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Using MatContM in the study of a nonlinear map in economics

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Abstract. MatContM is a MATLAB interactive toolbox for the numerical study of iterated smooth maps, their Lyapunov exponents, fixed points, and periodic, homoclinic and heteroclinic orbits as well as their stable and unstable invariant manifolds. The bifurcation analysis is based on continuation methods, tracing out solution manifolds of various types of objects while some of the parameters of the map vary. In particular, MatContM computes codimension 1 bifurcation curves of cycles and supports the computation of the normal form coefficients of their codimension two bifurcations, and allows branch switching from codimension 2 points to secondary curves. MatContM builds on an earlier command-line MATLAB package Cl_MatContM but provides new computational routines and functionalities, as well as a graphical user interface, enabling interactive control of all computations, data handling and archiving. We apply MatContM in our study of the monopoly model of T. Puu with cubic price and quadratic marginal cost functions. Using MatContM, we analyze the fixed points and their stability and we compute branches of solutions of period 5, 10, 13 17. The chaotic and periodic behavior of the monopoly model is further analyzed by computing the largest Lyapunov exponents.

1. Introduction

We consider dynamical systems generated by smooth nonlinear maps

$$x \mapsto f(x, \alpha), \quad x \in \mathbb{R}^n, \ \alpha \in \mathbb{R}^m \tag{1}$$

with state variable x and parameter vector α. The understanding of the dynamics of the map (1) requires most often numerical bifurcation analysis with the use of a dedicated software package. For this purpose we developed MatContM, a MATLAB continuation toolbox available at http://sourceforge.net/projects/matcont/. MatContM builds on the command line code Cl_MatContM described in [4], [5] and [6] but supports several new functionalities, as well as providing a uniform interface. A comprehensive tutorial is provided that illustrates the use of MatContM by investigating an example model. The tutorial can be found on http://.../cl_matcontm3p2/Tutorial_MatcontMGUI.pdf.

Experienced users can also use the code in MatContM via the MATLAB command line, which in some cases can extend the capabilities of the software.

MatContM computes bifurcations of fixed points and cycles in one parameter, continues bifurcation curves in two parameters and detects codimension 2 bifurcation points using
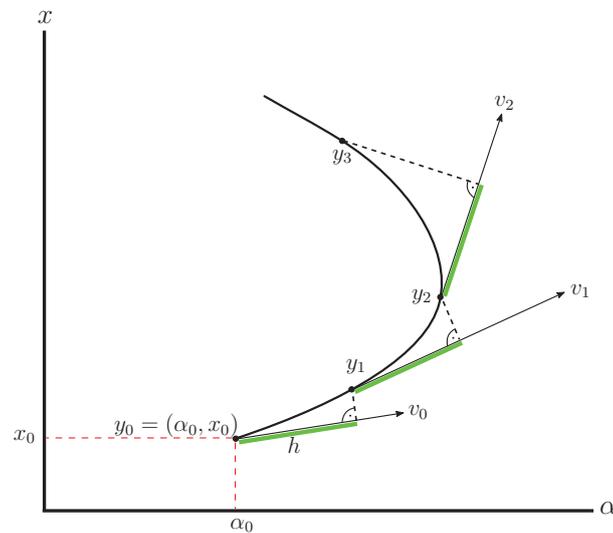
Figure 1: Schematic representation of pseudo-arclength continuation. The x represents the state space, the α represents the varying parameter, also referred to as the free parameter. MatContM uses a variant of Moore-Penrose continuation which builds upon pseudo-arclength continuation.

numerical continuation. A variant of pseudo-arclength continuation is used to approximate these curves (Figure 1). Bifurcations are detected by encountering zeros of test functions during the continuation process. The video file *animation1.mpg* demonstrates the continuation process for a fixed point continuation and *animation2.mpg* demonstrates how test functions are used to detect bifurcations (see Appendix A).

Critical normal form coefficients are computed at bifurcation points, using symbolic derivatives or if these are not available, finite differences or automatic differentiation is used. Automatic differentiation is usually slow, but becomes more competitive for cycles with high iteration number.

