JOURNAL OF GEOMETRIC MECHANICS
doi:10.3934/jgm.2016004
©American Institute of Mathematical Sciences
Volume **8**, Number **2**, June **2016**
pp. **199–220**

INFINITESIMALLY NATURAL PRINCIPAL BUNDLES

Bas Janssens*

Universiteit Utrecht, Budapestlaan 6
3584 CD Utrecht, The Netherlands

(Communicated by Rui L. Fernandes)

Abstract. We extend the notion of a natural fibre bundle by requiring diffeomorphisms of the base to lift to automorphisms of the bundle only infinitesimally, i.e. at the level of the Lie algebra of vector fields. We classify the principal fibre bundles with this property. A version of the main result in this paper (theorem 4.4) can be found in Lecomte's work [12]. Our approach was developed independently, uses the language of Lie algebroids, and can be generalized in several directions.

1. Introduction.

A smooth principal fibre bundle $\pi : P \to M$ with structure group G determines a sequence of groups

$$1 \to \Gamma_c(\mathrm{Ad}(P)) \to \mathrm{Aut}_c(P) \to \mathrm{Diff}_c(M) \to 1\,. \tag{1}$$

In this expression, $\mathrm{Diff}_c(M)$ is the group of compactly supported diffeomorphisms of M, and $\mathrm{Aut}_c(P)$ is the group of automorphisms of P that are trivial outside the preimage under π of a compact set. We have identified the gauge group of vertical automorphisms with the group of sections of the adjoint bundle $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} G$, and $\Gamma_c(\mathrm{Ad}(P))$ is the subgroup of compactly supported ones.

On the infinitesimal level, we have a corresponding exact sequence of Lie algebras

$$0 \to \Gamma_c(\mathrm{ad}(P)) \longrightarrow \Gamma_c(TP)^G \xrightarrow{\pi_*} \Gamma_c(TM) \to 0\,. \tag{2}$$

The last term, $\Gamma_c(TM)$, is the Lie algebra of smooth, compactly supported vector fields on M. The middle term, $\Gamma_c(TP)^G$, is the Lie algebra of G-invariant vector fields on P such that the support lies in the preimage under π of a compact set. The projection defines a Lie algebra homomorphism $\pi_* : \Gamma_c(TP)^G \to \Gamma_c(TM)$ because of G-invariance, and its kernel $\Gamma_c(TP)^G_v$ is the ideal of vertical, G-invariant vector fields with support in the preimage under π of a compact set. We identify $\Gamma_c(TP)^G_v$ with the Lie algebra $\Gamma_c(\mathrm{ad}(P))$ of smooth, compactly supported sections of the adjoint Lie algebra bundle $\mathrm{ad}(P) := P \times_{\mathrm{ad}} \mathfrak{g}$.

Definition 1.1. A *natural* principal fibre bundle is a principal fibre bundle P for which (1) is split exact, together with a distinguished splitting homomorphism $\Sigma : \mathrm{Diff}_c(M) \to \mathrm{Aut}_c(P)$. Moreover, Σ is required to be local in the sense that $\Sigma(\phi)(p)$ depends only on the germ of ϕ around $\pi(p)$.

2010 *Mathematics Subject Classification.* Primary: 53C27, 57R50, 22E65, 57R22.

Key words and phrases. Natural bundles, diffeomorphism group, integration of Lie algebroids, spin structures, Lie algebras of vector fields.

* Corresponding author.

Natural bundles were introduced[1] by Nijenhuis [18, 19], building on the theory of 'geometric objects' [24, 29, 17]. A classification theorem of Palais and Terng [20], completed by Epstein and Thurston [5] and based on earlier work of Salvioli [23], states that any natural fibre bundle is associated to the k^{th} order frame bundle $F^k(M)$.

Definition 1.2. An *infinitesimally natural* principal fibre bundle is a principal fibre bundle $\pi\colon P \to M$, together with a distinguished Lie algebra homomorphism $\sigma : \Gamma_c(TM) \to \Gamma_c(TP)^G$ that splits the exact sequence of Lie algebras (2).

This extends the notion of a natural fibre bundle in two separate ways. First of all, we do not *require* locality, but *prove* it. And secondly, we only require diffeomorphisms of the base to lift to automorphisms of the bundle *infinitesimally*, i.e., at the level of Lie algebras.

The outline of the paper is as follows. Sections 2, 3 and 4 are devoted to the classification of infinitesimally natural principal fibre bundles. The central result is theorem 4.4, which states the following.

Theorem. *Any infinitesimally natural principal fibre bundle is associated to the universal cover* $\widetilde{F}^{+k}(M)$ *of the connected component of the* k^{th} *order frame bundle.*

The number k is at most $\dim(G)$, unless $\dim(M) = 1$ and $\dim(G) = 2$, in which case also $k = 3$ may occur.

A version of this result was obtained earlier by Lecomte [12], using the theory of distributions and foliations. In the present paper, we use instead the theory of Lie algebroids and groupoids, which opens the door to generalizations in a context where additional structure or symmetry is present on the underlying manifold.

We extend Theorem 4.4 to fibre bundles with a finite dimensional structure group in Section 5, with special attention for vector bundles. Finally, in Section 6, we exhibit conditions under which a splitting of (2) gives rise to a flat connection.

2. Principal bundles as Lie algebra extensions.

We seek to classify infinitesimally natural principal fibre bundles. It is to this end that we study Lie algebra homomorphisms σ that split (2). In this section, we will prove that σ must be a differential operator of finite order.

The first step, to be taken in Section 2.1, is to show that maximal ideals in $\Gamma_c(TM)$ correspond precisely to points in M. Using this, we will prove in Section 2.2 that σ must be a local map. We will then prove, in Section 2.3, that σ is in fact a differential operator of finite order.

2.1. Ideals of the Lie algebra of vector fields.

The following proposition, due to Shanks and Pursell [25], constitutes the linchpin of the proof. It identifies the maximal ideals of the Lie algebra $\Gamma_c(TM)$ of smooth, compactly supported vector fields on M. The proof is taken from [25], with some minor clarifications.

For $q \in M$, define $I_q \subseteq \Gamma_c(TM)$ to be the ideal of vector fields $v \in \Gamma_c(TM)$ which are zero and flat at q, that is, $v(q) = 0$ and $(\mathrm{ad}(w_{i_1}) \dots \mathrm{ad}(w_{i_n})v)(q) = 0$ for all $w_{i_1}, \dots w_{i_n}, \in \Gamma_c(TM)$.

[1]Every natural principal bundle in the sense of Definition 1.1 induces a functor from the category of open subsets of M with local diffeomorphisms to the category of principal fibre bundles with bundle morphisms. To an object U it assigns the principal bundle $\pi^{-1}(U) \to U$ and to a morphism $\phi\colon U \to V$ it assigns the bundle morphism $\pi^{-1}(U) \to \pi^{-1}(V)$ defined by $p \mapsto \Sigma(\mathrm{germ}_{\pi p}(\phi))(p)$. Nijenhuis' definition of a natural fibre bundle is in terms of this induced functor.

Proposition 1. *The set of maximal ideals of* $\Gamma_c(TM)$ *is* $\{I_q\,;\,q \in M\}$.

To prove Proposition 1, we need the following, somewhat technical result.

Lemma 2.1. *Suppose that a maximal ideal* I *of* $\Gamma_c(TM)$ *contains a vector field which does not vanish at* $q \in M$. *Then* q *has a neighbourhood* U_q *such that for every* $w \in \Gamma_c(TM)$ *with* $\mathrm{Supp}(w) \subset U_q$, *there exist vector fields* $v \in I$ *and* $u \in \Gamma_c(TM)$ *such that* $w = [v, u]$.

Although one can easily find $u \in \Gamma_c(TM)$ and $v \in I$ such that $w = [u, v]$ locally, it is not always clear how to extend these to global vector fields while simultaneously satisfying $w = [u, v]$. For instance, with M the circle S^1 and $v = w = \partial_\theta$, the solution $u = \theta\partial_\theta$ does not globally exist. We need to do some work in order to define a proper cutoff procedure.

Proof. Choose $v \in I$ with $v(q) \neq 0$. There exist local co-ordinates $x_1, \ldots, x_n$ and an open neighbourhood W of q such that $v|_W = \partial_1$. Choose W to be a block centred around q, and nest two smaller blocks (in local co-ordinates) inside, so that $W \supset V \supset U$.

We take $W = (-\varepsilon, \varepsilon)^n$, $V = (-\frac{2}{3}\varepsilon, \frac{2}{3}\varepsilon)^n$, and $U = (-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon)^n$. Choose a smooth function g on M such that $g|_U = x_1$ and $g|_{M\setminus V} = 0$. Require also that $\partial_i g(x) = 0$ for $i \neq 1$ and $(x_2, \ldots, x_n) \in (-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon)^{n-1}$. Define $h := \partial_1 g$, and set $\tilde{v} := [v, g\partial_1] = h\partial_1$. Then $\tilde{v}|_U = \partial_1$, $\tilde{v}|_{M\setminus V} = 0$, and most importantly $\tilde{v} \in I$.

Now let $w \in \Gamma_c(TM)$ with $\mathrm{Supp}(w) \subset U$. We will find a global vector field $u \in \Gamma_c(TM)$ that realises $[\tilde{v}, u] = w$.

For $i \neq 1$, the i^{th} component of the above reads $h\partial_1 u^i = w^i$. In the region $x_1 \leq \frac{1}{3}\varepsilon$, we set $u^i(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} w^i(t, x_2 \ldots, x_n)\mathrm{d}t$. On U, where $\tilde{v} = \partial_1$, we then have $\partial_1 u^i = w^i$ and therefore $h\partial_1 u^i = w^i$. This is obviously also correct for points outside U with $x_1 \leq \frac{1}{3}\varepsilon$, as both u^i and w^i are zero.

For $\frac{1}{3}\varepsilon \leq x_1 \leq \frac{2}{3}\varepsilon$, where $w = 0$, we let $u^i(x_1, x_2, \ldots, x_n)$ be a constant function of x_1, so that $u^i(x_1, x_2, \ldots, x_n) = u^i(\frac{1}{3}\varepsilon, x_2, \ldots, x_n)$. We then have $\partial_1 u^i(x_1, \ldots, x_n) = 0$, guaranteeing $h\partial_1 u^i = w^i$. Note that this does not effect the smoothness of u^i.

Finally, for $\frac{2}{3}\varepsilon \leq x_1 \leq \varepsilon$, let u^i tend to zero, and let u^i be zero for $x_1 \geq \varepsilon$. This can be done in such a way that u^i remains smooth. Since both h and w are zero, we have $h\partial_1 u^i = w^i$ on all of M.

The case $i = 1$ is handled similarly, the only difference being that the first component of $[\tilde{v}, u] = w$ is now $h\partial_1 u^1 - \sum_j u^j \partial_j h = w^1$, the term $u^1\partial_1 h$ of which cannot be dispensed with. We have arranged that $u^i(x)$ with $i \neq 1$ is zero if $(x_2, \ldots, x_n) \notin (-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon)^{n-1}$, and that $\partial_i h(x)$ with $i \neq 1$ is zero if $(x_2, \ldots, x_n) \in (-\frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon)^{n-1}$. It follows that $u^i\partial_i h = 0$ for $i \neq 1$, and hence $h\partial_1 u^1 - u^1\partial_1 h = w^1$.

For $x_1 \leq \frac{1}{3}\varepsilon$, we once again set $u^1(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} w^1(t, x_2 \ldots, x_n)\mathrm{d}t$. For $\frac{1}{3}\varepsilon \leq x_1 \leq \frac{2}{3}\varepsilon$ however, one now has to define $u^1(x_1, \ldots, x_n) = h(x_1, \ldots, x_n)$ $u^1(\frac{1}{3}\varepsilon, x_2, \ldots, x_n)$ in order for $h\partial_1 u^1 - u^1\partial_1 h = w^1$ to hold. Since this renders u^1 zero on a neighbourhood of the boundary of V, one is then free to define u^1 to be zero on $M\setminus V$.

Thus, for every vector field w with support in U, we have constructed vector fields $\tilde{v} \in I$ and $u \in \Gamma_c(TM)$ such that $w = [\tilde{v}, u]$. □

Using Lemma 2.1, one readily proves the following.

Lemma 2.2. *Suppose that* I *is an ideal such that for all* $q \in M$, *there exists a* $v \in I$ *with* $v(q) \neq 0$. *Then* $I = \Gamma_c(TM)$.

