The Probability of Connectivity in a Hyperbolic Model of Complex Networks*

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Received 13 February 2015; accepted 30 July 2015
Published online 3 June 2016 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/rsa.20626

ABSTRACT: We consider a model for complex networks that was introduced by Krioukov et al. (Phys Rev E 82 (2010) 036106). In this model, N points are chosen randomly inside a disk on the hyperbolic plane according to a distorted version of the uniform distribution and any two of them are joined by an edge if they are within a certain hyperbolic distance. This model exhibits a power-law degree sequence, small distances and high clustering. The model is controlled by two parameters α and ν where, roughly speaking, α controls the exponent of the power-law and ν controls the average degree.

In this paper we focus on the probability that the graph is connected. We show the following results. For $\alpha > \frac{1}{2}$ and ν arbitrary, the graph is disconnected with high probability. For $\alpha < \frac{1}{2}$ and ν arbitrary, the graph is connected with high probability. When $\alpha = \frac{1}{2}$ and ν is fixed then the probability of being connected tends to a constant $f(\nu)$ that depends only on ν, in a continuous manner. Curiously, $f(\nu) = 1$ for $\nu \geq \pi$ while it is strictly increasing, and in particular bounded away from zero and one, for $0 < \nu < \pi$. © 2016 Wiley Periodicals, Inc. Random Struct. Alg., 49, 65–94, 2016

Keywords: random geometric graphs; complex networks

1. INTRODUCTION

In this paper we will study a random graph model that was introduced recently by Krioukov et al. in [11]. The aim of that work was the development of a geometric framework for the analysis of properties of the so-called complex networks. This term summarizes a large class of networks that emerge in a range of human activities which includes social networks, scientific collaborator networks as well as computer networks, such as the Internet, and the

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*Supported by a Marie Curie Career Integration Grant [PCIG09-GA2011-293619]

power grid—see for example [1]. These are networks that consist of a very large number of heterogeneous nodes (nowadays social networks such as the Facebook or the Twitter have billions of users), and they are locally sparse. This means that the number of neighbours of a typical node (also called the degree of the node) is much smaller than the total number of nodes in the network. However, this is not the case for all nodes. In fact, there are nodes that have a number of neighbours that is much larger than that of a typical node. These are the *hubs* of the network and in fact most of the typical nodes are within a small distance from them, keeping the distance between most pairs of nodes small. This phenomenon has come to be known as the *small world effect*. The existence of hubs is made possible by the distribution of the degrees. Measurements on several examples of networks suggest that this follows a *power law*—see [1] and the references therein. That is, the fraction of nodes of degree k scales like $k^{-\gamma}$, where γ is the exponent of the power law and in most cases this has been measured to be less than 3.

The basic hypothesis of Krioukov et al. [11] is that hyperbolic geometry underlies these networks. In particular, the power law degree distribution is in fact the expression of an underlying hyperbolic geometry. They defined an associated random graph model, which we will describe in detail shortly, and analysed some of its typical properties. More specifically, they observed a power law degree sequence as well as clustering properties. These characteristics were later verified rigorously by Gugelmann et al. [6] as well as by the second author [5] and Candellero and the second author [3] (the last work is on a different, but closely related model).

In a companion paper [2] we study the largest component of the random graph model introduced by Krioukov et al. [11]. In this work, we will focus on the probability that the graph is connected.

1.1. The Krioukov-Papadopoulos-Kitsak-Vahdat-Boguñá Model

We start by recalling some facts about the hyperbolic plane $\mathbb{H}$. The hyperbolic plane is an unbounded surface of constant negative curvature -1. There are several ways to represent it in two dimensions, including the half-plane model, the Beltrami-Klein disk model and the Poincaré disk model. In the Poincaré disk model, we equip the unit disk $\mathbb{D} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ with the metric determined[1] by the differential form $\mathrm{d}s^2 = 4\,\frac{dx^2+dy^2}{(1-x^2-y^2)^2}$. A readable introduction to hyperbolic geometry can for instance be found in the book of Stillwell [13]. In this paper we find it helpful to draw pictures in the *native model* of $\mathbb{H}$. This is obtained from the Poincaré disk model by multiplying each point $(x, y) \in \mathbb{D}$ by a scalar that equals the ratio of the distance to the origin in the hyperbolic metric, over the distance to the origin in the euclidean metric (in the case of the origin itself we define this ratio to equal one). In other words, the native model is obtained from the Poincaré disk model via the map $p \mapsto \frac{2\tanh^{-1}(\|p\|)}{\|p\|}p$. This produces a model of $\mathbb{H}$ that fills all of $\mathbb{R}^2$. It lacks many of the properties that make the classical models of $\mathbb{H}$ elegant to work with, but it does allow us to see more detail in visualizations of the graph model we are about to introduce. To the best of our knowledge the native model was first introduced by Krioukov et al. [11].

Basic facts about $\mathbb{H}$ that we will rely on heavily in the paper are that in $\mathbb{H}$ a disk of radius r (i.e. the set of points at hyperbolic distance at most r to a given point) has area equal to

[1]This means that the length of a curve $\gamma : [0, 1] \to \mathbb{D}$ is given by $2\int_0^1 \frac{\sqrt{(\gamma_1'(t))^2+(\gamma_2'(t))^2}}{1-\gamma_1^2(t)-\gamma_1^2(t)}\mathrm{d}t$.

$2\pi(\cosh(r)-1)$ and circumference length equal to $2\pi\sinh(r)$. Another important fact that we will rely on in the paper is the *hyperbolic cosine rule*. It states that if A, B, C are distinct points on the hyperbolic plane, and we denote by a the distance between B, C, by b the distance between A, C, by c the distance between A, B and by γ the angle (at C) between the shortest AC- and BC-paths, then $\cosh(c) = \cosh(a)\cosh(b) - \cos(\gamma)\sinh(a)\sinh(b)$.

We are now ready to introduce the model we will be studying in this paper. We call it the Krioukov-Papadopoulos-Kitsak-Vahdat-Boguñá-model, after its inventors. For convenience we will abbreviate this to KPKVB-model throughout the rest of the paper. The model has three parameters: the number of vertices N, which we think of as large, and $\alpha, \nu > 0$ which we think of as fixed. Given N, ν, α, we compute $R := 2\log(N/\nu)$. We now select N points independently at random from the disk of radius R centred at the origin O, which we denote by $\mathcal{D}_R$, according to the following probability distribution. If the random point u has polar coordinates (r, θ), then θ, r are independent, θ is uniformly distributed in $(0, 2\pi]$ and the cumulative distribution function of r is given by:

$$F_{\alpha,R}(r) = \begin{cases} 0 & \text{if } r < 0, \\ \frac{\cosh(\alpha r)-1}{\cosh(\alpha R)-1} & \text{if } 0 \le r \le R, \\ 1 & \text{if } r > R \end{cases} \tag{1}$$

Note that when $\alpha = 1$, then this is simply the uniform distribution on $\mathcal{D}_R$.

An alternative way to view this distribution is as follows. It can be seen that the above probability distribution corresponds precisely to the polar coordinates of a point taken uniformly at random from the disk of radius R around the origin in the hyperbolic plane of curvature[2] $-\alpha^2$. (We however treat these points as points of the ordinary hyperbolic plane.) The set of N points we have thus obtained will be the vertex set of our random graph and we denote it by V_N. The KPKVB-random graph, denoted $\mathcal{G}(N; \alpha, \nu)$, is formed when we join each pair of vertices, if and only if they are within (hyperbolic) distance R. Note this is precisely the radius of the disk $\mathcal{D}_R$ that the points live on. Figure 1 shows an example of such a random graph on $N = 1000$ vertices.

We should also mention that later in their paper, Krioukov et al. [11] in fact also defined a generalisation of this model (the model we study in this paper corresponds to the zero-temperature version of that one).

Let us remark that *edge-set of* $\mathcal{G}(N; \alpha, \nu)$ *is decreasing in* α *and increasing in* ν in the following precise sense. We remind the reader that a *coupling* of two random objects X, Y is a common probability space for a pair of objects (X', Y') whose marginal distributions satisfy $X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y$.

Lemma 1.1 ([2]). *Let* $\alpha, \alpha', \nu, \nu' > 0$ *be such that* $\alpha \ge \alpha'$ *and* $\nu \le \nu'$. *For every* $N \in \mathbb{N}$, *there exists a coupling such that* $G(N; \alpha, \nu)$ *is a subgraph of* $G(N; \alpha', \nu')$.

The proof can be found in our companion paper [2] on the largest component of the KPKVB-model.

We mention that Krioukov et al. in fact had an additional parameter in their definition of the model. However, it turns out that this parameter is obsolete in the following sense. Every

[2]That is the natural analogue of the hyperbolic plane, in which the Gaussian curvature equals $-\alpha^2$ at every point. One way to obtain (a model of) the the hyperbolic plane of curvature $-\alpha^2$ is to multiply the differential form in the Poincaré disk model by a factor $1/\alpha^2$.

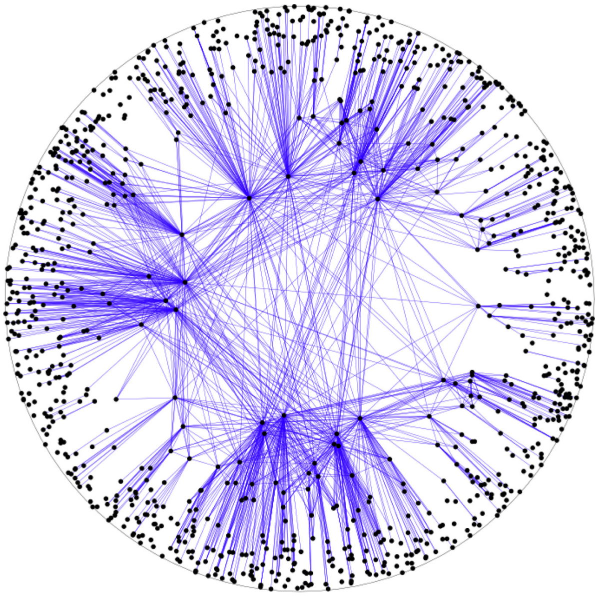

Fig. 1. Simulation of the KPKVB-model with $N = 1000, \alpha = .51, \nu = .2$, depicted in the native model of the hyperbolic plane. [color figure can be viewed in online issue, which is available at wileyonlinelibrary.com]

probability distribution (on labelled graphs) that is defined by some choice of parameters in the model with one extra parameter coincides with the probability distribution $\mathcal{G}(N; \alpha, \nu)$ for some N, α, ν. See [2] (Lemma 1.1) for more details.

Krioukov et al. [11] focus on the degree distribution of $\mathcal{G}(N; \alpha, \nu)$, showing that when $\alpha > \frac{1}{2}$ this follows a power law with exponent $2\alpha + 1$. They also discuss clustering on a smooth version of the above model. Their results have been verified rigorously by Gugelmann et al. [6]. Note that when $\alpha = 1$, that is, when the N vertices are uniformly distributed in $\mathcal{D}_R$, the exponent of the power law is equal to 3. When $\frac{1}{2} < \alpha < 1$, the exponent is between 2 and 3, as is the case in a number of networks that emerge in applications such as computer networks, social networks and biological networks (see for example [1]). When $\alpha = \frac{1}{2}$ then the exponent becomes equal to 2. This case has recently emerged in theoretical cosmology [10]. In a quantum-gravitational setting, networks between space-time events are considered where two events are connected (in a graph-theoretic sense) if they are causally connected, that is, one is located in the light cone of the other. The analysis of Krioukov et al. [10] indicates that the tail of the degree distribution follows a power law with exponent 2.

As observed by Krioukov et al. [11] and rigorously proved by Gugelmann et al. [6], the average degree of the random graph can be “tuned" through the parameter ν : for $\alpha > \frac{1}{2}$, the average degree tends to $2\alpha^2\nu/\pi(\alpha - \frac{1}{2})^2$. Also, recently Kiwi and Mitsche [9] showed that for $\alpha < 1$ the diameter is polylogarithmic in N.

1.2. The Probability That $\mathcal{G}(N; \alpha, \nu)$ is Connected

This paper focuses on the connectivity of $\mathcal{G}(N; \alpha, \nu)$. A straightforward consequence of the results on the degree sequence [6], is that the graph is a.a.s. disconnected whenever

$\alpha > \frac{1}{2}$ (since there will then be vertices of degree zero). Here and in the rest of the paper, the abbreviation a.a.s. stands for *asymptotically almost surely*, meaning "with probability tending to one as N tends to infinity". As we will show here, the probability that the graph is connected becomes bounded away from zero exactly when α crosses $\frac{1}{2}$. Let us also remark that the case $\alpha \leq \frac{1}{2}$ is a "dense" regime in the sense that the average degree of $\mathcal{G}(N; \alpha, \nu)$ is no longer constant, but grows with N. The main result of our paper is as follows.

Theorem 1.2. *Let $\alpha, \nu > 0$ be arbitrary. Then the following hold*

i. If $\alpha > \frac{1}{2}$ then $G(N; \alpha, \nu)$ is a.a.s. disconnected;
ii. If $\alpha < \frac{1}{2}$ then $G(N; \alpha, \nu)$ is a.a.s. connected.
iii. If $\alpha = \frac{1}{2}$ then

$$\lim_{N\to\infty} \mathbb{P}(G(N; \alpha, \nu) \text{ is connected }) = f(\nu),$$

where $f : (0, \infty) \to (0, 1]$ is a continuous function satisfying **(a)** *$f(\nu) = 1$ for all $\nu \geq \pi$;* **(b)** *$f(\nu)$ is strictly increasing for $0 < \nu < \pi$; and* **(c)** *$\lim_{\nu\downarrow 0} f(\nu) = 0$.*

This result highlights a strikingly different behaviour from all the other random graph models in the literature as far as we are aware, due the curious behaviour in part iii. When $\alpha = 1/2$ then the limiting probability of connectedness is bounded away from zero and one for all $0 < \nu < \pi$, while it equals one for $\nu \geq \pi$.

We also remark that, as we will see in the proof, unlike in the cases of the binomial random graph and Gilbert random graph (or random geometric graph), the probability of being connected is not simply governed by the presence of isolated vertices. In these two well-studied models, isolated vertices are in a sense that can be made precise the "last obstructions to connectivity". In contrast, in the KPKVB model the probability of being connected is governed by the presence or absence of a small "covering set". That is, a group of points C near the center of $\mathcal{D}_R$ with the property that every point of $\mathcal{D}_R$ is within distance R of some point of C.