We note that there is another MATLAB interactive toolbox called MatCont [3] that provides numerical continuation and bifurcation algorithms for the study of parameterized continuous-time dynamical systems (ODEs).

2. Features and functionalities of MatContM
In a typical use of MatContM, one starts with an initial fixed point or cycle, which may be obtained from analysis, simulations or previous continuations. One first computes curves of fixed points or cycles under variation of one parameter, and may detect bifurcation points on such curves. Starting from such bifurcation points, the continuer algorithm in MatContM can compute bifurcation curves. These curves are defined by a system of equations consisting of fixed point and bifurcation conditions. With one free parameter we can also compute curves of connecting orbits. By varying two system parameters we can compute bifurcation curves of limit points, period-doubling and Neimark-Sacker points as well as tangencies of homoclinic and heteroclinic orbits. The systems that define connecting orbits and their tangencies are fairly complicated as they involve the saddle equilibria at the endpoints of the orbits, their eigenspace structures and the whole connecting orbit. The following list contains functionalities that are provided by MatContM:

- Simulation (iteration) of maps, i.e. computation and visualization of orbits (trajectories).

- Computation of the Lyapunov exponents of long trajectories.
- Continuation of fixed points of maps and iterates of maps with respect to a control parameter.
- Detection of fold (limit point), flip (period-doubling point), Neimark-Sacker and branch points on curves of fixed points.
- Computation of normal form coefficients for fold, flip and Neimark-Sacker bifurcations.
- Continuation of fold, flip and Neimark-Sacker bifurcations in two control parameters. See *animation3.mpg* in Appendix A.
- Detection of all codimension 2 fixed point bifurcations on curves of fold, flip and Neimark-Sacker bifurcations.
- Computation of normal form coefficients for all codimension 2 bifurcations of fixed points.
- Switching to the period doubled branch in a flip point.
- Branch switching at branch points of fixed points.
- Switching to branches of codimension 1 bifurcations rooted in codimension 2 points. See *animation3.mpg* in Appendix A.
- Automatic differentiation for normal form coefficients of codimension 1 and codimension 2 bifurcations.
- Computation of one-dimensional invariant manifolds (stable and unstable) and in the two-dimensional case computing their transversal intersections to obtain initial homoclinic and heteroclinic connections.
- Continuation of homoclinic and heteroclinic orbits with respect to a control parameter and the detection of tangencies on the curve of orbits. See *animation4.mpg* in Appendix A.
- Continuation of homoclinic and heteroclinic tangencies in two control parameters. See *animation5.mpg* in Appendix A.

3. Practical use of MatContM

A simple but important feature is that MatContM has the ability to simulate maps, which means that it can compute orbits (trajectories). These orbits can also be visualized and their Lyapunov exponents can be computed.

The advanced use of MatContM relies on the ability to continue curves under variation of parameter(s) and apply advanced bifurcation theory on given examples of maps. The aim of the package is to allow a user to perform a bifurcation analysis of a map, without deep knowledge of the workings of MatContM or even the continuation software.

The user is able to enter a system, an initial fixed point and start computing with most of the settings left on default. This should lower the entry barrier for researchers from different research fields who want to investigate their models but do not want to be confronted with specification and implementation details.

The continuation curves can be visualized using the plot capabilities of MatContM; this can be done during and after the continuation.

Tools are provided to help with managing systems, diagrams and curves when generating large amount of data.

Specific features have been implemented to make branch switching between continuation curves fast and easy. An initial point can be selected out of a list of special points or it can be selected by double-clicking on a graph of the computed curve. A list of available curves for computation is shown depending on the type of the initial point.

