# Blow-Ups in Generalized Complex Geometry

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# Blow-Ups in Generalized Complex Geometry

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(met een samenvatting in het Nederlands)

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*voor Simone*

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# Introduction

Generalized complex geometry is a branch of differential geometry that was introduced around 2002 by Hitchin [28] and Gualtieri [23, 24]. Its main feature is the study of complex structures not on the ordinary tangent bundle of a manifold but instead on its so-called double tangent bundle, which is the sum of the tangent and cotangent bundle. This bigger bundle creates enough room to incorporate both complex and symplectic structures into one single framework, as well as other new kind of geometries. Intuitively, a generalized complex structure on a manifold can be visualized as a Poisson structure, together with a complex structure on the directions normal to the associated (singular) symplectic foliation. As such, they can be thought of as geometries that interpolate between the two extreme cases where the Poisson structure is zero or invertible, corresponding to complex and symplectic structures respectively. An interesting aspect of generalized geometry is the natural appearance of gauge-transformations, symmetries of the double tangent bundle generated by two-forms. This makes the theory rather appealing to certain branches of physics such as string theory, where these gauge transformations play an important role. Examples of generalized complex manifolds include complex and symplectic manifolds, holomorphic Poisson manifolds, six-dimensional nilmanifolds [13] and the connected sums $m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$ for $m$ odd [15]. The latter are known not to admit any complex or symplectic structures if $m$ is bigger than one, and the generalized complex structures on them are realized with the help of the so-called logarithmic transformation [14, 20]. As this example shows, in order to find new examples it is important to develop surgery tools that change the topology of the manifold, but are compatible with the generalized complex structure. Besides this, another main objective in the field is to investigate which properties or constructions in complex and symplectic geometry are actually features of generalized complex geometry. For instance, the concept of symplectic reduction has an analogue in the setting of generalized complex geometry [9]. Another example is the type-decomposition of forms in complex geometry [12]. This thesis is about such a generalization, namely that of the blow-up.

Blow-ups were invented by algebraic geometers in their study of birational transformations. It is unclear to the author when and by whom precisely the notion of blowing up was invented, but it dates back at least to Zariski [40], who introduced it in a modern language in order to study singularities. His work lead to results by many others, including the famous theorem by Hironaka [27] on the resolution of singularities. The study of singularities is a huge area of mathematical research, and we refer to [26] for a historical overview and further references. On the symplectic side, Gromov [22] pointed out that blow-ups can also be defined for symplectic manifolds. This was then worked out further by McDuff [36], who used it to produce examples of simply-connected non-Kählerian symplectic manifolds.

The first steps to introduce blow-ups in the setting of generalized complex geometry were made by Cavalcanti and Gualtieri [15], who showed that a blow-up exists for a so-called non-degenerate point in the complex locus of a generically symplectic four-manifold. Besides the logarithmic transformation, this formed the main tool in the construction of the generalized complex structures on the manifolds $m\mathbb{CP}^2\#n\overline{\mathbb{CP}^2}$ for $m$ odd. To go beyond this case of a point in four dimensions, we first need to understand which submanifolds are suitable for blowing up. In the complex and symplectic categories these are the complex, respectively, symplectic submanifolds. There are a number of different ways to define a generalized complex submanifold. One such definition was given in [23], which are now usually called branes due to their relation with string theory. In the symplectic category branes are generalizations of Lagrangian submanifolds, so they are not the right ones for blow-ups. Instead, a generalized complex submanifold in this thesis will be a submanifold which inherits a generalized complex structure from the ambient manifold. Even though this definition is the most natural one in the setting of blow-ups, it is a bit too general. For instance, any point forms a generalized complex submanifold. If the generalized complex structure at that point is a non-trivial mixture of complex and symplectic structures, it is unclear how to endow the blow-up with a generalized complex structure, and we will in fact give an example where such a blow-up does not exist (Section 3.3).

For this reason we restrict ourselves to two special subclasses. The first are called *generalized Poisson submanifolds*. They are submanifolds which look complex in transverse directions and they are Poisson submanifolds for the underlying Poisson structure (i.e. they are unions of open subsets of the symplectic leaves). An important ingredient to understand these manifolds is the local theory of generalized complex structures. In complex and symplectic geometry we have the Newlander-Nirenberg and Darboux theorems, telling us that all complex and symplectic structures look the same around a point. Partial results about the local structure of a generalized complex structure were obtained by Gualtieri [24] and Abouzaid and Boyarchenko [1], and the full description was obtained by Bailey [6]. It says that locally, every generalized complex structure is isomorphic to the product of a symplectic manifold and a holomorphic Poisson manifold. Using this result, we prove that generalized Poisson submanifolds come naturally equipped with a special ideal which gives them a holomorphic flavor, and we use that to construct the blow-up as a differentiable manifold. The question of whether this blow-up has a generalized complex structure for which the blow-down map is holomorphic then boils down to the analogous question in the context of holomorphic Poisson geometry. This has been answered by Polishchuk [37] and, building on that, we give necessary and sufficient conditions for blowing up a generalized Poisson submanifold (Theorem 3.1.7).

The second class of submanifolds are called *generalized Poisson transversals*. These look symplectic in transverse directions and form Poisson transversals for the underlying

Poisson structure (i.e. they intersect the symplectic leaves transversally and symplectically). The construction of their blow-ups is based on the symplectic construction of the blow-up, and it involves finding a normal form for the generalized complex structure in a neighborhood of the submanifold. Such a neighborhood theorem was already constructed by Frejlich and Mărcuţ [18] in the context of Poisson geometry, and it has a direct counterpart in our setting. We then use this normal form to blow up the submanifold, provided it is compact (Theorem 3.2.13). This last step uses reduction methods, just as the symplectic blow-up can be performed using symplectic cuts as shown by Lerman [34]. In contrast with the generalized Poisson submanifolds, there is no obstruction to blow-up, but the blow-up is not canonical. It depends on the specific choice of neighborhood, as well as the choice of level set for a specific moment map. The latter is analogous to the symplectic area of the exceptional divisor in symplectic geometry.

The above mentioned results for generalized Poisson submanifolds and generalized Poisson transversals are based on joint work with Bailey and Cavalcanti [4].

The blow-up construction also exists in Kähler geometry, and so it is natural to wonder whether the obtained results have analogues in generalized Kähler geometry. A generalized Kähler structure consists of a pair of generalized complex structures that are compatible with each other. They were introduced in [23, 25] in the language of generalized geometry, but they already existed for a long time in the guise of bi-Hermitian structures, as introduced by Gates, Hull and Roĉek [19]. Such a bi-Hermitian structure consists of a pair of complex structures that are compatible with a given metric and satisfy a certain integrability condition. Besides some results in dimension four (Pontecorvo [38], Apostolov, Gauduchon and Grantcharov [2]), bi-Hermitian structures proved difficult to work with. The language of generalized geometry provided an alternative point of view and allowed for some new developments. For instance, the before mentioned reduction theory now also became available for generalized Kähler structures, with applications to moduli spaces of instantons (Hitchin [29]; Burzstyn, Cavalcanti and Gualtieri [10, 11]). Another development was the deformation theorem of Goto [21], which states that on a compact manifold, a deformation of one of the two structures in a generalized Kähler pair can be coupled to a deformation of the second, provided the second is of a special type ("generalized Calabi-Yau"). This theorem can be applied to compact Kähler manifolds with a holomorphic Poisson structure, giving an important class of examples. These examples generalized an earlier construction by Hitchin on Del Pezzo surfaces [30]. Despite these developments the study of generalized Kähler manifolds remains difficult, and examples where the underlying manifold does not support a Kähler structure are scarce. Noteworthy examples of the latter include even dimensional compact Lie groups (Gualtieri [25]), and some specific solvmanifolds (Fino, Tomassini [17]).

On a generalized Kähler manifold, a generalized Poisson submanifold for one of the two generalized complex structures is automatically a generalized Poisson transversal for the other, and they are the natural candidates for the generalized Kähler blow-up. This question was first addressed in [16], where it was shown that a blow-up exists in the case of a non-degenerate point of complex type in a four-manifold. At first, one may think that this question is easy, since both separate blow-ups have already been developed.

However, the key ingredient of the generalized Kähler blow-up is to keep the generalized complex structures compatible with each other. Consequently, we are forced to abandon the blow-up techniques for generalized Poisson transversals all together. Instead, we blow up the submanifold using the techniques for generalized Poisson submanifolds, and then show that the bi-Hermitian structure that underlies the generalized Kähler structure lifts to a degenerate bi-Hermitian structure on the blow-up. Degeneracy here refers to the metric. Subsequently, we develop a deformation procedure that transforms such degenerate structures into non-degenerate ones. The idea underlying this deformation can be traced back to [30], and was also used in [16]. The deformation itself requires some geometrical input, which in the case of the blow-up boils down to geometric conditions on the submanifold. We obtain two concrete situations where a blow-up exists (Theorem 4.3.2). The first is when the exceptional divisor of the blow-up is again generalized Poisson, while the second is when the submanifold in question is contained in a Poisson divisor for one of the two complex structures in the bi-Hermitian picture. This will be the case for instance when the structure for which $Y$ is a generalized Poisson submanifold is generically symplectic. Finally, we apply this to generalized Kähler structures on even-dimensional compact Lie groups. We show that a maximal torus, which can be taken generalized Poisson for a suitably chosen generalized Kähler structure, can be blown up in a generalized complex way if and only if the Lie group equals $(S^1)^n \times (S^3)^m$, with $n + m$ even (Theorem 4.4.2). The result is then automatically generalized Kähler.

*Organization of the thesis:*
This thesis is organized as follows. In Chapter 1 we give an introduction to generalized complex geometry. Most of the material here is well-known and is presented in a way to make it accessible to the non-expert as well. In particular, we include proofs of most of the results. We discuss the natural pairing and the Courant bracket on the double tangent bundle, together with their symmetries. Then we cover the basic theory of Dirac structures, including a detailed description of the underlying linear algebra, as well as some functoriality properties. This is followed by the description of generalized complex structures, with particular emphasis on the notion of generalized complex submanifolds. We end the chapter with a description of generalized Kähler geometry.

In Chapter 2 we review the blow-up construction. We first discuss this in the general setting of smooth manifolds, where we give a definition of the blow-up by means of a certain universal property. It involves the concept of a so-called holomorphic ideal for a submanifold, and this way of defining the blow-up is basically the same as in algebraic geometry. We then proceed to describe a normal form for holomorphic ideals and use it to describe the manifold structure on the blow-up. This includes a well-known calculation of the fundamental group and the integral cohomology groups. After this we discuss how to perform the blow-up in the complex and symplectic settings. The latter mainly serves as a warm-up for the constructions in the chapters that follow.

Chapter 3 is about blow-ups of generalized complex structures. We investigate two classes of generalized complex submanifolds: generalized Poisson submanifolds and generalized

Poisson transversals. For the first we show that there exists a canonical holomorphic ideal and hence a canonical differentiable blow-up. We then give a necessary and sufficient condition for the blow-up to admit a generalized complex structure for which the blow-down map is generalized holomorphic. The resulting blow-up is, if it exists, always canonical. In contrast, for generalized Poisson transversals we show that a blow-up always exists, at least when the submanifold is compact. This is achieved by constructing a suitable normal form around the submanifold, which allows us to construct the blow-up in the easier setting of the zero section in a vector bundle. The resulting blow-up is however not canonical. Finally, we give an example of a generalized complex submanifold which is not of the above two mentioned types and show that it does not admit a generalized complex blow-up.