Overview of the Proof and the Structure of the Paper

In the next section, we spell out the short proofs of parts i and ii of Theorem 1.2. As mentioned earlier, part i follows directly from one of the main results in [6]. It turns out that when $\alpha \leq 1/2$ the probability that the graph is connected can be well approximated by the probability that there exists a set of points with small radii such that all of the disk $\mathcal{D}_R$ is covered by the disks of radius R around each of the points. We will call such a set of points a *cover*. In the case when $\alpha < 1/2$ it is relatively easy to show that a cover exists with probability $1 - o(1)$. In the case when $\alpha = 1/2$ determining the probability of the existence of a cover is much more involved. It turns out that this probability can be described in terms of a time-inhomogeneous branching process with infinitely many types.

In the next section we give the quick derivations of parts i and ii of Theorem 1.2. In Section 3, we review and extend some classical results on multitype branching processes that will be needed in the sequel. In Section 4, we describe an auxiliary random process that will help us to derive expressions for the probability of the graph being connected in the case when $\alpha = 1/2$. Finally, in Section 5, we derive part iii of Theorem 1.2 from the results in Section 4.

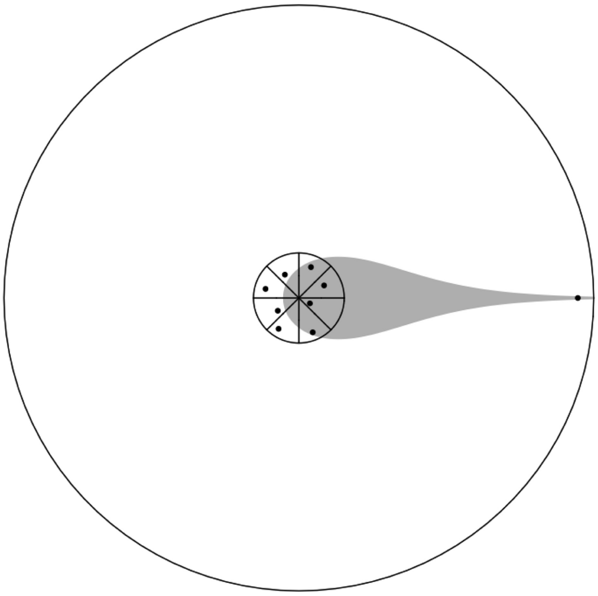

Fig. 2. The event E, depicted in the native model of the hyperbolic plane. The shaded area shows the disk of radius R around the point on the right.

2. PROOF OF THEOREM 1.2: PARTS I AND II

Part i is a direct corollary of Theorem 2.2 in [6], since this theorem implies that when $\alpha > \frac{1}{2}$ there are isolated vertices a.a.s.

For Part ii of Theorem 1.2 we argue as follows. Let us fix an arbitrary $\nu > 0$ and $0 < \alpha < \frac{1}{2}$. We partition the disk of radius one around the origin into eight equal slices $S_i := \{(r,\theta) : 0 \le r \le 1, (i-1)\pi/4 \le \theta \le i\pi/4\}, i = 1, \ldots, 8$. Now we let E denote the event that each S_i contains at least one point. See Fig. 2 for a depiction of the event E. Note that we have

$$\begin{aligned} \mathbb{P}(E) &\ge 1 - 8 \cdot \left(1 - \tfrac{\cosh(\alpha)-1}{8(\cosh(\alpha R)-1)}\right)^N \\ &\ge 1 - 8\exp\left[-N \cdot \tfrac{\cosh(\alpha)-1}{8(\cosh(\alpha R)-1)}\right] \\ &\ge 1 - \exp[-\Omega(e^{(1/2-\alpha)R})] \\ &= 1 - o(1), \end{aligned}$$

where the asymptotics are as $N \to \infty$ (and hence also $R \to \infty$). In the third line we have used that $N = \nu e^{R/2}$ and $\cosh(\alpha R) \sim \frac{1}{2}e^{\alpha R}$ as $N \to \infty$.

It remains to show that the event E implies that the graph is connected. To see this, suppose that E holds and let $u = (r,\theta) \in \mathcal{D}_R$ be arbitrary. Then there is a point $v = (r',\theta') \in V_N$ with $|\theta' - \theta| \le \pi/4$. Let us write $u' := (R,\theta)$ and $v' := (1,\theta')$. Now observe that

$$\begin{aligned} \cosh(1)\cosh(R) - \cos(|\theta'-\theta|)\sinh(1)\sinh(R) &\le \cosh(1)\cosh(R) \\ &\quad - \cos(\pi/4)\sinh(1)\sinh(R) \\ &\sim \left(\left(\frac{1}{2} - \frac{1}{4}\sqrt{2}\right)e + \left(\frac{1}{2} + \frac{1}{4}\sqrt{2}\right)e^{-1}\right)\cosh(R), \end{aligned}$$

where we used that $\cosh R \sim \sinh R$ as $R \to \infty$ in the last line. Since $(\frac{1}{2} - \frac{1}{4}\sqrt{2})e + (\frac{1}{2} + \frac{1}{4}\sqrt{2})e^{-1} < 1$, it follows from the hyperbolic cosine rule that $\mathrm{dist}_{\mathbb{H}}(u', v') \leq R$ provided that N is sufficiently large. We now need the following geometric fact, a proof of which can be found in our companion paper [2].

Lemma 2.1 ([2], Lemma B.1). *Suppose that* $p' = (r', \theta), q' = (s', \vartheta)$ *are two points in the hyperbolic plane satisfying* $\mathrm{dist}_{\mathbb{H}}(p', O), \mathrm{dist}_{\mathbb{H}}(q', O), \mathrm{dist}_{\mathbb{H}}(p', q') \leq R$ *and let* $p = (r, \theta), q = (s, \vartheta)$ *with* $r \leq r', s \leq s'$. *Then* $\mathrm{dist}_{\mathbb{H}}(p, q) \leq R$.

In our case, this last lemma gives that also $\mathrm{dist}_{\mathbb{H}}(u, v) \leq R$, provided N is sufficiently large. As $u \in \mathcal{D}_R$ was arbitrary, it follows that – when N is sufficiently large—the event E implies that every point of $\mathcal{D}_R$ is within distance R of a point of $V \cap B(0, 1)$. Hence, if E holds then the graph is certainly connected – in fact it will have diameter at most three.

This shows that when $0 < \alpha < \frac{1}{2}$ and $\nu > 0$ the graph is a.a.s. connected as claimed in part ii of Theorem 1.2. We now proceed with the proof of part iii.

3. MULTITYPE GALTON-WATSON PROCESSES

In preparation for the proof of part iii of Theorem 1.2, we will review and adapt a classical result on Galton-Watson branching processes with finitely many types. If there are $t < \infty$ types, then such a process is described by a sequence $Z_0, Z_1, \ldots$ of random vectors, where $Z_n := (Z_n^1, \ldots, Z_n^t)$ denotes the vector of the number of particles (individuals) of each type in the n-th generation. In each generation, each of the particles replaces itself with a random set of "children", independently of all other particles and the previous history of the process and according to a probability distribution that does not depend on the generation (but it typically does depend on the type of the particle). We denote

$$p(i; z_1, \ldots, z_t) := \mathbb{P}(Z_1 = (z_1, \ldots, z_t) | Z_0 = e_i).$$

Here and in the rest of the paper e_i denotes the i-th standard basis vector, i.e. the vector with a one in the i-th coordinate and zeroes everywhere else. That is, $p(i; z_1, \ldots, z_t)$ is the probability that a particle of type i fathers z_1 children of type 1, z_2 children of type of type 2, and so on until type t. We will say that "extinction" occurs if $Z_n = (0, \ldots, 0)$ for some n. Otherwise we say "survival" occurs.

We also set

$$m_{ij} := \mathbb{E}(Z_1^j | Z_0 = e_i).$$

That is, m_{ij} is equal to the expected number of children of type j of a particle of type i; and we write $M := (m_{ij})_{1 \leq i,j \leq t}$ for the "matrix of first moments". Let us also remark that, for every $k \in \mathbb{N}$ and $1 \leq i, j \leq t$ we have $(M^k)_{ij} = \mathbb{E}(Z_k^j | Z_0 = e_i)$ (the expected number of type-j particles in the k-th generation if we start with a single particle of type i). We say that the process is *positive regular* if there exists a $k \in \mathbb{N}$ such that every entry of M^k is positive. By the Perron-Frobenius theorem a positive regular matrix has a real, positive eigenvalue ρ that is larger in absolute value than all other eigenvalues (see for instance [7], Chapter II, Section 5, page 37). A multitype Galton-Watson process is called *singular* if each particle has exactly one child (with probability one). Otherwise it is *nonsingular*.

A proof of the following standard result can for instance be found in the book by Harris [7] (Theorem 7.1, Chapter II, page 41), who attributes it to Sevast'yanov [12] and independently Everett and Ulam [4].

Theorem 3.1. *Consider a positive regular, non-singular multitype Galton-Watson process with finitely many types, and let ρ denote the largest eigenvalue of its first moment matrix M. Then the following hold:*

i. If $\rho \leq 1$ then $\mathbb{P}(extinction|Z_0 = e_i) = 1$ for all types $1 \leq i \leq t$;
ii. If $\rho > 1$ then $\mathbb{P}(extinction|Z_0 = e_i) < 1$ for all types $1 \leq i \leq t$.

If $Z_0, Z_1, \ldots$ is as in Theorem 3.1 and ρ is the largest eigenvalue of M then we say the process is *subcritical* if $\rho < 1$, we say it is *critical* if $\rho = 1$ and we say it is *supercritical* if $\rho > 1$.

The following straightforward observation will be used in the sequel. For completeness we spell out a short proof.

Lemma 3.2. *Suppose that $Z_0, Z_1, \ldots$ is a positive regular, nonsingular, supercritical Galton Watson process with $t < \infty$ types. Then there exists another t-type Galton-Watson process $Y_0, Y_1, \ldots$ such that*

i. $p_Y(i; z_1, \ldots, z_t) = 0$ if $p_Z(i; z_1, \ldots, z_t) = 0$;
ii. $p_Y(i; z_1, \ldots, z_t) < p_Z(i; z_1, \ldots, z_t)$ if $p_Z(i; z_1, \ldots, z_t) > 0$ and $(z_1, \ldots, z_t) \neq (0, \ldots, 0)$;

and Y is positive regular, nonsingular and supercritical.

Proof. Let us fix a $0 < \delta < 1$, to be made specific later, and let us define the offspring distributions of Y by:

$$p_Y(i; z_1, \ldots, z_t) = \begin{cases} (1-\delta) \cdot p_Z(i; z_1, \ldots, z_t) & \text{if } (z_1, \ldots, z_t) \neq (0, \ldots, 0), \\ p_Z(i; 0, \ldots, 0) + \delta \cdot (1 - p_Z(i; 0, \ldots, 0)) & \text{if } (z_1, \ldots, z_t) = (0, \ldots, 0). \end{cases}.$$

It is easy to see that this way Y is nonsingular and that $m_{ij}^Y = (1-\delta) m_{ij}^Z$. So in particular Y is also positive regular, and the largest eigenvalue of its first moment matrix satisfies $\rho_Y = (1-\delta)\rho_Z$. Hence we can choose δ so that $\rho_Y > 1$, in which case Y is as required. ∎

Let us say that a Galton-Watson process $Z_0, Z_1, \ldots$ *stochastically dominates* a process $Y_0, Y_1, \ldots$ if there is a coupling such that $Z_n^i \geq Y_n^i$ for all $n \in \mathbb{N}$ and all types i. (Note that if the two processes do not have the same number of types then we can formally add types to the one with fewer types and redefine the offspring distributions in such a way that no particle ever gives birth to a child of the new types.) It is for instance easily seen that the process Y from the previous lemma is stochastically dominated by the original process Z.

We say that *explosion* occurs, if the total number of particles grows without bounds. In other words,

$$\{\text{explosion}\} = \left\{ \lim_{n\to\infty} \left(Z_n^1 + \cdots + Z_n^t \right) = \infty \right\}.$$

If $Z_0, Z_1, \dots$ is as in Theorem 3.1 above, then Theorem 6.1 on page 39 of [7] states that for every vector $z = (z_1, \dots, z_t)$ other than the all-zero vector there are only finitely many generations n for which $Z_n = z$ (with probability one). This has the following immediate corollary.

Theorem 3.3. *If $Z_0, Z_1, \dots$ is a positve regular, nonsingular multitype Galton-Watson process with finitely many types, then*

$$\mathbb{P}(extinction|Z_0 = z) + \mathbb{P}(explosion|Z_0 = z) = 1,$$

for every initial state z.

It is natural to also consider multitype Galton-Watson processes with countably many types. In this case the state of the i-th generation is of course a random vector $Z_i = (Z_i^1, Z_i^2, \dots)$ of countably many nonnegative numbers. We define $p(i; z_1, z_2, \dots)$ and m_{ij} analogously to the case of finitely many types. For $t \in \mathbb{N}$, the t-restriction of a Galton-Watson process $Z_0, Z_1, \dots$ with countably many types is the t-type Galton-Watson process $Y_0, Y_1, \dots$ with offspring distributions given by:

$$p_Y(i; z_1, \dots, z_t) := \begin{cases} p_Z(i; z_1, \dots, z_t, 0, 0, \dots) & \text{if } (z_1, \dots, z_t) \neq (0, \dots, 0), \\ 1 - \sum_{(z_1,\dots,z_t)\neq(0,\dots,0)} p_Y(i; z_1, \dots, z_t) & \text{if } (z_1, \dots, z_t) = (0, \dots, 0). \end{cases}$$

That is, the probability that a particle of type i in the Y process of a z_1 children of type 1, z_2 children of type 2 and so on up to type t, is the probability the a particle of type i under the Z process has exactly these children and none of type bigger than t. We can think of the t-restricted process as a version of the old process where a particle and its potential children die during labour if at least one of the potential children has a type $> t$.

Observe that the original process stochastically dominates the t-restricted process.

Lemma 3.4. *Suppose $Z_0, Z_1, \dots$ is a multitype Galton-Watson process with countably many types, that satisfies the following conditions:*

i. There exists a $c > 1$ such that, for every $i \in \mathbb{N}$, we have $\sum_{j=1}^{\infty} j \cdot m_{ij} \geq c \cdot i$;
ii. For every $i \in \mathbb{N}$ and $j \leq 2i$ we have $m_{ij} > 0$;
iii. Whenever $p(i; z_1, z_2, \dots) > 0$ we have $\sum_{j=1}^{\infty} j \cdot z_j \leq 2i$. (for every $i \in \mathbb{N}, z_1, z_2, \dots \geq 0$);
iv. We have

$$\lim_{i\to\infty} \sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1}+z_{i+2}+\dots>0}} p(i; z_1, z_2, \dots) = 0.$$

(That is, the probability that a particle of type i has at least one child of a strictly larger type is small for large i.)