An interface is also provided that allows the exchange of information between MatContM and the MATLAB command line. This enables the user to combine MatContM with other MATLAB software or simply to use computed data on the MATLAB command line.
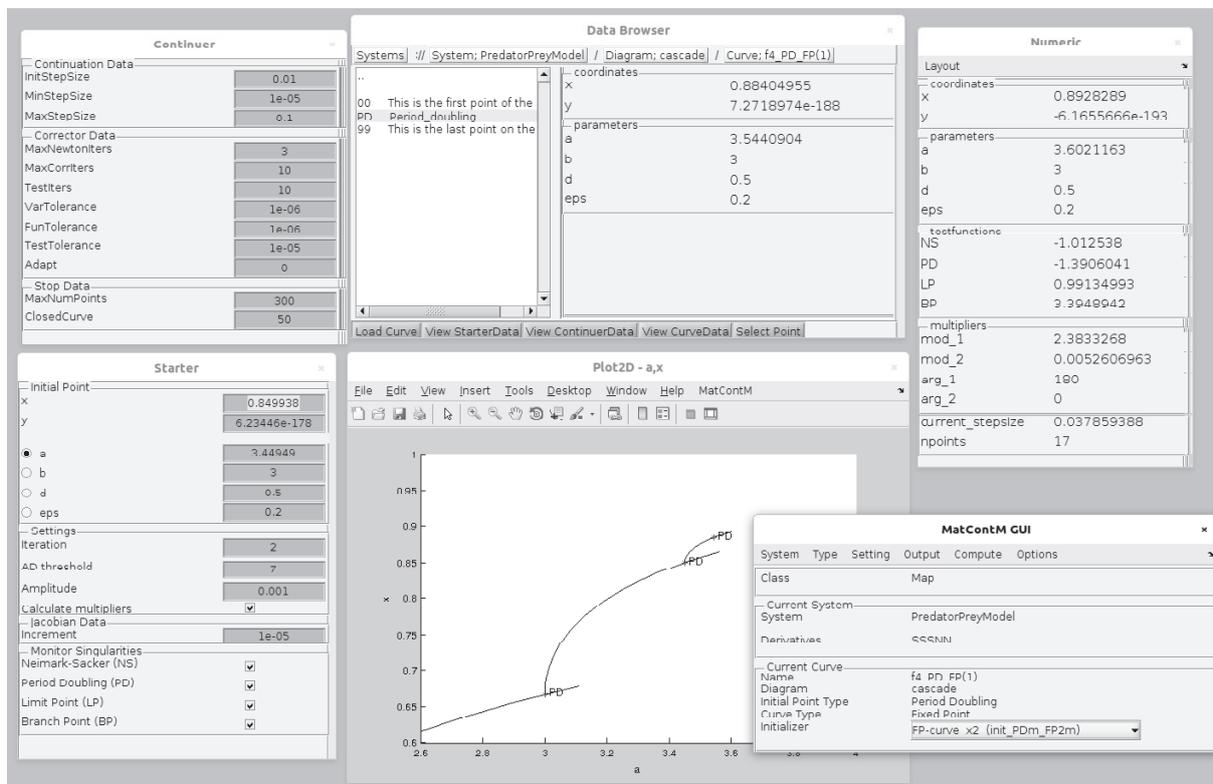
Figure 2: This action screenshot of MatContM presents the main windows that are opened during the computation of a cascade of period-doubling bifurcations in a predator-prey model [1]. Bottom right is the MatContM main window which contains general information on the model that is being used and the top-level commands. Top left is the Continuer window which shows the continuation variables. The fields of this window are independent of the particular curve that is being computed, though the numeric values can vary. Bottom left is the Starter window whose fields strongly depend on the curve that is being computed. Top middle is the Data Browser which allows to inspect all introduced systems, computed curves, etcetera. Top right is the Numeric window which during computation provides numerical values of computed quantities. Bottom middle is the 2D plot window which in this case plots several bifurcations curves; the horizontal axis presents a parameter of the system and the vertical axis is a state variable. The PD points are flip bifurcation points; the period-doubling cascade is clearly visible.

Figure 2 is an action screenshot of MatContM that gives a visual impression of the computation of a cascade of period-doubling bifurcations in a predator-prey model [1]. The video file *animation3.mpg* shows a two-parameter bifurcation analysis of the same model (Appendix A). The bifurcation diagram in *animation3.mpg* can be replicated using MatContM by following the tutorial available on the website.

