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Hausdorff dimension of the arithmetic sum of self-similar sets

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Abstract

Let $\beta > 1$. We define a class of similitudes

$$S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.$$

Taking any finite collection of similitudes $\{f_i(x)\}_{i=1}^m$ from S, it is well known that there is a unique self-similar set K_1 satisfying $K_1 = \cup_{i=1}^m f_i(K_1)$. Similarly, another self-similar set K_2 can be generated via the finite contractive maps of S. We call $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$ the arithmetic sum of two self-similar sets. In this paper, we prove that $K_1 + K_2$ is either a self-similar set or a unique attractor of some infinite iterated function system. Using this result we can calculate the exact Hausdorff dimension of $K_1 + K_2$ under some conditions, which partially provides the dimensional result of $K_1 + K_2$ if the IFS's of K_1 and K_2 fail the irrationality assumption, see Peres and Shmerkin (2009).

Keywords: Arithmetic sum; Self-similar sets; Hausdorff dimension

1. Introduction

Let $\{g_j\}_{j=1}^m$ be an iterated function system (IFS) of similitudes which are defined on $\mathbb{R}$ by

$$g_j(x) = r_j x + a_j,$$

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where the similarity ratios satisfy $0 < r_j < 1$ and the translation parameter $a_j \in \mathbb{R}$. It is well known that there exists a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^{m} g_j(K). \tag{1}$$

We call K the self-similar set or attractor for the IFS $\{g_j\}_{j=1}^m$, see [8] for further details. The IFS $\{g_j\}_{j=1}^m$ is called homogeneous if all the similarity ratios are equal. We say that $\{g_j\}_{j=1}^m$ satisfies the open set condition (OSC) [8] if there exists a non-empty bounded open set $V \subseteq \mathbb{R}$ such that

$$g_i(V) \cap g_j(V) = \varnothing, \quad i \neq j$$

and $g_j(V) \subseteq V$ for all $1 \leq j \leq m$. Under the open set condition, the Hausdorff dimension of K coincides with the similarity dimension which is the unique solution s of the equation $\sum_{j=1}^m r_j^s = 1$.

Let F_1 and F_2 be the self-similar sets with IFS's $\{r_i x + a_i\}_{i=1}^n$ and $\{r'_j x + b'_j\}_{j=1}^m$ respectively. We call $F_1 + F_2 = \{x + y : x \in F_1, y \in F_2\}$ the arithmetic sum of self-similar sets. The arithmetic sum of Cantor sets appears naturally in dynamical systems. Palis [15] posed the following problem which is currently known as the Palis' conjecture. Whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero or contains an interval. This conjecture was solved in [2]. However, for the general self-similar sets this conjecture is still open. In [9], Mendes and Oliveira proved that for the homogeneous Cantor sets, there are five possible structures for the sum. For the fractal structure, i.e. the similarity of the sum of self-similar sets, there are few results regarding this aspect. This is the first reason why we study the sum of self-similar sets. Another natural question concerning the sum of self-similar sets is to consider the Hausdorff dimension or Hausdorff measure of $F_1 + F_2$. Many papers have been devoted to this aspect. Let C_a be the central Cantor set generated by removing a central interval of length $1 - 2a$ from [0, 1], and then continuing this process inductively on each remaining two intervals. Denote $\gamma(a) = \dim_H(C_a) = \frac{\log 2}{-\log a}$. Peres and Solomyak [17] proved that

Theorem 1.1. *Given a fixed compact set $K \subset \mathbb{R}$, the following two statements hold for almost every $a \in (0, \frac{1}{2})$:*

if $\gamma(a) + \dim_H(K) \leq 1$, then $\dim_H(K + C_a) = \gamma(a) + \dim_H(K)$;
if $\gamma(a) + \dim_H(K) > 1$, then the Lebesgue measure of $C_a + K$ is positive.

Motivated by this result Eroğlu [3] considered the Hausdorff measure of the arithmetic sum of two Cantor sets, and gave a necessary and sufficient condition such that the Hausdorff measure of the sum of Cantor sets is positive. Peres and Solomyak's main idea is using the potential theory. This is the main reason why their result is the almost-type result. An important progress of the dimensional problem is due to Peres and Shmerkin. In [16], Peres and Shmerkin showed that

Theorem 1.2. *If there exist i, j satisfying $\frac{\log r_i}{\log r'_j} \notin \mathbb{Q}$, then*

$$\dim_H(F_1 + F_2) = \min\{1, \dim_H(F_1) + \dim_H(F_2)\}.$$

The hypothesis of this theorem is called the irrationality assumption. It is easy to see that many pairs of iterated function systems satisfy this assumption. Peres and Shmerkin's formula gives a sufficient condition under which the expected dimension of the sum of self-similar sets can be obtained. Their main idea is to project the product of two one-dimensional self-similar sets into the real line and to show that under the irrationality assumption the expected dimension of F_1+F_2 can be achieved. Later, Nazarov et al. [12] investigated similar problem for the convolutions of Cantor measures without resonance.

Motivated by Peres and Shmerkin's result and Palis' conjecture, we consider the IFS's of F_1 and F_2 failing the irrationality assumption. With a little effort, it can be shown that the IFS's $\{r_i x + a_i\}_{i=1}^n$ and $\{r'_j x + b'_j\}_{j=1}^m$ do not satisfy the irrationality assumption if and only if there exist $\beta > 1, n_i$ and $m_j \in \mathbb{N}$ such that $r_i = \frac{1}{\beta^{n_i}}, 1 \le i \le n$ and $r'_j = \frac{1}{\beta^{m_j}}, 1 \le j \le m$. Unless stated otherwise, in what follows we always assume that the similitudes of K_1 and K_2 are from

$$S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.$$

We suppose without loss of generality that the IFS's of K_1 and K_2 are $\{f_i(x) = \frac{x}{\beta^{n_i}} + a_i\}_{i=1}^n$ and $\{g_j(x) = \frac{x}{\beta^{m_j}} + b_j\}_{j=1}^m$, respectively.

We shall prove that $K_1 + K_2$ is either a self-similar set or an attractor of some infinite iterated function system (IIFS) [1,5]. Therefore, calculating the Hausdorff dimension of $K_1 + K_2$ is reduced to considering the dimension of the attractor of some IFS (IIFS). It is well known that generally it is difficult to calculate the Hausdorff dimension of a self-similar set, especially when overlaps occur. It is much more difficult to find the dimension of the attractor of some IIFS even if the IIFS satisfies certain separation condition. Here the attractor of the IIFS is in the sense of Definition 2.1, we will introduce this definition in the next section. In fact, Peres and Shmerkin's dimensional formula implies that we may not find the exact Hausdorff dimension of $F_1 + F_2$ generally. In this paper, we shall consider some cases which allow us to calculate the dimension of K_1+K_2 explicitly. An important difference between our main result and Peres and Shmerkin's formula is that we may not obtain the expected dimension for the sum of self-similar sets, see the first example in Section 4. Peres and Shmerkin gave a uniform formula while we emphasize on the individual example. In other words, our method is analyzing single example rather than giving a uniform formula for the dimension of the sum of self-similar sets. When $K_1 + K_2$ is a self-similar set with overlapping IFS, the techniques of the paper [7] could be useful. However, this is beyond our discussion, and we do not give further details.

For the topological structure of K_1+K_2, e.g. connected property and so on, generally we may not easily get further information. The main reasons are that the IFS (IIFS) of $K_1 + K_2$ may vary from each other and that discussing these two cases needs different techniques.

The structure of the paper is as follows. In Section 2, we introduce some basic results of infinite iterated function systems and define some necessary terminology. Next, we prove the similarity of $K_1 + K_2$. In Section 3, we concentrate on the Hausdorff dimension of $K_1 + K_2$. We consider both cases, i.e. $K_1 + K_2$ is a self-similar set or a unique attractor of some IIFS, and give some dimensional results. In Section 4, we offer some examples for which we can explicitly calculate the Hausdorff dimension of $K_1 + K_2$. Finally, we give some further remarks.