Then there exists a $t \in \mathbb{N}$ such that the t-restricted process is positive regular, nonsingular and supercritical.

Proof. Observe that, by condition ii, the t-restricted process is positive regular and nonsingular for every $t \geq 1$. Let $\varepsilon > 0$ be arbitrary, to be determined later. By condition iv, there exists a t_0 such that the probability that a particle of type $i \geq t_0$ has a child of type greater than i amongst its children is at most ε. That is:

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1}+z_{i+2}+\dots > 0}} p(i; z_1, z_2, \dots) < \varepsilon \quad \text{(for all } i \geq t_0\text{)}.$$

We now set $t := 2t_0$. Then we have that

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{t+1}+z_{t+2}+\dots > 0}} p(i; z_1, z_2, \dots) = 0 \quad \text{if } i < t_0,$$

by condition iii of the lemma. And, if $t_0 \leq i \leq t$ then we have:

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{t+1}+z_{t+2}+\dots > 0}} p(i; z_1, z_2, \dots) \leq \sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1}+z_{i+2}+\dots > 0}} p(i; z_1, z_2, \dots) < \varepsilon. \tag{2}$$

Let $M = (m_{ij})_{i,j \geq 1}$ denote the matrix of first moments of the original process, and let $M' = (m'_{ij})_{1 \leq i,j \leq t}$ denote that of the t-restricted process. We have that, for every $1 \leq i \leq t$:

$$\sum_{j=1}^{t} j \cdot m'_{ij} \geq \sum_{j=1}^{\infty} j \cdot m_{ij} - \varepsilon \cdot 2i \geq (c - 2\varepsilon)i,$$

using conditions i, iii of the lemma and (2). Thus, if we chose ε small enough so that $c' := c - 2\varepsilon > 1$, then we see that if $v := (1, 2, \dots, t)$ then $(M')^k v \geq (c')^k v$ coordinatewise. Since $(c')^k$ grows without bounds, it follows that M' must have an eigenvalue that is strictly larger than one in absolute value. So in particular (invoking Perron-Frobenius) the eigenvalue of largest absolute value is a real number strictly larger than one. This concludes the proof of the lemma. ∎

In a time-inhomogeneous multitype Galton-Watson process, the offspring distibutions depend on n, the generation. We now denote by $p_n(i; z_1, z_2, \dots) := \mathbb{P}(Z_{n+1} = (z_1, z_2, \dots) | Z_n = e_i)$ the probability that a particle of type i, in generation n, fathers exactly z_j children of type j (for $j = 1, 2, \dots$).

Lemma 3.5. *Suppose that $Z_0, Z_1, \dots$ is a time-inhomogeneous multi-type Galton-Watson process with countably many types such that the limits*

$$\lim_{n \to \infty} p_n(i; z_1, z_2, \dots) =: p(i; z_1, z_2, \dots),$$

exist for all $i \in \mathbb{N}$ and $z_1, z_2, \dots \geq 0$. Suppose further that the limits p belong to a (time-homogeneous) multitype Galton-Watson process satisfying the conditions of Lemma 3.4. Then

$$\liminf_{n \to \infty} \mathbb{P}(\textit{explosion} | Z_n = e_1) > 0.$$

Proof. Let $Z'_0, Z'_1, \dots$ denote the Galton-Watson process belonging to the limiting probabilities $p(i; z_1, z_2, \dots)$ and let us pick t according to Lemma 3.4 with respect to Z'. Let $Y_0, Y_1, \dots$ denote the t-restricted process.

Let $X_0, X_1, \ldots$ denote a process that Lemma 3.2 provides if we apply it to $Y_0, Y_1, \ldots$. Let $\mathcal{I} := \{(z_1, \ldots, z_t) \neq (0, \ldots, 0) : p_X(i; z_1, \ldots, z_t) > 0\}$. Observe that $\mathcal{I}$ is finite, so that there is an n such that $p_{n+m}(i; z_1, \ldots, z_t, 0, 0, \ldots) \geq p_X(i; z_1, \ldots, z_t)$, for all $m \geq 0$ and all $(z_1, \ldots, z_t) \in \mathcal{I}$. This means that $Z_n, Z_{n+1}, \ldots$ stochastically dominates $X_0, X_1, \ldots$, if we condition on $Z_n = X_0 = e_1$. So in particular:

$$\liminf_{n\to\infty} \mathbb{P}(Z \text{ explodes} | Z_n = e_1) \geq \mathbb{P}(X \text{ explodes} | X_0 = e_1) > 0.$$

This concludes the proof of the lemma. ∎

4. AN AUXILIARY COVERAGE PROCESS

In this section, we consider an auxiliary random process that is closely related to the KPKVB random graph with $\alpha = 1/2$. In the rest of the paper, $\mathcal{P} = \mathcal{P}_\nu$ will be a Poisson process on the *entire* hyperbolic plane with intensity function:

$$g\big(r, \theta\big) = g_\nu\big(r, \theta\big) := (\nu/4\pi) \cdot \sinh(r/2), \tag{3}$$

where (r, θ) represents a point of $\mathbb{H}$ in polar coordinate notation. We let $\mathbb{P}_\nu(.)$ denote the associated probability measure. $\mathbb{E}_\nu(.)$ denotes the expected values of random variables over the probability space. We say that an event $E(N)$ is realized with high probability (w.h.p.), if $\mathbb{P}_\nu(E(N)) \to 1$ as $n \to \infty$.

We set

$$\gamma(r) = \gamma_\lambda(r) := \lambda \cdot \arccos\left(\frac{\cosh(r) - 1}{\sinh(r)}\right), \tag{4}$$

where $\lambda > 0$ is a parameter. We will see in the proof of Lemma 5.2 that if two points $x_1 = (r_1, \theta_1), x_2 = (r_2, \theta_2) \in \mathcal{D}_R$ have $|\theta_1 - \theta_2|_{2\pi} \leq \gamma(r_1)$ with $\lambda < 1$, then x_1 and x_2 are within distance R (provided N is large). Here and in the rest of the paper we use the notation $x_r := \min(x, r - x)$ for $r > 0$ and $x \in [0, r]$. Let us remark that $\gamma(r)$ is strictly decreasing in r. (This can be easily seen from the facts that arccos(.) is strictly decreasing and that $(\cosh(r) - 1)/\sinh(r) = 1 - \frac{2}{e^r+1}$ is strictly increasing.) Let us say that an angle $\vartheta \in [0, 2\pi)$ is *covered* by a point $(r, \theta) \in \mathbb{H}$ if

$$|\vartheta - \theta|_{2\pi} \leq \gamma_\lambda(r).$$

We say that a set $A \subseteq \mathbb{H}$ is a *cover* if every angle is covered by some point of A. For $s > 0$, we denote by $\mathcal{C}_s(\lambda)$ the event that $\mathcal{P} \cap B_{\mathbb{H}}(O, s)$ is a cover. The event $\mathcal{C}(\lambda)$ will denote that $\mathcal{C}_s(\lambda)$ is realized for some (finite) $s < \infty$. Note that $\mathcal{C}(\lambda) = \bigcup_{s>0} \mathcal{C}_s(\lambda)$. We now define:

$$\Psi(\nu, \lambda) := \mathbb{P}_\nu(\mathcal{C}(\lambda)). \tag{5}$$

As we will see, $f(\nu) := \Psi(\nu, 1)$ has the properties claimed in Theorem 1.2 and the probability that $G(N; 1/2, \nu)$ is connected tends to $f(\nu)$ as $N \to \infty$. The following theorem is crucial for the proof of part iii of Theorem 1.2.

Theorem 4.1. *The function Ψ defined in* (5) *has the following properties:*

i. $\Psi(\nu, \lambda)$ is continuous in both parameters;

ii. $\Psi(\nu, \lambda) = 1$ *if* $\nu \cdot \lambda \geq \pi$*;*
iii. $\Psi(\nu, \lambda)$ *is strictly increasing in* ν *for* $0 < \nu < \pi/\lambda$*;*
iv. For every fixed $\lambda > 0$ *we have* $\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) = 0$.

The remainder of this section is devoted to the rather involved proof of this theorem. We will split the proof up into a sequence of lemmas.

Lemma 4.2. $\Psi(\nu, \lambda) > 0$ *for all* $\nu, \lambda > 0$.

Proof. Let us set $m := \min\{4, \lceil 4/\lambda \rceil\}$, and let E be the event that each of the $2m$ sets $[0, \pi/m) \times [0, 1], \ldots, [(2m-1)\pi/m, 2\pi) \times [0, 1]$ contains at least one point of $\mathcal{P}$. The expected number of points of $\mathcal{P}$ in each of these sets is $\frac{1}{2m} \cdot \int_0^{2\pi} \int_0^1 g(r, \theta) \mathrm{d}r \mathrm{d}\theta = (\nu/2m) \cdot (\cosh(1/2) - 1)$.

It is easily checked that $\arccos((\cosh(1) - 1)/\sinh(1)) > \pi/4$, so that $\gamma(r) > \lambda\pi/4$ for all $r \leq 1$. We claim the event E implies $\mathcal{C}(\lambda)$. To see this, suppose E is realized and pick an arbitrary angle $\theta \in [0, 2\pi)$. By symmetry, we can assume without loss of generality $\theta \in [0, \pi/m)$. Since E holds, there is a point $(r, \vartheta) \in \mathcal{P} \cap [0, \pi/m) \times [0, 1]$. We find that $|\theta - \vartheta|_{2\pi} < \pi/m \leq \lambda\pi/4 < \gamma(r)$. Thus, the event E indeed inplies $\mathcal{C}(\lambda)$.

We therefore have

$$\Psi(\nu, \lambda) \geq \mathbb{P}_\nu(E) = \left(1 - e^{-(\nu/m)\cdot(\cosh(1/2)-1)}\right)^m > 0,$$

as required. ∎

Lemma 4.3. *For all* $a, b, \lambda > 0$ *we have* $\Psi(a+b, \lambda) \geq \Psi(a, \lambda) + (1 - \Psi(a, \lambda)) \cdot \Psi(b, \lambda)$.

Proof. Since $\mathcal{P}_{a+b}$ can be seen as a *superposition* of $\mathcal{P}_a$ and $\mathcal{P}_b$ for every $a, b > 0$ (see for instance [8]), the probability that $\mathcal{C}(\lambda)$ occurs in $\mathcal{P}_{a+b}$ is at least the probability that it occurs in $\mathcal{P}_a$ plus the probability that is does not occur in $\mathcal{P}_a$ and it occurs in $\mathcal{P}_b$. ∎

Note that the previous two lemmas show that $\Psi(\nu, \lambda)$ is strictly increasing in ν whenever $\Psi(\nu, \lambda) < 1$.

Corollary 4.4. *If* $\nu, \lambda > 0$ *are such that* $\Psi(\nu, \lambda) < 1$ *then* Ψ *is strictly increasing in* ν *at* (ν, λ).

It will be helpful to consider a process where we reveal $\mathcal{P}$ in “discrete steps". For $n \in \mathbb{N}$ let us denote

$$r_n := n \cdot 2 \ln 2. \tag{6}$$

Let us denote $\mathcal{B}_n := \mathcal{P} \cap B_{\mathbb{H}}(0, r_n)$ and $\mathcal{A}_n := \mathcal{B}_n \setminus \mathcal{B}_{n-1}$. ($\mathcal{B}_n$ is the set of points of $\mathcal{P}$ with radii at most r_n and $\mathcal{A}$ is the set of points with radii between r_{n-1} and r_n.)

Before we continue, it will be helpful to derive some asymptotics. Observe that

$$\frac{\cosh(r) - 1}{\sinh(r)} = 1 - 2e^{-r}\left(\frac{1 - e^{-r}}{1 - e^{-2r}}\right). \tag{7}$$

Recall that $\cos(y) = 1 - y^2/2 + O(y^4)$. This implies that if $y = \arccos(1 - x)$ then $y = \sqrt{2x} \cdot \left(1 + O(x^2)\right)$. Combining this with (7) gives:

$$\gamma(r) = \lambda \cdot \arccos\left(\frac{\cosh(r) - 1}{\sinh(r)}\right) = 2\lambda e^{-r/2}(1 + O(e^{-r})) \quad \text{as } r \to \infty. \tag{8}$$

Let us also recall that $\gamma(r)$ is strictly decreasing in r. (As $(\cosh(r) - 1)/\sinh(r) = 1 - 2/(e^r + 1)$ is strictly increasing and $\arccos(.)$ is strictly decreasing.) Using Eq. (8) we can now derive the following.

Lemma 4.5. *For every fixed $\nu, \lambda > 0$ we have that*

$$\mathbb{E}_\nu|\{p \in \mathcal{A}_n : p \text{ covers the angle } 0\}| = (1 + O((1/4)^n)) \cdot (\nu\lambda/\pi) \cdot \ln 2,$$

and

$$\mathbb{P}_\nu(\mathcal{A}_n \text{ does not cover } 0) = (1 + O((1/4)^n)) \cdot (1/2)^{\nu\lambda/\pi},$$

where the $O(.)$-notation refers to $n \to \infty$.

