The infrared sector of quantum fields on cosmological space-times

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Cover Illustration: A picture of the nebula N44C, image credit: NASA and The Hubble Heritage Team (STScI/AURA). Merged with the silhouette of the windows of the Little Gidding church (chosen as a reference to the poem of Eliot) .

The infrared sector of quantum fields on cosmological space-times

De infrarode sector van kwantum velden op kosmologische ruimte-tijden

(met een samenvatting in het Nederlands)

Proefschrift

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We shall not cease from exploration
And the end of all our exploring
Will be to arrive where we started
And know the place for the first time.
Through the unknown, unremembered gate
When the last of earth left to discover
Is that which was the beginning;
At the source of the longest river
The voice of the hidden waterfall
And the children in the apple-tree
Not known, because not looked for
But heard, half-heard, in the stillness
Between two waves of the sea.
Quick now, here, now, always –
A condition of complete simplicity
(Costing not less than everything)
And all shall be well and
All manner of thing shall be well
When the tongues of flame are in-folded
Into the crowned knot of fire
And the fire and the rose are one.

- T.S. Eliot

Four Quartets: Little Gidding

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1 Introduction

One of the fascinating consequences of Einstein's general theory of relativity is that space-time itself is in fact a dynamical object. This dynamical nature has led to many important insights of which with little doubt the most important is the understanding of the gravitational attraction. According to the general theory of relativity, a massive object curves the space-time around it. In this curved space-time test particles will not any more move in straight lines, but instead follow the geodesics of the geometry. It turns out that an observer far away will now see the test particles falling towards the massive object, to a first approximation in exact accordance with Newton's law of gravity.
The dynamical nature of space-time has led to many other conclusions as well. When higher order corrections to Newton's law are taken into account, the theory correctly describes the precession of the perihelion of Mercury. This phenomenon could not be explained by Newton's theory alone and required for example an additional hypothetical planet, named Vulcan, extremely close to the sun. Since this planet was never found, the general theory of relativity not only *reproduces* Newtonian gravity, but actually *improves* it.
Another important example is the discovery by Hubble's observations on galaxy redshift in 1923 that all galaxies appear to move away from us, with a velocity proportional to their distance. While one could of course postulate that, for some unknown reason, there is this large scale motion of galaxies, centered around us, this can be most naturally understood from the dynamical nature of space-time. The fact that the galaxies appear to move away, is actually due to the expansion of the universe. This expansion is in fact an expansion of space-time itself, and Einstein's general theory of relativity explains how this expansion depends on the energy density of all matter and radiation in the universe. Hubble's observations and the general theory of relativity give an exact agreement.
Given the fact that space-time is dynamical it is an immediate question how this would affect other known theories. In particular we will be interested in how quantum fields will react to a dynamical background. The problem to construct a unified, fundamental, theory that includes both general relativity and quantum fields turns out to be extremely challenging [1, 2] and see e.g. [3] for an overview of the problems. The two most promising candidates for such a unified theory today are string theory (see e.g. [4] for an introduction) and loop quantum gravity (for a review see e.g. [5]). While these approaches are extremely interesting, it is also possible to consider the interplay between quantum field theory and general relativity in a perturbative manner [6], without knowledge of the underlying unified theory. In such an approach we consider the quantum fields to be propagating on a classical, dynamical, background space-time. Questions of this nature were already asked by Schrödinger in 1932 [7]. But the real breakthrough came with the work of Parker at the end of the 1960's [8, 9]. What Parker studied was a scalar field in a cosmological space-time. In this work it was realized that the expansion of space-time causes the particle number of the theory to be non-constant. Although not many particles where created in the case Parker studied, his work did show explicitly that interesting and non-trivial effects could occur when one considers quantum fields on a cosmological space-time. Following the work of Parker, Grishchuk showed in 1974 that similar effects are present for gravitational waves [10]. In the same year Hawking showed that due to the dynamical nature of space-time

during the collapse of a black hole, a black hole is actually not black, but emits radiation [11]. From that time onward an enormous amount of research has been done on the study of quantum fields on a dynamical background space-time [6, 18, 19, 17]. It was however not until the work of Starobinsky [25] in 1979, Mukhanov and Chibisov in 1981 [13] and subsequent works by Hawking [14, 12] and others that it was realized that cosmological particle production might actually lead to readily *observable* effects. The creation of particles in the early universe might be sufficient to leave an imprint in the form of temperature fluctuations and polarization on the remnant radiation from the early universe, known as the cosmic microwave background [15]. The experimental discovery of the temperature fluctuations in the microwave background by the WMAP satellite [16] provide good evidence that the fascinating effects first found by Parker in theory, might be truly realized in nature.
The property of curved space-time that leads to these effects is the fact that one works on a time dependent background. Amongst other things this leads to the lack of Poincaré symmetry [17, 18, 19]. Poincaré invariance in Minkowski space-time plays an interesting role, since it leads to a unique definition of the vacuum of the theory and therefore also to a natural notion of particles [20, 21]. When, on the other hand, one considers quantum fields on a curved background, a state which at a certain time appears to be the vacuum, will in general not be the vacuum at later times. Moreover, two inertial observers will in general not agree on the question, whether a certain given state is the vacuum of the theory. The concepts vacuum state or single particle state are therefore not uniquely and unambiguously defined in curved space-time.
It is this effect that has led to many interesting applications of quantum field theory on curved space-time, including the cosmological particle creation and the Hawking effect described above. Let us look now at those two examples in somewhat more detail, starting with the Hawking effect [11], that describes particle emission by black holes. To analyze this effect one considers a spherically symmetric distribution of matter, that is initially sufficiently distended, such that space-time is well described by Minkowski space-time. Therefore initially, the standard flat space description of the vacuum is meaningful. However after a sufficient amount of time, the gravitational attraction has caused the matter to collapse into a black hole. The geometry will now be described by the Schwarzschild metric [22], and more importantly, the system will not anymore be in its vacuum state. The fact that the system is not anymore in its vacuum state, implies that particles have been created out of the vacuum. In fact one can show that near the horizon of a black hole a thermal spectrum of particles is produced. A heuristic explanation of the physics involved is the following [6]. Due to the Heisenberg uncertainty relation, there is a constant creation and subsequent annihilation of virtual particles. Since pairs are created with all wavelengths, it can happen that a pair is created with a wavelength roughly the Schwarzschild radius of the black hole. It is in such a case that one particle might escape the black hole, while its partner stays trapped within the Schwarzschild radius. The presence of a horizon around the black hole has prevented the annihilation of the pair and consequently there is particle emission by the black hole. Notice however that since the whole notion of a particle is not well defined on a curved background, this heuristic argument should not be taken too literally.
Something similar happens on cosmological space-times where the expansion of the universe leads to the existence of cosmological horizons [15]. These cosmological horizons imply that if two points are sufficiently separated, and the expansion of the

universe is sufficiently fast, these points are not anymore causally connected. In other words a signal sent between these two points cannot catch up with the expansion of the universe. We can thus apply the heuristic argument presented for the black hole in a similar way here (with the same caution concerning the notion of particles!). If a virtual particle-antiparticle pair is created with a sufficiently long wavelength, the expansion of the universe prevents its subsequent annihilation [23]. Thus we are led to conclude – and this can also be shown in a rigorous way – that in expanding space-times there is a constant creation of long wavelength particles. The production of particles in the context of cosmology is more then just an interesting theoretical observation. It is nowadays believed that during a phase of a nearly exponential expansion in the early universe, known as inflation [26, 12, 27] and see e.g. [24] for a review , it is precisely these particles that provide the otherwise almost perfectly homogeneous universe with small inhomogeneities [13, 28]. These initial inhomogeneities will then, through their coupling to gravity, cause a fluctuation in the gravitational potential. These gravitational potentials become frozen during during inflation and thus remain practically unaltered. During the early phases of the universe following inflation, the energy density is dominated by radiation. The internal pressure of this radiation fluid prevents matter from falling in these gravitational potentials. As soon as the universe becomes dominated by matter however, the Jeans length becomes much less than the Hubble radius. Matter can now fall into the gravitational potentials and, due to gravitational collapse, eventually form all the structure we see in the universe today. Nowadays there are several ways of experimentally studying these inhomogeneities. First of all, we can directly study the large scale structures of the universe today, through baryon acoustic oscillations [29, 30], Lyman-α clouds studies [31] and the 21 cm transition in neutral hydrogen [32]. At the moment however, the most precise measurements still come from the cosmic microwave background[16]. The presence of the cosmic microwave background allows us to look at the structure of the very early universe. The inhomogeneities due to particle creation have left a tiny ($\mathcal{O}(10^{-5})$) imprint on the temperature of the cosmic microwave background, because photons that need to climb out of the gravitational potentials lose a bit of energy (and thus become colder). Theory predicts several features of the spectrum of these photons. First of all, the spectrum is expected to be nearly scale invariant. Second of all one expects the presence of acoustic oscillations and thirdly the fluctuations are expected to be nearly Gaussian. The spectrum of these fluctuations has been precisely measured and the results confirm these predictions based on particle production on super Hubble scales during inflation [16]. The constant creation of particles in the infrared however also presents a possible problem. The creation of particles constantly adds (or subtracts) a little bit of energy to the universe. However, one should also take into account that due to the expansion of the universe this energy is red-shifted, but *a priori* it is not clear which of these two effects dominates. To study this question in more detail, one needs to specify a particular ground state. Unfortunately in an expanding space-time, there is no unique state with minimal energy at all times, as in Minkowski space-time and obviously the result will depend on the choice of state. In cosmology, the state that minimizes the energy in the asymptotic past, and resembles the Minkowski space-time vacuum is known as the Bunch-Davies vacuum [33][1]. If we now assume that the state of the

[1] The Bunch-Davies vacuum is originally defined only in de Sitter space-time. As we shall see there is a

universe is the Bunch-Davies vacuum, we find that for a large class of cosmological models, the two point function (or equivalently the propagator) of a massless non-conformally coupled field, diverges in the infrared [34]. In particular this divergence is found if the pressure driving the universe's expansion is negative. This is precisely due to the effect described above. The constant creation of infrared modes causes a pile-up of these modes at large scales. The fact that the two point correlation diverges in the infrared implies that these infrared modes add an infinite amount of energy to the expectation value of the stress-energy tensor [6]. Notice that for a proper understanding of the conditions wether an infrared divergence occurs, one has to take several effects into account. We like to point out here that also the propagator for a massless, free scalar field in $D = 2$ space-time dimensions is infrared divergent even in Minkowski space-time [35]. For $D > 2$ this divergence disappears because of the phase-space suppression. The number of modes in a shell with thickness dk and radius k scales as $k^{D-2}dk$ and thus for $D > 2$ this decreases as k decreases. The cosmological particle production described above counteracts this phase-space suppression. To see whether this particle production is enough to compensate the phase-space suppression, one has to explicitly calculate the two-point correlation.
It is not difficult however to understand why the infrared divergence is only prominent for massless non-conformally coupled fields. A mass would, just like in flat space-time, automatically regulate any infrared divergence and thus remove the effect. While for conformally coupled fields, particles are created out of the vacuum, their production rate is suppressed by the expansion of the universe. This suppression is sufficient to make the infrared finite [23]. Many familiar fields thus present no problems, since for example massless gauge fields and massless fermions are conformally coupled. Some examples of massless non-conformally coupled fields are a massless scalar, the graviton and a massless antisymmetric tensor field [36]. For this thesis we shall mostly consider the massless scalar field. Understanding the physics of that field can then be translated to the more complicated cases, like the graviton.
Since the presence of an infrared divergence of the two point correlations is dependent on the choice of the ground state and the ground state in curved space-time is not uniquely defined, a natural solution to the infrared problem is to consider a different ground state. In this thesis we shall present a method to change the ground state in such a way that the infrared divergencies are resolved. In section 6 we shall do this by working on a spatially compact manifold [38, 37]. Physically one can see this as putting the universe in a co-moving box, with periodic boundary conditions. If the radius of this manifold is large enough, this approach in effect simply puts an infrared cut-off in the integral over the modes contributing to the two point correlator.
While this approaches make the two point correlations infrared finite, there still is particle production. This production of particles will give a contribution to the stress energy tensor and although the result will be finite, this effect is cumulative. At the same time, the expansion of the universe will cause these modes to redshift and therefore the contribution of one mode to the energy density will decrease in time. One has to study each case separately to resolve which effect is dominant.
Supposing that there is a growing contribution to the two point correlation function due to these modes, interesting physical effects can be considered. Although any effect due

natural extension of the Bunch-Davies vacuum to any cosmological space-time with constant deceleration. We shall use the term Bunch-Davies vacuum to describe a state in this general sense.

to particle production is initially small, the growth might make it large after a sufficient amount of time. This fact has been studied by several authors in the context of cosmological inflation [42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54]. The fascinating idea here is that, on the one hand one has either a cosmological constant, or an inflaton field causing the universe to expand exponentially fast (such a geometry is known as a de Sitter geometry), while on the other hand the growing infrared contributions counteract this exponential expansion. If one then waits long enough, the growing contributions eventually *might* end inflation naturally and – when inflation is driven by a cosmological constant – in addition explain the smallness of the cosmological constant. In de Sitter space however, at one loop order, the production of particles and their redshift cancels each other, leading not to a significant effect. However, at higher loop order, the interactions between different infrared modes become relevant and it is these interactions that do appear to exhibit the secular growth. Detailed calculations show that this appears to happen at two-loop order for gravitons [48] and at three-loop order for a massless minimally coupled scalar with quartic self interactions [54].
Apart from these effects, which are all computed during inflation, the growth of correlations might also be relevant in the context of dark energy [55, 56], or see [57] for a recent review. The reason for this is that although the growth is small, it is cumulative. Therefore it only becomes relevant at late enough times. This is exactly what one wants for dark energy, which has only very recently become the dominating contribution to the energy density of the universe. If the energy density in these correlations has an equation of state compatible with a dark energy component, the delayed effect we obtain through the cumulative growth might then give insight in the 'why now?' or cosmic coincidence problem of dark energy (for a review see e.g. [58]): why is the energy density in dark energy and in dark matter today of the same order of magnitude?
In this thesis we shall not go deeper into these higher loop calculations, instead we shall use the infrared divergence free propagator calculated in section 6 to calculate one loop effects in a much more general class of cosmological models than just the inflationary models considered before. In section 7 we shall calculate the expectation value of the stress energy tensor of a massless free scalar field. The calculation of the propagators and the stress energy tensor is based on references [71] and [38]. In section 8, which is based on [73], we shall consider the one-loop effective action for a massless scalar field with a quartic self interaction. Finally in section 9, which is based upon [74], we shall calculate the one loop effective action for a theory which contains both graviton and scalar matter fluctuations.

2 Cosmology

The field of cosmology deals with the dynamics of the universe on the largest of scales [15]. Since the universe is electrically neutral, on the largest scales the only relevant force is the gravitational interaction. This implies that the dynamics of the universe on the largest scales is determined by the Einstein equations [22]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T_{\mu\nu}, \tag{1}$$

here R and $R_{\mu\nu}$ are the Ricci scalar and tensor defined in (431) respectively, $g_{\mu\nu}$ is the metric and G_N is Newton's constant. The energy momentum tensor $T_{\mu\nu}$ describes the influence of all the matter in the universe on the geometry. The Einstein equations (1) follow from the variation with respect to the metric of the action

$$S = \int d^4x\sqrt{-g}\frac{1}{16\pi G_N}R + S_M \tag{2}$$

where S_M is the action associated to the matter content and g is the determinant of the metric. We find that we recover the Einstein equations if we make the identification

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{3}$$

Because of the Bianchi identity, which follows from the symmetries of the curvature tensors imposed by general coordinate invariance, $\nabla^\mu(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) = 0$, with ∇^μ the covariant derivative, we find that the energy momentum tensor is covariantly conserved

$$\nabla^\mu T_{\mu\nu} = 0. \tag{4}$$

Because of the complexity of the Einstein equations it is unfortunately a hopeless endeavor to try to solve (1) for all but the simplest models (and even if we had a solution, it would still be far from obvious how to construct physical observables!). Fortunately it turns out that observations show that the universe becomes spatially homogeneous on the largest of scales. Measurements of the large scale structure of the universe [59] show that at scales larger than roughly 200 Mpc the relative fluctuations in the energy density $\frac{\delta\rho}{\rho}$ are less than 0.1, while these fluctuations in the cosmic microwave background [16], which probes scales of roughly the Hubble radius or 4300 Mpc, are 10^{-4}. This leads then to the so-called cosmological principle, which states that on the largest scales the universe is homogeneous and isotropic. Imposing this additional symmetry on the Einstein equations, will make it possible to construct exact cosmological solutions.

The cosmological principle implies that we can model all the constituents of the universe with a homogeneous fluid. If such a fluid has an energy density ρ and a pressure p, it has an energy momentum tensor given by

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + g^{\mu\nu}p, \tag{5}$$

where u^μ is the fluid's four velocity. We can always choose our frame to be the rest frame of the fluid, where we have $u^\mu = a^{-1}\delta^\mu_0$. We furthermore will assume that the fluid obeys an equation of state

$$p = w\rho \tag{6}$$

with w constant. Typical values for w are 0 for non-relativistic dust and $w = 1/3$ for a gas of radiation. Because of the cosmological principle, space-time itself should also become spatially homogeneous on the largest of scales. The most general metric obeying this constraint is the Friedmann-Lemaître-Robertson-Walker metric, defined by the line element

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a(t)^2 \Big(\frac{dr^2}{1-kr^2} + d\theta^2 + \sin(\theta)^2 d\phi^2 \Big). \end{aligned} \tag{7}$$

Here (r, θ, ϕ) are spherical coordinates, $a(t)$ is called the scale factor and it specifies the expansion of the universe. k specifies the curvature of space-time. If $k > 0$, spatial sections are positively curved, if $k < 0$ spatial sections are negatively curved and if $k = 0$ spatial sections are flat. In the following we shall choose $k = 0$, conform observations and the predictions of cosmic inflation [24]. In that case the spatial part of the metric is simply Euclidean and we can thus write the line element as

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{8}$$

If we make the coordinate transformation to so-called conformal time η, defined by

$$dt = a d\eta, \tag{9}$$

the metric (8) becomes conformal to the Minkowski metric

$$g_{\mu\nu} = a(\eta)^2 \eta_{\mu\nu}. \tag{10}$$

In the following a dot will always indicate a derivative with respect to time t, while a prime will indicate a derivative with respect to conformal time η. By taking derivatives of the scale factor we construct two important parameters

$$H(t) \equiv \frac{\dot{a}}{a} \qquad ; \qquad \epsilon(t) \equiv -\frac{\dot{H}}{H^2} \tag{11}$$

H is known as the Hubble parameter and specifies the expansion of space-time. The parameter ϵ specifies the acceleration of space time, if $\epsilon > 1$, the expansion of the universe is decelerating, while for $\epsilon < 1$ it is accelerating.
Let us now consider the Einstein equations (1) for the specific case at hand where the energy momentum tensor and metric are given by (5) and (10). The Christoffel connection is given by

$$\Gamma^\alpha_{\mu\nu} = Ha\Big(\delta^0_\mu \delta^\alpha_\nu + \delta^0_\nu \delta^\alpha_\mu + \delta^\alpha_0 \eta_{\mu\nu}\Big) \tag{12}$$

and the curvature invariants, given for later use in D space-time dimensions, we find from (431)

$$\begin{aligned} R^\alpha{}_{\mu\beta\nu} =& -H^2 a^2 \Bigg(\Big(\delta^\alpha_\nu \eta_{\mu\beta} - \delta^\alpha_\beta \eta_{\mu\nu}\Big) + \epsilon\Big(\delta^\alpha_\nu \delta^0_\mu \delta^0_\beta - \delta^\alpha_0 \delta^0_\nu \eta_{\mu\beta} - \delta^\alpha_\beta \delta^0_\nu \delta^0_\mu + \delta^0_\beta \delta^\alpha_0 \eta_{\mu\nu}\Big)\Bigg) \\ R_{\mu\nu} =& H^2 a^2 \Bigg((D-1)\eta_{\mu\nu} - \epsilon\Big(\eta_{\mu\nu} - (D-2)\delta^0_\mu \delta^0_\nu\Big)\Bigg) \\ R =& H^2 (D - 2\epsilon)(D-1) \end{aligned} \tag{13}$$

With these expressions we can rewrite the Einstein equations in terms of H and ϵ. Because of the extra symmetry imposed by the cosmological principle, there are only two independent Einstein equations, known as the Friedmann equations (we set $D = 4$)

$$\begin{aligned} H^2 &= \frac{8\pi G_N}{3}\rho \\ (1-\epsilon)H^2 &= -\frac{4\pi G_N}{3}(\rho + 3p). \end{aligned} \tag{14}$$

The conservation equation (4) reduces to the single requirement

$$\dot{\rho} + 3H(\rho + p) = 0, \tag{15}$$

which is, because of the Bianchi identity, not independent of (14). Notice that if the universe is filled with more than one fluid, each with energy ρ_i and pressure p_i, ρ and p in (14) and (15) should be read as $\sum_i \rho_i$ and $\sum_i p_i$.
If the universe is filled with only one fluid, which obeys the equation of state (6) (with w constant and $w \neq -1$), we can find from (14) and (15) the following scalings

$$\begin{aligned} &\rho \propto a^{-3(1+w)} \\ &a \propto t^{\frac{2}{3(1+w)}} \end{aligned} \tag{16}$$

Moreover we find that ϵ is a constant, given by

$$\epsilon = \frac{3}{2}(1+w) \tag{17}$$

which allows us to write

$$H(t) = \frac{H_0}{1 + \epsilon H_0 t} \qquad ; \qquad a(t) = \Big(1 + \epsilon H_0 t\Big)^{\frac{1}{\epsilon}}, \tag{18}$$

where we chose our initial conditions such that $H(0) = H_0$ and $a(0) = 1$. In the conformal coordinates (10), if ϵ is constant, we find

$$H(\eta) = \frac{H_0}{\Big(-(1-\epsilon)H_0\eta\Big)^{\frac{-\epsilon}{1-\epsilon}}} \qquad ; \qquad a(\eta) = \frac{1}{\Big(-(1-\epsilon)H_0\eta\Big)^{\frac{1}{1-\epsilon}}} \tag{19}$$

where the constants are chosen such that for $\epsilon = 0$ we have $H(\eta) = H_0$ and $a(\eta) = -\frac{1}{H_0\eta}$. Since a and H are positive in an expanding universe, these choices imply that

$$\begin{aligned} -\infty < &\eta < 0 \qquad\qquad \text{if} \qquad \epsilon < 1 \\ 0 < &\eta < \infty \qquad\qquad \text{if} \qquad \epsilon > 1. \end{aligned} \tag{20}$$

Furthermore, we find using (19) that

$$(1-\epsilon)H\eta = -\frac{1}{\eta}. \tag{21}$$

In many cases, we shall consider a matter content given by a scalar field $\Phi(x)$, with a potential $V(\Phi(x))$. If we want to preserve the spatial homogeneity of the background, we need to assume that the scalar field is spatially homogeneous, thus $\Phi(x) = \Phi(t)$. The matter action is given by

$$S_M = \int d^4x\sqrt{-g}\Big(-\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - V(\Phi)\Big). \tag{22}$$

We find that the energy momentum tensor is given by

$$T_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - g_{\mu\nu}\Big(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + V(\Phi)\Big) \tag{23}$$

Because of the homogeneity of the field, we find that the energy and pressure are given by

$$\begin{aligned}\rho_\Phi &= \frac{1}{2}\dot{\Phi}^2 + V(\Phi)\\ p_\Phi &= \frac{1}{2}\dot{\Phi}^2 - V(\Phi).\end{aligned} \tag{24}$$

The conservation equation (15) becomes

$$\ddot{\Phi} + 3H\dot{\Phi} + \frac{dV(\Phi)}{d\Phi} = 0, \tag{25}$$

which is nothing but the equation of motion for the scalar field Φ. We like to end this section with a list of possible scenarios, ordered with respect to the value of ϵ.

- $\epsilon < \mathbf{0}$: in such a case the expansion of the universe is extremely fast, in the sense that the scale factor, measured with respect to $t = 0$ will go to infinity in finite time (see 18). Such an expansion requires $w < -1$, which breaks the weak energy condition ($\rho \geq 0$ and $\rho + p \geq 0$). Such matter is sometimes called phantom matter.

- $\epsilon = \mathbf{0}$: in this case the expansion of the universe is exponential in time. A geometry of this type is known as de Sitter space. The expansion can be driven by matter which has $w = -1$, which is known as a cosmological constant.

- $\mathbf{0} < \epsilon < \mathbf{1}$: in those cases the expansion of the universe is accelerating. This requires that $w < -1/3$ and thus breaks the strong energy condition ($\rho + p \geq 0$ and ($\rho + 3p \geq 0$).

- $\epsilon = \mathbf{1}$: the acceleration of the expansion is in this case zero, and $w = -1/3$.

- $\epsilon = \mathbf{3/2}$: this implies $w = 0$, or in other words, the pressure of the fluid is zero. This is the equation of state for non-relativistic matter.

- $\epsilon = \mathbf{2}$: this implies $w = 1/3$ and thus the trace of the stress-energy tensor is zero. This is the equation of state for relativistic matter. Notice that for $\epsilon = 2$ the Ricci scalar is zero.

- $\epsilon = \mathbf{3}$: in this case $w = 1$ and we see from (24) that this can be obtained by a free scalar field, whose energy is given purely by its kinetic energy

- $\epsilon \to \infty$: in this limit the scale factor becomes a constant and this can thus be seen as the limit to Minkowski space-time.

2.1 Horizons, scales and the Hubble parameter

In cosmology there are several length scales that play an important role. First of all there is the distance between two points [60]. If an observer in the coordinates (8) at a certain time t_0 measures a certain spatial distance between two points to be λ, an observer at a different time will measure a distance

$$\frac{a(t)}{a(t_0)}\lambda, \tag{26}$$

with a the scale factor. We call $\lambda/(a(t_0))$ the co-moving distance. This scaling is easily understood, as the universe expands, the distance between two points increases with the scale factor. Since all physical scales grow in this way, also the wavelength of radiation grows in time. If the scale factor today is $a(t_0)$, we define the redshift $z(t)$ of a photon emitted at time $t < t_0$ as

$$1 + z(t) = \frac{a(t_0)}{a(t)} \tag{27}$$

Since the redshift causes the radiation to lose energy as the universe expands, we conversely find that in the early universe the radiation was more energetic, or hotter. The second length scale we would like to mention is the causal horizon. This horizon is, just as in Minkowski space-time due to the fact that in a finite time, a signal can propagate only a finite distance. A signal sent at some time t_*, will at a later time t have travelled a distance

$$\ell_H(t) = a(t)\int_{t_*}^{t}\frac{dt'}{a(t')}. \tag{28}$$

If we choose t_* to be some initial time, $\ell_H(t)$ gives the size of the causally connected universe at time t. Let us foremost state what this length is *not*. It is not the 'size' of the universe. If we take two points, separated at time t by a distance less then $\ell_H(t)$ it also does *not* mean that these two points will be able to communicate in the future. What it *does* mean is that these points have been in causal contact in the past. Taking the constant ϵ expressions for H and a we find from (18)

$$\ell_H(t) = a(t)\Big(\frac{1}{a(t)H(t)(\epsilon-1)} - \frac{1}{a(t_*)H(t_*)(\epsilon-1)}\Big). \tag{29}$$

Notice that for $t > t_*$ this expression is always positive, since aH grows in time if $\epsilon < 1$ while it decreases in time for $\epsilon > 1$. This therefore means that in an accelerated expanding universe, the co-moving causal horizon is bounded by $\frac{1}{a(t_*)H(t_*)(1-\epsilon)}$. Thus two points, at time t_* separated by a larger distance than that, will not be able to communicate in the future. The reverse is true in a decelerating universe. At a certain time $t \gg t_*$, only those points with a co-moving distance less than $\frac{1}{a(t)H(t)(1-\epsilon)}$ have been in causal contact in the past. The third length scale we would like to introduce is the Hubble radius, given by the inverse Hubble parameter

$$r_H(t) = \frac{1}{H(t)}. \tag{30}$$

The meaning of this parameter is not always clear from the literature. First of all we saw from (29) that in an accelerating universe, the co-moving Hubble radius, r_H/a is apart from a factor $(1-\epsilon)$ the distance over which causal physics can act in the future, while in a decelerating universe it is the distance over which causal physics has acted starting from the (infinite) past. The most important interpretation for the Hubble radius however comes from its influence on the dynamics. We shall see in many cases that one can distinguish between super-Hubble and sub-Hubble physics. To see this we look at the equation of motion of a free scalar field in a cosmological background [60]

$$\ddot{\Phi} + 3H\dot{\Phi} - \frac{\vec{\nabla}^2}{a^2}\Phi = 0 \tag{31}$$

If the spatial gradients $\vec{\nabla}^2$ are small, we find that in a constant ϵ background $\dot{\Phi} \propto H\Phi$. This approximation therefore makes sense if the spatial gradients $\frac{\vec{\nabla}^2}{a^2}\Phi \ll r_H^{-2}\Phi$, thus if the fluctuations in Φ are super-Hubble. In this limit we can easily solve the equation of motion and obtain that $\Phi(t) = c_1 + c_2 \int \frac{dt}{a^3}$, with c_1 and c_2 constants. In the opposite limit, thus if fluctuation in Φ are sub-Hubble, we find after a simple rescaling by a that $a\Phi(\eta) = c_3 \int \frac{d^3k}{\sqrt{2k(2\pi)^3}} \exp(i(\vec{k}\cdot\vec{x} \pm k\eta))$, where η is conformal time, defined in (9). In both limits physics is completely different. On sub-Hubble scales, we essentially have a harmonic oscillator with a time dependent frequency and the field Φ fluctuates. On super-Hubble scales however, the field stops fluctuating. This property of the field has important consequences for the generation of cosmological perturbations, as we shall briefly describe below [28].
To finish this discussion of scales, we note the important distinction between accelerating and decelerating space-times, that in a decelerating space-time, the Hubble radius grows *faster* in time than physical length scales, while in an accelerating space-time, it grows *slower*. This means that in an accelerating space-time, length scales that are initially sub-Hubble, become super-Hubble after some time and vice versa for decelerating space-times.

2.2 Overview of the universe

In this section we give a short chronological overview of the standard big bang model. We will however not go into details on the experimental evidence for this model. Nor will we give much explicit calculations. This can all be found in the following introductions and reviews [15, 61, 60, 24, 62, 63]. Although much of the following relies heavily on assumptions regarding high energy physics, we shall for definiteness assume the standard picture that the universe starts out at very high temperatures and cools during its expansion.
A proper chronological overview should of course start at the beginning. However in the case of the big bang model we do not know what happens at the beginning. If one takes the limit to the infinite past of (18), one finds that the scale factor is 0 and the Hubble parameter is infinite. However, since the Hubble parameter is a measure for the energies involved, general relativity is expected to break down in this regime and we need to consider a quantum theory of gravity, which is expected to become relevant at the Planck scale $m_p = 1/\sqrt{G_N} \sim 10^{19}$GeV [64]. More in particular, the effective coupling we expect for quantum gravity will be $G_N H^2$, with H the Hubble parameter. Therefore if $H \ll m_p$ we can expect quantum gravity effects to be irrelevant. Thus we simply do not know what happens extremely close to the big bang. As the universe expands however, quantum gravity effects will become less and less relevant.

It is highly speculative what is the state of the universe at these high energies. It might be that all gauge fields (and fermions) of the standard model are unified in one representation of a large grand unified symmetry. During the expansion of the universe, the universe cools down. When the temperature reaches approximately $k_B T \sim 10^{16} GeV$ at $t \sim 10^{-35}$ seconds, any grand unified symmetry is typically assumed to be broken and the universe is thought to enter a phase of inflation [25, 26, 27]. The mechanism that is responsible for this phase is still not known, but the idea is the following. During the period of inflation, the universe is dominated by a fluid which has an equation of state parameter $w \approx -1$. From (15) we see that this implies that $\dot{\rho} \approx 0$. We call a quantity that strictly has the property $\dot{\rho} = 0$ a cosmological constant. If $w = -1$ we find from (17) that $\epsilon = 0$ and thus H is constant and we can easily solve the Friedmann equation (14) to find that the scale factor grows exponentially with time

$$a = e^{Ht}. \tag{32}$$

Such a geometry is known as de Sitter geometry and we shall study it in much more detail in section 4. The fluid responsible for this expansion is usually taken to be a scalar field, known as the inflaton. We see from (24) that for a scalar field w is close to -1 if the potential energy of the scalar field dominates over the kinetic energy. This can typically be achieved if the scalar field potential is such that that the field rolls slowly down the potential. This approach is therefore known as 'slow roll' inflation.
If the expansion of the universe is accelerating ($\epsilon < 1$), we see from (18) that physical scales grow faster than the Hubble volume. This implies that during inflation the universe becomes extremely homogeneous, in the sense that any initial sub-Hubble fluctuation is stretched far beyond Hubble scales. A similar thing happens to the curvature k of the universe. If k is not strictly zero, a sufficient amount of inflation will make any initial curvature insignificant. The length of inflation is typically measured in e-foldings

$$N = \ln\left(\frac{a_e}{a_b}\right), \tag{33}$$

where the scale factor at the beginning and the end of inflation are indicated by a_b and a_e respectively. To make the universe flat enough, typical estimates imply that inflation should last *at least* for about $N = 60$ e-foldings [24]. After at least 60 e-foldings inflation ends, for example because of the inflaton reaching a minimum of its potential where $V = 0$. The energy density of all normal matter has become negligible due to the fast expansion. Although during inflation the universe becomes very homogeneous, there are always inhomogeneities due to vacuum fluctuations of the various matter and gravitational fields. If these fluctuations cross the Hubble radius they stop fluctuating (for a review on this subject see e.g. [60]), as we saw from (31). It is important to note here that we can distinguish between two types of fluctuations. First of all there are scalar fluctuations. These can be simply seen as fluctuations in the energy density. By the Einstein equation, a fluctuation in the energy density can then be related to a fluctuation in the gravitational potential. Apart from scalar perturbations, there are tensor perturbations. These are due to fluctuating gravitons. Their effect on the geometry is a quadrupolar distortion of space (for a review on graviton perturbations see e.g. [65]).
When inflation ends, the inflaton will typically start to oscillate quickly. During these oscillations the inflaton decays to relativistic particles and thus the universe is again

filled with radiation. This process is known as reheating. After reheating the universe is essentially a spatially homogeneous mixture of radiation ($w = 1/3$) and normal matter ($w = 0$). At high temperatures, the energy density in radiation is much larger, and we can effectively describe the universe to be dominated by radiation. At $t \sim 10^{-8}$ seconds, or $k_B T \sim 100 GeV$ the universe undergoes an electroweak phase transition at which the gauge symmetry $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{EM}$ (the $SU(3)_c$ symmetry of QCD is unaffected). During this process the W and Z gauge bosons and the fermions acquire their mass, while the photon remains massless. Probably between inflation and the electroweak phase transition, the symmetry between baryons and anti-baryons is broken during the process of baryogenesis. The precise mechanism for baryogenesis however is still unknown. As the universe cools further down it undergoes the QCD transition at $k_B T \sim 160 MeV$. At this scale one enters the strongly coupled regime of QCD. At this point therefore the initial quark-gluon plasma disappears and quarks become confined through gluon exchange in mesons and baryons. Properties of the QCD transition have been studied at the RHIC (Relativistic Heavy Ion Collider) [66] and will be studied in the future at the LHC (Large Hadron Collider).

At temperatures between $k_B T \sim 1 MeV$ and $k_B T \sim 0.05 MeV$, or from $t \sim 1$ second to $t \sim 5$ minutes, the light nuclei (D, He, Li,...) are formed by the process of nucleosynthesis. The universe is however still hot enough to prevent atoms to form. The Compton scattering of the radiation from electrons and nuclei tightly couples the radiation and matter fluids. During radiation domination, physical scales grow *less* fast than the Hubble radius. This implies that during radiation domination, the fluctuations that crossed the Hubble radius during inflation re-enter the Hubble volume. The scalar perturbations, being simply perturbations in the energy density, will cause the homogeneous fluid to obtain inhomogeneities in its energy density, or temperature. The tensor perturbations however, cause the fluid to obtain a magnetic (or 'B mode') polarization [65].

We see from (16) that the energy density in radiation dilutes faster than the energy density in matter. This means that at a certain moment the universe will become matter dominated. This happens when the temperature is $k_B T \sim 1 eV$, or at a redshift $z \sim 3230$. As the universe reaches a temperature of roughly $k_B T \sim 0.3 eV$, at a redshift of $z \sim 1090$, hydrogen atoms can form effectively. Scattering of the radiation from atoms is much less efficient and thus the radiation and matter fluids effectively decouple. From this point onwards, the photons from the radiation fluid travel freely. Since the universe is homogeneous, this means we are still able to see these photons as a homogeneous background radiation. Because of the redshift due to the expansion of the universe, their wavelength is nowadays in the microwave region ($\lambda \sim 2$ mm)and therefore this fluid of photons is known as the cosmic microwave background radiation (CMBR). The CMBR has been experimentally found [67] and extensively studied [68, 16], and is indeed extremely homogeneous. The spectrum of the CMBR is, also consistent with predictions, an almost perfect black body spectrum with a temperature of 2.725 K. While very homogeneous, the temperature of the CMBR does have inhomogeneities on a relative scale of 10^{-5}. The scenario described to generate small inhomogeneities in the energy density produces an almost scale invariant spectrum of scalar density perturbations and thus of the gravitational potentials causing the temperature fluctuations in the CMBR. These predictions have been extensively studied and are completely consistent with the observations of the inhomogeneities in the CMBR [16]. The Planck satellite, which is

currently scheduled to be launched later this year, will further improve these measurements and might also be able to detect the B-mode polarization [69], thus possibly confirming the existence of primordial gravitational waves.
Eventually at a redshift of $z = 10 - 30$ (or $T \sim 100K$) the initial inhomogeneities, imprinted on the fluid, begin to form stars, galaxies and clusters of galaxies, by gravitational collapse thus forming the large scale structures of the universe as we see it today.
Observations of distant supernovae have shown however that this is not the end of the story [55]. There is also a dark energy component in the universe. This dark energy has an effective equation of state $w \approx -1$, and might therefore be described by a nonzero cosmological constant. From (16) it is clear that if one waits long enough, the energy density of such a fluid will eventually dominate and it is nowadays believed that the energy density of our universe today is made up of 72% dark energy, 23% dark matter and 4.6 % normal, baryonic matter [16]. The remaining energy density is mostly in neutrinos.

3 Quantum fields on a curved background space-time

One of the great problems in fundamental theoretical physics today is with little doubt the merging of general relativity with quantum field theory [1, 2]. Although there are several promising candidates for such a theory, most notably string theory and loop quantum gravity, it is currently not known which of these gives an accurate description of nature.
While this certainly is a fundamental problem, this does not prevent one from studying interesting physics [6]. As long as one considers processes that occur at energies smaller than the typical energy scale where quantum gravity effects become important it is perfectly consistent to treat all processes perturbatively. Since this scale is set by the Planck scale

$$m_p = \frac{1}{\sqrt{G_N}} \sim 10^{19} GeV, \tag{34}$$

we can describe most 'everyday' processes despite our lack of knowledge of the underlying theory. One is in such a case effectively using general relativity as a low energy effective action for the unknown quantum theory of gravity. The principle idea here is that in physics we always find a separation of scales. *Each* theory we know of, is expected to be valid only for a certain range of scales. New physical effects then emerge at higher energy scales, but as long as the processes one is considering are at lower energies, one can neglect these effects. Examples are abundant, for example classical mechanics is perfectly fine on macroscopic scales where $\hbar$ is irrelevant, or Newtonian gravity works excellent for weak gravitational fields. The fact that quantum gravity is not renormalizable is a problem that only presents problems near the Planck scale. This is similar to the Fermi theory for the weak interaction [20, 21]. This theory allows for the direct coupling of four fermions, in order to explain β-decay. While the theory is not renormalizable, it does produce correct answers for processes at energies below the W mass: $M_W \approx 80.4 GeV$. For higher energies, the propagation of the vector bosons becomes important and taking this into account makes the theory renormalizable.
This approach necessarily fails for questions that are non-perturbative. This includes for example important fundamental questions like the nature of space-time singularities (including the Big Bang), or the origins of the microstates that contribute to the black hole entropy.
Many of the problems of formulating quantum field theory on a curved background space-time have to do with the lack of Poincaré symmetry [6, 18], which is a natural consequence of the non-trivial time dependence of the background. When formulating quantum field theory on Minkowski space-time, Poincaré invariance, in particular invariance under time translations, allows one to define a unique vacuum state. In the sense that the notion of a lowest energy state is unique. Given this state the annihilation and creation operators then define uniquely the notion of particles. As we shall see this is in general not possible on a curved background. That is, one can certainly define a ground state, which at a certain time is the state of lowest energy, but this ground state will typically *not* be the state of lowest energy anymore at any later time. Notice that in this thesis we shall use both the concepts vacuum state and ground state interchangeably. We use these concepts to indicate the state $|0\rangle$, defined in (48), which is (in general) not the state of minimal energy.

While this presents all kinds of difficulties, it is also the source of many interesting phenomena like Hawking radiation and the generation of primordial density fluctuations. Because of the troubles with the notion of particles, we will eventually study quantum fields on curved space-time through the correlators associated to the field. From the correlators we can find, without needing the notion of particles, for example the contribution of quantum corrections to the energy density. To see how this all comes about, lets start with the basis: ordinary quantum field theory on a Minkowski background.