Proof. Let $w \in \Gamma_c(TM)$. Cover the support $\mathrm{Supp}(w)$ of w by finitely many neighbourhoods $U_{q_1} \dots U_{q_N}$ with the properties described in Lemma 2.1. Using a partition of unity, write $w = \sum_{j=1}^{N} w_j$ with $\mathrm{Supp}(w_j) \subseteq U_{q_j}$. By Lemma 2.1, there exist $v_j \in I$ and $u_j \in \Gamma_c(TM)$ such that $w_j = [v_j, u_j]$. In particular, every w_j is an element of I. It follows that also $w = \sum_{j=1}^{N} w_j$ is in I. □

Using Lemma 2.2, we now prove Proposition 1.

Proof of Proposition 1. For every proper ideal $I \subset \Gamma_c(TM)$, there exists a $q \in M$ such that $I \subseteq I_q$. Indeed, Lemma 2.2 guarantees the existence of a $q \in M$ such that $v(q) = 0$ for all $v \in I$. Since $\mathrm{ad}(w_1) \dots \mathrm{ad}(w_n) v \in I$ for all $v \in I$ and $w_1, \dots, w_n \in \Gamma_c(TM)$, it follows that also $(\mathrm{ad}(w_1) \dots \mathrm{ad}(w_n) v)(q) = 0$. Every $v \in I$ is therefore not only zero at q, but also flat. We conclude that $I \subseteq I_q$.

It remains to show that for every $q \in M$, the ideal I_q is indeed maximal. Suppose that $I_q \subseteq I$ for a proper ideal $I \subset \Gamma_c(TM)$. Then there exists a point $\tilde{q} \in M$ such that $I \subseteq I_{\tilde{q}}$, and hence $I_q \subseteq I_{\tilde{q}}$. It follows that $\tilde{q} = q$. Since $I_q \subseteq I \subseteq I_{\tilde{q}}$, it follows that $I = I_q$, so that the ideal I_q is maximal. □

A maximal subalgebra $\mathcal{A}$ of a Lie algebra L is either self-normalizing or ideal. Indeed it is contained in its normaliser, which therefore equals either $\mathcal{A}$ or L. A theorem of Barnes [2] states that a finite-dimensional Lie algebra is nilpotent if and only if[2] every maximal subalgebra is an ideal. On the other extreme:

Proposition 2. *Let L be a Lie algebra over a field $\mathbf{K}$, and let $\mathcal{S}$ be the set of subspaces $\mathcal{A} \subset L$ such that $\mathcal{A}$ is both an ideal and a maximal subalgebra. Then*

$$[L, L] = \bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$$

if $\mathcal{S} \neq \emptyset$, and $[L, L] = L$ if $\mathcal{S} = \emptyset$. In particular, L is perfect ($[L, L] = L$) if and only if every maximal subalgebra is self-normalizing.

Proof. Let $X \notin [L, L]$. Choose $[L, L] \subseteq \mathcal{A} \subsetneq L$ where $\mathcal{A}$ has codimension 1 in L, and $X \notin \mathcal{A}$. Then $\mathcal{A}$ is an ideal maximal subalgebra, which does not contain X. Thus $\bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A} \subseteq [L, L]$.

Let $\mathcal{A}$ be an ideal maximal subalgebra, and $X \notin \mathcal{A}$. Then $\mathcal{A} + \mathbf{K}X$ is a subalgebra strictly containing $\mathcal{A}$, so that it must equal L. Thus $[L, L] = [\mathcal{A} + \mathbf{K}X, \mathcal{A} + \mathbf{K}X] \subseteq \mathcal{A}$, whence $[L, L] \subseteq \bigcap_{\mathcal{A} \in \mathcal{S}} \mathcal{A}$. □

As a corollary, we have the following well known statement (cf. e.g. [1, Thm. 1.4.3])

Corollary 1. *The Lie algebra $\Gamma_c(TM)$ is perfect;*

$$[\Gamma_c(TM), \Gamma_c(TM)] = \Gamma_c(TM)\,.$$

Proof. According to lemma 1, the maximal ideals are precisely the ideals I_q of vector fields in $\Gamma_c(TM)$ which are zero and flat at q. I_q is strictly contained in the subalgebra $\mathcal{A}_q$ of vector fields which are zero at q, so that no ideal is a maximal subalgebra. So every maximal subalgebra is self-normalizing, and the result follows from Proposition (2). □

[2] Actually, the 'only if' part in Barnes' theorem is not written down in [2], but this is immediately clear from the proof of Engel's theorem. (See e.g. [9]).

2.2. **The splitting as a local map.** With the main technical obstacles out of the way, we turn our attention to the sequence (2). We now prove that σ is a local map.

Recall that the Lie algebra $\Gamma_c(TP)^G_v$ of vertical, G-invariant vector fields on P with support in the preimage of a compact subset of M is identified with the Lie algebra $\Gamma_c(\mathrm{ad}(P))$ of compactly supported sections of the adjoint Lie algebra bundle $\mathrm{ad}(P)$. Concretely, the section $m \mapsto [p_m, X]$ of $\mathrm{ad}(P) := P \times_{\mathrm{ad}} \mathfrak{g}$ is identified with the vertical, G-invariant vector field $v_p = \frac{d}{dt}|_0 p_m e^{tX}$ on P.

Lemma 2.3. *Let $P \to M$ be a principal G-bundle over M, with G any Lie group. Let $\sigma : \Gamma_c(TM) \to \Gamma_c(TP)^G$ be a Lie algebra homomorphism splitting the exact sequence of Lie algebras*

$$0 \to \Gamma_c(\mathrm{ad}(P)) \to \Gamma_c(TP)^G \to \Gamma_c(TM) \to 0\,. \tag{3}$$

Then σ is local in the sense that $\pi(\mathrm{Supp}(\sigma(v))) \subseteq \mathrm{Supp}(v)$.

Proof. Let $\mathcal{A}_m := \{v \in \Gamma_c(TM) \,|\, v(m) = 0\}$ be the subalgebra of vector fields that vanish at the point $m \in M$. Since $\pi_* \circ \sigma(v) = v$, the lift $\sigma(v) \in \Gamma_c(TP)^G$ of $v \in \mathcal{A}_m$ is vertical on the fibre $P_m = \pi^{-1}(m)$. Identifying the G-equivariant vector field $\sigma(v)|_{P_m} \in \Gamma(TP_m)^G$ with an element of the Lie algebra $\mathrm{ad}(P)_m \simeq \mathfrak{g}$, we obtain a Lie algebra homomorphism

$$\tilde{\sigma}_m \colon \mathcal{A}_m \to \mathfrak{g}\,.$$

Denote by $\widehat{\sigma}_m \colon \Gamma_c(T(M-\{m\})) \to \mathcal{A}_m$ its restriction to the ideal $\Gamma_c(T(M-\{m\})) \subseteq \mathcal{A}_m$. Since $\hat{\sigma}_m$ is a Lie algebra homomorphism into a finite dimensional Lie algebra, its kernel $\mathrm{Ker}(\hat{\sigma}_m)$ is an ideal in $\Gamma_c(T(M-\{m\}))$ of finite codimension. According to Lemma 1, however, all proper ideals are of infinite codimension. Therefore $\mathrm{Ker}(\hat{\sigma}_m) = \Gamma_c(T(M-\{m\}))$, and $\widehat{\sigma}_m$ is identically zero for all $m \in M$. Since $\sigma(v)|_{P_m} = 0$ if $m \notin \mathrm{Supp}(v)$, the section σ is local, in the sense that $\pi(\mathrm{Supp}(\sigma(v))) \subseteq \mathrm{Supp}(v)$. □

We identify $\Gamma_c(TP)^G$ with $\Gamma_c(TP/G)$, the compactly supported sections of the vector bundle $TP/G \to M$ that arises as the quotient of TP by the pushforward of the right G-action on P. This allows us to consider the splitting σ as a map $\sigma \colon \Gamma_c(TM) \to \Gamma_c(TP/G)$. Since σ is local by Lemma 2.3, it defines a morphism from the sheaf of smooth sections of $TM \to M$ to the sheaf of smooth sections of $TP/G \to M$. By Peetre's Theorem ([21]), the map $\sigma \colon \Gamma_c(TM) \to \Gamma_c(TP/G)$ must be a differential operator of locally finite order.

2.3. **The splitting as a differential operator.** In this section, we will prove that σ is a differential operator of finite order. Since we already know that it is of locally finite order, it remains to find a global bound on the order.

Recall that $\mathcal{A}_m \subseteq \Gamma_c(TM)$ is the subalgebra of vector fields that vanish at m, and $\tilde{\sigma}_m : \mathcal{A}_m \to \mathfrak{g}$ is the restriction of σ to $\mathcal{A}_m$, followed by the map $\Gamma(\mathrm{ad}(P)) \to \mathfrak{g}$ that picks out the fibre over m and identifies it with $\mathfrak{g}$.

The fact that σ is a differential operator of locally finite order implies that for each $m \in M$, the Lie algebra homomorphism $\tilde{\sigma}_m \colon \mathcal{A}_m \to \mathfrak{g}$ factors through the jet Lie algebra $J^{r,0}_m(TM) := \mathcal{A}_m/H^r_m$, with $H^r_m = \{v \in \mathcal{A}_m \,|\, j^r_m(v) = 0\}$ the ideal of vector fields that vanish up to order r.

Local co-ordinates provide one with a basis $x^{\vec{\alpha}}\partial_i$ of $J^{r,0}_m(TM)$, where $x^{\vec{\alpha}}$ is shorthand for $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. With $\mathrm{Vec}^k_n = \mathrm{Span}\{x^{\vec{\alpha}}\partial_i \,|\, |\vec{\alpha}| = k+1\,,\, i = 1 \ldots n\}$, we have

$J^{r,0}_m(TM) \simeq \bigoplus_{k=0}^{r} \mathrm{Vec}^k_n$. Since we do not know r, we define

$$\mathrm{Vec}_n = \bigoplus_{k=0}^{\infty} \mathrm{Vec}^k_n\,,$$

and remark that $\tilde{\sigma}_m$ induces a Lie algebra homomorphism $\mathrm{Vec}_n \to \mathfrak{g}$.

The Lie algebra Vec_n depends on M only through its dimension n. Note that Vec^k_n is the k-eigenspace of the Euler vector field $E := \sum_{i=1}^n x^i\partial_i$, and that each element of Vec_n can be uniquely written as a *finite* sum of homogeneous vector fields. If I is an ideal containing $v = \sum_{k=0}^N v_k$, then $\mathrm{ad}(E)^j v = \sum_{k=0}^N k^j v_k \in I$ for all j. By taking suitable linear combinations, one sees that $v_k \in I$. Thus any ideal splits into homogeneous components

$$I = \bigoplus_{k=0}^{\infty} I^k$$

with $I^k = I \cap \mathrm{Vec}^k_n$. This renders the ideal structure of Vec_n more or less tractable, enabling us to prove the following bound on the order of $\tilde{\sigma}_m$.

Lemma 2.4. *The order of the differential operator σ is at most $\dim(\mathfrak{g})$ unless $\dim(M) = 1$ and $\dim(\mathfrak{g}) = 2$, in which case the order is at most 3.*

Proof. We closely follow Epstein and Thurston [5]. One checks by hand that the only ideals of the Lie algebra $\mathrm{Vec}_1 = \mathrm{Span}\{x^k\partial \,|\, k \geq 1\}$ are $\mathrm{Span}\{x^2\partial, x^k\partial \,|\, k \geq 4\}$ and $\mathrm{Span}\{x^k\partial \,|\, k \geq N\}$ with $N \geq 1$.

Consider Vec_1 as a subalgebra of Vec_n, define K to be the kernel of the homomorphism $\mathrm{Vec}_n \to \mathfrak{g}$ induced by $\tilde{\sigma}_m$, and let $K_1 := K \cap \mathrm{Vec}_1$. We then have injective homomorphisms

$$\mathrm{Vec}_1/K_1 \hookrightarrow \mathrm{Vec}_n/K \hookrightarrow \mathfrak{g}\,,$$

so that $\dim(\mathrm{Vec}_1/K_1) \leq \dim(\mathfrak{g})$. As K_1 is an ideal, it must be of the form mentioned above. This leads us to conclude that $x_1^k\partial_1 \in K$ for all $k > \dim(\mathfrak{g})$ unless $\dim(\mathfrak{g}) = 2$, in which case $x_1^k\partial_1 \in K$ for all $k > 3$, and $x_1^2\partial_1 \in K$.