2. Preliminaries and main results

2.1. *Infinite iterated function systems*

Before stating our main results, we introduce some definitions and results of infinite iterated function systems (IIFS). Infinite iterated function systems behave differently from IFS's [1,5].

There are two definitions of the invariant set of IIFS, see for example, [5,1,6]. We adopt Fernau's definition [5].

Definition 2.1. Let $\mathcal{A} = \{\phi_i(x) = r_i x + a_i : i \in \mathbb{N}, 0 < r_i < 1, a_i \in \mathbb{R}\}$. If there exists $0 < s < 1$ such that for every $\phi_i \in \mathcal{A}$, $|\phi_i(x) - \phi_i(y)| \leq s|x - y|$, then $\mathcal{A}$ is called an infinite iterated function system, abbreviated as IIFS. A unique non-empty compact set J is called the attractor of $\mathcal{A}$ if

$$J = \overline{\bigcup_{i\in\mathbb{N}} \phi_i(J)},$$

where $\overline{A}$ denotes the closure of A.

Remark 2.2. The existence and uniqueness of J can be found in [5]. In [1], Mauldin and Urbański gave another definition of the attractor of IIFS, i.e. $J_0 = \bigcup_{i\in\mathbb{N}} \phi_i(J_0)$. However, for their definition the attractor J_0 may not be unique or compact, see example 1.3 from [5]. Evidently, $\overline{J_0} = J$.

An infinite iterated function system $\mathcal{A} = \{\phi_i : i \in \mathbb{N}\}$ satisfies the open set condition if there exists a non-empty bounded open set $O \subseteq \mathbb{R}$ such that

$$\phi_i(O) \cap \phi_j(O) = \varnothing, \quad i \neq j,$$

and $\phi_j(O) \subseteq O$ for all $j \in \mathbb{N}$. Under this separation condition, we can find the Hausdorff dimension of J_0. The following result can be found in [1,10] or [6].

Theorem 2.3. *For any IIFS satisfying the open set condition, we have*

$$\dim_H(J_0) = \inf\left\{t : \sum_{i\in\mathbb{N}} r_i^t \leq 1\right\}.$$

On the other hand, generally the Hausdorff dimension of J is more complicated. One of the difficulties is to analyze $J \setminus J_0$, see [6, Corollary 2]. For the most cases, we shall prove that $J = K_1 + K_2$ is an attractor of some IIFS in the sense of Definition 2.1. This makes the dimension of $K_1 + K_2$ complicated. We mentioned above that $\overline{J_0} = J$. If J_0 and J coincide except for a countable set, then by the countable stability of the Hausdorff dimension we have that $\dim_H(J_0) = \dim_H(J)$. We will give a sufficient condition under which we can identify J_0 with J apart from a countable set. This is the main idea we will implement, provided $K_1 + K_2$ is the unique attractor of some IIFS.

2.2. *Some definitions*

In this section, we introduce some definitions which make our discussion far more succinct. Given any finite reals $s_1, s_2, s_3, \ldots, s_n$. Let $\sum = \{s_1, s_2, \ldots, s_n\}^{\mathbb{N}}$ be a symbolic space. We say $c_1c_2\cdots c_m \in \{s_1, s_2, \ldots, s_n\}^m$ is a block with length m, and we use capital letters with hats to denote the finite blocks of Σ. For instance, we denote $c_1c_2\cdots c_m$ by $\hat{P}$, i.e. $\hat{P} = c_1c_2\cdots c_m$.

Definition 2.4. Let $\hat{P}_1 = d_1d_2\cdots d_m$ and $\hat{P}_2 = c_1c_2\cdots c_m$ be two blocks of $\{s_1, s_2, \ldots, s_n\}^m$. We define the concatenation of $\hat{P}_1$ and $\hat{P}_2$ by $\hat{P}_1 * \hat{P}_2 = d_1d_2\cdots d_m c_1c_2\cdots c_m$. The sum of $\hat{P}_1$

and $\hat{P}_2$ is defined by $\hat{P}_1 + \hat{P}_2 = (d_1 + c_1)(d_2 + c_2) \cdots (d_m + c_m)$. Concatenating $k \in \mathbb{N}$ blocks of $\hat{P}_1$ is denoted by

$$\hat{P}_1^k = \underbrace{\hat{P}_1 * \hat{P}_1 * \cdots * \hat{P}_1}_{k \text{ times}}.$$

The value of the block $\hat{P}_1 = d_1 d_2 \cdots d_m$ with respect to $\beta > 1$ is

$$(d_1 d_2 \cdots d_m)_\beta = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_m}{\beta^m}.$$

Similarly, we can define the value of an infinite sequence $(d_n) \in \sum$ by $(d_n)_\beta = \sum_{n=1}^{\infty} \frac{d_n}{\beta^n}$.

Remark 2.5. In this definition, when we define the summation of two blocks, we assume that these two blocks have the same length. However, in some cases we may need to consider the concatenation of infinite blocks. For instance, let $\{\hat{P}_i\}_{i=1}^{\infty}$ and $\{\hat{Q}_i\}_{i=1}^{\infty}$ be two block sets, the concatenations of $\hat{P}_1 * \hat{P}_2 * \cdots$ and $\hat{Q}_1 * \hat{Q}_2 * \cdots$ are two infinite sequences in Σ, we denote them by (a_n) and (b_n) respectively. The summation of $\hat{P}_1 * \hat{P}_2 * \cdots$ and $\hat{Q}_1 * \hat{Q}_2 * \cdots$ is $(a_n + b_n)_{n=1}^{\infty}$. We shall emphasize this case in the proofs of some results.

Now we give the definition of the codings of the points in the self-similar sets. It is sightly different from the usual way. Recall the IFS's of K_1 and K_2 are $\{f_i(x) = \frac{x}{\beta^{n_i}} + a_i\}_{i=1}^{n}$ and $\{g_j(x) = \frac{x}{\beta^{m_j}} + b_j\}_{j=1}^{m}$, where n_i, a_i, m_j, b_j are determined by the IFS's of K_1 and K_2. It is well known that for any $x \in K_1$, there exists $(i_k)_{k=1}^{\infty}$ such that

$$x = \lim_{k\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(0).$$

Usually, $(i_k)_{k=1}^{\infty}$ is called a coding of x. Nevertheless, we may make use of another representation.

Note that

$$f_i(x) = \frac{x}{\beta^{n_i}} + a_i = \frac{x + \beta^{n_i} a_i}{\beta^{n_i}} = \frac{x}{\beta^{n_i}} + \frac{0}{\beta} + \frac{0}{\beta^2} + \cdots + \frac{0}{\beta^{n_i - 1}} + \frac{\beta^{n_i} a_i}{\beta^{n_i}},$$

therefore, we can identify $f_i(x)$ with a block $(\underbrace{000\cdots0}_{n_i-1} a_i')$, where $a_i' = \beta^{n_i} a_i$. In fact, $f_i(x)$ and $(\underbrace{000\cdots0}_{n_i-1} a_i')$ can be determined mutually. Given $(\underbrace{000\cdots0}_{n_i-1} a_i')$ with length n_i and $a_i' = \beta^{n_i} a_i$, we can find a similitude

$$f_i(x) = \frac{x}{\beta^{n_i}} + \frac{0}{\beta} + \frac{0}{\beta^2} + \cdots + \frac{0}{\beta^{n_i - 1}} + \frac{\beta^{n_i} a_i}{\beta^{n_i}} = \frac{x + \beta^{n_i} a_i}{\beta^{n_i}} = \frac{x}{\beta^{n_i}} + a_i.$$

For simplicity we denote this block by $\hat{P}_i = (\underbrace{000\cdots0}_{n_i-1} a_i')$ if there is no fear of ambiguity.

