synthetic-geometric thinking (GST), analytic-arithmetic thinking (AAT) and analytic-structural thinking (AST). The GST mode refers to the geometric properties of the action, but does not concern their construction. The AAT mode refers to associated computations and numerical representations; the AST mode refers to the dialectics between epistemic and semantic aspects of the mathematical object. Consequently, synthetic thinking is related to visual perception, while analytical thinking refers to mathematical reasoning.

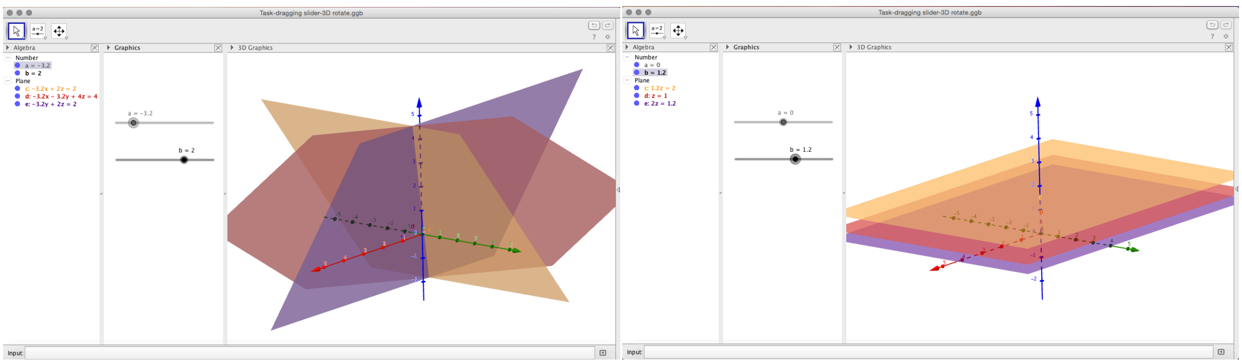
Theory of Semiotic Mediation (TSM)

TSM (Bartolini Bussi & Mariotti, 2008) is based on two principal notions; semiotic potential of an artefact and didactic cycles. The first requires an epistemological analysis of an artefact's evocative power for the construction of mathematical meanings. Didactic cycles refer to the design of teaching-learning activities. The signs that emerge are categorized: (i) the signs related to the activity and specifically referring to the use of the artefact are named as artefact signs (aS), (ii) math signs they are used in the math community (i.e., students' construction of a definition, a conjecture, a generalization or appearance of a proof and so on) and constitute the didactic goal are named mathematical signs (mS), (iii) the signs where their potential polysemy are used for the construction of the (interpretative) links between personal signs and math signs are named pivot signs (pS).

Mathematical context and semiotic potential of DGE

Solving linear equations not only corresponds to finding the intersection of lines in a plane, but also to intersecting planes or hyperplanes in three or higher dimensional spaces, which is needed for generalization towards linear systems. The key element in such computations is the notion of parameter[1], in relation to the different kinds of representations; algebraic, geometric and matrix forms of the associated system of linear equations in $\mathbf{R}^3$. To articulate these views, GeoGebra provides simultaneous Algebra and 3D Graphics windows (Turgut, 2015). In this case, we distinguish the following tools and functions providing semiotic potential for teaching and learning the notion of parameter: (i) the *Dragging* function enables the user to manipulate given objects, but also to explore dependency and co-variance of objects; (ii) the *Slider* tool provides the means for dynamic (co)variation, meaning the user can observe the variations synchronously in the Algebra and in other windows; (iii) the *Rotate 3D view* tool is included in the 3D Graphics window, providing user spatial orientation, i.e., to explore the 3D Graphics window from a different perspective.

As an exemplary task, think of the following system of parametrized linear equations (Anton, 1981, p. 54); $ax+bz=2$, $ax+by+4z=4$, $ay+2z=b$. The number of solutions of the system (i.e., no solution, one single solution or infinitely many solutions) depends on the values of the parameters a and b, since such parameters effect the position of the corresponding planes. In terms of the tools and functions summarized above, we postulate that students' dragging of the slider(s) would provide a dynamic variation on the system of linear equations, its associated augmented matrix, the corresponding planes and the number of solutions. As such, this may evoke an understanding of the role of parameters. Therefore, the task delivered to the students was, 'Move the sliders a and b, and systematically explore and explain what is happening'. Some exemplary cases of the task are presented in Figure 1. In this case study, two linear algebra students worked on this task. The students were 20 year olds at undergraduate sophomore level, with a background in coordinate free geometry, discrete mathematics and fundamental calculus. The session took place after a course on the topic of the system of linear equations and lasted 40 minutes.