Finally, in Chapter 4, we discuss blow-ups of generalized Kähler manifolds. We consider generalized Poisson submanifolds which can be blown-up for one of the two generalized complex structures and show that the underlying bi-Hermitian structure lifts to a so-called degenerate bi-Hermitian structure on the blow-up. We then introduce a certain deformation procedure that transforms degenerate structures into non-degenerate ones, which can then be applied to the blow-up. This deformation however requires some geometric input that is not always available. We identify explicitly two situations where the deformation can be applied, leading to a generalized Kähler blow-up. We end with an investigation of blow-ups on generalized Kähler Lie groups.

# Chapter 1

# Generalized Complex Geometry

This chapter is meant as a short introduction to generalized complex geometry, focussing mainly on the concepts that are relevant for later chapters. Most of the material here is well-known, and the main reference for it is [24]. We include the proofs of most statements in order to make this chapter self contained, and to establish the language and notation. In Section 1.1 we provide the framework of generalized geometry. We introduce the double tangent bundle and the Courant bracket, describe its symmetries and more general kind of morphisms, and discuss spinors and generalized metrics. In section 1.2 we define Dirac structures. We give a detailed description of the linear algebra behind them, in particular about the space of all Dirac structures on a vector space. We then discuss how to pull-back and push-forward Dirac structures along smooth maps. In Section 1.3 we discuss generalized complex structures. We define generalized holomorphic maps and generalized complex submanifolds, with particular emphasis on generalized Poisson submanifolds and generalized Poisson transversals, and discuss the notion of a generalized complex brane. We also give a short description of generalized Hermitian structures. In the final section of this chapter we consider the theory of generalized Kähler geometry. We describe the associated bi-Hermitan picture and discuss in some detail the pair of holomorphic Poisson structures underlying the bi-Hermitian structure.

## 1.1 The double tangent bundle

Let $M$ be a smooth $m$-dimensional manifold equipped with a closed real three-form $H$. The central object of interest in generalized geometry is the *double tangent bundle* $\mathbb{T}M := TM \oplus T^*M$, replacing the role of the ordinary tangent bundle. Elements of $\mathbb{T}M$ are denoted by $X + \xi, Y + \eta, \ldots$, where $X, Y \in TM$ and $\xi, \eta \in T^*M$, or simply by $u, v, \ldots$, if the distinction between vectors and forms is not really necessary.

The bundle $\mathbb{T}M$ is endowed with two structures. The first is a symmetric bilinear form

$$\langle X+\xi, Y+\eta\rangle := \frac{1}{2}\big(\xi(Y)+\eta(X)\big), \tag{1.1}$$

called the *natural pairing*. It is non-degenerate of signature $(m, m)$, and both $TM$ and $T^*M$ are isotropic[1] subspaces.

The second structure on $\mathbb{T}M$ is a bracket on its space of sections called the *Courant bracket*:

$$[\![X+\xi, Y+\eta]\!] := [X, Y] + \mathcal{L}_X\eta - \iota_Y d\xi - \iota_Y\iota_X H. \tag{1.2}$$

Some authors refer to (1.2) as the *Dorfman bracket*, reserving the name Courant for its skew-symmetrization. Often we will write $[\![\cdot,\cdot]\!]_H$ to emphasize which three-form is being used.

In the lemma below we list the main properties of the natural pairing and the Courant bracket on $\mathbb{T}M$. We will denote by $\pi : \mathbb{T}M \to TM$ the projection map, also called the *anchor*.

**Lemma 1.1.1.** For $u, v, w \in \Gamma(\mathbb{T}M)$ and $f \in C^\infty(M)$ we have

i) $[\![u, [\![v, w]\!]]\!] = [\![[\![u, v]\!], w]\!] + [\![v, [\![u, w]\!]]\!]$,

ii) $\pi([\![u, v]\!]) = [\pi(u), \pi(v)]$,

iii) $[\![u, fv]\!] = f[\![u, v]\!] + (\pi(u)\cdot f)v$,

iv) $[\![u, u]\!] = d\langle u, u\rangle$,

v) $\langle [\![u, v]\!], w\rangle + \langle v, [\![u, w]\!]\rangle = \pi(u)\cdot\langle v, w\rangle$.

*Remark* 1.1.2. A vector bundle $E$ over $M$ equipped with a bracket $[\![\cdot,\cdot]\!]$, pairing $\langle\cdot,\cdot\rangle$ and anchor $\pi : E \to TM$ satisfying the above axioms is called a *Courant algebroid*. Property iv) makes sense in this more abstract setting as well, for there is an induced map $\pi^* : T^*M \to E^* \cong E$, the last isomorphism using the pairing on $E$. Axioms i), ii) and iii) are precisely the axioms of a *Lie algebroid* over $M$, were it not for the failure of skew-symmetry of $[\![\cdot,\cdot]\!]$ as described by axiom iv).

### 1.1.1 Symmetries

In order to do geometry on $\mathbb{T}M$ it is important to understand its symmetries. If $\varphi$ is a diffeomorphism of $M$ we will denote by $\varphi_* : TM \to TM$ the corresponding tangent map. There is an induced map on $\mathbb{T}M$, again denoted by $\varphi_*$, given by

$$\varphi_* := \begin{pmatrix} \varphi_* & 0 \\ 0 & (\varphi^{-1})^* \end{pmatrix} : \mathbb{T}M \to \mathbb{T}M. \tag{1.3}$$

[1] A subbundle $V \subset \mathbb{T}M$ is called *isotropic* if $\langle u, v\rangle = 0$ for all $u, v \in V$.

Here the block form of the matrix refers to the splitting $\mathbb{T}M = TM \oplus T^*M$, and $(\varphi^{-1})^*$ denotes the pull-back of forms along the map $\varphi^{-1}$. Throughout we will denote by $\varphi_*$ both the bundle map on $\mathbb{T}M$ as well as the induced map on sections thereof. Note that $\varphi_*$ preserves the pairing but not the bracket:

$$\varphi_*([\![u,v]\!]_H) = [\![\varphi_* u, \varphi_* v]\!]_{\varphi_*(H)}. \tag{1.4}$$

We will denote by $O(\mathbb{T}M)$ the isomorphisms of $(\mathbb{T}M, \langle\cdot,\cdot\rangle)$ that cover the identity on $M$, whose Lie algebra is given by $\mathfrak{so}(\mathbb{T}M)$, the endomorphisms of $\mathbb{T}M$ that are skew-symmetric with respect to $\langle\cdot,\cdot\rangle$. Elements of $\mathfrak{so}(\mathbb{T}M)$ are of the form

$$\begin{pmatrix} A & \beta \\ B & -A^* \end{pmatrix}, \tag{1.5}$$

where $A : TM \to TM$ is an arbitrary endomorphism and $\beta : T^*M \to TM$ and $B : TM \to T^*M$ are both skew-symmetric. We can thus regard $\beta \in \Gamma(\Lambda^2 TM)$ as a bivector and $B \in \Omega^2(M)$ as a two-form. In particular, we obtain special subgroups of $O(\mathbb{T}M)$ generated by the transformations

$$e_*^A(X+\xi) := e^A(X) + e^{-A^*}(\xi), \tag{1.6}$$

$$e_*^B(X+\xi) := X + \xi - \iota_X B, \tag{1.7}$$

$$e_*^\beta(X+\xi) := X + \iota_\xi\beta + \xi. \tag{1.8}$$

These automorphisms all lie in $SO(\mathbb{T}M)$, the connected component of the identity of $O(\mathbb{T}M)$. Symmetries of the form (1.7) play an important role in the theory and are called *B-field transformations*. The reason for introducing the minus sign is conventional and will be motivated later (c.f. the discussion above (1.19)). Just as with diffeomorphisms, $B$-field transformations do not always preserve the bracket:

$$e_*^B([\![u,v]\!]_H) = [\![e_*^B u, e_*^B v]\!]_{H-dB}. \tag{1.9}$$

**Proposition 1.1.3** ([24, Proposition 2.5])**.**

*i) Let $F : TM \to TM$ be an automorphism that satisfies $F[X,Y] = [FX,FY]$ for all $X,Y \in \Gamma(TM)$. Then $F = \varphi_*$ for some diffeomorphism $\varphi$. In particular, Aut$(TM, [\cdot,\cdot]) =$ Diff$(M)$.*

*ii) Let $F : \mathbb{T}M \to \mathbb{T}M$ be an automorphism that satisfies $F[\![u,v]\!]_H = [\![Fu,Fv]\!]_H$ and $\langle Fu, Fv\rangle = \langle u,v\rangle$ for all $u,v \in \Gamma(\mathbb{T}M)$. Then $F = \varphi_* e_*^{-B}$ for some diffeomorphism $\varphi$ and two-form $B$, satisfying $\varphi^* H = H + dB$. In particular, we have an exact sequence*

$$0 \to \Omega^2_{cl}(M) \to \mathit{Aut}(\mathbb{T}M, \langle\cdot,\cdot\rangle, [\![\cdot,\cdot]\!]_H) \to \mathit{Diff}_{[H]}(M) \to 0 \tag{1.10}$$

*where Diff$_{[H]}(M)$ denotes the diffeomorphisms of $M$ that preserve the cohomology class of $H$.*

*Proof.* i): If $F$ covers $\varphi$ on $M$ then $(\varphi_*)^{-1} \circ F$ covers the identity on $M$, so it suffices to show that if $F$ covers the identity then $F = \text{Id}$. Let $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$. We have

$$F[X, fY] = F\big(f[X,Y] + (X \cdot f)Y\big) = f[FX, FY] + (X \cdot f)FY,$$

but also

$$F[X, fY] = [FX, fFY] = f[FX, FY] + (FX \cdot f)FY.$$

Since this holds for all $X, Y$ and $f$, we get $FX = X$, i.e. $F = \text{Id}$.
ii): If $F$ covers $\varphi$ we consider again $F' := (\varphi_*)^{-1} \circ F$, now with $\varphi_*$ given by (1.3). This new automorphism $F'$ covers the identity on $M$, but it will take $[\![\cdot,\cdot]\!]_H$ to $[\![\cdot,\cdot]\!]_{\varphi^* H}$. However, both brackets satisfy Lemma 1.1.1, which is all we need to repeat the trick applied in part i). We obtain $\pi = \pi \circ F'$, where $\pi : \mathbb{T}M \to TM$ is the anchor. It is readily verified that any $F' \in O(\mathbb{T}M)$ that satisfies $\pi = \pi \circ F'$ is of the form $F' = e_*^{-B}$ for some two-form $B$, and so $F = \varphi_* e_*^{-B}$. From (1.4) and (1.9) combined we see that $F$ preserves $[\![\cdot,\cdot]\!]_H$ if and only if $\varphi^* H = H + dB$. □

*Remark* 1.1.4. The above proposition reveals an asymmetry in the roles that $TM$ and $T^*M$ play in the double tangent bundle which is not immediately apparent from the definitions. Indeed, symmetries of $\mathbb{T}M$ include $B$-field transformations, but not their dual versions given by (1.8). Moreover, $B$-field transformations do not respect the splitting $\mathbb{T}M = TM \oplus T^*M$, for they take $TM$ into another isotropic subspace complementary to $T^*M$. As such, the splitting itself is not a natural thing to consider, and the only thing that is invariant under symmetries is the exact sequence

$$0 \to T^*M \xrightarrow{\pi^*} \mathbb{T}M \xrightarrow{\pi} TM \to 0. \tag{1.11}$$

An *isotropic splitting* of (1.11), i.e. a section $s : TM \to \mathbb{T}M$ of $\pi$ with isotropic image, recovers the identification $\mathbb{T}M = TM \oplus T^*M$, with $H$ measuring the lack of involutivity[2] of $s(TM)$:

$$H(X, Y, Z) = -2\langle [\![s(X), s(Y)]\!], s(Z)\rangle.$$

Different splittings give rise to different three-forms, all cohomologous to each other by (1.9). Therefore, it is really the cohomology class of $H$ that is an invariant of $\mathbb{T}M$, and it classifies the isomorphism class of $\mathbb{T}M$ as an exact Courant algebroid[3].