The following short calculation shows that if $\dim(\mathfrak{g}) = 2$ and $\dim(M) > 1$, then also $x_1^3\partial_1 \in K$. As K contains $x_1^2\partial_1$, it also contains $[x_1^2\partial_1, x_1\partial_2] = x_1^2\partial_2$, and thus $[x_1^2\partial_2, x_1x_2\partial_1] = x_1^3\partial_1 - 2x_1^2x_2\partial_2$. But by bracketing with $x_1^2\partial_2$ and $x_2\partial_1$ respectively, we see that $x_1^3\partial_1 - 3x_1^2x_2\partial_2$ is in K, ergo $x_1^3\partial_1 \in K$.

The next step is to show that if $x_1^s\partial_1 \in K$, then K also contains all $x^{\vec{\alpha}}\partial_i$ with $|\vec{\alpha}| = s$. First of all, we remain in K if we repeatedly apply $\mathrm{ad}(x_i\partial_1)$ to $x_1^s\partial_1$, to the effect of replacing x_1 by x_i up to a nonzero factor. This shows that $x^{\vec{\alpha}}\partial_1 \in K$. Then the relation $x^{\vec{\alpha}}\partial_i = [x^{\vec{\alpha}}\partial_1, x_1\partial_i] + x_1\partial_i x^{\vec{\alpha}}\partial_1$ transfers membership of K from right to left.

In the generic case $\dim(\mathfrak{g}) \neq 2$, $\dim(M) \neq 1$, we may conclude that the order of σ is at most $\dim(\mathfrak{g})$, because $H_m^{\dim(\mathfrak{g})} \subset K$. In the exceptional case that $\dim(\mathfrak{g}) = 2$ and $\dim(M) = 1$, the order of σ is at most 3. □

In particular, σ is a differential operator of finite rather than locally finite order. We summarise our progress so far in the following proposition.

Proposition 3. *Let P be an infinitesimally natural principal G-bundle. Then $\sigma : \Gamma_c(TM) \to \Gamma_c(TP)^G$ factors through the bundle of k-jets, where $k = 3$ if $\dim(M) = 1$, $\dim(\mathfrak{g}) = 2$ and $k = \dim(\mathfrak{g})$ otherwise. We can therefore define a bundle map $\nabla : J^k(TM) \to TP/G$ by $\nabla(j^k_m(v)) := \sigma(v)_m$. It makes the following diagram commute:*

$$\begin{array}{ccc} \Gamma(TM) & \xrightarrow{\sigma} & \Gamma(TP/G) \\ & {\scriptstyle j^k}\searrow \quad \nearrow{\scriptstyle \nabla} & \\ & \Gamma(J^k(TM))\,. & \end{array}$$

The point is that although σ is defined only on sections, ∇ comes from a veritable bundle map $J^k(TM) \to TP/G$.

2.4. **The compact support condition.** We defined infinitesimally natural principal fibre bundles in terms of a splitting $\sigma\colon \Gamma_c(TM) \to \Gamma_c(TP)^G$ of the sequence (2) of compactly supported sections. We now see that, equivalently, an infinitesimally natural principal fibre bundle can be defined as a bundle with a continuous splitting $\tau\colon \Gamma(TM) \to \Gamma(TP)^G$ of the exact sequence of Fréchet Lie algebras (with the usual topology of uniform convergence of all derivatives on compact subsets, cf. [7, Def. 3.8])

$$0 \to \Gamma(\mathrm{ad}(P)) \to \Gamma(TP)^G \to \Gamma(TM) \to 0\,, \tag{4}$$

the non-compactly supported version of (2).

Indeed, the extension $\tilde{\sigma}\colon \Gamma(TM) \to \Gamma(TP)^G$ of σ is a splitting of (4), which is local (in the sense that $\pi(\mathrm{supp}(\tilde{\sigma}(v))) \subseteq \mathrm{supp}(v)$) and continuous because it is a differential operator. Conversely, one sees from the proof of Lemma 2.3 that every Lie algebra homomorphism $\tau\colon \Gamma(TM) \to \Gamma(TP)^G$ that splits (4) maps $\Gamma_c(TM)$ to $\Gamma_c(TP)^G$, and hence restricts to a splitting of (2). If, furthermore, $\tau\colon \Gamma(TM) \to \Gamma(TP)^G$ is continuous, then it is uniquely determined by its restriction to the dense subalgebra $\Gamma_c(TM)$, and hence local. For later use, we formulate this observation as a proposition.

Proposition 4. *Every splitting of (2) induces a continuous splitting of (4). Conversely, every splitting of (4), continuous or not, induces a splitting of (2). Consequently, an infinitesimally natural principal fibre bundle can be alternatively characterised as a principal fibre bundle P, together with a distinguished continuous Lie algebra homomorphism $\tau\colon \Gamma(TM) \to \Gamma(TP)^G$ that splits the exact sequence of Fréchet Lie algebras (4).*

3. **Lie groupoids and algebroids of jets.** The bundles $J^k(TM)$ and TP/G are Lie algebroids, and it will be essential for us to prove that $\nabla : J^k(TM) \to TP/G$ is a homomorphism of Lie algebroids. In order to do this, we will first have a closer look at $J^k(TM)$ and TP/G, and at their corresponding Lie groupoids.

Let us first set some notation. The jet group $G^k_{0,0}(\mathbb{R}^n)$ is the group of k-jets of diffeomorphisms of $\mathbb{R}^n$ that fix 0. It is the semi-direct product of $\mathrm{GL}(\mathbb{R}^n)$ and the connected, simply connected, unipotent Lie group of k-jets that equal the identity to first order.

The subgroup $G^{+k}_{0,0}(\mathbb{R}^n)$ of orientation preserving k-jets is connected, but not simply connected. As $G^{+k}_{0,0}(\mathbb{R}^n)$ retracts to $\mathrm{SO}(\mathbb{R}^n)$, its homotopy group is isomorphic to $\{1\}$ if $n = 1$, to $\mathbb{Z}$ if $n = 2$, and to $\mathbb{Z}/2\mathbb{Z}$ if $n > 2$. For brevity, we introduce the following notation.

Definition 3.1. If $k > 0$, we denote $\pi_1(G^{+k}_{0,0}(\mathbb{R}^n))$ by Z.

Thus for $n > 2$, the universal cover $\tilde{G}^{+k}_{0,0}(\mathbb{R}^n) \to G^{+k}_{0,0}(\mathbb{R}^n)$ is $2:1$, and restricts to the spin group over $\mathrm{SO}(\mathbb{R}^n)$.

3.1. **The Lie groupoid of k-jets.** In this section, we define the Lie groupoid $G^k(M)$ of k-jets, its maximal source-connected Lie subgroupoid $G^{+k}(M)$, and the k^{th} order frame bundle $F^k(M)$.

Denote by $G^k_{m',m}(M)$ the manifold of k-jets at m of diffeomorphisms of M which map m to m', and denote by $G^k(M) = \cup_{M\times M} G^k_{m',m}(M)$ the groupoid of k-jets. If $j^k_m(\alpha)$ is a k-jet at m of a diffeomorphism α, then its source is $s(j^k_m(\alpha)) = m$, its target is $t(j^k_m(\alpha)) = \alpha(m)$, and multiplication is given by composition.

Remark. For any Lie groupoid $G \rightrightarrows M$, one can form the groupoid $J^kG \rightrightarrows M$ of k-jets of smooth, local bisections of $G \rightrightarrows M$. The groupoid $G^k(M) \rightrightarrows M$ defined above is precisely $J^k(G)$ for the pair groupoid $G = M \times M$, whose bisections are diffeomorphisms, cf. [13, §II.5].

Denote by $G^k_{*,m}(M)$ the manifold $s^{-1}(m)$ of k-jets with source m. The target map $t : G^k_{*,m}(M) \to M$ endows it with a structure of principal fibre bundle, the structure group $G^k_{m,m}(M) \simeq G^k_{0,0}(\mathbb{R}^n)$ acting freely and transitively on the right. As $G^1(M)_{*,m}$ is isomorphic to the frame bundle $F(M)$, one calls $G^k(M)_{*,m}$ the k^{th} order frame bundle, sometimes denoted $F^k(M)$.

Lemma 3.2. *Let M be connected and let $k \geq 1$. Then $G^k(M)$ is source-connected if and only if M is not orientable.*

Proof. We may as well consider $k = 1$, because the fibres of $G^k(M) \to G^1(M)$ can be contracted. Each source fibre $G^1(M)_{*,m}$ of $G^1(M)$ is isomorphic to the frame bundle. By definition, M is oriented precisely when the frames can be grouped into positively and negatively oriented ones. □

Definition 3.3. We define $G^{+k}(M)$ to be the maximal source-connected Lie subgroupoid of $G^k(M)$, and denote its source fibre by $F^{+k}(M)$.

In the light of the previous lemma, this means that $G^{+k}(M)$ is the Lie groupoid of k-jets of orientation preserving diffeomorphisms if M is orientable, and simply $G^k(M)$ if M is not.

Note that the map $D : \mathrm{Diff}(M) \to \mathrm{Diff}(G^k(M))$ defined by $D\alpha : j^k_m(\gamma) \mapsto j^k_m(\alpha\gamma)$ is a homomorphism of groups. We will call it the k^{th} order derivative. Because $D\alpha$ is source-preserving and right invariant, it defines a homomorphism $\mathrm{Diff}(M) \to \mathrm{Aut}^{G^k_{m,m}(M)}(G^k_{*,m}(M))$, splitting the exact sequence of groups (1). This makes $G^k_{*,m}(M) = F^k(M)$ into a natural bundle. Note that $F^{+k}(M)$ is infinitesimally natural.

3.2. **The Lie algebroid of k-jets.** The bundle $J^k(TM)$ possesses a structure of Lie algebroid, induced by the Lie groupoid $G^k(M)$. We now describe the Lie bracket on $\Gamma(J^k(TM))$ explicitly. Later, in section 3.4, we will use this to show that ∇ is a Lie algebroid homomorphism.

Remark. If $G \rightrightarrows M$ is a Lie groupoid with Lie algebroid $A \to M$, then the groupoid $J^kG \rightrightarrows M$ of k-jets of local bisections has Lie algebroid $J^k(A) \to M$. The following construction makes this Lie algebroid structure explicit in the case of the pair groupoid $G = M \times M$, where $A = TM$ and $J^k(A) = J^k(TM)$.

The Lie algebroid of $G^k(M)$ is a vector bundle $A \to M$. Its fibre A_m is by definition the subspace of the tangent space of $G^k(M)$ at $j^k_m(\mathrm{id})$ which is annihilated by ds. Sections of A therefore correspond to right-invariant vector fields on $G^k(M)$ tangent to the source fibres.

Each curve in $G^k_{*,m}(M)$ through $j^k_m(\mathrm{id})$ takes the shape $c(t) = j^k_m(\alpha_t)$ with $\alpha_0 = \mathrm{id}$, so that its tangent vector $a \in A_m$ takes the form $a = j^k_m(v)$, with $v = \partial_t|_0\alpha_t$. This shows that $A \simeq J^k(TM)$.

The anchor $dt : J^k(TM) \to TM$ is easily seen to be the canonical projection, so we shall denote it by π. The Lie bracket on $\Gamma(J^k(TM))$ however, which is defined as the restriction of the commutator bracket on $\Gamma(TG^k(M))$ to the right invariant source preserving vector fields, perhaps deserves some comment.

Define $J^{k,0}(TM)$ to be the kernel of π, and consider the exact sequence of Lie algebras

$$0 \to \Gamma(J^{k,0}(TM)) \to \Gamma(J^k(TM)) \xrightarrow{\pi} \Gamma(TM) \to 0\,.$$

It is split by $j^k : \Gamma(TM) \to \Gamma(J^k(TM))$, the infinitesimal version of the k^{th} order derivative. (The sequence of Lie algebroids of course does not split, as differentiation is not linear over $C^\infty(M)$.)

We will describe the Lie bracket on $\Gamma(J^k(TM))$ by giving it on $\Gamma(TM)$ and $\Gamma(J^{k,0}(TM))$ separately, and then giving the action of $\Gamma(TM)$ on $\Gamma(J^{k,0}(TM))$.

Proposition 5. *Let u and u' be sections of TM, and let $\tau : m \mapsto j^k_m(v_m)$ and $\tau' : m \mapsto j^k_m(v'_m)$ be sections of $J^{k,0}(TM)$, where v_m and v'_m are different local sections for each m. Then*

$$\begin{aligned}
[j^k(u), j^k(u')]_m &= j^k_m([u,u'])\,,\\
[\tau,\tau']_m &= j^k_m([v_m, v'_m])\,,\\
[j^k(u),\tau]_m &= j^k_m([u,v_m]) + j^k_m(d_u|_m(x \mapsto v_x))\,,
\end{aligned}$$

where $d_u|_m(x \mapsto v_x)$ is the ordinary derivative at m along u of a map from M to $\Gamma(TM)$. Although both terms on the right hand side depend on the choice of $m \mapsto v_m$, their sum does not.