We identify f_i with $f_{\hat{P}_i}$. The only difference between f_i and $f_{\hat{P}_i}$ is the symbol as both of them represent the map $f_i(x) = f_{\hat{P}_i}(x) = \frac{x}{\beta^{n_i}} + a_i$. Similarly, we may define blocks in terms of the IFS of K_2. Let $D_1 = \{\hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n\}$ and $D_2 = \{\hat{Q}_1, \hat{Q}_2, \ldots, \hat{Q}_m\}$, where $\hat{P}_i = (\underbrace{000\cdots0}_{n_i-1} a_i')$, $a_i' = \beta^{n_i} a_i$, $\hat{Q}_j = (\underbrace{000\cdots0}_{m_j-1} b_j')$ and $b_j' = \beta^{m_j} b_j$. We say D_1 and D_2

are the digit sets of K_1 and K_2 respectively. The elements of D_i are called the blocks. We emphasize that different blocks may stand for the same similitude, for example let $\hat{R}_1 = (08)$ and $\hat{R}_2 = (22)$ be two blocks with respect to base 3, since their associated similitudes coincide, i.e. $\varphi_{\hat{R}}(x) = \frac{x}{3^2} + \frac{0}{3} + \frac{8}{3^2} = \frac{x}{3^2} + \frac{2}{3} + \frac{2}{3^2}$, we can choose either of them if we want to find the digit sets of K_i, $1 \leq i \leq 2$. This replacement does not affect our main result. Usually, we pick the simpler blocks which facilitate our calculation. Once we choose the blocks, we fix them. With this new representation, we have the following simple lemma.

Lemma 2.6.

$$K_1 = \{x = \lim_{n\to\infty} f_{\hat{P}_{i_1}} \circ f_{\hat{P}_{i_2}} \circ \cdots \circ f_{\hat{P}_{i_n}}(0) : \hat{P}_{i_j} \in D_1\}.$$

$$K_2 = \{y = \lim_{n\to\infty} g_{\hat{Q}_{i_1}} \circ g_{\hat{Q}_{i_2}} \circ \cdots \circ g_{\hat{Q}_{i_n}}(0) : \hat{Q}_{i_j} \in D_2\}.$$

We call the concatenation $\hat{P}_{i_1} * \hat{P}_{i_2} * \cdots$ $(\hat{Q}_{i_1} * \hat{Q}_{i_2} * \cdots)$ *a coding of* x (y).

Proof. For any $x \in K_1$, we know that there exists $(i_n)_{n=1}^{\infty}$ such that

$$x = \lim_{n\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0).$$

The lemma is a restatement of this fact. □

Remark 2.7. Although the lemma above is very simple, the significance of this lemma is that we can translate over the problem, i.e. in order to study the sum of two numbers from K_1 and K_2 respectively, it is sufficient to consider the sum of the blocks from D_1 and D_2.

Motivated by this lemma, we define a crucial definition of this paper.

Definition 2.8. Take s blocks

$$\hat{P}_{i_1},\ \hat{P}_{i_2},\ \hat{P}_{i_3},\ \ldots,\ \hat{P}_{i_s}$$

from D_1 with lengths $p_1,\ p_2,\ p_3,\ \ldots,\ p_s$, t blocks

$$\hat{Q}_{j_1},\ \hat{Q}_{j_2},\ \hat{Q}_{j_3},\ \ldots,\ \hat{Q}_{j_t}$$

from D_2 with lengths $q_1,\ q_2,\ q_3,\ \ldots,\ q_t$. If there exist integers $k_1,\ k_2,\ k_3, \ldots, k_s, l_1,\ l_2,\ l_3, \ldots, l_t$ such that

$$\sum_{i=1}^{s} k_i p_i = \sum_{j=1}^{t} l_i q_i,$$

then the block $(\hat{P}_{i_1}^{k_1} * \hat{P}_{i_2}^{k_2} * \cdots * \hat{P}_{i_s}^{k_s}) + (\hat{Q}_{j_1}^{l_1} * \hat{Q}_{j_2}^{l_2} * \cdots * \hat{Q}_{j_t}^{l_t})$ is called a Matching with respect to β.

Remark 2.9. Let A and B be two concatenations of some blocks from D_1 and D_2 respectively. If A and B have the same length, then the summation of A and B is a Matching, i.e. $A + B$ is a Matching. We call the elements of D_i blocks. However, a Matching, in fact, is also a block which is the sum of concatenated blocks from D_1 and D_2 respectively. In what follows, we still call a Matching a block if there is no fear of ambiguity. Clearly, in this definition the blocks $\hat{P}_{i_j}$ and $\hat{P}_{i_k}$ $(j \neq k)$ could coincide. Given a Matching we may find its associated similitude. For instance, let (abc) be a Matching with respect to β, then the corresponding similitude is $\varphi(x) = \frac{x}{\beta^3} + \frac{a}{\beta} + \frac{b}{\beta^2} + \frac{c}{\beta^3}$.

We show that D_1 and D_2 generate countably many Matchings.

Lemma 2.10. *The cardinality of Matchings which are generated by D_1 and D_2 is at most countable.*

Proof. The proof is constructive. Firstly, we find out all the possible Matchings which have length 1. The cardinality of Matchings with length 1 is finite due to the finite cardinalities of D_1 and D_2. If there are no such Matchings (see Example 2.14), we then consider the Matchings with length 2. Similarly, we can find finite Matchings which are of length 2. If there do not exist such Matchings, then we may consider the Matchings with length 3. We continue this procedure and prove the lemma. However, the following adjustment is helpful to reduce some unnecessary Matchings, i.e. if the new born Matchings can be concatenated by the old Matchings, then we do not choose these new Matchings. In the remaining paper we always abide by this rule. In some cases, after some steps, all the new Matchings can be concatenated by the former old Matchings (see Example 2.13), then we stop the procedure. For this case, the cardinality of Matchings is finite. If the procedure can be continued for infinitely many times, then the cardinality of Matchings is infinitely countable. Hence, the cardinality of Matchings is either finite or countably infinite. □

Remark 2.11. We shall prove that if the cardinality of Matchings is finite, then K_1+K_2 is a self-similar set while $K_1 + K_2$ is the unique attractor of some IIFS if the cardinality of Matchings is infinitely countable.

Example 2.12. Let $K_1 = K_2$ be the attractor of the IFS $\{g_1(x) = \frac{x}{3}, g_2(x) = \frac{x+8}{9}\}$. All the possible Matchings are

$$\{(0), (22), (44), (242), (2442), (24442), (244442), (2444442), (24444442), \dots\},$$

where $D_1 = D_2 = \{(0), (08) = (22)\}$. Here, for simplicity we assume that $\hat{R} = (08) = (22)$ as their corresponding similitudes are the same, i.e. $\varphi_{\hat{R}}(x) = \frac{x}{3^2} + \frac{0}{3} + \frac{8}{3^2} = \frac{x}{3^2} + \frac{2}{3} + \frac{2}{3^2}$.

Example 2.13. Let $\{f_1(x) = \frac{x}{3}, f_2(x) = \frac{x+2}{3}\}$ be the IFS of K_1, K_2 is generated by $\{g_1(x) = \frac{x}{3}, g_2(x) = \frac{x+8}{9}\}$. Then the Matchings generated by D_1 and D_2 are $\{(0), (2), (24), (42), (44)\}$, where $D_1 = \{(0), (2)\}$ and $D_2 = \{(0), (22)\}$.

Example 2.14. Let $\{f_1(x) = \frac{x}{9}, f_2(x) = \frac{x}{3^3} + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3}\}$ be the IFS of $K_1 = K_2$, where $D_1 = D_2 = \{(00), (222)\}$. For this example, there is no Matching with length 1.