3.1 Quantum field theory on a Minkowski background

For concreteness we shall consider a massive, real, scalar field ϕ, with the action given by

$$S = \frac{1}{2}\int d^4x\big(- \partial^\mu \phi \partial_\mu \phi - m^2\phi^2\big) \tag{35}$$

and equation of motion

$$\big(\Box - m^2\big)\phi = 0. \tag{36}$$

Here the d'Alembertian is $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ and $\eta^{\mu\nu}$ is the Minkowski metric, which in 4 space-time dimensions is given by

$$\eta^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1). \tag{37}$$

We define the momentum conjugate to ϕ

$$\pi = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi \tag{38}$$

We quantize the field by imposing the commutation relation

$$[\phi(t,\vec{x}), \pi(t,\vec{x}')] = i\delta^3(\vec{x} - \vec{x}') \tag{39}$$

and all other commutators are equal to zero. To circumvent problems with the normalization, we assume that space is described by a three dimensional torus, with circumference L and volume $V = L^3$. To solve for the scalar field, we expand it in creation, $a^\dagger_{\vec{k}}$, and annihilation, $a_{\vec{k}}$, operators

$$\phi(t,\vec{x}) = \sum_{\vec{k}} \Big(a_{\vec{k}}\psi_{\vec{k}}(t,\vec{x}) + a^\dagger_{\vec{k}}\psi^*_{\vec{k}}(t,\vec{x})\Big) \tag{40}$$

where the wavevector $\vec{k}$ is restricted to values $\frac{2\pi}{L}\vec{n}$, with $\vec{n} = \{n_1, n_2, n_3\}$ and n_i integer. To find the mode functions $\psi_{\vec{k}}(t,\vec{x})$, we consider the following complete set of solutions

$$u_{\vec{k}}(t,x) = \frac{1}{\sqrt{2\omega V}} e^{i(\vec{k}\cdot\vec{x} - \omega t)} \qquad ; \qquad v_{\vec{k}}(t,x) = \frac{1}{\sqrt{2\omega V}} e^{i(\vec{k}\cdot\vec{x} + \omega t)}. \tag{41}$$

where $\omega = \sqrt{||k||^2 + m^2}$. $u_{\vec{k}}$ is known as the positive frequency solution and v_k as the negative frequency solution, since the eigenvalues of u and v with respect to the time translation operator $i\partial_t$ are positive and negative respectively. The normalization of (41) has been chosen such that the scalar product defined as

$$(\phi_1, \phi_2) = -i \int_t d^3x \Big(\phi_1 \partial_t \phi_2^* - (\partial_t \phi_1) \phi_2^* \Big), \tag{42}$$

where t denotes a spacelike hypersurface, is

$$(u_{\vec{k}}, u_{\vec{p}}) = \delta_{\vec{k}\vec{p}} \qquad ; \qquad (v_{\vec{k}}, v_{\vec{p}}) = -\delta_{\vec{k}\vec{p}} \qquad ; \qquad (u_{\vec{k}}, v_{\vec{p}}) = 0 \tag{43}$$

We can now define our mode solution to be

$$\psi_{\vec{k}}(t,x) = \alpha(\vec{k}) u_{\vec{k}}(t,x) + \beta(\vec{k}) v_{\vec{k}}(t,x), \tag{44}$$

where α and β do not depend on time. In order for $\psi_{\vec{k}}$ to have unit norm, we need

$$|\alpha|^2 - |\beta|^2 = 1, \tag{45}$$

such that (44) and its complex conjugate define a complete orthonormal basis.
The definition of the scalar product (42) implies that

$$a_{\vec{k}} = (\phi, \psi_{\vec{k}}) \tag{46}$$

Combining (40) with (39) and using the fact that ψ forms an orthonormal basis we find that

$$[a_{\vec{k}}, a_{\vec{p}}^\dagger] = \delta_{\vec{k}\vec{p}}, \tag{47}$$

and all other commutators zero. Notice that the requirement that $\psi_{\vec{k}}(t,\vec{x})$ has unit norm, the condition (45) and the normalization of the commutator (47) are all equivalent. We define the ground state $|0\rangle$ to be the state that is annihilated by all $a_{\vec{k}}$.

$$a_{\vec{k}}|0\rangle = 0 \qquad \forall \qquad \vec{k}. \tag{48}$$

Notice that this state will *not* necessarily be the state with lowest energy. From the ground state we can construct excited ('particle') states by applying the creation operator $a_{\vec{k}}^\dagger$.

$$a_{\vec{k}}^\dagger|0\rangle = |1_{\vec{k}}\rangle \qquad ; \qquad ; a_{\vec{k}}^\dagger a_{\vec{p}}^\dagger|0\rangle = |1_{\vec{k}} 1_{\vec{p}}\rangle \qquad (\vec{k} \neq \vec{p}) \tag{49}$$

To take the correct statistics into account (remember we are considering bosonic particles)

$$\begin{aligned} a_{\vec{k}}^\dagger |n_{\vec{k}}\rangle &= \sqrt{n+1}\,|(n+1)_{\vec{k}}\rangle \\ a_{\vec{k}} |n_{\vec{k}}\rangle &= \sqrt{n}\,|(n-1)_{\vec{k}}\rangle \end{aligned} \tag{50}$$

such that the operator $N_{\vec{k}} = a_{\vec{k}}^\dagger a_{\vec{k}}$ has an eigenvalue equal to the number of excitations with momentum k. Now an important point is that this definition of both the vacuum and of the notion of a particle is not unique. The reason is simply that the coefficients α and β are not uniquely fixed by the requirement (45) and therefore the mode function (44) is not unique. We can see this even more clearly by calculating the Hamiltonian [70]

$$H = \frac{1}{2} \int d^3x \Big(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2 \Big), \tag{51}$$

such that

$$
\begin{aligned}
H\big|n\rangle &= (|\alpha|^2+|\beta|^2)\omega(N+\frac{1}{2})\big|n\rangle \\
&= (1+2|\beta|^2)\omega(N+\frac{1}{2})\big|n\rangle.
\end{aligned}
\tag{52}
$$

So we see explicitly that a single 'particle' adds an energy $(1+2|\beta|^2)\omega$ to the Hamiltonian and the vacuum (48) has an energy of $(\frac{1}{2}+|\beta|^2)\omega$. Now $|\beta|^2$ can run from zero to infinity, consistent with (45), so we see that depending on our choice for the mode functions (44), we consider different types of excitations. Naturally, also the interpretation of the ground state changes. Now it is important to stress that none of these choices are incorrect. If we want however our ground state to be the state of minimal energy, or the vacuum, we see that we must choose $\alpha = 1$ and $\beta = 0$. Notice that so far we are considering pure states. If one wants to expand this for statistical field theory, one can employ a description using the density matrix.
Now the important thing of Minkowski space-time is that it is also this particular choice of the ground state that is invariant under Poincaré transformations. The condition that the ground state defined by (48) is invariant under the Poincaré group implies that all inertial observers will agree what the ground state is. And moreover they will agree on what a single excitation is. The reason for this is that in Minkowski space-time the set of inertial observers is invariant under the Poincaré group. The algebra of the Poincaré group is given by

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
[M_{\mu\nu}, P_\rho] &= \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu \\
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\nu\rho}M_{\mu\sigma}.
\end{aligned}
\tag{53}
$$

Here P_μ is the generator of space-time translations and $M_{\mu\nu}$ is the generator of Lorentz transformations. In particular we consider infinitessimal time translations

$$
t' \to t+\delta, \tag{54}
$$

with δ small. Under this transformation we find that the mode (44) transforms as

$$
\psi_{\vec{k}}(t') \to \alpha(\vec{k})(1-i\omega\delta)u_{\vec{k}}(t) + \beta(\vec{k})(1+i\omega\delta)v_{\vec{k}}(t). \tag{55}
$$

The field ϕ is after the transformation given by

$$
\phi(t') = \sum_{\vec{k}} \Big(a_{\vec{k}}\psi_{\vec{k}}(t') + a^\dagger_{\vec{k}}\psi^*_{\vec{k}}(t')\Big). \tag{56}
$$

The change in the modes ψ can be expressed equivalently in a change in the annihilation and creation operators, so we can write

$$
\phi(t') = \sum_{\vec{k}} \Big(a'_{\vec{k}}\psi_{\vec{k}}(t) + a^{\dagger'}_{\vec{k}}\psi^*_{\vec{k}}(t)\Big), \tag{57}
$$

such that we find up to first order in δ

$$
\Big(1-\frac{|\beta(\vec{k})|^2}{|\alpha(\vec{k})|^2}\Big)a_{\vec{k}} = \Bigg(\Big(1-\frac{|\beta(\vec{k})|^2}{|\alpha(\vec{k})|^2}\Big) + \Big(1+\frac{|\beta(\vec{k})|^2}{|\alpha(\vec{k})|^2}\Big)i\delta\omega\Bigg)a'_{\vec{k}} + 2i\delta\omega\frac{\beta(\vec{k})^*}{\alpha(\vec{k})}a^{\dagger'}_{\vec{k}}. \tag{58}
$$

We thus find that the ground state $|0\rangle$ defined by $a(\vec{k})|0\rangle = 0$ is only the ground state after the transformation, defined by $a'_{\vec{k}}|0'\rangle = 0$, if $\beta = 0$. If β is nonzero, creation and annihilation operators will be mixed and thus the definition of the ground state is changed. Thus we find that the unique Poincaré invariant vacuum state is given by

$$\alpha = 1 \qquad ; \qquad \beta = 0. \tag{59}$$

Let us emphasize again that this is not the unique ground state of Minkowski space, there is not such a thing. However it is the agreed ground state for all inertial observers, which minimizes the energy. Thus if we choose our mode functions to be the positive frequency solutions, we obtain the same vacuum, independent of the Lorentz frame where t is the time coordinate. To make this somewhat more clear, suppose we have two observers A and B, both in their own coordinate frame x^μ_A and x^μ_B. Both observers define a vacuum state by $\alpha_A = \alpha_B = 1$ and $\beta_A = \beta_B = 0$. The importance of Poincaré invariance is that when observer A concludes that a certain state $|0\rangle$ is in fact the vacuum, observer B will make the same conclusion. The ground state defined by (59) is moreover the state of minimum energy. Single excitations then add an energy ω to the vacuum. Notice also that only for this choice the modes (44) are eigenfunctions of the energy operator $i\partial_0$, with eigenvalue ω.
The above discussion dealt with inertial observers. It turns out that non inertial observers, even in Minkowski space-time, do not see the same vacuum state. This effect is known as the Unruh effect [6].

3.2 Quantum field theory on a curved background

In a curved space-time this analysis breaks down. The lack, in general, of invariance under time translations means that inertial observers will typically see different vacua and there is no way of making a unique, well motivated choice for α and β. Moreover, the lack of a time-like Killing vector makes the notion of energy dependent on the time slicing. It is therefore not possible to unambiguously define a state of minimum energy globally.
This lack of symmetry makes it impossible to define a unique state of minimum energy and hence makes the notion of the vacuum, and of the concept of a particle ambiguous. Therefore we will eventually abandon the whole notion of particles and consider only the correlators. An insightful way to show what goes on is to consider a space-time which has asymptotically flat regions in the past and the future. In these asymptotic regions, we can without any trouble define the positive frequency solutions, like in Minkowski space. We indicate the positive and negative frequency solution in the past, or *in* region with $u^{in}_{\vec{k}}$ and $v^{in}_{\vec{k}}$, analogous to (41). In the future or *out* region we use $u^{out}_{\vec{k}}$ and $v^{out}_{\vec{k}}$. We choose our modes to be pure positive frequency in their respective region

$$\psi^{in}_{\vec{k}} = u^{in}_{\vec{k}} \qquad ; \qquad \psi^{out}_{\vec{k}} = u^{out}_{\vec{k}} \tag{60}$$

and we normalize our modes in the same way as (43). Although these modes are defined by their properties in some asymptotic region, they are necessarily solutions to the field equations on the whole space-time. Now since (41) forms a complete set of solutions, we can always write

$$\psi^{in}_{\vec{k}} = \sum_{\vec{j}} \alpha_{\vec{j}\vec{k}} \psi^{out}_{\vec{j}} + \beta_{\vec{j}\vec{k}} (\psi^{out}_{\vec{j}})^* \tag{61}$$

and from the normalization of u and v it follows that

$$\sum_{\vec{k}} \Big(\alpha_{\vec{j}\vec{k}}\alpha^*_{\vec{k}\vec{\ell}} - \beta_{\vec{j}\vec{k}}\beta^*_{\vec{k}\vec{\ell}}\Big) = \delta_{\vec{j}\vec{\ell}}, \tag{62}$$

such that

$$\psi^{out}_{\vec{k}} = \sum_{\vec{j}} \alpha^*_{\vec{j}\vec{k}}\psi^{in}_{\vec{j}} - \beta_{\vec{j}\vec{k}}(\psi^{in}_{\vec{j}})^*. \tag{63}$$

The field operator now may be expanded both in terms of the *in* or the *out* modes.

$$\begin{aligned} \phi &= \sum_{\vec{k}} \big(a_{\vec{k}}\psi^{in}_{\vec{k}} + a^\dagger_{\vec{k}}(\psi^{in}_{\vec{k}})^*\big) \\ &= \sum_{\vec{k}} \big(b_{\vec{k}}\psi^{out}_{\vec{k}} + b^\dagger_{\vec{k}}(\psi^{out}_{\vec{k}})^*\big). \end{aligned} \tag{64}$$

The *in* vacuum is defined by $a_{\vec{k}}|0\rangle_{in} = 0$ while the *out* vacuum is defined by $b_{\vec{k}}|0\rangle_{out} = 0$. We can calculate the annihilation operators

$$\begin{aligned} a_{\vec{k}} &= (\phi, \psi^{in}_{\vec{k}}) = \sum_{\vec{j}} \big(\alpha^*_{\vec{j}\vec{k}} b_{\vec{j}} - \beta^*_{\vec{j}\vec{k}} b^\dagger_{\vec{j}}\big) \\ b_{\vec{k}} &= (\phi, \psi^{out}_{\vec{k}}) = \sum_{\vec{j}} \big(\alpha_{\vec{j}\vec{k}} a_{\vec{j}} + \beta_{\vec{j}\vec{k}} a^\dagger_{\vec{j}}\big), \end{aligned} \tag{65}$$

where the scalar product is the generalized form of (42)

$$(\phi_1, \phi_2) = -i \int_\Sigma \sqrt{-g_\Sigma}\, d\Sigma^\mu \Big(\phi_1 \partial_\mu \phi_2^* - (\partial_t \phi_1)\phi_2^*\Big). \tag{66}$$

Here $d\Sigma^\mu = n^\mu d\Sigma$ and n^μ is a timelike unit vector, normal to the hypersurface Σ, with metric g_Σ. Equation (65) describes the so-called Bogoliubov transformation. It describes how the creation and annihilation operators for different mode decompositions are related. We see that if the description of the modes in one region in terms of the modes in another region mixes positive and negative frequencies, (i.e. $\beta_{\vec{j}\vec{k}} \neq 0$), the vacuum described by one set of modes is not the same as the vacuum described by the other set of modes. In particular, if we start in the vacuum $|0\rangle_{in}$, an observer in the out region will see

$${}_{in}\langle 0|b^\dagger_{\vec{k}} b_{\vec{k}}|0\rangle_{in} = \sum_{\vec{j}} |\beta_{\vec{j}\vec{k}}|^2 \tag{67}$$

particles. An example of a Bogoliubov transformation we have already seen in (58).

3.2.1 Simple cosmological particle creation

Let us now consider as an example a spatially flat FLRW space-time. We consider a scalar field with an action

$$S = \frac{1}{2}\int d^4x \sqrt{-g}\Big(-\partial^\mu\phi\partial_\mu\phi - m^2\phi^2 - \xi R\phi^2\Big) \tag{68}$$

and the metric is given by (8). The scalar field obeys the equation of motion

$$\Big(\Box - m^2 - \xi R\Big)\phi = 0 \tag{69}$$

Here $\Box$ denotes the covariant d'Alembertian

$$\Box = \frac{1}{\sqrt{-g}}\partial_\mu g^{\mu\nu}\sqrt{-g}\partial_\nu. \tag{70}$$

With the metric (8) the equations of motion become

$$\Big(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} - \frac{\partial^2}{a^2\partial\vec{x}^2} + m^2 + \xi R\Big)\phi = 0. \tag{71}$$

If we now write

$$\phi = \frac{1}{a^{3/2}}e^{i\vec{k}\cdot\vec{x}}\chi \tag{72}$$

the equation of motion reduces to

$$\Big(\frac{\partial^2}{\partial t^2} + \omega^2(t)\Big)\chi = 0, \tag{73}$$

where

$$\omega^2(t) = \frac{k^2}{a^2} + m^2 + \xi R + \frac{3}{4}(2\epsilon - 3)H^2 \tag{74}$$

Thus we see that we have a very similar mode function as in Minkowski space. Only now the frequency has become time dependent. To make a sensible definition of the ground state in the *in* and *out* regions, we assume that in those regions the scale factor is constant. This automatically makes ω constant. We shall from now on work on an infinite volume and replace the sums with integrals

$$\begin{aligned} \frac{1}{V}\sum_{\vec{k}} \quad &\rightarrow \quad \int\frac{d^3\vec{k}}{(2\pi)^3} \\ V\delta_{\vec{k}\vec{p}} \quad &\rightarrow \quad (2\pi)^3\delta^3(\vec{k}-\vec{p}) \end{aligned} \tag{75}$$

We write our field in terms of its annihilation and creation operators

$$\phi = \frac{1}{a^{3/2}}\int\frac{d^3k}{(2\pi)^3}\Big(a_{\vec{k}}\psi_{\vec{k}}, + a^\dagger_{\vec{k}}\psi^*_{\vec{k}}\Big) \tag{76}$$

the positive frequency modes in the past and future are

$$\begin{aligned} \psi_{\vec{k}}(t\rightarrow-\infty) \equiv \psi^{in}_{\vec{k}} &= \sqrt{\frac{1}{2\omega_{in}}}e^{i(\vec{k}\cdot\vec{x}-\omega_{in}t)} \\ \psi_{\vec{k}}(t\rightarrow+\infty) \equiv \psi^{out}_{\vec{k}} &= \sqrt{\frac{1}{2\omega_{out}}}e^{i(\vec{k}\cdot\vec{x}-\omega_{out}t)} \\ &= \int\frac{d^3\vec{k}'}{(2\pi)^3}\Big(\alpha^*_{\vec{k}'\vec{k}}\psi^{in}_{\vec{k}'} - \beta_{\vec{k}'\vec{k}}(\psi^{in}_{\vec{k}'})^*\Big), \end{aligned} \tag{77}$$

where in the last line we used the fact that the modes form a complete set of solutions to write the *out* modes in terms of the *in* modes, similar to (61) and we defined

$\omega_{in} =\equiv \omega(t \to -\infty)$ and $\omega_{out} =\equiv \omega(t \to +\infty)$. As an example, suppose that the scale factor $a(t)$ is constant everywhere, but changes discontinuously at $t = 0$. So we have $a = a_{in}$ at $t < 0$ and $a = a_{out}$ at $t > 0$. Notice that the Hubble parameter is zero in both the in and out regions. We can determine the Bogoliubov coefficients by requiring first of all that at $t = 0$ the decomposition (77) is correct and require that $\int |\alpha_{k'k}|^2 - |\beta_{k'k}|^2 = 1$ to find

$$\begin{aligned} \alpha_{\vec{k}'\vec{k}} &= \frac{1}{2}\Big(\sqrt{\frac{\omega_{in}}{\omega_{out}}} + \sqrt{\frac{\omega_{out}}{\omega_{in}}}\Big)(2\pi)^3\delta^3(\vec{k'} - \vec{k}) \\ \beta_{\vec{k}'\vec{k}} &= \frac{1}{2}\Big(\sqrt{\frac{\omega_{in}}{\omega_{out}}} - \sqrt{\frac{\omega_{out}}{\omega_{in}}}\Big)(2\pi)^3\delta^3(\vec{k'} + \vec{k}). \end{aligned} \tag{78}$$

If the scalar field was initially in its vacuum state $|0\rangle_{in}$, the number of particles observed by an observer in the *out* region is[2]

$$_{in}\langle 0|N_{\vec{k}}|0\rangle_{in} = \int \frac{d^3k'}{(2\pi)^3}|\beta_{\vec{k}'\vec{k}}|^2 = \frac{1}{4}\frac{(\omega_{in} - \omega_{out})^2}{\omega_{in}\omega_{out}}. \tag{79}$$

As a particular case, suppose that $a_{out} = (1 + \sqrt{2})a_{in}$. Now we have

$$\begin{aligned} _{in}\langle 0|N_{\vec{k}}|0\rangle_{in} &= \frac{k^4}{(k^2 + m^2a_{in}^2)(k^2 + (3 + 2\sqrt{2})m^2a_{in}^2)} \\ &\sim 0 \qquad (k \to 0) \\ &\sim 1 \qquad (k \to \infty) \end{aligned} \tag{80}$$

In other words, an observer in the new vacuum will see approximately one excitation for each large value of $\vec{k}$. In the infrared (small $\vec{k}$) however an observer in the *out* region will not see any excitations if there were none initially. An interesting observation is that (79) is invariant under the interchange of ω_{in} and ω_{out}. So if we had chosen $a_{out} = (1 + \sqrt{2})^{-1}a_{in}$, we would have obtained the same result.

3.2.2 Adiabatic approximation

In the previous example, the rate of change in the background is infinite at the matching point $t = 0$. This large change caused the excitation of modes for all large values of k. More generally, if the rate of change in the background is characterized by H (such that in the previous example $H \to \infty$), then only modes will be excited with $\omega \lesssim H$. Now if we do not have asymptotically flat regions, but if the expansion rate is slow enough, one could still envision a state where particle production is minimal. Such a state would be the one 'closest' to Minkowski space and the notion of a well-defined vacuum and particle becomes at least approximately meaningful. One remarkable thing of such an adiabatic approximation is that, as long as the rate of change in the geometry is small, particle production is exponentially suppressed, independent of the total amount of change. To be more precise, we require that

$$\frac{\dot{\omega}}{\omega} \ll \omega \tag{81}$$

[2]This result corresponds to the $\rho \to \infty$ limit of the similar example presented in [6]

and we suppose we have rewritten the equations of motion in the form (73). We consider the WKB-type solution for the modes

$$\psi = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{2W}} e^{-i\int^t W(t')dt'}. \tag{82}$$

from which it follows that

$$W^2(t) = \omega^2(t) - \frac{1}{2}\Big(\frac{\ddot{W}}{W} - \frac{3}{2}\Big(\frac{\dot{W}}{W}\Big)^2\Big). \tag{83}$$

Thus we see that to the lowest order in the adiabatic approximation we have $W_0(t) = \omega(t)$ from which we can solve (83) by subsequent iterations. It is clear that this is a derivative expansion and around the classical solution (a simple exponential) and thus it is an expansion in powers of $\hbar$. Moreover we see that if ω is constant, this solution reduces to the standard Minkowski solution. Now if we write the A'th order adiabatic approximation (thus the solution for ψ after A iterations) for the mode function as ψ_A, we can always choose coefficients α_A and β_A, such that

$$\psi = \alpha_A \psi_A + \beta_A \psi_A^*. \tag{84}$$

is an *exact* solution to the field equation. Now α_A and β_A must consequently be constant up to adiabatic order A, since ψ_A is a solution to the field equations of this order and ψ also must be a solution. Now we can at a certain time t_0 choose

$$\begin{aligned} \alpha_A(t_0) &= 1 + \mathcal{O}(\partial_t^{A+1}) \\ \beta_A(t_0) &= 0 + \mathcal{O}(\partial_t^{A+1}). \end{aligned} \tag{85}$$

The vacuum defined by ψ is now said to be the adiabatic vacuum. It is the state obtained by an expansion, based on the limit (81) around the Minkowski vacuum. In that sense it is 'close' to the Minkowski vacuum, if the adiabatic condition (81) holds. The adiabatic vacuum defined by (85) is not unique, since we could have chosen a different time t_0 to match the exact solution to the adiabatic solution. However, suppose that we would have done the matching at some different time t_1, the mode (84) would only differ from the one defined at t_0 at higher adiabatic order. And therefore, the β Bogoliubov coefficient will be $0 + \mathcal{O}(\partial_t^{A+1})$. Since this coefficient determines the production of particles,we find that particle production is suppressed if the adiabatic condition (81) holds. Since the adiabatic approximation is an expansion in powers of $\hbar$, we thus find that particle creation is essentially a nonperturbative effect that cannot be correctly described by the adiabatic approximation. Notice that for a typical ω, like (74), with $\dot{\omega}$ finite, the adiabatic approximation will be fulfilled for the high k modes. Notice however that this does *not* mean that the ultraviolet sector of the theory reduces simply to the equivalent theory in Minkowski space-time, as we shall see in section 3.4.

3.3 Semi-classical Einstein equations, the effective action and back-reaction

So far we have looked at the dynamics of the quantum fields propagating on a certain curved space-time. Let us now look at the dynamics of space-time itself. At the classical level, the dynamics of the geometry are given by the variation with respect to the metric of the action (2)

$$S = \int \sqrt{-g} d^D x \frac{1}{16\pi G_N} \Big(R - (D-2)\Lambda \Big) + S_M, \tag{86}$$

where S_M is the action associated with the matter fields and D is the number of space-time dimensions. The variation with respect to the metric leads to the Einstein equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{D-2}{2} \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{87}$$

such that the energy momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}. \tag{88}$$

The semi-classical approach to quantum field theory on a curved background now is to consider all our matter fields to be quantum fields. These fields then propagate on a certain classical background metric. However, there is no reason not to consider the graviton field as a quantum field as well. Thus we promote all fields to operators and expand them around a certain classical background value (indicated with a superscript (b))

$$\begin{aligned} &R^{(b)}_{\mu\nu} - \frac{1}{2} R^{(b)} g^{(b)}_{\mu\nu} + \frac{D-2}{2} \Lambda g^{(b)}_{\mu\nu} \\ &+ \langle 0 | \delta \hat{R}_{\mu\nu} - \frac{1}{2} \delta \hat{R} \delta \hat{g}_{\mu\nu} + \frac{D-2}{2} \Lambda \delta \hat{g}_{\mu\nu} | 0 \rangle = 8\pi G_N \Big(T^{(b)}_{\mu\nu} + \langle 0 | \delta \hat{T}_{\mu\nu} | 0 \rangle \Big), \end{aligned} \tag{89}$$

where $\langle 0 | \ldots | 0 \rangle$ indicates the expectation value with respect to some state $|0\rangle$ and hats indicate that we are considering quantum operators. $\delta \hat{R}_{\mu\nu} = \hat{R}_{\mu\nu} - R^{(b)}_{\mu\nu}$ etc. Thus we see that the left hand side (LHS) of the Einstein equation obtains quantum corrections due to graviton fluctuations and the right hand side (RHS) of the Einstein equation obtains corrections due to matter fluctuations. Of course, because of general coordinate invariance there are gauge issues in quantizing the graviton, implying that the distinction between what is the background and what is a small perturbation is coordinate dependent. However in principle one can do this and deal with the gauge issues in a standard way by adding a gauge fixing term to the action. Of course a quantum theory for the graviton, being a quantum theory for gravity, will be non-renormalizable. However, when we consider processes at a certain energy E, higher order quantum effects will be suppressed by the Planck mass as E/m_p. Thus while the theory is not renormalizable, it does make sense as a perturbative effective theory for energies much smaller than the Planck scale.

Nothing prevents us from considering the graviton fluctuations as part of the matter sector of the theory (or at least the non geometric sector) and write the so-called semi-classical Einstein equation

$$R^{(b)}_{\mu\nu} - \frac{1}{2} R^{(0)} g^{(b)}_{\mu\nu} + \frac{D-2}{2} \Lambda g^{(b)}_{\mu\nu} = 8\pi G_N \langle T_{\mu\nu} \rangle. \tag{90}$$

All quantities on the left hand side are now pure geometric and of course constructed from the classical background metric $g^{(b)}_{\mu\nu}$ and $\langle T_{\mu\nu} \rangle$ includes all matter, including the

background and quantum fluctuations, coming from both the matter fields and the graviton. In the following we will omit the superscript (b).
The semi-classical approach can also be successfully implemented in electromagnetism. Then one couples a classical electromagnetic field to the expectation value of the current, i.e. $\nabla_\mu F^{\mu\nu} = \langle J^\nu \rangle$. Before discussing the implications of (90), we shall look somewhat closer on the meaning of $\langle T_{\mu\nu} \rangle$. Being a stress energy tensor, we should be able to construct a quantity $\Gamma[g_{\mu\nu}]$, known as the effective action, such that [6]

$$\langle T_{\mu\nu} \rangle = -\frac{2}{\sqrt{-g}} \frac{\Gamma[g_{\mu\nu}]}{\delta g^{\mu\nu}}. \tag{91}$$

Notice that the effective action will typically also depend on the background field. To give a meaning to $\Gamma[g_{\mu\nu}]$, we consider the generating functional used in path integral quantization. To be somewhat more precise, we shall consider as an example the functional for a scalar field φ. For definiteness we consider here the standard in-out formalism.

$$\begin{aligned} Z[J, g_{\mu\nu}] &= {}_{out}\langle 0|0\rangle_{in} \\ &= \int \mathcal{D}[\varphi] \exp\Big(iS_M[\varphi] + i\int \sqrt{-g} d^D x J(x)\varphi(x)\Big). \end{aligned} \tag{92}$$

Here $|0\rangle_{out}$ and $|0\rangle_{in}$ are the asymptotic vacuum states of the theory and J is an external current that can produce particles in this vacuum. By taking n functional derivatives of Z and then putting J to zero, we can find the expectation value of n-point functions. In the specific case of Minkowski space-time, we of course have that, without an external current, the *in* and *out* vacua are the same, such that we can choose

$$Z[0, g_{\mu\nu}]_{Mink} = 1. \tag{93}$$

However, we have seen that in curved space-time asymptotic states are in general not equivalent. We consider the variation of $Z[0, g_{\mu\nu}]$

$$\begin{aligned} \delta Z[0, g_{\mu\nu}] &= \int \mathcal{D}[\varphi](i\delta S_M) \exp\Big(iS_M[\varphi]\Big) \\ &= i\, {}_{out}\langle 0|\delta S_M|0\rangle_{in} \end{aligned} \tag{94}$$

Notice that if we consider the variation with respect to φ, this quantity is zero by the equation of motion. However if we consider the variation with respect to the metric this is not the case, since the Einstein-Hilbert action is not part of S_M. We find

$$-\frac{2}{\sqrt{-g}} \frac{\delta Z[0, g_{\mu\nu}]}{\delta g^{\mu\nu}} = i\, {}_{out}\langle 0|T_{\mu\nu}|0\rangle_{in}. \tag{95}$$

After performing the functional integral over all paths for $Z[0, g_{\mu\nu}]$, we essentially obtain again the exponential of an action, but now with all quantum effects included. Thus we define the effective action as

$$\Gamma[g_{\mu\nu}] = -i \ln(Z[0, g_{\mu\nu}]) \tag{96}$$

such that from (91) we find

$$-\frac{2}{\sqrt{-g}} \frac{\delta\Gamma[g_{\mu\nu}]}{\delta g^{\mu\nu}} = \frac{{}_{out}\langle 0|T_{\mu\nu}|0\rangle_{in}}{{}_{out}\langle 0|0\rangle_{in}} \equiv \langle T_{\mu\nu} \rangle. \tag{97}$$

We now briefly sketch how the generating functional $Z[0, g_{\mu\nu}]$ can be calculated using the background field method. We consider a scalar field operator $\varphi(x)$, with the action $S[\varphi]$. We now can in general define the background field Φ as

$$\langle \Psi_1 | \varphi(x) | \Psi_2 \rangle \equiv \Phi(x) \tag{98}$$

where Ψ_1 and Ψ_2 are two vectors in the Hilbert space. The quantum field ϕ is now the difference between the background field and the field operator

$$\varphi(x) \equiv \Phi(x) + \phi(x) \tag{99}$$

Now we assume that the background field is such that it is a stationary solution for the effective action

$$\begin{aligned} \frac{\delta \Gamma[\Phi]}{\delta \Phi(x)} &= \left\langle \Psi_1 \left| \frac{\delta S[\varphi]}{\delta \varphi(x)} \right| \Psi_2 \right\rangle \\ &= 0\ . \end{aligned} \tag{100}$$

This might at first seem to be a pointless definition. The field φ of course obeys its own equation of motion, making the RHS automatically zero. However with Γ given by the path integral as in (92) we can expand φ around the background solution. The resulting integral can now be expanded as a perturbation series in $\hbar$. The integrals over the quantum field ϕ in the resulting series are then Gaussian and can be performed. The result is given by (putting explicitly the factors of $\hbar$)

$$\begin{aligned} \Gamma[\Phi] = S[\Phi] &+ \frac{i\hbar}{2} \ln \left\{ \det \left[\frac{\delta^2 S[\Phi]}{\delta\Phi\delta\Phi} \right] \right\} - \frac{\hbar^2}{8} \int d^D w\, d^D x\, d^D y\, d^D z \\ &\times \frac{\delta^4 S[\Phi]}{\delta\Phi(w)\delta\Phi(x)\delta\Phi(y)\delta\Phi(z)} \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(w)\delta\Phi(x)} \right]^{-1} \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(y)\delta\Phi(z)} \right]^{-1} \\ &+ \frac{\hbar^2}{12} \int d^D u\, d^D v\, d^D w\, d^D x\, d^D y\, d^D z\, \frac{\delta^3 S[\Phi]}{\delta\Phi(u)\delta\Phi(v)\delta\Phi(w)} \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(u)\delta\Phi(x)} \right]^{-1} \\ &\times \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(v)\delta\Phi(y)} \right]^{-1} \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(w)\delta\Phi(z)} \right]^{-1} \frac{\delta^3 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)\delta\Phi(z)} + O(\hbar^3)\ . \end{aligned} \tag{101}$$

Here the exponent $^{-1}$ denotes the functional inverse,

$$\int d^D y\, \frac{\partial^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(y)} \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(y)\delta\Phi(z)} \right]^{-1} = \delta^D(x-z)\ . \tag{102}$$

Multiplying by i (times $\hbar$ which we henceforth will set back to one) shows that this is simply the propagator in the presence of background Φ,

$$i \left[\frac{\delta^2 S[\Phi]}{\delta\Phi(x)\delta\Phi(\tilde{x})} \right]^{-1} \equiv i\Delta(x; \tilde{x})\ , \tag{103}$$

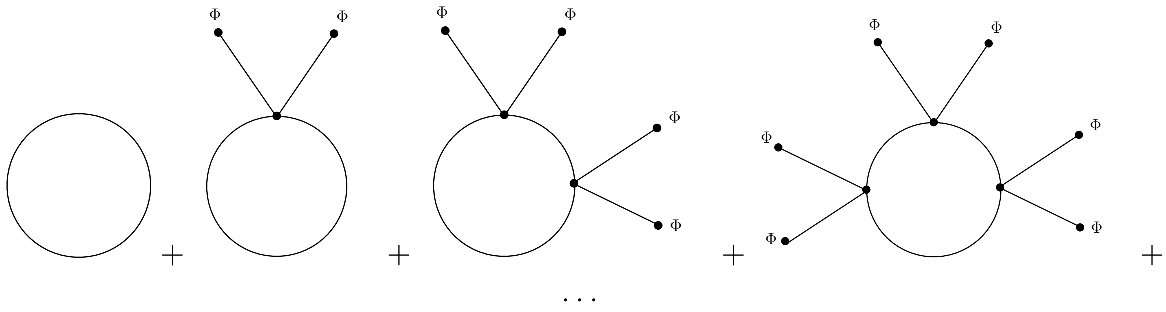

Figure 1: Feynman graphs corresponding to (105) (we assumed that $V(\Phi) = \frac{\lambda}{4!}\Phi^4$)

where the propagator $i\Delta(x;\tilde{x})$ depends also on the background field(s). Let us look slightly more carefully on what this expansion means. We focus on the $\mathcal{O}(\hbar)$ correction. We use that

$$\det(A) = e^{\mathrm{Tr}\ln(A)}, \tag{104}$$

where the trace here includes both the integration over space-time and the trace over possible Lorentz indices.
Suppose we have an action for the scalar field,
$S[\Phi] = \frac{1}{2}\int d^Dx\sqrt{-g}\Big(-\partial^\mu\Phi\partial_\mu\Phi - \frac{m^2}{2}\Phi^2 - V(\Phi)\Big)$. We can then write the contribution $\mathcal{O}(\hbar)$ as

$$-\frac{i\hbar}{2}\mathrm{Tr}\ln(\Delta) = \frac{i\hbar}{2}\Big(\mathrm{Tr}\ln(1+\Delta_F V''(\Phi)) - \mathrm{Tr}\ln(\Delta_F)\Big) \tag{105}$$

where Δ_F is the propagator associated with the free field with $V(\Phi) = 0$ and a prime denotes a derivative with respect to Φ . If we now assume that $V''(\Phi)$ is small, the first term of (105) can be expanded to give

$$\begin{aligned} i\hbar\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n}\int d^4z_1d^4z_n\Delta_F(z_1;z_2)V''(\Phi(z_2))\Delta_F(z_2;z_3)\times\dots \\ \dots\times V''(\Phi(z_n))\Delta_F(z_n-z_1)V''(\Phi(z_1)). \end{aligned} \tag{106}$$

Thus we see that we have one closed loop of fields, with n V'' insertions. The second term of (105) has no external Φ lines and can thus be interpreted as the contribution to the vacuum due to a closed loop. Thus for example if $V(\Phi) = \frac{\lambda}{4!}\Phi^4$, we have the expansion in graphs as shown in figure 1.
The $\mathcal{O}(\hbar^2)$ terms can be seen in a similar way to yield the two loop diagrams in figure 2, with insertions. Notice also that for the free field, that we shall discuss often in the following, the first graph of figure 1 is simply all there is. Notice that the legless diagrams in figures 1 and 2 do not contribute to the equations of motion in Minkowski space-time, since they effectively decouple from anything else. In curved space-time this is not the case. These diagrams explicitly depend on the metric and thus enter the Einstein equation through the variation with respect to the metric.
The scheme described above obviously depends upon the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ in (98). For the familiar, in-out effective field equations the state $|\Psi_2\rangle$ is the free vacuum in the infinite past, while $|\Psi_1\rangle$ is the free vacuum in the infinite future. However as we have seen before, it might not always make sense to talk about asymptotic vacuum states

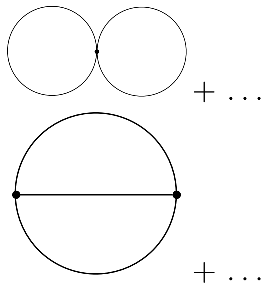

Figure 2: Feynman graphs corresponding to the $\mathcal{O}(\hbar^2)$ contribution of (101)

.

when considering fields on curved background space-time. In such a case, one can use the Schwinger-Keldysh effective field equations [75, 76]. In this case one chooses the two states to be the same, $|\Psi_1\rangle = |\Psi_2\rangle$, and they are taken to be free vacuum either in the infinite past or at some finite time[77]. This is especially convenient for cosmological settings in which the initial singularity precludes reaching the infinite past and in which particle creation makes it typically not possible to select a natural vacuum for the infinite future.
We shall consider in subsequent chapters how to explicitly calculate this formal expression. But for now we shall suppose one has calculated (101) either exactly, or in a certain approximation. What then can we learn from the semi-classical Einstein equation (90)? We see that to calculate $\langle T_{\mu\nu}\rangle$ we need to calculate the propagator, on a certain curved, classical background, such that equation (90) is satisfied. However, how to do this in general is not known. Using the Schwinger-deWitt proper time expansion one can study the propagator in the adiabatic regime[6]. This works excellent to determine the ultraviolet divergent structure of the propagator, but it fails for the ultraviolet finite contributions, while it is precisely these contributions which are sensitive to the effects of particle creation! This is in agreement with what we saw in section 3.2.2, where we found that the adiabatic approximation does not capture the effect of particle creation. Because of the lack of any general construction, we shall in practice consider the classical Einstein equation (87), calculate the metric and, using that specific metric, calculate $\langle T_{\mu\nu}\rangle$. However, by the semi-classical Einstein equations, the metric will subsequently change, making the calculation of $\langle T_{\mu\nu}\rangle$ inconsistent! This phenomenon is known as back-reaction. Typically however the classical background will only change insignificantly, because of the smallness of quantum effects in general, and the additional suppression by G_N in this particular case. In those cases we can then approximately neglect back-reaction. However, as we shall see later, there are also cases where the difference between the background $T_{\mu\nu}$ and the quantum corrected $\langle T_{\mu\nu}\rangle$ is initially small, but grows in time. In these examples, if one waits long enough, one cannot neglect the back-reaction. Since, as soon as back-reaction becomes significant, we do not even have an approximate solution to the semi-classical Einstein equations (90), all calculations in this direction remain on a certain level speculative, however it certainly *is* possible to try to qualify in what sort of 'direction' the metric is changing. Especially if the back-reaction is still small, the background metric can be treated to change adiabatically. In such a case firm predictions can be made.

3.4 Ultraviolet behavior

In Minkowski space it is well known that the expectation value of any product of field operators, when evaluated at the same point, is infinite. Physically this can be attributed to an infinite amount of fluctuating ultraviolet modes [20]. These divergences need then to be regulated using for example a momentum cut-off or by employing dimensional regularization. Because dimensional regularization respects all symmetries of the background space-time, we shall use that approach in this thesis. After regularization we can then absorb the divergencies in the renormalized parameters of the theory. Apart from divergencies that lead to the renormalization of the couplings of the theory, one also finds an infinity due to the vacuum energy. This energy is due to the fact that the hamiltonian (52) is nonzero, even if $N = 0$. When calculating an expectation value, one integrates over all these 'zero point energies' and the final answer diverges. In terms of Feynman diagrams, this contribution is the contribution to figures 1 and 2 with no external lines. In Minkowski space any process involving external fields decouples from these diagrams and thus this energy is unobservable. The zero point energy will lead to an infinite but field independent extra term to the potential of the field. In Minkowski space-time physics is only sensitive to energy differences and thus we are free to shift the 'zero point' of the energy, to simply subtract this vacuum energy. In practice this can be done by *normal ordering*, where one by hand puts in any expectation value the annihilation operators to the right of the creation operators.
In curved space-time the situation is similar, but there are some important differences. Intuitively it is clear that, if space-time is smooth enough, the ultraviolet modes should be almost insensitive to space-time curvature. We also saw that for high enough k the adiabatic approximation makes sense. To consider precisely how insensitive the ultraviolet is to space-time curvature, we consider the metric in Riemann normal coordinates around a point x_0

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{3}R_{\mu\alpha\nu\beta}(x_0)(x-x_0)^\alpha(x-x_0)^\beta + \mathcal{O}(x-x_0)^3. \tag{107}$$

Thus we see that the higher the momenta are that we consider, the more we can treat the metric in the mode equation as flat. Indeed we then find that we can locally define a Fock space that will resemble (in the sense of the adiabatic approximation) Minkowski space-time. However, as we have seen, the notion of states in a Fock space becomes ambiguous in general curved space-times. If we want to make more general statements, we need to consider field observables. In particular we shall consider the expectation value of the stress energy tensor. Since this object describes how energy couples to gravity and since all energy gravitates, all relevant information concerning divergences is stored in this object. We consider as an example the stress energy tensor for a scalar field with an action given by (68)

$$\begin{aligned} T_{\mu\nu} &\equiv -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} \\ &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\Big(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + (m^2+\xi R)\phi^2\Big) + \xi\Big(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big)\phi^2, \end{aligned} \tag{108}$$

where the last part comes from the variation of $R\phi^2$ with respect to the metric. The expectation value with respect to the vacuum state $|0\rangle$ can be written as

$$\begin{aligned}\langle 0|T_{\mu\nu}|0\rangle &= \Big(\delta^\rho_\mu\delta^\sigma_\nu(1-2\xi) - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}(1-4\xi)\Big)\langle 0|\partial_\rho\phi\partial_\sigma\phi|0\rangle \\ &\quad - 2\xi\Big(\delta^\rho_\mu\delta^\sigma_\nu - g_{\mu\nu}g^{\rho\sigma}\Big)\langle 0|\phi\nabla_\rho\partial_\sigma\phi|0\rangle + \Big(\xi R_{\mu\nu} - \frac{1}{2}(m^2+\xi R)g_{\mu\nu}\Big)\langle 0|\phi^2|0\rangle \\ &= \Bigg(\Big(\delta^\rho_\mu\delta^\sigma_\nu(1-2\xi) - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}(1-4\xi)\Big)\partial_\rho\tilde{\partial}_\sigma \\ &\quad - 2\xi\Big(\delta^\rho_\mu\delta^\sigma_\nu - g_{\mu\nu}g^{\rho\sigma}\Big)\nabla_\rho\partial_\sigma - \frac{1}{2}(m^2+\xi R)g_{\mu\nu} + \xi R_{\mu\nu}\Bigg) i\Delta(x;\tilde{x})\Big|_{x=\tilde{x}},\end{aligned} \tag{109}$$

where $\tilde{\partial}_\mu \equiv \frac{\partial}{\partial\tilde{x}^\mu}$ and $\Delta(x;\tilde{x})$ is the (time ordered) Feynman propagator, obeying

$$\sqrt{-g}\Big(\Box - m^2 - \xi R\Big)i\Delta(x;\tilde{x}) = i\delta^D(x-\tilde{x}), \tag{110}$$

where δ^D is the D-dimensional delta function. Now, if we consider $\langle 0|T_{\mu\nu}|0\rangle$ in Riemann normal coordinates, we first of all expect it to have all the properties present in flat space, in other words, ultraviolet divergences, which lead to a renormalization of the coupling constants and an infinite vacuum energy. Now while in flat space the vacuum energy is unobservable, this is not true on a general curved space. The vacuum energy couples to the metric through the volume factor $\sqrt{-g}$ in the action, and thus couples to gravity. The vacuum energy thus needs to be taken into account and will lead to a renormalization of the cosmological constant. However, we also see that extra effects will occur. The propagator obeys a second order differential equation and will therefore in general contain two derivatives of the metric. The stress energy tensor (109) will thus in general depend on terms containing up to four derivatives of the metric. Now in Riemann normal coordinates the first derivative of the metric is zero, but higher order derivatives are nonzero and will depend on space-time curvature as can be seen from (107). Thus we see that independent of how high the momenta of the modes under consideration are –or how 'local' one probes space-time– the expectation value of the stress-energy tensor will always depend on the structure of space-time. This property of the expectation value of the stress energy tensor (109) will lead to additional terms in the effective action, not present in Minkowski space-time. In principle these terms are just as sensitive to the ultraviolet divergences as the other coupling constants in the theory, and we thus expect that we also need to renormalize them. Because of general covariance, the most general action corresponding to those terms (thus terms with up to four derivatives of the metric) is

$$\int d^Dx\sqrt{-g}\Bigg(a_0 + a_1 R + a_2 R^2 + a_3 R^{\mu\nu}R_{\mu\nu} + a_4 R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\Bigg) \tag{111}$$

The constants a_0 and a_1 are renormalizations of the cosmological constant and Newton constant respectively. In four dimensions one can show that the so-called Gauss-Bonnet term

$$\sqrt{-g}G^2 \equiv \sqrt{-g}\Big(R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}\Big) \tag{112}$$

is a topological invariant and does not contribute to the equations of motion. We can use this to eliminate a_4. However, we still in general need to renormalize the couplings a_2 and a_3. This can be done using similar techniques as in flat space and we find that, with the inclusion of those terms, the theory will be renormalizable, provided that the theory was renormalizable in Minkowski space-time.