4. Study of a nonlinear map in economics

We use MatContM to numerically analyze a monopoly model first suggested by Puu [7] which assumes a cubic price and quadratic marginal cost function when maximizing profits using as strategic variable the produced quantity. The price function

$$p(x) = A - Bx + Cx^2 - Dx^3$$

is kept monotonically decreasing by requiring that A, B, C and D are strictly positive and $C^2 < 3BD$. The revenue of the monopolist is

$$R(x) = p(x)x.$$

The marginal revenue is then given by

$$MR = \frac{\mathrm{d}R}{\mathrm{d}x} = p + x\frac{\mathrm{d}p}{\mathrm{d}x}.$$

The marginal cost is assumed to be

$$MC = E - 2Fx + 3Gx^2,$$

where the parameters E, F and G are positive constants. The profit is maximized when $MR = MC$ using standard results of economic theory. The profit function is

$$\Pi(x) = (A - E)x - (B - F)x^2 + (C - G)x^3 - Dx^4. \tag{2}$$

The profit function (2) is assumed to be not explicitly known by the monopolist. In [7] a simple Newton-Raphson-like algorithm is used to find the maximum of (2). The last two visited points x_t and y_t are evaluated and using a step size δ we obtain the next point y_{t+1} as

$$
\begin{aligned}
y_{t+1} &= y_t + \delta\frac{\Pi(y_t) - \Pi(x_t)}{y_t - x_t} \\
&= y_t + \delta\Big((A - E) - (B - F)(x_t + y_t) + \\
&\quad (C - G)(x_t^2 + x_ty_t + y_t^2) - D(x_t^3 + x_t^2y_t + x_ty_t^2 + y_t^3)\Big)
\end{aligned}
\tag{3}
$$

The iteration of this procedure may lead to convergence to a profit maximum, an oscillating process or chaos depending on the value of the step size δ and the parameters. Following [7] we assume $A = 5.6$, $B = 2.7$, $C = 0.62$, $D = 0.05$, $E = 2$, $F = 0.3$ and $G = 0.02$. Using these parameter values, the profit function (2) is symmetric about the point $(3, 3)$. We keep the step size δ as a parameter.

The two step update process (3) can be interpreted as the two dimensional map:

$$M : \begin{pmatrix} x_t \\ y_t \end{pmatrix} \mapsto \begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} y_t \\ y_t + \delta P(x_t, y_t) \end{pmatrix}, \tag{4}$$

with one parameter δ, where

$$P(x, y) = 3.6 - 2.4(x + y) + 0.6(x^2 + xy + y^2) - 0.05(x^3 + x^2y + xy^2 + y^3). \tag{5}$$

We refer to [2] for a detailed analysis of (4) which discusses the chaotic behavior of this monopoly model and investigates solutions of period 4, 5, 10, 13 and 17.

We implement (4) in MatContM for numerical analysis. To start a fixed point continuation we first need an initial fixed point.

The fixed points of (4) are obtained by solving

$$x = y, \tag{6}$$
$$y = y + \delta P(x, y). \tag{7}$$

to get

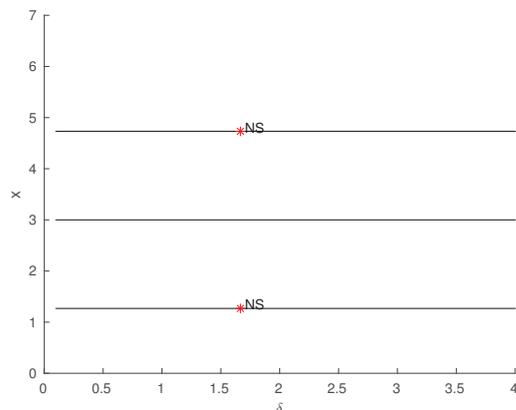$$P(x, x) = 3.6 - 4.8x + 1.8x^2 - 0.2x^3 = 0. \tag{8}$$

Figure 3: Results of fixed point continuations using the three fixed points of (4).