There is an infinitesimal version of (1.10) as well. If $\varphi_{t*} e_*^{-B_t}$ is a family of automorphisms with $\varphi_0 = \text{Id}$ and $B_0 = 0$, we obtain an infinitesimal symmetry of $\mathbb{T}M$ given by

$$Y + \eta \mapsto \left.\frac{d}{dt}\right|_{t=0} \varphi_{t*} e_*^{-B_t}(Y + \eta) = -\mathcal{L}_X(Y + \eta) + \iota_Y b, \tag{1.12}$$

[2]A subbundle $V \subset \mathbb{T}M$ is called *involutive* if $\Gamma(V)$ is closed under the Courant bracket.
[3]A Courant algebroid $E$ (see Remark 1.1.2) is called *exact* if (1.11), with $\mathbb{T}M$ replaced by $E$, is exact.

where $X$ is the vector field on $M$ induced by $\varphi_t$, and $b := \dot{B}_0$. The pair $(X, b)$ satisfies $\mathcal{L}_X H = db$, as a consequence of the identity $\varphi_t^* H = H + dB_t$. Hence, infinitesimal symmetries of $\mathbb{T}M$ give rise to elements of

$$\text{Der}(\mathbb{T}M) := \{(X, b) \in \Gamma(TM) \oplus \Omega^2(M) |\ \mathcal{L}_X H = db\}, \tag{1.13}$$

acting on $\Gamma(\mathbb{T}M)$ via (1.12). There is an adjoint map ad : $\Gamma(\mathbb{T}M) \to \text{Der}(\mathbb{T}M)$ given by $X + \xi \mapsto (X, \iota_X H + d\xi)$, and the induced action of ad$(X + \xi)$ on $\Gamma(\mathbb{T}M)$ is the adjoint action $Y + \eta \mapsto -[\![X + \xi, Y + \eta]\!]_H$. If $(X, b) \in \text{Der}(\mathbb{T}M)$ then $b - \iota_X H$ is closed. If it is exact, say $b - \iota_X H = d\xi$, we have $(X, b) = \text{ad}(X + \xi)$. Consequently, there is an exact sequence

$$0 \to \Omega^1_{\text{cl}}(M) \to \Gamma(\mathbb{T}M) \overset{\text{ad}}{\to} \text{Der}(\mathbb{T}M) \to H^2(M; \mathbb{R}) \to 0.$$

The last map in this sequence is given by $(X, b) \mapsto [b - \iota_X H]$. Given $u = X + \xi \in \Gamma(\mathbb{T}M)$ there is a concrete formula for the family of automorphisms that integrates ad$(u)$. Let $\varphi_t$ be the flow that integrates $X$ and define

$$\psi_t := \varphi_{t*} e_*^{-B_t} : \mathbb{T}M \to \mathbb{T}M, \tag{1.14}$$

where $B_t := \int_0^t \varphi_s^*(d\xi + \iota_X H) ds$. Then we have

$$\frac{d}{dt}\psi_t(v) = -[\![u, \psi_t(v)]\!]_H.$$

We remark here that (1.14) makes sense for all elements $(X, b) \in \text{Der}(\mathbb{T}M)$, which implies that (1.13) describes precisely the set of infinitesimal symmetries of $\mathbb{T}M$. We will however, only need (1.14) for infinitesimal symmetries of the form ad$(X + \xi)$.

Note that $u$ is not always preserved by its own flow:

$$\psi_t(u) = u - d\int_0^t \varphi_{s*}\langle u, u\rangle ds.$$

The following fact will be useful later.

**Lemma 1.1.5.** Let $V \subset \mathbb{T}M$ be a subbundle and $u \in \Gamma(\mathbb{T}M)$ with $\langle u, u\rangle = 0$. If $\psi_t$ denotes the flow of $u$, then $\psi_t(V) = V$ if and only if $[\![u, \Gamma(V)]\!] \subset \Gamma(V)$.

*Proof.* By differentiation with respect to $t$ we immediately see that $\psi_t(V) = V$ implies that $[\![u, \Gamma(V)]\!] \subset \Gamma(V)$, so let us prove the converse. Since $\langle u, u\rangle = 0$ we have $\psi_t(u) = u$. Let $v_i$ be a local frame[4] for $V$, and let $\alpha^j$ be a frame for the annihilator of $V$ in $(\mathbb{T}M)^*$. By assumption we have $[\![u, v_i]\!] = -\sum_j A_{ij} v_j$ for certain functions $A_{ij}$. Then,

$$\begin{aligned}\frac{d}{dt}\alpha^j(\psi_t(v_i)) &= \alpha^j(-[\![u, \psi_t(v_i)]\!]) = \alpha^j(-\psi_t[\![u, v_i]\!]) = \sum_k \alpha^j(\psi_t(A_{ik} v_k)) \\ &= \sum_k (\varphi_{-t}^* A_{ik})\alpha^j(\psi_t(v_k)),\end{aligned}$$

[4]A *local frame* for a vector bundle is a collection of local sections that form a basis in every fiber.

where $\varphi_t$ is the diffeomorphism of $M$ induced by $\psi_t$. Fix an index $j$ and a point $p \in M$ and define $f : \mathbb{R} \to \mathbb{R}^{\mathrm{rank}(V)}$ by $f_i(t) := \alpha^j(\psi_t(v_i))(p)$. If we set $B_{ik}(t) := A_{ik}(\varphi_{-t}(p))$, then the above equation gives $\frac{d}{dt}f(t) = B(t) \cdot f(t)$. Since $f(0) = 0$, we get $f(t) = 0$ for all $t$ and we deduce that $\alpha^j(\psi_t(v_i)) = 0$ for all $i$ and $j$. Consequently, $\psi_t(V) = V$. □

### 1.1.2 Morphisms

We now discuss functoriality in generalized geometry. Given $(M_1, H_1)$ and $(M_2, H_2)$, we need the notion of a map between them, which in addition should induce some kind of "derivative" on the level of double tangent bundles.

**Definition 1.1.6.** A *generalized map* from $(M_1, H_1)$ to $(M_2, H_2)$ is a pair $\Phi := (\varphi, B)$, where $\varphi : M_1 \to M_2$ is a smooth map and $B \in \Omega^2(M_1)$, satisfying $\varphi^* H_2 = H_1 + dB$.

We will often abbreviate $(\varphi, 0)$ by $\varphi$ and drop the prefix "generalized". Composition is given by $(\varphi, B) \circ (\psi, C) = (\varphi \circ \psi, C + \psi^* B)$, and $(\mathrm{Id}, 0)$ acts as the identity. In case $(M_1, H_1) = (M_2, H_2)$ and $\varphi$ is invertible, we recover the same maps of Proposition 1.1.3 ii). Such an invertible generalized map induces on $\mathbb{T}M$ the bundle map $\varphi_* e_*^{-B}$. However, if $\varphi$ is not invertible, it is not possible to push-forward forms, just as we can not pull-back vectors. So in general there is no bundle map associated to a map $\Phi : (M_1, H_1) \to (M_2, H_2)$. Nevertheless, it does give rise to a relation. We say that $X + \xi \in \mathbb{T}M_1$ is $\Phi$-*related* to $Y + \eta \in \mathbb{T}M_2$, and write $X + \xi \sim_{\Phi} Y + \eta$, if

$$\varphi_* X = Y, \;\; \xi = \varphi^* \eta - \iota_X B.$$

If $\varphi$ is invertible this is equivalent to $Y + \eta = \varphi_* e_*^{-B}(X + \xi)$. We denote by $\Gamma_\Phi$ the *graph* of $\Phi$, defined by

$$\Gamma_\Phi := \{(u, v) \in \mathbb{T}M_1 \oplus \varphi^*\mathbb{T}M_2 | u \sim_{\Phi} v\} \subset \mathbb{T}M_1 \oplus \varphi^*\mathbb{T}M_2. \tag{1.15}$$

This is not to be confused with the graph of the underlying map $\varphi$, which would be a subspace of $M_1 \times M_2$. There is a short exact sequence

$$0 \to \varphi^*(T^*M_2) \to \Gamma_\Phi \to TM_1 \to 0,$$

where the first map is given by $\eta \mapsto (\varphi^*\eta, \eta)$ and the second by $(u, v) \mapsto \pi(u)$.

An important property of this relation is that it is compatible with Courant brackets. To explain this properly we first remark that the notion of being $\Phi$-related is a pointwise condition, and really takes place between elements $u \in \mathbb{T}M_1$ and $v \in \varphi^*\mathbb{T}M_2$, as indicated also in the definition of $\Gamma_\Phi$. A section $v \in \Gamma(\mathbb{T}M_2)$ gives rise to a section $\varphi^* v$ of $\varphi^*\mathbb{T}M_2$, and when we say that $u$ and $v$ are $\Phi$-related we really mean that $u$ and $\varphi^* v$ are $\Phi$-related.

**Lemma 1.1.7.** If $u_1, u_2 \in \Gamma(\mathbb{T}M_1)$ and $v_1, v_2 \in \Gamma(\mathbb{T}M_2)$ are $\Phi$-related, i.e. $u_i \sim_{\Phi} v_i$, then also $[\![u_1, u_2]\!]_{H_1} \sim_{\Phi} [\![v_1, v_2]\!]_{H_2}$.

*Proof.* Write $u_i = X_i + \xi_i$ and $v_i = Y_i + \eta_i$. Then, by assumption, $\varphi_* X_i = Y_i$ and $\varphi^* \eta_i = \xi_i + \iota_{X_i} B$. Consequently, $\varphi_*[X_1, X_2] = [Y_1, Y_2]$, and

$$\begin{aligned}\varphi^*(\mathcal{L}_{Y_1}\eta_2 - \iota_{Y_2} d\eta_1 - \iota_{Y_2}\iota_{Y_1} H_2) =& \mathcal{L}_{X_1}\varphi^*\eta_2 - \iota_{X_2} d\varphi^*\eta_1 - \iota_{X_2}\iota_{X_1}\varphi^* H_2 \\ =& \mathcal{L}_{X_1}(\xi_2 + \iota_{X_2} B) - \iota_{X_2} d(\xi_1 + \iota_{X_1} B) \\ & - \iota_{X_2}\iota_{X_1}(H_1 + dB) \\ =& \mathcal{L}_{X_1}\xi_2 - \iota_{X_2} d\xi_1 - \iota_{X_2}\iota_{X_1} H_1 + \iota_{[X_1, X_2]} B,\end{aligned}$$

where we used that $[\mathcal{L}_{X_1}, \iota_{X_2}] = \iota_{[X_1, X_2]}$. So indeed, $[\![u_1, u_2]\!]_{H_1} \sim_\Phi [\![v_1, v_2]\!]_{H_2}$. □

*Remark* 1.1.8. Note that the symbol $\varphi^*$ has different interpretations: it can denote the pull-back of a form, the pull-back of a vector bundle, or the pull-back of a section of a bundle to a section of the pull-back bundle. It should be clear in each context what is meant.

### 1.1.3 Spinors

A pleasant aspect of metrics with split signature is that there is a concrete description of their spinor bundles in terms of exterior algebras. First, recall that if $(E, \langle\cdot,\cdot\rangle)$ is any vector bundle equipped with a non-degenerate metric[5], there is an associated bundle of *Clifford algebras* denoted by $\mathrm{Cl}(E, \langle\cdot,\cdot\rangle)$, or simply $\mathrm{Cl}(E)$ if the metric is clear from the context. It is defined as the free tensor algebra $\oplus_{n\geq 0} E^{\otimes n}$ modulo the ideal generated by elements of the form $u \otimes u - \langle u, u\rangle$. As vector bundles $\mathrm{Cl}(E) \cong \Lambda^\bullet E$, but this does not respect the algebraic structure. A *Clifford bundle* for $\mathrm{Cl}(E)$ is a vector bundle $V$ together with a multiplication $\mathrm{Cl}(E) \otimes V \to V$ that turns $V$ into a $\mathrm{Cl}(E)$-module. Concretely, to give such a bundle we need to specify how elements $u \in E$ act on elements $\rho \in V$, in such a way that $u \cdot (u \cdot \rho) = \langle u, u\rangle \rho$.