3.5 Conformal coupling and the trace anomaly

We consider the action (68) for a scalar field but generalize it to D space-time dimensions

$$S = -\frac{1}{2}\int d^D x\sqrt{-g}\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R\phi^2 + m^2\phi^2\Big) \tag{113}$$

and perform a conformal transformation of the metric

$$g_{\mu\nu}(x) \to \bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x). \tag{114}$$

We shall first perform the calculation and afterwards comment on the implications of such a transformation. Under the transformation (114) we find that

$$\begin{aligned}
\bar{g}^{\mu\nu} &= \Omega^{-2} g^{\mu\nu}\\
\sqrt{-\bar{g}} &= \Omega^D\sqrt{-g}\\
\bar{\Gamma}^\alpha_{\mu\nu} &= \Gamma^\alpha_{\mu\nu} + \Big(\delta^\alpha_\nu\partial_\mu + \delta^\alpha_\mu\partial_\nu - g^{\alpha\beta}g_{\mu\nu}\partial_\beta\Big)\ln(\Omega)\\
\bar{R}_{\mu\nu} &= R_{\mu\nu} + \Big((2-D)\partial_\mu\partial_\nu - g_{\mu\nu}\Box\Big)\ln(\Omega)\\
&\qquad + (D-2)\Big((\partial_\mu\ln(\Omega))(\partial_\nu\ln(\Omega)) - g^{\alpha\beta}g_{\mu\nu}(\partial_\alpha\ln(\Omega))(\partial_\beta\ln(\Omega))\Big)\\
\bar{R} &= \Omega^{-2}\Big(R - 2(D-1)\Box\ln(\Omega) - (D-2)(D-1)g^{\mu\nu}(\partial_\mu\ln(\Omega))(\partial_\nu\ln(\Omega))\Big).
\end{aligned} \tag{115}$$

Because the scalar field has a mass dimension of $\frac{D-2}{2}$, we rescale it as

$$\phi \to \bar{\phi} = \Omega^{\frac{2-D}{2}}\phi \tag{116}$$

and thus the kinetic term of the scalar rescales, up to a total derivative, as

$$\begin{aligned}
-\frac{1}{2}\sqrt{-\bar{g}}\bar{g}^{\mu\nu}(\partial_\mu\bar{\phi})(\partial_\nu\bar{\phi}) =& \sqrt{-g}\Bigg(-\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\\
&- \frac{D-2}{4}\phi^2\Big(\Box\ln(\Omega) + \frac{D-2}{2}g^{\mu\nu}(\partial_\mu\ln(\Omega))(\partial_\nu\ln(\Omega))\Big)\Bigg).
\end{aligned} \tag{117}$$

This means that the action (113) rescales as

$$\begin{aligned}
\bar{S} =& -\frac{1}{2}\int d^D x\sqrt{-g}\Bigg(g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + \xi R\phi^2 + \Omega^2 m^2\phi^2\\
&+ \Big(\frac{D-2}{2} - 2(D-1)\xi\Big)\phi^2\Big(\Box\ln(\Omega) + \frac{D-2}{2}g^{\mu\nu}(\partial_\mu\ln(\Omega))(\partial_\nu\ln(\Omega))\Big)\Bigg)
\end{aligned} \tag{118}$$

and we see that the action is invariant, $\bar{S} = S$, if $m^2 = 0$ and $\xi = \frac{D-2}{4(D-1)}$. If this is the case, we call such a scalar field conformally coupled to gravity. One important implication of this lies in cosmology, where we know that we can write $\bar{g}_{\mu\nu} = a(\eta)^2\eta_{\mu\nu}$. Thus the metric is after a conformal transformation equivalent to the metric of Minkowski space-time. Now if the scalar field is conformally coupled, that means that the rescaled field behaves exactly the same as a scalar field would do in flat space-time. The conformally coupled scalar thus does not 'feel' the expansion of space-time, apart from some trivial rescaling.
Another property of conformally coupled fields is that the trace of their energy momentum tensor is zero. This can be seen from

$$\begin{aligned}\bar{S} &= S + \int \frac{\delta \bar{S}}{\delta \bar{g}^{\mu\nu}} \delta \bar{g}^{\mu\nu} d^D x \\ &= S - \int \sqrt{-g}\bar{T}_{\mu\nu}\bar{g}^{\mu\nu} \frac{\delta\Omega}{\Omega} d^D x.\end{aligned} \tag{119}$$

Thus if $S = \bar{S}$, we must have that $\bar{T}_{\mu\nu}\bar{g}^{\mu\nu} = 0$. This property however turns out not to be conserved under quantum corrections. For a conformally coupled scalar one finds that after renormalization the trace of the stress energy tensor has acquired a contribution [6]

$$\langle 0|T^{\mu}{}_{\mu}|0\rangle = -\frac{1}{2880\pi^2}\Big(R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} - R_{\mu\nu}R^{\mu\nu} - \Box R\Big). \tag{120}$$

Especially the appearance of $\Box R$ is potentially interesting and several authors have looked at the implications of such a term on the cosmological evolution [78, 79, 80]. As a final remark we like to point out that the conformally coupled scalar is not the only conformally coupled field. Other examples include the photon field in 4 dimensions or the massless fermion field in D dimensions.

4 de Sitter space

de Sitter space is of special interest for mainly two reasons. First of all it is interesting because of the high amount of symmetry. In fact de Sitter space is the unique maximally symmetric space with positive curvature. The equivalent maximally symmetric space with negative curvature is known as anti de Sitter space. A maximally symmetric space has 10 Killing vectors, which is also the number of Killing vectors of Minkowski space. This high amount of symmetry makes it relatively easy to work on de Sitter space and many insights on quantum field theory on curved backgrounds, come from calculations performed on this background [13, 6, 43, 81, 82, 83, 84, 85, 10, 9, 40, 86, 33, 42]. The second reason of interest for de Sitter space is the fact that during inflation, see section 2, the universe can be very well approximated by de Sitter space [24].
D-dimensional de Sitter space can be represented by the hyperboloid [87]

$$\eta_{AB}X^AX^B = \frac{1}{H^2} \tag{121}$$

embedded in $D+1$-dimensional Minkowski space-time, with metric η_{AB}, where the indices A and B run from 0 to D. Here H is a constant, which we shall see can be related to the Hubble parameter (11). The isometry group of de Sitter space, $SO(1,D)$, is manifest in this embedding. We shall use flat coordinates, which cover only half of the de Sitter manifold, given by $(i = 1, 2, \ldots, D-2, D-1)$

$$\begin{aligned} X_0 &= \frac{1}{H}\sinh(Ht) + \frac{H}{2}x_ix^ie^{Ht}, \\ X_i &= e^{Ht}x_i, \\ X_D &= \frac{1}{H}\cosh(Ht) - \frac{H}{2}x_ix^ie^{Ht}, \\ -\infty &< t, x_i < \infty. \end{aligned} \tag{122}$$

In these coordinates the metric can be seen to be a special case of the spatially flat FLRW metric (8) with the scale factor a given by

$$a = e^{Ht} \tag{123}$$

and thus indeed H is the Hubble parameter, $\frac{\dot{a}}{a}$.
We can write the metric in conformal form by changing coordinates to conformal time η defined as $a d\eta = dt$:

$$g_{\mu\nu} = a^2\eta_{\mu\nu}, \qquad a = -\frac{1}{H\eta}, \qquad \eta < 0\,. \tag{124}$$

This metric covers only half of the de Sitter manifold. The other half is covered by $\eta > 0$ and corresponds to a contracting universe.
For this metric we find the Ricci tensor and scalar to be

$$R_{\mu\nu} = (D-1)H^2 g_{\mu\nu} \qquad ; \qquad R = D(D-1)H^2 \tag{125}$$

And since H is a constant, we see this is a solution to the Einstein equation with a cosmological constant Λ

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{D-2}{2}\Lambda g_{\mu\nu} = 0, \tag{126}$$

if

$$H^2 = \frac{\Lambda}{D-1}. \tag{127}$$

We define the de Sitter invariant distance function

$$1 - \frac{1}{2}Y(X;\tilde{X}) = H^2 \eta_{AB} X^A \tilde{X}^B \,. \tag{128}$$

In the conformal coordinates (122) this function reads

$$\begin{aligned} y(x;\tilde{x}) &= \frac{\Delta x^2(x;\tilde{x})}{\eta\tilde{\eta}} \\ \Delta x^2(x;\tilde{x}) &= -(\eta-\tilde{\eta})^2 + ||\vec{x}-\vec{\tilde{x}}||^2, \end{aligned} \tag{129}$$

where $a = a(\eta)$ and $\tilde{a} = a(\tilde{\eta})$. The function $y = y(x;\tilde{x})$ is related to the geodesic length $\ell = \ell(x;\tilde{x})$ between points x and $\tilde{x}$ as,

$$y(x;\tilde{x}) = 4\sin^2\left(\frac{1}{2}H\ell(x;\tilde{x})\right) \,. \tag{130}$$

The distance function y can be used to characterize the causal relation between points. If $y < 0$, points $\tilde{x}$ are time-like separated to x, and if $y > 0$, they are space-like separated. For $y = 0$ they are light-like separated. We define the antipodal point $\bar{x}$ of x by the map $\eta \to -\eta$.

$$\bar{x}^\mu = (-\eta, \vec{x}) \tag{131}$$

Notice that, since η in (124) is strictly negative, this point is not covered by the coordinates. If $y = 4$, $\tilde{x}$ lies on the lightcone of an (unobservable) image source at the antipodal point $\bar{x}$, see figure 3.

Let us now consider a nonminimally coupled massive scalar field on this background. The equation of motion

$$\Big(\Box - m^2 - \xi R\Big)\phi = 0 \tag{132}$$

can be written in conformal coordinates as

$$\Big(\frac{\partial^2}{(\partial\eta)^2} + \omega(\eta)^2\Big)\chi = 0 \tag{133}$$

with

$$\begin{aligned} \chi(\vec{k},\eta) &= \int d^{D-1}x e^{-i\vec{k}\cdot\vec{x}} a^{D/2-1}\phi \\ \omega(\eta)^2 &= k^2 + a^2\left(m^2 + \Big(\xi - \frac{D-2}{4(D-1)}\Big)R\right) \end{aligned} \tag{134}$$

The equation of motion can be exactly solved and the mode functions can be written in terms of

$$\begin{aligned} u(k,\eta) &= \sqrt{\frac{-\pi\eta}{4}} H_\nu^{(1)}(-k\eta) \\ \nu^2 &= \frac{1}{4} - \frac{\omega^2 - k^2}{(aH)^2} \\ &= \Big(\frac{D-1}{2}\Big)^2 - \Big(\frac{m^2}{H^2} + D(D-1)\xi\Big), \end{aligned} \tag{135}$$

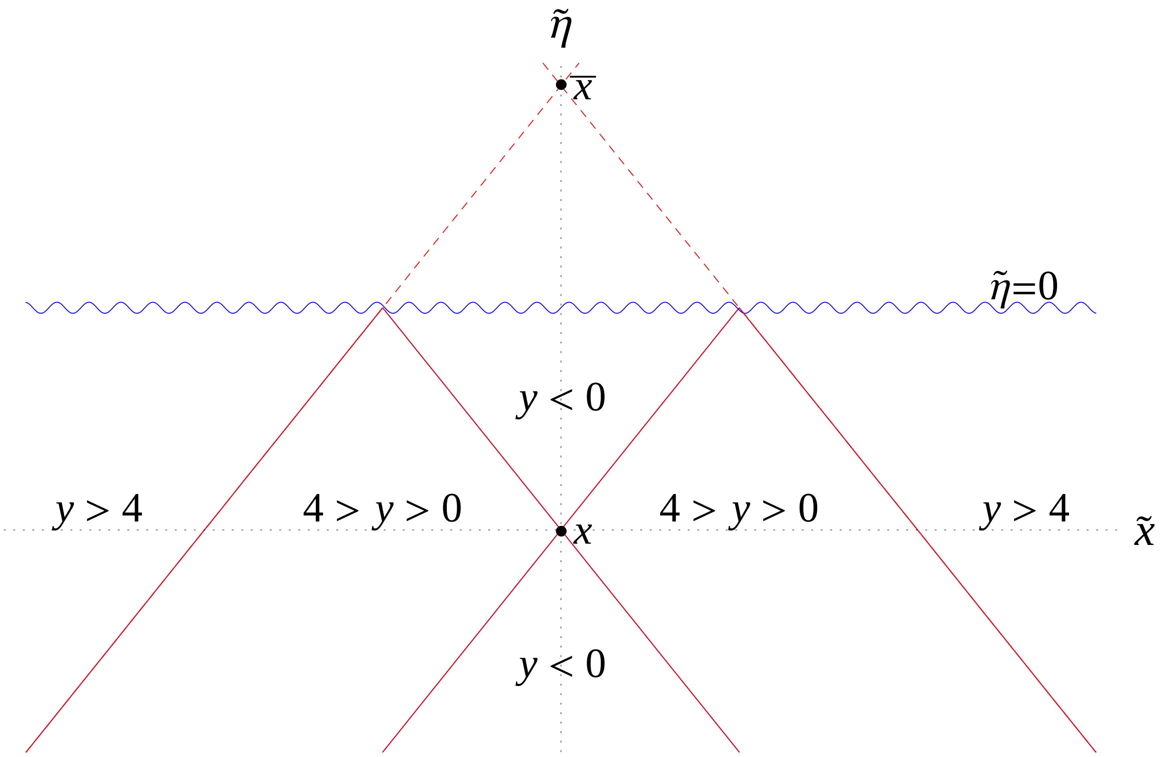

Figure 3: The causal structure in the conformal coordinates (124). The coordinates cover only the region $\eta < 0$. The wavy line at $\eta = 0$ indicates future infinity. The lightcone of the point x is given by $y = 0$. If $y = 4$, the point $\tilde{x}$ lies on the light cone of an unobservable image source at the antipodal point $\bar{x}$.

as

$$\psi(\vec{k},\eta) = \alpha_{\vec{k}} u(\vec{k},\eta) + \beta_{\vec{k}} u^*(\vec{k},\eta). \tag{136}$$

Here $H^{(1)}_\nu(z)$ is the Hankel function of the first kind. It is related to the better known Bessel and Neumann functions as

$$H^{(1)}_\nu(z) \equiv J_\nu(z) + iN_\nu(z) = \frac{i}{\sin(\nu\pi)}\Big\{e^{-i\nu\pi}J_\nu(z) - J_{-\nu}(z)\Big\}\,. \tag{137}$$

The mode functions allow us to decompose the field as

$$\phi = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{1-\frac{D}{2}} \Big(b(\vec{k})e^{i\vec{k}\cdot\vec{x}}\psi(\vec{k},\eta) + b^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\psi^*(\vec{k},\eta) \Big) \tag{138}$$

where we indicated the annihilation and creation operators with b and $b^\dagger$ to avoid confusion with the scale factor. To fix the vacuum, we see that in de Sitter space time we have

$$\lim_{\eta\to-\infty} \frac{\omega'}{\omega} = 0, \tag{139}$$

where a prime denotes a derivative with respect to conformal time. From (139) and (81) we find that in the infinite past the adiabatic vacuum defines a natural initial state. We take the lowest order WKB approximation and then the positive frequency adiabatic vacuum is for $\eta \to -\infty$

$$\psi^A(\eta\to-\infty) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \tag{140}$$

In this same limit we find for the mode functions (136)

$$\psi(\eta\to-\infty) = \frac{1}{\sqrt{2k}} \Big(\alpha e^{-ik\eta} e^{-i\pi(\frac{1}{4}+\frac{1}{2}\nu)} + \beta e^{ik\eta} e^{i\pi(\frac{1}{4}+\frac{1}{2}\nu)} \Big) + \mathcal{O}\Big(\frac{1}{\eta}\Big). \tag{141}$$

Therefore we see that (up to an irrelevant phase factor) the vacuum that has the correct form in the asymptotic past has $\alpha = 1$ and $\beta = 0$. This choice is known as the Bunch-Davies vacuum and, as we shall show later, it is a de Sitter invariant vacuum. Notice that the limits we took are equivalent to the $k \to \infty$ limit. Thus the reduction in this regime to the adiabatic vacuum means that the ultraviolet modes are equivalent as in Minkowski space-time.

4.1 de Sitter propagators and vacua

In this section we shall construct the Feynman propagator for a scalar field on de Sitter space. First we shall solve the Klein-Gordon equation with a δ source and look at the divergent structure. After that we shall consider the integral over the mode functions. From the discussion, the dependence of the propagator on the choice of α and β will become clear.
The time ordered Feynman propagator is given by

$$\begin{aligned} i\Delta(x;\tilde{x}) &\equiv \langle 0\big|T\Big(\phi(x)\phi(\tilde{x})\Big)\big|0\rangle \\ &\equiv \langle 0\big|\theta(t-\tilde{t})\phi(x)\phi(\tilde{x}) + \theta(\tilde{t}-t)\phi(\tilde{x})\phi(x)\big|0\rangle, \end{aligned} \tag{142}$$

where

$$\begin{aligned}\theta(x) &= 1 \qquad ; \qquad x > 0 \\ &= 0 \qquad ; \qquad x < 0\end{aligned} \tag{143}$$

is the Heaviside step function and T indicates the time ordering. It satisfies the following Klein-Gordon equation

$$\sqrt{-g}(\Box - m^2 - \xi R)\imath\Delta(x;\tilde{x}) = \imath\delta^D(x-\tilde{x}), \tag{144}$$

and $\delta^D(x-\tilde{x})$ is the D-dimensional Dirac delta function. If the state $|0\rangle$ is invariant under the de Sitter group, it follows that also the propagator should be invariant under the de Sitter group. If this is the case, it can, up to a pole prescription, only depend [81, 82] on the space-time points x and $\tilde{x}$ via the de Sitter invariant distance function y given in (129). We rewrite the left hand side of (144) in terms of y:

$$\begin{aligned}a^{D-2}\Bigg[&\Big(-(\partial_0 y)^2 + (\partial_i y)^2\Big)\Big(\frac{d}{dy}\Big)^2 \\ &+ \Big[\Big(-\partial_0^2 y + \partial_i^2 y\Big) - \big(D-2\big)\Big(\frac{a'}{a}\Big)(\partial_0 y)\Big]\frac{d}{dy} - a^2\mu^2\Bigg]\imath\Delta(y)\end{aligned} \tag{145}$$

where $i\Delta(y) = i\Delta(x;\tilde{x})$ and

$$\mu^2 \equiv m^2 + \xi R \tag{146}$$

The several terms in (145) evaluate to

$$\begin{aligned}a^{-2}\Big[-(\partial_0 y)^2 + (\partial_i y)^2\Big] &= H^2 y(4-y) \\ a^{-2}\Big[\Big(-\partial_0^2 y + \partial_i^2 y\Big) - \big(D-2\big)\Big(\frac{a'}{a}\Big)(\partial_0 y)\Big] &= -H^2 D(y-2)\end{aligned} \tag{147}$$

The δ-function in (144) is only sourced by the most singular term in $i\Delta$ for $y \to 0$. Putting everything together we find for the non-singular terms that

$$\left[y(4-y)\Big(\frac{d}{dy}\Big)^2 - \big(D(y-2)\big)\frac{d}{dy} - \frac{\mu^2}{H^2}\right]\imath\Delta(y) = 0. \tag{148}$$

Since in de Sitter space, the term μ^2/H^2 is a constant, we see that the Klein Gordon equation has reduced to an ordinary differential equation in terms of y. The resulting equation is a hypergeometric equation with a general solution [71]

$$\begin{aligned}\imath\Delta(y) = A\,{}_2F_1\Big[&\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; \frac{y}{4}\Big] \\ + B\,{}_2F_1\Big[&\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y}{4}\Big],\end{aligned} \tag{149}$$

where

$$\nu^2 = \Big(\frac{D-1}{2}\Big)^2 - \frac{\mu^2}{H^2} \tag{150}$$

is the D-dimensional generalization of (135). The hypergeometric function has a branch point if the argument is 1 or $\tilde{\infty}$ and it has a branch cut along the interval $(1,\infty)$, where it is continuous from below. [3] We see therefore that (149) has in general two singular regions. First of all, if the points x and $\tilde{x}$ are null separated, we have $y \to 0$ and the second hypergeometric function is singular. This is of course expected, since for $y \to 0$ we expect the two-point function to be similar to the Minkowski result. If we require that in this limit the most singular solution reduces to the standard Minkowski result, we can fix the constant B. The most singular term for $y \to 0$ from (149) is

$$B\Big(\frac{y}{4}\Big)^{1-D/2}\frac{\Gamma(D/2)\Gamma(D/2-1)}{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}. \tag{151}$$

The correct time-ordered ε pole prescription for y turns out to be, as we shall see from (164)

$$y_{++}(x,\tilde{x}) = \frac{1}{\eta\tilde{\eta}}\Big(-\big(|\eta-\tilde{\eta}|-\imath\varepsilon\big)^2+r^2\Big) \tag{152}$$

Now since

$$\partial^2\frac{1}{(\eta\tilde{\eta}y_{++})^{D/2-1}} = \frac{4\pi^{D/2}}{\Gamma(\frac{D}{2}-1)}\imath\delta^D(x-\tilde{x}) \tag{153}$$

and the δ function in (144) is sourced by the action of $\sqrt{-g}\Box$ on (151), we find that the δ function is correctly sourced if

$$B = \frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(D/2)}\frac{H^{D-2}}{(4\pi)^{D/2}}. \tag{154}$$

The constant A however is in principle unfixed. For non-zero A we see that the propagator has an additional singularity if $y = 4$, or in other words, if $\tilde{x}$ is null related to the antipodal point of x, see figure 3. The propagator therefore appears to give a response to an image source located at the antipodal point of x. The vacuum of a mode decomposition that leads to such an additional singularity is known as an α-vacuum. The appearance of the image charge is outside the lightcone [88] and hence the 'unphysical' singularity is unobservable, and moreover, the image charge itself is not covered by the coordinates. Despite of this, several authors have argued that the interpretation of such a state as a vacuum state leads to unacceptable physics [89, 90, 91]. However the situation changes when one considers excited states [92, 93]. Notice that excited states, with β k independent require an infinite amount of energy and are thus very unlikely. Excited states with a k dependent β necessarily break de Sitter invariance.

Let us now consider the construction of the propagator by performing the integral over the modes. We consider the propagator in terms of the field (142). Using the mode

[3] For a proper definition of (149) we need a certain ε prescription for y. This prescription shall be given explicitly later in this section.

functions (136) we obtain

$$
\begin{aligned}
i\Delta(x;\tilde{x}) =& \frac{\pi}{4}\sqrt{\eta\tilde{\eta}}(a\tilde{a})^{1-\frac{D}{2}}\int\frac{d^{D-1}k}{(2\pi)^{D-1}}e^{ik(\vec{x}-\vec{\tilde{x}})} \\
&\times\Bigg\{\theta(\eta-\tilde{\eta})\Bigg(|\alpha|^2H_\nu^{(1)}(-k\eta)H_\nu^{(1)}(-k\tilde{\eta})^*+|\beta|^2H_\nu^{(1)}(-k\eta)^*H_\nu^{(1)}(-k\tilde{\eta})\Bigg) \\
&+\theta(\tilde{\eta}-\eta)\Bigg(|\alpha|^2H_\nu^{(1)}(-k\eta)^*H_\nu^{(1)}(-k\tilde{\eta})+|\beta|^2H_\nu^{(1)}(-k\eta)H_\nu^{(1)}(-k\tilde{\eta})^*\Bigg) \\
&+\Bigg(\alpha\beta^*H_\nu^{(1)}(-k\eta)H_\nu^{(1)}(-k\tilde{\eta})+\alpha^*\beta H_\nu^{(1)}(-k\eta)^*H_\nu^{(1)}(-k\tilde{\eta})^*\Bigg)\Bigg\}.
\end{aligned}
\tag{155}
$$

In performing the integral we shall assume that α and β are k independent. This in general does not have to be the case. However, if we require that our state is de Sitter invariant, this must be the case. The reason for this is given in [94]. The argument is that, if we consider the Bogolibov transformation between the Bunch-Davies vacuum ($\alpha=1$, $\beta=0$) and any other vacuum, the number of particles created with momentum k is $|\beta_k|^2$. Since the de Sitter group includes boosts, an immediate consequence of de Sitter invariance is that, if we have any quanta of any given momentum k, we need the same amount of quanta for every k, and thus N_k cannot depend on k. Moreover we have the normalization as before

$$|\alpha|^2-|\beta|^2=1 \tag{156}$$

and thus we immediately deduce that both α and β cannot be k dependent.
Now recall the $D=4$ angular integral,

$$\int\frac{d^3k}{(2\pi)^3}\,e^{i\vec{k}\cdot\vec{r}}f(\|\vec{k}\|)=\frac{1}{2\pi^2}\int_0^\infty dk\,k^2\,\frac{\sin(kr)}{kr}\,f(k)\;. \tag{157}$$

Here and henceforth we define $r\equiv\|\vec{x}-\vec{\tilde{x}}\|$. Generalizing to D spacetime dimensions we have $d^{D-1}k=k^{D-2}dk\,d\Omega_{D-2}$, $k=\|\vec{k}\|$, where

$$d\Omega_{D-2}=\sin^{D-3}(\theta_{D-3})d\theta_{D-3}\sin^{D-4}(\theta_{D-4})d\theta_{D-4}\ldots d\phi, \tag{158}$$

where θ_{D-3}, θ_{D-4}, …, and ϕ are the angles on the sphere $\mathbb{S}^{D-2}$. Making use of

$$\int d\Omega_{D-2}=\frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}=\frac{2(4\pi)^{\frac{D}{2}-1}\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(D-1\right)}\,, \tag{159}$$

and Eq. (8.411.7) in [95] gives,

$$\int\frac{d^{D-1}k}{(2\pi)^{D-1}}\,e^{i\vec{k}\cdot\vec{r}}f(\|\vec{k}\|)=\frac{1}{2^{D-2}\pi^{\frac{D-1}{2}}}\int_0^\infty dk\,k^{D-2}\,\frac{J_{\frac{D-3}{2}}(kr)}{(\frac{1}{2}kr)^{\frac{D-3}{2}}}\,f(k)\;. \tag{160}$$

To perform the integral, we rewrite the Hankel functions in terms of the modified Bessel function of the second kind (also known as the Macdonald function).

$$H_\nu^{(1)}(z)=\frac{2}{\pi}e^{-\frac{i\pi}{2}(\nu+1)}K_\nu(-iz) \tag{161}$$

Now we can show using several relations from [95] that, if the integral converges, we have

$$\int_0^\infty k^{\lambda+1} K_\nu(ak)K_\nu(bk)J_\lambda(ck)dk = \frac{\sqrt{\pi}c^\lambda\Gamma(\lambda+1+\nu)\Gamma(\lambda+1-\nu)}{2^{3/2}\Gamma(3/2+\lambda)} 2^{-\lambda-1/2}\frac{1}{(ab)^{\lambda+1}} \times {}_2F_1\Big(\lambda+1+\nu, \lambda+1-\nu, \lambda+\frac{3}{2}, \frac{1-u}{2}\Big), \tag{162}$$

where $u = \frac{a^2+b^2+c^2}{2ab}$, provided that

$$Re(a) > 0 \qquad ; \qquad Re(b) > 0 \qquad ; \qquad c > 0. \tag{163}$$

The integral (155) can be evaluated to yield

$$\begin{aligned} i\Delta(x;\tilde{x}) = &\Big((a\eta)(\tilde{a}\tilde{\eta})\Big)^{1-D/2} \frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{(4\pi)^{D/2}\Gamma(D/2)} \\ &\Big\{ |\alpha|^2 {}_2F_1\Big(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y_{++}(x,\tilde{x})}{4}\Big) \\ &+ |\beta|^2 {}_2F_1\Big(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y_{--}(\bar{x},\bar{\tilde{x}})}{4}\Big) \\ &- \alpha^*\beta e^{i\pi(\frac{D-1}{2}+\nu)} {}_2F_1\Big(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y_{+-}(\bar{x},\tilde{x})}{4}\Big) \\ &- \alpha\beta^* e^{-i\pi(\frac{D-1}{2}+\nu)} {}_2F_1\Big(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y_{-+}(x,\bar{\tilde{x}})}{4}\Big)\Big\}. \end{aligned} \tag{164}$$

Here a bar indicates the antipodal point $\bar{x} = \overline{(\eta,\vec{x})} = (-\eta,\vec{x})$ and the y functions with the different ε prescriptions are given by (152) and

$$\begin{aligned} y_{+-}(x,\tilde{x}) &= \frac{1}{\eta\tilde{\eta}}\Big(-\big(\eta-\tilde{\eta}+\imath\varepsilon\big)^2+r^2\Big) \\ y_{-+}(x,\tilde{x}) &= \frac{1}{\eta\tilde{\eta}}\Big(-\big(\eta-\tilde{\eta}-\imath\varepsilon\big)^2+r^2\Big) \\ y_{--}(x,\tilde{x}) &= \frac{1}{\eta\tilde{\eta}}\Big(-\big(|\eta-\tilde{\eta}|+\imath\varepsilon\big)^2+r^2\Big) \end{aligned} \tag{165}$$

Thus we see that the first two lines of (164) are divergent when $\tilde{x}$ lies on the light-cone of x, while the third and fourth lines are divergent when $\tilde{x}$ lies on the light-cone of the antipodal point of x.
Since

$$\partial^2 \frac{1}{(\eta\tilde{\eta}y_{--})^{D/2-1}} = -\frac{4\pi^{D/2}}{\Gamma(\frac{D}{2}-1)}\imath\delta^D(x-\tilde{x}) \tag{166}$$

and because of the normalization (156), we find, similar to the discussion that led to the determination of B in (154), that when $\tilde{x}$ lies on the light cone of x, the δ-function in (144) is correctly sourced. While the sourcing of the δ-function is fixed, the small scale behavior is in general dependent on α and β. However, if we require that there is no additional singularity, we need $\alpha = 1$ and $\beta = 0$. However, for any other choice, we will

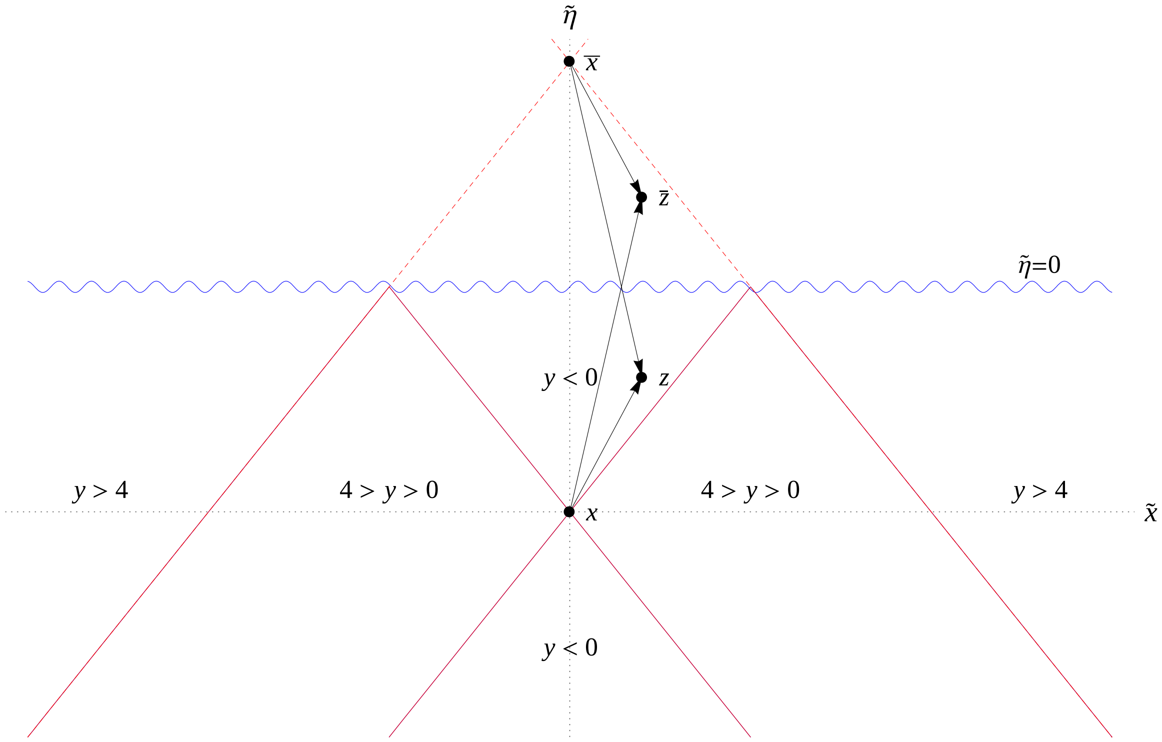

Figure 4: The causal structure as in figure 3, but with a schematic representation of the four different contributions to (164).

have not only a change in the small scale behavior, but also the additional image source at the antipodal point. The presence of such a source indeed seems inevitable as soon as the state under consideration is not precisely the Bunch-Davies vacuum.
The hypergeometric functions in (164) have been written somewhat suggestively. This is done such that each contribution to the full time ordered propagator can be thought of as the propagation of Bunch-Davies type states between different combinations of x, $\tilde{x}$ and their antipodal points. We can see this nicely in figure 4.

4.2 The massless minimally coupled case

In the massless minimally coupled (MMC) case ($m = \xi = 0$), the above construction breaks down. In this case the parameter ν is equal to $\frac{D-1}{2}$ and we see that the propagator (149) (or (164)) reduces to a constant, which is of course a trivial solution to the MMC Klein-Gordon equation. From (148) we find that the second independent solution now is given by

$$y^{1-D/2}{}_2F_1\Big(1-\frac{D}{2},\frac{D}{2},2-\frac{D}{2},\frac{y}{4}\Big), \tag{167}$$

which is singular both for $y = 0$ and for $y = 4$. So if we demand that the propagator obeys the de Sitter invariant differential equation (148), we are forced to accept that the only possible nontrivial state has two singular regions. And thus we can define nothing like a Minkowski-like adiabatic state. It was shown by Allen [82] that this propagator cannot correspond to a de Sitter invariant Fock vacuum $|0\rangle$. In other words, the 'propagator' (167) cannot be a correct propagator. What physically goes wrong can be seen more clearly from the mode functions. If we consider the small argument expansion of the Bessel function,

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}z)^{\nu+2n}}{n!\Gamma(\nu+n+1)} \,, \tag{168}$$

we can easily find the small argument expansion for the Hankel function using (137). We now consider contributions to (155) of the form

$$k^{\frac{D-1}{2}} J_{\frac{D-3}{2}}(kr) H^{(1)}_{\frac{D-3}{2}}(-k\eta) H^{(1)}_{\frac{D-3}{2}}(-k\tilde{\eta})^*. \tag{169}$$

We find that

$$\begin{aligned} k^{\frac{D-1}{2}} J_{\frac{D-3}{2}}(kr) &= \Big(\frac{r}{2}\Big)^{\frac{D-3}{2}} \frac{k^{D-2}}{\Gamma(\frac{D-1}{2})} \Big(1 + \mathcal{O}(k^2)\Big) \\ H^{(1)}_{\frac{D-3}{2}}(-k\eta) H^{(1)}_{\frac{D-3}{2}}(-k\tilde{\eta})^* &= \Big(\frac{\eta\tilde{\eta}}{4}\Big)^{\frac{1-D}{2}} \frac{\Gamma((D-1)/2)}{\pi^2} k^{1-D} \Big(1 + \mathcal{O}(k^2)\Big) \end{aligned} \tag{170}$$

and thus the contribution (169) is for small k proportional to $\frac{1}{k}$. We thus find that the MMC scalar in de Sitter space is logarithmically infrared divergent. This divergence should be taken care of and we shall consider this question in the next sections in much more detail. The basic idea will however always be to in some way suppress the long wavelength contributions to the propagator. This can be immediately translated to a k-dependence of the coefficients α and β. But we showed in the paragraph before (156) that any k dependence of α and β necessarily breaks de Sitter invariance. Thus we indeed find that for the MMC scalar, a physically viable, infrared finite state necessarily breaks de Sitter invariance.

We like to point out however, that there is no necessity, from a physical point of view, to demand invariance under the full de Sitter group [44]. Indeed, while de Sitter space provides an excellent framework to study properties of inflation, it is never truly realized in nature. In de Sitter space the Hubble parameter is per definition globally constant, therefore it follows that if the universe was once de Sitter it will always be de Sitter. From the fact that the universe is not de Sitter today, it follows that it was never de Sitter. The deviation of the inflationary universe from de Sitter space is even measurable through the deviation of the spectral index of scalar cosmological perturbations, n_s, from exact scale invariance [24, 16]. Here the spectral index is defined as $n_s - 1 \equiv \frac{d\ln(P)}{d\ln(k)}$, with P the spectrum of the perturbations. Scale invariance implies $n_s = 1$. There thus appears to be no reason to demand invariance under the full de Sitter group. The only symmetry present in the universe are the symmetries of a general FLRW metric: spatial translations and rotations. The boosts present in the de Sitter group that where crucial in the argument that α and β are k-independent are therefore no symmetries of a realistic FLRW geometry. Violating this requirement, for example to fix the infrared problem as noted above, is therefore in a realistic setting allowed.

5 Infrared divergencies in cosmology

We shall now focus on the main subject of this thesis, the infrared divergencies present in many cosmological models. Before going into a more rigorous derivation, we give a simple heuristic description of the physics of cosmological particle production. We consider a simple application of the energy-time uncertainty relation [23, 43] (notice however that the precise meaning of both energy and time is ambiguous in curved space-time) that states that a virtual particle-anti particle pair with energy E and momenta $\pm k$ can exist a time δt given by

$$\int_t^{t+\delta t} dt' E(t', k) \lesssim 1. \tag{171}$$

While in flat space δt is very small, it can grow large in curved space-time. The reason is that the energy of the virtual pair decreases due to the expansion of the universe. If the redshift is sufficiently fast, δt can grow very large. In particular if the field is massless it can in principle become infinite if the momentum k of the pair is small enough. If we define the energy as

$$E = \frac{k}{a} \tag{172}$$

and we consider a FLRW space-time with constant ϵ (11), we find from (171)

$$\frac{k}{a(t)H(t)(1-\epsilon)}\Big(1 - \frac{a(t)H(t)}{a(t+\delta t)H(t+\delta t)}\Big) \lesssim 1. \tag{173}$$

If the expansion of space-time is accelerating, aH is a growing function in time and we see that δt can grow to infinity if at time t we have that

$$\frac{k}{a(t)H(t)(1-\epsilon)} \lesssim 1. \tag{174}$$

In other words, there is a continuous production of particles, with a physical wavelength, a/k much larger than the Hubble radius due to the Heisenberg uncertainty relation. Although intuitively appealing, the above argument is only of limited use. Since the whole notion of a particle is not well-defined in curved space, it is not obvious how to interpret the single particle uncertainty relation (171). Moreover, because of the lack of a time-like Killing vector, the notion of energy is also not properly defined. However, for a given coordinate frame, the above discussion at least gives a reasonable intuitive idea. For a proper analysis on what happens in the infrared, we need to consider the propagator. Since a propagator is essentially a correlation function of the field between two points, we can interpret a growth of the propagator at large distances, as an excess of long wavelength modes. Thus we expect that the super-Hubble particle production described above leads to an enhancement of the infrared contribution to the propagator. For concreteness we once again consider a scalar field ϕ in D dimensions, with a mass m and conformal coupling ξ, whose action is given in (68). The background metric is, at this point, a completely general FLRW metric and we work in conformal coordinates (10). The equations of motion are exactly equivalent as in de Sitter (133) with (134). However, the crucial difference is that, unlike in de Sitter, we cannot solve the massive equations. To see this, we write (133) as

$$\Big(\partial_\eta^2 + k^2 + \frac{\frac{1}{4} - \nu^2}{\eta^2}\Big)\chi = 0 \tag{175}$$

with

$$\nu^2 = \Big(\Big(\frac{D-1-\epsilon}{2}\Big)^2 - (D-1)(D-2\epsilon)\xi - \frac{m^2}{H^2}\Big)(Ha\eta)^2 \tag{176}$$

and we immediately see the difference with the de Sitter case. There the parameter ν was constant, while here it is in general not. Clearly if ν is not constant, (175) is in general not analytically solvable. In a general FLRW space-time, ϵ and H have a non-trivial time dependence, making it next to impossible to find a solution. Therefore we restrict ourselves to the simpler case, where ϵ is a constant. Using (21) we find that

$$\nu^2 = \Big(\frac{D-1-\epsilon}{2(1-\epsilon)}\Big)^2 - \frac{(D-1)(D-2\epsilon)}{(1-\epsilon)^2}\xi - \frac{m^2}{(1-\epsilon)^2 H^2} \tag{177}$$

Unfortunately, the presence of the mass still makes it impossible to solve (175) analytically for most values of ϵ. Exceptions are $\epsilon = 0$ where we recover the de Sitter result, and $\epsilon = 2$, where the solution can be written in terms of the ${}_1F_1$ confluent hypergeometric function. We shall not consider these two exceptions, and will from now on consider only massless scalar fields, in a background where ϵ is constant. As we explained in section (2) this is in many cosmological models a very reasonable constraint, valid if the expansion of the universe is driven by a single fluid, with a constant equation of state parameter w. Thus ν is given by

$$\nu^2 = \Big(\frac{D-1-\epsilon}{2(1-\epsilon)}\Big)^2 - \frac{(D-1)(D-2\epsilon)}{(1-\epsilon)^2}\xi \tag{178}$$

In this case the positive frequency solution for the mode functions can be written in the same functional form as in the de Sitter case and is given by

$$u(\vec{k},\eta) = \sqrt{\frac{-\pi\eta}{4}} H^{(1)}_\nu(-\vec{k}\eta), \tag{179}$$

allowing us to write, similar as before

$$\begin{aligned} \phi &= \int \frac{d^{D-1}\vec{k}}{(2\pi)^{D-1}} a^{1-\frac{D}{2}} \Big(b(\vec{k})e^{i\vec{k}\cdot\vec{x}}\psi(\vec{k},\eta) + b^\dagger(\vec{k})e^{-i\vec{k}\cdot\vec{x}}\psi^*(\vec{k},\eta)\Big) \\ \psi(\vec{k},\eta) &= \alpha u(k,\eta) + \beta u^*(\vec{k},\eta). \end{aligned} \tag{180}$$

We impose the canonical quantization conditions

$$\begin{aligned} [b(k), b^\dagger(\tilde{k})] &= (2\pi)^{D-1}\delta^{D-1}(k-\tilde{k}) \\ [\phi(x), a^{D-2}\partial_\eta\phi(\tilde{x})] &= i\delta^{D-1}(x-\tilde{x}) \end{aligned} \tag{181}$$

and we remark that the conjugate momentum to ϕ is $a^{D-2}\partial_\eta\varphi$. We find that we obtain the following condition on the mode functions

$$\psi\partial_\eta\psi^* - \psi^*\partial_\eta\psi = i, \tag{182}$$

which implies for α and β (180)

$$|\alpha|^2 - |\beta|^2 = 1. \tag{183}$$

The state that asymptotically reduces to the adiabatic vacuum is once again given by $\alpha = 1$ and $\beta = 0$. We shall consider the propagator (142) associated to this state. Making use of (160) we find that the propagator is given by

$$\begin{aligned} i\Delta(x;\tilde{x}) &= \frac{\sqrt{\eta\tilde{\eta}}\,(a\tilde{a})^{1-\frac{D}{2}}}{2^D \pi^{\frac{D-3}{2}}} \int_0^\infty dk\, k^{D-2}\, \frac{J_{\frac{D-3}{2}}(kr)}{(\frac{1}{2}kr)^{\frac{D-3}{2}}} \\ &\times \Big\{ \theta(\eta-\tilde{\eta}) H_\nu^{(1)}(-k\eta) H_\nu^{(1)}(-k\tilde{\eta})^* + \theta(\tilde{\eta}-\eta) H_\nu^{(1)}(-k\eta)^* H_\nu^{(1)}(-k\tilde{\eta}) \Big\}. \end{aligned} \tag{184}$$

Thus we see that the integral we have is completely similar to the one we encountered in the de Sitter case. Therefore, if the integral converges, we can use (162) to evaluate the propagator. We are at this point primarily interested in the infrared behavior. Using (168) we find that if $\nu > 0$ for small k, the leading order contribution to the integrand is given by

$$\frac{2^{2\nu-D}\Gamma(\nu)^2}{\pi^{(D+1)/2}\Gamma(\frac{D-1}{2})} \big((a\eta)(\tilde{a}\tilde{\eta})\big)^{1-\frac{D}{2}} \sqrt{\eta\tilde{\eta}} \int dk (\sqrt{\eta\tilde{\eta}}k)^{D-2-2\nu}. \tag{185}$$

Thus we see that the integrand is infrared divergent for all

$$\nu \geq \frac{D-1}{2}. \tag{186}$$

Similar we find that if $\nu < 0$ we have an infrared divergence if

$$\nu \leq -\frac{D-1}{2}. \tag{187}$$

Looking at (178) we recover the result that the minimally coupled scalar ($\xi = 0$) in de Sitter ($\epsilon = 0$) is divergent for any D. For the specific case of $\xi = 0$ and $D = 4$, we see that the propagator is infrared divergent for all

$$0 \leq \epsilon \leq \frac{3}{2} \qquad ; \qquad \xi = 0 \tag{188}$$

In terms of the equation of state parameter (6) we find from (17) that this implies $w \leq 0$. In other words, as soon as the pressure contributing to the expansion of the universe becomes negative, the two point correlator is infrared divergent. Notice that this is a larger range of values than expected from the discussion based on the uncertainty relation (171), there the argument only holds for $\epsilon < 1$.
For nonzero ξ, we find that there are infrared divergencies if

$$\epsilon_- \leq \epsilon \leq \epsilon_+, \tag{189}$$

where

$$\begin{aligned} \epsilon_\pm = &\frac{1}{D(D-2)} \Big((D-1)(D-2+4\xi) \\ &\pm \sqrt{(D-1)\Big((D-1)(D-2)-4\xi\Big)\Big(D-2-4(D-1)\xi\Big)} \Big). \end{aligned} \tag{190}$$

From this equation we find that for all $\xi < \frac{D-2}{4(D-1)}$ (conformal coupling) there is an infrared divergence. The smaller ξ is, the larger the range of values for ϵ for which there

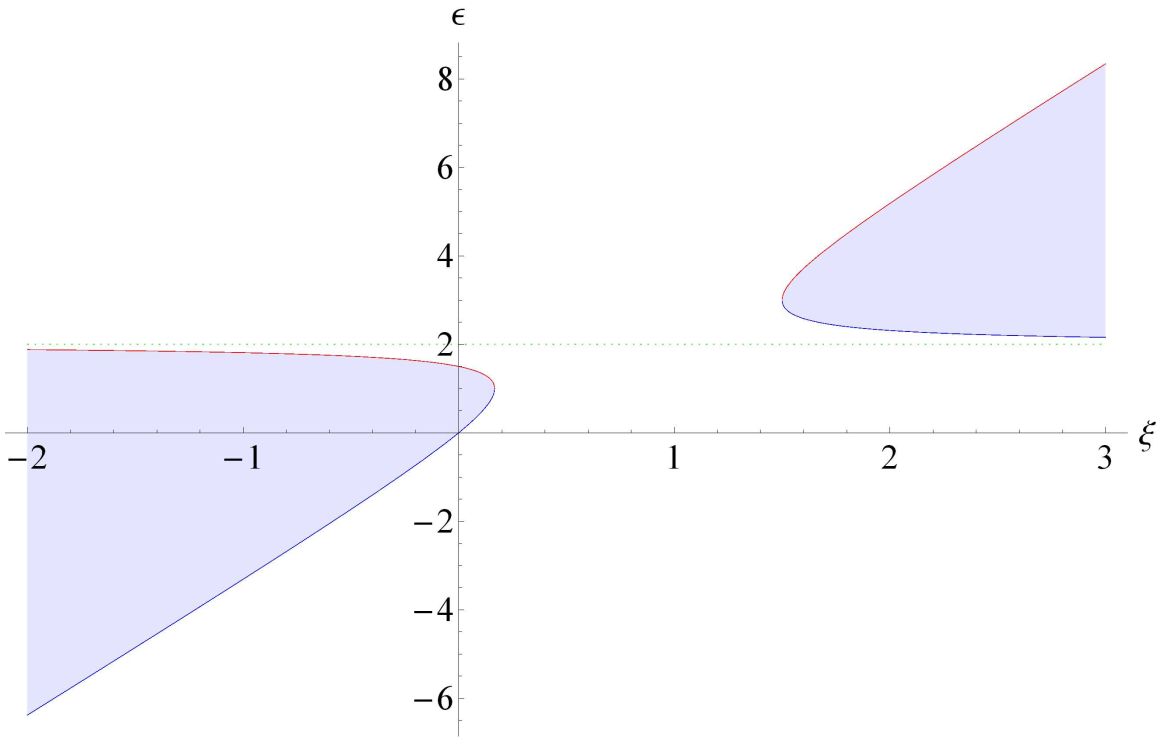

Figure 5: The range of values of ϵ where there is an infrared divergence versus ξ. The upper red curve is ϵ_+ and the lower blue curve is ϵ_- as given in (190). There is an infrared divergence for all values in the shaded region. The green dotted line indicates $\epsilon = 2$

is such a divergence. Moreover, for all $\xi > \frac{(D-1)(D-2)}{4}$ there is also a divergence, and the larger the value of ξ, the larger the range of ϵ. We can see this in figure 5
This is all physically very reasonable. The presence of a nonzero ξ acts effectively as a (time-dependent) mass, proportional to ξR. A positive mass regulates the infrared, while a negative mass destabilizes the solution and thus enhances the long range correlations. This is precisely what we find, if one remembers that $R > 0$ if $\epsilon < 2$ and $R < 0$ if $\epsilon > 2$.If $\xi = \frac{D-2}{4(D-1)}$ we now from section 3.5 that we should effectively recover the Minkowski space-time, so the infrared divergences should disappear there.
The presence of the infrared divergencies is due to the creation of modes at super-Hubble scales. The constant creation of these modes makes the universe highly correlated on large scales, causing the propagator to diverge. It is important to emphasize here the meaning of infrared divergencies, in contrast to the ultraviolet divergencies discussed in section 3.4. Ultraviolet divergences indicate that loop corrections have made an infinite change between observed parameters and the corresponding parameters of the Lagrangian [20, 21, 38]. They can be cancelled by expressing the parameters of the Lagrangian in terms of observed quantities plus counterterms which subtract the divergences. Infrared divergences however do not mean anything about parameters in the Lagrangian. Instead, they signify that there is something unphysical about the computation being done. One does not deal with an infrared divergence by subtracting a counterterm; the correct procedure is rather to compute physically well-defined quantities. The classic example is the Bloch-Nordsieck switch from infrared divergent, exclusive processes to infrared finite, inclusive processes in quantum electrodynamics [20].
The unphysical thing about the propagator calculated above is that due to the particle creation, the super Hubble modes cannot be described by a coherent Bunch-Davies vacuum. Several plausible fixes have been proposed:

- One could work on a compact spatial manifold such as a torus T^{D-1} for which there are no initially super-horizon modes [37, 38]. In this case the free field expansion becomes a sum rather than an integral but it is generally valid to make the integral approximation to this sum, with a nonzero lower limit. When this was done for the graviton propagator on de Sitter background ($\epsilon = 0$) there is no disturbance to powerful consistency checks such as the one loop Ward identity [96] and the nature of allowed counterterms [97, 98]. The renormalization of scalar field theories is not even affected at two loop order [45, 46, 99, 100].