Proof. If μ_n denotes the expected number of points in $\mathcal{A}_n$ that cover the angle 0, then

$$\begin{aligned}
\mu_n &= \int_0^{2\pi} \int_{r_{n-1}}^{r_n} 1_{\{|\theta|_{2\pi} < \gamma(r)\}} \cdot g(r,\theta)\mathrm{d}r\mathrm{d}\theta \\
&= \int_{r_{n-1}}^{r_n} 2\gamma(r) \cdot g(r,\theta)\mathrm{d}r \\
&= \int_{r_{n-1}}^{r_n} 4\lambda(1 + O(e^{-r}))e^{-r/2} \cdot (\nu/4\pi) \cdot \sinh(r/2)\mathrm{d}r \\
&= \int_{r_{n-1}}^{r_n} 4\lambda(1 + O(e^{-r}))e^{-r/2} \cdot (\nu/4\pi) \cdot (1 + O(e^{-r}))\frac{1}{2}e^{r/2}\mathrm{d}r \\
&= (1 + O(e^{-r_n})) \cdot (\nu\lambda/2\pi) \int_{r_{n-1}}^{r_n} 1 \,\mathrm{d}r \\
&= (1 + O(4^{-n})) \cdot (\nu\lambda/\pi) \cdot \ln 2.
\end{aligned} \tag{9}$$

Here we used that $\sinh(x) = (1 + O(e^{-x})) \cdot \frac{1}{2}e^x$ for large x. This proves the first statement of the lemma. The second statement follows immediately from the fact that $\mathbb{P}_\nu(\mathcal{A}_n \text{ covers } 0) = e^{-\mu_n}$. ∎

Lemma 4.6. *We have $\gamma(r_n) > \lambda \cdot 2^{-n}$, for all $n \in \mathbb{N}$.*

Proof. It suffices to prove that

$$\varphi(r) := e^{r/2} \cdot \gamma(r)/\lambda = e^{r/2} \cdot \arccos\left(\frac{\cosh(r) - 1}{\sinh(r)}\right),$$

is strictly larger than one for all $r \geq r_1 = 2\ln 2$. Observe that $\cos(y) \geq 1 - y^2/2$ for all $y \in \mathbb{R}$. This implies that if $y = \arccos(1 - x)$ then $y \geq \sqrt{2x}$. Combining this with (7) shows that

$$\varphi(r) \geq e^{r/2} \cdot 2e^{-r/2}\left(\frac{1 - e^{-r}}{1 - e^{-2r}}\right)^{1/2} = 2\left(\frac{1 - e^{-r}}{1 - e^{-2r}}\right)^{1/2} \geq 2\sqrt{1 - e^{-r}} \geq \sqrt{3} > 1,$$

using that $r \geq 2\ln 2$ for the penultimate inequality. ∎

Lemma 4.7. *For every $\nu, \lambda > 0$ there exists a $c = c(\nu, \lambda) > 0$ such that*

$$\mathbb{P}_\nu\big[\mathcal{A}_n \text{ covers } [0, \lambda 2^{-n})\big] \geq c,$$

(i.e., the probability that $[0, \lambda 2^{-n})$ is covered in its entirety by the points of $\mathcal{P}$ with radii between r_{n-1} and r_n is at least c) for all $n \in \mathbb{N}$.

Proof. It follows from Lemma 4.6 and the monotonicity of $\gamma(r)$ that if (r, θ) covers 0 and furthermore $\theta \in [0, \pi)$ and $r \leq r_n$ then (r, θ) in fact covers all of $[0, \lambda 2^{-n})$. It follows that

$$\begin{aligned}\mathbb{P}_\nu\big[\mathcal{A}_n \text{ covers } [0, \lambda 2^{-n})\big] &\geq \frac{1}{2} \cdot \mathbb{P}_\nu\big[\mathcal{A}_n \text{ covers } 0\big] = \frac{1}{2} \cdot \big(1 - (1 + O((1/4)^n)) \cdot (1/2)^{\nu\lambda/\pi}\big) \\ &= \Omega(1),\end{aligned}$$

using Lemma 4.5. ∎

Let us write $\mathcal{U}_n \subseteq [0, 2\pi)$ for the set of angles *not* covered by the points of $\mathcal{B}_n$. Then $\mathcal{U}_n$ clearly consists of a finite number of intervals. Let $\mathcal{U}_n^{\text{long}} \subseteq \mathcal{U}_n$ denote the union of all intervals of length at least $\lambda 2^{-n}$, and let $\mathcal{U}_n^{\text{short}} := \mathcal{U}_n \setminus \mathcal{U}_n^{\text{long}}$ denote the union of all intervals strictly shorter than $\lambda 2^{-n}$.

We now also define

$$\begin{aligned} L_n = L_n(\lambda) := \text{length}(\mathcal{U}_n) \cdot \lambda^{-1} \cdot 2^n, \quad L_n^{\text{long}} &= L_n^{\text{long}}(\lambda) := \text{length}(\mathcal{U}_n^{\text{long}}) \cdot \lambda^{-1} \cdot 2^n, \\ L_n^{\text{short}} &= L_n^{\text{short}}(\lambda) := \text{length}(\mathcal{U}_n^{\text{short}}) \cdot \lambda^{-1} \cdot 2^n. \end{aligned} \tag{10}$$

The λ is omitted when it is clear from the context.
That is, L_n denotes total length of $\mathcal{U}_n$, multiplied by $\lambda^{-1}2^n$ and $L^{\text{long}}, L^{\text{short}}$ are defined analogously. We let $\mathcal{N}_n^{\text{short}}$ denote the number of components of $\mathcal{U}_n^{\text{short}}$ (i.e. the number of intervals of length strictly less than $\lambda 2^{-n}$), and we set

$$Y_n := \mathcal{N}_n^{\text{short}} + L_n^{\text{long}}. \tag{11}$$

Recall that if $(E_n)_n$ is a sequence of events then we say the event "E_n *almost always*" holds if E_n holds for all but finitely many n. In other words $\{E_n \text{ almost always}\} = \liminf E_m = \bigcup_n \bigcap_{m>n} E_m$. We can for instance write

$$\{\mathcal{C}(\lambda)\} = \{L_n = 0 \text{ almost always}\} = \{Y_n = 0 \text{ almost always}\}.$$

Also recall that we say that the event "E_n *infinitely often*" holds if E_n holds for infinitely many n. In other words $\{E_n \text{ infinitely often}\} = \bigcap_n \bigcup_{m>n} E_m$.

Lemma 4.8. *For every $\nu, \lambda, K > 0$ we have $\mathbb{P}_\nu(Y_n > K$ almost always$) = 1 - \Psi(\nu, \lambda)$.*

Proof. Observe that $\mathbb{P}_\nu(Y_n = 0 \text{ almost always}) = \Psi(\nu, \lambda)$. Let us also observe that, for every $K > 0$:

$$\begin{aligned}\mathbb{P}_\nu(Y_n = 0 \text{ almost always}) + \mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often}) \\ + \mathbb{P}_\nu(Y_n > K \text{ almost always}) = 1.\end{aligned}$$

Hence, it suffices to show that $\mathbb{P}_\nu(Y_n \in (0, K]$ infinitely often $) = 0$ for every $K > 0$. Observe that if $Y_n = y$, then $\mathcal{U}_n$ can be covered by at most $2\lceil y \rceil$ intervals of length $\lambda 2^{-n}$. By Lemma 4.7, and positive correlation, there exists a $c > 0$ such that for all $y > 0$:

$$\mathbb{P}_\nu(Y_{n+1} = 0 | Y_n = y, Y_{n-1} = y_{n-1}, \ldots, Y_1 = y_1) \geq c^{2\lceil y \rceil}, \tag{12}$$

for all $n \in \mathbb{N}$ and all $y, y_1, \ldots, y_{n-1} > 0$. Now let N_1 be the (random) $n \in \mathbb{N}$ for which $Y_n \in (0, K]$ for the first time. Similarly, let N_i be the i-th index n for which $Y_n \in (0, K]$. (Here we set $N_i = \infty$ if $Y_n \in (0, K]$ for less than i indices n.) It follows from (12) that $\mathbb{P}_\nu(N_{i+1} < \infty | N_i < \infty) \leq 1 - c^{2\lceil K \rceil} =: x$. But then we also have that, for every $M \in \mathbb{N}$:

$$\begin{aligned}
\mathbb{P}_\nu(Y_n \in (0, K] \text{ infinitely often }) &\leq \mathbb{P}_\nu(N_i < \infty \text{ for all } 1 \leq i \leq M) \\
&= \mathbb{P}_\nu(N_1 < \infty) \cdot \textstyle\prod_{i=1}^{M-1} \mathbb{P}_\nu(N_{i+1} < \infty | N_i < \infty) \\
&\leq 1 \cdot x^{M-1}.
\end{aligned}$$

Sending $M \to \infty$ shows that $\mathbb{P}_\nu(Y_n \in (0, K]$ infinitely often$) = 0$, as required. ∎

Lemma 4.9. *If $I \subseteq \mathcal{U}_n$ is an interval then $I \cap \mathcal{U}_{n+1}$ consists of at most $\left\lfloor \frac{\text{length}(I)}{\lambda 2^{-n}} \right\rfloor + 1$ intervals.*

Proof. Notice that, if the interval I is cut into $k + 1$ disjoint, non-empty intervals by $\mathcal{A}_{n+1}$ then there must be k points $(\rho_1, \theta_1), \ldots, (\rho_k, \theta_k) \in A_{n+1}$ such that the intervals $(\theta_i - \gamma(\rho_i), \theta_i + \gamma(\rho_i))$ are disjoint and completely contained in I. Hence we must have that

$$\text{length}(I) > \sum_{i=1}^{k} 2\gamma(\rho_i) \geq 2k\gamma(r_n) > k\lambda 2^{-n},$$

using Lemma 4.6. The lemma follows. ∎

Corollary 4.10. *If $I \subseteq \mathcal{U}_n$ is an interval of length less than $\lambda 2^{-n}$ then $I \cap \mathcal{U}_{n+1}$ is either empty or a single interval.*

Another relatively obvious, but key, observation is the following.

Lemma 4.11. *If $I, J \subseteq [0, 2\pi)$ are two sets such that $|x-y|_{2\pi} \geq 2\gamma(r_n)$ for all $x \in I, y \in J$, then $I \cap \mathcal{A}_m$ and $J \cap \mathcal{A}_m$ are independent for all $m > n$.*

Proof. This follows immediately from the fact that a point of radius bigger than r_n cannot simultaneously cover two angles that are more than $2\gamma(r_n)$ apart, and the fact that $\mathcal{P}_\nu \cap A$ and $\mathcal{P}_\nu \cap B$ are independent if $A, B \subseteq \mathbb{H}$ are disjoint. ∎

Lemma 4.12. *For every $\nu, \lambda, K > 0$ we have $\mathbb{P}_\nu(L_n^{long} > K$ infinitely often$) = 1 - \Psi(\nu, \lambda)$.*

Proof. Recall that $\Psi(\nu, \lambda) = \mathbb{P}_\nu(L_n = 0$ almost always$)$, so that $\mathbb{P}_\nu(L_n > 0$ almost always$) = 1 - \Psi(\nu, \lambda)$. It thus suffices to show that $\mathbb{P}_\nu(L_n > 0$ and $L_n^{\text{long}} < K$ almost always$) = 0$, for every $K > 0$. Suppose that, on the contrary, for some $K > 0$ it holds that

$$\mathbb{P}_\nu(L_n > 0 \text{ and } L_n^{\text{long}} < K \text{ almost always}) > 0.$$

By Lemma 4.8 it must then also be the case that $\mathbb{P}_\nu(Y_n > K'$ and $L_n^{\text{long}} < K$ almost always$) > 0$, for every constant K'. And, since $Y_n = \mathcal{N}_n^{\text{short}} + L_n^{\text{long}}$, we must then also have that $\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K'$ and $L_n^{\text{long}} < K$ almost always$) > 0$, for every constant K'. Let us remark that, if E_n almost always holds, then there is a (random) N such that E_n holds for all $n \geq N$. Hence, to prove the lemma it suffices to show that for every $K > 0$ there exists a $K' = K'(K) > 0$ such that $\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K'$ and $L_n^{\text{long}} < K$ for all $n \geq n_0) = 0$, for all $n_0 \in \mathbb{N}$.

Let $K > 0$ thus be arbitrary. Let $c = c(\nu, \lambda)$ be as provided by Lemma 4.7, and let us choose K' such that $K' > 8K/c$ and

$$\mathbb{P}(\text{Bi}(a, c) > ac/2) \geq 2/3,$$

for all $a \geq K'$. (The existence of such a K' follows for instance from the Chebyschev bound.)

Observe that, by Lemma 4.9, if $L_n^{\text{long}} \leq K$ then the long components (intervals) of generation n will split into no more than $2K$ components in generation $n+1$. On the other hand, the short intervals of generation n each disappear with probability $\geq c$ and if they don't disappear then they cannot split into two or more intervals by Lemma 4.9.

This shows that for all $a \geq K', b \leq K$ we have

$$\mathbb{P}_\nu(\mathcal{N}_{n+1}^{\text{short}} < (1 - c/4)\mathcal{N}_n^{\text{short}} | \mathcal{N}_n^{\text{short}} = a, L_n^{\text{long}} = b) \geq 2/3.$$

(To see this note that, with probability 2/3, no more than $(1 - c/2)a$ short intervals survive to the next generation, while the long intervals generate at most $2b \leq 2K < K' \cdot c/4 \leq ac/4$ short ones.)

On the other hand, if $\mathcal{N}_n^{\text{short}} = a$ and $L_n^{\text{long}} \leq K$ then a (deteministic) upper bound is $\mathcal{N}_{n+1}^{\text{short}} \leq a + 2K \leq (1 + c/4)\mathcal{N}_n^{\text{short}}$.