We now specify to the case of $(\mathbb{T}M, \langle\cdot,\cdot\rangle)$. Consider the action of $\mathbb{T}M$ on $\Lambda^\bullet T^*M$ given by

$$(X + \xi) \cdot \rho := \iota_X \rho + \xi \wedge \rho. \tag{1.16}$$

It satisfies $u \cdot (u \cdot \rho) = \langle u, u\rangle \rho$, hence induces an action of $\mathrm{Cl}(\mathbb{T}M)$ on $\Lambda^\bullet T^*M$. The special feature of this particular Clifford bundle is the following.

**Lemma 1.1.9.** The induced map $cl : \mathrm{Cl}(\mathbb{T}M) \to \mathrm{End}(\Lambda^\bullet T^*M)$ is an isomorphism.

*Proof.* Both $\mathrm{Cl}(\mathbb{T}M)$ and $\mathrm{End}(\Lambda^\bullet T^*M)$ have rank $2^{2m}$, so it suffices to show that $cl$ is injective. Fix a basis $e_1, \ldots, e_m$ for $TM$ with dual basis $e^1, \ldots, e^m$ for $T^*M$, and denote by $e_I := e_{i_1} \cdot \ldots \cdot e_{i_k}$ for $I = \{1 \leq i_1 < \cdots < i_k \leq m\}$ and similarly for $e^I$. Then the elements $e_I e^J$, where $I, J$ run over all strictly increasing subsets of $\{1, \ldots, m\}$, form a basis for $\mathrm{Cl}(\mathbb{T}M)$. Suppose that $cl(\sum_{I,J} \lambda_{IJ} e_I e^J) = 0$. Fix a $J_0$ and let $J_0^c$ denote its complement in $\{1, \ldots, m\}$. Acting on the form $e^{J_0^c}$, we obtain

[5] The Clifford construction actually makes sense for arbitrary bilinear forms, non-degenerate or not.

$\sum_I \lambda_{IJ_0}\iota_{e_I}(e^{J_0} \wedge e^{J_0^c}) = 0$. Since the terms $\iota_{e_I}(e^{J_0} \wedge e^{J_0^c})$ are all linearly independent, it follows that $\lambda_{IJ_0} = 0$. As $J_0$ was arbitrary, $cl$ is indeed injective. □

We will call $\Lambda^\bullet T^*M$ the *spinor bundle* for $\mathbb{T}M$, and refer to differential forms as *spinors*.

### Derived brackets

Even though the construction of the spinor bundle depends only on the natural pairing, it gives rise to an intriguing relationship between the Courant bracket on $\Gamma(\mathbb{T}M)$ and the operator $d^H$ on $\Gamma(\Lambda^\bullet T^*M)$. The latter is defined by

$$d^H\rho := d\rho + H \wedge \rho. \tag{1.17}$$

This relation follows from the theory of *derived brackets*. For the general set-up we refer to [32], here we will be rather specific. We regard $\Gamma(\mathrm{Cl}(\mathbb{T}M)) \cong \Gamma(\mathrm{End}(\Lambda^\bullet T^*M))$ as a subset of $\mathcal{A} := \mathrm{End}\big(\Gamma(\Lambda^\bullet T^*M)\big)$, the algebra of *all* linear endomorphisms of $\Gamma(\Lambda^\bullet T^*M)$. Inside $\mathcal{A}$, $\Gamma(\mathrm{Cl}(\mathbb{T}M))$ is characterized as those linear maps that are $C^\infty(M)$-linear, i.e. those that commute with the action of multiplication by functions. The algebra $\mathcal{A}$ carries a natural $\mathbb{Z}_2$-grading, defined by declaring an operator $A \in \mathcal{A}$ even if it preserves the parity of a form and odd if it reverses it. So in this sense, elements of $\Gamma(\mathbb{T}M) \subset \Gamma(\mathrm{Cl}(\mathbb{T}M))$ are odd. As for any $\mathbb{Z}_2$-graded algebra, we obtain a graded Lie bracket $\{\cdot,\cdot\}$ on $\mathcal{A}$, given by

$$\{A, B\} := AB - (-1)^{|A||B|}BA$$

on homogeneous[6] elements $A, B$, and extended bilinearly to all of $\mathcal{A}$. Here $|A|$ denotes the degree of $A$. For $u, v \in \Gamma(\mathbb{T}M)$ we have the relation

$$\{u, v\} = 2\langle u, v\rangle,$$

which follows immediately from the Clifford relation (1.16). Now, given an odd element $D \in \mathcal{A}$ satisfying $D^2 = 0$, its *derived bracket* on $\mathcal{A}$ is defined by

$$[A, B]_D := \{\{A, D\}, B\}.$$

By the Jacobi identity for $\{\cdot,\cdot\}$ and the fact that $D^2 = 0$, we obtain a Jacobi identity for $[\cdot,\cdot]_D$:

$$[A, [B, C]_D]_D = [[A, B]_D, C]_D + (-1)^{(|A|+1)(|B|+1)}[B, [A, C]_D]_D.$$

Note the shift of degrees on the right-hand side. In general, $[\cdot,\cdot]_D$ will not be skew-symmetric. For example if $A$ is odd we have

$$[A, A]_D = \{D, \{A, A\}\}.$$

The interesting aspect of this construction is that for some operators $D \in \mathcal{A}$ the derived bracket preserves $\Gamma(\mathrm{Cl}(\mathbb{T}M)) \subset \mathcal{A}$, or $\Gamma(V)$ for some subbundle $V \subset \mathrm{Cl}(\mathbb{T}M)$. Here are the main examples.

[6] An element of $\mathcal{A}$ is *homogeneous* if it is either even or odd.

**Example 1.1.10.** Consider $D = d$, the usual exterior derivative on $\Gamma(\Lambda^\bullet T^*M)$. Viewing $TM \subset \mathrm{Cl}(\mathbb{T}M)$, a vector field $X \in \Gamma(TM)$ is identified with the operator $\iota_X \in \mathcal{A}$. Then, using the Cartan formula, we obtain

$$[\iota_X, \iota_Y]_D = \{\{\iota_X, d\}, \iota_Y\} = \{\mathcal{L}_X, \iota_Y\} = \iota_{[X,Y]}.$$

In other words, $\Gamma(TM)$ is closed under the derived bracket of $d$, and the induced bracket coincides with the ordinary Lie bracket on $TM$.

**Example 1.1.11.** Again take $D = d$, but this time consider $\Lambda^\bullet TM \subset \mathrm{Cl}(\mathbb{T}M)$. Since $TM$ is isotropic, this is in fact an inclusion of algebras. Sections of $\Lambda^\bullet TM$ are called *polyvector fields* and, as in Example 1.1.10, the space of polyvector fields is closed under the bracket derived from $d$. The induced bracket on $\Gamma(\Lambda^\bullet TM)$ is called the *Schouten-Nijenhuis* bracket, and is explicitly given by

$$[X_0 \dots X_p, Y_0 \dots Y_q] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] X_0 \dots \widehat{X_i} \dots X_p Y_0 \dots \widehat{Y_j} \dots Y_q,$$

together with $[X, f] = X \cdot f$ for $f \in C^\infty(M)$. Here a hat means we omit the underlying symbol.

**Example 1.1.12.** Now consider $D = d^H$, defined by (1.17). This time $\Gamma(TM)$ is not closed under the derived bracket, because $H$ transforms two vectors into a one-form. However, $\Gamma(\mathbb{T}M)$ is closed and the induced bracket is precisely the Courant bracket. In other words, we have

$$[\![u, v]\!] \cdot \rho = \{\{u, d^H\}, v\} \cdot \rho. \tag{1.18}$$

This equation gives some insight on the definition of the Courant bracket, and serves as an efficient tool for computations.

**Example 1.1.13.** Let $F$ be a closed $p$-form, where $p \geq 2$, and consider the operator $D = d^F := d + F\wedge$. Then $TM \oplus \Lambda^{p-2}T^*M$ is closed under the derived bracket, which is given by

$$[\![X + \xi, Y + \eta]\!]_F = [X, Y] + \mathcal{L}_X \eta - \iota_Y d\xi + (-1)^p \iota_Y \iota_X F.$$

The bundle $TM \oplus \Lambda^{p-2}T^*M$ shares some properties with $\mathbb{T}M$. For instance, there are gauge-transformations of the form $X + \xi \mapsto X + \xi - \iota_X A$ for $A \in \Omega^{p-1}(M)$, changing $F$ to $F + (-1)^{p+1} dA$.

### Symmetries and the Chevalley pairing

In general, if $(V, \langle \cdot, \cdot \rangle)$ is a vector space endowed with a metric, one can form the so-called *Spin group* $\mathrm{Spin}(V, \langle \cdot, \cdot \rangle)$ or $\mathrm{Spin}(V)$ for short, which is a subgroup of $\mathrm{Cl}^\times(V)$, the group of invertible elements of the Clifford algebra. The group $\mathrm{Spin}(V)$ acts on $\mathrm{Cl}(V)$ by conjugation and this action preserves the subspace $V \subset \mathrm{Cl}(V)$. This induces a double cover

Spin$(V) \to SO(V)$, where the latter equals the group of automorphisms of $(V, \langle\cdot,\cdot\rangle)$ that preserve orientation. In particular, the Lie algebras of both groups are isomorphic. Note that since Spin$(V) \subset$ Cl$(V)$, Spin$(V)$ acts on any module for Cl$(V)$ in a way that is compatible with the action on Cl$(V)$ itself, i.e. $g \cdot (x \cdot \rho) = (g \cdot x) \cdot (g \cdot \rho)$ for every $g \in$ Spin$(V)$, $x \in$ Cl$(V)$ and $\rho$ an element of the module. In particular, the Lie algebra of $SO(V)$ acts naturally on any Clifford module.

Applying this to the Clifford bundle of $(\mathbb{T}M, \langle\cdot,\cdot\rangle)$, we obtain an action of the infinitesimal symmetries $\mathfrak{so}(\mathbb{T}M)$ on the space of spinors, i.e. the space of differential forms. For the specific infinitesimal symmetries given by two-forms and bivectors, the exponentiation of this action is given by

$$e^B\rho := \sum_{n\geq 0} \frac{1}{n!} B^n \wedge \rho, \qquad e^\beta\rho := \sum_{n\geq 0} \frac{1}{n!}\iota_{\beta^n}\rho,$$

where we follow the convention that $\iota_{X_1\wedge\ldots\wedge X_p}(\rho) := \iota_{X_p}\ldots\iota_{X_1}(\rho)$. Note that if we did not insert the minus sign in (1.7), then it would have appeared in the formula for $e^B\rho$. For a proof of these formulas we refer to [24]. The actions on $\mathbb{T}M$ and on $\Lambda^\bullet T^*M$ are compatible in the sense that

$$e^B(u\cdot\rho) = (e^B_* u)\cdot e^B\rho, \qquad e^\beta(u\cdot\rho) = (e^\beta_* u)\cdot e^\beta\rho. \tag{1.19}$$

The spinor bundle is equipped with the so-called *Chevalley pairing*[7]:

$$(\gamma,\rho)_{\text{Ch}} := (\gamma\wedge\rho^T)_{\text{top}}, \qquad \gamma,\rho\in\Gamma(\Lambda^\bullet T^*M). \tag{1.20}$$

Here the superscript $T$ stands for transposition, acting on a degree $l$-form by

$$(\beta_1\ldots\beta_l)^T := \beta_l\ldots\beta_1 = (-1)^{\frac{1}{2}l(l-1)}\beta_1\ldots\beta_l, \tag{1.21}$$

and the subscript top stands for the highest degree component. The Chevalley pairing takes values in the determinant line bundle $\Lambda^m T^*M$. Explicitly, if $\gamma = \sum_i \gamma_i$ denotes the decomposition in terms of degree and similarly for $\rho$, we have

$$(\gamma,\rho)_{\text{Ch}} = \sum_{i=0}^m (-1)^{\frac{1}{2}i(i-1)}\gamma_{m-i}\wedge\rho_i.$$

Furthermore, the pairing is compatible with the Clifford action in the sense that $(u\cdot\gamma,\rho)_{\text{Ch}} = (-1)^{m+1}(\gamma, u\cdot\rho)_{\text{Ch}}$ for all $u\in\mathbb{T}M$. In particular, both operations $e^B$ and $e^\beta$ are orthogonal for it.