- One could choose the coefficients α and β in the mode functions (180) such that the super-horizon modes are less singular than they would have been in the Bunch-Davies vacuum [40, 41]. Because only the super-horizon modes change there would be no effect on the Hadamard short distance behavior of the propagator. Of course the time dependence of the mode functions is determined by the scalar field equation but their initial values and the initial values of their first time derivatives can be freely specified. For example, if the initial values for the infrared modes were chosen to be those of the Bunch-Davies mode functions for $\nu = 1/2$ (regardless of the actual value of ν) then there would be no infrared divergence, either initially or at any later time [39].

- Instead of splitting the ultraviolet and the infrared sector, one could also consider the matching between the Bunch-Davies vacuum in an infrared safe space-time and the space-time one wishes to study[72] for all modes. Also in this case the initial state does not lead to infrared divergences and thus also the final state will be safe. In this approach the ultraviolet mode functions will differ from the ultraviolet Bunch-Davies mode functions, but the ultraviolet divergent structure will not change.

5.1 Infrared divergence in de Sitter and back-reaction

In this thesis we shall implement the first of these fixes. But let us now first consider what we naively might expect to happen. Both approaches effectively correspond to a choice of the initial vacuum, such that the long range correlations are suppressed. The creation of infrared modes however, is *not* suppressed. Therefore, as time goes on, we expect the universe to become more and more correlated on super-Hubble scales. Of course, within a finite amount of time, the amount of correlation can only be finite, leading to an infrared finite propagator. However we do expect a growing contribution coming from the continuously produced infrared modes. For example lets consider the case of de Sitter inflation ($\epsilon = 0$, $\nu = 3/2$) and suppose we have implemented the first fix. Effectively this means we impose an infrared cut-off at some scale k_0. The leading order contribution coming from the lower limit of the integral (185) is in $D = 4$ proportional to

$$\begin{aligned} i\Delta(x,\tilde{x})_{dS} &\propto \sqrt{\eta\tilde{\eta}} \int_{k_0} dk (\sqrt{\eta\tilde{\eta}}k)^{2-2\nu} \\ &\propto -\ln(k\sqrt{\eta\tilde{\eta}}) + \mathcal{O}(2\nu - 3) \end{aligned} \tag{191}$$

We see thus that what we obtain is a term that grows logarithmically with conformal time [41]. This indeed signifies that, although we have made the infrared finite, the long range correlations still do grow. Notice however that the contribution due to (191) to the energy density will also be redshifted in time, countering the growth due to the long range correlations.

The growth of (191) is exactly what we anticipated when we discussed back-reaction in section 3.3. The logarithm in (191) can at least in principle cause the expectation value of the stress-energy tensor (109) to grow in time. Therefore, though initially small, quantum effects might have a significant effect at late times [37]. Of course, as soon as back-reaction becomes significant, the geometry of the universe is no longer de Sitter and the analysis breaks down. Still this logarithmic growth has led to a lot of interesting physics [86, 45, 46, 44, 37, 96, 49, 50]. In many back-reaction calculations the idea is roughly the following. Suppose inflation is driven by a large cosmological constant (or alternatively by an inflaton field). In this era, the universe is approximately de Sitter and if non-conformally coupled massless fields are present (like the scalar discussed above, or perhaps more realistically the graviton [101]) we can expect a growing contribution (191) to the energy density. As long as this growth is not too fast, it will effectively behave as a contribution to the cosmological constant, and if the sign of this contribution is correct, it will decrease the cosmological constant. Thus at each moment in time, we effectively have a de Sitter universe, but at each subsequent time step, the effective cosmological constant is slightly smaller. Eventually the growing contribution

becomes non-perturbatively large and the calculation cannot be trusted anymore. However, if one extrapolates the behavior described above, one might be led to the conclusion that the cosmological constant eventually will be completely screened by quantum corrections. The hope is then that such a mechanism might simultaneously end inflation in a natural way and describe why the present day value of the cosmological constant is so small. If correct, such a mechanism would solve two important problems in theoretical physics, without the need for any exotic new ingredients. For example, if the graviton is responsible for this effect, it was found in [47, 48, 37, 96] that at two loop order, interactions between virtual graviton pairs appear to slow down inflation. However appealing, this work has also been criticized [102, 103, 52, 51]. Similar mechanisms have also been studied in the context where the growing infrared modes are due to a scalar field [49, 50, 104, 53, 54]. It was found in [54] that at three loop order the contribution due to the infrared modes grows in time.
Other effects on which the effect of the growing two point function has been studied include for example a massless minimally coupled scalar with a quartic self-interaction [45, 46, 99]. The resulting model shows a violation of the weak energy condition on cosmological scales. A non-minimally coupled, massive scalar with quartic self-interaction has also been studied [105]. In this model the radiative corrections to slow roll inflation where calculated and found to be unobservable.
Also scalar electrodynamics has been studied extensively [23, 106, 107, 108, 100, 109]. The vacuum polarization has been studied and it has been shown that in such a model super-Hubble photons acquire a mass. The contribution to the zero-point energy of these photons sources cosmological magnetic fields.

5.2 The relevance of massless scalar fields

So far we only considered (massless) scalar fields, for the simple reason that those fields are typically the easiest to study. However the conclusions made above can be generalized to many other massless fields.
For example we consider the graviton, whose action is obtained from the Einstein Hilbert action (2) by decomposing the metric in a background piece $g^{(0)}_{\mu\nu}$ plus a small perturbation

$$\begin{aligned} g_{\mu\nu} &= g^{(0)}_{\mu\nu} + h_{\mu\nu} \\ g^{\mu\nu} &= g^{(0)\mu\nu} + \delta g^{\mu\nu} \end{aligned} \tag{192}$$

and we assume that the matter degrees of freedom do not fluctuate (the more general case will be treated in section 9). We require that $g^{\mu\alpha}g_{\mu\beta} = \delta^{\alpha}_{\beta}$ and we raise and lower indices on the perturbation with the background metric: ${h^{\mu}}_{\alpha} = g^{\mu\nu}h_{\nu\alpha}$ and find

$$\delta g^{\mu\nu} = -h^{\mu\nu} + {h^{\mu}}_{\alpha}h^{\alpha\nu} + \mathcal{O}(h^3). \tag{193}$$

After a long calculation we find that [71] the gauge fixed propagator of the pseudo graviton field $\psi_{\mu\nu}$, defined by

$$h_{\mu\nu} \equiv \sqrt{16\pi G_N}a^2\psi_{\mu\nu} \tag{194}$$

can be written in a general FLRW space-time as

$$
\begin{aligned}
[_{\rho\sigma}\Delta^{\alpha\beta}] =& \Big(2\bar{\delta}_\rho^{(\alpha}\bar{\delta}_\sigma^{\beta)} - \frac{2}{D-3}\bar{\eta}_{\rho\sigma}\bar{\eta}^{\alpha\beta}\Big)\Delta_0 + 4\delta^0_{(\rho}\bar{\delta}^{(\alpha}_{\sigma)}\delta^{\beta)}_0\Delta_1 \\
&+ \left[\frac{2}{(D-2)(D-3)}(\eta_{\rho\sigma} + (D-2)\delta^0_\sigma\delta^0_\rho)(\eta^{\alpha\beta} + (D-2)\delta^\beta_0\delta^\alpha_0)\right]\Delta_2,
\end{aligned}
\tag{195}
$$

where

$$
\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} + \delta^0_\mu\delta^0_\nu. \tag{196}
$$

and the individual propagators are given by

$$
\sqrt{-g}\Big(\Box_S - n(D-n-1)(1-\epsilon)H^2\Big)\Delta_n(x;\tilde{x}) = \imath\delta^D(x-\tilde{x}), \tag{197}
$$

here $n = 0, 1, 2$ and $\Box_S$ is the d'Alembertian as it acts on a scalar field, irrespective of what it actually acts on

$$
\Box_S \equiv a^{-D}\partial_\alpha a^{D-2}\eta^{\alpha\beta}\partial_\beta. \tag{198}
$$

Thus we see we can write the graviton propagator in terms of massless scalar field propagators with a certain amount of conformal coupling. Thus we can apply all the conclusions we made for the scalar field also in this case. Similar situations have been shown to arise, at least in de Sitter space, for the photon [44] and an antisymmetric tensor field [36].

6 Resolving the infrared divergencies using a momentum cut-off

The purpose of this section is to resolve the infrared divergence with the first of the three fixes proposed in the previous section. Thus we shall construct the propagator on a spatially compact manifold, such that we effectively cut away all modes with momenta less than some cut-off k_0 [38]. We shall perform this calculation by first simply neglecting the infrared divergences and calculate the propagator on $\mathbb{R}^{D-1}$ (which is incorrect). We can then afterwards subtract the infrared divergent contribution. The infinite volume propagator is given by the integral (184) and is of the same form as (162). We can thus immediately find the result that

$$\begin{aligned} i\Delta_\infty(x;\tilde{x}) =& \frac{\left[(1-\epsilon)^2 H\tilde{H}\right]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{D}{2})} \\ & \times {}_2F_1\Big(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; 1-\frac{y}{4}\Big), \end{aligned} \tag{199}$$

where we use a subscript ∞ to indicate we work on a non-compact manifold. Here and henceforth we define the quantity $y = y_{++}(x;\tilde{x})$ as defined in (152). The propagator (199) is the generalization of the Chernikov-Tagirov propagator for de Sitter space to space-times with constant, but arbitrary ϵ [81]. The constant ϵ propagator was already found for $D=4$ in [33].
Expression (199) has also been obtained by solving the propagator equation (144) with the Ansatz of $(HH')^{\frac{D}{2}-1}$ times a function of $y(x;x')$ [110, 74]. If we employ the transformation formulae for hypergeometric functions and then their series expansion we can write this as,

$$\begin{aligned} i\Delta_\infty(x;\tilde{x}) \;=\;& \frac{\left[(1-\epsilon)^2 H\tilde{H}\right]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \Bigg\{ \Gamma\Big(\frac{D}{2}-1\Big)\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} {}_2F_1\Big(\frac{1}{2}+\nu, \frac{1}{2}-\nu; 2-\frac{D}{2}; \frac{y}{4}\Big) \\ & + \frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} {}_2F_1\Big(\frac{D-1}{2}+\nu, \frac{D-1}{2}-\nu; \frac{D}{2}; \frac{y}{4}\Big)\Bigg\}, \end{aligned} \tag{200}$$

$$\begin{aligned} =& \frac{\left[(1-\epsilon)^2 H\tilde{H}\right]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \Gamma\Big(\frac{D}{2}-1\Big) \Bigg\{ \Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{\Gamma(2-\frac{D}{2})}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \\ & \times \sum_{n=0}^{\infty} \Bigg[\frac{\Gamma(\frac{3}{2}+\nu+n)\Gamma(\frac{3}{2}-\nu+n)}{\Gamma(3-\frac{D}{2}+n)\,(n+1)!} \Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} \\ & - \frac{\Gamma(\frac{D-1}{2}+\nu+n)\Gamma(\frac{D-1}{2}-\nu+n)}{\Gamma(\frac{D}{2}+n)\,n!} \Big(\frac{y}{4}\Big)^{n} \Bigg]\Bigg\}. \end{aligned} \tag{201}$$

In the context of dimensional regularization, we can automatically subtract all D dependent powers of y. We then see however that in (201) the gamma functions on the last line diverge for certain values of ν, irrespective of whether or not $\tilde{x}^\mu = x^\mu$ and with the dimensional regularization still in effect. We immediately find that divergences occur for

$$\begin{aligned}\nu = \frac{D-1}{2} + N &\Longrightarrow \Gamma\Big(\frac{D-1}{2} - \nu + n\Big) = \Gamma(-N+n) \\ \nu = \frac{D-1}{2} - N &\Longrightarrow \Gamma\Big(\frac{D-1}{2} + \nu + n\Big) = \Gamma(-N+n),\end{aligned} \tag{202}$$

where $N = 0, 1, 2, \ldots$. Based on the discussion following (185) however we conclude that the integral (184) has infrared divergences for *all* $\nu > (D-1)/2$ and $\nu < -(D-1)/2$, while the final result (201) only diverges for the discrete values of ν given by expressions (202). The reason for this is because dimensional regularization [111, 112] automatically subtracts power law divergences and only registers logarithmic divergences. For most values of ν the infrared divergence is a power law, and dimensional regularization — quite incorrectly — sets it to zero. It is only for the discrete values (202) that a logarithmic divergence occurs and causes expression (201) to become ill-defined. To see this, consider the integral (184) and note that the logarithmic divergence could derive from any of the order k^{2N} series corrections to the leading small argument expansion of the Bessel function (168). We immediately find that logarithmic infrared divergences occur precisely for the values given in (202).
It is important to understand that the infinite space propagator has physical problems for *every* value of ν in the infrared divergent range $|\nu| \geq \frac{D-1}{2}$ whether or not ν happens to take one of the critical values (202) necessary for a logarithmic divergence. This is because ultraviolet and infrared divergences mean different things, as was also emphasized before. The automatic subtraction of dimensional regularization is not an error for ultraviolet divergences; it merely saves one the trouble of defining and subtracting the appropriate counterterm to cancel a power law divergence. Employing the automatic subtraction of dimensional regularization to remove a power law infrared divergence however, corresponds to adding an illegal counterterm. The result that an unphysical question now returns a finite answer does of course not make the answer physical.

6.1 Finite Space Mode Sum

The purpose of this section is to resolve the infrared problems present in the infinite space propagator by working on a finite-sized spatial manifold [37, 38]. We show how this changes the mode sum for the propagator. We also derive the corrections it makes to the integrated, position-space form. Explicit demonstrations are given that the correction terms cure the $N = 0$ and $N = 1$ divergences in expressions (202). And certain special cases are checked against known results [45, 46, 113].
We work on T^{D-1}, which supports the spatially flat FLRW geometry (10). If the coordinate radius in each direction is $2\pi/k_0$ then the integral approximation for the free field expansion of the operator is the same as (180) except that the integral is cut off at $\|\vec{k}\| = k_0$,

$$\phi = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} a^{1-\frac{D}{2}} \theta(k-k_0) \Big(b(k) e^{i\vec{k}\cdot\vec{x}} \psi(k,\eta) + b^\dagger(k) e^{-i\vec{k}\cdot\vec{x}} \psi^*(k,\eta) \Big). \tag{203}$$

The mode functions ψ are as in (180), with $\alpha = 1$ and $\beta = 0$. This ensures that we recover normal flat space-time physics in the ultraviolet.

Of course the same cutoff works its way into the mode sum for the propagator (184),

$$
\begin{aligned}
&i\Delta(x;\tilde{x}) = \\
&\quad \frac{\Big[(1-\epsilon)^2 H\tilde{H}\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\pi^{\frac{3}{2}}\, 2^{\frac{D-3}{2}}}{\Big(\frac{r}{\sqrt{\eta\tilde{\eta}}}\Big)^{\frac{D-3}{2}}} \int_{z_0}^{\infty} dz\, z^{\frac{D-1}{2}} J_{\frac{D-3}{2}}\Big(\frac{r}{\sqrt{\eta\tilde{\eta}}}\, z\Big) \\
&\qquad \times \Bigg\{ \theta(\eta-\tilde{\eta}) H^{(1)}_{\nu}\Big(\sqrt{\frac{\eta}{\tilde{\eta}}}\, z\Big) H^{(1)}_{\nu}\Big(\sqrt{\frac{\tilde{\eta}}{\eta}}\, z\Big)^{*} + \theta(\tilde{\eta}-\eta) \times \Big(\text{conjugate}\Big) \Bigg\}. \qquad (204)
\end{aligned}
$$

Here and subsequently

$$
z \equiv \sqrt{\eta\tilde{\eta}}k \qquad ; \qquad z_0 \equiv \sqrt{\eta\tilde{\eta}}k_0. \qquad (205)
$$

We can obviously break the integral over z up into two parts,

$$
\int_{z_0}^{\infty} dz = \int_{0}^{\infty} dz - \int_{0}^{z_0} dz\ . \qquad (206)
$$

This means that the result for (204) is what we already have (201) from the work of [71, 110, 74], minus the finite range integral. It would be simple enough to expand the integrand of this second contribution and then integrate termwise, but we really only need the most infrared singular parts.

To be more specific and to simplify the calculation somewhat we shall assume that $\xi = 0$, and the parameter ν in the remainder of this section will be given by

$$
\hat{\nu} = \frac{D-1-\epsilon}{2(1-\epsilon)}. \qquad (207)
$$

Notice the difference between this parameter and the one used before is that $\hat{\nu}$ becomes negative if the expansion of the universe is decelerating ($\epsilon > 1$). At the end of this section, we shall see that we can easily generalize the result to nonzero ξ and the original definition of ν.

With this in mind, we see that for the inflationary case $0 \leq \epsilon < 1$ divergences occur at,

$$
\epsilon = \frac{2N}{D-2+2N} \Longrightarrow \Gamma\Big(\frac{D-1}{2} - \hat{\nu} + n\Big) = \Gamma(-N+n) \quad \text{for} \quad N = 0, 1, 2, \dots \qquad (208)
$$

For the decelerating case of $1 < \epsilon$ divergences are found at,

$$
\epsilon = 2\frac{D-1+N}{D+2N} \Longrightarrow \Gamma\Big(\frac{D-1}{2} + \hat{\nu} + n\Big) = \Gamma(-N+n) \quad \text{for} \quad N = 0, 1, 2, \dots \qquad (209)
$$

For the inflationary case of $0 \leq \epsilon < 1$ the index $\hat{\nu}$ is positive and the most infrared singular parts of the integrand derive from the $J_{-\hat{\nu}}$ contributions to the Hankel functions (137). For the decelerating case of $1 < \epsilon \leq 2(D-1)/D$ the index $\hat{\nu}$ is negative and it is the $J_{+\hat{\nu}}$ parts of the Hankel functions that are the most infrared singular. We shall work out the series of leading corrections in each case.

6.1.1 Accelerating case

Let us begin with the most infrared singular correction for the inflationary case of $0 \leq \epsilon < 1$. From the small argument expansion of the Bessel functions (168) and (185) we see that the desired correction is,

$$\begin{aligned}\delta i\Delta_0(x;\tilde{x}) &= -\frac{\Big[(1-\epsilon)^2 H\tilde{H}\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{2^{2\hat{\nu}}\Gamma^2(\hat{\nu})}{\pi^{\frac{1}{2}}\Gamma(\frac{D-1}{2})} \int_0^{z_0} dz\, z^{D-2-2\hat{\nu}} \\ &= \frac{\Big[(1-\epsilon)^2 H\tilde{H}\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2})} \frac{2(1-\epsilon)}{\epsilon(D-2)} \Big(\frac{1}{k_0^2\eta\tilde{\eta}}\Big)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} . \end{aligned} \qquad (210)$$

To reach the last form (210) we have used the doubling formula for the Gamma function,

$$\Gamma(2x) = \frac{2^{2x-1}}{\pi^{\frac{1}{2}}}\, \Gamma(x)\Gamma\Big(x+\frac{1}{2}\Big) . \qquad (211)$$

Of course the infrared divergence at $z = 0$ in $\delta i\Delta_0$ was dimensionally regulated, the same way as in the infinite space result $i\Delta_\infty$. This is wrong for $\delta i\Delta_0$, just as it was for $i\Delta_\infty$, but expression (206) implies that the two errors must cancel.
We shall use the notation $\delta i\Delta_N$ to indicate the N-th order correction in the case of an inflationary universe ($\epsilon < 1$). For the correction in the decelerating case ($1 < \epsilon \leq 2(D-1)/D$) we use the notation $\delta i\Delta^N$. While the power law divergences now are removed with a physical reason, let us also show that the logarithmic divergences are also correctly removed. The addition of (210) should eliminate the $N = 0$ logarithmic divergence (208). Both (210) and the $n = 0$ term from the last line of (201) have a common factor that we may as well omit,

$$\frac{\Big[(1-\epsilon)^2 H\tilde{H}\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \times \frac{1}{\Gamma(\frac{1}{2}+\hat{\nu})} . \qquad (212)$$

The remaining contributions are,

$$\begin{aligned}&\frac{2(1-\epsilon)\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\epsilon(D-2)\Gamma(\frac{D-1}{2})} \Big(\frac{1}{k_0^2\eta\tilde{\eta}}\Big)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} - \frac{\Gamma(\frac{D}{2}-1)\Gamma(2-\frac{D}{2})}{\Gamma(\frac{1}{2}-\hat{\nu})} \frac{\Gamma(\frac{D-1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2}-\hat{\nu})}{\Gamma(\frac{D}{2})} \\ &\quad = \frac{2(1-\epsilon)\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\epsilon(D-2)\Gamma(\frac{D-1}{2})} \Bigg\{ \Big(\frac{1}{k_0^2\eta\tilde{\eta}}\Big)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} \\ &\qquad\qquad + \frac{\Gamma(\frac{D-1}{2})}{\Gamma(\hat{\nu})} \frac{\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}-\hat{\nu})} \frac{\Gamma(\frac{D-1}{2}+\hat{\nu})}{\Gamma(2\hat{\nu})} \frac{\Gamma(\frac{D-1}{2}-\hat{\nu})}{\frac{2(1-\epsilon)}{\epsilon(D-2)}} \Bigg\}. \end{aligned} \qquad (213)$$

Near $N = 0$ we can define a small parameter

$$\alpha \equiv \frac{D-1}{2} - \hat{\nu} = \epsilon(D-2)/[2(1-\epsilon)] \qquad (214)$$

and we expand (213) in this parameter. The limit $\alpha \to 0$ is equivalent to the limit when ϵ vanishes and expression (213) reduces to

$$\lim_{\epsilon\to 0} \frac{2(1-\epsilon)\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\epsilon(D-2)\Gamma(\frac{D-1}{2})} \Bigg\{ \Big(\frac{1}{k_0^2\eta\tilde{\eta}}\Big)^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} + \frac{\Gamma(\frac{D-1}{2})}{\Gamma(\hat{\nu})} \frac{\Gamma(1-\frac{D}{2})}{\Gamma(\frac{1}{2}-\hat{\nu})} \frac{\Gamma(\frac{D-1}{2}+\hat{\nu})}{\Gamma(2\hat{\nu})} \frac{\Gamma(\frac{D-1}{2}-\hat{\nu})}{\frac{2(1-\epsilon)}{\epsilon(D-2)}} \Bigg\} \tag{215}$$
$$= \Gamma(D-1)\Bigg\{ \ln(a\tilde{a}) - \pi\cot\Big(\frac{\pi D}{2}\Big) + 2\ln\Big(\frac{H_0}{k_0}\Big) + \psi\Big(\frac{D-1}{2}\Big) - \psi\Big(\frac{D}{2}\Big) + \psi(D-1) - \gamma \Bigg\} ,$$

where $\psi(z) = (d/dz)\ln(\Gamma(z))$ indicates the digamma function, we used $a = -1/H_0\eta$ (valid for $\epsilon = 0$), $\psi(1) = -\gamma$ and the reflection formula for the digamma function,

$$\psi(1-x) = \psi(x) + \pi\cot(\pi x) \,. \tag{216}$$

Multiplying (215) by the common factor (212), and then adding the rest of (201) — which is not singular for $\epsilon = 0$ — gives the following result,

$$\lim_{\epsilon\to 0} i\Delta(x;\tilde{x}) = \frac{H_0^{D-2}}{(4\pi)^{\frac{D}{2}}} \Bigg\{ \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Big[\ln(a\tilde{a}) - \pi\cot\Big(\frac{\pi D}{2}\Big) + 2\ln\Big(\frac{H_0}{k_0}\Big) + \psi\Big(\frac{D-1}{2}\Big) - \psi\Big(\frac{D}{2}\Big) + \psi(D-1) - \gamma\Big]$$
$$+ \sum_{n=1}^{\infty}\Big[\frac{\Gamma(D-1+n)}{n\,\Gamma(\frac{D}{2}+n)}\Big(\frac{y}{4}\Big)^n - \frac{\Gamma(\frac{D}{2}+1+n)}{(2-\frac{D}{2}+n)\,(n+1)!}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}\Big] + O(k_0^2)\Bigg\}. \tag{217}$$

We see that we indeed recover a term that grows logarithmically with a, as we anticipated in (191). Except for the order k_0^2 corrections, and for some constant, finite factors on the second line, expression (217) agrees precisely with the result first obtained in [45] and used subsequently in many one and two loop computations [114, 88, 99, 100, 115, 116, 117] on de Sitter background.
It is straightforward to work out the next contributions from the lower limit. We merely add up the three first order corrections from the Bessel and Hankel functions for the case of $\hat{\nu}$ positive,

$$J_{\frac{D-3}{2}}\Big(\frac{r}{\sqrt{\eta\tilde{\eta}}}z\Big) = \Big(\frac{r}{2\sqrt{\eta\tilde{\eta}}}\Big)^{\frac{D-3}{2}} \times \frac{z^{\frac{D-3}{2}}}{\Gamma(\frac{D-1}{2})}\Bigg\{1 - \frac{\frac{r^2}{\eta\tilde{\eta}}\frac{z^2}{4}}{(\frac{D-1}{2})} + O(z^4)\Bigg\} , \tag{218}$$

$$H_{\hat{\nu}}^{(1)}\Big(\sqrt{\frac{\eta}{\tilde{\eta}}}\,z\Big) = \frac{-i(\frac{1}{2}z)^{-\hat{\nu}}(\eta/\tilde{\eta})^{-\hat{\nu}/2}}{\sin(\hat{\nu}\pi)\Gamma(1-\hat{\nu})}\Bigg\{1 - \frac{\frac{\eta}{\tilde{\eta}}\frac{z^2}{4}}{1-\hat{\nu}} + O(z^4)\Bigg\} , \tag{219}$$

$$H_{\hat{\nu}}^{(1)}\Big(\sqrt{\frac{\tilde{\eta}}{\eta}}\,z\Big)^{*} = \frac{i(\frac{1}{2}z)^{-\hat{\nu}}(\tilde{\eta}/\eta)^{-\hat{\nu}/2}}{\sin(\hat{\nu}\pi)\Gamma(1-\hat{\nu})}\Bigg\{1 - \frac{\frac{\tilde{\eta}}{\eta}\frac{z^2}{4}}{1-\hat{\nu}} + O(z^4)\Bigg\} . \tag{220}$$

The resulting lower limit term is,

$$\delta i\Delta_1 = -\frac{\Bigl[(1\!-\!\epsilon)^2 H H'\Bigr]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{2\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac12\!+\!\hat{\nu})\Gamma(\frac{D-1}{2})} \Biggl[\frac{(\eta^2\!+\!\tilde{\eta}^2)}{4(\hat{\nu}\!-\!1)\eta\tilde{\eta}} - \frac{\Delta x^2}{2(D\!-\!1)\eta\tilde{\eta}}\Biggr] \int_0^{z_0} dz\, z^{D-2\hat{\nu}}$$

$$= \frac{\Bigl[(1\!-\!\epsilon)^2 H H'\Bigr]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{2\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac12\!+\!\hat{\nu})\Gamma(\frac{D-1}{2})} \times \Biggl[\frac{(\eta^2\!+\!\tilde{\eta}^2)}{4(\hat{\nu}\!-\!1)\eta\tilde{\eta}} - \frac{\Delta x^2}{2(D\!-\!1)\eta\tilde{\eta}}\Biggr] \frac{-z_0^{D+1-2\hat{\nu}}}{D\!+\!1\!-\!2\hat{\nu}} \ . \qquad (221)$$

This should cancel the $N=1$ divergence of (208) at $\epsilon = 2/D$, which affects the $n=0$ and $n=1$ terms on the last line of (201),

$$\frac{\Bigl[(1\!-\!\epsilon)^2 H \tilde{H}\Bigr]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(\frac{D+1}{2}\!+\!\hat{\nu})\Gamma(\frac{D+1}{2}\!-\!\hat{\nu})\Gamma(-\frac{D}{2})}{\Gamma(\frac12\!+\!\hat{\nu})\Gamma(\frac12\!-\!\hat{\nu})} \Biggl[\frac{-\frac{D}{2}}{(\frac{D-1}{2})^2\!-\!\hat{\nu}^2} - \frac{y}{4}\Biggr] \ . \qquad (222)$$

The key to seeing that the infrared divergence of (221) cancels that in (222) is to express both in terms of the small parameter,

$$\alpha \equiv \hat{\nu} - \Bigl(\frac{D\!+\!1}{2}\Bigr) = \frac{(D\epsilon\!-\!2)}{2(1\!-\!\epsilon)} \ . \qquad (223)$$

As before, we extract the common factor of

$$\frac{\Bigl[(1\!-\!\epsilon)^2 H \tilde{H}\Bigr]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{1}{\Gamma(\frac12\!+\!\hat{\nu})} \ . \qquad (224)$$

When expanded in terms of α, the lower limit contribution from (221) is this factor times,

$$\frac{2\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac{D-1}{2})} \Biggl[\frac{(\eta^2\!+\!\tilde{\eta}^2)}{4(\hat{\nu}\!-\!1)\eta\tilde{\eta}} - \frac{r^2}{2(D\!-\!1)\eta\tilde{\eta}}\Biggr] \frac{-z_0^{D+1-2\hat{\nu}}}{D\!+\!1\!-\!2\hat{\nu}}$$

$$= \frac{\Gamma(D\!+\!1\!+\!2\alpha)\Gamma(\frac{D+1}{2}\!+\!\alpha)}{4\alpha\Gamma(\frac{D+1}{2})} \Biggl[\frac{1}{1\!+\!\frac{2\alpha}{D-1}}\Bigl(\frac{\eta^2\!+\!\tilde{\eta}^2}{\eta\tilde{\eta}}\Bigr) - \frac{r^2}{\eta\tilde{\eta}}\Biggr] \Bigl(\frac{1}{k_0^2 \eta\tilde{\eta}}\Bigr)^{\alpha} , \qquad (225)$$

$$= \frac{\Gamma(D\!+\!1)}{4} \Biggl\{\frac{2\!-\!y}{\alpha} + (2\!-\!y)\Biggl[2\psi(D\!+\!1) + \psi\Bigl(\frac{D\!+\!1}{2}\Bigr) + \ln\Bigl(\frac{1}{k_0^2\eta\tilde{\eta}}\Bigr)\Biggr] - \frac{2}{D\!-\!1}\Bigl(\frac{\eta^2\!+\!\tilde{\eta}^2}{\eta\tilde{\eta}}\Bigr) + O(\alpha)\Biggr\}. \qquad (226)$$

In contrast, the contribution from the infinite space propagator (222) is the factor (224) times,

$$\frac{\Gamma(\frac{D+1}{2}+\hat{\nu})\Gamma(\frac{D+1}{2}-\hat{\nu})\Gamma(-\frac{D}{2})}{\Gamma(\frac{1}{2}-\hat{\nu})}\left[\frac{-\frac{D}{2}}{(\frac{D-1}{2})^2-\hat{\nu}^2}-\frac{y}{4}\right]$$

$$= -\frac{\Gamma(D+1+\alpha)\Gamma(1-\alpha)\Gamma(-\frac{D}{2})}{4\alpha\Gamma(-\frac{D}{2}-\alpha)}\left[\frac{2}{(1+\frac{\alpha}{D})(1+\alpha)}-y\right], \tag{227}$$

$$= \frac{\Gamma(D+1)}{4}\Bigg\{-\Big(\frac{2-y}{\alpha}\Big)+(2-y)\Big[-\psi(D+1)+\psi(1)-\psi\Big(-\frac{D}{2}\Big)\Big] +2\Big(\frac{D+1}{D}\Big)+O(\alpha)\Bigg\}. \tag{228}$$

Adding (226) to (228) and taking $\alpha = 0$ (which implies $\epsilon = 2/D$) gives,

$$\frac{\Gamma(D+1)}{4}\Bigg\{(2-y)\Big[\ln\Big(\frac{1}{k_0^2\eta\tilde{\eta}}\Big)+\psi(D+1)+\psi\Big(\frac{D+1}{2}\Big)-\gamma-\psi\Big(-\frac{D}{2}\Big)\Big] -\frac{2}{D-1}\Big(\frac{\eta^2+\tilde{\eta}^2}{\eta\tilde{\eta}}\Big)+2\Big(\frac{D+1}{D}\Big)\Bigg\}, \tag{229}$$

$$= \frac{\Gamma(D+1)}{4}\Bigg\{(2-y)\Big[\Big(\frac{D-2}{D}\Big)\ln(a\tilde{a})+2\ln\Big[\frac{(D-2)H_0}{Dk_0}\Big]-\pi\cot\Big(\frac{\pi D}{2}\Big) +\psi(D+1)+\psi\Big(\frac{D+1}{2}\Big)-\gamma-\psi\Big(\frac{D}{2}+1\Big)\Big] -\frac{2}{D-1}\Big(\frac{\eta^2+\tilde{\eta}^2}{\eta\tilde{\eta}}\Big)+2\Big(\frac{D+1}{D}\Big)\Bigg\}. \tag{230}$$

Thus we find indeed that the result has become finite. The $N = 0$ correction (210) is now also finite, but it does contribute a term that grows as $\frac{1}{(\eta\tilde{\eta})}$. We can get the full propagator for $\epsilon = 2/D$ by multiplying (230) by the common factor (224), and then adding the rest of (201) with the now finite $N=0$ correction (210),

$$\lim_{\epsilon\to\frac{2}{D}} i\Delta(x;\tilde{x}) =$$

$$\frac{[(1-\frac{2}{D})^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}\Bigg\{\frac{\Gamma\big(\frac{D}{2}+1\big)}{\big(1-\frac{D}{2}\big)\big(-\frac{D}{2}\big)}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1}+\frac{\Gamma(\frac{D}{2}+2)}{(2-\frac{D}{2})(1-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2}$$

$$+\frac{1}{2!}\frac{\Gamma(\frac{D}{2}+3)}{(3-\frac{D}{2})(2-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-3}+\frac{\Gamma(D+1)}{4\Gamma(\frac{D}{2}+1)}\Bigg[\frac{2(D-1)}{k_0^2\eta\tilde{\eta}}-\frac{2}{D-1}\Big(\frac{\eta}{\tilde{\eta}}+\frac{\tilde{\eta}}{\eta}\Big)$$

$$+2+\frac{2}{D}+(2-y)\Big\{\Big(1-\frac{2}{D}\Big)\ln(a\tilde{a})-\pi\cot\Big(\frac{\pi D}{2}\Big)+K_D\Big\}\Bigg]$$

$$+\sum_{n=2}^{\infty}\Bigg[\frac{\Gamma(\frac{D}{2}+2+n)}{(2-\frac{D}{2}+n)(1-\frac{D}{2}+n)\,(n+1)!}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2}$$

$$-\frac{\Gamma(D+n)}{n(n-1)\,\Gamma(\frac{D}{2}+n)}\Big(\frac{y}{4}\Big)^{n}\Bigg]+O(k_0^2)\Bigg\}. \tag{231}$$

Here the constant K_D is,

$$K_D \equiv 2\ln\Big[\Big(1-\frac{2}{D}\Big)\frac{H_0}{k_0}\Big]+\psi(D+1)+\psi\Big(\frac{D+1}{2}\Big)-\gamma-\psi\Big(\frac{D}{2}+1\Big)\,. \tag{232}$$

As far as we know the literature contains no result against which we can check (231) but its limit in $D=4$ dimensions has been worked out,

$$\lim_{D\to4}\lim_{\epsilon\to\frac{2}{D}} i\Delta(x;\tilde{x}) = \frac{H\tilde{H}}{64\pi^2}\Bigg\{\frac{4}{y}+\frac{18}{k_0^2\eta\tilde{\eta}}-2\Big(\frac{\eta}{\tilde{\eta}}+\frac{\tilde{\eta}}{\eta}\Big)-11y+16$$

$$+3(2-y)\Big[-\ln\big(y\big)+2\ln\Big(\frac{H_0}{2k_0}\Big)+\frac{1}{2}\ln(aa')-2\gamma\Big]+O(k_0^2)\Bigg\}. \tag{233}$$

This agrees perfectly with equation (3.82) of [113].
We have seen that the lower limit term which corrects the $N=0$ problem in (208) is given by (210). For the $N=1$ problem the corresponding lower limit correction is (221). To see the general pattern, first substitute the relation for H in terms of η,

$$H = H_0\Big[-(1-\epsilon)H_0\eta\Big]^{\frac{\epsilon}{1-\epsilon}}\,. \tag{234}$$

This reveals the $N=0$ correction (210) *to be constant*,

$$\delta i\Delta_0 \equiv \frac{\Big[(1-\epsilon)^2 H_0^2\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2})}\frac{2(1-\epsilon)}{\epsilon(D-2)}\Bigg[\frac{(1-\epsilon)^2H_0^2}{k_0^2}\Bigg]^{\frac{\epsilon(D-2)}{2(1-\epsilon)}}\,. \tag{235}$$

The same substitution reveals that the $N=1$ correction (221) is quadratic in the coordinates,

$$\delta i\Delta_1 \equiv \frac{\Big[(1-\epsilon)^2 H_0^2\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2})} \\ \times \left[\frac{k_0^2(\eta^2+\tilde{\eta}^2)}{4(\hat{\nu}-1)} - \frac{k_0^2 r^2}{2(D-1)}\right] \frac{2(1-\epsilon)}{(D\epsilon-2)} \left[\frac{(1-\epsilon)^2 H_0^2}{k_0^2}\right]^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} . \qquad (236)$$

Both corrections are homogeneous solutions of the propagator equation (144),

$$\partial_\mu \Big(\sqrt{-g} g^{\mu\hat{\nu}} \partial_{\hat{\nu}} \delta i\Delta_N\Big) = 0 \; . \qquad (237)$$

Note that each lower limit correction $\delta i\Delta_N$ must separately solve (237) because each goes like a distinct power of k_0. *The freedom to add such homogeneous terms is precisely what is not fixed by just solving the propagator equation rather than using the mode sum.* We could work out the N-th lower limit correction $\delta i\Delta_N$ from the mode sum but that would involve tedious multiplications of corrections from the Bessel function and the two Hankel functions. A simpler technique is to use the fact that the correction must have the form,

$$\delta i\Delta_N = \frac{\Big[(1-\epsilon)^2 H_0^2\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2})} \frac{2(1-\epsilon)}{(D-2+2N)\epsilon-2N} \\ \times \left[\frac{(1-\epsilon)^2 H_0^2}{k_0^2}\right]^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} k_0^{2N} \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} a_{k\ell} r^{2k} \eta^{2\ell} \tilde{\eta}^{2(N-k-\ell)} . \qquad (238)$$

Then we determine the coefficients $a_{k\ell}$ by three requirements:

1. The coefficient a_{N0} derives entirely from the z^{2N} correction of the Bessel function and, by direct examination of (204), we can see that it is,

$$a_{N0} = \frac{(-1)^N \Gamma(\frac{D-1}{2})}{N! 4^N \Gamma(\frac{D-1}{2}+N)} \; ; \qquad (239)$$

2. Symmetry under $\eta \leftrightarrow \tilde{\eta}$ implies,

$$a_{k\ell} = a_{k(N-k-\ell)} \; ; \qquad (240)$$

3. The series must solve the homogeneous Klein-Gordon equation (237).

The differential equation (237) implies,

$$0 = \Big[\partial^2 + \frac{2\hat{\nu}-1}{\eta}\,\partial_0\Big] \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} a_{k\ell} r^{2k} \eta^{2\ell} \tilde{\eta}^{2(N-k-\ell)} \; , \tag{241}$$

$$= \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} 2k(2k+D-3) a_{k\ell} r^{2k-2} \eta^{2\ell} \tilde{\eta}^{2(N-k-\ell)} - \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} 4\ell(\ell-\hat{\nu}) a_{k\ell} r^{2k} \eta^{2\ell-2} \tilde{\eta}^{2(N-k-\ell)} \; , \tag{242}$$

$$= \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1-k} 2(k+1)(2k+D-1) a_{k+1\,\ell} r^{2k} \eta^{2\ell} \tilde{\eta}^{2(N-1-k-\ell)} - \sum_{k=0}^{N-1} \sum_{\ell=0}^{N-1-k} 4(\ell+1)(\ell+1-\hat{\nu}) a_{k\,\ell+1} r^{2k} \eta^{2\ell} \tilde{\eta}^{2(N-1-k-\ell)}. \tag{243}$$

Hence the coefficients must obey,

$$(k+1)(2k+D-1) a_{k+1\,\ell} = 2(\ell+1)(\ell+1-\hat{\nu}) a_{k\,\ell+1} \; . \tag{244}$$

The unique solution consistent with the other two of the three properties is,

$$a_{k\ell} = \Big(-\frac{1}{4}\Big)^N \frac{1}{k!\,\ell!\,(N-k-\ell)!} \frac{\Gamma(\frac{D-1}{2})\,\Gamma^2(1-\hat{\nu})}{\Gamma(k+\frac{D-1}{2})\Gamma(\ell+1-\hat{\nu})\Gamma(N-k-\ell+1-\hat{\nu})} \; . \tag{245}$$

For $N=0$ this gives the known result

$$N=0 \qquad \Longrightarrow \qquad a_{00} = 1 \; . \tag{246}$$

A less trivial check is that it also works for $N=1$,

$$N=1 \qquad \Longrightarrow \qquad a_{00} = a_{01} = \frac{1}{4(\hat{\nu}-1)} \qquad \text{and} \qquad a_{10} = -\frac{1}{2(D-1)} \; , \tag{247}$$

thus we see we indeed correctly recover (236).