(a): the case $a \neq 0$, $b=2$ (infinite solutions) (b): the case $a=0$, $b \neq 2$ (no solution)

Figure 1. Two exemplary cases of students' parameter values of the task

RESULTS

Analysis from a TM perspective

Students' GST was apparent in understanding the relationships between the movements of the sliders and the positions of the planes. The students only focused on the movements of the planes; they did not focus on the planes' equations and the algebraic meanings of a and b, although they

[1] For example, in the same context, the intersection of two non-parallel hyperplanes in $\mathbf{R}^4$ can be either a hyperline or a hyperplane. It depends on the number of parameters in the solution.

knew the algebraic equations of planes from previous lectures. While moving the sliders, they discovered the parameters' effects on the position of the planes. As a next step, AAT appeared when they showed paradigmatic reasoning of the equations in the Algebra window, and on the algebraic meaning of *a* and *b* in the system of linear equations. Consequently, they tried to solve the system of linear equations through the Gauss-Jordan elimination method to understand how *a* and *b* are related in the system's solution. Their study of the rows of the matrix and the movements of the sliders opened a door to connecting particular solutions of the system in terms of 'completely zeros' in the rows of the augmented matrix and intersections and positions of the planes. In this way, they established a connection among *completely zero rows*, *number of free variables* in the solution and the *dimension of the intersection space*, which is considered a manifestation of AST.

Analysis from a TSM perspective

Students began their work by moving the sliders. They first expressed some phrases including: "they are moving", "slider's movement" and "positions of the planes" which can be labelled aS. Next, they considered a variation of *a*, mentioning that 'when $a = 0$, all planes coincide' and 'if $a \neq 0$ they intersect along a line'. They tested their conjectures through the 3D rotate tool, but not with a particular focus on the Algebra window. An emergence of pS was in the students' exploration of the dialectic relationship between *a* and *b* and their role in the solution of system of linear equations:

18 Student A:	"Just a second. These [referring sliders] are same here [referring Algebra window]."
…	[They are forming linear equations step by step checking movements of sliders]
38 Student A:	"Hımm. I remember, and now understand, these equations are corresponding planes' equations, but they depend on *a* and *b*, so affecting the movement of planes or positions. In fact this is a system of linear equations."

An emergence of mS was found when the students chose to solve the augmented matrix of the system of linear equations by the Gauss-Jordan elimination method. In this way, they connected the rows of the matrix, the parameters *a* and *b*, and the solution of the system. Next, the students made conjectures on the relationships between completely zero rows in the matrix, the dimension of the intersection space and the number of free variable in the solution. Finally, they validated their conjectures and constructed a meaning of the parameters as free variables. There was a chain of (mathematical) signs (Turgut, 2015); "consistent or inconsistent system", "a completely zero row", "a free variable", "we have a one-dimensional thing: a line", "two completely zero rows", "two free variables", "they intersect along a plane, a two-dimensional thing", "no free variable" and "they intersect along a point, zero-dimensional thing".

Comparing the analyses and conclusion

Table 1 provides an overview of the data from both perspectives. The joint analyses reveal the students' conceptualization process *step by step*, but from different viewpoints. The view on the classification of the TM and the identification of the emergence of signs were largely complementary.

In short, the combined analysis is promising. Meanwhile, we need *an integrated analysis tool* to better understand students' learning of linear algebra in a DGE. For example, our case was limited to the notion of parameter, whereas other main topics of (abstract) linear algebra, such as linear combination, span and linear independency, awaiting such an analysis.

TM Analysis		Semiotic Analysis	
GST	*– Focus on the movement of the planes and sliders* *– Find the key values' effects on the position of the planes through spatial perception*	**aS**	– "*… they are moving…*" – "*… slider's movement…*" – "*… position of the planes…*" – "*… all planes coincide…*" – "*… they intersect along a line…*"
AAT	*– Focus on the algebraic meaning of a and b* *– Paradigmatic reasoning on the (linear) equations* *– Use of Gauss-Jordan elimination method* *– Analysing relationships between completely zero rows and a and b*	**pS**	– "*… these…*" – "*… here…*" – "*… this is…*" – "*… these zeros…*"
AST	*– Reasoning on the solutions of the system, a and b values, and position of the planes* *– Making connection among completely zero rows, number of free variables (parameters), and dimension of intersection space of the planes (as a set of points)*	**mS**	– "*… a completely zero row…*" "*…in relation to…*" – "*…a free variable…*" – "*…one-dimensional thing: a line…*" – "*… two completely zero rows…*" – "*… two free variables…*" – "*… they intersect along a plane, a two-dimensional thing…*" – "*… no free variable…*" – "*… they intersect along a point, zero-dimensional thing…*"

Table 1: An overview of two analyses

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References

Anton, H. (1981). *Elementary Linear Algebra* (3rd ed.). US: John Wiley & Sons.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosh (Eds.), *Handbook of International Research in Mathematics Education* (2. ed., pp. 746-783). Mahwah: Lawrence Erlbaum.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 209–246). The Netherlands: Kluwer Academic Publishers.

Turgut, M. (2015). Theory of semiotic mediation in teaching-learning linear algebra: In search of a viewpoint in the use of ICT. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the 9th Congress of European Research in Mathematics Education*. Prague: Charles University.