The fixed points of M are $x = 3 \pm \sqrt{3}$ (maxima of (2)) and $x = 3$ (minimum of (2)).

Next, we investigate the stability of these fixed points numerically using `MatContM`. We select a fixed point continuation and enter $x = y = 3 + \sqrt{3}$ in the `Starter` window and we select δ as the free system parameter with initial value 0.1. All other options remain on default. We select **Compute Forward** to start the continuation. We also enable the Numeric window for monitoring the multipliers. The results for all three fixed points are shown in Figure 3.

The fixed points $x = y = 3 \pm \sqrt{3}$ are initially stable for $\delta = 0.1$. The `Numeric` window indicates that the multipliers are within the unit circle. The fixed points lose stability around $\delta = 1.6666\ldots$ through a Neimark-Sacker (NS) bifurcation. This fixed point $x = y = 3$ is initially unstable and remains unstable throughout the continuation. This confirms the theoretical results in §2 of [2].

One of the remarkable properties of (4) is that a stable cycle of period-4 is born. We can obtain this cycle using simulation. We set $x = y = 4$ and $\delta = 1.8$. We select `Point` as an initial point type and select `Orbit (simulation)`. After computing hundreds of points we see the convergence to a period-4 cycle, see Figure 4. When we start from $x = y = 2$ we get another period-4 cycle.

We compute over 4000 points and select the last computed point. We can now continue the period-4 cycle by continuation of a fixed point of the iterated map $M^{(4)}$. Near $\delta \approx 2.62$ we start to detect many branch points (BP), this is caused by coexistence with other cycles with higher periods. In [2] it is also shown that the period-4 cycle loses stability at $\delta \approx 3.8647$.

Figure 5 repeats the computation of orbits for different values of δ. Notice how the fixed points evolve into period-4 cycles when δ passes the Neimark-Sacker bifurcation (upper-right). The two lower images show chaos setting in and we observe the two stable objects merge into a strange attractor.

`MatContM` allows to compute Lyapunov exponents. Select `Point` as `Point Type` and then select `compute Lyapunov exponents` as an initializer. The starter window allows to enter a range of parameter values. Figure 6 shows the largest Lyapunov exponents for δ between 1.5 and 4. A bifurcation diagram, obtained through simulation, is also included.