After we find all the possible Matchings, we denote this set by

$$D = \{\hat{R}_1, \hat{R}_2, \dots, \hat{R}_{n-1}, \hat{R}_n, \dots\},$$

the lengths of these Matchings are increasing. By Remark 2.9, D uniquely determines a set of similitudes $\Phi^\infty \triangleq \{\phi_1, \phi_2, \phi_3, \phi_4, \dots\}$. We define $E \triangleq \bigcup_{\{\phi_n\}\in\Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \cdots \circ \phi_n([0, 1])$ and have $E = \bigcup_{i\in\mathbb{N}} \phi_i(E)$, see Section 2 from [1].

Now we state the first main result.

Theorem 2.15. *$K_1 + K_2$ is either a self-similar set or an attractor of some infinite iterated function system. More precisely, if the cardinality of Matchings is finite, then $K_1 + K_2$ is a self-similar set. When the cardinality is infinitely countable, we have*

$$K_1 + K_2 = \overline{\bigcup_{\phi_i\in\Phi^\infty} \phi_i(K_1 + K_2)}.$$

Remark 2.16. A minor modification enables us to prove the following stronger result: for any $n \in \mathbb{N}^+$ and any $\{K_i\}_{i=1}^n$, $K_1 + K_2 + \cdots + K_n = \{\sum_{i=1}^n x_i : x_i \in K_i\}$ is either a self-similar set or a unique attractor of some IIFS, where $\{K_i\}_{i=1}^n$ are generated by the similitudes of S. In [9], Mendes and Oliveira proved that for the homogeneous Cantor sets, there are five possible structures for the sum. However, in our setting we may find only two structures, i.e. $K_1 + K_2$ is either a self-similar set or an attractor of some IIFS.

We have an interesting corollary of Theorem 2.15.

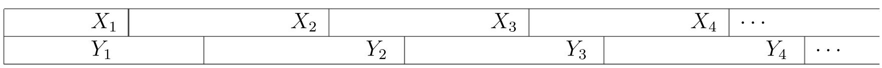
Corollary 2.17. *Let F_1 and F_2 be the self-similar sets with IFS's $\{r_i x + a_i\}_{i=1}^n$ and $\{r'_j x + b'_j\}_{j=1}^m$, if $0 < r_i, r'_j < 1$ for any $1 \le i \le n$ and $1 \le j \le m$, then*

$$\dim_P(F_1 + F_2) = \overline{\dim}_B(F_1 + F_2).$$

2.3. Proofs of Theorem 2.15 and Corollary 2.17

To begin with we assume that the cardinality of all Matchings is infinitely countable. Before we prove the main results, we need some preliminaries. In Lemma 2.6 we give the definition of the codings of K_i, $1 \le i \le 2$. Here we define the coding of $x + y \in K_1 + K_2$ in a natural way, i.e. we denote the coding of $x + y$ by $(x_n + y_n)_{n=1}^\infty$, where (x_n) and (y_n) are the codings of x and y respectively.

We know that (x_n) $((y_n))$ can be decomposed into infinite blocks from D_1 (D_2), see the following figure



There are two floors in this figure. By Remark 2.5, the concatenation of $X_1 * X_2 * \cdots$ $(Y_1 * Y_2 * \cdots)$ is (x_n) $((y_n))$, and we can define the summation of the concatenated infinite blocks.

We call the top floor (bottom floor) the x-floor (y-floor). In other words, in the x-floor the concatenation of each block X_i is the coding of x. We shall use this diagram representing the blocks in the proofs of some lemmas. Let $(a_n)_{n=1}^\infty$ be a coding of some point $x + y \in K_1 + K_2$, i.e., $(a_n) = (x_n + y_n)$, where $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are the codings of $x \in K_1$ and $y \in K_2$ respectively. Given $k > 0$, we say $(c_{i_1} c_{i_2} \cdots c_{i_k})$ is a segment of $(a_i)_{i=1}^\infty$ with length k if there exists $j > 0$ such that $c_{i_1} c_{i_2} \cdots c_{i_k} = a_{j+1} \cdots a_{j+k}$. We define

$$C = \Big\{(a_n) = (x_n + y_n) : \text{there exists } N \in \mathbb{N}^+ \text{ such that any segment of } (a_{N+i})_{i=1}^\infty \text{ is not a Matching}\Big\}.$$

Lemma 2.18. *Let $(a_n) \in C$, for any $\epsilon > 0$ we can find a coding $(b_n)_{n=1}^\infty$ which is the concatenation of infinite Matchings such that*

$$|(a_n)_\beta - (b_n)_\beta| < \epsilon.$$

Proof. Let $(a_n) \in C$ and $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ satisfying $\beta^{-n_0} < \epsilon$. Now, we choose $(b_n)_{n=1}^\infty$ such that its value in base β is a point of E. Let $b_1 b_2 b_3 \cdots b_{n_0} = a_1 a_2 a_3 \cdots a_{n_0}$. If

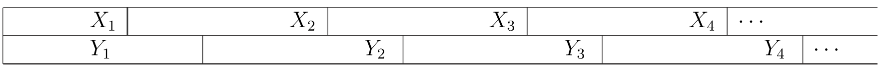
$a_1a_2a_3\cdots a_{n_0}$ is a Matching or a concatenation of some Matchings, then we can choose arbitrary tail $(b_{n_0+i})_{i=1}^{\infty}$ which is the concatenation of infinite Matchings. Subsequently we have that

$$\begin{aligned}|(a_n)_\beta - (b_n)_\beta| &= |(a_{n_0+1}a_{n_0+2}a_{n_0+3}\cdots)_\beta - (b_{n_0+1}b_{n_0+2}b_{n_0+3}\cdots)_\beta| \\ &\le M\sum_{i=n_0+1}^{\infty}\beta^{-i} < M(\beta-1)^{-1}\epsilon,\end{aligned}$$

where M is a positive constant which depends on β and the translations of the IFS's of K_1 and K_2. Hence we prove that there exists a point $b \in E$, i.e. $b = (b_n)_\beta$, such that

$$|(a_n)_\beta - (b_n)_\beta| < \epsilon.$$

If $a_1a_2a_3\cdots a_{n_0}$ is not a concatenation of some Matchings, by virtue of the definition of (a_n), $(a_n) = (x_n + y_n)$, where (x_n), (y_n) are the codings of some points in K_1 and K_2, respectively. However, (x_n) $((y_n))$ can be decomposed into the concatenation of $X_1 * X_2 * \cdots$ $(Y_1 * Y_2 * \cdots)$. We use the following diagram to represent this.



From this figure, we know that the summation of $X_1 * X_2 * \cdots$ and $Y_1 * Y_2 * \cdots$ is precisely the coding (a_n). Suppose that there exist p, q such that $a_1a_2a_3\cdots a_{n_0}$ is a prefix of $(X_1 * X_2 * \cdots * X_p) + (Y_1 * Y_2 * \cdots * Y_q)$, here we should emphasize that the lengths of $X_1 * X_2 * \cdots * X_p$ and $Y_1 * Y_2 * \cdots * Y_q$ may not coincide. However, we can still define the summation of their prefixes. Since $X_1 * X_2 * \cdots * X_p$ and $Y_1 * Y_2 * \cdots * Y_q$ do not have the same length, we assume that $\sum_{i=1}^{p}|X_i| < \sum_{i=1}^{q}|Y_i|$, where $|X_i|$ denotes the length of the block X_i, then the first n_0 digits of the "summation" $(X_1 * X_2 * \cdots * X_p) + (Y_1 * Y_2 * \cdots * Y_q)$ are $a_1a_2a_3\cdots a_{n_0}$. Let $k_1 = \sum_{i=1}^{p}|X_i|$ and $k_2 = \sum_{i=1}^{q}|Y_i|$. Then $(X_1 * X_2 * \cdots * X_p)^{k_2} + (Y_1 * Y_2 * \cdots * Y_q)^{k_1}$ is a Matching or a concatenation of some Matchings as $(X_1 * X_2 * \cdots * X_p)^{k_2}$ and $(Y_1 * Y_2 * \cdots * Y_q)^{k_1}$ have the same length. Moreover, the initial n_0 digits of $(X_1 * X_2 * \cdots * X_p)^{k_2} + (Y_1 * Y_2 * \cdots * Y_q)^{k_1}$ are $a_1a_2a_3\cdots a_{n_0}$. Now the remaining proof is the same as the first case. □

Remark 2.19. The main idea of this lemma is that any $(a_n) \in C$ can be approximated by a sequence (c_n) which is the concatenation of infinite Matchings.