Let us now fix arbitrary $n_0 \in \mathbb{N}, a_0 > K', b_0 \leq K$. If $\mathcal{N}_{n_0}^{\text{short}} = a_0, L_{n_0}^{\text{long}} = b_0$ and $\mathcal{N}_n^{\text{short}} > K', L_n^{\text{long}} \leq K$ for all $n \geq n_0$ then, for every $m \geq 2\log(K'/a_0)/\log(1-c^2/16)$, there are more than $m/2$ indices $n \leq i \leq n+m-1$ such that $\mathcal{N}_{i+1}^{\text{short}} > (1-c/4)\mathcal{N}_i^{\text{short}}$. (Otherwise we would have that $\mathcal{N}_m^{\text{short}} < ((1 - c/4)(1 + c/4))^{m/2} \cdot a_0 = (1 - c^2/16)^{m/2} \cdot a_0 < K'$.) Thus, we have

$$\begin{aligned}
&\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K', L_n^{\text{long}} \leq K \text{ for all } n \geq n_0 | \mathcal{N}_{n_0}^{\text{short}} = a_0, L_{n_0}^{\text{long}} = b_0) \\
&\leq \lim_{m\to\infty} \mathbb{P}(\text{Bi}(m, 1/3) \geq m/2) = 0.
\end{aligned}$$

(The last inequality follows for instance from the weak law of large numbers.) Since n_0, a_0, b_0 were arbitrary, it follows that

$$\mathbb{P}_\nu(\mathcal{N}_n^{\text{short}} > K' \text{ and } L_n^{\text{long}} < K \text{ for all } n \geq n_0) = 0 \quad \text{ for all } n_0 \in \mathbb{N},$$

as required. ∎

Lemma 4.13. *If $\nu \cdot \lambda = \pi$ then there exists a constant $C = C(\nu, \lambda)$ such that $\mathbb{E}_\nu L_n \leq C$ for all n.*

Proof. For every $\nu, \lambda > 0$, we have that

$$\begin{aligned}
\mathbb{E}_\nu L_n &= 2^n \cdot \lambda^{-1} \cdot \int_0^{2\pi} \mathbb{P}_\nu(\text{the angle } \theta \text{ is covered by } \mathcal{B}_n)\mathrm{d}\theta \\
&= 2^n \cdot \lambda^{-1} \cdot 2\pi \cdot \mathbb{P}_\nu(\text{ the angle 0 is covered by } \mathcal{B}_n).
\end{aligned}$$

Hence, when $\nu \cdot \lambda = \pi$, we have

$$\begin{aligned}\mathbb{E}_\nu L_n &= 2^n \cdot \lambda^{-1} \cdot 2\pi \cdot \exp\left[-\sum_{i=1}^{n}(1+O((1/4)^i)) \cdot \ln 2\right] \\ &= 2^n \cdot \lambda^{-1} \cdot 2\pi \cdot \exp[-n\ln 2 + O(1)] = O(1),\end{aligned}$$

using Lemma 4.5. ∎

Lemma 4.14. *Let $\nu \cdot \lambda \leq \pi$ and suppose that $\Psi(\nu,\lambda) < 1$ then $\mathbb{E}_\nu L_n \to \infty$ as $n \to \infty$.*

Proof. It follows from Lemma 4.12 that $\mathbb{P}_\nu(L_n > K$ infinitely often $) = 1 - \Psi(\nu,\lambda)$, for every constant $K > 0$. Let us thus pick a K (to be made explicit later), and let N be the (random) first index n such that $L_n > K$. (Here $N = \infty$ if no such n exists. Note $N < \infty$ with probability $1 - \Psi(\nu,\lambda) > 0$.) Let n_0 be such that $\mathbb{P}_\nu(N < n_0) > (1-\Psi(\nu,\lambda))/2$. By conditioning on the value of N, we find that for $n \geq n_0$:

$$\begin{aligned}\mathbb{E}L_n &\geq \sum_{m=0}^{n_0} \mathbb{E}(L_n|N=m)\mathbb{P}_\nu(N=m) \\ &\geq \sum_{m=0}^{n_0} K \cdot 2^{n-m} \cdot \lambda^{-1} \cdot \exp[-\sum_{i=m+1}^{n}(1+O((1/4)^i)) \cdot (\nu\lambda/\pi) \cdot \ln 2] \cdot \mathbb{P}_\nu(N=m) \\ &= \sum_{m=0}^{n_0} K \cdot 2^{n-m} \cdot \lambda^{-1} \cdot \exp[-(n-m) \cdot (\nu\lambda/\pi) \cdot \ln 2 + O(1)] \cdot \mathbb{P}_\nu(N=m) \\ &= \Omega\left(K \cdot \sum_{m=0}^{n_0} 2^{(n-m)(1-\nu\lambda/\pi)}\mathbb{P}_\nu(N=m)\right) \\ &= \Omega\left(K \cdot \sum_{m=0}^{n_0} \mathbb{P}_\nu(N=m)\right) \\ &= \Omega(K \cdot (1-\Psi(\nu,\lambda))/2).\end{aligned}$$

Sending $K \to \infty$ proves the lemma. ∎

It follows immediately from Lemmas 4.13 and 4.14 that:

Corollary 4.15. *If $\nu\lambda = \pi$ then $\Psi(\nu,\lambda) = 1$.*

This last corollary of course also implies that $\Psi(\nu,\lambda) = 1$ for all $\nu \cdot \lambda \geq \pi$.

Lemma 4.16. *For every $\nu, \lambda > 0$ with $\nu \cdot \lambda < \pi$ there exists an $\eta_0 = \eta(\nu,\lambda)$ such that for every $0 < \eta < \eta_0$ we have*

$$\liminf_{n\to\infty} \mathbb{P}_\nu\big([0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_n\big) > 1/2.$$

Proof. Let μ_n denote the expected number of points $(r,\theta) \in \mathcal{A}_n$ that cover 0, and let $\tilde{\mu}_n$ denote expected number of points $(r,\theta) \in \mathcal{A}_n$ that cover *some* point of $[0, \eta \cdot 2^{-n})$. Then we have, similar to the proof of Lemma 4.5:

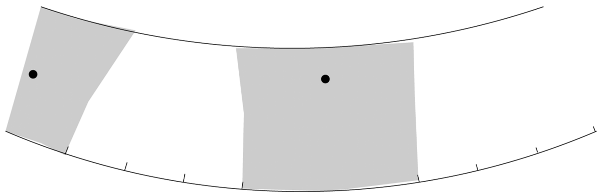

Fig. 3. Depiction of the "particles" of the process.

$$\begin{aligned}
\tilde{\mu}_n &= \int_{r_{n-1}}^{r_n} \big(\eta \cdot 2^{-n} + 2\gamma(r)\big) \cdot g(r,\theta)\mathrm{d}r \\
&= \eta \int_{r_{n-1}}^{r_n} 2^{-n} \cdot (1 + O(e^{-r})) \cdot (\nu/4\pi)e^{r/2}\mathrm{d}r + \mu_n \\
&= (1 + o(1)) \cdot (\eta\nu/8\pi + (\nu\lambda/\pi) \cdot \ln 2),
\end{aligned} \tag{13}$$

reusing the computations (9) in the second line. Since $\nu\lambda < \pi$ we can choose $\eta > 0$ such that $\eta\nu/8\pi + (\nu\lambda/\pi) \cdot \ln 2 < \ln 2$. In that case we have

$$\liminf_{n\to\infty} \mathbb{P}_\nu\big([0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \subseteq \mathcal{U}_n\big) = \liminf_{n\to\infty} e^{-\tilde{\mu}_n} > 1/2,$$

as required. ∎

For the remainder of the section, we fix $\eta > 0$ such that the conclusion of the last lemma holds. Let us now consider the following random process. We start by dissecting $[0, 2\pi)$ into intevals $[0, \eta)$, $[\eta, 2\eta)$, $\dots$, $[2\pi - \eta, 2\pi)$ of length η. (We assume without loss of generality that $\eta = \frac{2\pi}{k}$, for some k.) Each of these intervals "survives" if none of its points is covered by points of $\mathcal{P}$ of radius at most r_1. In each subsequent "generation", we split the surviving intervals in two, and these survive if none of their points are covered by a point of $\mathcal{P}$ of radius between r_{n-1} and r_n. This does produce a kind of branching process, but with the unfortunate property that the offspring of different intervals in generation n are not always independent (e.g., if two intervals share an endpoint then their offspring are dependent, or more generally if they are close enough for a point of radius bigger than r_n to cover a point in each of the two intervals.)

To deal with this problem, we group the surviving intervals into "particles" consisting of (maximal) sequences of intervals each sharing an endpoint with the next. The type of a particle will be the number of intervals it consists of. See Fig. 3 for a depiction.
Note that, in generation n, the gap between different particles is at least $2 \cdot \gamma(r_n)$. So no point of radius $> r_n$ can cover points in two different particles of generation n. This implies that the offspring distributions are independent.

Thus, we have defined a time-inhomogeneous multitype Galton-Watson process $Z_0^\lambda, Z_1^\lambda, \dots$ with countably many types. Again, we drop the superscript if it is clear from the context. Let $p_n(i; z_1, z_2, \dots)$ denote the probability that a particle of type i in generation n produces z_1 children of type 1, z_2 children of type 2 and so on. (Note that strictly speaking we would also need to introduce types for the case when $\mathcal{U}_n = [0, 2\pi)$ in which case there is one particle that "wraps around". This situation however does not occur as soon as there is at least one point with radius $\leq r_n$. So this is not a real issue. We leave it to the reader to check that the proofs below can be adapted to work also with this more proper but also more cumbersome definition of the process.)

Lemma 4.17. *For every $i, z_1, z_2, \dots$ the limits*

$$p(i; z_1, z_2, \dots) := \lim_{n\to\infty} p_n(i; z_1, z_2, \dots),$$

exist.

Proof. Let us fix $i, z_1, z_2, \dots,$ and let E_n denote the event that $[0, i \cdot \eta \cdot 2^{-n})$ is split into a groups of intervals of length $2^{-(n+1)}$ in the required way by $\mathcal{A}_n$, i.e. among $[0, \eta \cdot 2^{-(n+1)}), \dots, [(2i-1)\cdot\eta\cdot 2^{-(n+1)})$ there are z_1 intervals such that none of their points are covered by $\mathcal{A}_n$ but some point in each of the neighbouring intervals were covered, and so on.

Let $A_n \subseteq \mathbb{H}$ denote the set of all points (r, θ) with $r_{n-1} < r \le r_n$ and $\theta \in (-10\cdot\lambda\cdot 2^n, (i\cdot\eta + 10\lambda)\cdot 2^{-n})$; and let $W_n := |\mathcal{P} \cap A_n|$ denote the number of points of $\mathcal{P}$ that fall inside A_n. By (8), for large enough n, whether or not E_n holds will only depend on the points of $\mathcal{P}$ that fall inside A_n. We have

$$p_n(i; z_1, z_2, \dots) = \mathbb{P}_\nu(E_n) = \sum_{t=0}^{\infty} \mathbb{P}_\nu(E_n | W_n = t)\mathbb{P}_\nu(W_n = t). \tag{14}$$

Let us observe that

$$\begin{aligned}
\mathbb{E}W_n &= \int_{A_n} g(r,\theta)\mathrm{d}r\mathrm{d}\theta = 2^{-n}\cdot(i\cdot\eta + 20\lambda)\cdot(\nu/2\pi)\cdot 2(\cosh(r_n/2) - \cosh(r_{n-1}/2)) \\
&= 2^{-n}\cdot(i\cdot\eta + 20\lambda)\cdot(\nu/2\pi)\cdot(e^{r_n/2} + e^{-r_n/2} - e^{r_{n-1}/2} + e^{-r_{n-1}/2}) \\
&= (1+o(1))\cdot(i\cdot\eta + 20\lambda)\cdot(\nu/2\pi).
\end{aligned}$$

It follows also that W_n converges in distribution to a $\mathrm{Po}\big((i\cdot\eta + 20\lambda)^{-1}\cdot(\nu/2\pi)\big)$-distributed random variable. Therefore, in the light of (14), in order to prove that $p_n(i; z_1, z_2, \dots)$ converges, it suffices to prove that the conditional probability $\mathbb{P}_\nu(E_n|W_n = t)$ converges for every fixed $t \in \mathbb{N}$. Let us thus fix a $t \in \mathbb{N}$.

Observe that if we condition on $W = t$ then $\mathcal{P} \cap A$ behaves like t i.i.d. random vectors $X_1 = (\rho_1, \theta_1), \dots, X_t = (\rho_t, \theta_t)$ with common probability density:

$$\tilde{g}(\rho, \theta) = \frac{g(\rho,\theta)}{\int_A g(r', \theta')\mathrm{d}r'\mathrm{d}\theta'} = (1 + o_n(1))\cdot(i\cdot\eta + 20\lambda)^{-1}\cdot e^{\rho/2},$$

where we used that $g(\rho, \theta) = (\nu/4\pi)\sinh(\rho/2) = (1 + O(e^{-\rho}))\cdot(\nu/4)\cdot e^{\rho/2}$.

For notational convenience we write $I_j := [j\cdot\eta\cdot 2^{-(n+1)}, (j+1)\cdot\eta\cdot 2^{-(n+1)})$. For $0 \le j < 2i$ and $1 \le s \le t$ we set $F_n^{j,s} := \big\{(\rho_s, \theta_s) \text{ covers a point of } I_j\big\}$ and for $J \subseteq \{0, \dots, 2i-1\} \times \{1, \dots, t\}$ we define

$$F_n^J := \left(\bigcap_{(j,s)\in J} F_n^{j,s}\right) \cap \left(\bigcap_{(j,s)\notin J} (F_n^{j,s})^c\right).$$

I.e., the event F_n^J prescribes precisely which of the t points covers which of the $2i$ intervals. Clearly there is some family of sets $\mathcal{J} \subseteq 2^{\{0,\dots,2i-1\}\times\{1,\dots,t\}}$ such that

$$\mathbb{P}_\nu(E_n|W_n = t) = \mathbb{P}_\nu\left(\bigcup_{J\in\mathcal{J}} F_n^J\right) = \sum_{J\in\mathcal{J}} \mathbb{P}_\nu(F_n^J).$$