Proof. The first equality is clear, as j^k is a homomorphism of Lie algebras. The second equality can be seen as follows. Consider the bundle of groups $G^k(M)_{*,*} := \{j^k_m(\alpha) \in G^k(M) \,|\, \alpha(m) = m\}$, with bundle map $s = t$. Its sections $\Gamma(G^k(M)_{*,*})$ form a group under pointwise multiplication, whose Lie algebra is $\Gamma(J^{k,0}(TM))$ with the pointwise bracket. As $\Gamma(G^k(M)_{*,*})$ acts from the left on $G^k(M)$ by $j^k_m(\gamma) \mapsto j^k_m(\alpha_m) \circ j^k_m(\gamma)$, respecting both the source map and right multiplication, the inclusion $\Gamma(J^{k,0}(TM)) \to \Gamma(J^k(TM))$ is a homomorphism of Lie algebras. This proves the second line.

To verify the third line, we must choose a smooth map $x \mapsto v_x$ from M to $\Gamma(TM)$ such that $\tau_x = j^k_x(v_x)$ in a neighbourhood of m. Each v_x necessarily has a zero at x. If we denote $m(s) := \exp(su)m$, then the bracket $[j^k(u),\tau]_m$ is by definition[3] minus the mixed second derivative along s and t at 0 of the groupoid commutator

$$j^k_{m(s)}(\exp(-su))j^k_{m(s)}(\exp(-tv_{m(s)}))j^k_m(\exp(su))j^k_m(\exp(tv_m))\,,$$

which is just $j^k_m\left(\exp(-su)\exp(-tv_{m(s)})\exp(su)\exp(tv_m)\right)$. The terms not involving derivatives of $s \mapsto m(s)$ yield $j^k_m([u,v_m])$, and the terms which do provide the extra $j^k_m(d_u|_m(x \mapsto v_x))$. □

[3]The Lie algebra of the diffeomorphism group is the Lie algebra of vector fields, but the exponential map is given by $v \mapsto \exp(-v)$, where exp denotes the unit flow along v. This is why the groupoid commutator might seem odd at first sight.

3.3. **The gauge groupoid and its algebroid.** Given a principal G-bundle $\pi : P \to M$, one can define the gauge groupoid $(P \times P)/G$, that is the quotient of the pair groupoid by the diagonal action [13, §II.1]. Source and target come from projection on the second and first term respectively, $M \hookrightarrow (P \times P)/G$ as $\mathrm{id}_{\pi(p)} = [(p,p)]$, and multiplication is well defined by $[(r,q)] \circ [(q,p)] = [(r,p)]$. An element $[(q,p)]$ corresponds precisely to a G-equivariant diffeomorphism $\pi^{-1}(p) \to \pi^{-1}(q)$, and the product to composition of maps.

Its Lie algebroid TP/G is called the Atiyah algebroid. Indeed, the subspace of $T_{\mathrm{id}_m}((P \times P)/G)$ which annihilates ds is canonically $(TP/G)_m$. A section of TP/G can be identified with a G-equivariant section of TP, endowing $\Gamma(TP/G)$ with the Lie bracket that comes from $\Gamma(TP)^G$.

3.4. **The splitting as a homomorphism of Lie algebroids.** The point of considering the Lie algebroid structure of $J^k(TM)$ was of course to prove the following.

Lemma 3.4. *The map $\nabla : J^k(TM) \to TP/G$ is a homomorphism of Lie algebroids.*

Proof. The fact that ∇ respects the anchor is immediate. As $\pi_* \circ \nabla(j^k_m(v_m)) = \pi_* \circ \sigma(v_m)(m) = v_m(m)$, it equals $\pi \circ j^k(v_m)$ in the point m.

We now show that $\nabla : \Gamma(J^k(TM)) \to \Gamma(TP/G)$ is a homomorphism of Lie algebras. First of all, the restriction of ∇ to $j^k(\Gamma(TM))$ is a homomorphism. Indeed, $[\nabla(j^k(v)), \nabla(j^k(w))] = [\sigma(v), \sigma(w)]$, which equals $\sigma([v,w])$ because σ is a homomorphism. This in turn is just $\nabla(j^k([v,w]))$, so that $[\nabla(j^k(v)), \nabla(j^k(w))] = \nabla([j^k(v), j^k(w)])$.

Secondly, its restriction to $\Gamma(J^{k,0}(TM))$ is a homomorphism. If $\tau_x = j^k_x(v_x)$ and $\upsilon_x = j^k_x(w_x)$ are sections of $J^{k,0}(TM)$, then $\nabla\tau$ and $\nabla\upsilon$ are in the kernel of the anchor. This implies that their commutator at a certain point m depends only on their values at m, not on their derivatives. To find the commutator at m, we may therefore replace $j^k_x(v_x)$ by $j^k_x(v_m)$ and likewise $j^k_x(w_x)$ by $j^k_x(w_m)$. We then see that $[\nabla j^k_x(v_x), \nabla j^k_x(w_x)]_m = [\nabla j^k_x(v_m), \nabla j^k_x(w_m)]_m$. We already know that this is $\nabla(j^k_x([v_m, w_m]))_m$, so that $[\nabla(\tau), \nabla(\upsilon)]_m = \nabla([\tau, \upsilon])_m$.

The last step is to show that ∇ respects the bracket between $j^k(\Gamma(TM))$ and $\Gamma(J^{k,0}(TM))$. Again, let $j^k(v)$ be an element of the former and $\tau_x = j^k_x(w_x)$ of the latter. Considered as an equivariant vector field on P, the vertical vector field $\nabla(\tau)$ takes the value $\sigma_p(w_{\pi(p)})$ at $p \in P$. Then $[\nabla(j^k(v)), \nabla(\tau)]$ is the Lie derivative along $\sigma(v)$ of the vertical vector field $\sigma_p(w_{\pi(p)})$. Differentiating along $\sigma(v)$ is done by considering $(p, p') \mapsto \sigma_p(w_{\pi(p')})$, differentiating w.r.t. p and p' separately, and then putting $p = p'$. This results in $[\sigma(v), \sigma_p(w_{\pi(p)})]_{p_0} = [\sigma(v), \sigma(w_{\pi(p_0)})]_{p_0} + \sigma(d_v|_{\pi(p_0)}(x \mapsto w_x))$, which is in turn the same as $\sigma_{p_0}([v, w_{\pi(p_0)}] + d_v|_{\pi(p_0)}(m \mapsto w_m)))$, so that $[\nabla(j^k(v)), \nabla(\tau)] = \nabla([j^k(v), \tau])$ as required. Therefore ∇ must be a homomorphism on all of $\Gamma(J^k(TM))$. □

Definition 3.5. A connection ∇ of a Lie algebroid A on a vector bundle E is by definition a bundle map of A into $\mathrm{DO}^1(E)$, the first order differential operators on E, which respects the anchor, cf. [13, § III.5]. If moreover it is a morphism of Lie algebroids, then the connection is called flat. A flat connection of A on E is also called a representation of A on E.

This explains our notation for the map ∇ induced by σ. Given a representation V of G, one may form the associated vector bundle $E := P \times_G V$. The map ∇ then defines a Lie algebroid homomorphism of $\Gamma(J^k(TM))$ into the Lie algebroid of first order differential operators on E. (Simply consider a section of E as a G-equivariant function $P \to V$, and let $\Gamma^G(TP)$ act by Lie derivative.) By definition, this is a flat connection, or equivalently a Lie algebroid representation.

3.5. **Infinitesimally natural transitive Lie algebroids.** The above is readily generalized to transitive Lie algebroids $L \to M$, whose anchor $\rho\colon L \to TM$ is surjective. The kernel $K \subseteq L$ of ρ is then a Lie algebroid $K \to M$ with trivial anchor. In particular, every fibre K_m has a Lie algebra structure. The anchor gives rise to the exact sequence of Lie algebras

$$0 \to \Gamma_c(K) \longrightarrow \Gamma_c(L) \xrightarrow{\rho} \Gamma_c(TM) \to 0\,. \tag{5}$$

Definition 3.6. An infinitesimally natural transitive Lie algebroid is a transitive Lie algebroid $L \to M$ with a Lie algebra homomorphism $\sigma\colon \Gamma_c(TM) \to \Gamma_c(L)$ that splits the exact sequence (5) of Lie algebras.

Remark. Every infinitesimally natural principal bundle P gives rise to an infinitesimally natural structure on the Atiyah Lie algebroid $TP/G \to M$. However, these are not the only examples of infinitesimally transitive Lie algebroids; the present setting also covers transitive Lie algebroids which are not integrable.

Theorem 3.7. *Let $L \to M$ be an infinitesimally natural transitive Lie algebroid. Then the splitting σ factors through a Lie algebroid homomorphism*

$$\nabla\colon J^k(TM) \to L\,,$$

that is, $\sigma(v)_m = \nabla(j^k_m v)$ for all $m \in M$ and $v \in \Gamma_c(TM)$. The order k of the differential operator σ satisfies $k \leq \mathrm{rank}(K)$, except when $n = 1$ and $\mathrm{rank}(K) = 2$, in which case $k \leq 3$.

Proof. Let $\mathcal{A}_m \subseteq \Gamma_c(TM)$ be the subalgebra of compactly supported vector fields v that vanish at the point $m \in M$. Since $v_m = \rho \circ \sigma(v)_m$ vanishes for $v \in \mathcal{A}_m$, we have $\sigma(v)_m \in K_m$. The map $\tilde{\sigma}_m\colon \mathcal{A}_m \to K_m$ defined by $\tilde{\sigma}_m(v) = \sigma(v)_m$ is a Lie algebra homomorphism. Consider its restriction $\hat{\sigma}_m\colon \Gamma_c(T(M-\{m\})) \to K_m$. Since K_m is finite dimensional, its kermel $\mathrm{Ker}(\hat{\sigma}_m)$ is an ideal of finite codimension. By Proposition 1, the maximal ideals of $\Gamma_c(T(M-\{m\}))$ are all of infinite codimension. It follows that the kernel of $\hat{\sigma}_m$ is the full Lie algebra $\Gamma_c(T(M-\{m\}))$, and that $\hat{\sigma}_m = 0$. Since $\sigma(v)_m = 0$ if $m \notin \mathrm{Supp}(v)$, the map σ defines a morphism from the sheaf of smooth sections of TM to the sheaf of smooth sections of L. By Peetre's Theorem, it is a differential operator of locally finite order. If we choose local coordinates $x_1, \dots x_n$ around m, then the Lie algebra homomorphism $\tilde{\sigma}_m\colon \mathcal{A}_m \to K_m$ corresponds to a Lie algebra homomorphism $\mathrm{Vec}_n \to K_m$. The global bound on k follows from Lemma 2.4. This shows that σ is a differential operator of order k, and factors through a vector bundle map $\nabla\colon J^k(TM) \to L$. The proof that ∇ is a morphism of Lie algebroids is analogous to that of Lemma 3.4. □

4. **The classification theorem.** We use the fact that $\nabla : J^k(TM) \to TP/G$ is a homomorphism of Lie algebroids to find a corresponding homomorphism of Lie groupoids. This will give us the desired classification of infinitesimally natural principal fibre bundles.

4.1. **Integrating a homomorphism of Lie algebroids.** The following theorem states that homomorphisms of Lie algebroids induce homomorphisms of Lie groupoids if the initial groupoid is source-simply connected.

Theorem 4.1 (Lie II for algebroids)**.** *Let G and H be Lie groupoids, with corresponding Lie algebroids A and B respectively. Let $\nabla : A \to B$ be a homomorphism of Lie algebroids. If G is source-simply connected, then there exists a unique homomorphism $G \to H$ of Lie groupoids which integrates ∇.*

Remark. The result was probably announced first in [22], and proofs have appeared e.g. in [16] and [14]. We follow [4], which the reader may consult for details.