The negative values for σ in Figure 6 indicate the presence of stable cycles. Through simulations, we indeed find cycles of period 5, 10, 13 and 17 of (4). See Table 1. We then perform a numerical stability analysis. Once a cycle has been discovered for certain parameter value, it can be used as a starting point for continuation.
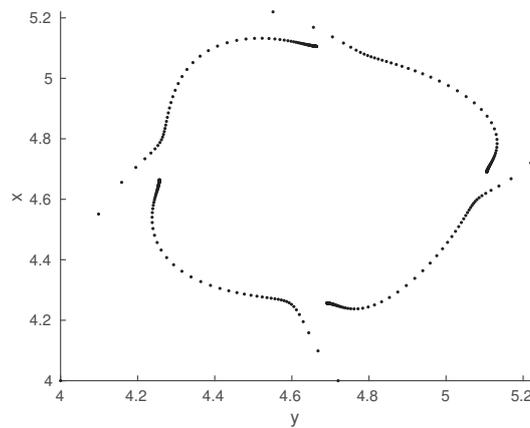
Figure 4: An orbit starting from $(4, 4)$ converges to a period-4 cycle when $\delta = 1.8$. We obtain $((5.10, 4.66), (4.66, 4.25), (4.25, 4.68), (4.68, 5.10))$.
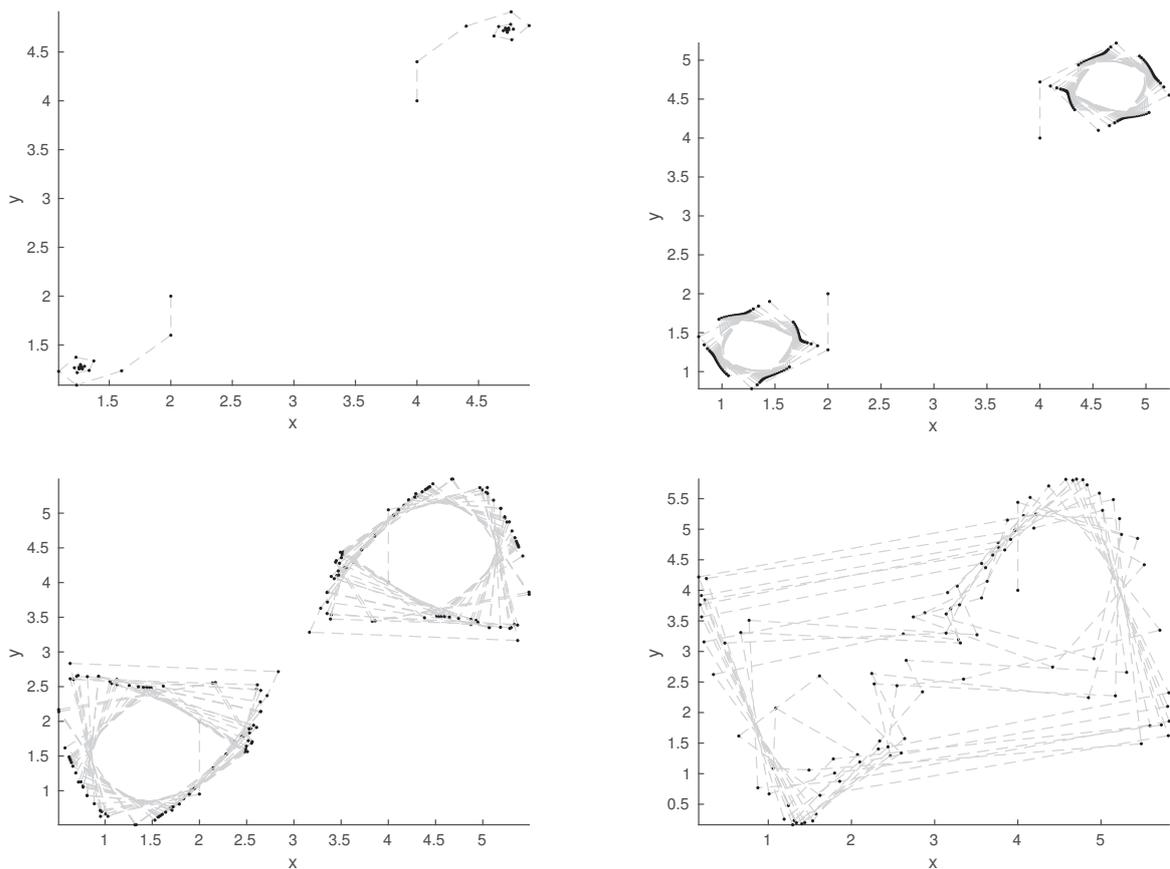