It thus suffices to prove that the probabilities $\mathbb{P}_\nu(F_n^J)$ converge. Let us thus fix some $J \subseteq \{0, \ldots, 2i-1\} \times \{1, \ldots, t\}$. Setting

$$\varphi_n^j(\rho, \theta) := \begin{cases} 1 & \text{if } \theta \in \big(j \cdot \eta \cdot 2^{-(n+1)} - \gamma(\rho), (j+1) \cdot \eta \cdot 2^{-(n+1)} + \gamma(\rho)\big); \\ 0 & \text{otherwise.} \end{cases},$$

and $\ell := -10 \cdot \lambda \cdot 2^{-n}, u := (i \cdot \eta + 10 \cdot \lambda) \cdot 2^{-n}$, we can write

$$\begin{aligned}
&\mathbb{P}_\nu(F_n^J) \\
= &\int_\ell^u \int_{r_{n-1}}^{r_n} \cdots \int_\ell^u \int_{r_{n-1}}^{r_n} \prod_{(j,s)\in J} \varphi_n^j(\rho_s, \theta_s) \cdot \prod_{(j,s)\notin J} (1 - \varphi_n^j(\rho_s, \theta_s)) \cdot \prod_{s=1}^t \tilde{g}(\rho_s, \theta_s)\, \mathrm{d}\rho_1 \mathrm{d}\theta_1 \ldots \mathrm{d}\rho_t \mathrm{d}\theta_t \\
= &\int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \cdots \int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \prod_{(j,s)\in J} \varphi_n^j(r_{n-1} + x_s, 2^{-n} \cdot \vartheta_s) \cdot \\
&\qquad \prod_{(j,s)\notin J} (1 - \varphi_n^j(r_{n-1} + x_s, 2^{-n}\vartheta_s)) \cdot \prod_{s=1}^t \tilde{g}(r_{n-1} + x_s, 2^{-n}\vartheta_s) \cdot 2^{-t\cdot n}\, \mathrm{d}x_1 \mathrm{d}\vartheta_1 \ldots \mathrm{d}x_t \mathrm{d}\vartheta_t \\
= &\int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \cdots \int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \prod_{(j,s)\in J} \varphi_n^j(r_{n-1} + x_s, 2^{-n} \cdot \vartheta_s) \cdot \prod_{(j,s)\notin J} (1 - \varphi_n^j(r_{n-1} + x_s, 2^{-n}\vartheta_s)) \cdot \\
&\qquad (1 + o_n(1)) \cdot (i \cdot \eta + 20)^{-t} \cdot e^{\sum_{s=1}^t (r_{n-1} + x_i)/2} \cdot 2^{-t\cdot n}\, \mathrm{d}x_1 \mathrm{d}\vartheta_1 \ldots \mathrm{d}x_t \mathrm{d}\vartheta_t \\
= &\int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \cdots \int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \prod_{(j,s)\in J} \varphi_n^j(r_{n-1} + x_s, 2^{-n} \cdot \vartheta_s) \cdot \prod_{(j,s)\notin J} (1 - \varphi_n^j(r_{n-1} + x_s, 2^{-n}\vartheta_s)) \cdot \\
&\qquad (1 + o_n(1)) \cdot (i \cdot \eta + 20)^{-t} \cdot 2^{-t} \cdot e^{(x_1 + \cdots + x_t)/2}\, \mathrm{d}x_1 \mathrm{d}\vartheta_1 \ldots \mathrm{d}x_t \mathrm{d}\vartheta_t,
\end{aligned}$$

applying the substitutions $r_s = r_{n-1} + x_s, \theta_s = 2^{-n}\vartheta_s$ in the second line. Let us now define, for $0 \le x \le 2\ln 2$ and $-10 \cdot \lambda \le \vartheta \le i \cdot \eta + 10 \cdot \lambda$:

$$\psi^j(x, \vartheta) := \begin{cases} 1 & \text{if } \vartheta \in (j \cdot \eta/2 - \lambda \cdot e^{-x/2}, (j+1) \cdot \eta/2 + \lambda \cdot e^{-x/2}), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (8) that

$$\lim_{n\to\infty} \varphi_n^j(r_{n-1} + x, 2^{-n}\vartheta) = \psi^j(x, \vartheta) \quad \text{almost everywhere.}$$

(Recall that *almost everywhere* means "for all (x, ϑ) except for a set of Lebesgue measure zero".) Using the dominated convergence theorem we can now conclude that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}_\nu(F_n^J) = (2i \cdot \eta + 40\lambda)^{-t} &\int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \cdots \int_{-10\cdot\lambda}^{i\eta+10\cdot\lambda} \int_0^{2\ln 2} \prod_{(j,s)\in J} \psi^j(x_s, \vartheta_s) \cdot \\
&\prod_{(j,s)\notin J} (1 - \psi^j(x_s, \vartheta_s)) \cdot e^{(x_1 + \cdots + x_t)/2}\, \mathrm{d}x_1 \mathrm{d}\vartheta_1 \ldots \mathrm{d}x_t \mathrm{d}\vartheta_t.
\end{aligned}$$

The lemma follows. ▪

Lemma 4.18. *The limits $p(i; z_1, z_2, \ldots)$ from Lemma 4.17 satisfy the conditions of Lemma 3.4.*

Proof. Let us first note that the expression $\sum_j m_{ij}$ simply counts the expected (total) number of intervals of length $\eta \cdot 2^{-(n+1)}$ in the offspring of a type i particle. An uncovered interval I of length $\eta \cdot 2^{-n}$ in generation n will get split into two uncovered intervals of length $\eta \cdot 2^{-(n+1)}$ in generation $n+1$ if no point of $\mathcal{A}_n$ covers a point of I. It thus follows immediately from the choice of η (cf. Lemma 4.16) that

$$\sum_j m_{ij} \geq \liminf_{n\to\infty} 2i \cdot \mathbb{P}_\nu([0, \eta \cdot 2^{-n}) \in \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \in \mathcal{U}_n) = c \cdot i,$$

where $c := 2 \cdot \liminf_{n\to\infty} \mathbb{P}_\nu([0, \eta \cdot 2^{-n}) \in \mathcal{U}_{n+1} | [0, \eta \cdot 2^{-n}) \in \mathcal{U}_n) > 1$. This verifies the first condition of Lemma 3.4.

The third condition follows immediately from the fact that the total length of the offspring of a particle is never more than the length of the particle.

To see that the second condition holds, it suffices to show that the probability that a particle of type i gives birth to at least one particle of type j is bounded away from zero whenever $j \leq i$. To this end, let $\mu_n^{(i)}$ denote the expected number of points $(r, \theta) \in \mathcal{A}_n$ that cover some angle of $[0, j \cdot \eta \cdot 2^{-(n+1)})$. By an almost verbatim repeat of the computations (13) we have

$$\begin{aligned}\mu_n^{(i)} &= \int_{r_{n-1}}^{r_n} \big(j \cdot \eta \cdot 2^{-(n+1)} + 2\gamma(r)\big) \cdot g(r,\theta)\mathrm{d}r \\ &= (1 + o(1)) \cdot (j \cdot \eta\nu/4\pi + (\nu\lambda/\pi) \cdot \ln 2),\end{aligned}$$

Let E denote the event that that $\mathcal{A}_n$ covers no angle of $[0, j \cdot \eta \cdot 2^{-(n+1)})$ but some angle of $[0, (j+1) \cdot \eta \cdot 2^{-(n+1)})$. Since the probability that a particle of type j is born among the offspring of a type i particle is at least the probability that E holds, we have that

$$\begin{aligned}m_{ij} &\geq \mathbb{P}_\nu(E) \\ &= \lim_{n\to\infty} \mathbb{P}(\mathrm{Po}(\mu_n^{(j)}) = 0) \cdot \mathbb{P}(\mathrm{Po}(\mu_n^{(j+1)} - \mu_n^{(j)}) > 0) \\ &= \lim_{n\to\infty} (\mu_n^{(j+1)} - \mu_n^{(j)}) \cdot e^{-\mu_n^{(j+1)}} \\ &= (\lambda \cdot e^{x/2}) \cdot e^{-(j+1)\cdot\eta\nu/4\pi + (\nu\lambda/\pi)\cdot\ln 2} \\ &> 0.\end{aligned}$$

It remains to check that the fourth condition holds. To this end, observe that if we cut an interval of length $i \cdot \eta \cdot 2^{-n}$ into four equal parts, then if $\mathcal{A}_n$ covers at least one point in each part, then the offspring of the original type-I particle will consist of particles of types $\leq i$. Hence, we have:

$$\sum_{\substack{z_1, z_2, \dots \geq 0, \\ z_{i+1} + z_{i+2} + \dots > 0}} p(i; z_1, z_2, \dots) \leq 1 - \liminf_{n\to\infty} (1 - e^{-\mu_n^{(\lfloor i/4 \rfloor)}})^4 = 1 - \big(1 - e^{-(\lfloor i/4 \rfloor \cdot \eta\nu/4\pi + (\nu\lambda/\pi)\cdot\ln 2)}\big)^4.$$

It is clear that if we send $i \to \infty$ then this last expression approaches zero. This proves that the fourth condition holds, and finishes the proof of the lemma. ∎

Invoking Lemma 3.5, we have the following immediate corollary.

Corollary 4.19. *If* $\nu \cdot \lambda < \pi$ *then* $\liminf_{n\to\infty} \mathbb{P}_\nu(Z$ *explodes* $| Z_n = e_1) > 0$.

We are now also able to deduce:

Lemma 4.20. *If* $\nu\lambda < \pi$ *then* $\Psi(\nu, \lambda) < 1$.

Proof. Observe that the event that Z explodes is contained in the event that $\mathcal{C}(\lambda)$ does not occur. By Corollary 4.19 we can pick $n \in \mathbb{N}$ such that $\mathbb{P}_\nu(Z \text{ explodes } |Z_n = e_1) > 0$. Let E denote the event that $\mathcal{B}_n = \emptyset$, i.e. no point of $\mathcal{P}_\nu$ has radius $\leq r_n$. Then we have that

$$\mathbb{P}_\nu(E) = \exp[-(\nu/2) \cdot (\cosh(r_n/2) - 1)] > 0.$$

We have

$$\begin{aligned} 1 - \Psi(\nu, \lambda) &= \mathbb{P}_\nu(\text{not } \mathcal{C}(\lambda)) \\ &\geq \mathbb{P}_\nu(E) \cdot \mathbb{P}_\nu(Z \text{ explodes}|E) \\ &\geq \mathbb{P}_\nu(E) \cdot \mathbb{P}_\nu(Z \text{ explodes}|Z_n = e_1) \\ &> 0, \end{aligned}$$

where the penultimate inequality holds by obvious monotonicity. ∎

Lemma 4.21. *For every $\lambda > 0$ it holds that* $\lim_{\nu\downarrow 0} \Psi(\nu, \lambda) = 0$

Proof. The proof is very similar to the previous lemma. Let us first observe that for every fixed n the conditional probability $\mathbb{P}_\nu(Z \text{ explodes } |Z_n = e_1)$ is nonincreasing in ν. (This can for instance be seen by noting that a Poisson process with intensity function $g_{\nu+\delta}(r, \theta)$ is the superposition of one with density function g_ν and one with density function g_δ.) Hence we can find an $n_0 \in \mathbb{N}$ and $c > 0$ such that $\mathbb{P}_\nu(Z \text{ explodes}|Z_n = e_1) \geq c$ for all $n \geq n_0$ and all $0 < \nu < 1$. Now note that for every $K > 0$, there exists an n such that among $[0, \eta \cdot 2^{-n}), \ldots, [2\pi - \eta \cdot 2^{-n}, 2\pi)$ there are at least K intervals that are separated by pairwise distance of at least $2\gamma(r_n)$. Fix such an n, and let E denote the event that no point fell inside $\mathcal{B}_n$.

Then we have

$$1 - \Psi(\nu, \lambda) \geq \mathbb{P}_\nu(E) \cdot \left(1 - \mathbb{P}_\nu(Z \text{ dies out } |Z_n = e_1)^K\right) \geq e^{-(\nu/2)\cdot(\cosh(r_n/2)-1)} \cdot (1 - (1 - c)^K).$$

Let $\varepsilon > 0$ be arbitrary. By choosing K sufficiently large, we can ensure that $(1 - c)^K < \varepsilon$. It follows that

$$\limsup_{\nu\downarrow 0} \Psi(\nu, \lambda) \leq 1 - \lim_{\nu\downarrow 0} e^{-(\nu/2)\cdot(\cosh(r_n/2)-1)} \cdot (1 - \varepsilon) = \varepsilon.$$

Sending ε to zero finishes the proof. ∎

Let $L_n^\eta = L_n^\eta(\lambda)$ denote the total length of all components of $\mathcal{U}_n$ that have length at least $\eta \cdot 2^{-n}$. As usual, when λ is clear from the context we omit it. A similar proof to that of the previous lemma also gives the following.

Lemma 4.22. *If $\nu\lambda < \pi$ and $K > 0$ arbitrary then $\mathbb{P}_\nu(L_n^\eta > K$ almost always $) =$* $1 - \Psi(\nu, \lambda)$.

Proof. Observe that if $L_n^\eta > K$ almost always, then $\mathcal{C}(\lambda)$ certainly does not occur. This shows that

$$\mathbb{P}_\nu(L_n^\eta > K \text{ almost always }) \leq 1 - \Psi(\nu, \lambda).$$

Also observe that if Z explodes, then we also have that $L_n^\eta > K$ almost always.

Now let $\varepsilon > 0$ be arbitrary and let us fix a $K' = K'(\varepsilon, K)$, to be made precise later. By Lemma 4.12, we have that $\mathbb{P}_\nu(L_n^{\text{long}} > K'$ infinitely often $) = 1 - \Psi(\nu, \lambda)$. As in the proof of the previous lemma, we can pick $n_0, c > 0$ such that $\mathbb{P}_\nu(Z$ explodes $|Z_n = e_1) \geq c$ for all $n \geq n_0$.

Observe that if $L^{\text{long}} > K'$ then we can find a family of at least

$$M := \left\lceil \frac{K' \cdot 2^{-n}}{\eta \cdot 2^{-n} + 2\gamma(r_n)} \right\rceil,$$

intervals of length $\eta \cdot 2^{-n}$ in $\mathcal{U}_n$ that are separated by pairwise distance $2\gamma(r_n)$. By (8), we have that $M = \Omega(K')$.

Now consider the following setup. We let N denote the (random) first integer after n_0 for which $L_n^{\text{long}} > K'$, where $N = \infty$ if there is no such N. Note that the event $N = n$ is independent of $\mathcal{P} \setminus B_{\mathbb{H}}(O; r_n)$. This shows that

$$\begin{aligned}
\mathbb{P}_\nu(Z \text{ explodes}) &\geq \sum_{n=n_0}^{\infty} \mathbb{P}_\nu(N = n) \cdot \left(1 - \mathbb{P}_\nu(Z \text{ dies out}|Z_n = e_1)^M\right) \\
&\geq \sum_{n=n_0}^{\infty} \mathbb{P}_\nu(N = n) \cdot \left(1 - (1 - c)^M\right) \\
&\geq \sum_{n=n_0}^{\infty} \mathbb{P}_\nu(N = n) \cdot (1 - \varepsilon) \\
&= \mathbb{P}_\nu(N < \infty) \cdot (1 - \varepsilon) \\
&\geq \mathbb{P}_\nu(L_n^{\text{long}} > K' \text{ infinitely often }) \cdot (1 - \varepsilon) \\
&= (1 - f(\nu)) \cdot (1 - \varepsilon).
\end{aligned}$$

Sending ε to zero gives the lemma. ∎

Let us define

$$\Psi_n(\nu, \lambda) := \mathbb{P}_\nu(\mathcal{C}_{r_n}(\lambda)).$$

In other words, Ψ_n is the probability that $\mathcal{B}_n$ is a cover.