Sketch of proof. The idea is that ∇ allows one to lift piecewise smooth paths of constant source in G to piecewise smooth paths of constant source in H. Source-preserving piecewise smooth homotopies in G of course do not affect the endpoint of the path in H, so that, if G is source-simply connected, one obtains a map $G \to H$ by identifying elements g of G with equivalence classes of source preserving paths from $\mathrm{id}_{s(g)}$ to g. One checks that this is the unique homomorphism of Lie groupoids integrating ∇. □

Unfortunately, $G^k(M)$ is not always source connected, let alone source-simply connected. Recall that $G^{+k}(M)$ is the maximal source-connected Lie subgroupoid of $G^k(M)$, and therefore has the same Lie algebroid $J^k(TM)$.

We define $\tilde{G}^{+k}(M)$ to be the set of piecewise smooth, source preserving paths in $G^{+k}(M)$ beginning at an identity, modulo piecewise smooth, source preserving homotopies. It is a smooth manifold because $G^{+k}(M)$ is, and a Lie groupoid under the unique structure making the projection on the endpoint $\tilde{G}^{+k}(M) \to G^{+k}(M)$ into a morphism of groupoids. Explicitly, the multiplication is given as follows. If $g(t)$ a path from id_m to $g(1)_{m'm}$, and $h(t)$ a path from $\mathrm{id}_{m'}$ to $h(1)_{m''m'}$, then the product $[h] \circ [g]$ is $[(h \cdot g(1)) * g]$, where the dot denotes groupoid multiplication and the star concatenation of paths. The proof of associativity is the usual one.

Note that the source fibre $\tilde{G}^{+k}(M)_{*,m}$ is precisely the universal cover of the connected component of the k^{th} order frame bundle $G^{+k}(M)_{*,m} = F^{+k}(M)$. In order to cut down on the subscripts, we introduce new notation for $\tilde{G}^{+k}(M)_{*,m}$ and its structure group $\tilde{G}^{+k}(M)_{m,m}$.

Definition 4.2. We denote the universal cover of the connected component of the k^{th} order frame bundle by $\widetilde{F}^{+k}(M)$, and its structure group by $G(k, M)$.

It is an infinitesimally natural bundle because $F^{+k}(M)$ is. Note that $G(k, M)$ is not the universal cover of $G^{+k}_{m,m}(M)$, but rather its extension by $\pi_1(F^k(M))$. As $\pi_1(F^k(M)) = \pi_1(F(M))$, we have the exact sequence of groups

$$1 \to \pi_1(F(M)) \to G(k, M) \to G^{+k}_{m,m}(M) \to 1\,.$$

The group $G^{+k}_{m,m}(M)$ in turn is isomorphic to $G^{+k}_{0,0}(\mathbb{R}^n)$ if M is orientable, and to $G^{k}_{0,0}(\mathbb{R}^n)$ if it is not.

4.2. **Classification.** Now that we have found a source-simply connected Lie groupoid with $J^k(TM)$ as Lie algebroid, we can finally apply Lie's second theorem for algebroids to obtain the following.

Proposition 6. *If σ splits the exact sequence of Lie algebras (2), then it induces a morphism of groupoids $\exp \nabla : \tilde{G}^{+k} \to (P \times P)/G$ such that the following diagram commutes, with $\exp_m$ the flow along a vector field starting at id_m.*

$$
\begin{array}{ccccccc}
\Gamma(TM) & & \xrightarrow{\ \sigma\ } & & \Gamma(TP/G) & \xrightarrow{\ \exp_m\ } & (P\times P)/G \\
& {\scriptstyle j^k}\searrow & & \nearrow{\scriptstyle \nabla} & & \nearrow{\scriptstyle \exp\nabla} & \\
& & \Gamma(J^k(TM)) & \xrightarrow{\ \exp_m\ } & \tilde{G}^{+k}(M) & &
\end{array}
$$

Proof. As $\tilde{G}^{+k}(M)$ is a source-simply connected Lie groupoid with $J^k(TM)$ as Lie algebroid, we can apply Lie's second theorem for algebroids. □

It is perhaps worthwhile to formulate this for general transitive Lie groupoids, as it clarifies the link with the recent work of Grabowski, Kotov and Poncin [8]. For the Atiyah algebroid A of a principal fibre bundle with a connected, reductive structure group, they classify Lie algebra isomorphisms of $\Gamma(A)$ in terms of the Lie algebroid isomorphisms of A.

Theorem 4.3. *Let $\mathcal{G} \rightrightarrows M$ be a transitive Lie groupoid, with Lie algebroid A. The kernel of the anchor K is then a bundle of Lie algebras with fixed dimension d. Suppose that the sequence*

$$0 \to \Gamma(K) \to \Gamma(A) \to \Gamma(TM) \to 0\,,$$

with K the kernel of the anchor, splits as a sequence of Lie algebras. Then this splitting is induced by a morphism of Lie algebroids $\nabla : J^k(TM) \to A$, and there is a corresponding morphism of Lie groupoids $\tilde{G}^{+k}(M) \to \mathcal{G}$. The number k is at most 3 if d is 2 and $\dim(M) = 1$, and at most d otherwise.

Sketch of proof. Using Theorem 3.7, this is analogous to the case of the gauge groupoid. □

We have paved the way for a classification of infinitesimally natural principal fibre bundles.

Theorem 4.4. *Let $\pi : P \to M$ be an infinitesimally natural principal G-bundle with a splitting σ of (2). Then there exists a group homomorphism $\rho : G(k, M) \to G$ such that the bundle P is associated to $\widetilde{F}^{+k}(M)$ through ρ, i.e.*

$$P \simeq \widetilde{F}^{+k}(M) \times_\rho G\,.$$

Moreover, σ is induced by the canonical splitting for $\widetilde{F}^{+k}(M)$.

Proof. Fix a base point m on M. The map $\exp\nabla$ yields a homomorphism of groups $\rho : \tilde{G}^{+k}_{m,m}(M) \to ((P\times P)/G)_{m,m}$, the latter isomorphic to G, the former to $G(k, M)$.

The map $\tilde{G}^{+k}_{*,m}(M) \times_\rho ((P\times P)/G)_{m,m} \to ((P\times P)/G)_{*,m}$ that takes $(g_{m',m}, p_{m,m})$ to $(\exp\nabla(g_{m',m}))\cdot p_{m,m}$ is well defined and injective because two pairs share the same image if and only if they are equivalent modulo $\tilde{G}^{+k}_{m,m}(M)$. It is also surjective and G-equivariant, and hence an isomorphism of principal G-bundles. As $((P\times P)/G)_{*,m} \simeq P$ and $\tilde{G}^{+k}_{*,m}(M) \times_\rho ((P\times P)/G)_{m,m} \simeq \widetilde{F}^{+k}(M) \times_\rho G$, the equivalence is proven. The remark on σ follows from the construction. □

This classifies the infinitesimally natural principal fibre bundles. They are all associated (via a group homomorphism) to the bundle $\tilde{G}^{+k}_{*,m} = \widetilde{F}^{+k}(M)$.

The classification of natural principal fibre bundles is now an easy corollary. The following well known result ([20, 28]) states that they are precisely the ones associated to $G^k_{*,m}(M) = F^k(M)$.

Corollary 2. *Let $\pi : P \to M$ be a natural principal G-bundle with local splitting Σ of (1). Then P is associated to $F^k(M)$. That is, there exists a homomorphism $\rho : G^k_{0,0}(\mathbb{R}^n) \to G$ such that*

$$P \simeq F^k(M) \times_\rho G\,.$$

Moreover, Σ is induced by the canonical one for $F^k(M)$.

Proof. As the homomorphism $\Sigma : \mathrm{Diff}_c(M) \to \mathrm{Aut}_c(P)$ is local, it induces a homomorphism of groupoids $\Sigma : \mathrm{Germ}(M) \to (P \times P)/G$, with $\mathrm{Germ}(M)$ the groupoid of germs of diffeomorphisms of M. We need but show that Σ factors through $j^k : \mathrm{Germ}(M) \to G^k(M)$ for some $k > 0$, cf. the proof of theorem 4.4.

The Lie algebra homomorphism $\sigma : \Gamma_c(TM) \to TP/G$ defined by $\sigma(v) := \partial_t|_0 \Sigma(\exp(tv))$ is local by assumption, and according to proposition 3 it factors through the k-jets for some $k > 0$. It suffices to show that $\Sigma(\phi)_{m,m} = \mathrm{id}_{m,m}$ for any $\phi \in \mathrm{Germ}_{m,m}(M)$ that agrees with the identity to k^{th} order at m.

In local co-ordinates $\{x^i\}$, we write $\phi^i(x) = x^i + v^i(x)$, where $v : \mathbb{R}^n \to \mathbb{R}^n$ vanishes to k^{th} order. We define the one parameter family of germs of diffeomorphisms $\phi^i_t(x) := x^i + tv^i(x)$. Then $\partial_t|_\tau \Sigma(\phi_\tau)^{-1}\Sigma(\phi_t)_m = 0$, as it equals $\sigma_m(\partial_t|_\tau \phi_\tau^{-1}\phi_t)$, the image of a vector field that vanishes to order k at m. Therefore $t \mapsto \Sigma(\phi_t)_{m,m}$ is constant, and $\Sigma(\phi)_{m,m} = \mathrm{id}_{m,m}$ as required. □

To summarise: natural principal fibre bundles are associated to a higher frame bundle, whereas infinitesimally natural principal fibre bundles are associated to the universal cover of a higher frame bundle.

4.3. **The Bundle $\widetilde{F}^{+k}(M)$.** The above considerations prompt a few remarks on the universal cover of the connected component of the frame bundle $\widetilde{F}^{+k}(M)$, and on its (disconnected) structure group $G(k, M)$. Recall that they are just the source fibre $\tilde{G}^{+k}_{*,m}(M)$ and isotropy group $\tilde{G}^{+k}_{m,m}(M)$ of $\tilde{G}^{+k}(M)$.

4.3.1. *General manifolds.* If $\pi_1(M)$ is the homotopy groupoid of M, define the homomorphism of groupoids $\mathrm{Pr} : \pi_1(M)_{m',m} \to \pi_0(G_{m',m}(M))$ by lifting a path in M to a path in $G^k(M)$ with fixed source, and taking the connected component of its end point. It makes

$$\begin{array}{ccccc} & & \pi_1(M)_{m',m} & & \\ & \nearrow & & \searrow & \\ \tilde{G}^{+k}_{m',m}(M) & & & & \pi_0(G^k_{m',m}(M)) \\ & \searrow & & \nearrow & \\ & & G^k_{m',m}(M) & & \end{array}$$

into a commutative diagram.

Define $(G^k(M) \times \pi_1(M))^{\mathrm{Pr}}$ to be the groupoid of pairs $(g, [f])$ such that $\pi_0(g) = \mathrm{Pr}([f])$. If M is orientable, this is simply $G^{+k}(M) \times \pi_1(M)$. The map of groupoids $\tilde{G}^{+k}(M) \to (G^k(M) \times \pi_1(M))^{\mathrm{Pr}}$ is well defined and surjective. It restricts to a covering map of principal fibre bundles

$$\tau : \tilde{G}^{+k}_{*,m}(M) \to (G^k(M) \times \pi_1(M))^{\mathrm{Pr}}_{*,m}\,. \tag{6}$$

The kernel of the corresponding cover of groups is precisely $i_*\pi_1(G^{+k}_{m,m}(M))$, with $i : G^{+k}_{m,m}(M) \to G^{+k}_{*,m}(M)$ the inclusion. Note that i_* has a nonzero kernel precisely when a vertical loop is contractible in $G^{+k}_{*,m}(M)$, but not by a homotopy which stays inside the fibre. Denoting $\pi_1(G^{+k}_{m,m}(M))$ by Z, we obtain the exact sequence

$$1 \to Z/\mathrm{Ker}(i_*) \xrightarrow{i_*} \tilde{G}^{+k}_{m,m}(M) \xrightarrow{\tau} (G^k(M) \times \pi_1(M))^{\mathrm{Pr}}_{m,m} \to 1\,. \tag{7}$$

A moment's thought reveals that this extension is central: if $g(t)$ is a path in $G^{+k}_{m,m}(M)$ and $h(t)$ one in $G^{+k}_{*,m}(M)$, then both $h * (i \circ g)$ and $(i \circ g) \cdot h(1) * h$ can be homotoped into $t \mapsto h(t)g(t)$.

We may as well restrict attention to the case $k = 1$, in which $G^{1}_{*,m}(M)$ is the frame bundle $F(M)$. Indeed, as $G^{+k}_{*,m}(M) \to G^{1,+}_{*,m}(M)$ has contractible fibres, $\tilde{G}^{+k}_{*,m}(M)$ is just the pullback of $G^{+k}_{*,m}(M)$ along $\tilde{G}^{1,+}_{*,m}(M) \to G^{1,+}_{*,m}(M)$.