Figure 5: Orbits for $\delta = 1.0$ (upper-left), $\delta = 1.8$ (upper-right), $\delta = 2.62$ (lower-left) and $\delta = 3.6$ (lower-right). When $\delta = 1.0$ there is convergence to the stable fixed points. When $\delta = 1.8$ there is convergence to the stable period-4 cycles. For $\delta = 2.62$ and $\delta = 3.6$ we notice chaotic behavior due to coexistence with other cycles of different periods. The stable period-4 cycles persist but have a very small stability region.
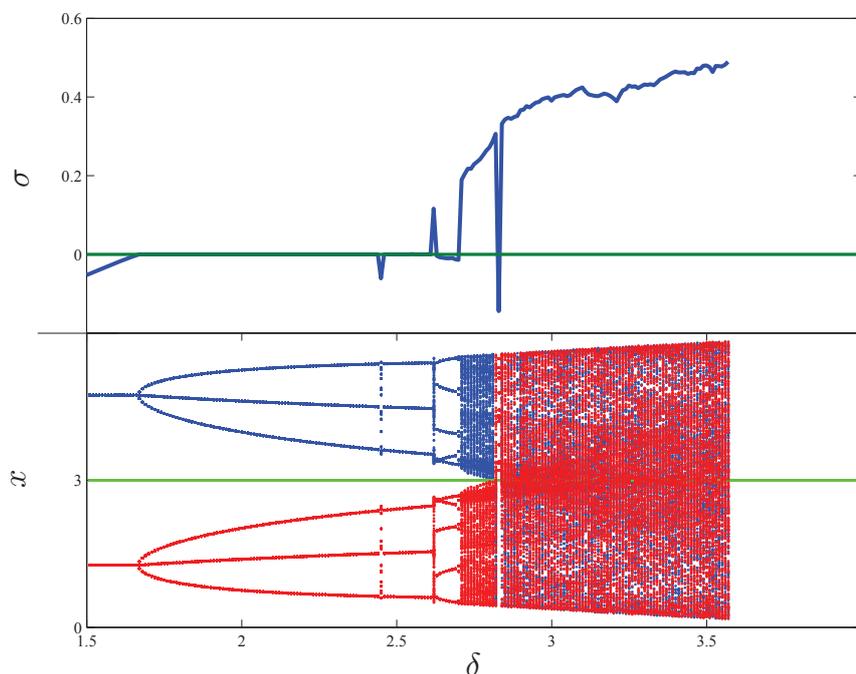
Figure 6: Top: the largest Lyapunov exponent, σ, as a function of δ. Negative values correspond to stable cycles, zero corresponds to 4-cycles with one multiplier equal to one, and positive values indicate chaotic behaviour. Bottom: bifurcation diagram obtained by simulation.

For the cycles of period 5, 10, 13 and 17, and using the initial data in Table 1, we continue each cycle with free parameter δ. The continuation of each cycle leads to a closed curve of cycles. Limit point (LP), Branch point (BP) and period-doubling (PD) bifurcations are found along these curves. Figure 7 shows the cycle of period 13 for $\delta = 2.83$, A part of the branch of 13-cycles is shown in Figure 8.

	5-Cycle	10-Cycle	13-Cycle	17-Cycle
δ	2.63	2.7	2.83	2.45
x	5.39438	3.99216	5.47699	4.24294
y	3.38445	4.86040	4.26739	5.16145

Table 1: One point on the 5,10,13,17-cycles.

During continuation, `MatContM` can also compute the multipliers which determine the stability of the 5, 10, 13 and 17-cycles.

Thus, the stability regions of the 5, 10, 13 and 17-cycles are bounded by the LP and PD points. The 10-cycles are stable in the regions bounded by the BP and PD points. The 5, 10, 13 and 17-cycles are unstable between two successive LP points or two successive PD points. Table 2 shows the stability regions for each cycle. The existence of PD bifurcations on the 10, 13, and 17-cycles indicates the existence of cycles of higher periods as well.

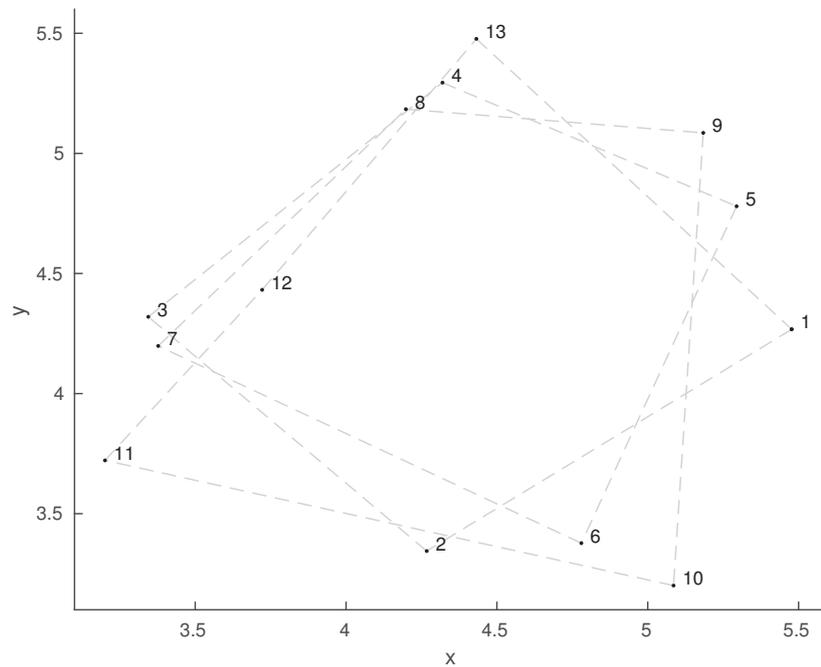In *animation6.mpg* we show the continuation process of a 5-cycle in the state space and in