Lemma 2.20. $\overline{E} = K_1 + K_2$.

Proof. For every $\epsilon > 0$ and $x + y \in K_1 + K_2$, we can find a coding (a_n) satisfying $x + y = \sum_{n=1}^{\infty} a_n\beta^{-n}$. If there exists a subsequence of integers $n_k \to \infty$ such that $(a_1, a_2, a_3, \ldots, a_{n_k})$ is a concatenation of some Matchings, then by the definition of $E \triangleq \bigcup_{\{\phi_n\}\in\Phi^\infty}\bigcap_{n=1}^{\infty}\phi_1\circ\phi_2\cdots\phi_n([0,1])$ we have $x + y \in E$. If $(a_n) \in C$, by Lemma 2.18 there exists $b \in E$ such that $|b - x - y| < \epsilon$. □

Lemma 2.21. $\overline{\bigcup_{i\in\mathbb{N}}\phi_i(K_1 + K_2)} = K_1 + K_2$.

Proof. On the one hand, $E = \bigcup_{i\in\mathbb{N}}\phi_i(E)$, this equality implies that

$$\overline{E} = \overline{\bigcup_{i\in\mathbb{N}}\phi_i(E)} = \overline{\overline{\bigcup_{i\in\mathbb{N}}\phi_i(E)}} \supseteq \overline{\bigcup_{i\in\mathbb{N}}\overline{\phi_i(E)}} = \overline{\bigcup_{i\in\mathbb{N}}\phi_i(K_1+K_2)},$$

i.e. we have

$$\overline{\bigcup_{i\in\mathbb{N}} \phi_i(K_1+K_2)} \subseteq K_1+K_2.$$

On the other hand, $E = \bigcup_{i\in\mathbb{N}} \phi_i(E) \subseteq \bigcup_{i\in\mathbb{N}} \phi_i(K_1+K_2)$, therefore we prove the converse inclusion in terms of Lemma 2.20. □

Proof of Theorem 2.15. Using Lemma 2.10, we know that there are at most countably many Matchings generated by D_1 and D_2. If the cardinality of Matchings is infinitely countable, then by Lemma 2.21, K_1+K_2 is an attractor of Φ^∞. If the cardinality is finite, then K_1+K_2 is a self-similar set. The proof is similar with Lemmas 2.18 and 2.20. The only difference is that it is not necessary to approximate the coding of $x+y \in K_1+K_2$. In fact, we can directly find a coding which is the concatenation of infinite Matchings such that the value of this infinite coding is $x+y$. In other words, we have $E = K_1+K_2$. □

Now, we can prove Corollary 2.17. When the IFS's of F_1 and F_2 satisfy the irrationality assumption, it is easy to prove Corollary 2.17 due to Peres and Shmerkin [16]. In fact, we can prove a stronger result. Let us recall their main result.

Theorem 2.22. *Let F_1 and F_2 be the attractors of $\{r_i x + a_i\}_{i=1}^n, \{r'_j x + b_j\}_{j=1}^m$ respectively. If there exist i, j such that $\frac{\log r_i}{\log r'_j} \notin \mathbb{Q}$, then* $\dim_H(F_1+F_2) = \min\{\dim_H F_1 + \dim_H F_2, 1\}$.

Proof of Corollary 2.17. Firstly, we prove under the irrationality assumption that

$$\dim_H(F_1+F_2) = \dim_P(F_1+F_2) = \dim_B(F_1+F_2) = \min\{\dim_H F_1 + \dim_H F_2, 1\}.$$

Using the theorem above, if $\dim_H(F_1+F_2) = 1$, then

$$1 = \dim_H(F_1+F_2) \le \dim_P(F_1+F_2) \le \overline{\dim}_B(F_1+F_2) \le 1.$$

Suppose $\dim_H(F_1+F_2) = \dim_H(F_1) + \dim_H(F_2)$. We note that for any $A, B \subseteq \mathbb{R}$, we have $B - A = P_{\frac{\pi}{4}}(A \times B)$, where $P_{\frac{\pi}{4}}(A \times B)$ denotes the projection of $A \times B$ on the y axis along lines having 45° angle with the x axis. Therefore,

$$\begin{aligned} \dim_H(F_1+F_2) &\le \overline{\dim}_B(F_1+F_2) \\ &\le \overline{\dim}_B((-F_2)\times F_1) \\ &\le \overline{\dim}_B(F_1) + \overline{\dim}_B(F_2) \\ &= \dim_H(F_1) + \dim_H(F_2). \end{aligned}$$

The second inequality holds as the projection is a Lipschitz map, the third inequality is due to the property of product of fractal sets, see the product formula 7.5, page 102, [4]. For the last equality, we use the fact that for any self-similar set, its Hausdorff dimension and the Box dimension coincide.

If K_1 and K_2 are generated by the similitudes of S and the cardinality of Matchings is infinitely countable, then we have $\dim_P(K_1+K_2) = \overline{\dim}_B(K_1+K_2) = \dim_P(E) = \overline{\dim}_B(E)$ due to Lemma 2.20 and Theorem 3.1 from [1]. By Theorem 2.15, we know that K_1+K_2 is a self-similar set if the cardinality of Matchings is finite. Hence, whether the irrationality assumption holds or not we always have $\dim_P(K_1+K_2) = \overline{\dim}_B(K_1+K_2)$. □

3. Dimension of $K_1 + K_2$

3.1. IFS case

Let $\sharp D$ be the cardinality of all Matchings generated by D_1 and D_2. In this section we give a necessary and sufficient condition for the finiteness of $\sharp D$. We know that $K_1 + K_2$ is a self-similar set if $\sharp D$ is finite. Hence, in this case we may make use of various techniques finding the Hausdorff dimension of $K_1 + K_2$.

We say that $D_i, 1 \leq i \leq 2$, is homogeneous if the length of all the blocks is equal. For simplicity we may identify the blocks with the lengths of the blocks. There is one point we should keep in mind, namely different blocks of D_i may have the same length. Hence we should count the multiplicity when some blocks have the same length, see the following example.

Example 3.1. Let $\{f_1(x) = \frac{x}{3},\ f_2(x) = \frac{x+2}{3}\}$ be the IFS of K_1, K_2 is generated by $\{g_1(x) = \frac{x}{3},\ g_2(x) = \frac{x+8}{9}\}$. The digit sets are $D_1 = \{(0),\ (2)\}$ and $D_2 = \{(0),\ (22)\}$. We can denote D_1 by $D'_1 = \{1, 1\}$. For simplicity we still use D_1. Similarly, $D_2 = \{1, 2\}$. It is clear that D_1 is homogeneous and that two 1's in the set refer to different similitudes.

It is easy to find that the digits in D_i stand for the length of the blocks and the similarity ratios, see the following example.

Example 3.2. Let $\{f_1(x) = \frac{x}{\beta^6} + a_1,\ f_2(x) = \frac{x}{\beta^{10}} + a_2\}$ be the IFS of K_1. We know that $D_1 = \{6, 10\}$. 6 represents the length of the block $(00000\,(a_1\beta^6))$ and stands for the similarity ratios $\frac{1}{\beta^6}$.