4.3.2. *Orientable manifolds.* For orientable manifolds, the situation simplifies. If we identify the connected component of $G^{1}_{m,m}(M)$ with $\mathrm{GL}^+(\mathbb{R}^n)$, we obtain a homomorphism i_* of $\widetilde{\mathrm{GL}}^+(\mathbb{R}^n)$ into $\tilde{G}^{1,+}_{m,m}(M)$. There is a second homomorphism $\pi_1(F(M)) \to \tilde{G}^{1,+}_{m,m}(M)$. Their images intersect in $Z/\mathrm{Ker}(i_*)$, and commute by an argument similar to the one on centrality of (7). The group $(\widetilde{\mathrm{GL}}^+(\mathbb{R}^n) \times \pi_1(F(M)))_Z$ is defined as the quotient of $\widetilde{\mathrm{GL}}^+(\mathbb{R}^n) \times \pi_1(F(M))$ by the equivalence relation $(gz, h) \sim (g, zh)$, and can be regarded as a subgroup of $\tilde{G}^{1,+}_{m,m}(M)$. Note that if $\mathrm{Ker}(i_*)$ is nonzero, the above equivalence relation sets it to 1.

If M is orientable, we may restrict our attention to $F^+(M)$, which then has connected fibres. Any path in $F^+(M)$ which starts and ends in the same fibre can therefore be obtained by combining a closed loop with a path in $\mathrm{GL}^+(\mathbb{R}^n)$. For orientable manifolds, we thus have $\tilde{G}^{1,+}_{m,m}(M) \simeq (\widetilde{\mathrm{GL}}^+(\mathbb{R}^n) \times \pi_1(F^+(M)))_Z$, and in the same vein

$$G(k, M) \simeq (G(k, \mathbb{R}^n) \times \pi_1(F^+(M)))_Z\,. \tag{8}$$

Remark. Note that an infinitesimally natural bundle P is natural, i.e. associated to $F^k(M)$ rather than $\widetilde{F}^k(M)$, if and only if the group homomorphism $\rho : G(k, M) \to G$ of Theorem 4.4 is trivial on $\pi_1(F^+(M))$.

4.3.3. *Spin manifolds.* Let M be an orientable manifold, equipped with a pseudo-Riemannian metric g of signature $\eta \in \mathrm{Bil}(\mathbb{R}^n)$. Then

$$OF^+_g := \{f \in F^+(M) \,|\, f^*g = \eta\}$$

is the bundle of positively oriented orthogonal frames. A spin structure is then by definition an $\widetilde{\mathrm{SO}}(\eta)$-bundle[4] Q over M, plus a map $u : Q \to OF^+_g$ such that the following diagram commutes, with κ the canonical homomorphism $\widetilde{\mathrm{SO}}(\eta) \to \mathrm{SO}(\eta)$.

$$\begin{array}{ccc}
\widetilde{\mathrm{SO}}(\eta) & \xrightarrow{\kappa} & \mathrm{SO}(\eta) \\
\circlearrowleft & & \circlearrowleft \\
Q & \xrightarrow{u} & OF^+_g \\
& \searrow \quad \swarrow & \\
& M &
\end{array}$$

[4] There is a subtlety here. Suppose η has indefinite signature, say $(3, 1)$. The group $\mathrm{SO}(3, 1)$ has 2 connected components, so that a universal cover does not exist. As it is a subgroup of the simply connected group $\mathrm{GL}^+(\mathbb{R}^4)$, we simply define $\widetilde{\mathrm{SO}}(3, 1)$ to be $\kappa^{-1}(\mathrm{SO}(3, 1))$ with $\kappa : \widetilde{\mathrm{GL}}^+(\mathbb{R}^4) \to \widetilde{\mathrm{GL}}^+(\mathbb{R}^4)$ the covering map. Thus $\widetilde{\mathrm{SO}}(3, 1)$ is, perhaps surprisingly, not isomorphic to the 2-component spin group $\mathrm{Spin}(3, 1)$. Indeed, if T is time inversion and P is the inversion of 3 space co-ordinates, then $(PT)^2 = \mathbf{1}$ in $\widetilde{\mathrm{SO}}(3, 1)$, as opposed to $(PT)^2 = -\mathbf{1}$ in $\mathrm{Spin}(3, 1)$. Therefore $\pi^{-1}(\pm\mathbf{1}) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ in $\widetilde{\mathrm{SO}}(3, 1)$, whereas $\pi^{-1}(\pm\mathbf{1}) = \mathbb{Z}/4\mathbb{Z}$ in $\mathrm{Spin}(3, 1)$ (see [3]).

Of course the connected component of unity of $\widetilde{\mathrm{SO}}(3, 1)$ and that of $\mathrm{Spin}(3, 1)$ are both isomorphic to $\mathrm{SL}(\mathbb{C}^2)$, so that none of this is relevant if M is both orientable and time-orientable, i.e. if the structure group of the frame bundle reduces to $\mathrm{SO}^{\uparrow}(3, 1)$.

A manifold is called spin if it admits a spin structure.

Define $\hat{Q} := Q \times_{\widetilde{\mathrm{SO}}(\eta)} \widetilde{\mathrm{GL}}^+(\mathbb{R}^n)$, and let us again denote the induced map $\hat{Q} \to F^+(M)$ by u. As any cover of $F^+(M)$ by a $\widetilde{\mathrm{GL}}^+(\mathbb{R}^n)$-bundle can be obtained in this way, there is a 1:1 correspondence between spin covers of $OF_g^+(M)$ and $F^+(M)$. In particular, whether or not M is spin does not depend on the metric.

The Serre homotopy exact sequence gives rise to the exact sequence

$$1 \to Z/\mathrm{Ker}(i_*) \to \pi_1(F^+(M)) \to \pi_1(M) \to 1\,. \tag{9}$$

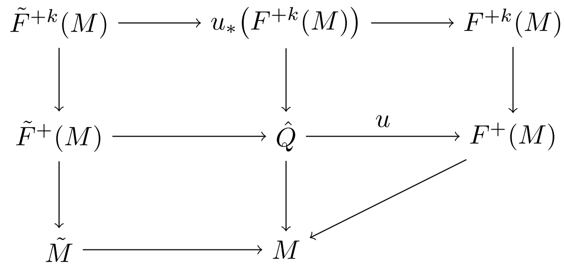
The following proposition is well known.

Proposition 7. *A spin structure exists if and only if $i_* : Z \to \pi_1(F^+(M))$ is injective and (9) splits as a sequence of groups. If spin structures exist, then equivalence classes of spin covers correspond to splittings of (9).*

Proof. See for example [15]. Our criterion for M to be spin is equivalent to the vanishing of the second Stiefel-Whitney class, see e.g. [11]. □

Remark. In terms of group cohomology, one can consider the sequence (9) as a cohomology class $[\omega]$ in $H^2(\pi_1(M), Z/\mathrm{Ker}(i_*))$. Spin bundles exist if and only if both $\mathrm{Ker}(i_*)$ and $[\omega]$ are trivial, in which case they are indexed by $H^1(\pi_1(M), Z)$.

If a spin structure exists, then $\widetilde{F}^+$ is simply the pullback along the universal cover $\tilde{M} \to M$ of $\hat{Q} \to M$. The picture then becomes



with each of the three squares a pullback square.

5. More general fibre bundles.

In this section, we will prove a version of theorem 4.4 for fibre bundles which are not principal. It would however be overly optimistic to expect an analogue of of theorem 4.4 to hold for arbitrary smooth fibre bundles, so we will restrict ourselves to those bundles that carry a sufficiently rigid structure on their fibres, such as vector bundles.

5.1. Structured fibre bundles.

We start by making this statement more precise. The following definition is modelled after the definition of a vector bundle, to which it reduces in the case that $\mathbf{C}$ is the category of vector spaces.

Definition 5.1. Let $\mathbf{C}$ be a category with a faithful functor $\mathcal{F}$ to the category of manifolds. Let C_0 be an object in $\mathbf{C}$ such that $\mathrm{Aut}(C_0)$ is a finite dimensional Lie group, and $\mathcal{F}$ induces a smooth action on $F_0 := \mathcal{F}(C_0)$.

Then a 'structured fibre bundle' with fibre C_0 is a smooth fibre bundle $\pi : F \to M$ with generic fibre $F_0 = \mathcal{F}(C_0)$, and for each $m \in M$ a choice of structure $C_m \in \mathrm{ob}(\mathbf{C})$ such that $\mathcal{F}(C_m) = \pi^{-1}(m)$. We also require that for each $m \in M$, there exist a trivialisation $\phi : \pi^{-1}(U) \to F_0 \times U$ over some neighbourhood U of m which is structure preserving in the sense that for each $x \in U$, there exists an isomorphism $\hat{\phi}_x : C_x \to C_0$ such that $\mathcal{F}(\hat{\phi}_x) = \phi|_{\pi^{-1}(x)}$.

Every vector bundle is associated to its frame bundle. We generalise this to structured fibre bundles.

Proposition 8. *Let* $\pi : F \to M$ *be a structured fibre bundle with fibre* C_0*. Then* $\mathrm{Fr}(F) := \bigsqcup_{x\in M} \mathrm{Iso}(C_0, C_x)$ *is a smooth principal fibre bundle over* M *with structure group* $\mathrm{Aut}(C_0)$*, and* F *is isomorphic to* $\mathrm{Fr}(E) \times_{\mathrm{Aut}(C_0)} F_0$ *as a smooth fibre bundle.*

Proof. If ϕ is a local trivialisation of F over U, then the bijection

$$\hat{\phi} : \bigsqcup_{x\in U} \mathrm{Iso}(C_0, C_x) \xrightarrow{\sim} U \times \mathrm{Aut}(C_0)$$

given by $\hat{\phi} : \alpha_x \mapsto (x, \hat{\phi}_x \circ \alpha_x)$ is an $\mathrm{Aut}(C_0)$-equivariant local trivialisation of $\mathrm{Fr}(E)$. We need to prove that for any two local trivialisations ϕ and ψ over U and V respectively, the equivariant bijection

$$\hat{\psi} \circ \hat{\phi}^{-1} : U \cap V \times \mathrm{Aut}(C_0) \xrightarrow{\sim} U \cap V \times \mathrm{Aut}(C_0)$$

is a smooth map, or equivalently that the map $\hat{\gamma} : U \cap V \to \mathrm{Aut}(C_0)$ defined by $\hat{\gamma}(x) := \mathrm{pr}_2\hat{\psi}\hat{\phi}^{-1}(x, \mathrm{id})$ is smooth. Once we have established this, smoothness of $\hat{\phi}\circ\hat{\psi}^{-1}$ will imply that $\hat{\psi}\circ\hat{\phi}^{-1}$ is a diffeomorphism, that $\mathrm{Fr}(F)$ is a smooth principal fibre bundle, and that the map $\mathrm{Fr}(F) \times_{\mathrm{Aut}(C_0)} F_0 \to F$ given by $[\alpha, f] \mapsto \mathcal{F}(\alpha)f$ is an isomorphism of smooth fibre bundles.

The fact that $\hat{\gamma}$ is continuous w.r.t. the topology of pointwise convergence on $\mathrm{Aut}(C_0)$ (induced by its action on F_0) follows directly from the fact that $\psi \circ \phi^{-1} : U \cap V \times F_0 \xrightarrow{\sim} U \cap V \times F_0$ is a homeomorphism.

The action of $\mathrm{Aut}(C_0)$ on F_0 is smooth by assumption, and effective because $\mathcal{F}$ is faithful. We can therefore choose $(f_1, \ldots, f_k) \in {F_0}^k$ with $k = \dim(\mathrm{Aut}(C_0))$ such that the map $\mathrm{Lie}(\mathrm{Aut}(C_0)) \to T_{f_1}F_0 \times \ldots \times T_{f_k}F_0$ is injective. On a neighbourhood $N \subset \mathrm{Aut}(C_0)$ of the identity, the map $A : \mathrm{Aut}(C_0) \to F_0^k : \alpha \mapsto (\alpha(f_1), \ldots, \alpha(f_k))$ is thus a diffeomorphism onto its image, and if R_β is right multiplication by $\beta \in \mathrm{Aut}(C_0)$, the same goes for $A \circ R_\beta : R_{\beta^{-1}}(N) \to A(N)$. Since $\hat{\gamma}$ is continuous, we can choose $W \subset U \cap V$ and $\beta \in \mathrm{Aut}(C_0)$ such that $\hat{\gamma}(x) \in R_{\beta^{-1}}(N)$ for all $x \in W$. Since the map $A \circ R_\beta \circ \hat{\gamma} : W \to A(N)$ is given by

$$x \mapsto (\mathrm{pr}_2\psi \circ \phi^{-1}(x, \mathcal{F}(\beta)(f_1)), \ldots, \mathrm{pr}_2\psi \circ \phi^{-1}(x, \mathcal{F}(\beta)(f_k)))\,,$$

it is certainly smooth. Since $A \circ \beta^{-1}$ is a diffeomorphism, $\hat{\gamma}$ is smooth as well. □

If $\pi : F \to M$ is any smooth fibre bundle, then an automorphism of π is by definition a diffeomorphism α of F such that $\pi(f) = \pi(f')$ implies $\pi(\alpha(f)) = \pi(\alpha(f'))$. It is called vertical if it maps each fibre to itself.