### 1.1.4 Generalized metrics

The natural pairing has split signature and as a consequence its automorphism group, $O(m,m)$, is non-compact. On $TM$, we can reduce the non-compact structure group $GL(m)$ to its compact subgroup $O(m)$ by choosing a metric. This has an analogue in generalized geometry.

[7] Often also called the *Mukai pairing*.

**Definition 1.1.14.** A *generalized metric* on $M$ is a map $\mathcal{G} : \mathbb{T}M \to \mathbb{T}M$ which satisfies $\mathcal{G}^2 = 1$ and for which the pairing

$$(u, v) \mapsto \langle \mathcal{G}u, v \rangle =: \mathbb{G}(u, v) \tag{1.22}$$

defines a positive definite metric on $\mathbb{T}M$.

Note that for $\mathbb{G}$ to be symmetric we need $\mathcal{G}^* = \mathcal{G}$. There are a couple of different descriptions of generalized metrics, summarized in the following lemma.

**Lemma 1.1.15.** There is a one-to-one correspondence between

i) Generalized metrics $\mathcal{G}$.

ii) Subbundles $V_+, V_- \subset \mathbb{T}M$ which are orthogonal to each other with respect to the natural pairing, which itself is positive definite on $V_+$ and negative definite on $V_-$.

iii) Isotropic splittings of $\mathbb{T}M$ and metrics $g$ on $TM$.

*Proof.* i) $\Leftrightarrow$ ii): From $\mathcal{G}^2 = 1$ we obtain a decomposition $\mathbb{T}M = V_+ \oplus V_-$, where $V_\pm$ is the $(\pm 1)$-eigenspace of $\mathcal{G}$. On $V_+$ and $V_-$ the natural pairing is then positive and negative definite respectively, and $V_+$ and $V_-$ are orthogonal to each other because $\mathcal{G}^* = \mathcal{G}$. Conversely, any decomposition of $\mathbb{T}M$ into subbundles $V_\pm$ with the last mentioned properties defines a generalized metric, by declaring $\mathcal{G}$ to be $\pm 1$ on $V_\pm$.

i)$\Leftrightarrow$ iii): Given $\mathcal{G}$, the subspace $\mathcal{G}(T^*M) \subset \mathbb{T}M$ is isotropic and has zero intersection with $T^*M$, hence gives an isotropic splitting of $\mathbb{T}M$. In this splitting $\mathcal{G}$ is anti-diagonal, and the fact that $\mathcal{G}^2 = 1$ and that $\mathbb{G}$ is a metric on $\mathbb{T}M$ force $\mathcal{G}$ to be of the form

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{1.23}$$

for some metric $g$ on $M$. Conversely, given a splitting $\mathbb{T}M = TM \oplus T^*M$ and a metric $g$ on $TM$, the above formula gives a generalized metric $\mathcal{G}$. □

*Remark* 1.1.16. a) The description of $\mathcal{G}$ in terms of $V_\pm$ in ii) shows that a generalized metric on $\mathbb{T}M$ is the same as a reduction of the structure group $O(m, m)$ to its maximal compact subgroup $O(m) \times O(m)$. Note that $V_-$ is determined by $V_+$ as its orthogonal complement, so a generalized metric is the same as a maximal, positive definite subbundle of $\mathbb{T}M$.

b) If we are given an arbitrary isotropic splitting of $\mathbb{T}M$ then $\mathcal{G}$ need not be of the form (1.23). However, since any two splittings are related to each other by a $B$-field transformation, there is a two-form $b$ on $M$ such that

$$\mathcal{G} = e_*^{-b} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e_*^{b} \tag{1.24}$$

for some metric $g$. In terms of $g$ and $b$ the bundles $V_\pm$ are given by

$$V_\pm = \{X \pm g(X) + b(X) | X \in TM\} \subset \mathbb{T}M.$$

Given a generalized metric $\mathcal{G}$ we will refer to the unique splitting in which $\mathcal{G}$ is anti-diagonal as the *metric splitting*. We can always pass from a given splitting to the metric splitting, but doing so might require a non-closed two-form $b$ and hence induce a change in the three-form.

## 1.2 Dirac structures

The main reason for studying $\mathbb{T}M$ is that it unifies a class of geometric structures into one single framework. This unifying concept is that of Dirac structures.

**Definition 1.2.1.** A *Dirac structure* on $(M, H)$ is a subbundle $L \subset \mathbb{T}M$ which is Lagrangian, i.e. isotropic and of maximal rank $m = \dim(M)$, and involutive;

$$[\![\Gamma(L), \Gamma(L)]\!]_H \subset \Gamma(L).$$

A Lagrangian subbundle which is not necessarily involutive is called an *almost Dirac structure*. It is customary to call an almost Dirac structure *integrable* if it is Dirac, i.e. if the involutivity condition is satisfied.

**Example 1.2.2.** Let $B$ be a two-form on $M$ and consider $e_*^B(TM) = \{X - \iota_X B | X \in TM\} \subset \mathbb{T}M$. It is isotropic because $B$ is skew and since its rank equals that of $TM$, it is Lagrangian. To see when it is integrable, we use (1.9) to compute

$$[\![e_*^B(X), e_*^B(Y)]\!]_H = e_*^B([\![X, Y]\!]_{H+dB}) = e_*^B([X, Y]) - \iota_Y \iota_X (H + dB).$$

Hence, integrability is equivalent to $H + dB = 0$. Note that any Dirac structure $L$ satisfying $L \cap T^*M = 0$ is of this type.

**Example 1.2.3.** Dual to the previous example, let $\pi \in \Gamma(\Lambda^2 TM)$ be a bivector on $M$, giving rise to the subbundle $e_*^\pi(T^*M) = \{\pi(\xi) + \xi | \xi \in T^*M\}$. As in the previous example, this is Lagrangian because $\pi$ is skew. To understand the integrability conditions in this case we recall that for a function $f$ we can form its *Hamiltonian vector field* $X_f := \pi(df)$, and the *Poisson bracket* induced by $\pi$ is defined by $\{f, g\} = \pi(df, dg) = dg(X_f)$. Then, we have

$$[\![X_f + df, X_g + dg]\!]_H = [X_f, X_g] + d\{f, g\} - \iota_{X_g} \iota_{X_f} H.$$

In particular, by Remark 1.2.5 below, $e_*^\pi(T^*M)$ is integrable if and only if $[X_f, X_g] = X_{\{f,g\}} - \pi(\iota_{X_g} \iota_{X_f} H)$. Since this is an identity of vector fields, we can test its validity by acting on an arbitrary third function $h$. We obtain that $\pi$ is *Poisson*, i.e. $e_*^\pi(T^*M)$ is integrable, if and only if

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = H(X_f, X_g, X_h).$$

In other words, for $\pi$ to be Poisson we need the failure of the Jacobi identity for the Poisson bracket to be governed by $H$. One often calls $\pi$ *twisted-Poisson* in case $H \neq 0$, reserving the name Poisson for the case when $H = 0$. Dual to the previous example, any Dirac structure $L$ with $L \cap TM = 0$ is of this type.

**Example 1.2.4.** Let $\Delta \subset TM$ be a subbundle and consider $L = \Delta \oplus \mathrm{Ann}(\Delta)$, where $\mathrm{Ann}(\Delta) := \{\alpha \in T^*M | \ \alpha|_\Delta = 0\}$. This is clearly a Lagrangian subbundle and we see from (1.2) that $L$ is integrable if and only if $\Delta$ is involutive for the Lie bracket and $H|_\Delta = 0$. By Frobenius' theorem, involutivity of $\Delta$ is equivalent to it defining a *foliation* on $M$; a decomposition of $M$ into immersed connected $p$-dimensional submanifolds called *leaves*, where $p$ is the rank of $\Delta$, and which locally fit together nicely. The latter means that locally $M$ is given by $\mathbb{R}^p \times \mathbb{R}^{m-p}$, and the leaves of the foliation are unions of the plaques $\mathbb{R}^p \times$ constant. The bundle $\Delta$ is then given by the tangent spaces of the leaves. The condition on $H$ is that it vanishes when evaluated on triples of vectors tangent to the foliation.

*Remark* 1.2.5. Integrability of an almost Dirac structure $L$ is encoded in its so-called *Nijenhuis tensor*:

$$N_L(u, v, w) := \langle [\![u, v]\!], w\rangle, \qquad u, v, w \in L.$$

Using Lemma 1.1.1 and the fact that $L$ is isotropic we see that $N_L$ is tensorial (i.e. linear over $C^\infty(M)$) and skew-symmetric. In other words, $N_L \in \Gamma(\Lambda^3 L^*)$, and $L$ is integrable if and only if $N_L$ is zero. The fact that $N_L$ is tensorial means that it suffices to verify integrability of $L$ on an arbitrary collection of sections that span $L$ at all points. This justifies why in example 1.2.3 we investigated integrability only on sections of $e^\pi_*(T^*M)$ of the form $X_f + df$.

There is also a complexified version of Dirac structures, which in fact are the ones we are really interested in. To define these we remark that both the pairing and the bracket on $\mathbb{T}M$ can be transported to $\mathbb{T}M_\mathbb{C} := \mathbb{T}M \otimes \mathbb{C}$ by complex linear extension.

**Definition 1.2.6.** A *complex Dirac structure* on $(M, H)$ is a Lagrangian, involutive complex subbundle $L \subset \mathbb{T}M_\mathbb{C}$.

Note that the three-form $H$ is still taken to be real. The space of real Dirac structures sits inside the space of complex Dirac structures as those $L$ which satisfy $L = \overline{L}$. At the other extreme, we have

**Definition 1.2.7.** A *generalized complex structure* on $(M, H)$ is a complex Dirac structure $L$ which satisfies $L \cap \overline{L} = 0$.

We will study these in more detail later from a slightly different perspective.

### 1.2.1 Linear algebra

In order to understand Dirac structures on a manifold properly we first need to take a step back and study them at the level of linear algebra. Hence, in this section we ignore any integrability issues, and focus only on how Dirac structures look inside $\mathbb{T}_xM$ for $x \in M$. To this end we fix a real vector space $V$ of dimension $m$, playing the role of $T_xM$, and consider its double $\mathbb{V} = V \oplus V^*$. The natural pairing $\langle\cdot,\cdot\rangle$ on $\mathbb{V}$ is defined as before, and the corresponding Clifford algebra acts again on $\Lambda^\bullet V^*$.