6.1.2 Decelerating case

Let us turn now to the decelerating case of $1 < \epsilon \leq 2(D-1)/D$ for which the infinite space propagator (201) diverges at the discrete values given in (209). The leading order contribution to the propagator is, equivalent to (185), but now for negative $\tilde{\nu}$

$$\frac{2^{-2\tilde{\nu}-D}\Gamma(-\tilde{\nu})^2}{\pi^{(D+1)/2}\Gamma(\frac{D-1}{2})} \big((a\eta)(\tilde{a}\tilde{\eta})\big)^{1-\frac{D}{2}} \sqrt{\eta\tilde{\eta}} \int dk (\sqrt{\eta\tilde{\eta}}k)^{D-2+2\tilde{\nu}}. \tag{248}$$

By paralleling what we did for the inflationary case one can show that the lower limit term which corrects the $N=0$ problem is,

$$\delta i\Delta^0 = -\frac{\Big[(1-\epsilon)^2 H\tilde{H}\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(-2\hat{\nu})\Gamma(-\hat{\nu})}{\Gamma(\frac{1}{2}-\hat{\nu})\Gamma(\frac{D-1}{2})}\, 2\int_0^{z_0} dz\, z^{D-2+2\hat{\nu}} \; ,$$

$$= \frac{\Big[(1-\epsilon)^2 H_0^2\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(-2\hat{\nu})\Gamma(-\hat{\nu})}{\Gamma(\frac{1}{2}-\hat{\nu})\Gamma(\frac{D-1}{2})} \frac{2(\epsilon-1)}{2(D-1)-D\epsilon} \tag{249}$$

$$\times \left[\frac{(1-\epsilon)^2 H_0^2}{k_0^2}\right]^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} \Big(k_0^2 \eta\tilde{\eta}\Big)^{2\hat{\nu}} \; . \tag{250}$$

One can easily check that $(k_0^2\eta\tilde{\eta})^{2\hat{\nu}}$ indeed solves again the homogeneous equation (237). So the full series of these lower limit corrections should take the form,

$$\delta i\Delta^N = \frac{\Big[(1-\epsilon)^2 H_0^2\Big]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(-2\hat{\nu})\Gamma(-\hat{\nu})}{\Gamma(\frac{1}{2}-\hat{\nu})\Gamma(\frac{D-1}{2})} \frac{2(\epsilon-1)}{2(D-1+N)-(D+2N)\epsilon}$$

$$\times \left[\frac{(1-\epsilon)^2 H_0^2}{k_0^2}\right]^{\frac{\epsilon(D-2)}{2(1-\epsilon)}} (k_0^2\eta\tilde{\eta})^{2\hat{\nu}} k_0^{2N} \sum_{k=0}^{N} \sum_{\ell=0}^{N-k} b_{k\ell} r^{2k} \eta^{2\ell} \tilde{\eta}^{2(N-k-\ell)} \; . \tag{251}$$

We determine the coefficients $b_{k\ell}$ by the same three requirements as the $a_{k\ell}$, although the solution will be different because the Ansatz (251) is.
We need to commute the differential operator in (237) through the prefactor of $(k_0^2\eta\tilde{\eta})^{2\hat{\nu}}$ in the Ansatz (251),

$$\Big[\partial^2 + \frac{2\hat{\nu}-1}{\eta}\,\partial_0\Big] \Big(k_0^2\eta\tilde{\eta}\Big)^{2\hat{\nu}} = \Big(k_0^2\eta\tilde{\eta}\Big)^{2\hat{\nu}} \Big[\partial^2 + \frac{-2\hat{\nu}-1}{\eta}\,\partial_0\Big] \; . \tag{252}$$

That is a highly significant result because it means the equation the bare series obeys is the same as we already solved for the lower series but with the replacement $\hat{\nu} \to -\hat{\nu}$. So we can write down the answer immediately,

$$b_{k\ell} = \Big(-\frac{1}{4}\Big)^N \frac{1}{k!\,\ell!\,(N-k-\ell)!} \frac{\Gamma(\frac{D-1}{2})\,\Gamma^2(1+\hat{\nu})}{\Gamma(k+\frac{D-1}{2})\Gamma(\ell+1+\hat{\nu})\Gamma(N-k-\ell+1+\hat{\nu})} \; . \tag{253}$$

It is worth explicitly checking that the lowest N corrections $\delta i\Delta^N$ cancel the ϵ poles in (201) from the upper series (209). From (251) and (253) we see that the $N=0$ correction is,

$$\delta i\Delta^0 = \frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{\Gamma(-\hat{\nu})\Gamma(-2\hat{\nu})}{\Gamma(\frac{D-1}{2})\Gamma(\frac{1}{2}-\hat{\nu})} \frac{2(\epsilon-1)}{\Big[2(D-1)-D\epsilon\Big]} \left(\frac{1}{k_0^2\eta\tilde{\eta}}\right)^{\frac{2(D-1)-D\epsilon}{2(\epsilon-1)}} \; . \tag{254}$$

This should cancel the divergence at $\epsilon = 2(D-1)/D$ in the $n=0$ term on the last line of (201),

$$\frac{[(1-\epsilon)^2 HH']^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D-1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2}-\hat{\nu})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{1}{2}-\hat{\nu})} \; . \tag{255}$$

The relevant small parameter is,

$$\alpha \equiv \frac{2(D-1)-D\epsilon}{2(\epsilon-1)} = -\frac{D-1}{2} - \hat{\nu} \; . \tag{256}$$

Adding (254) to (255) and taking the limit that α vanishes gives,

$$\lim_{\alpha\to 0} \frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D-1}{2}+\alpha)\Gamma(D-1+2\alpha)}{\Gamma(\frac{D}{2}+\alpha)\Gamma(\frac{D-1}{2})\,\alpha} \Biggl\{ \Bigl(\frac{1}{k_0^2\eta\tilde{\eta}}\Bigr)^{\alpha} - \frac{\Gamma(\frac{D-1}{2})}{\Gamma(\frac{D-1}{2}+\alpha)} \frac{\Gamma(D-1+\alpha)}{\Gamma(D-1+2\alpha)} \frac{\Gamma(1-\frac{D}{2})}{\Gamma(1-\frac{D}{2}-\alpha)} \frac{\Gamma(1-\alpha)}{\Gamma(1)} \Biggr\} . \tag{257}$$

$$= \frac{[(1-\frac{2}{D})^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \Biggl\{ 2\ln\Bigl[\Bigl(1-\frac{2}{D}\Bigr)\frac{H_0}{k_0}\Bigr] - \Bigl(1-\frac{2}{D}\Bigr)\ln(a\tilde{a}) -\pi\cot\Bigl(\frac{D\pi}{2}\Bigr) - \gamma - \psi\Bigl(\frac{D}{2}\Bigr) + \psi(D-1) + \psi\Bigl(\frac{D-1}{2}\Bigr) \Biggr\} . \tag{258}$$

By using $\epsilon-1 = 1-2/D$ the final result for the propagator can be expressed in a form that is identical with the de Sitter case (217),

$$\lim_{\epsilon\to\frac{2(D-1)}{D}} i\Delta(x;\tilde{x}) = \frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \Biggl\{ \frac{\Gamma(\frac{D}{2})}{\frac{D}{2}-1}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+1)}{\frac{D}{2}-2}\Bigl(\frac{4}{y}\Bigr)^{\frac{D}{2}-2} + \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})}\Bigl[(1-\epsilon)\ln(a\tilde{a}) -\pi\cot\Bigl(\frac{\pi D}{2}\Bigr) + 2\ln\Bigl[|1-\epsilon|\frac{H_0}{k_0}\Bigr] + \psi\Bigl(\frac{D-1}{2}\Bigr) - \psi\Bigl(\frac{D}{2}\Bigr) + \psi(D-1) - \gamma\Bigr] + \sum_{n=1}^{\infty}\Bigl[\frac{\Gamma(D-1+n)}{n\,\Gamma(\frac{D}{2}+n)}\Bigl(\frac{y}{4}\Bigr)^n - \frac{\Gamma(\frac{D}{2}+1+n)}{(2-\frac{D}{2}+n)\,(n+1)!}\Bigl(\frac{y}{4}\Bigr)^{n-\frac{D}{2}+2}\Bigr] + O(k_0^2) \Biggr\}. \tag{259}$$

We find that we have a similar logarithmic growth as in the de Sitter case. However, there is a crucial difference here. In de Sitter the prefactor H_0 is constant, while here H decreases with time. The total factor $H\tilde{H}\ln(a\tilde{a})$ can also be seen to decrease in time. We will content ourselves with working out one more propagator. From (251) and (253) we see that the $N=1$ correction is,

$$\frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{2\Gamma(-\hat{\nu})\Gamma(-2\hat{\nu})}{\Gamma(\frac{1}{2}-\hat{\nu})\Gamma(\frac{D-1}{2})} \Bigl(\frac{-1}{4\eta\tilde{\eta}}\Bigr) \Bigl[\frac{2r^2}{D-1} + \frac{\eta^2+\tilde{\eta}^2}{1+\hat{\nu}}\Bigr] \frac{(k_0^2\eta\tilde{\eta})^{-\alpha}}{2\alpha} \; , \tag{260}$$

where we define the small parameter α as,

$$\alpha \equiv \frac{2D-(D+2)\epsilon}{2(\epsilon-1)} = -\frac{D+1}{2} - \hat{\nu} \; . \tag{261}$$

This should cancel the divergences from the $n=0$ and $n=1$ terms on the last line of (201),

$$\frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{\Gamma(\frac{D+1}{2}+\hat{\nu})\Gamma(\frac{D+1}{2}-\hat{\nu})\Gamma(-\frac{D}{2})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{1}{2}-\hat{\nu})} \Biggl\{ -\frac{y}{4} - \frac{\frac{D}{2}}{(\frac{D-1}{2})^2-\hat{\nu}^2} \Biggr\} . \tag{262}$$

Adding (260) to (262) and taking α to zero gives,

$$\frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D+1)}{4\Gamma(\frac{D}{2}+1)} \Bigg\{ -\frac{2}{D-1}\Big(\frac{\eta}{\tilde{\eta}} + \frac{\tilde{\eta}}{\eta}\Big) + 2 + \frac{2}{D} \\ +(2-y)\Bigg[-\Big(\frac{D-2}{D+2}\Big) \ln(aa') + 2\ln\Big[\Big(\frac{D-2}{D+2}\Big)\frac{H_0}{k_0}\Big] - \pi\cot\Big(\frac{D\pi}{2}\Big) \\ -\gamma - \psi\Big(\frac{D}{2}+1\Big) + \psi\Big(\frac{D+1}{2}\Big) + \psi(D+1)\Bigg]\Bigg\} \,. \qquad (263)$$

By taking advantage of the fact that $(1-\epsilon) = -(D-2)/(D+2)$ we can express the full propagator in a form identical to the $N=1$ result (231) from the lower series,

$$\lim_{\epsilon\to\frac{2D}{D+2}} i\Delta(x;\tilde{x}) \\ = \frac{[(1-\epsilon)^2 H\tilde{H}]^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}} \Bigg\{ \frac{\Gamma(\frac{D}{2}+1)}{(1-\frac{D}{2})(-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-1} + \frac{\Gamma(\frac{D}{2}+2)}{(2-\frac{D}{2})(1-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-2} \\ +\frac{1}{2!}\frac{\Gamma(\frac{D}{2}+3)}{(3-\frac{D}{2})(2-\frac{D}{2})}\Big(\frac{4}{y}\Big)^{\frac{D}{2}-3} + \frac{\Gamma(D+1)}{4\Gamma(\frac{D}{2}+1)}\Bigg[\frac{2(D-1)}{k_0^2\eta\tilde{\eta}} - \frac{2}{D-1}\Big(\frac{\eta}{\tilde{\eta}}+\frac{\tilde{\eta}}{\eta}\Big) \\ +2 + \frac{2}{D} + (2-y)\Big\{(1-\epsilon)\ln(a\tilde{a}) - \pi\cot\Big(\frac{\pi D}{2}\Big) + C_D\Big\}\Bigg] \\ +\sum_{n=2}^{\infty}\Bigg[\frac{\Gamma(\frac{D}{2}+2+n)}{(2-\frac{D}{2}+n)(1-\frac{D}{2}+n)\,(n+1)!}\Big(\frac{y}{4}\Big)^{n-\frac{D}{2}+2} \\ -\frac{\Gamma(D+n)}{n(n-1)\,\Gamma(\frac{D}{2}+n)}\Big(\frac{y}{4}\Big)^{n}\Bigg] + O(k_0^2)\Bigg\}. \qquad (264)$$

Here the constant C_D is,

$$C_D \equiv 2\ln\Big[|1-\epsilon|\frac{H_0}{k_0}\Big] + \psi(D+1) + \psi\Big(\frac{D+1}{2}\Big) - \gamma - \psi\Big(\frac{D}{2}+1\Big) \,. \qquad (265)$$

6.1.3 Final result

We thus see that if we write the propagator $i\Delta$ in terms of the infinite space propagator (199) and the corrections (238) and (251),

$$i\Delta(x;\tilde{x}) = i\Delta_\infty(x;\tilde{x}) + \sum_{N=0}^{\infty}\delta i\Delta_N(x;\tilde{x}) + \sum_{N=0}^{\infty}\delta i\Delta^N(x;\tilde{x}) \,, \qquad (266)$$

we obtain an infrared finite result. The corrections we rewrite as

$$\begin{aligned}
\delta i\Delta_N(x;\tilde{x}) &= -\frac{\Big(H\tilde{H}(1-\epsilon)^2\Big)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}\frac{2z_0^{2N+(D-1)-2\hat{\nu}}}{2N+(D-1)-2\hat{\nu}} \\
&\quad\times\frac{\Gamma(2\hat{\nu})\Gamma(\hat{\nu})}{\Gamma(\frac{1}{2}+\hat{\nu})\Gamma(\frac{D-1}{2})}\sum_{k=0}^{N}\sum_{\ell=0}^{N-k}a_{k\ell}\Big(\frac{r}{\tilde{\eta}}\Big)^{2k}\Big(\frac{\eta}{\tilde{\eta}}\Big)^{2\ell-N} \\
\delta i\Delta^N(x;\tilde{x}) &= -\frac{\Big(H\tilde{H}(1-\epsilon)^2\Big)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}\frac{2z_0^{2N+(D-1)+2\hat{\nu}}}{2N+(D-1)+2\hat{\nu}} \\
&\quad\times\frac{\Gamma(-2\hat{\nu})\Gamma(-\hat{\nu})}{\Gamma(\frac{1}{2}-\hat{\nu})\Gamma(\frac{D-1}{2})}\sum_{k=0}^{N}\sum_{\ell=0}^{N-k}b_{k\ell}\Big(\frac{r}{\tilde{\eta}}\Big)^{2k}\Big(\frac{\eta}{\tilde{\eta}}\Big)^{2\ell-N}
\end{aligned} \tag{267}$$

Since $a_{k\ell}(\hat{\nu}) = b_{k\ell}(-\hat{\nu})$, we find that (266) is exactly symmetric under $\hat{\nu} \leftrightarrow -\hat{\nu}$. Since if $\xi = 0$ the only difference between ν (as defined in (178)) and $\hat{\nu}$ (207) is a potential minus sign, we can essentially drop all the hats in the corrections how they appear in (266). Moreover, because of the role ν plays, one is guaranteed that even if $\xi \neq 0$ (267) stays the same, when expressed in terms of the more general ν given in (178). Thus we can write the corrections as

$$\begin{aligned}
\delta i\Delta_N(x;\tilde{x}) &= -\frac{\Big(H\tilde{H}(1-\epsilon)^2\Big)^{\frac{D}{2}-1}}{(4\pi)^{\frac{D}{2}}}\frac{2z_0^{2N+(D-1)-2\nu}}{2N+(D-1)-2\nu} \\
&\quad\times\frac{\Gamma(2\nu)\Gamma(\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{D-1}{2})}\sum_{k=0}^{N}\sum_{\ell=0}^{N-k}a_{k\ell}\Big(\frac{r}{\tilde{\eta}}\Big)^{2k}\Big(\frac{\eta}{\tilde{\eta}}\Big)^{2\ell-N} \\
\delta i\Delta^N(x;\tilde{x}) &= \delta i\Delta_N(x;\tilde{x})(\nu\leftrightarrow -\nu).
\end{aligned} \tag{268}$$

When we construct the stress energy tensor, based on this propagator, we shall discuss its properties in more detail. For now let us just mention that z_0 approaches zero in an accelerating universe, while it grows to infinity in a decelerating universe. Thus we see from (268), as expected that if $\nu > \frac{D-1}{2}$ the Δ_N correction has growing terms. These growing terms are due to the fact that although we cut away initial super-Hubble modes, the universe becomes more and more correlated on large scales as time goes on. What happens in the decelerating case is difficult to say at this point. Both corrections Δ_N and Δ^N contain in principle infinitely fast growing terms, since the sum over N goes to infinity. Moreover these terms appear to be there for all values of ν, while the infrared is only divergent for $\nu < (D-1)/2$ as expected, based on (186). We shall come back to this issue when we discuss the stress energy tensor.

7 The stress energy tensor

In this section we shall calculate the expectation value of the scalar stress-energy tensor using the propagators constructed in the preceding two sections. The energy momentum tensor for a massless scalar field is (conform Eq. (109))

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\Big(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\Big) + \xi\Big((R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) - \nabla_\mu\nabla_\nu + g_{\mu\nu}\Box\Big)\phi^2. \tag{269}$$

When written in terms of the propagator we obtain

$$\begin{aligned}\langle 0|T_{\mu\nu}|0\rangle = \Bigg(&\Big(\delta^\rho_\mu\delta^\sigma_\nu(1-2\xi) - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}(1-4\xi)\Big)\partial_\rho\tilde{\partial}_\sigma \\ &- 2\xi\Big(\delta^\rho_\mu\delta^\sigma_\nu - g_{\mu\nu}g^{\rho\sigma}\Big)\nabla_\rho\partial_\sigma + \xi\Big(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\Big)\Bigg)i\Delta(x;\tilde{x})\Big|_{x=\tilde{x}}.\end{aligned} \tag{270}$$

Alternatively we can take the trace of (269) to obtain

$$T^\mu{}_\mu = \Big(1 - \frac{D}{2}\Big)\Big((\partial^\mu\phi)(\partial_\mu\phi) + R\xi\phi^2\Big) + (D-1)\xi\Box\phi^2 \tag{271}$$

and using the scalar equation of motion (132) for ϕ we find that the expectation value for the trace is given by

$$T_q \equiv \langle 0|T^\mu{}_\mu|0\rangle = \Big(\frac{2-D}{4} + (D-1)\xi\Big)\Box i\Delta(x;x). \tag{272}$$

Since any quantum correction will respect the symmetries of the underlying space-time, we know that we should be able to write the energy momentum tensor conform Eq. (5) as

$$\langle 0|T^\mu{}_\nu|0\rangle = \mathrm{diag}\Big(\rho_q, \underbrace{-p_q, -p_q, .., -p_q}_{D-1}\Big) \tag{273}$$

where we used a subscript q to indicate we are considering the quantum corrections to the stress energy tensor. Next we can calculate the individual components using the conservation equation (15)

$$\begin{aligned}\frac{d}{dt}(a^D\rho_q) &= a^D H(\rho_q - (D-1)p_q) \\ &= -a^D H T_q\end{aligned}, \tag{274}$$

such that in $D=4$ we obtain from (272) and (274)

$$\rho_q - 3p_q = \frac{1}{2}\Big(1 - 6\xi\Big)\Box i\Delta(x;x). \tag{275}$$

Now an important question is whether the energy density in the quantum corrections calculated here can dominate over the energy density in the classical background. The background Friedmann equations (14) tell us that the equation of state parameter of the background w_b is given by

$$w_b \equiv \frac{p_b}{\rho_b} = \frac{1}{3}(2\epsilon - 3) \tag{276}$$

and moreover we know from the conservation equation (274), which holds for any stress energy tensor, that the energy density scales in general with the scale factor as

$$\begin{aligned} \rho_b &= \rho_b^\star a^{-3(1+w_b)} \\ \rho_q &= \rho_q^\star a^{-3(1+w_q)}, \end{aligned} \tag{277}$$

where $\rho_{b,q}^\star$ indicate the energy density for $a = 1$. The energy density due to the quantum fluctuations will typically be proportional to $\rho_q \sim H^4$ and we thus find from the Friedmann equation (14) that $\frac{\rho_q^\star}{\rho_b^\star} \sim \left(\frac{H^\star}{m_p}\right)^2$, where $H^\star$ is H at $a = 1$. We therefore find that

$$\frac{\rho_q}{\rho_b} \sim \left(\frac{H^\star}{m_p}\right)^2 a^{3(w_b - w_q)} \tag{278}$$

which implies that the quantum contribution to the stress energy grows with respect to the background contribution whenever

$$w_q < w_b\,. \tag{279}$$

If this is the case, the quantum correction can dominate over the classical background energy density after a sufficient amount of time, despite the smallness of $\frac{H^\star}{m_p}$. *If* this happens, we find using (18) that this happens at a time

$$t \sim \frac{1}{H^\star}\left(\frac{m_p}{H^\star}\right)^{\frac{1+w_b}{w_b - w_q}} \qquad ; \qquad (w_q < w_b). \tag{280}$$

Notice that the dominance of the quantum corrections therefore might happen at late times. This is interesting in the context of the 'why now?' problem of dark energy. The cumulative growth is simply only relevant at late enough times. Of course one then also needs to show that the equation of state is that of dark energy (which it will not be in the present analysis).

7.1 Calculation of the stress-energy tensor

In this section we shall calculate the expectation value of the scalar stress-energy tensor using the propagator obtained in section (6). We shall use (270) to calculate the full stress energy tensor, using the propagator given in (266) and (268).

7.1.1 The infinite space contribution

We first consider the contribution to (270) coming from $i\Delta_\infty$. From (165) we find that at coincidence ($y \to 0$) the following identities hold

$$\begin{aligned} \partial_\rho y\Big|_{y=0} &= 0 \\ \partial_\rho \tilde{\partial}_\sigma y\Big|_{y=0} &= -\frac{2}{\eta^2}\eta_{\rho\sigma} = -2(1-\epsilon)^2 H^2 g_{\rho\sigma} \\ \nabla_\rho \partial_\sigma y\Big|_{y=0} &= \frac{2}{\eta^2}\eta_{\rho\sigma} = 2(1-\epsilon)^2 H^2 g_{\rho\sigma}\,. \end{aligned} \tag{281}$$

Moreover, since in dimensional regularization all D dependent powers of y can be automatically subtracted, we find using (6.131.2) in [95] that the contributions from the hypergeometric function appearing in $i\Delta_\infty$ relevant for this calculation are

$$
\begin{aligned}
{}_2F_1\Big(\frac{D-1}{2}+\nu,\frac{D-1}{2}-\nu;\frac{D}{2};1-\frac{y}{4}\Big)\Big|_{y=0} &= \frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D}{2})}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \\
\frac{d}{dy}\,{}_2F_1\Big(\frac{D-1}{2}+\nu,\frac{D-1}{2}-\nu;\frac{D}{2};1-\frac{y}{4}\Big)\Big|_{y=0} & \\
&= -\frac{1}{2D}\Big(\nu^2-\Big(\frac{D-1}{2}\Big)^2\Big)\frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D}{2})}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}.
\end{aligned}
\tag{282}
$$

Using these identities we immediately find that

$$
\begin{aligned}
\partial_\rho\tilde{\partial}_\sigma i\Delta_\infty(x;\tilde{x})\Big|_{x=\tilde{x}} &= H^D|1-\epsilon|^D\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \\
&\times\Big[\Big(\frac{D}{2}-1\Big)^2\frac{\epsilon^2}{(1-\epsilon)^2}a^2\delta^0_\rho\delta^0_\sigma+\frac{1}{D}\Big(\nu^2-\Big(\frac{D-1}{2}\Big)^2\Big)g_{\rho\sigma}\Big] \\
\nabla_\rho\partial_\sigma i\Delta_\infty(x;\tilde{x})\Big|_{x=\tilde{x}} &= H^D|1-\epsilon|^D\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \\
&\times\Big[\frac{(D-2)\epsilon}{4(1-\epsilon)^2}(2+D\epsilon)a^2\delta^0_\rho\delta^0_\sigma+\Big(\frac{(D-2)\epsilon}{2(1-\epsilon)^2}-\frac{1}{D}\big(\nu^2-\Big(\frac{D-1}{2}\Big)^2\big)\Big)g_{\rho\sigma}\Big].
\end{aligned}
\tag{283}
$$

Making use of Eq. (270) and (13) the one-loop contribution to the stress-energy from $i\Delta_\infty$ can be written as,

$$
\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle_\infty &= \frac{H^D|1-\epsilon|^D}{(4\pi)^{D/2}}\frac{\Gamma(1-\frac{D}{2})\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} \\
&\times\Bigg[\Big(\frac{(D-2)\epsilon}{2(1-\epsilon)}\Big)^2\Big(1-\frac{4(D-1)}{D-2}\xi\Big)a^2\delta^0_\rho\delta^0_\sigma \\
&+\Bigg(\frac{(D-2)^2(1-2\epsilon+2\epsilon^2)}{8(1-\epsilon)^2}\Big(1-\frac{4(D-1)}{D-2}\xi\Big)+\frac{D-2}{8D}(1-4\nu^2)\Bigg)g_{\rho\sigma}\Bigg]
\end{aligned}
\tag{284}
$$

.

7.1.2 The Δ_N correction

To calculate the contribution to the stress-energy tensor due to the correction terms (268) it is easiest to first calculate the expectation value of the trace, and then use the conservation equation to calculate the full energy momentum tensor. For this we need the correction at coincidence. Using (268) we find

$$
\begin{aligned}
\delta i\Delta_N(x;x) &= -\frac{1}{4\pi^{5/2})}\frac{1}{3+2N-2\nu}\frac{\Gamma(\nu-N)\Gamma(2\nu-N)}{\Gamma(\frac{1}{2}+\nu-N)\Gamma(N+1)}H^2(1-\epsilon)^2z_0^{2N+3-2\nu} \\
&\equiv A_N H^2 z_0^{2N+3-2\nu}
\end{aligned}
\tag{285}
$$

where we used that

$$
\sum_{\ell=0}^{N}\frac{N!}{\ell!(N-\ell)!}\frac{\Gamma(N+1-2\nu)}{\Gamma(\ell+1-\nu)\Gamma(N+1-\ell-\nu)} = \frac{\Gamma(2N+1-2\nu)}{\Gamma^2(N+1-\nu)}\,. \tag{286}
$$

We have put $D = 4$, since the corrections do not contain an ultraviolet singularity. From (275) we find that we need

$$\frac{1}{2}(1-6\xi)\Box H^2 z_0^{2N+3-2\nu} = -H^4 z_0^{2N+3-2\nu}(1-6\xi)(1-\epsilon)(N-\nu)\big(2(1-\epsilon)(N-\nu)+3-\epsilon\big). \quad (287)$$

Furthermore we have

$$\frac{d}{d\eta}\big(a^4 H^4 z_0^{2N+3-2\nu}\big) = a^5 H^5 z_0^{2N+3-2\nu}(1-\epsilon)(1-2N+2\nu) \quad (288)$$

and we find immediately that

$$\begin{aligned}
\rho_N &= -A_N H^4 z_0^{2N+3-2\nu}(1-6\xi)(N-\nu)\frac{2(1-\epsilon)(N-\nu)+3-\epsilon}{1-2N+2\nu}\\
p_N &= -\frac{1}{3}A_N H^4 z_0^{2N+3-2\nu}(1-6\xi)(N-\nu)\\
&\quad\times\Big(2(1-\epsilon)(N-\nu)+3-\epsilon\Big)\Big(\frac{1}{1-2N+2\nu}-(1-\epsilon)\Big)\\
w_N &= \frac{1-\epsilon}{3}\big(2(N-\nu)+\frac{\epsilon}{1-\epsilon}\big).
\end{aligned} \quad (289)$$

Now we can write the full correction to the stress energy tensor in this case as

$$\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle_N &= (\rho_N + p_N)a^2\delta^0_\mu\delta^0_\nu + p_N g_{\mu\nu}\\
&= \frac{1}{4\pi^{5/2}}\frac{\big(2(1-\epsilon)(N-\nu)+3-\epsilon\big)}{3+2N-2\nu}\frac{\Gamma(\nu-N)\Gamma(2\nu-N)}{\Gamma(\frac{1}{2}+\nu-N)\Gamma(N+1)}\\
&\quad\times H^4(1-\epsilon)^2 z_0^{2N+3-2\nu}(1-6\xi)(N-\nu)\\
&\quad\times\Bigg(\frac{1}{3}\Big(\frac{4}{1-2N+2\nu}-(1-\epsilon)\Big)a^2\delta^0_\mu\delta^0_\nu + \frac{1}{3}\Big(\frac{1}{1-2N+2\nu}-(1-\epsilon)\Big)g_{\mu\nu}\Bigg).
\end{aligned} \quad (290)$$

When $\xi = 0$ this reduces to the correction found in [38]. Because of the symmetry in the propagators (268), we immediately find the second correction to be

$$\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle^N &= \frac{1}{4\pi^{5/2}}\frac{\big(2(1-\epsilon)(N+\nu)+3-\epsilon\big)}{3+2N+2\nu}\frac{\Gamma(-\nu-N)\Gamma(-2\nu-N)}{\Gamma(\frac{1}{2}-\nu-N)\Gamma(N+1)}\\
&\quad\times H^4(1-\epsilon)^2 z_0^{2N+3+2\nu}(1-6\xi)(N+\nu)\\
&\quad\times\Bigg(\frac{1}{3}\Big(\frac{4}{1-2N-2\nu}-(1-\epsilon)\Big)a^2\delta^0_\mu\delta^0_\nu + \frac{1}{3}\Big(\frac{1}{1-2N-2\nu}-(1-\epsilon)\Big)g_{\mu\nu}\Bigg).
\end{aligned} \quad (291)$$

7.1.3 Renormalization

The total one-loop stress energy tensor is the sum of the three contributions (284), (290) and (291),

$$\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle &\equiv (\rho_q + p_q)a^2\delta^0_\mu\delta^0_\nu + p_q g_{\mu\nu}\\
&= \langle 0|T_{\mu\nu}|0\rangle_\infty + \sum_{N=0}^{\infty}\langle 0|T_{\mu\nu}|0\rangle_N + \sum_{N=0}^{\infty}\langle 0|T_{\mu\nu}|\Omega\rangle^N\,.
\end{aligned} \quad (292)$$

Note that the ultraviolet divergence (which in dimensional regularization appears as a term multiplying $1/(D-4)$) is confined to (284). When expanded around $D=4$, Eq. (284) gives,

$$
\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle_\infty = & -\frac{\epsilon(2-\epsilon)(1-6\xi)^2}{8\pi^2(D-4)}H^4\Big(\epsilon a^2\delta^0_\mu\delta^0_\nu + \big(\epsilon - \frac{3}{4}\big)g_{\mu\nu}\Big) \\
& - \frac{\epsilon(2-\epsilon)(1-6\xi)^2}{16\pi^2}H^4\Bigg(\gamma_E + \ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big) + \psi(\frac{1}{2}-\nu) + \psi(\frac{1}{2}+\nu) \\
& \qquad + 2\frac{1-2\nu'}{1-2\nu} + 2\frac{1+2\nu'}{1+2\nu}\Bigg)\Big(\epsilon a^2\delta^0_\mu\delta^0_\nu + (\epsilon - \frac{3}{4})g_{\mu\nu}\Big) \\
& - \frac{(2-\epsilon)(1-6\xi)}{16\pi^2}\Bigg((1-4\xi)\epsilon^2a^2\delta^0_\mu\delta^0_\nu + \frac{1}{8}\Big(-7+8\epsilon(1-4\xi)+30\xi\Big)g_{\mu\nu}\Bigg) \\
& + \mathcal{O}(D-4)
\end{aligned}
\tag{293}
$$

here we have defined $\nu' = (\frac{d}{dD}\nu)\Big|_{D=4}$ and we expanded H^D as,

$$
H^D = H^4\mu^{D-4}\left[1 + \frac{D-4}{2}\ln\Big(\frac{H^2}{\mu^2}\Big)\right] + \mathcal{O}((D-4)^2)\,, \tag{294}
$$

where μ is an arbitrary renormalization scale. It is known that this theory can be renormalized by the R^2 counterterm only. Indeed, taking a functional derivative with respect to $g^{\mu\nu}$ of the counterterm action results in

$$
-\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^Dx\sqrt{-g}\alpha R^2 = \alpha(4\nabla_\mu\nabla_\nu R - 4g_{\mu\nu}\Box R + g_{\mu\nu}R^2 - 4RR_{\mu\nu})\,. \tag{295}
$$

Making use of the corresponding expressions for R and $R_{\mu\nu}$ in FLRW spaces (13) this evaluates to

$$
\begin{aligned}
-\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^Dx\sqrt{-g}R^2 \;=\; & 144(2-\epsilon)\epsilon H^4\left[\epsilon\, a^2\delta^0_\mu\delta^0_\nu + \Big(\epsilon - \frac{3}{4}\Big)g_{\mu\nu}\right] \\
& -H^4\Big(48\epsilon(1-4\epsilon+\epsilon^2)a^2\delta^0_\mu\delta^0_\nu - 12(3-22\epsilon+22\epsilon^2-4\epsilon^3)g_{\mu\nu}\Big)(D-4) \\
& + \mathcal{O}\Big((D-4)^2\Big)\,.
\end{aligned}
\tag{296}
$$

From Eqs. (293) and (296) we see that the divergence in (293) is canceled by

$$
\alpha = \frac{\mu^{D-4}(1-6\xi)^2}{1152\pi^2(D-4)}\,, \tag{297}
$$

where μ controls the undetermined finite part of α. The renormalized stress-energy tensor can be now easily obtained

$$
\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle =& -\frac{\epsilon(2-\epsilon)(1-6\xi)^2}{16\pi^2}H^4\Bigg(\gamma_E+\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)+\psi(\frac{1}{2}-\nu)+\psi(\frac{1}{2}+\nu)\\
&+2\frac{1-2\nu'}{1-2\nu}+2\frac{1+2\nu'}{1+2\nu}\Bigg)\Big(\epsilon a^2\delta^0_\mu\delta^0_\nu+(\epsilon-\frac{3}{4})g_{\mu\nu}\Big)\\
&+\frac{H^4(1-6\xi)}{16\pi^2}\Bigg(\Big(-(1-4\xi)(2-\epsilon)\epsilon^2-\frac{2}{3}(1-6\xi)(1-4\epsilon+\epsilon^2)\epsilon\Big)a^2\delta^0_\mu\delta^0_\nu\\
&+\Big(-\frac{1}{8}\big(-7+8\epsilon(1-4\xi)+30\xi\big)-\frac{1-6\xi}{6}\big(3-22\epsilon+22\epsilon^2-4\epsilon^3\big)\Big)g_{\mu\nu}\Bigg)\\
&+\sum_{N=0}^{\infty}\big(\langle 0|T_{\mu\nu}|0\rangle_N+\langle 0|T_{\mu\nu}|0\rangle^N\big)\,,
\end{aligned}
\tag{298}
$$

where the terms in the last line are given in Eqs. (290) and (291).

7.1.4 Resolving the divergencies of the digamma functions

Even though the ultraviolet divergences have been removed by dimensional renormalization, the renormalized stress-energy (298) still seems to diverge at the poles of the (di)gamma functions (208–209) (see also Eq. (202)). For general ν these poles are at $\nu = 3/2+M$, where M is an integer ≥ 0. We shall now show that these divergences are only apparent however, and that they are cancelled by the correction terms in (298) given by (290) and (291), precisely as they were designed to do [4]. We expand

$$\nu = \frac{3}{2}+M-\delta \tag{299}$$

and realizing that

$$(2-\epsilon)(1-6\xi) = (\nu^2-\frac{1}{4})(1-\epsilon)^2 \tag{300}$$

we see that the contribution from the digamma function to the stress energy tensor is given by

$$\frac{(1+M)(2+M)(1-\epsilon)^2\epsilon(1-6\xi)}{16\pi^2\delta}H^4\Big[\epsilon a^2\delta^0_\mu\delta^0_\nu+\Big(\epsilon-\frac{3}{4}\Big)g_{\mu\nu}\Big]+\mathcal{O}(\delta^0) \tag{301}$$

where we used

$$\psi(-M-1+\delta) = -\frac{1}{\delta}+\mathcal{O}\big(\delta^0\big). \tag{302}$$

To check that our construction works, we next calculate the $N = M$ contribution from the sum over (290)

$$
\begin{aligned}
\langle 0|T_{\mu\nu}|0\rangle_N =& -\frac{(1+N)(2+N)(1-\epsilon)^2\epsilon(1-6\xi)}{16\pi^2}H^4\\
&\times\Big[\epsilon a^2\delta^0_\mu\delta^0_\nu+\Big(\epsilon-\frac{3}{4}\Big)g_{\mu\nu}\Big]\Big(\frac{1}{\delta}+\ln(z_0^2)+\mathcal{O}(\delta^0)\Big)
\end{aligned}
\tag{303}
$$

[4]One might think that the second digamma function in (298) has a simple pole also at $\nu = 1/2$, but this term will be cancelled by the $(\nu^2-1/2)$ prefactor.

By comparing Eq. (303) with (301) we see that the $\mathcal{O}(1/\delta)$ terms cancel, as required. The resulting leading order contribution to the one-loop stress-energy tensor is finite and depends logarithmically on the scale factor as $\propto H^4[\ln(a)+\text{const.}]$, where the constant term contains a logarithm of the cutoff k_0. Notice that there will typically also be terms that grow as a power of z_0 due to the fact that we included only one term of the sum over N.

7.1.5 The leading order contribution in the accelerating case

As we mentioned before, in an inflationary space-time z_0 approaches zero, while in a decelerating space-time z_0 grows to infinity. In both cases we shall construct the leading order contribution from the corrections to the stress energy tensor. We start with the accelerating case. If z_0 approaches zero, we see that the leading order contribution comes from the $N = 0$ term of (290). This term is growing if $\nu > 3/2$, as is expected, since this is the requirement for an infrared divergence (186). The case $\nu = 3/2$ leads to the logarithmic growth described above The leading order contribution is thus

$$\begin{aligned}\langle 0|T_{\mu\nu}|0\rangle_{L.O,\nu=3/2} - &\frac{(1-\epsilon)^2\epsilon(1-6\xi)}{8\pi^2}H^4 \\ &\times\Big[\epsilon a^2\delta^0_\mu\delta^0_\nu + \Big(\epsilon-\frac{3}{4}\Big)g_{\mu\nu}\Big]\ln(z_0^2) + \mathcal{O}(z_0^0)\end{aligned} \tag{304}$$

If $\nu > 3/2$, the contribution is

$$\begin{aligned}\langle 0|T_{\mu\nu}|0\rangle_{L.O} =& \frac{4H^4(1-\epsilon)^4}{3\pi^3(1+2\nu)(2\nu-3)}\Big(\frac{z_0}{2}\Big)^{3-2\nu}(1-6\xi)\Gamma(\nu)\Gamma(\nu+1)\Big(\frac{3-\epsilon}{2(1-\epsilon)}-\nu\Big) \\ &\left(\Big(\frac{3+\epsilon}{2(1-\epsilon)}-\nu\Big)a^2\delta^0_\mu\delta^0_\nu + \Big(\frac{\epsilon}{2(1-\epsilon)}-\nu\Big)g_{\mu\nu}\right) + \mathcal{O}(z_0^{5-2\nu}).\end{aligned} \tag{305}$$

Notice that this expression is zero if $\xi = 0$, since then $\nu = \frac{3-\epsilon}{2(1-\epsilon)}$. In that specific case we need the next to leading order contribution, which is proportional to $z_0^{5-2\nu}$.

$$\langle 0|T_{\mu\nu}|0\rangle_{N.L.O} = \frac{2H^4(1-\epsilon)^4}{\pi^3(2\epsilon-1)}\Big(\frac{z_0}{2}\Big)^{5-2\nu}\Gamma(\nu)^2\left(\frac{1-\epsilon}{3}a^2\delta^0_\mu\delta^0_\nu - \frac{1-\epsilon}{6}g_{\mu\nu}\right)+\mathcal{O}(z_0^{7-2\nu}), \tag{306}$$

where we explicitly used the fact that $\xi = 0$ and $\nu = \frac{3-\epsilon}{2(1-\epsilon)}$. Finally we see that in this case, the specific point $\epsilon = 1/2$ is special. If $\epsilon = 1/2$, we have that $\nu = 5/2$ and thus this point corresponds again to a logarithmic contribution and we find from (303)

$$\langle 0|T_{\mu\nu}|0\rangle_{L.O,\xi=0,\epsilon=1/2} - \frac{3}{128\pi^2}H^4\Big[a^2\delta^0_\mu\delta^0_\nu - \frac{1}{2}g_{\mu\nu}\Big]\ln(z_0^2) + \mathcal{O}(z_0^0). \tag{307}$$

We are now interested in the question whether these contributions can ever dominate over the background contribution, thus we want to now according to (279) if $w_q < w_b = \frac{1}{3}(2\epsilon - 3)$. We can immediately read of the leading order late time contribution to w_q from the expressions above and we find from (304), (305), (306) and

(307)

$$
\begin{aligned}
w_q &= \frac{4}{3}\epsilon - 1 \qquad ; \qquad \nu = 3/2 \\
w_q &= \frac{1}{3}\epsilon - 2(1-\epsilon)\nu \qquad ; \qquad \nu > 3/2 \quad ; \quad \xi \neq 0 \\
w_q &= -\frac{1}{3} \qquad ; \qquad \nu > 3/2 \quad ; \quad \xi = 0 \quad ; \epsilon \neq 1/2 \\
w_q &= -\frac{1}{3} \qquad ; \qquad \nu > 3/2 \quad ; \quad \xi = 0 \quad ; \epsilon = 1/2
\end{aligned}
\tag{308}
$$

Which can actually be summarized as

$$
\begin{aligned}
w_q &= \frac{1}{3}\epsilon - 2(1-\epsilon)\nu \\
&= w_b + \frac{3-\epsilon}{3}\Big(1 - \sqrt{1 - \frac{24(2-\epsilon)}{(3-\epsilon)^2}\xi}\Big) \qquad ; \qquad \xi \neq 0 \\
w_q &= -\frac{1}{3} \\
&= w_b + \frac{2}{3}(1-\epsilon) \qquad ; \qquad \xi = 0.
\end{aligned}
\tag{309}
$$

From this we immediately see that if $\xi \geq 0$ we always have that $w_q > w_b$ (remember that we are considering an accelerated universe, thus $\epsilon < 1$). On the other hand we have for all $\xi < 0$ that $w_q < w_b$. Thus in this case we find that the energy density in quantum effects grows faster in time than the energy density in the background. That this happens for negative ξ is not surprising, since a negative ξ effectively induces a (time dependent) negative mass.
We can compare these results with results obtained in de Sitter space ($\epsilon = 0$) in [122] (see also [123]). There the late time contribution to $\langle 0|T_{\mu\nu}|0\rangle$ is calculated for a field with mass m and conformal coupling ξ. The inclusion of a mass in de Sitter space leads only to a modification of ν. In practice one can substitute ξR with $m^2 + \xi R$. The calculation is performed independent of the state, as long as the result is infrared finite and possesses the correct ultraviolet divergences. In this paper it is found that if $\nu < 3/2$, $\langle 0|T_{\mu\nu}|0\rangle$ asymptotically approaches at late times the Bunch-Davies value. If $\nu = 3/2$ and $\xi = m = 0$, $\langle 0|T_{\mu\nu}|0\rangle$ approaches a constant and if $\nu > 3/2$, contributions to $\langle 0|T_{\mu\nu}|0\rangle$ grow as $\eta^{3-2\nu}$. This is exactly what we find. If $\nu < 3/2$, all infrared correction terms decay in time and we are simply left with $\langle 0|T_{\mu\nu}|0\rangle_\infty$. This is of course in our case nothing but the Bunch-Davies contribution. If $\nu = 3/2$ and $\epsilon = 0$, we see from (304) that we indeed obtain that $\langle 0|T_{\mu\nu}|0\rangle$ approaches a constant (although we did not explicitly calculate that constant). Finally for $\nu > 3/2$ (which in de Sitter if $m = 0$ implies that $\xi < 0$) we also find a growth proportional to $(k_0\eta)^{3-2\nu}$.