Definition 5.2. An automorphism of a structured fibre bundle $F \to M$ is an automorphism of the smooth bundle such that for every $m \in M$, there exists an isomorphism $\hat{\alpha}_m : C_m \to C_{m'}$ such that $\mathcal{F}(\hat{\alpha}_m) = \alpha|_{\pi^{-1}(m)}$. The group of automorphisms is denoted $\mathrm{Aut}^{\mathbf{C}}(F)$.

One can then construct a sequence of groups

$$1 \to \mathrm{Aut}_c^{\mathbf{C}}(F)^V \to \mathrm{Aut}_c^{\mathbf{C}}(F) \to \mathrm{Diff}_c^F(M) \to 1 \tag{10}$$

and its corresponding exact sequence of Lie algebras

$$0 \to \Gamma_c^{\mathbf{C}}(TF)^V \to \Gamma_c^{\mathbf{C}}(TF)^P \to \Gamma_c(TM) \to 0\,, \tag{11}$$

where 'V' is for vertical, 'P' for projectable, and c again stands for '0 outside a compact subset of M'. The proof of the following corollary of theorem 4.4 is now a formality.

Corollary 3. *Let $\pi : F \to M$ be a structured fibre bundle with fibre C_0 such that (11) splits as a sequence of Lie algebras. Then there exists a homomorphism $\rho : G(k, M) \to \mathrm{Aut}(C_0)$ such that*

$$F \simeq \widetilde{F}^{+k}(M) \times_\rho F_0 \,.$$

Proof. Using proposition 8, one constructs a natural isomorphism from $\mathrm{Aut}^{\mathbf{C}}(F)$ to $\mathrm{Aut}(\mathrm{Fr}(F))$ under which the vertical subgroups of the two correspond, so that the exact sequence (10) is isomorphic to (1), and therefore (11) to (2). In view of the isomorphism $F \simeq \mathrm{Fr}(F) \times_{\mathrm{Aut}(C_0)} F_0$, we can apply theorem 4.4 to $\mathrm{Fr}(F)$ in order to substantiate our claim. □

5.2. **Vector bundles.** We specialise to the case of vector bundles. These are precisely structured fibre bundles in the category of finite dimensional vector spaces.

The exact sequence of Lie algebras (11) for a vector bundle E with fibre V is then

$$1 \to \mathrm{DO}^0_c(E) \to \mathrm{DO}^1_c(E) \to \Gamma_c(TM) \to 0 \,, \tag{12}$$

where $\mathrm{DO}^1_c(E)$ is the Lie algebra of compactly supported 1^{st} order differential operators on $\Gamma(E)$, and $\mathrm{DO}^0_c(E)$ is the ideal of 0^{th} order ones, that is to say $\mathrm{DO}^0_c(E) \simeq \Gamma_c(E \otimes E^*)$.

Corollary 3 then says that (12) splits as a sequence of Lie algebras if and only if there is a representation ρ of $G(k, M)$ on V such that $E \simeq \widetilde{F}^{+k}(M) \times_\rho V$.

But thanks to the fact that all finite dimensional representations of the universal cover of $\mathrm{GL}^+(\mathbb{R}^n)$ factor through $\mathrm{GL}^+(\mathbb{R}^n)$ itself, we can even say something slightly stronger.

Proposition 9. *Let $E \to M$ be a vector bundle for which (12) splits as a sequence of Lie algebras. Then there exists a representation ρ of the group $(G^k \times \pi_1(M))^{\mathrm{Pr}}_{m,m}$ on V such that*

$$E \simeq (G^k(M) \times \pi_1(M))^{\mathrm{Pr}}_{*,m} \times_\rho V \,.$$

Remark. If M is orientable, this reads $E \simeq \pi^* F^{+k}(M) \times_\rho V$. In this expression, $\pi^* F^{+k}(M)$ is the pullback of $F^{+k}(M)$ along $\pi : \tilde{M} \to M$, considered as a principal $G^{+k}_{0,0}(\mathbb{R}^n) \times \pi_1(M)$-bundle over M.

Proof. Consider the restriction of the map τ in equation (6) to the group $\tilde{G}^{+k}_{m,m}(M)$. In order to prove the proposition, we need but show that its kernel Z acts trivially on V. For $k = 0$, this is clear.

If k is at least 1, the homomorphism $\widetilde{\mathrm{GL}}^+(\mathbb{R}^n) \to \tilde{G}^{+k}_{m,m}(M)$ makes V into a finite dimensional representation space for $\widetilde{\mathrm{GL}}^+(\mathbb{R}^n)$. But it is known (see [10, p. 311]) that all finite dimensional representations of its cover factor through $\mathrm{GL}^+(\mathbb{R}^n)$ itself. This implies that the subgroup Z which covers the identity must act trivially on V, and we may consider

$$\tilde{G}^{+k}_{*,m}(M)/Z \simeq (G^k(M) \times \pi_1(M))^{\mathrm{Pr}}_{*,m}$$

to be the underlying bundle, as announced. □

This reduces the problem of classifying vector bundles with split sequence (12) to the representation theory of $(G^k \times \pi_1(M))^{\mathrm{Pr}}_{m,m}$.

The above extends a result [28] of Terng, in which she classifies vector bundles which allow for a local splitting of the sequence of groups (10). It is an extension first of all in the sense that we prove, rather than assume, that the splitting is local. Secondly, we have shown that in classifying vector bundles with split sequence (12) of Lie algebras rather than groups, one encounters only slightly more. Intuitively speaking, the extra bit is the representation theory of $\pi_1(M)$. We refer to [28] for a thorough exposition of the representation theory of $G^k_{0,0}(\mathbb{R}^n)$.

6. **Flat connections.** In this section, we investigate splittings that come from a flat equivariant connection on a principal G-bundle $P \to M$. We will prove that if the Lie algebra $\mathfrak{g}$ of G does not contain $\mathfrak{sl}(\mathbb{R}^n)$ as a subalgebra, then the sequence of Lie algebras (2) splits if and only if P admits a flat equivariant connection. In other words, the sequence (2) then splits as a sequence of Lie algebras if and only if it splits as a sequence of Lie algebras and $C^\infty(M)$-modules.

Note that this is certainly not the case for general groups G. The frame bundle for example always allows for a splitting of (2), but usually not for a flat connection.

Remark. For general Lie groups G, Lecomte has shown ([12, Thm. 3.1]) that the existence of a splitting is obstructed by the image of the Chern-Weil homomorphism under the map from de Rham cohomology to the Lie algebra cohomology of $\Gamma(TM)$ with values in $C^\infty(M)$. Combined with results of Shiga–Tsujishita [26], this implies that the characteristic classes of $P \to M$ are contained in the ideal generated by the Pontrjagin classes of M.

6.1. **Lie algebras that do not contain $\mathfrak{sl}(\mathbb{R}^n)$.** Although lemma 2.4 exhibits σ as a differential operator of finite order, the bound on the order is certainly not optimal. With full knowledge of the Lie algebras at hand, sharper restrictions can be put on the kernel of σ. In particular, if $\mathfrak{g}$ does not contain $\mathfrak{sl}(\mathbb{R}^n)$, there is only a single relevant ideal, and σ is of order at most 1. For notation, see section 2.2.

Lemma 6.1. *Let $n = 1$, and let $\mathfrak{g}$ be such that it does not contain two nonzero elements such that $[X, Y] = Y$. Or let $n \geq 2$, and let $\mathfrak{g}$ be such that it does not admit $\mathfrak{sl}(\mathbb{R}^n)$ as a subalgebra. Then the kernel of the homomorphism $\check{f}_m : \mathrm{Vec}_n \to \mathfrak{g}$ contains $\{v \in \mathrm{Vec}_n \,|\, \mathrm{Div}_m(v) = 0\}$.*

Proof. We start with the case $n = 1$. Again, we note that the only ideals of $\mathrm{Vec}_1 = \mathrm{Span}\{x^k\partial_x \,|\, k \geq 1\}$ are $\mathrm{Span}\{x^2\partial_x, x^k\partial_x \,|\, k \geq 4\}$, and for each $N \geq 1$ an ideal $\mathrm{Span}\{x^k\partial_x \,|\, k \geq N\}$. The corresponding quotients all contain two elements X and Y with $[X, Y] = Y$, except the ideals corresponding to $N = 1, 2$. This means that also $\mathfrak{g}$, containing $\mathrm{Vec}_1/\ker(\check{f}_m)$ as the image of $\check{f}_m$, will possess X and Y such that $[X, Y] = Y$ unless the kernel of $\check{f}_m$ contains the ideal $\mathrm{Span}\{x^k\partial_x \,|\, k \geq 2\} = \{v \in \mathrm{Vec}_1 \,|\, \mathrm{Div}_m(v) = 0\}$.

Now for $n \geq 2$. Under the identification $\mathrm{Vec}^0_n \simeq \mathfrak{gl}(\mathbb{R}^n)$ given by $x_i\partial_j \mapsto e_{ij}$, the Euler vector field is the identity $\mathbf{1}$ and Div_m becomes the trace. As $\ker(\check{f}_m)^0$ is an ideal in $\mathfrak{gl}(\mathbb{R}^n)$, it can be either 0, $\mathbb{R}\mathbf{1}$, $\mathfrak{sl}(\mathbb{R}^n)$ or $\mathbb{R}\mathbf{1} \oplus \mathfrak{sl}(\mathbb{R}^n)$. In the former two cases, $\mathrm{Im}(\check{f}_m) \simeq \mathrm{Vec}_n/\ker(\check{f}_m)$, and hence $\mathfrak{g}$, would contain $\mathfrak{sl}(\mathbb{R}^n)$ as a subalgebra, contradicting the hypothesis. Hence $\mathfrak{sl}(\mathbb{R}^n) \subseteq \ker(\check{f}_m)$. If we now show that $[\mathrm{Vec}_n, \mathfrak{sl}(\mathbb{R}^n)] = \mathfrak{sl}(\mathbb{R}^n) \bigoplus_{k=1}^\infty \mathrm{Vec}^k_n$, the proof will be complete.

Let $i \neq j$. We then have $[x_i\partial_j, x_j x^{\vec{\alpha}}\partial_j] = (\alpha_j + 1)x_i x^{\vec{\alpha}}\partial_j$, showing that $x_i x^{\vec{\alpha}}\partial_j \in [\mathrm{Vec}_n, \mathfrak{sl}(\mathbb{R}^n)]$. The only basis elements not of this shape are of the form $x^k_j\partial_j$. But

$[x_j\partial_i, x_i x_j^{k-1}\partial_j] = x_j^k\partial_j - x_i x_j^{k-1}\partial_i$. If $k \geq 2$, the latter part was just shown to be in $[\mathrm{Vec}_n, \mathfrak{sl}(\mathbb{R}^n)]$, so that also $x_j^k\partial_j \in [\mathrm{Vec}_n, \mathfrak{sl}(\mathbb{R}^n)]$. If $k = 1$, the elements $x_j\partial_j - x_i\partial_i$ join $x_i\partial_j$ to form a basis of $\mathfrak{sl}(\mathbb{R}^n)$. □

This rather limits the possibilities. Not only can we restrict to first order, but also the Lie algebroid map $\nabla : J^1(TM) \to TP/G$ vanishes on the trace-zero jets $K_m = \{\, j^1_m(v) \in J^1(TM) \,|\, v(m) = 0 \text{ and } \mathbf{tr}(v) = 0 \,\}$, so that it factors through the 'trace Lie algebroid' $\mathbf{tr}_m(M) := J^1_m(TM)/K_m$.