In this context a Dirac structure on $V$ is simply a Lagrangian subspace of $(\mathbb{V}, \langle\cdot,\cdot\rangle)$. We will denote the space of all of them by $\mathrm{Dir}(V)$, and similarly $\mathrm{Dir}_{\mathbb{C}}(V)$ will denote the space of all complex Dirac structures on $V$ (i.e. Lagrangian subspaces of $\mathbb{V}_{\mathbb{C}}$ with respect to the complexified natural pairing). Given $L \in \mathrm{Dir}(V)$, we can consider its image $E := \pi(L) \subset V$, where $\pi : \mathbb{V} \to V$ denotes the projection. There is an exact sequence

$$0 \to \mathrm{Ann}(E) \to L \xrightarrow{\pi} E \to 0,$$

where $\mathrm{Ann}(E) = \{\alpha \in V^* |\, \alpha|_E = 0\} = L \cap V^*$. Moreover, given a splitting $s : E \to L$ of this sequence, we obtain a two-form $\varepsilon \in \Lambda^2 E^*$ defined by

$$\varepsilon(X, Y) := -2\langle s(X), Y\rangle.$$

Phrased differently, for $X \in E$ there is a $\xi \in V^*$ with $s(X) = X + \xi \in L$, and $\varepsilon(X, Y) = -\xi(Y)$. Since $L$ is isotropic, $\varepsilon$ is skew, and since the difference of two splittings takes values in $\mathrm{Ann}(E)$, $\varepsilon$ is independent of $s$. Then $L$ is completely determined by the data $(E, \varepsilon)$ because

$$L = L(E, \varepsilon) := \{X + \xi \in E \oplus V^* |\, \xi|_E = -\iota_X \varepsilon\}.$$

Indeed, both $L$ and $L(E, \varepsilon)$ are Lagrangian and $L \subset L(E, \varepsilon)$ by construction, hence they must be equal for dimensional reasons. Consequently, there is a one-to-one correspondence between $\mathrm{Dir}(V)$ and pairs $(E, \varepsilon)$, with $E \subset V$ and $\varepsilon \in \Lambda^2 E^*$. For complex Dirac structures we have the same description, but with $E \subset V_{\mathbb{C}}$.

**Definition 1.2.8.** The *type* of a Dirac structure $L$ is the real codimension of $\pi(L)$ in $V$. Similarly, the type of a complex Dirac structure $L$ is the complex codimension of $\pi(L)$ in $V_{\mathbb{C}}$.

The description of Dirac structures in terms of pairs $(E, \varepsilon)$ does not shed much light on the topology of $\mathrm{Dir}(V)$, mainly because the type is not the same for all Dirac structures. To gain additional insight into $\mathrm{Dir}(V)$ we study it as a homogeneous space for $O(\mathbb{V})$, the group of orthogonal transformations with respect to $\langle\cdot,\cdot\rangle$. First, we claim that $O(\mathbb{V})$ acts transitively on $\mathrm{Dir}(V)$. Indeed, given any $L \in \mathrm{Dir}(V)$, choose a Dirac structure $\widetilde{L}$ which is complementary to $L$, i.e. $L \oplus \widetilde{L} = \mathbb{V}$. We will see later that such an $\widetilde{L}$ always exists. The natural pairing then identifies $\widetilde{L} \cong L^*$, and given any vector space isomorphism $A : V \to L$ the induced map

$$g : \mathbb{V} = V \oplus V^* \xrightarrow{A \oplus (A^{-1})^*} L \oplus L^* \cong L \oplus \widetilde{L} = \mathbb{V}$$

lies in $O(\mathbb{V})$ and satisfies $g(V) = L$. In particular, $\mathrm{Dir}(V) \cong O(\mathbb{V})/\mathrm{Stab}(L)$ for any $L \in \mathrm{Dir}(V)$, where $\mathrm{Stab}(L) = \{g \in O(\mathbb{V})|\, gL = L\}$. This description is still not very insightful, especially since the groups involved are rather large (e.g. they are non-compact). It is possible to give a more economical description using compact groups, but before doing so we first explain how the special subgroups of $SO(\mathbb{V})$ given by (1.6)-(1.8) act on $\mathrm{Dir}(V)$. Here $SO(\mathbb{V})$ is the connected component of the identity in $O(\mathbb{V})$. For $A \in GL_+(V)$, we denote by $\mathbb{A} \in SO(\mathbb{V})$ the map given by

$$\mathbb{A} : V \oplus V^* \xrightarrow{A \oplus (A^{-1})^*} V \oplus V^*.$$

Then, for $L = L(E, \varepsilon)$ we have

$$\mathbb{A}(L(E, \varepsilon)) = L(AE, (A^{-1})^* \varepsilon). \tag{1.25}$$

Next, for $B \in \Lambda^2 V^*$ we have

$$e_*^B L(E, \varepsilon) = L(E, \varepsilon + B|_E). \tag{1.26}$$

Combining (1.25) and (1.26), we deduce that for any $L, L' \in \mathrm{Dir}(V)$ with the same type there is a $g \in SO(\mathbb{V})$ with $gL = L'$. It remains to study the effect of a transformation $e_*^\beta$ for $\beta \in \Lambda^2 V$. For this it is convenient to take a dual point of view. Let $\pi_{V^*} : \mathbb{V} \to V^*$ denote the other projection and consider $F := \pi_{V^*}(L)$. As above, we obtain a two-form $\gamma \in \Lambda^2 F^*$ such that

$$L = L(F, \gamma) := \{X + \xi \in V \oplus F | \; X|_F = \iota_\xi \gamma\}.$$

Then, similar to (1.26), we have $e_*^\beta L = L(F, \gamma + \beta|_F)$. Note that if both $L = L(F, \gamma)$ and $L = L(E, \varepsilon)$, we have $E \supset \mathrm{Ann}(F) = L \cap V$ and

$$E/\mathrm{Ann}(F) \subset V/\mathrm{Ann}(F) \cong F^*$$

coincides with the image of $\gamma : F \to F^*$. In particular, $\dim(E) = \mathrm{codim}(F) + \mathrm{rank}(\gamma)$. Since a transformation $e_*^\beta$ fixes $F$ but changes $\gamma$, we see that the type of $L$ is not fixed by $e_*^\beta$. As skew-symmetric maps have even rank, this change of type will only happen in even amounts. Consequently, for any two $L, L' \in \mathrm{Dir}(V)$ whose types agree modulo 2, there is a $g \in SO(\mathbb{V})$ with $gL = L'$. In particular, they lie in the same connected component of $\mathrm{Dir}(V)$. The same statement holds for complex Lagrangians as well. We will see below that Dirac structures whose types have different parity are in different components of $\mathrm{Dir}(V)$.

### Dirac structures in terms of generalized metrics

To obtain an efficient description of $\mathrm{Dir}(V)$ and $\mathrm{Dir}_{\mathbb{C}}(V)$ we will use a fixed generalized metric on $\mathbb{V}$, i.e. a decomposition $\mathbb{V} = V_+ \oplus V_-$ where $V_+$, $V_-$ are mutually orthogonal subspaces on which $\langle \cdot, \cdot \rangle$ is positive and negative definite respectively. At this point the story is a little different for real and complex Dirac structures and we start with the real case. Since Dirac structures are isotropic they have zero intersection with both $V_+$ and $V_-$, so they can be written uniquely as the graph of a map $a : V_+ \to V_-$. Such a graph is Lagrangian if and only if $a$ is an isometry with respect to the metrics $\pm\langle \cdot, \cdot \rangle$ on $V_\pm$. Hence $\mathrm{Dir}(V) \cong O(m)$, albeit non-canonically. Let us be more specific and choose a generalized metric of the form (1.24), for $g$ a metric on $V$ and with $b = 0$. Then $V_\pm = \mathrm{graph}(\pm g)$, and via the isomorphism $\pi : (V_\pm, \pm\langle\cdot,\cdot\rangle) \to (V, g)$ we obtain a bijection between isometries $a : (V_+, \langle\cdot,\cdot\rangle) \to (V_-, -\langle\cdot,\cdot\rangle)$ and isometries $A : (V, g) \to (V, g)$. The Lagrangian corresponding to $a$ is then, in terms of $A$, given by

$$L(A) := \{X + AX + g(X - AX) |\ X \in V\}.$$

Note that $L(\mathrm{Id}) = V$ and $L(-\mathrm{Id}) = V^*$. Moreover, $L(-A)$ is complementary to $L(A)$, showing that indeed all Dirac structures have a Dirac complement. Since $\pi(L(A)) = \mathrm{Im}(1 + A)$, we obtain

$$\mathrm{type}(L(A)) = \dim(\ker(1 + A)).$$

This is precisely the multiplicity of $-1$ as an eigenvalue for $A$, which modulo 2 is constant on the connected components of $O(m)$. We summarize the above discussion in the following corollary.

**Corollary 1.2.9.** $\mathrm{Dir}(V)$ is diffeomorphic to $O(m)$, where $m = \dim(V)$. It has two connected components, corresponding to Dirac structures of even and odd types.

A similar trick can be done for complex Lagrangians but we have to be a bit careful, for when we complexify the natural pairing we lose its split signature; all non-degenerate bilinear forms over $\mathbb{C}$ are equivalent. Start again with a generalized metric $\mathcal{G}$ induced by a metric $g$ on $V$, so that $\mathbb{V} = V_+ \oplus V_-$. Inside $\mathbb{V}_\mathbb{C}$, the spaces $(V_+)_\mathbb{C}$ and $(V_-)_\mathbb{C}$ contain isotropic subspaces. However, $V_+ \oplus iV_-$ and $iV_+ \oplus V_-$ are positive and negative definite respectively. The same reasoning as before thus applies, and we can write any complex Lagrangian $L$ as the graph of a map from $V_+ \oplus iV_-$ to $iV_+ \oplus V_-$. For convenience we rewrite this into a map from $V_+ \oplus V_-$ to itself, i.e. simply an endomorphism of $\mathbb{V}$. We obtain

$$L = L(\mathcal{J}) := \left\{ \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} (u - i\mathcal{J}u) |\ u \in \mathbb{V} \right\} \tag{1.27}$$

for some endomorphism $\mathcal{J} : \mathbb{V} \to \mathbb{V}$. The matrix in this expression is taken with respect to the decomposition $\mathbb{V}_\mathbb{C} = (V_+)_\mathbb{C} \oplus (V_-)_\mathbb{C}$. For (1.27) to define a complex subspace of $\mathbb{V}_\mathbb{C}$ we need $\mathcal{J}^2 = -1$, i.e. $\mathcal{J}$ has to be a complex structure on $\mathbb{V}$. Then, for $L(\mathcal{J})$ to be Lagrangian we need

$$0 = \langle \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} (u - i\mathcal{J}u), \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} (u - i\mathcal{J}u) \rangle = \langle \mathcal{G}(u - i\mathcal{J}u), u - i\mathcal{J}u \rangle \quad \forall u \in \mathbb{V},$$

which is equivalent to $\mathcal{J}$ being orthogonal with respect to the positive definite metric $\mathbb{G}$ induced by $\mathcal{G}$ on $\mathbb{V}$ (see (1.22)). Hence, $\mathrm{Dir}_\mathbb{C}(V)$ is diffeomorphic to the space of complex structures on $\mathbb{V}$ which are compatible with $\mathbb{G}$. Note that the Dirac structure $V_\mathbb{C}$ corresponds to the complex structure

$$\mathcal{J}_V : (X + gX) \oplus (Y - gY) \mapsto -(Y + gY) \oplus (X - gX), \tag{1.28}$$

in terms of the decomposition $\mathbb{V} = V_+ \oplus V_-$. Dually, $V_{\mathbb{C}}^*$ corresponds to $-\mathcal{J}_V$. Again, for arbitrary $\mathcal{J}$, $L(-\mathcal{J})$ is complementary to $L(\mathcal{J})$. In general, the space of all complex structures on a $2m$-dimensional vector space which are compatible with a given positive definite metric is diffeomorphic to $O(2m)/U(m)$. This has two connected components, distinguished from each other by the induced orientation on the vector space. Specifically, a complex structure $\mathcal{J}$ on $\mathbb{V}$ induces an orientation on $\mathbb{V}$ by declaring a basis of the form $u_1, \mathcal{J}u_1, \ldots, u_m, \mathcal{J}u_m$ to be positive. But $\mathbb{V}$ already comes with a natural orientation, defined by declaring a basis of the form $e_1, \ldots, e_m, e^1, \ldots e^m$ to be positive, where $e_i, e^i$ is a dual basis for $V$ and $V^*$. We say that the orientation induced by $\mathcal{J}$ is $+1$ if it agrees with the canonical orientation and $-1$ otherwise. Accordingly, $\mathrm{Dir}_{\mathbb{C}}(V)$ decomposes into two connected components. Because structures whose types have the same parity are in the same component, there must be a direct relation between the parity of $L(\mathcal{J})$ and the orientation induced by $\mathcal{J}$. To understand this, it suffices to consider one specific Dirac structure, say $V_{\mathbb{C}} \in \mathrm{Dir}_{\mathbb{C}}(V)$. This has type 0, and the associated complex structure is given (1.28). To see what orientation it induces we pick an orthonormal basis $e_1, \ldots, e_m$ on $V$, with dual basis $e^1, \ldots, e^m$. Then, a positive basis with respect to $\mathcal{J}_V$ is given by $e_1 + e^1, e_1 - e^1, \ldots, e_m + e^m, e_m - e^m$. A quick calculation yields that this basis differs from the canonical orientation of $\mathbb{V}$ by the sign $(-1)^{\frac{1}{2}m(m+1)}$.