For this example, by the definition of K_1 we have $K_1 = f_1(K_1) \cup f_2(K_1)$. Iterating this equation, then we have that

$$K_1 = f_1 \circ f_1(K_1) \cup f_1 \circ f_2(K_1) \cup f_2 \circ f_1(K_1) \cup f_2 \circ f_2(K_1).$$

Hence we obtain 4 similitudes $\{f_1 \circ f_1,\ f_1 \circ f_2,\ f_1 \circ f_2,\ f_2 \circ f_2\}$. Their associated digit set which consists of some blocks can also be denoted by a simpler set $D'' = \{12, 16, 16, 20\}$. Similarly, we can iterate the original IFS for any finite times. For the sake of convenience, we still use the set of the lengths of the blocks as it not only stands for the new iterated blocks but also refers to the similarity ratios under new IFS.

Definition 3.3. Let $D_1 = \{k, k, \ldots, k\}$ be a homogeneous set with l digits. We say D_2 is a multiplier set of D_1 if we iterate the IFS of K_2 for finite times, all the numbers of the new digit set D' are the multipliers of k,i.e., $D' = \{l_1 k, l_2 k, \ldots, l_t k\}$, where $l_i \in \mathbb{N}^+$. Similarly, if D_2 is homogeneous, we can also define D_1 as the multiplier set of D_2 if D_1 satisfies similar property.

Theorem 3.4. *$\sharp D$ is finite if and only if D_1 (D_2) is homogeneous and D_2 is a multiplier set of D_1 (D_1 is a multiplier set of D_2).*

We partition the proof of this theorem into several lemmas.

Lemma 3.5. *If D_1 is homogeneous and D_2 is a multiplier set of D_1, then $\sharp D$ is finite.*

Proof. Let $D_1 = \{k, k, \ldots, k\}$ be a homogeneous set and D_2 be a multiplier set of D_1. By the definition of multiplier set, after finite iterations of the IFS of K_2, say t times, $D'_2 = \{l_1 k, l_2 k, \ldots, l_m k\}$, where $l_i \in \mathbb{N}^+$. Now we prove that $\sharp D$ is finite. Let $D_2 = \{s_1, s_2, \ldots, s_p\}$, where $s_p \in \mathbb{N}^+$. If we take any t digits from D_2, each time we can pick any numbers, which means we can pick s_i for any $1 \le k \le t$ times, then by the definition of multiplier set, $s_{i_1} + s_{i_2} + \cdots + s_{i_t}$ is a multiplier of k. Since $D_1 = \{k, k, \ldots, k\}$ is homogeneous and the cardinality of $D_2' = \{l_1 k, l_2 k, \ldots, l_m k\}$ is finite, it follows that $\sharp D$ is finite. □

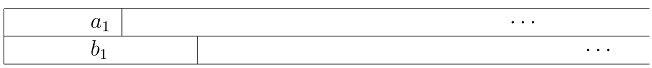
Lemma 3.6. *If $\sharp D$ is finite, then either D_1 or D_2 is homogeneous.*

Proof. We have proved that if $\sharp D$ is finite, then $K_1 + K_2$ is a self-similar set. This fact implies that for any coding of $x + y \in K_1 + K_2$, say $(a_n) = (x_n + y_n)$, its associated value in base β is $x + y$, where (x_n) and (y_n) are the codings of x and y respectively. Moreover, (a_n) is the infinite concatenation of some Matchings. In other words, there exists a sequence $N_k \to \infty$ such that $(a_1 a_2 \cdots a_{N_k})$ is a concatenation of some Matchings.

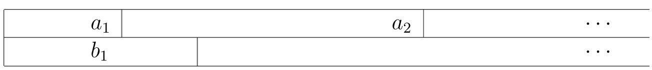
If neither D_1 nor D_2 is homogeneous, we may find a coding of some point in $K_1 + K_2$ which does not contain any Matchings in its arbitrary long prefix. This contradicts with the assumption that $\sharp D$ is finite.

Now we find a coding which satisfies the property we mentioned above. Without loss of generality, we assume that $D_1 = \{a_1, a_2, \ldots, a_p\}$ and $D_2 = \{b_1, b_2, \ldots, b_q\}$, where $a_1 \neq a_2$ and $b_1 \neq b_2$.

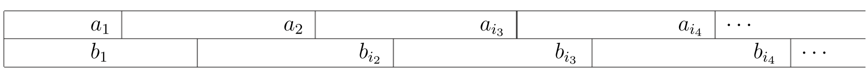
We demonstrate how we can construct the coding we need. Recall the definition of x-floor and y-floor, we know that summation of the concatenation of the blocks of x-floor and y-floor is the coding of some point of $K_1 + K_2$. Since $a_1 \neq a_2$ and $b_1 \neq b_2$, we may suppose $a_1 \neq b_1$ and put them in the x-floor and y-floor respectively, see the following figure



Here we identify the block with its length. Since $a_1 \neq b_1$, it follows that no Matching appears. Next, for the x-floor, we pick a_2 which satisfies that $a_1 + a_2 \neq b_1$. If $a_1 + a_2 = b_1$, then we pick a_1 again. The Matching cannot appear as $a_1 \neq a_2$ and $a_1 + a_2 = b_1$ imply that $a_1 + a_1 \neq b_1$. Now the x-floor and y-floor become the following:



For the y-floor, we repeat the same procedure. Finally we have



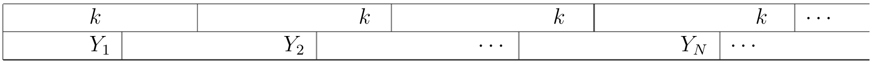
For each step, the Matching does not appear as the length of the concatenations of blocks from x and y-floor is not matched. The summation of the infinite concatenated blocks from x and y-floor is the coding we need. □

Now we may set $D_1 = \{k, k, \ldots, k\}$, if D_2 is not a multiplier set of D_1, we implement similar idea constructing a coding such that its arbitrary long prefix is not a concatenation of some Matchings.

Hence, in order to prove Theorem 3.4, it remains to prove the following lemma.

Lemma 3.7. *Let $D_1 = \{k, k, \ldots, k\}$, if D_2 is not a multiplier set of D_1, then $\sharp D$ is not finite.*

Proof. If $\sharp D$ is finite, then any coding of $x + y \in K_1 + K_2$, say $(a_n) = (x_n + y_n)$, is the infinite concatenation of some Matchings. Namely there exists a sequence $N_k \to \infty$ such that $(a_1 a_2 \cdots a_{N_k})$ is a concatenation of some Matchings. If we can find a coding (a_n) such that for any n $(a_1 a_2 \cdots a_n)$ is not a concatenation of some Matchings, then we prove this lemma. Since D_2 is not a multiplier set of D_1, it follows that for any finite iterations of the IFS of K_2, there always exists one block which is the concatenation of some blocks from D_2 such that its length is not a multiplier of k. We let this block be $(b_1 b_2 \cdots b_t)$, see the following figure:



We may assume that $(b_1 b_2 \cdots b_t) = Y_1 * Y_2 * \cdots * Y_N$ for some N, where each Y_i is some block from D_2. By the assumption we know that its length is not a multiplier of k. Hence we can find such coding (a_n) (sum of the x and y-floor) satisfying that for any n, $(a_1 a_2 \cdots a_n)$ is not a concatenation of some Matchings. □

Remark 3.8. When $K_1 + K_2$ is a self-similar set, we do not know whether $\sharp D$ is finite or not.

If $\sharp D$ is finite, then $K_1 + K_2$ is a self-similar set. In this case, we can explicitly find all the similitudes of the IFS. Therefore we can implement many ideas calculating $\dim_H(K_1 + K_2)$. We do not discuss this problem in detail.