7.1.6 The leading order contribution in the decelerating case

In a decelerating space-time we have a different situation, the sums over N run to infinity and since z_0 grows, this leads in principle to immensely fast growing terms. Moreover we see that this happens, independent of ν. Thus also infrared perfectly finite space-times become dominated by the cut-off. Since this will essentially spoil the use of this approach for decelerating space-times, we will not bother to construct the leading order stress energy tensor explicitly. Instead we shall limit ourselves to calculating the leading order terms in a decelerating space-time of the trace of the stress energy tensor. After taking the trace of (290) we can perform the sum over N to obtain

$$\sum_{N=0}^{\infty}\langle 0|T^{\mu}{}_{\mu}|0\rangle_N = \frac{2\Gamma(\nu)^2}{\pi^3(5-2\nu)(3-2\nu)}(1-6\xi)H^4(1-\epsilon)^4\Big(\frac{z_0}{2}\Big)^{3-2\nu} \times\Bigg[\nu(5-2\nu)\Big(\frac{3-\epsilon}{2(1-\epsilon)}-\nu\Big)\,{}_2F_3\Big(\frac{1}{2}-\nu,\frac{3}{2}-\nu;1-2\nu,\frac{5}{2}-\nu,-\nu;-z_0^2\Big) + \frac{z_0^2}{2}(3-2\nu)\,{}_2F_3\Big(\frac{3}{2}-\nu,\frac{5}{2}-\nu;2-2\nu,\frac{7}{2}-\nu,1-\nu;-z_0^2\Big)\Bigg] \tag{310}$$

The leading order can be studied by considering the asymptotic expansion of the ${}_2F_3$ hypergeometric functions. We have in general

$$\begin{aligned}{}_2F_3\Big(a_1,a_2;b_1,b_2,b_3;-z\Big) &\propto \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}{\Gamma(a_1)\Gamma(a_2)} \\ &\times\Bigg[\frac{\Gamma(a_1)\Gamma(a_2-a_1)}{\Gamma(b_1-a_1)\Gamma(b_2-a_1)\Gamma(b_3-a_1)}z^{-a_1}\Big(1+\mathcal{O}\big(\frac{1}{z}\big)\Big) \\ &+\frac{\Gamma(a_2)\Gamma(a_1-a_2)}{\Gamma(b_1-a_2)\Gamma(b_2-a_2)\Gamma(b_3-a_2)}z^{-a_2}\Big(1+\mathcal{O}\big(\frac{1}{z}\big)\Big) \\ &+\frac{(z)^{\chi}}{\sqrt{\pi}}\Bigg(\cos(\pi\chi+2\sqrt{z})\Big(1+\mathcal{O}\big(\frac{1}{z}\big)\Big) \\ &\frac{1}{16\sqrt{z}}\Big((3a_1+3a_2+b_1+b_2+b_3-2)(8\chi-2) \\ &+16(b_1b_2+b_1b_3+b_2b_3-a_1a_2)-3\Big)\sin(\pi\chi+2\sqrt{z})\Bigg)\Big(1+\mathcal{O}\big(\frac{1}{z}\big)\Big)\Bigg],\end{aligned} \tag{311}$$

with

$$\chi=\frac{1}{2}\Big(a_1+a_2-b_1-b_2-b_3+\frac{1}{2}\Big). \tag{312}$$

Using this we find that the leading order terms are

$$\begin{aligned}\sum_{N=0}^{\infty}\langle 0|T^{\mu}{}_{\mu}|0\rangle_N &= \frac{1}{\pi^2\sin(\pi\nu)^2}(1-6\xi)(1-\epsilon)^4H^4\Bigg[\frac{1}{4(1-\epsilon)}\Big(\frac{z_0}{2}\Big)^2 \\ &\frac{1}{8}\Big(\frac{z_0}{2}\Big)^2\Big(\frac{5\epsilon-1}{\epsilon-1}+4\nu^2\Big)\sin(2z_0+\pi\nu) \\ &-\Big(\frac{z_0}{2}\Big)^3\cos(2z_0+\pi\nu)\Bigg]+\mathcal{O}\Big(z_0\Big)^0.\end{aligned} \tag{313}$$

The second series is obtained by interchanging ν with $-\nu$ and can be easily added. We see from (313) that there are indeed growing contributions for $z_0\to\infty$. Let us compare the scaling in time of the quantum contributions with the background energy density. It is easiest in this case to consider the Friedmann equations (14). We see that the trace of the background stress energy tensor scales in time as

$$\rho_b - 3p_b \propto H^2. \tag{314}$$

From this we immediately find that the contribution (313) decays in time with respect to the background contribution to the Friedmann equations in most cases. Only if $\epsilon > 3$, we see that the term $H^2 z_0^3 \cos(2z_0 + \pi\nu)$ will start to dominate. This behavior is independent of ν, as long as ν is not half integer. In that case this contribution is cancelled by the second contribution obtained by replacing ν with $-\nu$.
It would be perfectly fine if we had a growing contribution for all $\nu > 3/2$, since these values correspond to the cases with an infrared divergence. This is however not what we find. What happens here is the following. In an inflationary space-time the physical wavelength associated with the infrared cutoff, $a(t)/k_0$, grows faster than the Hubble radius. Thus a super-Hubble cutoff stays super-Hubble at all times. In contrast, for a decelerating universe the Hubble radius grows faster than $a(t)/k_0$ and therefore an initial super-Hubble cutoff will eventually enter the Hubble radius. Hence the effect of the cutoff becomes more and more profound as time evolves. This effect moreover is larger, for larger values of ϵ, and we indeed find that for $\epsilon > 3$, the effect is strong enough to dominate over the classical background.

8 The one loop effective potential

In the previous section the focus was on how the energy density of quantum corrections to the scalar field influence the dynamics of the background space-time. However these corrections are always suppressed by a factor of $(m_p)^{-2}$ with respect to the background contribution. Therefore these corrections can only become significant if the quantum energy density grows in time with respect to the background energy density. However, as we have seen, this will typically not happen in the cases considered.
Alternatively one can consider how the quantum loop corrections change the dynamics of the scalar field itself [73]. This can have interesting consequences for example for the dynamics of the inflaton field [24, 119], or on electroweak phase transitions due to quantum correction to the potential of the Higgs field.
Such a discussion however is not very useful in the context of the free field we have discussed so far. In order to obtain interesting dynamics, we need some interaction term. However in the presence of such an interaction, it becomes much more complicated to construct the propagator.
Suppose we have a scalar field action

$$S[\varphi] = \int \sqrt{-g}\left[-\frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}\xi R\varphi^2 - \frac{1}{4!}\lambda\varphi^4\right]. \tag{315}$$

Conform the discussion leading to (101) we write the field as a background contribution Φ and a quantum field ϕ

$$\varphi = \Phi + \phi \tag{316}$$

and expand the action up to second order in ϕ. The linear term will be zero by the equations of motion and we obtain

$$\begin{aligned} S[\varphi] &= S[\Phi] + \delta S[\phi] + \mathcal{O}(\phi^3) \\ \delta S &= \int \sqrt{-g}\left[-\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}\xi R\phi^2 - \frac{1}{4}\lambda\Phi^2\phi^2\right]. \end{aligned} \tag{317}$$

When we integrate out the quadratic quantum field we obtain the effective action at one loop order

$$\Gamma = S[\Phi] + \frac{i}{2}\mathrm{Tr}\ln\Big[\sqrt{-g}(\Box - \xi R - \frac{\lambda}{2}\Phi^2)\Big]. \tag{318}$$

We are interested in the dynamics of the background field Φ, whose one loop corrected equation of motion is

$$\sqrt{-g}\left[\Box - \xi R - \frac{\lambda}{3!}\Phi^2 - \frac{\lambda}{2}i\Delta(x;x)\right]\Phi = 0\,, \tag{319}$$

where $i\Delta(x;x)$ is the coincident propagator, obeying

$$\sqrt{-g}\left(\Box - \xi R - \frac{\lambda}{2}\Phi^2\right)i\Delta(x;\tilde{x}) = i\delta^D(x-\tilde{x})\,, \tag{320}$$

Calculating this propagator is typically not possible, unless one specializes Φ to a certain background value, around which one is then considering the quantum fluctuations. For example in Minkowski space-time one typically chooses $\Phi(x) = \Phi_0$ =constant [21, 120]. In that case the $\lambda\Phi^2$ contribution to (320) behaves like a mass term and since we can solve the massive scalar field propagator in Minkowski space-time, we can solve this problem. The resulting effective action leads to a correction to the potential of the scalar field. This new potential is known as the effective potential. In Minkowski space-time this was first worked out by Coleman and Weinberg [120]. They obtained after renormalization at an arbitrary scale μ the following effective equation of motion of Φ

$$\begin{aligned} \Box\Phi - \frac{dV_{eff}}{d\Phi} &= 0 \\ V_{eff} &= \frac{\lambda}{4!}\Phi^4 + \frac{\lambda^2}{256\pi^2}\Phi^4\Big(\ln\Big(\frac{\Phi^2}{\mu^2}\Big) - \frac{25}{6}\Big) \end{aligned} \tag{321}$$

Now $\Phi(x) = \Phi_0$ is still a solution for the minimum of this potential, and thus we have self-consistently constructed the one loop contribution.

In curved space-time we will typically not be able to do this. If we focus on the particular case at hand, the cosmological backgrounds with constant ϵ, we saw in section 5 that we *cannot* solve for the scalar field propagator in the presence of a mass. Thus choosing Φ a constant will not help. Moreover, choosing Φ constant does not even solve the tree level equations of motion in the presence of a nonzero ξ, thus this certainly is not the way to go.

The solution however presents itself. If we choose $\Phi(x) = \Phi_0 H(t)$, where $H(t)$ is the Hubble parameter, the term $\lambda\Phi^2$ acts similar to the term ξR, as can be seen from (13). Moreover, since

$$\Box H = (D - 1 - 2\epsilon)\epsilon H^3, \tag{322}$$

this Ansatz does indeed solve the tree level equation of motion. There is however a problem. The ultraviolet logarithms will now contribute to the effective action as $\ln\Big(\frac{H^2}{\mu^2}\Big)$. Moreover, the infrared terms we encountered also have a nontrivial scaling with time. Since both these terms, will scale typically different than H^2, the Ansatz $\Phi = \Phi_0 H$ will not solve the one loop corrected equation of motion. We shall not worry too much about that now and come back to this issue later.

We rescale our field, as in (175) to write the propagator equation

$$\Big(\partial^2 + \eta^{-2}\big(\nu^2 - \frac{1}{4}\big)\Big)\Big(\big(a(\eta)a(\tilde\eta)\big)^{\frac{D}{2}-1} i\Delta(x;\tilde x)\Big) = i\delta^D(x - \tilde x)\,. \tag{323}$$

With the parameter ν now given by

$$\begin{aligned} \nu^2 &= \frac{(D-1-\epsilon)^2}{4(1-\epsilon)^2} - \frac{\rho}{(1-\epsilon)^2} \\ \rho &= \xi\frac{R}{H^2} + \frac{\lambda}{2}\frac{\Phi^2}{H^2} \\ &= \xi(D-1)(D-2\epsilon) + \frac{\lambda}{2}\Phi_0^2 \end{aligned} \tag{324}$$

thus we see that indeed ν is a constant and we can therefore use the previous construction of the propagator. We use for this section the propagator we constructed using an infrared cut-off in section 6, which is given in (266) and (268). We shall consider only an accelerating universe, thus $\epsilon < 1$, in the remainder of this section. The only relevant contributions are then expected to originate only from the Δ_N corrections. We will for now focus on the one loop contribution to (319),

$$-\frac{\lambda}{2}i\Delta(x;x) = -\frac{\lambda}{2}\left[\frac{(1-\epsilon)^{D-2}}{(4\pi)^{D/2}}H^{D-2}\frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\Gamma(1-\frac{D}{2}) + \delta i\Delta_N(x;x)\right], \tag{325}$$

where we used the coincident limit of (199), and dropped all D dependent powers of zero. We use the following expansion of the Γ functions around $D=4$

$$\begin{aligned}\frac{\Gamma(\frac{D-1}{2}+\nu)\Gamma(\frac{D-1}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)} &= \left[\left(\frac{D-3}{2}\right)^2-\nu^2\right]\frac{\Gamma(\frac{D-3}{2}+\nu)\Gamma(\frac{D-3}{2}-\nu)}{\Gamma(\frac{1}{2}+\nu)\Gamma(\frac{1}{2}-\nu)}\\ &= \frac{1}{(1-\epsilon)^2}\left\{(\xi-\frac{1}{6})(D-1)(D-2\epsilon)+\frac{\lambda}{2}\Phi_0^2+\frac{1}{6}(1-5\epsilon+3\epsilon^2)(D-4)+\mathcal{O}(D-4)^2\right\}\\ &\quad\times\left\{1+\left(\psi(\frac{1}{2}+\nu)+\psi(\frac{1}{2}-\nu)\right)\frac{D-4}{2}+\mathcal{O}(D-4)^2\right\}\end{aligned} \tag{326}$$

and using

$$\Gamma(1-\frac{D}{2}) = \frac{2}{D-4}+\gamma_E-1+\mathcal{O}(D-4) \tag{327}$$

we can write (325) as

$$\begin{aligned}-\frac{\lambda}{2}i\Delta(x;x) = -\frac{\lambda}{2}\Bigg\{&\frac{\left[(1-\epsilon)^2H^2\right]^{\frac{D-4}{2}}}{(4\pi)^{D/2}}\Gamma(1-\frac{D}{2})\left(\left(\xi-\frac{1}{6}\right)R+\frac{\lambda}{2}\Phi^2\right)\\ &+\frac{1}{16\pi^2}\left(\left(\xi-\frac{1}{6}\right)R+\frac{\lambda}{2}\Phi^2\right)\left(\psi(\frac{1}{2}+\nu)+\psi(\frac{1}{2}-\nu)\right)\\ &+\frac{1}{16\pi^2}\frac{1}{3}(1-4\epsilon+3\epsilon^2)H^2+\mathcal{O}(D-4)\Bigg\}\end{aligned} \tag{328}$$

To renormalize the theory, we add a the following counterterms

$$S_C = \int\sqrt{-g}\left[\frac{\alpha}{2}R\Phi^2+\frac{\beta}{4!}\Phi^4\right]. \tag{329}$$

Its contribution to the equations of motion (319) is simply

$$\sqrt{-g}\left[\alpha R+\frac{\beta}{3!}\Phi^2\right]\Phi\,, \tag{330}$$

and thus the theory is renormalized if

$$\alpha = \frac{\lambda}{2}(\xi-\frac{1}{6})\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{D/2}}\mu_1^{D-4} \tag{331}$$

$$\beta = \frac{3\lambda^2}{2}\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{D/2}}\mu_2^{D-4} \tag{332}$$

(333)

where we introduced the renormalization scales μ_1 and μ_2. The renormalized one loop corrected equation of motion is now

$$
\begin{aligned}
&\sqrt{-g}\Big(\Box - \frac{dV_{eff}}{d\Phi}\Big)\Phi \\
&= \sqrt{-g}\Bigg\{\Box - \Bigg[\xi + \frac{\lambda}{32\pi^2}(\xi - \frac{1}{6})\Bigg(\ln\Big(\frac{(1-\epsilon)^2H^2}{\mu_1^2}\Big) + \psi(\frac{1}{2}+\nu) + \psi(\frac{1}{2}-\nu)\Bigg)\Bigg]R\Phi \\
&\quad - \Bigg[\frac{\lambda}{6} + \frac{\lambda^2}{64\pi^2}\Bigg(\ln\Big(\frac{(1-\epsilon)^2H^2}{\mu_2^2}\Big) + \psi(\frac{1}{2}+\nu) + \psi(\frac{1}{2}-\nu)\Bigg)\Bigg]\Phi^3 \\
&\quad - \frac{\lambda}{96\pi^2}(1-5\epsilon+3\epsilon^2)H^2\Phi - \frac{\lambda}{2}\delta i\Delta_N(x;x) - \frac{\lambda}{2}\delta i\Delta^N(x;x)\Bigg\}\Phi = 0,
\end{aligned}
\tag{334}
$$

where the coincident limit of the correction terms Δ_N and Δ^N has been calculated in (285) and notice that in $D = 4$

$$
(\xi - \frac{1}{6})R + \frac{\lambda}{2}\Phi^2 = (\frac{1}{4} - \nu^2)(1-\epsilon)^2H^2. \tag{335}
$$

The equation of motion (334) is unfortunately only of limited use. The assumption underlying the calculation is that an expansion around $\Phi = \Phi_0 H$ makes sense. However we see that because of the presence of the logarithm and the presence of the growing z_0 terms this ansatz will not solve the equation of motion anymore. Thus we find the interesting result that quantum corrections ruin the scaling which is expected on classical grounds. This was to be expected, since the ultraviolet physics introduces a new scale μ, while the infrared physics introduces a scale z_0. Both scales introduce a different time dependence than H, and thus a solution that is only determined by the time dependence of H can never solve the one loop corrected effective action.
We for now ignore this issue and consider the effective potential (334) in various limits. We should then be aware that if the corrections to the scaling are too large, the results should only be seen as an indication of how the dynamics changes. In any accelerating space-time, the second sum over N quickly goes to zero. In the following discussion we shall therefore not consider these terms.

8.1 Analysis

In the previous section we obtained the one loop corrected equations of motion for the scalar field (334). In this section we shall now consider some approximations for the effective potential. We shall for simplicity set $\mu_1 = \mu_2$ and assume $0 < \epsilon < 1$. We shall in particular focus on the question whether the quantum corrections can cause the potential to obtain additional minima and thus induce a symmetry breaking. In this subsection we shall perform some approximations, in the next section we shall plot the exact (numerical) effective potential, confirming the conclusions made here.

8.1.1 Large field limit

If the field Φ is large enough, we see that ν becomes imaginary. One can easily find using techniques similar to those in section 5 that the integral over the modes in this case is again proportional to $\int dkk^{2\pm 2\nu}$, similar to (185). Thus for imaginary ν, this quantity is not infrared divergent and we do not expect the infrared corrections to be relevant. This is also expected, since the limit $\Phi \to \infty$ is equivalent to the limit $H \to 0$, or in other words, the Minkowski limit. Since the propagator in this regime recovers its standard Hadamard form, we expect to recover the Coleman-Weinberg potential (321). We obtain

$$\begin{aligned} \frac{dV_{eff}}{d\Phi} =& \xi R\Phi + \frac{\lambda}{3!}\Phi^3 + \frac{\lambda}{96\pi^2}(1-5\epsilon+3\epsilon^2)H^2\Phi \\ &+ \frac{\lambda}{32\pi^2}\Phi\Big((\xi-\frac{1}{6})R+\frac{\lambda}{2}\Phi^2\Big)\Bigg\{ \ln\Big(\frac{(\xi-\frac{1}{6})R+\frac{\lambda}{2}\Phi^2}{\mu^2}\Big) \\ &- \frac{1}{3}\Big(\frac{(1-\epsilon)^2H^2}{(\xi-\frac{1}{6})R+\frac{\lambda}{2}\Phi^2}\Big) - \frac{1}{10}\Big(\frac{(1-\epsilon)^2H^2}{(\xi-\frac{1}{6})R+\frac{\lambda}{2}\Phi^2}\Big)^2 + \mathcal{O}(\Phi^{-6})\Bigg\} \end{aligned} \tag{336}$$

When we take the limit $R \to 0$ we indeed recover the same logarithmic contribution to the effective potential as in the Coleman-Weinberg case (321). Moreover, for $\epsilon = 0$ we recover the same dependence on the logarithms as found in ([105]), where the effective potential in de Sitter space has been calculated. The finite part does differ however, since our renormalization scheme is different from the ones used in [120] and [105]. From (336) we see that symmetry breaking occurs when the logarithm becomes negative. This occurs for

$$(\xi - \frac{1}{6})R + \frac{\lambda}{2}\Phi^2 < \mu^2. \tag{337}$$

On the other hand, the large field expansion (336) is valid if

$$(\xi - \frac{1}{6})R + \frac{\lambda}{2}\Phi^2 > (1-\epsilon)^2H^2. \tag{338}$$

These two conditions are consistent, provided that

$$(1-\epsilon)^2H^2 < \mu^2. \tag{339}$$

This however seems highly unlikely if the theory is renormalized at some low energy scale, while we require the Hubble parameter to be near the inflationary scale.

8.1.2 Logarithmic corrections

It is unfortunately not very useful to consider the small field limit of (334). The highly non-polynomial dependence on ν makes it then complicated to make any analytic statement. Instead we shall consider an approximation in ν itself. If $\nu = 3/2$, the corrections to (334) are only logarithmic. We obtain

$$\sqrt{-g}\Bigg\{\Box - \xi R - \frac{\lambda}{3!}\Phi^2 + \frac{\lambda}{16\pi^2}(1-\epsilon)^2H^2\Bigg(-\frac{1}{2}+\ln\Big(\frac{2H^2(1-\epsilon)^2z_0^2}{\mu^2}\Big)\Bigg) \tag{340}$$

$$-\frac{\lambda}{96\pi^2}(1-5\epsilon+3\epsilon^2)H^2\Bigg\}\Phi = 0\,. \tag{341}$$

In this case we thus see that the effective potential exhibits symmetry breaking, if the logarithm is large enough. Or, alternatively, if we neglect the logarithm and put $\xi = 0$, we find a symmetry breaking for $\epsilon > 1/2$. A large logarithm is initially not very strange, indeed if H is at the inflationary scale, μ is a low energy renormalization scale and z_0 is not yet extremely small, one can easily envision this. However, as time goes on, z_0 will approach zero. Thus we see that eventually the logarithm *will* become negative. Therefore at late enough times, the symmetry will be restored.

8.1.3 Late times

We now consider the late time limit of (334). Remember that we are considering an accelerating universe, $\epsilon < 1$ and thus late times implies $\eta \to 0$. At late times, and for ν not too close to $3/2$, the leading order contribution to (334) is the $N = 0$ contribution from the first sum. Thus we have

$$\sqrt{-g}\left(\Box - \xi R - \frac{\lambda}{3!}\Phi^2 + \frac{\lambda H^2(1-\epsilon)^2}{2\pi^3}\frac{\Gamma(\nu)^2}{3-2\nu}\left(\frac{z}{2}\right)^{3-2\nu}\right)\Phi = 0. \tag{342}$$

Since ν is quadratic in Φ, the correction term is odd in Φ. The potential is given by

$$V = \frac{\xi}{2}R\Phi^2 + \frac{\lambda}{4!}\Phi^4 - \int \frac{\lambda H^2(1-\epsilon)^2}{2\pi^3}\frac{\Gamma(\nu)^2}{3-2\nu}\left(\frac{z}{2}\right)^{3-2\nu}\Phi d\Phi \tag{343}$$

and thus we see that the contribution of the potential is even in Φ and moreover for all $\nu > 3/2$, $\Phi > 0$ it will be positive. For $\nu < 3/2$ the correction terms will be suppressed at late times, and thus irrelevant. Since the minima of the potential are at $\frac{d}{d\Phi}V = 0$, this implies that as soon as the correction terms dominate over the other quantum contributions any symmetry broken by radiative corrections will be restored. The potential will have only one minimum, at $\Phi = 0$. Thus the late time symmetry restoration we saw in the $\nu \to 3/2$ approximation above is generic for any $\nu > 3/2$.

8.1.4 Numerical solutions

The properties of the potential are most easily seen by considering some plots. All plots are numerical plots of the full potential (334) and we defined $V_0 = \frac{V}{H^4}$. All plots are made with $\xi = 0$, $\lambda = 0.1$, $k_0/H_0 = 1$, $\mu/H_0 = 10^{-13}$. To factor out the time dependence of the Hubble parameter we have scaled the potential and the field by $H(t)$,

$$\Phi_0 \equiv \frac{\Phi(t)}{H(t)} \quad , \quad V_0(\Phi_0) \equiv \frac{V_{\rm eff}(\Phi)}{H^4(t)} \,. \tag{344}$$

Fig. 6 shows V_0 over the range $-2 < \Phi_0 < +2$ for three different conformal times,

$$H_0\eta = -10^{-1} \quad , \quad H_0\eta = -10^{-3} \quad , \quad H_0\eta = -10^{-5} \,. \tag{345}$$

The earliest time shows symmetry breaking, whereas the symmetry has been restored by the second time, and the third plot shows that further time evolution makes the potential steeper at the origin. Fig. 7 shows the same three curves but over the expanded scale $-8 < \Phi_0 < +8$. The large field expansion (336) is valid by $\Phi_0 = \pm 8$.

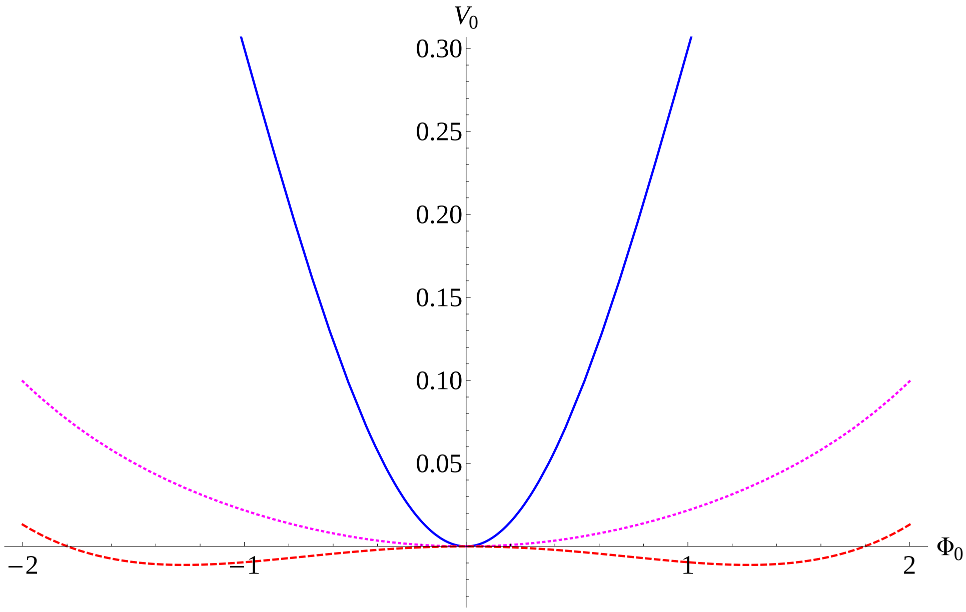

Figure 6: Three different snapshots of $V_0 \equiv V_{\text{hub}}/H^4$ as a function of Φ_0 for $\xi = 0$, $\lambda = 0.1$ with $\epsilon = 0.2$, $\mu/H_0 = 10^{-13}$ and $k_0/H_0 = 1$. The red dashed curve is for $H_0\eta = -10^{-1}$, the purple dotted curve is for $H_0\eta = -10^{-3}$ (about 3.7 e-foldings later), and the blue solid curve is for $H_0\eta = -10^{-5}$ (another 3.7 e-foldings later).

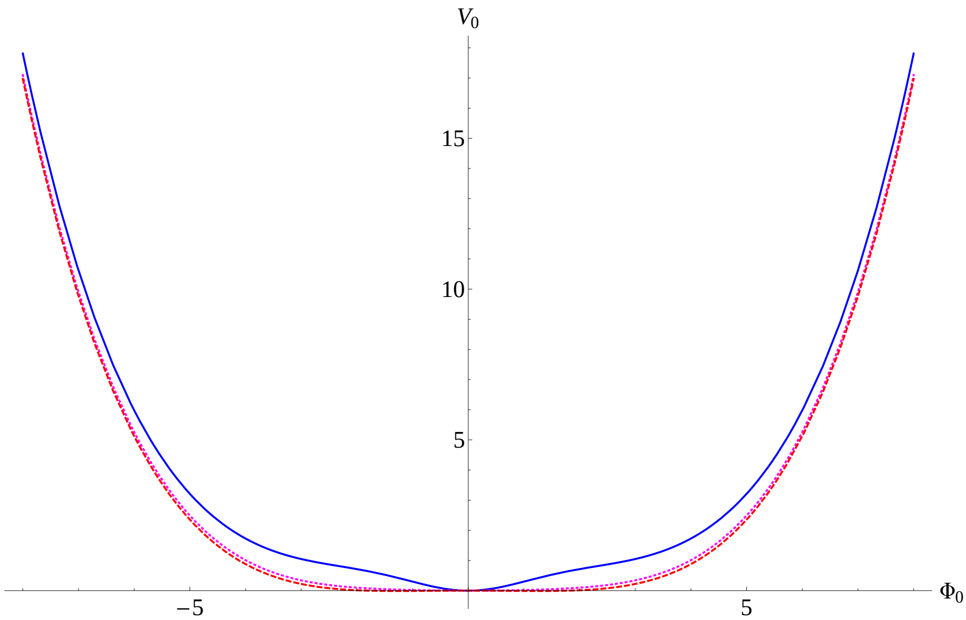

Figure 7: The same three snapshots of $V_0 \equiv V_{\text{hub}}/H^4$ as a function of Φ_0 as in Fig. 6 over a larger range of Φ_0. For these parameters ($\lambda = 0.1$ and $\epsilon = 0.2$) the index ν becomes imaginary for $\Phi_0 = \pm\frac{14}{\sqrt{5}} \approx \pm 6.3$ and the large field expansion (336) is valid by the end of the plotted range at $\Phi_0 = \pm 8$.

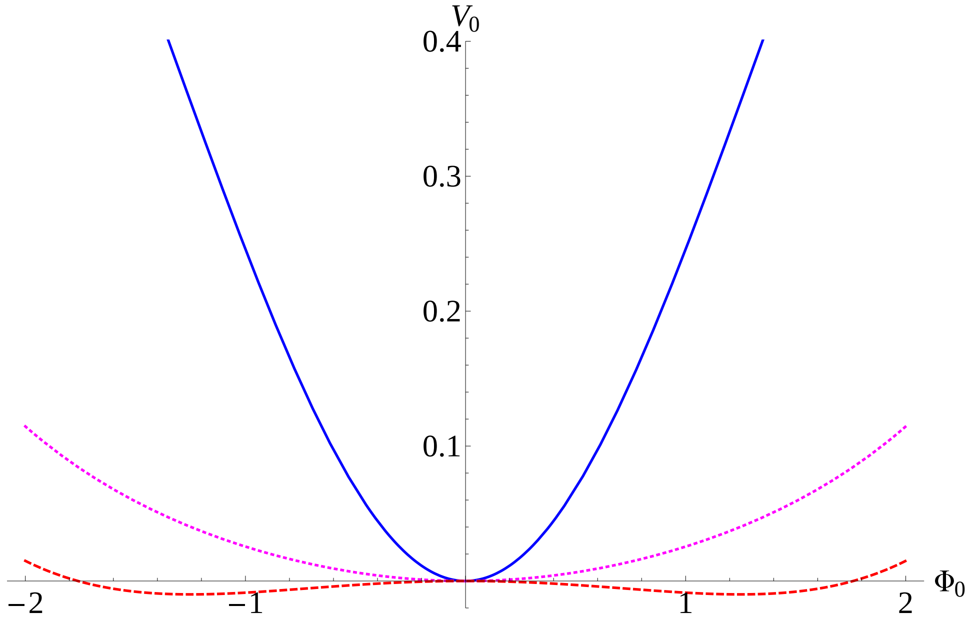

Figure 8: $V_0 \equiv V_{\rm hub}/H^4$ as a function of Φ_0 for three different values of ϵ. The other parameters are $\xi = 0$, $\lambda = 0.1$, $\mu/H_0 = 10^{-13}$, $k_0/H_0 = 1$ and $H_0\eta = -10^{-2}$. The red dashed curve is for $\epsilon = 0.1$, the purple dotted curve is for $\epsilon = 0.3$, and the blue solid curve is for $\epsilon = 0.4$.

Note that the steepening at late times is a small field effect; time evolution has very little impact in the large field or even the intermediate field regime.
We close with a comment on ϵ dependence. This is only very weak in the large field regime (336), as required by the need to agree with the result of Coleman and Weinberg [120]. In the small field regime however at late times, the value of ϵ controls the rate at which time evolution causes the $M = 0$ infrared contribution (343) dominate. One can see all this by plotting V_0 versus Φ_0 for different values of ϵ at the same time. In Fig. 8 and Fig. 9 we have chosen,

$$\xi = 0 \quad , \quad \lambda = .1 \quad , \quad \mu = 10^{-13} H_0 \quad , \quad k_0 = H_0 \quad , \quad H_0\eta = -10^{-2} \; , \tag{346}$$

for three different values of ϵ,

$$\epsilon = 0.1 \quad , \quad \epsilon = 0.3 \quad , \quad \epsilon = 0.4 \; . \tag{347}$$

In Fig. 8 the three curves are plotted over the range $-2 < \Phi_0 < +2$. The plot for $\epsilon = .1$ shows symmetry breaking, whereas the symmetry has been restored for $\epsilon = .3$, and the plot for $\epsilon = .4$ shows that the potential steepens with increasing ϵ. In Fig. 9 the same three curves are plotted over the expanded range $-8 < \Phi_0 < +8$. One can see that the curve for $\epsilon = .4$ is still noticeably above the other curves in the intermediate regime, but the three curves are almost indistinguishable in the large field regime.

8.2 Discussion

From the analysis so far we draw the following conclusions. First of all, we have seen that at late times, or for values of ϵ close to 1, the infrared correction terms start to

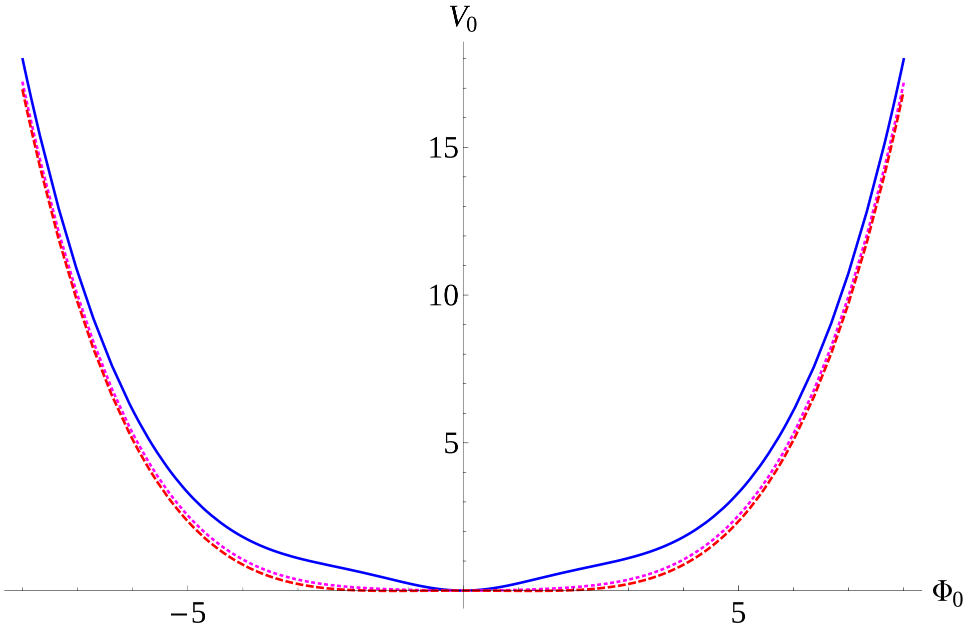

Figure 9: The same plots of $V_0 \equiv V_{\rm hub}/H^4$ as a function of Φ_0 as in Fig. 8 over a larger range of Φ_0. For these parameters the large field result (336) applies by $\Phi_0 = \pm 8$.

dominate the quantum contribution. We also saw that these corrections will cancel any possible symmetry breaking at earlier times. We could see this very explicitly in the specific case where $\nu = 3/2$. Here at early times symmetry could be broken due to quantum contributions and the potential develops two minima, while at later times, the growing infrared corrections restore the symmetry, but we find that the behavior is generic. Let us briefly discuss the implications of this behavior.
We shall consider the Higgs field, during inflation. In the standard model the Higgs mechanism is responsible for the masses of the particles. Although the Higgs is a complex doublet, the characteristics of the potential will be essentially the same as the way we calculated it (for a complex field you obtain simply a factor 2). The Higgs field has a nonzero mass, put by most estimates to be between 115 and 160 GeV. This mass is however much smaller than the inflationary scale $H_I \sim 10^{13}$ GeV. Therefore it is an excellent approximation to consider the Higgs to be effectively massless during inflation. Of course, one should be a bit careful, since the mass will regulate the infrared. However, assuming that our results at least give qualitatively reasonable results, we see that at the beginning of inflation it is perfectly possible that the one loop corrections cause the Higgs to obtain a nontrivial expectation value. This breaks the electroweak symmetry and particles become massive. We see from for example figure 6 that the expectation value of the Higgs in the broken phase will be roughly proportional to the Hubble parameter during inflation. Therefore the masses obtained by the fermions will be roughly a factor 10^{11} times their standard model values. Normally during inflation the production of fermions will be irrelevant, since massless fermions are conformally coupled. When this is not the case, they will be produced, and their asymptotic (after a

long enough time) particle number is given by [121]

$$n_\psi = \frac{1}{e^{2\pi m_\psi/H} + 1}. \tag{348}$$

Here $m_\psi \sim f_y \Phi \sim f_y H$ is the fermion mass, with Yukawa coupling f_y. For all fermions but the top quark, f_y is small, and in those cases we thus have

$$n_\psi = \frac{1}{2}(1 - \pi f_y + \mathcal{O}(f_y^2)), \tag{349}$$

or almost maximal production. Now as inflation goes on, $z_0 \to 0$, or when inflation ends, $\epsilon \to 1$, the electroweak symmetry is restored, and the effect disappears again. If this symmetry restoration occurs sufficiently close to the end of inflation, these fermions might still be observable. If the restoration however occurs earlier during inflation, these particles will be completely diluted away at the end of inflation.
Second of all we have seen that quantum corrections break the classical scaling of the background $\Phi \sim \Phi_0 H$. This is in contrast to for example the Coleman-Weinberg potential, which does obey the classical scaling in Minkowski space-time $\Phi \sim \Phi_0$. Unfortunately this means that the previous conclusion cannot be trusted completely, but it does give an indication of what might happen. The fact that the scaling is changed by the ultraviolet physics can be understood as follows. If we renormalize the theory at a fixed scale μ, this extra scale now introduces a different time dependence into the problem. μ is simply a constant energy scale, while the Hubble scale does change. The fact that the scaling is also affected by the infrared physics is due to similar reasons. The infrared physics introduces a typical scale $\propto k_0\eta$, where k_0 is the infrared cut-off. This scaling is also not the same as the scaling of the Hubble parameter and thus will also change the scaling of Φ.

9 The case of the graviton

In the previous sections we have concerned ourselves with the study of a scalar field. In particular the focus was mostly on a massless, free scalar field. Many interesting phenomena arose due to the presence of an infrared singularity for a wide class of cosmological backgrounds. However, no massless scalar fields have as of yet been observed. The only scalar field that is commonly recognized, the Higgs field, has a nonzero mass, but can be considered effectively massless at high energies (or high H in a cosmological setting). There are however many other scalar fields proposed in the literature, of which some might be light enough. For example, quintessence models might have light scalars. Also various modifications of gravity, like Brans-Dicke theory or TeVeS, require additional massless scalar fields [124]. And of course there is the possibility of considering light composite scalars.

Apart from these fields, also the graviton is extremely interesting in this context. As we shall see explicitly below, the kinetic operator for the graviton has the same structure as the kinetic operator for the massless scalar field, with different amounts of conformal coupling. Therefore we expect that the infrared effects we found for the massless scalar, will also occur in the case of the graviton. Notice that there are many ideas in the literature concerning massive gravitons [125, 126]. This would of course regulate the infrared sector for the gravitons. However all of these approaches are not without problems. Either one gets additional degrees of freedom, which survive even in the massless limit [127, 128] or one gets a propagating scalar ghost [129].

Because of the potential relevance for inflationary cosmology, the quantum behavior of gravitons on a (locally) de Sitter background has been a widely studied subject over the past years [130, 84, 131, 132, 101, 48, 47, 133, 134, 135, 85, 42, 49, 50, 104, 136, 51, 137, 140, 53, 54, 113, 71]. One line of research deals with the back-reaction of gravitational waves on the background spacetime [49, 50, 136, 51]. However of more interest for the present work is the one loop back-reaction by virtual gravitons on a de Sitter background, which has been calculated by several authors, using different techniques [135, 138, 9]. Since it is not clear whether in these works exactly the same quantity is calculated and the renormalization schemes differ, the numerical coefficients differ. However the main result is the same: one loop graviton contributions to the expectation value of the energy momentum tensor result in a finite, time independent shift of the effective cosmological constant. Since the contribution can always be absorbed in a counterterm [138], the exact numerical coefficient coming from such a calculation is scheme dependent and has no physical meaning.

We would like to apply our results, which are valid in a wider class of cosmological backgrounds, now also to the case of the graviton. One immediate problem with working with a more general space-time instead of de Siter space-time, is that for consistency of the Einstein equations the addition of matter fields is unavoidable. Whereas in de Sitter space, the only relevant metric fluctuations are the tensor modes (gravitational waves), in a more general setting one has to take the mixing of gravitational and matter degrees of freedom into account [28, 139, 140, 53, 54, 113].