**Corollary 1.2.10.** Let $\mathcal{G}$ be a generalized metric on $V$. Then, if $m = \dim(V)$,

$$\mathrm{Dir}_{\mathbb{C}}(V) \cong \{\mathcal{J} : \mathbb{V} \to \mathbb{V} |\ \mathcal{J}^2 = -1,\ \mathbb{G}(\mathcal{J}u, \mathcal{J}v) = \mathbb{G}(u, v)\} \cong O(2m)/U(m).$$

Moreover, the type of the Dirac structure $L(\mathcal{J})$ is even if and only if the orientation induced by $\mathcal{J}$ equals $(-1)^{\frac{1}{2}m(m+1)}$.

### Dirac structures in terms of spinors

We give yet another description of $\mathrm{Dir}(V)$ and $\mathrm{Dir}_{\mathbb{C}}(V)$, this time in terms of spinors. Given a nonzero spinor $\rho \in \Lambda^\bullet V^*$ we can consider its annihilator

$$L_\rho := \mathrm{Ann}(\rho) = \{u \in \mathbb{V} |\ u \cdot \rho = 0\}.$$

For $u, v \in L_\rho$ we have $0 = \{u, v\} \cdot \rho = 2\langle u, v\rangle\rho$, and since $\rho \neq 0$ we see that $L_\rho$ is isotropic. If $L_\rho$ is maximal, i.e. $\dim(L_\rho) = m$, we call $\rho$ a *pure spinor*. Note that two spinors that are proportional to each other define the same annihilator.

**Lemma 1.2.11.** For every Dirac structure there is a unique line of pure spinors in $\Lambda^\bullet V^*$ having $L$ as its annihilator. The same is true for $\mathrm{Dir}_{\mathbb{C}}(V)$ and complex lines of pure spinors in $\Lambda^\bullet V_{\mathbb{C}}^*$. If $L = L(E, \varepsilon)$, the pure spinors associated to $L$ are those of the form $\rho = e^B \wedge \Omega$ where $\Omega \in \det(\mathrm{Ann}(E)) := \Lambda^{\mathrm{top}}(\mathrm{Ann}(E))$ is nonzero and $B|_E = \varepsilon$.

*Proof.* Let $L = L(E, \varepsilon)$ be any Dirac structure. Choose $B \in \Lambda^2 V^*$ with $B|_E = \varepsilon$, so that $L = e_*^B L(E, 0)$. Then, by compatibility of the Clifford multiplication with symmetries, i.e. (1.19), it suffices to describe all spinors for $L(E, 0) = E \oplus \mathrm{Ann}(E)$. Let $\rho$ be any spinor and write $\rho = \rho_0 + \ldots + \rho_m$ for its decomposition into degrees. Then $\rho$ is annihilated by $L(E,0)$ if and only if it is annihilated by $E$ and $\mathrm{Ann}(E)$ separately, which amounts to $\iota_X\rho_i = 0 = \xi\wedge\rho_i$ for all $i$ and $X\in E$, $\xi\in\mathrm{Ann}(E)$. These two conditions together enforce $\rho\in\det(\mathrm{Ann}(E))$. In particular, $\det(\mathrm{Ann}(E))$ is the unique line of pure spinors that has $L(E,0)$ as its annihilator, and so $e^B\det(\mathrm{Ann}(E))$ is the unique line of pure spinors for $L$. □

Note that $\rho$ will have mixed degree in general, but it will always be either even or odd. In fact, the type of $L$ is given by the degree of $\Omega$, and so the parity of $L$ agrees with the parity of $\rho$. Observe that when writing $\rho = e^B\wedge\Omega$, the form $\Omega$ is unique up to scalar multiples, but we can add to $B$ any two-form $B'$ whose restriction to $E$ vanishes, i.e. $B'\wedge\Omega = 0$.

*Remark* 1.2.12. Let $L\in\mathrm{Dir}(V)$ with corresponding spinor line $K$, and fix a complementary $\widetilde{L}\in\mathrm{Dir}(V)$. We can act repeatedly with $\widetilde{L}$ on $K$, obtaining the subspaces $\Lambda^k\widetilde{L}\cdot K\subset\Lambda^\bullet V^*$. The natural pairing induces an isomorphism $\widetilde{L}\cong L^*$, and for $u_1,\dots,u_l\in L$, $v_1,\dots,v_k\in\widetilde{L}$ and $\rho\in K\backslash 0$ we have

$$u_l\cdot\ldots\cdot u_1\cdot v_1\cdot\ldots\cdot v_k\cdot\rho = \begin{cases}\det\big((2\langle u_i,v_j\rangle)_{i,j}\big)\rho & \text{if } l=k,\\ 0 & \text{if } l>k.\end{cases}$$

From this it follows that Clifford multiplication $\Lambda^\bullet\widetilde{L}\to\Lambda^\bullet\widetilde{L}\cdot K\subset\Lambda^\bullet V^*$ is injective, hence bijective for dimensional reasons. Consequently, we obtain a decomposition

$$\Lambda^\bullet V^* = \bigoplus_{k=0}^{m}\Lambda^k\widetilde{L}\cdot K. \tag{1.29}$$

Since forms in $K$ might have mixed degree this decomposition is not compatible with the $\mathbb{Z}$-grading of $\Lambda^\bullet V^*$. Nevertheless, there is a $\mathbb{Z}_2$-grading on the right-hand side, because the parity of forms in $K$ is either even or odd and the Clifford multiplication by elements of $\widetilde{L}$ changes parity by one. Although (1.29) depends on the choice of $\widetilde{L}$, the subspaces $\bigoplus_{l\le k}\Lambda^l\widetilde{L}\cdot K\subset\Lambda^\bullet V^*$ depend only on $L$, as can be seen from

$$\bigoplus_{l\le k}\Lambda^l\widetilde{L}\cdot K = \{\psi\in\Lambda^\bullet V^* \,|\, u_1\cdot\ldots\cdot u_{k+1}\cdot\psi = 0\ \forall u_1,\dots,u_{k+1}\in L\}. \tag{1.30}$$

We will use this in the next section to discuss the integrability of Dirac structures.

We now have a couple of different descriptions of Dirac structures and the following lemma describes how to recognize the generalized complex ones in each of these (see Definition 1.2.7).

**Lemma 1.2.13.** Let $L = L(E,\varepsilon) = L(\mathcal{J})$ be a complex Dirac structure with $\mathrm{type}(L) = k$, and let $\rho = e^{B+i\omega}\wedge\Omega$ be a spinor[8] for $L$. Then the following are equivalent.

i) $L$ is generalized complex.

[8]Here $B$ and $\omega$ are real two-forms and $\Omega$ is a complex form of degree $k = \mathrm{type}(L)$. See also Lemma 1.2.11.

ii) $E + \overline{E} = V_{\mathbb{C}}$ and $\mathrm{Im}(\varepsilon)|_{E\cap\overline{E}}$ is non-degenerate.

iii) $(\rho, \overline{\rho})_{\text{Ch}} = (-1)^{\frac{1}{2}k(k-1)}(2i\omega)^{\frac{m}{2}-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$.

iv) $\mathcal{J}\mathcal{G} + \mathcal{G}\mathcal{J}$ is invertible, where $\mathcal{G}$ is the generalized metric that is used to write $L = L(\mathcal{J})$ (see (1.27)).

*Proof.* i)⇔ ii): We have

$$L(E,\varepsilon) \cap L(\overline{E}, \overline{\varepsilon}) = \{X + \xi|\ X \in E \cap \overline{E},\ \xi|_E = -\iota_X\varepsilon,\ \xi|_{\overline{E}} = -\iota_X\overline{\varepsilon}\}.$$

For this to be zero we need the above three conditions on $X$ and $\xi$ to have 0 as their only common solution. Clearly $X = 0$ and $\xi \in \mathrm{Ann}(E + \overline{E})$ is always a solution, so we definitely need $E + \overline{E} = V_{\mathbb{C}}$. Then, on $E \cap \overline{E}$ we have $-\iota_X(\varepsilon - \overline{\varepsilon}) = 0$, and we need this to enforce $X = 0$. This is equivalent to saying that $\mathrm{Im}(\varepsilon)|_{E\cap\overline{E}}$ is non-degenerate.
ii)⇔ iii): $E + \overline{E} = V_{\mathbb{C}}$ if and only if $0 = \mathrm{Ann}(E + \overline{E}) = \mathrm{Ann}(E) \cap \mathrm{Ann}(\overline{E})$, which is equivalent to $\mathrm{Ann}(E)$ and $\mathrm{Ann}(\overline{E})$ being linearly independent, i.e. $\Omega \wedge \overline{\Omega} \neq 0$. In that case we have $E \cap \overline{E} = \ker(\Omega \wedge \overline{\Omega})$, whose complex dimension equals $m - 2k$. We have $(B + i\omega)|_E = \varepsilon$, so $\mathrm{Im}(\varepsilon) = \omega|_E$. Then $\omega|_{E\cap\overline{E}}$ is non-degenerate if and only if its top power $(\omega|_{E\cap\overline{E}})^{\frac{m-2k}{2}}$ is nonzero, which is equivalent to $\omega^{\frac{m}{2}-k} \wedge \Omega \wedge \overline{\Omega} \neq 0$ in $\Lambda^m V^*$.
i)⇔ iv): We have

$$L \cap \overline{L} = \left\{z \in \mathbb{V}_{\mathbb{C}}|\ z = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}(u - i\mathcal{J}u) = \begin{pmatrix}1 & 0\\ 0 & -i\end{pmatrix}(v + i\mathcal{J}v) \text{ for some } u, v \in \mathbb{V}\right\}.$$

The equation relating $u$ and $v$ can be rewritten as $u - i\mathcal{J}u = \mathcal{G}(v + i\mathcal{J}v)$, which implies that $u = \mathcal{G}v$ and so $-\mathcal{J}\mathcal{G}v = \mathcal{G}\mathcal{J}v$. Consequently, $L \cap \overline{L} = 0$ if and only if $\mathcal{J}\mathcal{G} + \mathcal{G}\mathcal{J}$ is invertible. □

*Remark* 1.2.14. Note that the existence of a generalized complex structure on $V$ implies that $m$ is even. The space $E \cap \overline{E}$ is invariant under complex conjugation, so it equals the complexification of a real subspace $(E \cap \overline{E})_{\mathbb{R}} \subset V$. The lemma tells us that this space comes equipped with a symplectic structure. Moreover, the quotient $V/(E\cap\overline{E})_{\mathbb{R}}$ inherits a complex structure from the decomposition $\big(V/(E\cap\overline{E})_{\mathbb{R}}\big)_{\mathbb{C}} = E/(E\cap\overline{E}) \oplus \overline{E}/(E\cap\overline{E})$. Hence, generalized complex structures on $V$ can be thought of as symplectic subspaces of $V$ together with a complex structure on the quotient.