Figure 7: The period-13 cycle represented in Table 1 and used as initial data for computing Figure 8.
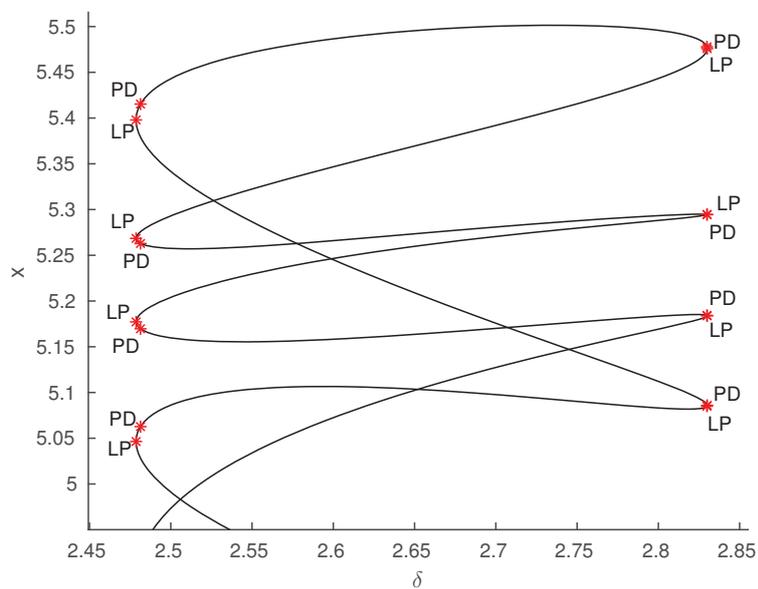

Figure 8: Zoom of a bifurcation diagram of period-13 cycles in (δ, x)-plane

the parameter space. We obtain a closed curve similar to Figure 8. We select a Period-Doubling bifurcation point and start a continuation of the doubled cycle of period 10. This continuation curve also has Period-Doubling bifurcations which allows us to start the continuation of a 20-

	Stable for δ in
5-Cycle	$(2.62813, 2.69111) \cup (3.52522, 3.52573)$
10-Cycle	$(2.69817, 2.70485) \cup (2.80134, 2.80214) \cup (3.12750, 3.12753) \cup (3.52149, 3.52522)$
13-Cycle	$(2.47864, 2.48136) \cup (2.82987, 2.83005)$
17-Cycle	$(2.44977, 2.45042) \cup (2.76425, 2.76432)$

Table 2: Stability regions of the 5,10,13 and 17-cycles of the monopoly model.

cycle.

Appendix A. Animations
The following video files were submitted:

 (i) *animation1.mpg*: Continuation of a fixed point

 (ii) *animation2.mpg*: Detection of bifurcations

(iii) *animation3.mpg*: Generating a bifurcation diagram

(iv) *animation4.mpg*: Continuation of connecting orbits

 (v) *animation5.mpg*: Continuation of tangencies

(vi) *animation6.mpg*: Period-doubling of cycles

These files are also available on http://sourceforge.net/projects/matcont/files/Documentation/MatcontM/files/NOMA15/.

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