9.1 The quadratic gravitational and matter action

The model we consider is the action of gravity plus a scalar field φ with an arbitrary potential $V(\varphi)$

$$S = \int \sqrt{-\hat{g}}\Big(\frac{\hat{R} - (D-2)\Lambda}{\kappa} + \frac{1}{2}\varphi\Box\varphi - V(\varphi)\Big), \tag{350}$$

where $\kappa = 16\pi G_N = 16\pi/m_{\rm P}^2$ denotes the (rescaled) Newton constant, $m_{\rm P} \simeq 1.2\times 10^{19}$ GeV is the Planck mass, Λ denotes the cosmological constant, and D is the number of space-time dimensions. By an appropriate choice of the potential V, such a model can mimic any mixture of fluids which are relevant for the evolution of the universe. We consider a conformally flat FLRW background (10) and split the fields in a background contribution and a quantum contribution [101, 113]

$$\begin{aligned}\hat{g}_{\mu\nu} &= g_{\mu\nu}(\eta) + \delta g_{\mu\nu} = a^2(\eta_{\mu\nu} + \sqrt{\kappa}\psi_{\mu\nu})\\ \hat{g}^{\mu\nu} &= g^{\mu\nu}(\eta) + \delta g^{\mu\nu} = a^{-2}(\eta^{\mu\nu} - \sqrt{\kappa}a^4\psi^{\mu\nu}) + \mathcal{O}(\psi^2)\\ \varphi &= \Phi(\eta) + \phi\,,\end{aligned} \tag{351}$$

where $\delta g_{\mu\nu} \equiv h_{\mu\nu}$ denotes the graviton field, $\psi_{\mu\nu}$ is the so called pseudo-graviton field and $\delta g^{\mu\nu} = -h^{\mu\nu} + h^{\mu}_{\ \alpha}h^{\alpha\nu} + \mathcal{O}(h^3)$. Notice that indices on the pseudo-graviton field $\psi_{\mu\nu}$ are raised and lowered with the full background metric $g_{\mu\nu}(\eta) = a^2\eta_{\mu\nu}$. The background scalar field Φ is homogeneous and thus only depends on (conformal) time. The background fields obey the tree level Friedmann equations (14) and the scalar field equation of motion:

$$\begin{aligned} H^2 - \frac{1}{D-1}\Lambda - \frac{\kappa}{(D-1)(D-2)}\Big(\frac{1}{2a^2}\Phi'^2 + V(\Phi)\Big) &= 0\\ a^{-1}H' + \frac{D-1}{2}H^2 - \frac{1}{2}\Lambda + \frac{\kappa}{2(D-2)}\Big(\frac{1}{2a^2}\Phi'^2 - V(\Phi)\Big) &= 0\\ \Phi'' + (D-2)aH\Phi' + a^2\frac{\partial V}{\partial\Phi}(\Phi) &= 0\,,\end{aligned} \tag{352}$$

from which one can derive the following identities

$$\begin{aligned}\sqrt{\kappa}\Phi' &= \sqrt{2(D-2)\epsilon}\,aH\\ \sqrt{\kappa}\Phi'' &= \sqrt{2(D-2)\epsilon}(1-\epsilon)a^2H^2 + \mathcal{O}(\epsilon')\\ \sqrt{\kappa}\frac{\partial V}{\partial\Phi}(\Phi) &= -\sqrt{2(D-2)\epsilon}(D-1-\epsilon)H^2\\ \frac{\partial^2 V}{\partial\Phi^2}(\Phi) &= 2(D-1-\epsilon)\epsilon H^2 + \mathcal{O}(\epsilon')\end{aligned} \tag{353}$$

In these equations a prime indicates a derivative with respect to conformal time. Notice that from section (8) we have learned that these simple scalings with H will be broken by the loop corrections.
For the purpose of this paper we are only interested in the quadratic perturbations in the

fields and we do not consider any interactions. After many partial integrations we find [5]

$$
\begin{aligned}
\mathcal{L}^{(2)} =& a^{D+4}\psi^{\mu\nu}\Bigg(\Big(\square_s - \mathcal{W}\Big)\Big(\frac{1}{4}\delta^\rho_\mu\delta^\sigma_\nu - \frac{1}{8}\eta_{\mu\nu}\eta^{\rho\sigma}\Big) + \mathcal{X}\delta^0_\mu\delta^\rho_\nu\delta^\sigma_0 - \mathcal{Y}\eta_{\mu 0}\delta^0_\nu\eta^{\rho\sigma}\Bigg)\psi_{\rho\sigma} \\
&- a^{D-2}\psi_{00}\Big(\sqrt{\kappa}\Phi''\Big)\phi + a^{D-2}\sqrt{\kappa}\eta^{\mu\nu}\psi_{\mu\nu}\mathcal{Z}\phi \\
&+ \frac{1}{2}a^D\phi\Big(\square_s - \frac{\partial^2 V}{\partial\varphi^2}(\Phi) + a^{-2}\Phi'^2\kappa\Big)\phi + \frac{1}{2}\sqrt{-g}g^{\alpha\beta}F_\alpha F_\beta\,,
\end{aligned}
\tag{354}
$$

where we defined

$$
\begin{aligned}
F_\alpha &= a^2\nabla_\mu\Big(\psi^\mu_\alpha - \frac{1}{2}\delta^\mu_\alpha g^{\rho\sigma}\psi_{\rho\sigma}\Big) - \phi\Phi'\sqrt{\kappa}\delta^0_\alpha \\
\mathcal{W} &= 2(D-2)\Big[a^{-1}H' + \frac{D-1}{2}H^2 - \frac{1}{2}\Lambda + \frac{\kappa}{2(D-2)}\Big(\frac{1}{2a^2}\Phi'^2 - V(\Phi)\Big)\Big] \\
\mathcal{X} &= \Big(\frac{1}{2}\kappa a^{-2}\Phi'^2 - \frac{D-2}{2}(H^2 - a^{-1}H')\Big) \\
\mathcal{Y} &= \Big(\frac{1}{4}\kappa a^{-2}\Phi'^2 + \frac{D-2}{2}a^{-1}H'\Big) \\
\mathcal{Z} &= -\frac{1}{2}\Bigg(\Phi'' + (D-2)aH\Phi' + \frac{\partial V}{\partial\varphi}(\Phi)a^2\Bigg)
\end{aligned}
\tag{355}
$$

and

$$
\square_s = \frac{1}{\sqrt{-g}}\partial_\mu\sqrt{-g}g^{\mu\nu}\partial_\nu \tag{356}
$$

is the d'Alembertian as it acts on a scalar field. We add a gauge fixing term

$$
\mathcal{L}_{GF} = -\frac{1}{2}\sqrt{-g}g^{\alpha\beta}F_\alpha F_\beta \tag{357}
$$

and therefore we also need to add a ghost lagrangian

$$
\mathcal{L}_{\text{ghost}} = -\sqrt{-g}a^{-2}\bar{V}^\mu\delta F_\mu\,, \tag{358}
$$

where we consider the change of F_μ under infinitesimal coordinate transformations $x'^\mu = x^\mu + \sqrt{\kappa}V^\mu$. From $\phi(x^\mu) = \phi(x'^\mu) + \delta\phi$ and $\psi_{\mu\nu}(x^\mu) = \psi_{\mu\nu}(x'^\mu) + \delta\psi_{\mu\nu}$, we find up to first order in V^μ:

$$
\begin{aligned}
\delta\phi &= -\sqrt{\kappa}V^0\Phi' \\
\delta\psi_{\mu\nu} &= -a^{-2}\Big(g_{\alpha\nu}\partial_\mu V^\alpha + g_{\alpha\mu}\partial_\nu V^\alpha + 2\Big(\frac{a'}{a}\Big)g_{\mu\nu}V^0\Big)
\end{aligned}
\tag{359}
$$

and thus

$$
\mathcal{L}_{\text{ghost}} = a^D\eta_{\alpha\beta}\bar{V}^\alpha\Bigg(\delta^\beta_\mu\square_s - (D-2)(H^2 - a^{-1}H')\delta^\beta_0\delta^0_\mu + a^{-2}\kappa\Phi'^2\delta^\beta_0\delta^0_\mu\Bigg)V^\mu\,, \tag{360}
$$

where V and $\bar{V}$ are the ghost and anti-ghost fields.

[5] An analogous result can be found in Ref. [113]. The main difference is that our result (354) includes also terms that vanish on-shell.

The quadratic lagrangian (354–355) contains mixing between the different components. The following field redefinition removes the mixing between ψ_{ij} and ψ_{00} and ϕ *on-shell* [6].

$$
\begin{aligned}
\psi_{ij} &= z_{ij} + \frac{\delta_{ij}}{D-3} z_{00} \\
\psi_{00} &= z_{00} \\
\psi_{0i} &= z_{0i} \, .
\end{aligned}
\tag{361}
$$

The resulting quadratic lagrangian (which still is valid off shell) can be written as

$$
\mathcal{L}^{(2)} + \mathcal{L}_{GF} + \mathcal{L}_{\rm ghost} = \frac{1}{2} X^T_{ij} G^{ijkl} X_{kl} + \frac{1}{2} z_{0i} \mathcal{D}^{ij}_{\rm vector} z_{oj} + \bar{V}^\alpha \mathcal{D}^{\rm ghost}_{\alpha\beta} V^\beta , \tag{362}
$$

where

$$
\begin{aligned}
X_{ij} &= \begin{pmatrix} z_{ij} \\ z_{00} \\ \phi \end{pmatrix} \\
G^{ijkl} &= \begin{pmatrix} \mathcal{D}^{ijkl}_{\rm tensor} & a^D \mathcal{Y} \delta^{ij} & a^{D-2}\sqrt{\kappa}\mathcal{Z}\delta^{ij} \\ a^D \mathcal{Y}\delta^{kl} & \mathcal{D}_{\rm scalar} & a^{D-2}\sqrt{\kappa}(\frac{2}{D-3}\mathcal{Z} - \Phi'') \\ a^{D-2}\sqrt{\kappa}\mathcal{Z}\delta^{kl} & a^{D-2}\sqrt{\kappa}(\frac{2}{D-3}\mathcal{Z} - \Phi'') & \mathcal{D}_\phi \end{pmatrix} \\
\mathcal{D}^{ijkl}_{\rm tensor} &= a^D \Big(\Box_s - \mathcal{W} \Big) \Big(\frac{1}{2}\delta^{ik}\delta^{jl} - \frac{1}{4}\delta^{ij}\delta^{kl} \Big) \\
\mathcal{D}^{ij}_{\rm vector} &= -a^D \Bigg(\Big(\Box_s - \mathcal{W} \Big) + 2\mathcal{X} \Bigg) \eta^{ij} \\
\mathcal{D}_{\rm scalar} &= a^D \Bigg(\frac{D-2}{2(D-3)} \Big(\Box_s - \mathcal{W} \Big) + 2\mathcal{X} + \frac{4}{D-3}\mathcal{Y} \Bigg) \\
\mathcal{D}_\phi &= a^D \Big(\Box_s - \frac{\partial^2 V}{\partial \varphi^2}(\Phi) + a^{-2}\Phi'^2 \kappa \Big) \\
\mathcal{D}^{\rm ghost}_{\alpha\beta} &= a^D \Big(\eta_{\alpha\beta}\Box_s + 2\mathcal{X}\delta^0_\alpha \eta_{\beta 0} \Big) .
\end{aligned}
\tag{363}
$$

Note that G^{ijkl} contains the tensor as well as the two scalar (gravitational and matter) kinetic operators.

9.2 The propagators

We shall now construct the propagators, associated to the various modes in (362). In general this will not be possible, due to the nontrivial dependence of the propagators on the background fields and the mixing between the different modes. Therefore we shall restrict ourselves in calculating the *on-shell* propagators. As will be clear from the discussion in section 9.3 this will actually be sufficient to calculate the one loop effective action. It will turn out that kinetic operators can be written in terms of

$$
\mathcal{D}_n \equiv \sqrt{-g} \Big[\Box_s - n \Big(D - n - 1 + \frac{n(n-1)}{2}\epsilon \Big) (1-\epsilon) H^2 \Big] \qquad (n = 0, 1, 2) \, , \tag{364}
$$

[6]In the special case when $D = 4$, this and Eq. (370) below agree with Ref. [113].

with an associated propagator

$$\mathcal{D}_n i\Delta_n(x;\tilde{x}) = i\delta^D(x-\tilde{x}) \qquad (n=0,1,2)\,. \tag{365}$$

The operator (364) is however nothing but the kinetic operator for the massless scalar field, with conformal coupling

$$\xi = \frac{n(D-n-1+\frac{n(n-1)}{2}\epsilon)}{(D-1)(D-2\epsilon)}(1-\epsilon) \tag{366}$$

such that we can use the propagators calculated in the previous sections with

$$\nu_n^2 = \Big(\frac{D-1-\epsilon}{2(1-\epsilon)}\Big)^2 - \frac{n\Big(D-n-1+\frac{n(n-1)}{2}\epsilon\Big)}{1-\epsilon}\,. \tag{367}$$

We find for the kinetic operators for the vector and the ghost *on-shell*

$$\begin{aligned} \mathcal{D}^{ij}_{\text{vector}}\,|_{\text{on shell}} &= -\mathcal{D}_1\delta^{ij} \\ \mathcal{D}^{\text{ghost}}_{\mu\nu}\,|_{\text{on shell}} &= \Big(\bar{\eta}_{\mu\nu}\mathcal{D}_0 + \delta^0_\mu\eta_{\nu_0}\mathcal{D}_1\Big) \end{aligned} \tag{368}$$

and their associated propagators:

$$\begin{aligned} i_j\Delta^k_{\text{vector}} &= -\delta^k_j i\Delta_1 \\ i_\alpha\Delta^\rho_{\text{ghost}} &= \Big(\bar{\delta}^\rho_\alpha i\Delta_0 + \delta^0_\alpha\delta^\rho_0 i\Delta_1\Big) \end{aligned} \tag{369}$$

There is still mixing between z_{00} and ϕ. The *on-shell* part of the mixing can be removed by the following rotation

$$\begin{aligned} X &= RY \\ Y_{ij} &= \begin{pmatrix} z_{ij} \\ \chi \\ \nu \end{pmatrix} \\ R &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\lambda}\cos(\theta) & -\sqrt{\lambda}\sin(\theta) \\ 0 & \frac{1}{\sqrt{\lambda}}\sin(\theta) & \frac{1}{\sqrt{\lambda}}\cos(\theta) \end{pmatrix} \\ \lambda &= \sqrt{\frac{2(D-3)}{D-2}} \\ \tan(2\theta) &= \frac{2\sqrt{(D-3)\epsilon}}{D-3-\epsilon}\,, \qquad \theta = \arccos\Big(-\sqrt{\frac{\epsilon}{D-3+\epsilon}}\Big). \end{aligned} \tag{370}$$

This rotation reduces the *on-shell* part of the term $\frac{1}{2}X^TGX$ to

$$\frac{1}{2}Y^T_{ij}G^{ijkl}_{diag}Y_{kl}\,|_{\text{on shell}} = \frac{1}{2}Y^T_{ij}\begin{pmatrix} \Big(\frac{1}{2}\delta^{ki}\delta^{lj} - \frac{1}{4}\delta^{ij}\delta^{kl}\Big)\mathcal{D}_0 & 0 & 0 \\ 0 & \frac{1}{\lambda}\mathcal{D}_0 & 0 \\ 0 & 0 & \frac{1}{\lambda}\mathcal{D}_2 \end{pmatrix} Y_{kl}\,. \tag{371}$$

Thus the associated propagator matrix $\mathcal{M}_{diag}$, defined by

$$G_{diag}\mathcal{M}_{diag}(x;\tilde{x})\,|_{\text{on shell}} = \mathbf{1}\delta^D(x-\tilde{x}) \tag{372}$$

is given by

$$\begin{aligned}\mathcal{M}_{diag}\,|_{\text{on shell}} &= \begin{pmatrix} {}_{rs}\Delta_{kl} & 0 & 0 \\ 0 & \lambda\Delta_0 & 0 \\ 0 & 0 & \lambda\Delta_2 \end{pmatrix} \\ i_{rs}\Delta_{kl} &= \Big(2\delta_{r(k}\delta_{l)s} - \frac{2}{D-3}\delta_{rs}\delta_{kl}\Big) i\Delta_0\,.\end{aligned} \tag{373}$$

To avoid confusion with the subscript *diag*, we have omitted the explicit Lorentz indices in $\mathcal{M}_{diag}$ and G_{diag}. It now follows that the non-diagonal propagator matrix ${}_{rs}\mathcal{M}_{kl}$ that inverts G^{ijkl} *on-shell* is

$$\begin{aligned}{}_{rs}\mathcal{M}_{kl}\,|_{\text{on shell}} &= R\mathcal{M}_{diag}R^T\,|_{\text{on shell}} \\ &= \begin{pmatrix} {}_{rs}\Delta_{kl} & 0 & 0 \\ 0 & \lambda^2(\cos^2(\theta)\Delta_0 + \sin^2(\theta)\Delta_2) & \lambda\cos(\theta)\sin(\theta)(\Delta_0-\Delta_2) \\ 0 & \lambda\cos(\theta)\sin(\theta)(\Delta_0-\Delta_2) & (\sin^2(\theta)\Delta_0+\cos^2(\theta)\Delta_2) \end{pmatrix}.\end{aligned} \tag{374}$$

This finishes the construction of all the propagators and we indeed find that all modes can be described in terms of Δ_n.

9.3 One-loop effective action

In this section we shall first sketch using a simple example how to calculate the correction to the Friedmann equation due to the one-loop effective action. Afterwards we shall apply it to the case at hand. We consider as an example a model with an action

$$S = S_0 + S_\chi, \tag{375}$$

where χ is a quantum scalar field with an action $S_\chi = S_\chi[\chi]$ and S_0 is the classical action of any background fields (including for example the Einstein-Hilbert action). If the action S_χ is quadratic in χ,

$$S_\chi = \int d^Dx \frac{1}{2}\chi\mathcal{D}_\chi\chi \tag{376}$$

We get the following effective action, conform (101)

$$\Gamma = S_0 - i\ln\Big(\frac{1}{\sqrt{\text{Det}(\mathcal{D}_\chi)}}\Big) = S_0 + \frac{i}{2}\text{Tr}\ln\big(\mathcal{D}_\chi\big) \tag{377}$$

Here the trace involves tracing over the Lorentz indices and space-time integration of the operator at coincidence [6].
While in principle one could – at least formally – evaluate the effective action, the object one is eventually interested in is the effective Friedmann equation, *i.e.* the equations of motion associated with the background metric. Moreover in the present

case we need to work under the constraint that ϵ is constant. As long as $\dot{\epsilon}$ remains small, there is no problem with imposing such a constraint on the equations of motion. On the other hand, imposing this constraint on the level of the action typically changes the dynamics substantially. Therefore we shall not attempt to explicitly construct the effective action, but instead we shall directly calculate the effective Friedmann equation. By taking the functional derivative of the action with respect to the scale factor $a = a(\eta)$, we obtain the Einstein trace equation, that is the $-(00) + (D-1)(ii)$ component of the Einstein equation. The second Friedmann equation can then always be obtained by imposing the Bianchi identity, similar to what we did when discussing the stress energy tensor. Thus we are interested in calculating

$$\begin{aligned}\frac{\delta\Gamma}{\delta a} &= \frac{\delta S_0}{\delta a} + \frac{i}{2}\frac{\delta}{\delta a}\mathrm{Tr}\ln\big(\mathcal{D}_\chi\big)\\ &= Va^{D-1}\Big[\frac{D(D-1)(D-2)}{\kappa}\Big(H^2 - \frac{1}{D-1}\Lambda + \frac{2}{D}a^{-1}H'\Big) + \sum_i\Big((D-1)p_i - \rho_i\Big)\Big]\\ &\qquad + \frac{i}{2}\frac{\delta}{\delta a}\mathrm{Tr}\ln\big(\mathcal{D}_\chi\big)\,,\end{aligned} \tag{378}$$

where $V = \int d^{D-1}x$ denotes the volume of space and we assumed that S_0 contains the Einstein-Hilbert action, and matter fields with an associated pressure and energy p_i and ρ_i. Notice that the quantum contribution $\frac{i}{2}\frac{\delta}{\delta a}\mathrm{Tr}\ln\big(\mathcal{D}_\chi\big)$ is by definition nothing but $Va^{D-1}g^{\mu\nu}\langle T_{\mu\nu}\rangle$. We now consider the calculation of this contribution. To be explicit, we shall assume that χ is a massless minimally coupled scalar and therefore

$$\mathcal{D}_\chi = \sqrt{-g}\Box. \tag{379}$$

with an associated propagator $i\Delta(x;\tilde{x})$ which obeys

$$\mathcal{D}_\chi i\Delta(x;\tilde{x}) = i\delta^D(x-\tilde{x})\,. \tag{380}$$

Instead of considering the χ field, it is convenient to use a rescaled field

$$\hat{\chi} = a^{\frac{D}{2}-1}\chi \tag{381}$$

with an associated kinetic operator and propagator

$$\begin{aligned}\hat{\mathcal{D}}_\chi &= \partial^2 + \frac{(D-2)(D-4)}{4}\frac{a'^2}{a^2} + \frac{D-2}{2}\frac{a''}{a}\\ i\hat{\Delta}(x;\tilde{x}) &= a^{\frac{D}{2}-1}\tilde{a}^{\frac{D}{2}-1}i\Delta(x;\tilde{x}).\end{aligned} \tag{382}$$

Now, as we shall see explicitly later on, up to a D dimensional divergent delta function that does not contribute in dimensional regularization, we have that

$$\frac{\delta}{\delta a}\mathrm{Tr}\ln\big(\mathcal{D}_\chi\big) = \frac{\delta}{\delta a}\mathrm{Tr}\ln\big(\hat{\mathcal{D}}_\chi\big)\,. \tag{383}$$

We shall now show how to calculate the trace logarithm contribution to (378) due to the field $\hat{\chi}$. To be precise we give the exact coordinate dependence of each term indicated by

x^μ, y^μ, and z^μ

$$
\begin{aligned}
\frac{i}{2}\frac{\delta}{\delta a(z^0)}\mathrm{Tr}\ln\big(\hat{\mathcal{D}}_\chi(x)\big) &= \frac{i}{2}\mathrm{Tr}\Big(\hat{\Delta}(x;y)\Big(\frac{\delta}{\delta a(z^0)}\hat{\mathcal{D}}_\chi(x)\Big)\Big)\\
&= \frac{i}{2}\int d^Dx\int d^Dy\hat{\Delta}(x;y)\delta^D(x-y)\\
&\quad\Bigg[\Big(-\frac{(D-2)(D-4)}{2}\frac{a'(x^0)^2}{a(x^0)^3}-\frac{D-2}{2}\frac{a''(x^0)}{a(x^0)^2}\Big)\delta(x^0-z^0)\\
&\quad+\frac{(D-2)(D-4)}{2}\frac{a'(x^0)}{a(x^0)^2}\partial_{x^0}\delta(x^0-z^0)+\frac{D-2}{2a(x^0)}\partial^2_{x^0}\delta(x^0-z^0)\Bigg] \qquad (384)\\
&= \frac{1}{2}V\Bigg[\Big(-\frac{(D-2)(D-4)}{2}\frac{a'(z^0)^2}{a(z^0)^3}-\frac{D-2}{2}\frac{a''(z^0)}{a(z^0)^2}\Big)i\hat{\Delta}(z^0;z^0)\\
&\quad-\frac{(D-2)(D-4)}{2}\partial_{z^0}\Big(\frac{a'(z^0)}{a(z^0)^2}i\hat{\Delta}(z^0;z^0)\Big)+\frac{D-2}{2}\partial^2_{z^0}\Big(\frac{1}{a(z^0)}i\hat{\Delta}(z^0;z^0)\Big)\Bigg]\\
&= -\frac{D-2}{4}Va^{D-1}\Box_z i\Delta(z;z)\,,
\end{aligned}
$$

where in going from the second to the third step we used the delta functions to change ∂_{x^0} to $-\partial_{z^0}$ and we used that the propagators at coincidence are a function of time only. In the last step we used (382) to rewrite $i\hat{\Delta}$ in terms of $i\Delta$. Now we see from (272) that indeed we have that

$$\frac{i}{2}\frac{\delta}{\delta a}\mathrm{Tr}\ln\big(\hat{\mathcal{D}}_\chi\big) = Va^{D-1}g^{\mu\nu}\langle T_{\mu\nu}\rangle\,. \qquad (385)$$

Comparing this with Eq. (378) we see that this is exactly what one expects, justifying thus our procedure. The above example shows that indeed one can use the functional derivatives with respect to a to evaluate the quantity that interests us. It also shows that the rescaling (381) does not influence the final result in the context of dimensional regularization. Notice that (378) is just an equation of motion. Therefore, *after* the variation is performed, in the second line of (384), we can safely evaluate all the quantities appearing *on-shell*. Hence we only need the propagators $i\Delta(y;z)$ *on-shell*. This justifies our on-shell diagonalization procedure in section IV, based on which we constructed the propagators.

Now we apply this technique to the case at hand. The only difference is that there are more quantum fields. This however does not change the procedure. Up to an irrelevant constant, the effective action can be obtained from

$$\exp[i\Gamma] = \int d^Dx\big(\mathcal{D}h_{\mu\nu}\big)\big(\mathcal{D}\phi\big)\big(\mathcal{D}U^\alpha\big)\big(\mathcal{D}\bar{U}^\alpha\big)\exp\Big(i\big(S^{(0)}+S^{(2)}\big)\Big)\,, \qquad (386)$$

where U and $\bar{U}$ denote the unrescaled ghost fields associated with the graviton field $h_{\mu\nu}$.

When written in terms of the rescaled fields this can be recast as [7]

$$\exp[i\Gamma] = \int d^Dx \big(\mathcal{D}z_{ij}\big)\big(\mathcal{D}z_{0i}\big)\big(\mathcal{D}z_{00}\big)\big(\mathcal{D}\phi\big)\big(\mathcal{D}V^\alpha\big)\big(\mathcal{D}\bar{V}^\alpha\big)\exp\Big(i\big(S^{(0)}+S^{(2)}\big)\Big)$$
$$= \exp\Big(iS^{(0)}\Big)\frac{\mathcal{D}^{\rm ghost}_{\alpha\beta}}{\sqrt{\det(\mathcal{D}^{ij}_{\rm vector})\det(G^{ijkl})}}\,, \tag{387}$$

where

$$S^{(0)} = \int d^Dx\sqrt{-g}\Big(\frac{1}{\kappa}(R-(D-2)\Lambda) - \frac{1}{2}(\partial\Phi)^2 - V(\Phi)\Big)$$
$$S^{(2)} = \int d^Dx\mathcal{L}^{(2)}\,, \tag{388}$$

$\mathcal{L}^{(2)}$ is given in (362) and $(\partial\Phi)^2 = -\Phi'^2$. From Eq. (387) we obtain

$$\Gamma = S^{(0)} + \frac{i}{2}{\rm Tr}\ln[\mathcal{D}^{ij}_{\rm vector}] + \frac{i}{2}{\rm Tr}\ln[G^{ijkl}] - i{\rm Tr}\ln[\mathcal{D}^{\rm ghost}_{\alpha\beta}]$$
$$\equiv S^{(0)} + \Gamma_{1L}\,. \tag{389}$$

From which we find the expression equivalent to (378) to be

$$\frac{\delta\Gamma}{\delta a} = \frac{\delta S_0}{\delta a} + \frac{\delta\Gamma_{1L}}{\delta a}$$
$$= Va^{D-1}\Big[\frac{D(D-1)(D-2)}{\kappa}\Big(H^2 - \frac{1}{D-1}\Lambda + \frac{2}{D}a^{-1}H'\Big) + (D-1)p_M - \rho_M\Big] + \frac{\delta\Gamma_{1L}}{\delta a}\,, \tag{390}$$

where p_M and ρ_M are the pressure and energy density associated to the background scalar field matter, given by $\rho_M = \frac{1}{2}\dot{\Phi}^2 + V(\Phi)$ and $p_M = \frac{1}{2}\dot{\Phi}^2 - V(\Phi)$.
The one loop contribution can be written analogously to (384) as

$$\frac{\delta\Gamma_{1L}}{\delta a} = \int d^Dx\left(\frac{i}{2}{}_i\Delta^{\rm vector}_j(x;x)\frac{\delta}{\delta a}\mathcal{D}^{ij}_{\rm vector} + \frac{i}{2}{}_{ij}\mathcal{M}_{kl}\frac{\delta}{\delta a}G^{ijkl} - i{}^\alpha\Delta^\beta_{\rm ghost}(x;x)\frac{\delta}{\delta a}\mathcal{D}^{\rm ghost}_{\alpha\beta}\right)\,. \tag{391}$$

The functional derivatives should naturally be taken of the *off-shell* kinetic operators. However, as soon as these derivatives are taken, we are simply left with an equation of motion. It is therefore completely valid to impose the background equations of motion at that point. This is exactly the same procedure that leads to (270). Therefore the propagators in (391) can be evaluated *on-shell* and thus we can use the propagators as they are calculated in the previous section.
Instead of using the kinetic operators as given in (363), we rescale all our fields as

$$z_{\mu\nu} \to a^{1-D/2}\hat{z}_{\mu\nu} \quad ; \qquad \phi \to a^{1-D/2}\hat{\phi}\,, \tag{392}$$

[7] Our field redefinition (361) has a Jacobian equal to one. Furthermore, the rescaling by a^2 of ψ with respect to the 'true' graviton will contribute to the effective action as a D-dimensional delta function $\delta^D(0)$. Such a term does not contribute in dimensional regularization.

which is identical to the rescaling in Eq. (381). This rescaling changes the effective action (387) by a D dimensional delta function that does not contribute in dimensional regularization. With the following identity

$$\phi\sqrt{-g}\Box_s\phi = \hat{\phi}\Big[\eta^{\alpha\beta}\partial_\alpha\partial_\beta + \frac{D-2}{2}\Big(\frac{D}{2}-\epsilon\Big)a^2H^2\Big]\hat{\phi} \tag{393}$$

we can easily calculate the kinetic operators of the rescaled fields. We will indicate these rescaled kinetic operators with a hat. The associated propagators are also easily obtained:

$$\Delta(x;\tilde{x}) = (a\tilde{a})^{1-D/2}\hat{\Delta}(x;\tilde{x})\,. \tag{394}$$

We now shall consider one functional derivative in detail. The others are calculated similarly. We follow the same procedure as in the example considered at the start of this section. Our calculation proceeds analogously as in Eq. (384). We once again insert the explicit arguments $\eta = x^0$ and $\tilde{\eta}$ for the two coordinates respectively

$$\begin{aligned}
&\int d^Dx\,{}_{ij}\hat{\Delta}_{kl}(x;x)\frac{\delta}{\delta a(\eta)}\hat{\mathcal{D}}^{ijkl}_{\rm tensor}(\tilde{\eta}) \\
&\quad= \int d^Dx\,{}_{ij}\hat{\Delta}_{kl}(x;x)\frac{\delta}{\delta a(\eta)}\Bigg(\eta^{\alpha\beta}\partial_\alpha\partial_\beta + \frac{D-2}{2}\big(\frac{D}{2}-\epsilon(\tilde{\eta})\big)a(\tilde{\eta})^2H(\tilde{\eta})^2 - a(\tilde{\eta})^2\mathcal{W}\Bigg) \\
&\qquad\times\ \Big(\frac{1}{2}\delta^{ik}\delta^{jl} - \frac{1}{4}\delta^{ij}\delta^{kl}\Big) \\
&\quad= \int d^Dx\,{}_{ij}\hat{\Delta}_{kl}(x;x)\frac{\delta}{\delta a(\eta)}\Bigg(\eta^{\alpha\beta}\partial_\alpha\partial_\beta - \frac{D-2}{4}\Big((3D-16)\big(\frac{a(\tilde{\eta})'}{a(\tilde{\eta})}\big)^2 + 6\frac{a(\tilde{\eta})''}{a(\tilde{\eta})}\Big) \\
&\qquad + (D-2)\Lambda a(\tilde{\eta})^2 - \kappa\Big(\frac{1}{2}\Phi'^2 - a(\tilde{\eta})^2V(\Phi)\Big)\Bigg)\Big(\frac{1}{2}\delta^{ik}\delta^{jl} - \frac{1}{4}\delta^{ij}\delta^{kl}\Big)
\end{aligned} \tag{395}$$

$$\begin{aligned}
&= \int d^Dx\,{}_{ij}\hat{\Delta}_{kl}(x;x)\Bigg(-\frac{D-2}{4}\Big(2(3D-16)\big(\frac{a'(\tilde{\eta})}{a(\tilde{\eta})^2}\frac{\partial}{\partial\tilde{\eta}} - \frac{a'(\tilde{\eta})^2}{a(\tilde{\eta})^3}\big) \\
&\qquad + 6\big(\frac{1}{a(\tilde{\eta})}\frac{\partial^2}{(\partial\tilde{\eta})^2} - \frac{a(\tilde{\eta})''}{a(\tilde{\eta})^2}\big)\Big) \\
&\qquad + 2(D-2)\Lambda a(\tilde{\eta}) + 2\kappa V(\Phi)a(\tilde{\eta})\Bigg)\Big(\frac{1}{2}\delta^{ik}\delta^{jl} - \frac{1}{4}\delta^{ij}\delta^{kl}\Big)\delta(\tilde{\eta}-\eta) \\
&= \int d^Dx\,\delta(\tilde{\eta}-\eta)\Bigg(-\frac{3(D-2)}{2}\frac{1}{a(\tilde{\eta})}\frac{\partial^2}{(\partial\tilde{\eta})^2} \\
&\qquad + \frac{1}{2}(D-2)(3D-10)\Big(H\frac{\partial}{\partial\tilde{\eta}} + (1-\epsilon)H(\tilde{\eta})^2a(\tilde{\eta})\Big) \\
&\qquad + 2(D-2)\Lambda a(\tilde{\eta}) + 2\kappa V(\Phi)a(\tilde{\eta})\Bigg)\Big(\frac{1}{2}\delta^{ik}\delta^{jl} - \frac{1}{4}\delta^{ij}\delta^{kl}\Big)_{ij}\hat{\Delta}_{kl}(x;x)
\end{aligned}$$

$$= V\Bigg(-\frac{3(D-2)}{2}\frac{1}{a}\partial_\eta^2 + \frac{1}{2}(D-2)(3D-10)H\partial_\eta$$
$$+ \frac{1}{2}(D-2)^2(7-3\epsilon)H^2 a\Bigg)\Big(\frac{1}{2}\delta^{ik}\delta^{jl} - \frac{1}{4}\delta^{ij}\delta^{kl}\Big)_{ij}\hat{\Delta}_{kl}(x;x)\,.$$

In step three we integrated by parts and dropped the boundary terms because of the delta function. In step four we used the background equations of motion (352) for Φ. This is justified, since corrections to those equations will be of order 1-loop and the above expression is already at order 1-loop. Therefore the error one is making is of 2-loop order.
The other functional derivatives we need are calculated similarly

$$\frac{1}{V}\int {}_i\hat{\Delta}_{j,\text{vector}}(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}^{ij}_{\text{vector}} = \Bigg(\frac{D-2}{2}\frac{1}{a}\partial_\eta^2 - \frac{1}{2}(D-2)(3D-2)H\partial_\eta$$
$$- \frac{1}{2}(D-2)\Big((3D+2)(1-\epsilon) + 4(D-2)\Big)aH^2\Bigg)\delta^{ij}{}_i\hat{\Delta}_{j,\text{vector}}(x;x)$$
$$\frac{1}{V}\int \hat{\mathcal{M}}_{(1,1)}(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}_{\text{scalar}} = \frac{D-2}{2(D-3)}\Bigg(\frac{D+2}{2}\frac{1}{a}\partial_\eta^2 + \frac{3}{2}(D^2-4)H\partial_\eta$$
$$+ \frac{1}{2}(D-2)\Big((7D+2) - (3D+10)\epsilon\Big)H^2 a\Bigg)\hat{\mathcal{M}}_{(1,1)}(x;x)$$
$$\frac{1}{V}\int \hat{\mathcal{M}}_{(2,2)}(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}_{\phi} = \Bigg(\frac{D-2}{2}\frac{1}{a}\partial_\eta^2 - \frac{1}{2}(D-2)^2H\partial_\eta - \Big(\frac{1}{2}(D-2)^2(1-\epsilon)$$
$$+ 4(D-1-\epsilon)\epsilon\Big)aH^2\Bigg)\hat{\mathcal{M}}_{(2,2)}(x;x) \tag{396}$$
$$\frac{1}{V}\int {}^\alpha\hat{\Delta}^\beta_{\text{ghost}}\frac{\delta}{\delta a}\hat{\mathcal{D}}^{\text{ghost}}_{\alpha\beta} = \Big(\frac{D-2}{2}\frac{1}{a}\partial_\eta^2 - \frac{1}{2}(D-2)^2H\partial_\eta$$
$$- \frac{1}{2}(D-2)^2(1-\epsilon)H^2 a\Big)\eta_{\alpha\beta}{}^\alpha\hat{\Delta}^\beta_{\text{ghost}}$$
$$+ \Big((D-2)\frac{1}{a}\partial_\eta^2 + 4(D-2)H\partial_\eta + 4(D-2)(1-\epsilon)H^2 a\Big)\delta^0_\alpha\eta_{\beta 0}{}^\alpha\hat{\Delta}^\beta_{\text{ghost}}$$
$$\frac{1}{V}\int \hat{\mathcal{M}}_{(1,2)}(x;x)\frac{\delta}{\delta a}\sqrt{\kappa}\Big(\frac{2}{D-3}\mathcal{Z} - \Phi''\Big)$$
$$= \frac{\sqrt{2(D-2)\epsilon}}{D-3}\Big((D-2)H\partial_\eta + (3D-4-D\epsilon)H^2 a\Big)\hat{\mathcal{M}}_{(1,2)}(x;x)\,.$$

Here we indicated with $\hat{\mathcal{M}}_{(n,m)}(x;x)$ the (n,m) component of the propagator matrix $\hat{\mathcal{M}}$ (374). The last thing we need before we can calculate the one loop contribution (391) are the propagators at coincidence and their derivatives. Since all propagators are related to the propagator $\hat{\Delta}_n$, we only need that one.

9.3.1 Infinite space contribution

We shall regulate the infrared with the cut-off as described in section 6. The propagators thus are expressed as (266)

$$i\Delta(x;\tilde{x}) = i\Delta_\infty(x;\tilde{x}) + \sum_{N=0}^{\infty}\delta i\Delta_N(x;\tilde{x}) + \sum_{N=0}^{\infty}\delta i\Delta^N(x;\tilde{x})\,, \tag{397}$$

We first consider the infinite space contribution (199). Taking the $y\to 0$ limit of (199) and dropping the D dependent powers of y that do not contribute in dimensional regularization we obtain (we drop the subscript ∞)

$$\begin{aligned}
i\hat{\Delta}_n(x;x) = a^{D-2}i\Delta_n(x;x) &= |1-\epsilon|^{D-2}(aH)^{D-2}\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma(\frac{D-1}{2}+\nu_n)\Gamma(\frac{D-1}{2}-\nu_n)}{\Gamma(\frac{1}{2}+\nu_n)\Gamma(\frac{1}{2}-\nu_n)}\\
\frac{\partial}{\partial\eta}i\hat{\Delta}_n(x;x) &= Ha(D-2)(1-\epsilon)i\hat{\Delta}_n(x;x)\\
\Big(\frac{\partial}{\partial\eta}\Big)^2 i\hat{\Delta}_n(x;x) &= H^2a^2(D-1)(D-2)(1-\epsilon)^2\, i\hat{\Delta}_n(x;x)
\end{aligned} \tag{398}$$

We can now collect all the terms of (391). Using (369), (374), (395), (396) and (398) we obtain for the vector and the ghost

$$\begin{aligned}
\frac{1}{V}\int\frac{i}{2}{}_i\hat{\Delta}^{\rm vector}_j(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}^{ij}_{\rm vector} &= \frac{1}{4}(D-1)(D-2)\Big[2(D-1)(D+2)\\
&\quad -\,(D^2+D+2)\epsilon-(D-1)(D-2)\epsilon^2\Big]aH^2 i\hat{\Delta}_1(x;x)\\
\frac{1}{V}\int -i^\alpha\hat{\Delta}^\beta_{\rm ghost}(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}^{\rm ghost}_{\alpha\beta} &= H^2a\frac{1}{2}(D-1)^2(D-2)^2\epsilon(1-\epsilon)i\hat{\Delta}_0(x;x)\\
&\quad -\frac{1}{2}(D-1)(D-2)(1-\epsilon)\Big(2(D+2)-3(D-2)\epsilon\Big)aH^2 i\hat{\Delta}_1(x;x)\,.
\end{aligned} \tag{399}$$

The last term evaluates to

$$\begin{aligned}
\frac{1}{V}\int\frac{i}{2}{}_{ij}\hat{\mathcal{M}}_{kl}(x;x)\frac{\delta}{\delta a}\hat{G}^{ijkl} = \frac{1}{V}\int\frac{i}{2}{}_{ij}\tilde{\Delta}_{kl}(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}^{ijkl}_{\rm tensor}&\\
+\;\frac{1}{2}\lambda^2\Big(\cos(\theta)^2 i\hat{\Delta}_0+\sin(\theta)^2 i\hat{\Delta}_2\Big)\frac{\delta}{\delta a}\hat{\mathcal{D}}_{\rm scalar}&\\
+\;\lambda\cos(\theta)\sin(\theta)(i\hat{\Delta}_0-i\hat{\Delta}_2)\frac{\delta}{\delta a}\Big(\sqrt{\kappa}\frac{2}{D-3}\mathcal{Z}-\sqrt{\kappa}\Phi''\Big)\Big)&\\
+\;\frac{1}{2}\Big(\sin(\theta)^2 i\hat{\Delta}_0+\cos(\theta)^2 i\hat{\Delta}_2\Big)\frac{\delta}{\delta a}\hat{\mathcal{D}}_\phi\,.&
\end{aligned} \tag{400}$$

The contribution from the tensor is

$$\frac{1}{8}D(D-1)(D-2)^2\Big[(1+3D)-3(D-1)\epsilon\Big]\epsilon aH^2 i\hat{\Delta}_0(x;x)\,, \tag{401}$$

while the terms in (400) multiplying $i\hat{\Delta}_0$ contribute as

$$\begin{aligned}
\frac{\epsilon}{4(D-3+\epsilon)}\Big[&-(D-1)(D-3)(D^2-8D+4)+[D(D-2)(D^2-11D+14)+8]\epsilon\\
&+(D-1)(D-2)(D+2)\epsilon^2\Big]aH^2 i\hat{\Delta}_0(x;x).
\end{aligned} \tag{402}$$

Finally the terms in (400) multiplying $i\hat{\Delta}_2$ contribute as

$$\frac{1}{4(D-3+\epsilon)}\Big[4(D-1)(D-2)(D-3)(D+3)-\Big(5D^4-20D^3-9D^2+68D-36\Big)\epsilon$$
$$+\Big(D^4-5D^3-4D^2+24D-32\Big)\epsilon^2+\Big(D^3-5D^2+8D+4\Big)\epsilon^3\Big]aH^2 i\hat{\Delta}_2(x;x)\,. \tag{403}$$

Putting (399–403) and (391) together and expanding the result around $D=4$ we obtain the following non renormalized one loop contributions to the Friedmann trace equation,

$$\begin{aligned}
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma^0_{1L}}{\delta a} &= \frac{\epsilon(63\epsilon^2+2\epsilon-105)}{64\pi^2(1+\epsilon)}(1-\epsilon)^2(4\nu_0^2-1)\Bigg(\frac{2}{D-4}+\gamma_E+\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)\\
&\qquad +\psi(\frac{1}{2}+\nu_0)+\psi(\frac{1}{2}-\nu_0)+4\frac{4\nu_0\nu_0'-1}{4\nu_0^2-1}\Bigg)\\
&\quad +\frac{\epsilon}{32\pi^2(1+\epsilon)^2}(1-\epsilon)^2(4\nu_0^2-1)\Big(93\epsilon^3+90\epsilon^2-169\epsilon-122\Big)\\
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma^1_{1L}}{\delta a} &= \frac{3(9\epsilon^2-7\epsilon-6)}{64\pi^2}(1-\epsilon)^2(4\nu_1^2-1)\Bigg(\frac{2}{D-4}+\gamma_E+\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)\\
&\qquad +\psi(\frac{1}{2}+\nu_1)+\psi(\frac{1}{2}-\nu_1)+4\frac{4\nu_1\nu_1'-1}{4\nu_1^2-1}\Bigg)\\
&\quad +\frac{1}{64\pi^2}(1-\epsilon)^2(4\nu_1^2-1)\Big(51\epsilon^2-17\epsilon-54\Big)\\
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma^2_{1L}}{\delta a} &= -\frac{(5\epsilon-6)(\epsilon^2-2\epsilon-7)}{64\pi^2(1+\epsilon)}(1-\epsilon)^2(4\nu_2^2-1)\\
&\qquad \Bigg(\frac{2}{D-4}+\gamma_E+\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)\\
&\qquad +\psi(\frac{1}{2}+\nu_2)+\psi(\frac{1}{2}-\nu_2)+4\frac{4\nu_2\nu_2'-1}{4\nu_2^2-1}\Bigg)\\
&\quad +\frac{1}{64\pi^2(1+\epsilon)^2}(1-\epsilon)^2(4\nu_2^2-1)\Big(3\epsilon^4+13\epsilon^3-83\epsilon^2+35\epsilon+30\Big)
\end{aligned} \tag{404}$$

here ν_n indicates ν_n as given in (367) in $D=4$ and ν_n' is $\frac{d}{dD}\nu_n\Big|_{D=4}$. γ_E is the Euler constant and we used the following expansions of the propagators (398)

$$\begin{aligned}
aH^2 i\hat{\Delta}_n(x;x) &= a^{D-1}|1-\epsilon|^{D-2}H^D\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\Big(\Big(\frac{D-3}{2}\Big)^2-\nu_n^2\Big)\\
&\times \Big[1+\frac{D-4}{2}\Big(\psi\Big(\frac{1}{2}+\nu_n\Big)+\psi\Big(\frac{1}{2}-\nu_n\Big)\Big)\Big] \qquad (405)\\
|1-\epsilon|^{D-2}H^D\frac{\Gamma(1-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}} &= \frac{(1-\epsilon)^2H^4}{16\pi^2}\\
&\times \Big(\frac{2\mu^{D-4}}{D-4}+\gamma_E-1+\ln\Big(\frac{H^2}{\mu^2}\Big)+\ln\Big(\frac{(1-\epsilon)^2}{4\pi}\Big)\Big), \qquad (406)
\end{aligned}$$

plus terms that vanish in $D=4$. Here $\psi(z)=(d/dz)\ln(\Gamma(z))$ is the digamma function and μ is an arbitrary renormalization scale introduced for later convenience. If we use the explicit expression for ν_n, we can add all terms together and obtain for the effective action

$$\begin{aligned}
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma_{1L}}{\delta a} &= \frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma^0_{1L}}{\delta a}+\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma^1_{1L}}{\delta a}+\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma^2_{1L}}{\delta a}\\
&= -\frac{\epsilon\Big(186-149\epsilon-11\epsilon^2+10\epsilon^3\Big)}{8\pi^2}\frac{\mu^{D-4}}{D-4}\\
&- \frac{\epsilon}{16\pi^2}\Big[\Big(108+62\epsilon-153\epsilon^2+27\epsilon^3\Big)+\Big(186-149\epsilon-11\epsilon^2+10\epsilon^3\Big)\\
&\qquad\times\Big(\gamma_E+\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)\Big)\Big]\\
&+ \frac{\epsilon(63\epsilon^2+2\epsilon-105)}{64\pi^2(1+\epsilon)}(1-\epsilon)^2(4\nu_0^2-1)\Big(\psi(\frac{1}{2}+\nu_0)+\psi(\frac{1}{2}-\nu_0)\Big)\\
&+ \frac{3(9\epsilon^2-7\epsilon-6)}{64\pi^2}(1-\epsilon)^2(4\nu_1^2-1)\Big(\psi(\frac{1}{2}+\nu_1)+\psi(\frac{1}{2}-\nu_1)\Big)\\
&- \frac{(5\epsilon-6)(\epsilon^2-2\epsilon-7)}{64\pi^2(1+\epsilon)}(1-\epsilon)^2(4\nu_2^2-1)\Big(\psi(\frac{1}{2}+\nu_2)+\psi(\frac{1}{2}-\nu_2)\Big),
\end{aligned}$$
(407)

We kept the digamma functions in terms of ν, since then it will be trivial to add the correction terms to regulate the infrared.