3.2. IIFS case

Comparing with IFS case, it is much more complicated when $K_1 + K_2$ is a unique attractor of some IIFS. We have mentioned the main reasons in the second section.

By Lemma 2.20, we know that when $\sharp D$ is infinitely countable, $\overline{E} = K_1 + K_2$. If $\overline{E} \setminus E$ is uncountable, we may not calculate the dimension of $K_1 + K_2$ in terms of the dimensional theory of IIFS. Hence, we need to find some class that can guarantee $\dim_H(E) = \dim_H(K_1 + K_2)$. In fact, even for calculating $\dim_H(E)$, it is not easy to find $\dim_H(E)$ when the IIFS has some overlaps [13,6].

Let $(a_n)_{n=1}^{\infty}$ be the coding of some point $x + y \in K_1 + K_2$, i.e., $(a_n) = (x_n + y_n)$, where (x_n) and (y_n) are the codings of x and y respectively. Recall the definition of C,

$$C = \{(a_n) : \text{there exists } N \in \mathbb{N}^+ \text{ such that any segment of } (a_{N+i})_{i=1}^{\infty} \text{ is not a Matching}\}.$$

We have

Lemma 3.9. *If C is countable, then we have that $E = K_1 + K_2$ apart from a countable set.*

Proof. By Lemma 2.20, $\overline{E} = K_1 + K_2$. It remains to prove that there are only countably many limit points of E which are not in E. For any $x + y \in K_1 + K_2 = \overline{E}$, there is a coding (a_n) such that the value of this coding is $x + y$. If there exists $n_k \to \infty$ satisfying that $(a_1 a_2 \cdots a_{n_k})$ is a Matching or a concatenation of some Matchings, by the definition of $E \triangleq \bigcup_{\{\phi_n\} \in \Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n([0, 1])$, we know that $x + y \in E$. If $(a_n) \in C$, then $\overline{E} \setminus E$ is countable as C and the cardinality of all the Matchings is countable. □

The following lemma gives a sufficient condition which implies that C is countable.

Lemma 3.10. *C is countable if there exists k such that $D_1 = \{k, k, \ldots, k, 2k\}$ and $D_2 = \{k, k, \ldots, k, 2k\}$, i.e. both D_1 and D_2 have only blocks with length k apart from the last block with length $2k$.*

Proof. If $D_1 = \{k, k, \ldots, k, 2k\}$ and $D_2 = \{k, k, \ldots, k, 2k\}$, we need to find all possible sequences of C. Without loss of generality, we assume that the prefix of the summation of the x and y-floor does not contain any Matchings. Firstly, we choose two blocks from D_1 and D_2 and stack on the x-floor and y-floor respectively. We can pick only k from D_1 and $2k$ from D_2 (or $2k$ from D_1 and k from D_2). Otherwise, a Matching will appear, see the following figure

k	$\cdots$
$2k$	$\cdots$

Then at the second step for the x-floor we cannot take any block of D_1 with length k as $k+k = 2k$ and a new Matching appears. Hence for the x-floor we can pick only the block with length $2k$. Similarly, for the y-floor we cannot take a block of D_2 with length k as $k + 2k = 2k + k$, which can generate a new Matching. Therefore, we must take a block with length $2k$ for the y-floor if we do not want a new Matching to appear. The figure now is

k	$2k$	$\cdots$
$2k$	$2k$	$\cdots$

It is easy to see that if we want to avoid the new Matchings in the summed blocks of two floors we cannot choose blocks freely from the second step on. The figure below illustrates this idea.

k	$2k$	$2k$	$2k$	$\cdots$
$2k$	$2k$	$2k$	$2k$	$\cdots$

From the analysis above, we see that the sequences in C are eventually periodic. Thus, we prove that C is countable. □

Remark 3.11. The condition of the lemma is not necessary, for instance, let $D_1 = \{k, 2k\}$ and $D_2 = \{k, 3k\}$. We can similarly prove that in this case C is countable. Generally it is not easy to find all the Matchings. However, for the case in this lemma we can find all possible Matchings without much calculation.

This lemma enables us to define the following IFS.

For any $k \in \mathbb{N}^+$, let the IFS's of K_1 and K_2 be

$$\left\{ f_i(x) = \frac{x}{\beta^k} + a_i,\, 1 \le i \le n-1,\, f_n(x) = \frac{x}{\beta^{2k}} + a_n \right\} \tag{2}$$

and

$$\left\{ g_j(x) = \frac{x}{\beta^k} + b_j,\, 1 \le j \le n-1,\, g_n(x) = \frac{x}{\beta^{2k}} + b_n \right\}, \tag{3}$$

where $a_i, b_j \in \mathbb{R}^+ \cup \{0\}$. We denote their attractors by K_1 and K_2 respectively. Without loss of generality, we let the convex hull of K_i be $[0, B_i], 0 \le i \le 2$. This assumption yields that $f_i([0, B_1]) \subset [0, B_1], 1 \le i \le n$ and $g_j([0, B_2]) \subset [0, B_2], 1 \le j \le n$.

Let $D = \{\hat{R}_1, \hat{R}_2, \ldots, \hat{R}_{n-1}, \hat{R}_n \cdots\}$ be all the Matchings generated by $D_1 = \{k, k, \ldots, 2k\}$ and $D_2 = \{k, k, \ldots, 2k\}$ and its associated IIFS be $\Phi^\infty \triangleq \{\phi_1, \phi_2, \phi_3, \phi_4, \ldots\}$. Define $E \triangleq \bigcup_{\{\phi_n\}\in\Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n([0, B_1 + B_2])$. We know that a Matching $\hat{R}_i$ is a block. Suppose $\hat{R}_i = (c_1 c_2, \ldots, c_p)$ for some $p \in \mathbb{N}$. We call each c_i the digit of $\hat{R}_i$. Since D_1 and D_2 have a finite number of blocks, it follows that the range of every possible digit c_j in each

Matching $\hat{R}_i$ is finite, i.e. c_j can take only finite numbers. Let c be the positive constant defined as follows:

$$c = \min\{|c_i - c_j| : c_i \text{ and } c_j \text{ are any digits which are from two Matchings}\}.$$

Similarly, we let A and B be the largest digits of the blocks of D_1 and D_2 respectively.

Theorem 3.12. *Let K_1 and K_2 be two self-similar sets with IFS's (2) and (3), respectively. Then $E = K_1 + K_2$ up to a countable set. If*

$$A + B + B_1 + B_2 < c(\beta - 1),$$

then Φ^∞ satisfies the open set condition and $\dim_H(K_1 + K_2)$ is computable.

Proof of Theorem 3.12. By Lemmas 3.9 and 3.10, we prove the first statement. For the second statement, given any two Matchings $(s_1 s_2 \cdots s_p), (t_1 t_2 \cdots t_q)$ with $p < q$, their associated similitudes are $\phi_{s_1 s_2 \cdots s_p}(x) = \beta^{-p}x + \sum_{i=1}^{p} s_i \beta^{-i}$ and $\phi_{t_1 t_2 \cdots t_q}(x) = \beta^{-q}x + \sum_{i=1}^{q} t_i \beta^{-i}$ respectively. Let $V = (0, B_1 + B_2)$, simple calculation implies that

$$\phi_{s_1 s_2 \cdots s_p}(V) = \left(\sum_{i=1}^{p} s_i \beta^{-i}, \sum_{i=1}^{p} s_i \beta^{-i} + (B_1 + B_2)\beta^{-p}\right)$$

$$\phi_{t_1 t_2 \cdots t_q}(V) = \left(\sum_{i=1}^{q} t_i \beta^{-i}, \sum_{i=1}^{q} t_i \beta^{-i} + (B_1 + B_2)\beta^{-q}\right).$$