This in turn is the Lie algebroid of the 'determinant groupoid' $\mathrm{Det}(M)$. An element $[\alpha]_{m',m}$ of $\mathrm{Det}(M)_{m',m}$ is by definition an equivalence class of diffeomorphisms mapping m to m', with $\alpha \sim \beta$ if and only if $\mathrm{Det}(\beta^{-1}\alpha) = 1$.

As $[\alpha]_{m',m}$ identifies $\wedge^n(T^*_m M)$ with $\wedge^n(T^*_{m'}M)$, the source fibre $\mathrm{Det}(M)_{*,m}$ is isomorphic to the determinant line bundle $\wedge^n(T^*M) \to M$. Its[5] connected component $\wedge^{n,+}(T^*M)$ is the the bundle of positive top forms if M is orientable, and the whole bundle otherwise.

Its universal covering space is the bundle $\wedge^{n,+}(T^*\tilde{M})$ of positive top forms on $\tilde{M}$. Indeed, $\tilde{M}$ is always orientable, regardless whether M is. This means that $\wedge^n(T^*\tilde{M})$ is a trivial bundle, and that its connected component $\wedge^{n,+}(T^*\tilde{M}) \simeq \tilde{M} \times \mathbb{R}^+$ is simply connected. The covering map is induced by the map $\tilde{M} \to M$. This leads to the following version of theorem 4.4.

Proposition 10. *Let P be a principal G-bundle over an n-dimensional manifold M. Let G be such that its Lie algebra $\mathfrak{g}$ does not contain $\mathfrak{sl}(\mathbb{R}^n)$ if $n > 1$, or $[X, Y] = Y$ if $n = 1$. Then there is a homomorphism $\pi_1(M) \times \mathbb{R}^+ \to G$ associating P to the principal $\pi_1(M) \times \mathbb{R}^+$-bundle $\wedge^{n,+}(T^*\tilde{M}) \to M$.*

$$P \simeq \wedge^{n,+}(T^*\tilde{M}) \times_{\pi_1(M)\times\mathbb{R}^+} G\,.$$

We may even classify the possible splittings.

Corollary 4. *Under the hypotheses of proposition 10, any Lie-algebra homomorphism $\sigma : \Gamma_c(TM) \to \Gamma_c(TP)^G$ which splits the sequence of Lie algebras (2) can be written*

$$\sigma = \nabla^\mu + \Lambda \mathrm{Div}_\mu\,, \tag{13}$$

where ∇^μ is a flat equivariant connection on P, and Λ a section of $\mathrm{ad}(P)$ which is constant w.r.t. the connection induced on $\mathrm{ad}(P)$ by ∇^μ.

Remark. In particular, this shows that there exists a flat connection which splits (2), even though most splittings are not flat connections.

Proof. First, we prove the case $P = \wedge^{n,+}(T^*\tilde{M})$. Pick a nonzero (pseudo-) density μ on M. This induces an honest density $\tilde{\mu}$ on $\tilde{M}$, which in turn identifies $\wedge^{n,+}(T^*\tilde{M})$ with $\tilde{M} \times \mathbb{R}^+$. The local trivialisations of $\wedge^{n,+}(T^*\tilde{M}) \to \tilde{M}$ and $\tilde{M} \to M$ combine to locally trivialise $\wedge^{n,+}(T^*\tilde{M}) \to M$. This yields a flat equivariant connection ∇^μ on $\wedge^{n,+}(T^*\tilde{M})$, which annihilates $\tilde{\mu}$.

The splitting σ is uniquely determined by the action of $\sigma(v)$ on local sections $\tilde{\nu}$, which reads $\sigma(v)(\tilde{\nu}) = \pi^* \circ \mathcal{L}_v \circ \pi^{*-1}\tilde{\nu}$, where π is the map from $\tilde{M}$ to M. If we

[5] An isomorphism $\wedge^n(T^*M) \simeq \mathrm{Det}(M)_{*,m}$ is only given after a choice of $\lambda_0 \in \wedge^n(T^*_m M)$. This determines the connected component.

define the divergence w.r.t. μ by the requirement that the Lie derivative $\mathcal{L}_v\mu$ equal $\mathrm{Div}_\mu(v)\mu$, then we have

$$\begin{aligned}\sigma(v)(f\tilde{\mu}) &= \pi^*(\mathcal{L}_v(f\mu)) \\ &= \pi^*(v(f)\mu + \mathrm{Div}_\mu(v)f\mu) \\ &= v(f)\tilde{\mu} + \mathrm{Div}_\mu(v)f\tilde{\mu} \\ &= \nabla^\mu_v(f\tilde{\mu}) + \mathrm{Div}_\mu(v)f\tilde{\mu}\,.\end{aligned}$$

This shows that $\sigma(v) = \nabla^\mu_v + \Lambda\mathrm{Div}_\mu(v)$, with $\Lambda = \partial_r$, the equivariant vertical vector field defined by the action of $\mathbb{R}^+$ on $\wedge^n(T^*\tilde{M})$. (Equivariant vertical vector fields on P correspond to sections of $\mathrm{ad}(P)$.) The general case follows by proposition 10. □

6.2. **Lie algebra cohomology.** If we specialise to the case of a trivial bundle over an abelian group G, we find ourselves in the realm of Lie algebra cohomology. The continuous cohomology of the Lie algebra of vector fields with values in the functions has already been unravelled in all degrees [6]. Corollary 5 describes this cohomology only in degree 1, but now with all cocycles rather than just the continuous ones.

Corollary 5. *Let H_{LA} denote Lie algebra cohomology and H_{dR} de Rham cohomology. Let $\mathfrak{g}$ be abelian, and consider the representation $C^\infty_c(M,\mathfrak{g})$ of $\Gamma_c(TM)$ where a vector field v acts by the Lie derivative $\mathcal{L}_v$. Then*

$$H^1_{LA}(\Gamma_c(TM), C^\infty_c(M,\mathfrak{g})) \simeq H^1_{dR}(M,\mathfrak{g}) \oplus \mathfrak{g}\,.$$

Proof. Consider the trivial bundle $M \times G \to M$ over an abelian Lie group G, which comes equipped with a flat connection ∇^0, which acts as Lie derivative. Note that abelian $\mathfrak{g}$ certainly satisfy the conditions of propositions 10 and 4. View $\Gamma_c(\mathrm{ad}(P)) \simeq C^\infty_c(M,\mathfrak{g})$ as a representation of $\Gamma_c(TM)$, and consider its Lie algebra cohomology. An n-cochain is an alternating linear map $\Gamma_c(TM)^n \to C^\infty_c(M,\mathfrak{g})$. For $f^1 \in C^1$, closure $\delta f^1 = 0$ amounts to

$$\mathcal{L}_v f^1(w) - \mathcal{L}_w f^1(v) - f^1([v,w]) = 0\,.$$

Due to this cocycle condition, $\sigma = \nabla^0 + f^1$ is once again a Lie algebra homomorphism splitting π_*. According to corollary 4, it must therefore take the shape $\sigma = \nabla^\mu + \Lambda\mathrm{Div}_\mu$, where $\Lambda \in \mathfrak{g}$ is constant. One can write $\nabla^\mu = \nabla^0 + \omega^1$ for some closed 1-form ω^1, so that $f^1 = \omega^1 + \Lambda\mathrm{Div}_\mu$. This classifies the closed 1-cocycles.

Exact 1-cocycles satisfy $f^1(v) = \delta f^0(v) = \mathcal{L}_v f^0 = df^0(v)$, with f^0 a 0-cocycle, that is an element of $C^\infty(M,\mathfrak{g})$.

Note that a change of density $\mu' = e^h\mu$ alters f^1 by a mere coboundary Λdh, so that the choice of μ is immaterial. The class of $\omega^1 + \Lambda\mathrm{Div}_\mu$ modulo δC^0 is thus determined by $[\omega^1] \in H^1_{dR}(M,\mathfrak{g})$ and $\Lambda \in \mathfrak{g}$. □

Continuity turns out to be implied by the closedness condition. A similar situation was encountered by Takens in [27], when proving that all derivations of $\Gamma_c(TM)$ are inner, i.e. $H^1_{LA}(\Gamma_c(TM), \Gamma_c(TM)) = 0$.

Acknowledgments. The author was supported by the NWO grant 613.001.214 'Generalised Lie algebra sheaves.' He is indebted to Marius Crainic, Johan van de Leur, Eduard Looijenga, Christoph Wockel and the late Hans Duistermaat for their kind advice.

REFERENCES

[1] A. Banyaga, *The Structure of Classical Diffeomorphism Groups*, volume 400 of Mathematics and its Applications, Kluwer Academic Publishers Group, Dordrecht, 1997.
[2] D. W. Barnes, Nilpotency of Lie algebras, *Math. Zeitschr.*, **79** (1962), 237–238.
[3] M. Berg, C. DeWitt-Morette, S. Gwo and E. Kramer, The pin groups in physics: C, P and T, *Rev. Math. Phys.*, **13** (2001), 953–1034.
[4] M. Crainic and R. L. Fernandes, Integrability of Lie brackets, *Ann. of Math.*, **157** (2003), 575–620.
[5] D. B. A. Epstein and W. P. Thurston, Transformation groups and natural bundles, *Proc. London Math. Soc.*, **38** (1979), 219–236.
[6] D. B. Fuks, *Cohomology of Infinite Dimensional Lie Algebras*, Contemporary Soviet Mathematics, Consultants Bureau, New York, 1986.
[7] H. Glöckner, Differentiable mappings between spaces of sections, 2002. arXiv:1308.1172.
[8] J. Grabowski, A. Kotov and N. Poncin, Geometric structures encoded in the Lie structure of an Atiyah algebroid, *Transform. Groups*, **16** (2011), 137–160.
[9] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1972.
[10] A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, first edition, 1996.
[11] H. B. Lawson and M.-L. Michelsohn, *Spin geometry*, Princeton University Press, second edition, 1994.
[12] P. B. A. Lecomte, Sur la suite exacte canonique associée à un fibré principal, *Bulletin de la S. M. F.*, **113** (1985), 259–271.
[13] K. Mackenzie, *Lie Groupoids and Lie Algebroids in Differential Geometry*, volume 124 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1987.
[14] I. Moerdijk and J. Mrčun, On integrability of infinitesimal actions, *Amer. J. Math.*, **124** (2002), 567–593.
[15] S. Morrison, *Classifying Spinor Structures*, Master's thesis, University of New South Wales, 2001.
[16] K. C. H. Mackenzie and P. Xu, Integration of Lie bialgebroids, *Topology*, **39** (2000), 445–467.
[17] A. Nijenhuis, *Theory of the Geometric Object*, 1952. Doctoral thesis, Universiteit van Amsterdam.
[18] A. Nijenhuis, Geometric aspects of formal differential operations on tensors fields, In *Proc. Internat. Congress Math. 1958*, pages 463–469. Cambridge Univ. Press, New York, 1960.
[19] A. Nijenhuis, Natural bundles and their general properties. Geometric objects revisited, In *Differential geometry (in honor of Kentaro Yano)*, pages 317–334. Kinokuniya, Tokyo, 1972.
[20] R. S. Palais and C. L. Terng, Natural bundles have finite order, *Topology*, **19** (1977), 271–277.
[21] J. Peetre, Réctification (*sic*) à l'article «une caractérisation abstraite des opérateurs différentiels», *Math. Scand.*, **8** (1960), 116–120.
[22] J. Pradines, Théorie de Lie pour les groupoides différentiables, relation entre propriétés locales et globales, *Comptes Rendus Acad. Sci. Paris A*, **263** (1966), 907–910.
[23] S. E. Salvioli, On the theory of geometric objects, *J. Diff. Geom.*, **7** (1972), 257–278.
[24] J. A. Schouten and J. Haantjes, On the Theory of the Geometric Object, *Proc. London Math. Soc.*, **S2-42** (1937), 356–376.
[25] M. E. Shanks and L. E. Pursell, The Lie algebra of a smooth manifold, *Proc. Amer. Math. Soc.*, **5** (1954), 468–472.
[26] Kōji Shiga and Toru Tsujishita, Differential representations of vector fields, *Kōdai Math. Sem. Rep.*, **28** (1976/77), 214–225.
[27] F. Takens, Derivations of vector fields, *Comp. Math.*, **26** (1973), 151–158.
[28] C. L. Terng, Natural vector bundles and natural differential operators, *Am. J. Math.*, **100** (1978), 775–828.
[29] A. Wundheiler, Objekte, Invarianten und Klassifikation der Geometrie, *Abh. Sem. Vektor Tenzoranal. Moskau*, **4** (1937), 366–375.

Received June 2015; revised April 2016.

E-mail address: B.Janssens@uu.nl