Lemma 4.23. *Let $s > 0$ be fixed, but arbitrary. Let F be any event that depends only on $\mathcal{P}_\nu \cap B_{\mathbb{H}}(0, s)$ (i.e. F depends only on the points of radius less than s), and set $\varphi(\nu) := \mathbb{P}_\nu(F)$. Then φ is a continuous function of ν.*

Proof. Let Y denote the number of points of $\mathcal{P}$ with radius at most s. Then Y is Poisson-distributed with mean $\mathbb{E}Y = \nu \cdot (\cosh(s/2) - 1)$. Let us remark that

$$a_t := \mathbb{P}_\nu(F|Y = t),$$

is independent of ν. (To see this, note that if we condition on $Y = t$ then the points of $\mathcal{P}$ with radius $\leq s$ behave like an i.i.d. sample $X_1, \ldots, X_t$ with common density function

$$h(r, \theta) = \frac{g(r, \theta)}{\int_0^{2\pi} \int_0^s g(t, \beta) \mathrm{d}t \mathrm{d}\beta} = \frac{\sinh(r/2)}{2\pi \cdot (\cosh(s/2) - 1)}.$$

The function h is clearly independent of ν.) We clearly have

$$\varphi(\nu) = \sum_{t=0}^{\infty} a_t \cdot \mathbb{P}_\nu(Y = t).$$

Let us now fix an arbitrary $\varepsilon > 0$. Set $K := 1000 \cdot \mathbb{E}_\nu Y/\varepsilon$. By Markov's inequality we have $\mathbb{P}_\mu(Y \geq K) \leq \mathbb{E}_\mu Y/K \leq \varepsilon/2$, for all $\mu < 500\nu$. Hence, for all $\mu < 500\nu$ we have

$$\left| \varphi(\mu) - \sum_{t=0}^{K} a_t \cdot p_t(\mu) \right| < \varepsilon/2,$$

where $p_t(\mu) := \mathbb{P}_\mu(Y = t) = (\mu \cdot (\cosh(s/2) - 1))^t \cdot e^{-\mu\cdot(\cosh(s/2)-1)}/t!$. Now observe that p_t is a continuous function of μ for every (fixed) t. It follows that there is a $\delta > 0$ such that if $|\mu - \nu| < \delta$ then $|p_t(\mu) - p_t(\nu)| < \varepsilon/2(K+1)$ for all $0 \leq t < K$. Hence we also have that $|\varphi(\mu) - \varphi(\nu)| < \varepsilon$ whenever $|\mu - \nu| < \min(\delta, 499\nu)$. This proves that φ is continuous as claimed. ∎

Corollary 4.24. *For every $n \in \mathbb{N}$, the function Ψ_n is continuous in its first parameter, ν.*

Lemma 4.25. *For every $n \in \mathbb{N}$, the function Ψ_n is continuous in its second parameter, λ.*

Proof. Let us fix ν. Let us take $\lambda_1 < \lambda_2$ and let us write $\gamma_i(r) = \lambda_i \arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)$ for $i = 1, 2$. Note that $\Psi_n(\nu, \lambda_2) - \Psi_n(\nu, \lambda_1)$ is precisely the probability of the event E that $\bigcup_{(r,\theta)\in\mathcal{B}_n}(\theta - \gamma_2(r), \theta + \gamma_2(r))$ covers all angles, but some angle is not covered by $\bigcup_{(r,\theta)\in\mathcal{B}_n}(\theta - \gamma_1(r), \theta + \gamma_1(r))$.

Next, let us observe that if E holds then there must exist two points $(r,\theta), (s,\vartheta) \in \mathcal{B}_n$ such that

$$\gamma_1(r) + \gamma_1(s) < |\theta - \vartheta|_{2\pi} < \gamma_2(r) + \gamma_2(s). \tag{15}$$

(Consider some component I of $\mathcal{U}_n$ under λ_1. The leftmost endpoint of this interval is the rightmost endpoint of $(\theta - \gamma_1(r), \theta + \gamma_2(r))$ for some $(r,\theta) \in \mathcal{B}_n$. Since $\mathcal{C}(\lambda)$ occurs at λ_2, it must be the case that $\theta + \gamma_2(r)$ is inside some interval $(\vartheta - \gamma_2(s), \vartheta + \gamma_2(s))$.) From this it follows that

$$\mathbb{P}_\nu(E) \leq (\mathbb{E}_\nu|\mathcal{B}_n|)^2 \cdot \mathbb{P}_\nu\left(|\theta - \vartheta|_{2\pi} \in (\gamma_1(r) + \gamma_1(s), \gamma_2(r) + \gamma_2(s)\right),$$

where $(r,\theta), (s,\vartheta)$ are chosen i.i.d. according to the distribution with density $g/\int_{B_{\mathbb{H}}(O,R)}\int_0^{2\pi} g$. (We used Palm Theory for counting the number of pairs with this property.)

Now note that the length of the interval $(\lambda_1(r) + \lambda_1(s), \lambda_2(r) + \lambda_2(s))$ is at most $2(\lambda_2 - \lambda_1)\lim_{x\downarrow 0}\arccos\left(\frac{\cosh(x)-1}{\sinh(x)}\right) = (\lambda_2 - \lambda_1)\cdot\pi$. It follows that

$$\mathbb{P}_\nu(E) \leq (\mathbb{E}_\nu|\mathcal{B}_n|)^2 \cdot \frac{\lambda_2 - \lambda_1}{2}.$$

Thus, by choosing λ_1, λ_2 such that $\lambda_2 - \lambda_1 < 2\varepsilon/(\mathbb{E}_\nu|\mathcal{B}_n|)^2$, we can ensure that $|\Psi_n(\nu,\lambda_2) - \Psi_n(\nu,\lambda_1)| \leq \mathbb{P}_\nu(E) < \varepsilon$. This proves that Ψ_n is indeed continuous in λ. ∎

Next, we define, for every $\eta, K > 0$ and $n \in \mathbb{N}$:

$$\Phi_{n,\eta,K}(\nu, \lambda) := \mathbb{P}_\nu(L_n^\eta > K).$$

By an application of Lemma 4.23, we find that:

Corollary 4.26. *$\Phi_{n,\eta,K}$ is continuous in its first parameter, ν. (For every $\eta, K > 0$ and $n \in \mathbb{N}$.)*

Lemma 4.27. *$\Phi_{n,\eta,K}$ is continuous in its second parameter, λ. (For every $\eta, K > 0$ and $n \in \mathbb{N}$.)*

Proof. To begin, we fix $\nu, \lambda, \eta, K > 0$ and $n \in \mathbb{N}$. Observe that there exists some $\delta > 0$ such that

$$\mathbb{P}_\nu(L_n^\eta \geq K + \delta) \geq \Phi_{n,\eta,K}(\nu, \lambda) - \varepsilon/3. \tag{16}$$

Similarly, we may assume that δ is small enough so that

$$\mathbb{P}_\nu(\mathcal{U}_n \text{ has a component of length} \in [\eta 2^{-n} - \delta, \eta 2^{-n} + \delta]) < \varepsilon/3. \tag{17}$$

(Arguing as in the proof of Lemma 4.25, but now considering pair of points whose distance is close to $\gamma(r) + \gamma(s) + \eta 2^{-n}$.)

Finally let us pick some $\lambda' \neq \lambda$, and let X denote the sum $\sum_{(r,\theta)\in\mathcal{B}_n} 2|\lambda' - \lambda| \arccos\left(\frac{\cosh(r)-1}{\sinh(r)}\right)$. (I.e., X is the sum over all points in $\mathcal{B}_n$ of the difference in the covered length under the two choices of the parameter λ.) Using Markov's inequality, we have that

$$\mathbb{P}_\nu(X > \delta) \leq \frac{\mathbb{E}_\nu X}{\delta} \leq \mathbb{E}_\nu|\mathcal{B}_n| \cdot \pi \cdot |\lambda' - \lambda| < \varepsilon/3, \tag{18}$$

we the last inequality holds for $|\lambda' - \lambda|$ sufficiently small.

Observe that if $L_n^\eta \geq K + \delta$ with respect to λ, there are no components in $\mathcal{U}_n$ of length $\in [\eta 2^{-n} - \delta, \eta 2^{-n} + \delta]$, and $X \leq \delta$, then $L_n^\eta > K$ with respect to λ'. Thus, combining (16), (17) and (18), we have proved the lemma. ∎

Lemma 4.28. *Ψ is continuous.*

Proof. Let $\nu, \lambda > 0$ be arbitrary. We first assume that $\nu\lambda \geq \pi$. In this case $\Psi(\nu, \lambda) = 1$ by Corollary 4.15. Note that, since $\mathcal{C}(\lambda) = \bigcup_n \mathcal{C}_{r_n}(\lambda)$, there exists an n such that $\Psi_n(\nu, \lambda) \geq 1 - \varepsilon/2$. Since Ψ_n is continuous, there is a $\delta > 0$ such that

$$\Psi(\nu', \lambda') \geq \Psi_n(\nu', \lambda') \geq \Psi_n(\nu, \lambda) - \varepsilon/2 \geq 1 - \varepsilon,$$

for all $\nu' \in (\nu - \delta, \nu + \delta)$ and $\lambda' \in (\lambda - \delta, \lambda + \delta)$. This shows Ψ is continuous at ν, λ.

Let us then assume that $\nu\lambda < \pi$. Let us pick $\nu' > \nu, \lambda' > \lambda$ such that still $\nu'\lambda' < \pi$; and let $n_0 \in \mathbb{N}, c > 0$ be such that

$$\mathbb{P}_{\nu'}(Z^{\lambda'} \text{ explodes } | Z_n^{\lambda'} = e_1) \geq c,$$

for all $n \geq n_0$. Note that, by obvious monotonicity, this inequality also holds for all $\nu'' < \nu', \lambda'' < \lambda'$ (here we keep η, used in the definition of the process Z, fixed).

Let $\varepsilon > 0$ be arbitrary and let $K = K(\varepsilon)$ be fixed to be made precise later. Since $\Psi(\nu, \lambda) = \lim_{n\to\infty} \Psi_n(\nu, \lambda)$, we can find an n_1 such that $|\Psi_n(\nu, \lambda) - \Psi(\nu, \lambda)| < \varepsilon/2$ for all $n \geq n_1$. Similarly, since

$$1 - \Psi(\nu, \lambda) = \mathbb{P}_\nu(L_n^\eta(\lambda) > K \text{ almost always }) = \lim_{n\to\infty} \mathbb{P}_\nu(L_m^\eta(\lambda) > K \text{ for all } m \geq n \text{ }),$$

we can fix an n_2 such that $\Phi_{n,\eta,K}(\nu, \lambda) = \mathbb{P}_\nu(L_n^\eta > K) \geq 1 - \Psi(\nu, \lambda) - \varepsilon/2$ for all $n \geq n_2$.

Let us now fix $n := \max\{n_0, n_1, n_2\}$ and put $\varphi(\nu) := \mathbb{P}_\nu(L_n = 0)$, $\psi(\nu) = \mathbb{P}_\nu(Z_n > K)$.

Since both Ψ_n and $\Phi_{n,\eta,K}$ are continuous, we can pick a $\delta > 0$ such that $|\Psi_n(\nu'', \lambda'') - \Psi_n(\nu, \lambda)| < \varepsilon/2$ and $|\Phi_{n,\eta,K}(\nu'', \lambda'') - \Phi_{n,\eta,K}(\nu, \lambda)| < \varepsilon/2$ for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$. We assume without loss of generality that $\delta < \min(\lambda' - \lambda, \nu' - \nu)$.

Now note that if $L_n^\eta(\lambda) > K$ then there are at least

$$M := \left\lceil \frac{K \cdot \eta \cdot 2^{-n}}{\eta \cdot 2^{-n} + 2\gamma(r_n)} \right\rceil = \Omega(K),$$

intervals of length at least $\eta \cdot 2^{-n}$ that are contained in $\mathcal{U}_n$ and that are separated by pairwise distance $2\gamma(r_n)$. It follows that, for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$, we have

$$\begin{aligned}\mathbb{P}_{\nu''}(L_m^\eta(\lambda'') > K \text{ almost always} | L_n^\eta(\lambda'') > K) &\geq 1 - \mathbb{P}_{\nu''}(Z(\lambda'') \text{ dies out } | Z_n(\lambda'') = e_1)^M \\ &\geq 1 - (1 - c)^M \\ &\geq 1 - \varepsilon/2,\end{aligned}$$

where the last inequality holds provided we chose K sufficiently large (which we can assume without loss of generality). We thus get that

$$\begin{aligned}1 - \Psi(\nu'', \lambda'') &= \mathbb{P}_{\nu'',\lambda''}(\text{not } \mathcal{C}(\lambda)) \\ &\geq \mathbb{P}_{\nu'',\lambda''}(Z \text{ explodes } | L_n^\eta > K)\Phi_{n,\eta,K}(\nu'', \lambda'') \\ &\geq (1 - \varepsilon/2) \cdot (1 - \Psi(\nu, \lambda) - \varepsilon/2) \\ &\geq 1 - \Psi(\nu, \lambda) - \varepsilon,\end{aligned}$$

for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$. In other words, $\Psi(\nu'', \lambda'') \leq \Psi(\nu, \lambda) + \varepsilon$ for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$. On the other hand we have

$$\Psi(\nu'', \lambda'') \geq \Psi_n(\nu'', \lambda'') \geq \Psi(\nu, \lambda) - \varepsilon,$$

for all $\nu'' \in (\nu - \delta, \nu + \delta)$ and $\lambda'' \in (\lambda - \delta, \lambda + \delta)$, by choice of n and δ. We have seen that Ψ is continuous at (ν, λ) as required. ∎

We have already proved Theorem 4.1, but for completeness we collect our findings from this Section in an explicit proof.

Proof of Theorem 4.1. That Ψ is continuous was just established in the previous lemma. That $\Psi(\nu, \lambda) = 1$ when $\nu\lambda \geq \pi$ was established in Corollary 4.15. That Ψ is strictly increasing at every point $(\nu, \lambda$ with $\nu\lambda < \pi$ follows from Corollary 4.4 together with Lemma 4.20. That $\lim_{\nu \downarrow 0} \Psi(\nu, \lambda) = 0$ was established in Lemma 4.21. ∎

5. THE PROOF OF PART III OF THEOREM 1.2

Here we finally prove the remaining part of Theorem 1.2, making use of Theorem 4.1

Lemma 5.1. *Let $\mathcal{P} = \mathcal{P}_\nu$ be as defined earlier. For every $\varepsilon > 0$ there is a coupling such that $\mathcal{P}_{\nu-\varepsilon} \cap B_{\mathbb{H}}(O; R) \subseteq V_N \subseteq \mathcal{P}_{\nu+\varepsilon} \cap B_{\mathbb{H}}(O; R)$ w.h.p. as $N \to \infty$.*

Proof. Let $X_1, X_2, \ldots$ be an infinite supply of i.i.d. points distributed according to (1). Then we can set $V = \{X_1, \ldots, X_N\}$. Now let $Z_1 \stackrel{d}{=} \mathrm{Po}((1-\delta)N), Z_2 \stackrel{d}{=} \mathrm{Po}((1+\delta)N)$ and set $V_i := \{X_1, \ldots, X_{Z_i}\}$ for $i = 1, 2$. It follows from the Chebyschev inequality that

$$\mathbb{P}_\nu(Z_1 \leq N \leq Z_2) = 1 - o(1).$$

Put differently, this proves that $V_1 \subseteq V_N \subseteq V_2$ w.h.p.