We assume that $(s_1 s_2 \cdots s_p) < (t_1 t_2 \cdots t_q)$, i.e, there exists $1 \le i_0 \le p$ such that $s_k = t_k$ for any $1 \le k \le i_0 - 1$ and $s_{i_0} < t_{i_0}$. By the definition of c, we can check that the two intervals above do not overlap, namely $\phi_{s_1 s_2 \cdots s_p}(V) \cap \phi_{t_1 t_2 \cdots t_q}(V) = \emptyset$. It remains to prove that $\phi(V) \subset V$ for any $\phi \in \Phi^\infty$. Let ϕ be generated by the Matching $\hat{R}_1 * \hat{R}_2 + \hat{T}_1 * \hat{T}_2$, the associated similitudes of $\hat{R}_i$ and $\hat{T}_i$ are $H_i(x)$ and $I_i(x)$ respectively. Let the length of $\hat{R}_1 * \hat{R}_2 + \hat{T}_1 * \hat{T}_2$ be k_0. It is easy to find that

$$\phi(x) = H_1 \circ H_2(x) + I_1 \circ I_2(0).$$

Hence,

$$\phi(V) = \left(H_1 \circ H_2(0) + I_1 \circ I_2(0), H_1 \circ H_2(0) + I_1 \circ I_2(0) + \frac{B_1 + B_2}{\beta^{k_0}}\right).$$

Recall the assumption of K_1 and K_2, the convex hull of K_i is $[0, B_i], 1 \le i \le 2$, i.e., $H_s([0, B_1]) \subset [0, B_1], 1 \le s \le 2$ and $I_t([0, B_2]) \subset [0, B_2], 1 \le t \le 2$. Therefore $0 < \phi(x) < B_1 + B_2$. Similarly, we can prove that $\phi(V) \subset V$ for any $\phi \in \Phi^\infty$. As such Φ^∞ satisfies the open set condition. The calculation of $\dim_H(K_1 + K_2)$ now is a straightforward application of Theorem 2.3. □

Generally we do not know how to calculate $\dim_P(K_1 + K_2)$ or when do we have following equality

$$\dim_H(K_1 + K_2) = \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2).$$

We finish this section by making some remarks on these two problems. Let F_n be the attractor of the first n similitudes of Φ^∞, i.e., F_n is the attractor of the IFS $\{\phi_i\}_{i=1}^n$. Clearly

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots.$$

Recall the definition of Hausdorff metric [4]. Given two compact sets $J_1, J_2 \subset \mathbb{R}$, then the Hausdorff metric of J_1 and J_2 is defined by

$$\mathcal{H}(J_1, J_2) = \inf\{s : J_1 \subset (J_2)_s, J_2 \subset (J_1)_s\},$$

where $(A)_s = \{x :$ there exists $y \in A$ such that $|x - y| \leq s\}$.

We have

Lemma 3.13. $\overline{\cup_{n=1}^{\infty} F_n} = K_1 + K_2$.

Proof. $0 \leq \mathcal{H}(\overline{\cup_{n=1}^{\infty} F_n}, K_1+K_2) \leq \mathcal{H}(F_n, K_1+K_2) \to 0$ as $n \to \infty$. Here $\mathcal{H}(F_n, K_1+K_2) \to 0$ can be found in [5]. □

Proposition 3.14. *If* $(\overline{\cup_{n=1}^{\infty} F_n}) \setminus (\cup_{n=1}^{\infty} F_n)$ *is a countable set, then*

$$\dim_H(K_1 + K_2) = \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2).$$

Proof. Since $(\overline{\cup_{n=1}^{\infty} F_n}) \setminus (\cup_{n=1}^{\infty} F_n)$ is countable, it follows by Lemma 3.13 that

$$\begin{aligned}\dim_P(K_1 + K_2) &= \dim_P(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \dim_P(F_n)\\ &= \dim_H(\cup_{n=1}^{\infty} F_n) = \dim_H(\overline{\cup_{n=1}^{\infty} F_n}) = \dim_H(K_1 + K_2).\end{aligned}$$

We finish the proof by Corollary 2.17. □

4. Examples

In this section, we give some examples for which Theorem 2.22 cannot calculate $\dim_H(K_1 + K_2)$.

Example 4.1. Let $K_1 = K_2$ be the self-similar sets with IFS $\{g_1(x) = \frac{x}{3}, g_2(x) = \frac{x+8}{3^2}\}$, then $\dim_H(K_1 + K_2) = \frac{\ln t_0}{-\ln 3}$, where t_0 is the smallest positive root of $t^3 - t^2 - 2t + 1 = 0$.

We know that $D_1 = D_2 = \{(0), (22)\}$, all the Matchings which are generated by D_1 and D_2 are

$$D = \{(0), (22), (44), (242), (2442), (24442), (244442)\cdots\}.$$

The corresponding IIFS of D is

$$\Phi^{\infty} = \{\varphi_1 = f_0, \varphi_2 = f_2 \circ f_2, \varphi_3 = f_4 \circ f_4, \varphi_4 = f_2 \circ f_4 \circ f_2, \ldots\},$$

where $f_0(x) = \frac{x}{3}, f_2(x) = \frac{x+2}{3}, f_4(x) = \frac{x+4}{3}$.

By Theorem 3.12, $\dim_H(K_1 + K_2) = \dim_H(E)$. Obviously this IIFS satisfies the OSC, i.e.

$$\varphi_i((0, 2)) \cap \varphi_j((0, 2)) = \varnothing$$

for any $i \neq j$ and $\varphi_i((0, 2)) \subseteq (0, 2)$ for any $i \in \mathbb{N}$. Now we can use Theorem 2.3 to calculate the dimension. It is easy to check that

$$\dim_H(K_1 + K_2) < \min\{1, \dim_H(K_1) + \dim_H(K_2)\}.$$

This example illustrates that without the irrationality assumption, the expected dimension of $K_1 + K_2$ may not be achieved. This differs from Peres and Shmerkin's result [16].

Example 4.2. Let $\{f_1(x) = \frac{x}{\beta}, f_2(x) = \frac{x+2}{\beta}\}$ and $\{g_1(x) = \frac{x}{\beta}, g_2(x) = \frac{x}{\beta^2} + \frac{2}{\beta} + \frac{2}{\beta^2}\}$ be the IFS's of K_1 and K_2 respectively. Then $K_1 + K_2$ is a self-similar set, the IFS is $\{\varphi_1(x) = \frac{x}{\beta}, \varphi_2(x) = \frac{x+2}{\beta}, \varphi_3(x) = \frac{x}{\beta^2} + \frac{2}{\beta} + \frac{4}{\beta^2}, \varphi_4(x) = \frac{x}{\beta} + \frac{4}{\beta} + \frac{2}{\beta^2}, \varphi_5(x) = \frac{x}{\beta^2} + \frac{4}{\beta} + \frac{4}{\beta^2}\}$. This IFS does not satisfy the OSC generally, in fact it is of finite type if β is a Pisot number, see [14, Theorem 2.5]. Hence, we can calculate the Hausdorff dimension of $K_1 + K_2$ in terms of the main result of [14]. We omit the details.

5. Final remarks

The main result of this paper is that $K_1 + K_2$ is either a self-similar set or a unique attractor of some IIFS. However, to calculate the dimension of $K_1 + K_2$ is difficult, especially the IIFS case. As in this case, we should consider the limit points of E as well as the separation condition. Ignoring either of them may hinder the calculation of the dimension of $K_1 + K_2$. In fact, even finding all the Matchings is not a trivial task. On the other hand, we may implement the Vitali process if the IIFS has overlaps, see [11, Theorem 3.1], this process is complicated. Ngai and Tong [13] gave a dimensional formula of J_0 under the so-called weak separation condition, but it is still not easy to check this condition generally. Some techniques of [7] are useful to analyze the Hausdorff dimension of self-similar sets.

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