9.3.2 Renormalization

The contribution (407) contains a $1/(D-4)$ divergence and therefore needs to be renormalized. If we take our approximation that $\epsilon=$ constant literally, the divergence is of the form const$\times H^4a^{D-1}$. However in a more realistic treatment, ϵ is a dynamical parameter and our result is expected to be correct up to zeroth order in $\dot{\epsilon}$. In this more realistic case, the ϵ structure of the divergent term should be taken into account and subtracted accordingly. Therefore if ϵ is varying slowly enough, such that (407) remains approximately valid, we still need a counter lagrangian that produces the same ϵ structure as the divergence in (407), in order for the theory to be renormalized at all times. For this purpose many terms can be used, and indeed many terms are reported in the literature [85, 111, 141]. However we cannot simply use these terms, since counterterms are dependent on the gauge fixing used [142] and as far as we know, no general calculations have been done using our gauge fixing (357). Calculations using (357) have been done however in the special case of de Sitter space [138] [98], considering both scalar and graviton loops. From these works and also e.g. from [135] it follows that the one loop contribution in this limit should be finite. Since the de Sitter limit is $\epsilon \to 0$, this agrees with (407). To ensure that the one loop contribution due to gravitons in the de Sitter case vanishes we need in that limit a counterterm

$$\sqrt{-g}\Big(H^2 - \frac{\Lambda}{D-1}\Big)^2 , \tag{408}$$

which in our more general case becomes

$$\mathcal{L}_{CT1} = \sqrt{-g}\, a_0\Big(R - \frac{D}{D-2}\Big(\kappa V(\Phi) + (D-2)\Lambda\Big)\Big)^2 , \tag{409}$$

where a_0 is a constant. This follows from the fact that the cosmological constant can always be seen as a part of the scalar potential and thus for an invariant counterterm they should come together. Moreover from [98] it follows that we also need a counterterm of the form $\sqrt{-\hat{g}}H^2\kappa\hat{g}^{\mu\nu}(\partial_\mu\hat{\phi})(\partial_\nu\hat{\phi})$ to vanish. From Eq. (13) it follows that for our case this generalizes to

$$\mathcal{L}_{CT2} = \sqrt{-g}\, a_1(Rg^{\mu\nu} - DR^{\mu\nu})\kappa(\partial_\mu\Phi)(\partial_\nu\Phi) , \tag{410}$$

with a_1 a constant. Finally, also from [98] it follows that a counterterm $\sqrt{-g}\kappa(\Box\Phi)^2$ should not appear.
Therefore a reasonable choice for the counter-lagrangian is

$$\begin{aligned}\mathcal{L}_c = \sqrt{-g}\Big[a_0\Big(R - \frac{D}{D-2}(\kappa V + (D-2)\Lambda)\Big)^2 + a_1\Big(Rg^{\mu\nu} - DR^{\mu\nu}\Big)\kappa\partial_\mu\Phi\partial_\nu\Phi \\ + a_2\frac{\partial^2 V(\Phi)}{\partial\Phi^2}R + a_3\kappa g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)\frac{\partial^2 V(\Phi)}{\partial\Phi^2}\Big] .\end{aligned} \tag{411}$$

We stress that the counter-lagrangian (411) *for the purpose of this calculation* could be chosen differently. There are many other terms with the correct dimensionality that could have been used [141]. Since the divergence (407) gives only four constraints (one for each power of ϵ), we at present can fix only 4 coefficients. This does not mean that the counter-lagrangian is arbitrary. Different types of calculations could fix the counter-lagrangian uniquely, as it is for example done in Ref. [111]. However apart from the two cases mentioned above, we do not know of any calculation in our gauge, which we could use to further specify our counter-lagrangian. Thus the 'true' counter-lagrangian corresponding to the theory will probably contain different counterterms than (411). However, these different counterterms do not change the conclusions of this section. The only effect would be that in Eq. (414) the origin of the β_i's changes, but not the fact that they are essentially arbitrary. Neither do different counterterms change the fact that the logarithmic terms, the digamma functions and the IR corrections cannot be subtracted by local counterterms. The terms in our counter-lagrangian (411) contribute as follows to the Friedmann trace equation

$$
\begin{aligned}
&\frac{1}{V}\frac{\delta}{\delta a}\int d^Dx\sqrt{-g}\Big(R-\frac{D}{D-2}(\kappa V+(D-2)\Lambda)\Big)^2\\
&\qquad = a^{D-1}H^4(D-2)\epsilon\Big(8\epsilon(2+3\epsilon)+D^2(4+9\epsilon)-2D(2+13\epsilon+12\epsilon^2)\Big)\\
&\qquad\quad +\mathcal{O}(\epsilon')\\
&\qquad = a^{D-1}H^4\Big(16\epsilon(6+7\epsilon-9\epsilon^2)+4\epsilon(26+37\epsilon-30\epsilon^2)(D-4)\\
&\qquad\quad +\mathcal{O}((D-4)^2,\epsilon')\Big)\\
&\frac{1}{V}\frac{\delta}{\delta a}\int d^Dx\kappa\sqrt{-g}(Rg^{\mu\nu}-DR^{\mu\nu})(\partial_\mu\Phi)(\partial_\nu\Phi)\\
&\qquad = 2a^{D-1}H^4(D-1)(D-2)^2\epsilon\Big((D-1)(D-6\epsilon)+6\epsilon^2\Big)+\mathcal{O}(\epsilon')\\
&\qquad = a^{D-1}H^4\Big(144\epsilon(1-\epsilon)(2-\epsilon)+24\epsilon(23-30\epsilon+8\epsilon^2)(D-4)\\
&\qquad\quad +\mathcal{O}((D-4)^2,\epsilon')\\
&\frac{1}{V}\frac{\delta}{\delta a}\int d^Dx\sqrt{-g}\frac{\partial^2V(\Phi)}{\partial\Phi^2}R\\
&\qquad = a^{D-1}H^4(D-1)(D-1-\epsilon)2\epsilon\Big(D(D-2)-2(3D-4)\epsilon+12\epsilon^2\Big)+\mathcal{O}(\epsilon')\\
&\qquad = a^{D-1}H^4\Big(24\epsilon(3-\epsilon)(2-4\epsilon+3\epsilon^2)\\
&\qquad\quad +4\epsilon\big(51-88\epsilon+53\epsilon^2-6\epsilon^3\big)(D-4)\Big)+\mathcal{O}\Big((D-4)^2,\epsilon'\Big)\\
&\frac{1}{V}\frac{\delta}{\delta a}\int d^Dx\kappa\sqrt{-g}g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)\frac{\partial^2V(\Phi)}{\partial\Phi^2}\\
&\qquad = -4a^{D-1}H^4(D-2)^2\epsilon^2(D-1-\epsilon)+\mathcal{O}(\epsilon')\\
&\qquad = a^{D-1}H^4\Big(-16\epsilon^2(3-\epsilon)-16\epsilon^2(4-\epsilon)(D-4)\Big)+\mathcal{O}\Big((D-4)^2,\epsilon'\Big)\,,
\end{aligned}
\tag{412}
$$

where we used (13) and once again we used the background equations of motion (352) and (353). We find that all divergencies cancel if

$$
\begin{aligned}
a_0 &= \frac{37}{960\pi^2}\frac{\mu^{D-4}}{D-4}+a_0^f \quad ; \quad a_1=\frac{49}{640\pi^2}\frac{\mu^{D-4}}{D-4}+a_1^f\\
a_2 &= -\frac{5}{288\pi^2}\frac{\mu^{D-4}}{D-4}+a_2^f \quad ; \quad a_3=-\frac{43}{480\pi^2}\frac{\mu^{D-4}}{D-4}+a_3^f\,,
\end{aligned}
\tag{413}
$$

where the a_i^f $(i=0,1,2,3)$ indicates a possible finite part. Adding the contribution from the counterterms (412) to the one loop contribution (407), we obtain the following renormalized contribution

$$\begin{aligned}\frac{1}{a^3V}\frac{\Gamma_{1L,\mathrm{ren}}}{\delta a} &= \frac{H^4}{16\pi^2}\Bigg[\beta_1\epsilon+\beta_2\epsilon^2+\beta_3\epsilon^3+\beta_4\epsilon^4 \\ &-\epsilon\Big(186-149\epsilon-11\epsilon^2+10\epsilon^3\Big)\Big(\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)\Big) \\ &+\frac{\epsilon(63\epsilon^2+2\epsilon-105)}{4(1+\epsilon)}(1-\epsilon)^2(4\nu_0^2-1)\Big(\psi(\frac{1}{2}+\nu_0)+\psi(\frac{1}{2}-\nu_0)\Big) \\ &+\frac{3(9\epsilon^2-7\epsilon-6)}{4}(1-\epsilon)^2(4\nu_1^2-1)\Big(\psi(\frac{1}{2}+\nu_1)+\psi(\frac{1}{2}-\nu_1)\Big) \\ &-\frac{(5\epsilon-6)(\epsilon^2-2\epsilon-7)}{4(1+\epsilon)}(1-\epsilon)^2(4\nu_2^2-1)\Big(\psi(\frac{1}{2}+\nu_2)+\psi(\frac{1}{2}-\nu_2)\Big)\Bigg]\,.\end{aligned} \tag{414}$$

Here the parameters β_i ($i=1,2,3,4$) are given by,

$$\begin{aligned}\beta_1 &= 16\pi^2\times 12\Big[8a_0^f+24a_1^f+12a_2^f\Big]-186\gamma_E+\frac{1727}{3} \\ \beta_2 &= 16\pi^2\times 4\Big[28a_0^f-108a_1^f-84a_2^f-12a_3^f\Big]+149\gamma_E-\frac{5969}{9} \\ \beta_3 &= 16\pi^2\times 4\Big[-36a_0^f+36a_1^f+78a_2^f+4a_3^f\Big]+11\gamma_E+\frac{10457}{45} \\ \beta_4 &= 16\pi^2\Big[-72a_2^f\Big]-10\gamma_E-\frac{61}{3}\,.\end{aligned} \tag{415}$$

All β_i's ($i=1,2,3,4$) in Eq. (414) are free parameters that remain undetermined until they are fixed by experiment. Other terms in Eq. (414), in particular the logarithm and polygamma functions, cannot be altered by local counterterms and hence these terms constitute the physical graviton plus massless scalar one loop contributions in homogeneous expanding spaces with a power law expansion characterized by a constant $\epsilon=-\dot{H}/H^2$.

9.3.3 Correction terms

The effective action (414) is divergent for half integer values of the parameters ν_i. The reason is that the propagators used do not describe the infrared physics correctly and we need to add the correction term $\delta i\Delta_N$ as given in (268). Since we will be considering accelerating space-times, we do not care for the $\delta i\Delta^N$ corrections, since they become quickly insignificant in that case. Since the correction term is ultraviolet finite, we can put $D=4$ in all terms. Following the same procedure as above, we find for the coincident limit and its derivatives

$$\begin{aligned}\delta i\hat{\Delta}_{N,n}(x;x) &= A_{N,n}(aH)^2z_0^{2N+3-2\nu_n} \\ \frac{\partial}{\partial\eta}\delta i\hat{\Delta}_{N,n}(x;x) &= -(1+2N-2\nu_n)(1-\epsilon)(aH)\delta i\hat{\Delta}_{N,n}(x;x) \\ \Big(\frac{\partial}{\partial\eta}\Big)^2\delta i\hat{\Delta}_{N,n}(x;x) &= (1+2N-2\nu_n)(2N-2\nu_n)(1-\epsilon)(aH)\delta i\hat{\Delta}_{N,n}(x;x)\end{aligned} \tag{416}$$

where ν_n is given in (367) and a subscript n implies that that quantity is evaluated with ν_n. $A_{N,n} = A_N(\nu_n)$ is defined in (285). We find for the corrections due to the vector and the ghost (396)

$$\begin{aligned}
&\frac{1}{V}\int \frac{i}{2}(\delta_i\hat{\Delta}_j^{\rm vector})_N(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}^{ij}_{\rm vector} = 3\Bigg(4+7(1-\epsilon)\\
&\quad -5(1-\epsilon)(1+2N-2\nu_1)-(1-\epsilon)^2(N-\nu_1)(1+2N-2\nu_1)\Bigg)aH^2\delta i\hat{\Delta}_{N,1}(x;x)\\
&\frac{1}{V}\int -i({}^\alpha\hat{\Delta}^\beta_{\rm ghost})_N(x;x)\frac{\delta}{\delta a}\hat{\mathcal{D}}^{\rm ghost}_{\alpha\beta} = 6(1-\epsilon)(N-\nu_0)\\
&\quad\times\Big((2\nu_0-2N)(1-\epsilon)+\epsilon-3\Big)aH^2\delta i\hat{\Delta}_{N,0}(x;x)\\
&\quad +6(1-\epsilon)(N-\nu_1)\Big((2\nu_1-2N)(1-\epsilon)+\epsilon+1\Big)aH^2\delta i\hat{\Delta}_{N,1}(x;x).
\end{aligned} \tag{417}$$

The tensor contribution to (400) is

$$-6\big(3(\nu_0-N)(1-\epsilon)+2\big)\big(2(\nu_0-N)(1-\epsilon)-3\big)aH^2\delta i\hat{\Delta}_{N,0}(x;x)\,, \tag{418}$$

while the terms in (400) multiplying $\delta i\hat{\Delta}_{N,0}$ contribute as

$$\begin{aligned}
&\frac{1}{1+\epsilon}\Bigg(-4(3-\epsilon)\epsilon+(1-\epsilon)^2(1+3\epsilon)(2N^2+2\nu_0^2)-(3+\epsilon)(1-\epsilon)(1-3\epsilon)\\
&\quad +N(1-\epsilon)\Big(3-4\nu_0-8\epsilon(1+\nu_0)-3\epsilon^2(1-4\nu_0)\Big)\Bigg)aH^2\delta i\hat{\Delta}_{N,0}(x;x).
\end{aligned} \tag{419}$$

Finally the terms in (400) multiplying $\delta i\hat{\Delta}_{N,2}$ contribute as

$$\begin{aligned}
&\frac{1}{1+\epsilon}\Bigg((1-\epsilon)^2(3+\epsilon)(2N^2+2\nu_1^2)+2(3+5\epsilon(1-\epsilon)+\epsilon^3)\\
&\quad +(1-\epsilon)(3+\epsilon)\Big((5+\epsilon)(1-N)-4(1-\epsilon)N\nu_1\Big)\Bigg)aH^2\delta i\hat{\Delta}_{N,2}(x;x).
\end{aligned} \tag{420}$$

We add the corrections together to obtain the following three contributions

$$\begin{aligned}
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma_N^{(0)}}{\delta a} &= \Bigg(-\frac{\epsilon(63\epsilon^2+2\epsilon-105)}{1+\epsilon} + \frac{1-\epsilon}{1+\epsilon}(3+2N-2\nu_0) \\
&\qquad \times \Big((\nu_0-N)(1-\epsilon)(23+21\epsilon) + (4+3\epsilon)(3-7\epsilon) \Big) \Bigg) \frac{\delta i\Delta_{N,0}}{H^2} \\
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma_N^{(1)}}{\delta a} &= \Bigg(-3(9\epsilon^2+7\epsilon+6) \\
&\qquad + 9(1-\epsilon)(3+2N-2\nu_1)\Big((\nu_1-N)(1-\epsilon) - \epsilon \Big) \Bigg) \frac{\delta i\Delta_{N,1}}{H^2} \\
\frac{1}{Va^{D-1}H^4}\frac{\delta\Gamma_N^{(2)}}{\delta a} &= \Bigg(\frac{(5\epsilon-6)(\epsilon^2-2\epsilon-7)}{1+\epsilon} \\
&\qquad - \frac{(1-\epsilon)(3+\epsilon)}{1+\epsilon}(3+2N-2\nu_2)\Big((\nu_2-N)(1-\epsilon) + 4 - \epsilon \Big) \Bigg) \frac{\delta i\Delta_{N,2}}{H^2},
\end{aligned} \tag{421}$$

where we have written $\delta\Gamma_N^{(n)}$ for the contribution to $\delta\Gamma_N$ that multiplies $\Delta_{N,n}$.
From these three contributions we see explicitly that indeed when $\nu = N + 3/2$, the corrections have the correct prefactor to add up correctly to cancel the divergence in the digamma functions (414). This gives the final one loop corrected Friedmann trace

equation

$$
\begin{aligned}
&\frac{24}{\kappa}\Big((1-\frac{1}{2}\epsilon)H^2-\frac{1}{3}\Lambda\Big)+3p_M-\rho_M+\frac{H^4}{16\pi^2}\Bigg\{\beta_1\epsilon+\beta_2\epsilon^2+\beta_3\epsilon^3+\beta_4\epsilon^4\\
&-\epsilon\Big(186-149\epsilon-11\epsilon^2+10\epsilon^3\Big)\ln\Big(\frac{(1-\epsilon)^2H^2}{4\pi\mu^2}\Big)\\
&+\frac{\epsilon(63\epsilon^2+2\epsilon-105)}{4(1+\epsilon)}(1-\epsilon)^2\\
&\qquad\times\Bigg((4\nu_0^2-1)\Big(\psi(\frac{1}{2}+\nu_0)+\psi(\frac{1}{2}-\nu_0)\Big)-\sum_N\frac{64\pi^2}{(H(1-\epsilon))^2}\delta i\Delta_{N,0}\Bigg)\\
&+\frac{3(9\epsilon^2-7\epsilon-6)}{4}(1-\epsilon)^2\\
&\qquad\times\Bigg((4\nu_1^2-1)\Big(\psi(\frac{1}{2}+\nu_1)+\psi(\frac{1}{2}-\nu_1)\Big)-\sum_N\frac{64\pi^2}{(H(1-\epsilon))^2}\delta i\Delta_{N,1}\Bigg)\\
&-\frac{(5\epsilon-6)(\epsilon^2-2\epsilon-7)}{4(1+\epsilon)}(1-\epsilon)^2\\
&\qquad\times\Bigg((4\nu_2^2-1)\Big(\psi(\frac{1}{2}+\nu_2)+\psi(\frac{1}{2}-\nu_2)\Big)-\sum_N\frac{64\pi^2}{(H(1-\epsilon))^2}\delta i\Delta_{N,2}\Bigg)\Bigg\}\\
&+\sum_N(3+2N-2\nu_0)\Bigg(\frac{1-\epsilon}{1+\epsilon}\Big((\nu_0-N)(1-\epsilon)(23+21\epsilon)+(4+3\epsilon)(3-7\epsilon)\Big)\Bigg)H^2\delta i\Delta_{N,0}\\
&+\sum_N(3+2N-2\nu_1)\Bigg(9(1-\epsilon)\Big((\nu_1-N)(1-\epsilon)-\epsilon\Big)\Bigg)H^2\delta i\Delta_{N,1}\\
&-\sum_N(3+2N-2\nu_2)\Bigg(\frac{(1-\epsilon)(3+\epsilon)}{1+\epsilon}\Big((\nu_2-N)(1-\epsilon)+4-\epsilon\Big)\Bigg)H^2\delta i\Delta_{N,2}=0\,.
\end{aligned}
\tag{422}
$$

This equation is our final result of this section. It presents the one loop quantum corrected Friedmann trace equation, in the presence of both graviton and scalar fluctuations.

9.4 Discussion

Having found the one loop corrected Friedmann trace equation, we can ask the question whether the quantum effects calculated will have any significant effect. We immediately see that the quantum contribution is suppressed by a factor H^2/m_p^2. Because of this suppression, the one loop contribution can only become relevant for the dynamics of the universe if there is a significant enhancement. This enhancement could in the case at hand come from the infrared growing terms. The question whether this enhancement is significant is equivalent to the question whether the quantity

$$H^2 z_0^{3+2N-2\nu} \tag{423}$$

grows in time. In an accelerating universe the fastest growing term has $N = 0$. Since $z_0 = -k\eta$ this term grows with η as

$$\propto \eta^{\frac{3-\epsilon}{1-\epsilon}-2\nu}. \tag{424}$$

Thus we see that for the different ν values we have in (422) this evaluates to

$$\begin{aligned} \eta^{\frac{3-\epsilon}{1-\epsilon}-2\nu_0} &= \eta^0 \\ \eta^{\frac{3-\epsilon}{1-\epsilon}-2\nu_1} &= \eta^2 \\ \eta^{\frac{3-\epsilon}{1-\epsilon}-2\nu_2} &= \eta^4. \end{aligned} \tag{425}$$

These general expressions fail near the $N = 0$ pole of the digamma functions (meaning $\nu_i = 3/2$). Here one will typically pick up an additional logarithm, $\ln(z_0)$ from the correction terms. Thus we see that the ν_0 contribution, in the presence of such an additional logarithm appears to grow in time. However $\nu_0 = 3/2$ implies $\epsilon = 0$ and we see from (422) that this contribution is cancelled by the pre-factor. Thus none of the contributions actually grows in time. Notice that this is similar to the conclusions made in section 7, but not exactly the same. In that case the pre-factor was such that if $\xi = 0$ (as is the case for ν_0), the leading order $N = 0$ contribution is zero for all ν. Indeed one immediately sees from (275) that if Δ_N is constant, it will not contribute in that case. In the calculation above, the relation between the trace of the stress energy tensor and the coincident propagators is much more complicated. For example in (395) we see that also undifferentiated factors of the coincident propagator contribute.

The fact that the pre-factor is such that near de Sitter space the logarithmic growth drops out, is also in concordance with results obtained in [140]. This property however does not have to stay true in general. For example if higher order loop corrections are taken into account, this structure might change such that while the correction terms are extremely small, their contribution grows in time, making them significant at late enough times. Also when considering a different background geometry, one might get growing effects even at one loop order. In [49, 50] for example one loop contributions have been considered in chaotic inflationary models. Although those works found a secular growth, in [53] it was concluded that this growth disappears if one considers truly gauge invariant quantities. Notice that in chaotic inflation models one typically has in the slow-roll approximation $\epsilon \propto \frac{m_p^2}{\phi(t)^2}$, with ϕ the inflaton field.

10 Conclusions

In this thesis we studied the infrared properties of quantum fields on a cosmological background. In particular we considered free massless scalar fields, with a possible coupling ξ to the Ricci scalar. The study of such a field is first of all motivated by the graviton, whose kinetic operator can be written in a similar form. Second of all, light scalar fields can be considered effectively massless in the early universe (where the mass has to be compared with the Hubble parameter H). This can for example include the Higgs field, quintessence fields, the axion or scalar fields arising in scalar-tensor theories of gravity. Notice however that, since the mass of a scalar field is not protected by any symmetry, loop corrections might generate a significant mass for these fields. For example in de Sitter space calculations have been done, for a massless minimally coupled scalar with quartic self interactions [99], or for a massless minimally coupled charged scalar coupled to the photon [88]. In both cases a generated mass squared is found to be proportional to H^2 and therefore such a field can not be considered light anymore. However flat directions in supersymmetric models can be considered massless, since they do not interact and hence cannot acquire quantum contributions. Also notice that the mass of the Higgs field needs to be protected anyway if one wants to solve the hierarchy problem.

Our background space-time is a homogeneous, flat FLRW space-time (8), with the additional constraint that the parameter $\epsilon \equiv -\frac{\dot{H}}{H^2}$ is a constant (11). Although this constraint, which implies that the acceleration of the universe is constant, is necessary in order to perform analytic calculations, it is quite reasonable. The Friedmann equations (14) imply that, if the energy density of the universe is dominated by one fluid, with a constant equation of state parameter w (6), one immediately finds that ϵ is constant [15]. It has been known for a long time that a massless scalar field on such a background possesses problems in the infrared [42, 82] if the parameter ν (178), given in D space-time dimensions by

$$\nu^2 = \Big(\frac{D-1-\epsilon}{2(1-\epsilon)}\Big)^2 - \frac{(D-1)(D-2\epsilon)}{(1-\epsilon)^2}\xi \tag{426}$$

is larger then $\frac{D-1}{2}$ (186). To be more precise, the propagator, which measures the correlation of the field between two different points, diverges at the lower end of the integration over the momenta when evaluated for the Bunch-Davies vacuum [33]. What physically happens is that the expansion of the universe implies the existence of a cosmological horizon. Virtual particles that are created due to the Heisenberg uncertainty relation with a wavelength longer than this horizon fail to annihilate each other within a finite time [23, 43]. Thus nonconformal scalar modes, with a super Hubble wavelength, couple strongly to the expanding background. This causes the creation of particles and the growth of large scale, super Hubble, field correlations. Such an effect is naturally the largest if the field is massless. The fact that the propagator is infrared divergent does not indicate that infrared particle creation is not a real physical processes, instead it means that the way the propagator is calculated is not physical. The unphysical thing in this specific case is that the Bunch-Davies vacuum cannot describe the super-Hubble vacuum correctly due to the particle creation. In this thesis we presented a fix to this problem. In this fix we effectively choose the initial vacuum to be less singular in the infrared. If this is done, such that the infrared contribution to the

propagator is finite initially, it will naturally stay finite at all later times.
The resolution we propose to the infrared problems we presented in section 6 [38]. The idea here is to work on a spatially compact universe, with a co-moving size $2\pi/k_0$. This effectively generates an infrared cut-off in the integral over the mode functions. The final result for the propagator of a massless scalar field in a constant ϵ background (266) then naturally is infrared finite.
To see what the physical effect of this approaches is, we calculated the expectation value of the stress energy tensor in section 7. The stress energy tensor couples to the metric through the Einstein equations and thus any quantum contribution to the stress energy tensor in principle also induces a change to the metric. In a cosmological setting this implies that quantum effects will either accelerate or slow the expansion of the universe. This is known as backreaction. Of course this effect is small, but because particle creation – which is a physical effect – is cumulative, it is at least in principle possible that the expectation value of the stress energy grows in time. Therefore an initially extremely small effect can significantly influence the expansion of the universe after a sufficient amount of time.
Using the infrared regulated propagator, we find in the ultraviolet that we correctly reproduce the Bunch-Davies result. This should of course happen; we do not want to change ultraviolet physics because of a problem with long wavelength particles. In the infrared the discussion can be split in two parts, depending on whether the expansion of the universe is accelerating or decelerating.
If the universe is accelerating ($\epsilon < 1$), we found that, although we see a growth in energy due to the production of infrared particles, this energy will always dilute away faster than the energy present in the background space-time. This dilution is due to the fact that any infrared mode ripped out of the vacuum will be redshifted and thus lose energy. We could however enhance the effect by allowing the coupling ξ between the scalar field and the Ricci scalar to become negative. This is however not very surprising, such a negative coupling will act as a (time dependent) negative mass and thus destabilize the field, leading to a growth.
In a decelerating universe the situation is quite different. We find that, since in a decelerating unverse, the Hubble radius grows faster then physical scales, the cut-off scale actually enters the Hubble volume. While the infrared divergence is determined by the parameter ν, the relative growth of the cut-off with respect to the Hubble parameter is determined by ϵ. We indeed find in this case that the correction terms due to the cut-off can grow faster in time with respect to the background energy density if $\epsilon > 3$. This effect is thus not due to particle creation, but it is due to the fact that the influence of the cut-off grows more and more profound as time goes on. The cut-off removes modes that would otherwise influence the expansion of the universe. Moreover it does this even in space-times where there is no infrared divergence. Of course this was expected. Requiring that the universe is compact is a physical change, which leads to observable effects. Notice also that during inflation, the physical cut-off $k_0\eta$ becomes much smaller than the Hubble scale as time goes on. Unless k_0 was initially very large, the cut-off $k_0\eta$ will very likely up to today stay small enough, similar to how inflation makes any initial curvature of the universe negligibly small at later times. Therefore it is unlikely that the late time behavior described above is already relevant for our universe, and thus this approach is perfectly consistent with observations.
It is natural to ask how general these conclusions are. Firstly let us comment on the

dependence on the infrared regularization scheme. As we mentioned in section 5, there are several other schemes proposed in the literature [40, 41, 72]. All these schemes effectively introduce some infrared scale κ one way or the other. Modes with momenta lower then this scale are then suppressed, making the infrared finite, while modes with momenta larger then this scale should be unaffected. In an accelerating universe we do not expect much difference between the approach presented here and any other approach. In this case physical length scales grow faster then the Hubble radius, such that any details on what happens at length scale larger then κ^{-1} become less and less visible. This is in agreement with the conclusions made in [122], where the one loop expectation value of the stress energy tensor in de Sitter space indeed showed qualitatively identical scaling with time, independent of the precise state chosen, as long as the state is infrared finite and possesses the correct ultraviolet behavior. In a decelerating space-time, after long enough time, the situation might be different. The cut-off length κ^{-1} eventually enters the Hubble volume and therefore might significantly influence physics. We saw that in the case studied in this thesis: the late time behavior in a decelerating space-time was dominated not by infrared effects, but by effects due to the finite size of the universe. What the effect of a different regularization is in a decelerating space-time remains to be studied.
Secondly, while in our one loop calculations the growth of energy is always cancelled by the redshift of the modes, this does not have to remain true in more general cases. In particular studies in de Sitter space show that, if higher order loop corrections are taken into account, the energy density might obtain a contribution that grows logarithmically (while in de Sitter space, the background energy is constant) [54, 37].
As a second application we used the cut-off regulated propagator to calculate the one-loop effective potential for a massless scalar field, with a coupling ξ to the Ricci scalar and a quartic self interaction. We calculated the effective potential by assuming that the background field Φ has the same time dependence as H. While this is true at the classical level, this scaling is broken by the quantum corrections. This can be understood from the fact that the ultraviolet sector of the theory introduces a renormalization scale μ, while the infrared introduces the cut-off scale k_0. While H depends on time, k_0 and μ are constants, and therefore requiring that Φ scales as H will not solve the one loop corrected equation of motion (with de Sitter space, where H is constant, as an exception). However, as long as quantum contributions are small, we can still trust our result. What we find is that initially, the quantum corrections can induce a phase transition, similar as in flat space [120], by generating minima in the potential for nonzero Φ. However the growing infrared contributions are such that after a sufficient amount of time the effective potential will always end up with only one minimum at $\Phi = 0$. Thus the infrared modes have restored the symmetry of the potential.
Finally we calculated in section 9 the one loop effective action in a theory where both gravity and a scalar field produce quantum fluctuations. The mixing between the dynamical degrees of freedom in such a case leads to more complicated expressions, but in the end, it turns out that – up to some tensor structure – all kinetic operators can be written as the kinetic operator of a massless free scalar field with a coupling to the Ricci scalar. Therefore we could use the propagators calculated in section 6 to calculate the one loop effective action, without infrared singularities. The final answer (422) is very similar to the obtained expectation value of the stress energy tensor for just a single massless scalar field. Also for the graviton we find that at one loop order all growth in

energy due to the creation of infrared modes is cancelled by the redshift of these modes. While in this thesis we constructed the propagator for a scalar field in a wide class of FLRW backgrounds, the problem is still unsolved for a general FLRW space-time. All our results are only valid if ϵ is constant and it would therefore be very interesting to see how the propagator is affected if the time-dependence of ϵ is taken into account. Solving this problem is important first of all to answer the question whether it remains true that there is no secular growth at one loop order. Second of all, it is important because *if* one finds a growing contribution, any significant back-reaction makes the constant ϵ calculation untrustworthy. Knowing the propagator for a more general FLRW space-time would solve this problem. One approach that might prove useful in this context has been initiated in [118, 143, 144]. The idea is to match two space-times both with a constant, but slightly different value of ϵ onto each other. By considering many subsequent matchings, one approaches in the limit where the difference between two steps is infinitesimal the limit where ϵ changes continuously in time. Although this approach is technically involving, in principle it can be used to calculate the infrared contributions to the propagator in a general FLRW space-time.

The study of quantum fields on cosmological space-times is a fascinating one, with a very rich structure. The models studied in this thesis already show interesting physics. Further work in this field will most probably lead to technical advances in the future, which will allow us to delve deeper into this rich structure. A better understanding of this structure can then eventually turn out to be crucial for a proper understanding of the universe itself.

A Conventions and definitions

Our metric convention is (-,+,+,+). and $\hbar = c = 1$. Most of this thesis uses the flat FLRW metric, with the metric given by

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2 \tag{427}$$

Or using conformal time η

$$\begin{aligned} ds^2 &= a(\eta)^2(-d\eta^2 + d\vec{x}^2) \\ &\equiv a(\eta)^2 \eta_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{428}$$

with $\eta_{\mu\nu}$ the Minkowski metric. We shall sometimes use for convenience a modification of the Minkowski metric, where the (00) component is zero

$$\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu. \tag{429}$$

Derivatives are indicated as

$$\dot{x} \equiv \frac{\partial}{\partial t} x \qquad ; \qquad x' \equiv \frac{\partial}{\partial \eta} x \tag{430}$$

The Christoffel connection and the curvature tensors are in general given by

$$\begin{aligned} \Gamma^\alpha_{\mu\nu} &= \frac{1}{2} g^{\alpha\beta}\Big(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\Big) \\ R^\rho{}_{\mu\sigma\nu} &= \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\rho_{\sigma\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\sigma\mu} \\ R_{\mu\nu} &\equiv R^\rho{}_{\mu\rho\nu} \qquad\qquad R \equiv g^{\mu\nu} R_{\mu\nu} \end{aligned} \tag{431}$$

In the specific case of of flat FLRW we have in D space-time dimensions

$$\begin{aligned} R^\alpha{}_{\mu\beta\nu} =& -H^2 a^2 \Bigg(\Big(\delta^\alpha_\nu \eta_{\mu\beta} - \delta^\alpha_\beta \eta_{\mu\nu}\Big) + \epsilon\Big(\delta^\alpha_\nu \delta^0_\mu \delta^0_\beta - \delta^\alpha_0 \delta^0_\nu \eta_{\mu\beta} - \delta^\alpha_\beta \delta^0_\nu \delta^0_\mu + \delta^0_\beta \delta^\alpha_0 \eta_{\mu\nu}\Big)\Bigg) \\ R_{\mu\nu} =& H^2 a^2 \Bigg((D-1)\eta_{\mu\nu} - \epsilon\Big(\eta_{\mu\nu} - (D-2)\delta^0_\mu \delta^0_\nu\Big)\Bigg) \\ R =& H^2 (D - 2\epsilon)(D-1) \end{aligned} \tag{432}$$

here

$$\begin{aligned} \delta^\beta_\alpha &\equiv \eta_{\alpha\rho}\eta^{\beta\rho} \\ H &\equiv \frac{\dot{a}}{a} \\ \epsilon &\equiv -\frac{\dot{H}}{H^2} \end{aligned} \tag{433}$$

In most of our treatment we used the following symbols (the numbers in parentheses refer to equations where a definition is given)

- ϕ : (quantum) scalar field (180)

- Φ Background scalar field, (99)
- φ The 'full' scalar field, (99)
- χ the rescaled and fourier transformed scalar field (134)
- ψ the mode functions (180)
- u positive frequency solutions (179)
- $v = u^*$ negative frequency solutions
- x, $\tilde{x}$ the coordinates points where the propagators are defined (142)
- $\bar{x}$ The antipodal point of x, (131)

In section 6 specifically we used

- k_0 Infrared cutoff
- $z = \sqrt{k\eta\tilde{\eta}}$
- $z_0 \equiv \sqrt{k_0\eta\tilde{\eta}}$

In section 7 we use a subscript q and b to indicate the quantum and background energy and pressure respectively.
In section 9 $\psi_{\mu\nu}$ indicates the pseudo-graviton field and the diagonal fields are defined in equations (361) and (370). In this section hats will indicate rescaled quantities, as given in (392).

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Samenvatting

Het is reeds lange tijd bekend dat de twee-punts correlatiefunctie, of equivalent de propagator, van een massaloos scalair veld op een kosmologische achtergrond kan divergeren in het infrarood. In het bijzonder treed deze divergentie op, wanneer de druk verantwoordelijk voor de uitdijing van het universum negatief is en wanneer de correlatie berekend wordt ten opzichte van het Bunch-Davies vacuüm. De fysische oorsprong van deze divergentie ligt in de creatie van deeltjes op super-Hubble schaal. Deze creatie vindt plaats doordat virtuele deeltjes die ontstaan door de Heisenberg onzekerheidsrelatie met een golflengte langer dan de Hubble straal, elkaar niet meer binnen een eindige tijd annihileren, als de uitdijing van het universum snel genoeg is. Dit effect is uiteraard het sterkst aanwezig bij massaloze, niet conform gekoppelde deeltjes, zoals een niet conform gekoppelde massaloos scalair veld of bijvoorbeeld het graviton. De continue creatie van deze deeltjes met golflengtes langer dan de Hubble straal zorgt voor een groei in de twee-punts correlatie over lange afstanden en het is precies dit effect dat zorgt voor de eerder genoemde infrarode divergentie.

In dit proefschrift wordt deze infrarode divergentie nader onderzocht. Een infrarode divergentie in de natuurkunde betekend in de regel dat de specifieke vraag die onderzocht wordt, of de aannames die in de berekening gemaakt worden, niet fysisch zijn. In dit specifieke geval is de niet fysische aanname die leidt tot de infrarode divergentie, de aanname dat het Bunch-Davies vacuüm een goede beschrijving geeft van het infrarood. Een resolutie voor de infrarode divergentie is dan ook snel gevonden: we kiezen een nieuwe grondtoestand, zodanig dat de infrarode contributie aan de twee-punts correlatiefunctie onderdrukt wordt in het infrarood. In dit proefschrift wordt een methode beschreven die dit bereikt. De methode wordt uitgewerkt op een kosmologische achtergrond, waarbij de versnelling van de uitdijing van het universum constant is.

In de methode die wordt beschreven wordt de infrarode contributie onderdrukt door aan te nemen dat het universum een eindige doos is, met periodieke randvoorwaarden. De grootte van de doos beweegt mee met de uitdijing van het heelal. Effectief betekend deze methode dat de integraal over de impuls die bedraagt aan de twee-punts correlatiefunctie dient te worden vervangen door een som. Als de doos groot genoeg is, wordt deze som echter goed benadert door een integraal met een eindige ondergrens. Deze eindige ondergrens zorgt er dan vanzelfsprekend voor dat de infrarode contributie aan de twee-punts correlatiefunctie ook eindig is.

Het feit dat de propagator nu infrarood eindig is, betekend echter niet dat er geen creatie van deeltjes meer plaatsvindt. Dit is simpelweg een fysisch proces, met reële gevolgen. Om het effect hiervan te bekijken, worden in dit proefschrift twee zaken bestudeerd. Ten eerste wordt de één lus kwantum contributie aan de verwachtingswaarde van de energie-impuls tensor berekend. Deze verwachtingswaarde koppelt via de Einstein vergelijkingen aan de metriek en zorgt er dus in principe voor dat de metriek gewijzigd wordt door kwantum contributies. Aangezien we steeds in een kosmologische ruimte werken, betekend dit dat kwantum effecten de uitdijing van het heelal kunnen versnellen dan wel vertragen. Dit verschijnsel wordt 'terugkoppeling' (backreaction) genoemd. Dit effect is uiteraard in beginsel onwaarneembaar klein, maar de creatie van deeltjes is een cumulatief effect, dat er voor kan zorgen dat de verwachtingswaarde van de energie-impuls tensor groeit in de tijd. Daardoor kan een initieel klein effect na verloop van tijd een significante bijdrage leveren aan de Einstein vergelijkingen en dus significant de uitdijing van het universum beïnvloeden. De expliciete berekening laat echter zien dat in een versneld uitdijend heelal de kwantum contributie altijd insignificant blijft. De expansie van het heelal zorgt voor roodverschuiving van de infrarode deeltjes, waardoor ze energie verliezen. Dit effect is groot genoeg om het cumulatieve effect van de deeltjes creatie te laten verdwijnen. In een vertraagd uitdijend heelal daarentegen is dit niet het geval. Hier vinden we dat als de uitdijing traag genoeg is, kwantum effecten kunnen domineren na een voldoende lange tijd. De reden hiervoor is dat in een vertraagd uitzettend heelal de grootte van de doos na verloop van tijd altijd kleiner zal worden dan de Hubble straal. Dus het effect van het feit dat we werken in een heelal met eindige grootte wordt steeds explicieter. Als we de doos echter groot genoeg nemen, zodat we voor huidige metingen kunnen stellen dat de grootte van de doos veel groter is dan de Hubble straal, geeft dit geen enkel conflict met experimenten. Desondanks blijft het de vraag of bovenstaand effect reëel is, en ook gerealiseerd wordt in een realistisch model van het universum.

De tweede toepassing die we bekijken is de één lus gecorrigeerde effectieve potentiaal voor het scalaire veld. In dit geval voegen we vierdegraads interactie toe. Om de effectieve potentiaal uit te rekenen, nemen we aan dat het achtergrondveld proportioneel is aan de Hubble parameter. Hoewel deze proportionaliteit gerealiseerd wordt in afwezigheid van kwantum effecten, blijkt dat de proportionaliteit gebroken wordt wanneer de één lus correcties in beschouwing worden genomen. Voorts vinden we dat de effectieve potentiaal, equivalent aan wat er gebeurd in Minkowski ruimte, twee niet triviale minima kan ontwikkelen. Wanneer het scalaire veld naar deze minima rolt is de oorspronkelijke symmetrie rond de oorsprong gebroken. Het effect van de infrarode deeltjes creatie is echter dat na verloop van tijd, de symmetrie weer hersteld wordt. Na een voldoende lange tijd heeft de effectieve potentiaal altijd maar één minimum, in de oorsprong. Wanneer een dergelijk effect wordt bekeken in de context van bijvoorbeeld het Higgs veld, zien we dat tijdens de periode van symmetrie breking fermionen een massa kunnen krijgen. Deze generatie van massa kan dan zorgen voor de creatie van fermionen in het vroege universum. Als de symmetrie weer hersteld wordt, verdwijnt dit effect weer, doordat massaloze fermionen conform gekoppeld zijn.

Na deze toepassingen van de infrarood geregulariseerde propagator voor een massaloos scalair veld, bestuderen we ook het graviton veld. Ook voor dit veld is de creatie van infrarode deeltjes een belangrijk effect. Voor een consistente benadering, moet men zowel graviton als materie fluctuaties in beschouwing nemen. Wanneer men zowel graviton als materie fluctuaties bekijkt, loopt men tegen allerhande moeilijkheden aan doordat deze fluctuaties koppelen aan elkaar. Het is echter mogelijk om nieuwe velden te definiëren die ontkoppeld zijn en het blijkt bovendien dat voor deze ontkoppelde velden de massaloze scalaire propagator gebruikt kan worden om lus correcties uit te rekenen. We berekenen ook in dit geval de één lus contributie aan de verwachtingswaarde van de energie impuls tensor en we vinden wederom dat de kwantum contributies in een versneld uitdijend heelal altijd irrelevant zijn.

Uit dit proefschrift blijkt dat het correct in beschouwing nemen van infrarode deeltjes creatie leidt tot interessante nieuwe fysica. Hoewel de één lus berekeningen van de energie impuls tensor niet tot een significant effect leiden, is het de vraag of dit ook het geval is wanneer twee of meer lus diagrammen worden bekeken. Berekeningen in de Sitter ruimte laten immers zien dat daar in het geval van gravitonen twee lus effecten en bij scalaire velden drie lus effecten tot een significant effect kunnen leiden. Het zou zeer

interessant zijn om te kijken of dit ook nog steeds het geval is in de meer algemene kosmologische ruimtes die in dit proefschrift bekeken worden.
Een ander aspect wat meer aandacht verdient is een studie naar de afhankelijkheid van de effecten op de gekozen regularisatie van het infrarood. Naar verwachting zal een andere regularisatie in een versneld uitzettend universum vergelijkbare resultaten geven. In een vertraagd uitdijend heelal echter zagen we dat specifieke gevolgen van de in dit proefschrift gekozen regularisatie zichtbaar worden na verloop van tijd. Het is niet ondenkbaar dat andere regularisaties tot andere conclusies leiden in dit geval.

Publications

This thesis is based on the following articles

- T. M. Janssen and T. Prokopec,
  "Implications of the graviton one-loop effective action on the dynamics of the Universe,"
  arXiv:0807.0447 [gr-qc].

- T. M. Janssen, S. P. Miao, T. Prokopec and R. P. Woodard,
  "Infrared Propagator Corrections for Constant Deceleration,"
  Class. Quant. Grav. **25** (2008) 245013
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- T. M. Janssen and T. Prokopec,
  "A graviton propagator for inflation,"
  Class. Quantum Grav. **25** (2008) 055007
  arXiv:0707.3919 [gr-qc].

- T. M. Janssen, S. P. Miao, T. Prokopec and R. P. Woodard,
  "The Hubble Effective Potential,"
  JCAP **05** (2009) 003
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Curriculum Vitae

Ik ben geboren op 11 mei 1981 te Rio de Janeiro, Brazilie. Mijn Gymnasium-opleiding genoot ik op het Gymnasium Haganum te Den Haag, waar ik in 1999 mijn diploma behaalde. Van 1999 tot 2005 heb ik aan de Rijksuniversiteit Groningen theoretische natuurkunde gestudeerd. Tijdens mijn studie heb ik meerdere sudentassistentschappen vervuld, heb mij ingezet voor de opleidingscommissie en ben actief geweest in verscheidene verenigingsbesturen. Mijn afstudeeronderzoek, getiteld 'Can the stringy moduli drive the cosmological inflation?', heb ik gedaan onder begeleiding van prof. dr. E. Bergshoeff.

Na mijn studie ben ik in 2005 aan de Universiteit van Utrecht begonnen aan mijn promotieonderzoek, onder begeleiding van mijn co-promotor dr. T. Prokopec. In de periode van 2005 tot heden heb ik aan mijn onderzoek gewerkt, met als resultaat dit proefschrift. Daarnaast heb ik meerdere onderzoeksscholen en conferenties bezocht, waar ik mijn onderzoek gepresenteert heb en geassisteerd bij het geven van onderwijs.