Now observe that V_1 is a Poisson process with intensity function:

$$\begin{aligned} h_1(r,\theta) &= (1-\delta)N \cdot (1/2\pi) \cdot \frac{(1/2)\cdot\sinh(r/2)}{\cosh(R/2)-1} \cdot 1_{\{r \leq R\}} \\ &= (1-\delta)\nu e^{R/2} \cdot (1/2\pi) \cdot \frac{(1/2)\cdot\sinh(r/2)}{\cosh(R/2)-1} \cdot 1_{\{r \leq R\}} \\ &= (1-\delta+o(1)) \cdot (\nu/4\pi) \cdot \sinh(r/2) \cdot 1_{\{r \leq R\}}. \end{aligned}$$

So, provided we chose $\delta = \delta(\varepsilon)$ sufficiently small, we have $h_1(r,\theta) \geq g_{\nu-\varepsilon}(r,\theta)1_{\{r\leq R\}}$ for all r, θ if N is sufficiently large (where g is the density of $\mathcal{P}$ defined in (3)). Similarly the density h_2 of V_2 satisfies $h_2 \leq g_{\nu+\varepsilon}1_{\{r\leq R\}}$ for N sufficiently large. The statement follows. ∎

Lemma 5.2. *For every $\nu > 0$ we have $\liminf_{N\to\infty} \mathbb{P}(G(N; 1/2, \nu)$ is connected $) \geq \Psi(\nu, 1)$.*

Proof. Let us pick a $\delta > 0$ such that $\Psi(\nu-\delta, 1-\delta) > \Psi(\nu, 1) - \varepsilon/3$. For convenience we write $\mu := \nu - \delta, \lambda := 1 - \delta$. Next, let us pick $s > 0$ such that $\mathbb{P}_\mu(\mathcal{C}_s(\lambda)) \geq \Psi(\mu, \lambda) - \varepsilon/3$. This is possible as $\mathcal{C}_s \subseteq \mathcal{C}_{s'}$ for $s < s'$, so $\mathbb{P}_\mu(\mathcal{C}_s(\lambda))$ is nondecreasing in s with limit $\mathbb{P}_\mu(\mathcal{C}(\lambda)) = \Psi(\mu, \lambda)$. Let us consider the coupling from the previous lemma. Taking N sufficiently large, we can assume that the probability that it fails is at most $\varepsilon/3$ and that $s < R/2$. (Recall that $R = R(N)$ depends on and is growing with N.)

We claim that, if $\mathcal{C}_s(\lambda)$ occurs with respect to μ, and the coupling succeeds (i.e. $\mathcal{P}_\mu \cap B_{\mathbb{H}}(O, R) \subseteq V_N$), then the graph $G(N; 1/2, \nu)$ will be connected. To see this suppose that $\mathcal{C}_s(\lambda)$ occurs with respect to μ, and pick an arbitrary point $X_i = (\rho_i, \theta_i) \in V_N$. There is some point $X_j = (\rho_j, \theta_j) \in V_N$ with $\rho_j \leq s$ such that $|\rho_i - \rho_j|_{2\pi} < \gamma(\rho_j) = \lambda \cdot \arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right)$. We claim that X_i and X_j must have distance less than R. To see this, note first that we are done when $\rho_i \leq R/2$ (using as $\rho_j \leq s < R/2$ and the triangle inequality). By the hyperbolic cosine rule we have that the distance between X_i and X_j is less than R if and only if

$$|\theta_i - \theta_j|_{2\pi} < \arccos\left(\frac{\cosh(\rho_i)\cosh(\rho_j) - \cosh(R)}{\sinh(\rho_i)\sinh(\rho_j)}\right).$$

Now notice that

$$\begin{aligned} \arccos\left(\frac{\cosh(\rho_i)\cosh(\rho_j) - \cosh(R)}{\sinh(\rho_i)\sinh(\rho_j)}\right) &\leq \arccos\left(\frac{\cosh(\rho_i)\cosh(\rho_j) - \cosh(\rho_i)}{\sinh(\rho_i)\sinh(\rho_j)}\right) \\ &= \arccos\left(\frac{\cosh(\rho_i)}{\sinh(\rho_i)} \cdot \frac{\cosh(\rho_j) - 1}{\sinh(\rho_j)}\right). \end{aligned}$$

Recall that $(\cosh(r)-1)/\sinh(r) = 1-2e^{-r}+o(e^{-r})$ and note that $\cosh(\rho_i)/\sinh(\rho_i) = 1+O(e^{-2\rho_i}) = 1+O(e^{-R})$. Using Taylor's expansion $\arccos(x+y) = \arccos(x) - y/\sqrt{1-x^2} + O(xy^2/(1-x^2)^{3/2})$, we see that

$$\arccos\left(\frac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) = \arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)} + O(e^{-R})\right) = \arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) + O(e^{\rho_j-R}).$$

Using eqs. (7) and (8), we find that

$$\arccos\left(\frac{\cosh(\rho_i)}{\sinh(\rho_i)}\cdot\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right) = (1+o(1))\cdot\arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right).$$

Since $|\theta_i-\theta_j|_{2\pi} \le \gamma(\rho_j) = (1-\delta)\cdot\arccos\left(\frac{\cosh(\rho_j)-1}{\sinh(\rho_j)}\right)$, we do find that X_i, X_j have distance at most R (for N sufficiently large).

This shows that, provided $\mathcal{C}_s(\lambda)$ occurs with respect to μ and the coupling succeeds (i.e. $\mathcal{P}_\mu \cap B_{\mathbb{H}}(O,R) \subseteq V_N$), then every vertex of $G(N;1/2,\nu)$ will be at distance less that R from some vertex of radius $< R/2$. So the graph will have diameter at most three, and in particular it will be connected. That is, we have shown

$$\liminf_{N\to\infty} \mathbb{P}(G(N;1/2,\nu) \text{ is connected }) \ge \mathbb{P}_\mu(\mathcal{C}_s(\lambda)) - \mathbb{P}(\text{the coupling fails}) \ge \Psi(\nu,1)-\varepsilon.$$

Sending ε to zero proves the lemma. ∎

Lemma 5.3. *For every $\nu > 0$ we have $\limsup_{N\to\infty} \mathbb{P}(G(N;1/2,\nu)$ is connected $) \le \Psi(\nu,1)$.*

Proof. If $\nu > \pi$ then there is nothing to prove as $\Psi(\nu,1) = 1$. Let us thus suppose that $\nu < \pi$ so that $\Psi(\nu,1) < 1$. Reformulating, it suffices to show that

$$\liminf_{N\to\infty} \mathbb{P}(G(N;1/2,\nu) \text{ is NOT connected }) \ge 1-\Psi(\nu,1).$$

Pick a $\delta > 0$ such that $\Psi(\nu+\delta,1+\delta) \le \Psi(\nu,1)+\varepsilon/2$ and write $\mu := \nu+\delta, \lambda := 1+\delta$. Let K be large but fixed, to be made more precise later; and let $\eta = \eta(\mu,\lambda)$ be as in Lemma 4.16. By Lemma 4.22, there exist an $n_0 \in \mathbb{N}$ such that, for all $n \ge n_0$:

$$\Phi_{n,\eta,K}(\mu,\lambda) = \mathbb{P}_\nu(L_n^\eta > K) \ge 1-\Psi(\mu,\lambda)-\varepsilon/2.$$

Now let $n := \lfloor R/2\ln 2\rfloor - 1$, and let F denote the event that $L_n^\eta > K$ (with respect to μ,λ). Given that F holds, we can pick $M = \Omega(K)$ intervals $I_1,\dots,I_M \subseteq \mathcal{U}_n$ of length $\eta 2^{-n}$ such that the angle between a point in I_i and a point in I_j is at least $1000\cdot 2^{-n}$ (for all $1 \le i \ne j \le M$). Now let F_i denote the event that there is *exacltly one* point $X_\ell = (\rho_\ell,\theta_\ell) \in V_{\text{Poi}}$ such that **1)** $R-\varepsilon < \rho_\ell \le R$ and $\theta_\ell \in I_i$ and **2)** there is no point of $X_m = (\rho_m,\theta_m) \in \mathcal{P}_\mu$ with $\rho_m > r_n$ and θ_m within angle $10\cdot 2^{-n}$ of one of the endpoints of I_i. Observe that

$$\mathbb{P}_\nu(F_i|F) = \mathbb{P}(\text{Po}(\mu_1)=1)\mathbb{P}(\text{Po}(\mu_2)=0) = \Theta(1),$$

where $\mu_1 := \eta \cdot 2^{-n} \cdot (\nu/4\pi) \cdot (\cosh(R/2) - \cosh((R-\varepsilon)/2))$ and $\mu_2 := 20 \cdot 2^{-n} \cdot (\nu/4\pi) \cdot (\cosh(R/2) - \cosh(r_n/2)) - \mu_1$. (That both μ_1, μ_2 are $\Theta(1)$ follows from the fact that $\cosh(R/2), \cosh((R-\varepsilon)/2), \cosh(r_n/2) = \Theta(2^n)$.) Note also that the event F_i-s are independent (given F). Hence we have

$$\mathbb{P}\left(\bigcup F_i | F\right) \geq 1 - (1 - \Theta(1))^M > 1 - \varepsilon/2,$$

provided we chose K sufficiently large.

We now claim that, if F and some F_i hold, then there is a point $X_j \in W := \mathcal{P}_\mu \cap B_{\mathbb{H}}(O; R)$ that is at distance $> R$ from all other points in W (namely the sole vertex $X_j = (\rho_j, \theta_j)$ with angle in $\theta_j \in I_i$ and radius $\rho_j > R - \delta$). To see this, let $X_k = (\rho_k, \theta_k) \in W$ be an arbitrary other point. If $\rho_k > r_n$ we have $|\theta_j - \theta_k|_{2\pi} > 10 \cdot 2^{-n}$. On the other hand, we have $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) \leq \mathrm{dist}_{\mathbb{H}}(X_j', X_k')$ where $X_j' = (r_n, \theta_j), X_k' = (r_n, \theta_k)$ by Lemma 2.1. Hence, by the hyperbolic cosine rule $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) \leq R$ only if the difference in angle $|\theta_j - \theta_k|_{2\pi}$ is at most

$$\begin{aligned}\arccos\left(\frac{\cosh^2(r_n) - \cosh(R)}{\sinh^2(r_n)}\right) &= \arccos\left(1 - O(e^{-r_n})\right)\\ &= (1+o(1))2e^{-r_n/2} = (1+o(1)) \cdot 2^{-(n-1)}.\end{aligned}$$

It follows $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) > R$.

Now suppose that $\rho_k < r_n$. Since $\theta_j \in \mathcal{U}_n$ it follows that

$$|\theta_j - \theta_k|_{2\pi} > (1+\delta)\arccos\left(\frac{\cosh(r_k) - 1}{\sinh(r_k)}\right).$$

Now observe that, for $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) < R$ to hold, the angle between them can be at most $\arccos\left(\frac{\cosh(r_j)\cosh(r_k) - \cosh(R)}{\sinh(r_j)\sinh(r_k)}\right)$, by the hyperbolic cosine rule. Since $r_j \in (R - \varepsilon, R)$ we have that $\cosh(r_j) = (1 + O(\varepsilon))\cosh(R)$ and $\sinh(r_j) = (1 + O(\varepsilon))\cosh(R)$. This also gives that

$$\frac{\cosh(r_j)\cosh(r_k) - \cosh(R)}{\sinh(r_j)\sinh(r_k)} = (1 + O(\varepsilon) \cdot \frac{\cosh(r_k) - 1}{\sinh(r_k)}.$$

Using Taylor's expansion $\arccos(x+y) = arccos(x) + O(y/(1-x^2)^{1/2})$, we now find

$$\begin{aligned}\arccos\left(\frac{\cosh(r_j)\cosh(r_k) - \cosh(R)}{\sinh(r_j)\sinh(r_k)}\right) &= \arccos\left(\frac{\cosh(r_k) - 1}{\sinh(r_k)}\right) + O(\varepsilon e^{-r_k/2})\\ &= (1 + O(\varepsilon)) \cdot \arccos\left(\frac{\cosh(r_k) - 1}{\sinh(r_k)}\right).\end{aligned}$$

(Using that $(\cosh(r_k) - 1)/\sinh(r_k) = 1 - O(e^{-r})$. It follows that $\mathrm{dist}_{\mathbb{H}}(X_j, X_k) > R$, as claimed. Hence if $(\bigcup F_j) \cap F$ has been realized, then at least one point of W will have distance larger than R to all other points of W.

We wish now to deduce that in such a case, $G(N; 1/2, \nu)$ will have an isolated vertex, but as it happens V_N is a strict subset of W. To get around this problem, we use the coupling from Lemma 5.1, and symmetry. Suppose that $(\bigcup F_j) \cap F$ holds, and choose a point X_j of distance $> R$ to all other points (uniformly at random from all such points, say). By symmetry considerations, under the coupling from Lemma 5.1 the probability that X_j is also a point of $\mathcal{P}_{\nu-\delta}$ is $\frac{\nu-\delta}{\nu+\delta} = 1 - O(\delta)$. Putting everything together, we find that

$$\begin{aligned}\mathbb{P}(G(N;1/2,\nu) \text{ has an isolated vertex}) &\geq \mathbb{P}\left(\bigcup F_i | F\right) \mathbb{P}_\nu(F) - O(\delta) - \mathbb{P}(\text{coupling fails}) \\ &\geq (1-\varepsilon/2)\cdot(1-\Psi(\mu,\lambda)-\varepsilon/2) - O(\delta) - o(1) \\ &\geq (1-\varepsilon/2)\cdot(1-\Psi(\nu,1)-\varepsilon) - O(\delta) - o(1).\end{aligned}$$

Sending ε, δ to zero gives the lemma. ▪

To conclude, let us point out that Theorem 4.1 implies that $f(\nu) := \Psi(\nu, 1)$ has the properties described in Theorem 1.2iii.

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