### Functoriality

We end this linear algebra discussion by explaining how to transport Dirac structures from one vector space to another by linear maps. Let $A : V \to W$ be a linear map. It induces maps

$$\mathfrak{F}A : \mathrm{Dir}(V) \to \mathrm{Dir}(W), \qquad \mathfrak{B}A : \mathrm{Dir}(W) \to \mathrm{Dir}(V), \tag{1.31}$$

called the *forward*- and *backward image by* $A$, respectively. They are defined as follows. For $L_V \in \mathrm{Dir}(V)$ and $L_W \in \mathrm{Dir}(W)$ we set

$$\mathfrak{F}A(L_V) := \{AX + \eta|\ X + A^*\eta \in L_V\}, \quad \mathfrak{B}A(L_W) := \{X + A^*\eta|\ AX + \eta \in L_W\}.$$

To see that the outcomes are again Dirac, write $L_V = L(F, \gamma)$ and $L_W = L(E, \varepsilon)$, for $F \subset V^*$, $\gamma \in \Lambda^2 F^*$ and $E \subset W$, $\varepsilon \in \Lambda^2 E^*$. We can regard $A$ as a map $A : A^{-1}E \to E$ and $A^*$ as a map $A^* : (A^*)^{-1}F \to F$, and as such consider the elements $A^*\epsilon \in \Lambda^2(A^{-1}E)^*$ and $A\gamma = (A^*)^*\gamma \in \Lambda^2((A^*)^{-1}F)^*$. Then we have

**Lemma 1.2.15.** $\mathfrak{F}A(L_V) = L((A^*)^{-1}F, A\gamma)$ and $\mathfrak{B}A(L_W) = L(A^{-1}E, A^*\varepsilon)$.

*Proof.* Both statements are dual to each other so it suffices to prove one of them, say the second. We have

$$\mathfrak{B}A(L_W) = \{X + A^*\eta |\ AX \in E,\ \eta|_E = -\iota_{AX}\varepsilon\},$$

and we need to show that this coincides with

$$L(A^{-1}E, A^*\varepsilon) = \{X + \xi |\ X \in A^{-1}E,\ \xi|_{A^{-1}E} = -\iota_X(A^*\varepsilon)\}.$$

From these two expressions it is clear that $\mathfrak{B}A(L_W) \subset L(A^{-1}E, A^*\varepsilon)$. Conversely, for $X + \xi \in L(A^{-1}E, A^*\varepsilon)$ we first observe that $\xi \in \mathrm{Ann}(\mathrm{Ker}(A))$, hence $\xi = A^*\tilde{\eta}$ for some $\tilde{\eta} \in W^*$. This $\tilde{\eta}$ need not satisfy $\tilde{\eta}|_E = -\iota_{AX}\varepsilon$, but it will do so on the smaller subspace $E \cap \mathrm{Im}(A)$. In particular, $\tilde{\eta}|_E + \iota_{AX}\varepsilon \in \mathrm{Ann}(E \cap \mathrm{Im}(A)) \subset E^*$, and we can choose an extension $\alpha \in \mathrm{Ann}(\mathrm{Im}(A)) = \mathrm{Ker}(A^*)$. Then $\xi = A^*\eta$, where $\eta := \tilde{\eta} - \alpha$ does satisfy $\eta|_E = -\iota_{AX}\varepsilon$, hence $X + \xi \in \mathfrak{B}A(L_W)$. □

**Example 1.2.16.** If $L_W = e_*^{\omega}(W)$ is the graph of a two-form $\omega \in \Lambda^2 W^*$, it follows from the lemma that $\mathfrak{B}A(L_W) = e_*^{A^*\omega}(V)$, the graph of the pull-back of $\omega$. Dually, if $L_V = e_*^{\pi}(V^*)$ is the graph of a bivector $\pi \in \Lambda^2 V$, we have $\mathfrak{F}A(e_*^{\pi}(V^*)) = e_*^{A\pi}(W^*)$, the graph of the push-forward $A\pi \in \Lambda^2 W$ of $\pi$.

*Remark* 1.2.17. If $A : V \to W$ and $B : W \to Z$ are linear maps, we have

$$\mathfrak{B}A(\mathfrak{B}B(L_Z)) = \mathfrak{B}(BA)(L_Z) \quad \text{and} \quad \mathfrak{F}B(\mathfrak{F}A(L_V)) = \mathfrak{F}(BA)(L_V).$$

This follows immediately from the definition, but also from the previous lemma.

If $L_1$ and $L_2$ are Dirac structures on $V$, their product $L_1 \times L_2 \subset \mathbb{V} \times \mathbb{V}$ is a Dirac structure on $V \times V$. We can take the backward image along the diagonal map $\Delta : V \to V \times V$ to obtain a new Dirac structure called the *Baer sum*;

$$L_1 \boxtimes L_2 := \mathfrak{B}\Delta(L_1 \times L_2) = \{X + \xi + \eta |\ X + \xi \in L_1,\ X + \eta \in L_2\}.$$

The Baer sum is associative, and we have $V \boxtimes L = L$ and $V^* \boxtimes L = V^*$. So, $(\mathrm{Dir}(V), \boxtimes)$ is a semi-group with identity given by $V$, and an "infinity" given by $V^*$. Moreover, $L(E, \varepsilon) \boxtimes L(E, \varepsilon') = L(E, \varepsilon + \varepsilon')$, hence for these Dirac structures $\boxtimes$ acts as some kind of addition. It is also possible to multiply Dirac structures by nonzero numbers. Given $t \neq 0$ we can form $t \cdot L := \{X + t\xi |\ X + \xi \in L\}$, and clearly $t \cdot L(E, \varepsilon) = L(E, t\varepsilon)$. Define the *transpose* of $L$ by

$$L^T := (-1) \cdot L = \{X - \xi |\ X + \xi \in L\}. \tag{1.32}$$

From $L(E, \varepsilon)^T = L(E, -\varepsilon)$ we see that if $\rho$ is a spinor for $L$, then $\rho^T$ is a spinor for $L^T$. Here $\rho^T$ is the transpose of $\rho$, defined in (1.21).

### 1.2.2 Integrability

We return to the context of Dirac structures on smooth manifolds. Let $L$ be a real or complex almost Dirac structure on $M$. We can regard $L$ as a family of Dirac structures in $\mathbb{T}_x M$, varying smoothly with $x$. Associated to $L$ is its type, which now is a function on $M$. From Corollaries 1.2.9 and 1.2.10 we deduce

**Corollary 1.2.18.** The type of $L$ is, modulo 2, constant on connected components of $M$.

If the type of $L$ is constant, $E := \pi(L) \subset TM$ is of constant rank and hence a smooth subbundle. Choosing a smooth section $s : E \to L$ we obtain a two-form $\varepsilon \in \Gamma(\Lambda^2 E^*)$ as before, and again $L = L(E, \varepsilon)$. Then $L(E, \varepsilon)$ is integrable if and only if $E \subset TM$ is involutive and $d\varepsilon + H|_E = 0$. Here $d\varepsilon \in \Gamma(\Lambda^3 E^*)$ denotes the *leafwise derivative* of $\varepsilon$, defined by

$$d\varepsilon(X, Y, Z) := X(\varepsilon(Y, Z)) - \varepsilon([X, Y], Z) + \text{cyclic permutations}, \qquad X, Y, Z \in \Gamma(E).$$

The proof of this integrability criterion is straightforward but it is not very useful in general, as a lot of interesting Dirac structures exhibit type change.

A more convenient description can be given in terms of spinors. We know that at each point $x \in M$ there is a unique spinor line in $\Lambda^\bullet T_x^* M$ corresponding to $L_x$. These spinor lines fit together into a smooth line subbundle of $\Lambda^\bullet T^* M$. One way to see this is as follows. The connected component of $\mathrm{Dir}(T_x M)$ containing $L_x$, for $x \in M$ fixed, has the form $SO(\mathbb{T}_x M)/\mathrm{Stab}(L_x)$. We can choose a local section $s$ of the submersion $\mathrm{Spin}(\mathbb{T}_x M) \to SO(\mathbb{T}_x M)/\mathrm{Stab}(L_x)$ that maps $L_x$ to Id. Here the group $\mathrm{Spin}(\mathbb{T}_x M)$, together with its double cover to $SO(\mathbb{T}_x M)$ and its action on spinors was discussed in Section 1.1.3. Then, regarding $L$ on a neighborhood $U$ of $x$ as a smooth map $L : U \to \mathrm{Dir}(T_x M)$, we have $L_y = s(L(y)) \cdot L_x$ for $y \in U$. In particular, we can define $\rho_y := s(L(y)) \cdot \rho_x$, for $\rho_x$ a nonzero spinor for $L_x$, and this defines a smooth spinor on $U$ for $L$.

**Lemma 1.2.19.** Let $L$ be a (complex) almost Dirac structure. Then $L$ is integrable if and only if for every local nonzero spinor $\rho$ we have $d^H \rho = u \cdot \rho$ for some $u \in \Gamma(\mathbb{T}M)$ (respectively $u \in \Gamma(\mathbb{T}M_{\mathbb{C}})$).

*Proof.* The almost Dirac structure $L$ is integrable if and only if for all $u, v \in \Gamma(L)$ we have $[\![u, v]\!] \in \Gamma(L)$, i.e. $[\![u, v]\!] \cdot \rho = 0$. Using (1.18) and the fact that $u \cdot \rho = v \cdot \rho = 0$, we get

$$[\![u, v]\!] \cdot \rho = \{\{u, d^H\}, v\} \cdot \rho = -v \cdot u \cdot d^H \rho.$$

We see that $L$ is integrable precisely when $v \cdot u \cdot d^H \rho = 0$ for all $u, v \in L$. If $\widetilde{L}$ is a complementary almost Dirac structure, not necessarily integrable, it induces the splitting (1.29). From (1.30) we deduce that $L$ is integrable if and only if $d^H \rho \in \mathbb{R}\rho \oplus \widetilde{L} \cdot \rho$. Since $d^H \rho$ and $\rho$ have different parity, this is equivalent to $d^H \rho \in \widetilde{L} \cdot \rho$. Since $\mathbb{T}M = L \oplus \widetilde{L}$ and $L$ annihilates $\rho$ we have $\widetilde{L} \cdot \rho = \mathbb{T}M \cdot \rho$, which proves the lemma. $\square$

### 1.2.3 Functoriality

Given a map $\Phi = (\varphi, B) : (M_1, H_1) \to (M_2, H_2)$, we would like to know to what extent Dirac structures can be pulled-back or pushed-forward under $\Phi$. We already know how to do that on the level of linear algebra, and now we study this on the level of smooth manifolds. Throughout we will be talking about real Dirac structures, but all statements hold equally well in the complex case.

In order to avoid potential confusion between the pull-back as an ordinary vector bundle we will use the terminology from [8], which is also the reference for more information about this topic. Let $L_2$ be a Dirac structure on $(M_2, H_2)$. We define the *backward image* of $L_2$ along $\Phi$ by

$$\begin{aligned} \mathfrak{B}\Phi(L_2) :&= \{u \in \mathbb{T}M_1 | u \sim_{\Phi} v \text{ for some } v \in L_2\} \\ &= \{X + \varphi^*\eta - \iota_X B | \ \varphi_* X + \eta \in L_2\}. \end{aligned} \tag{1.33}$$

Note that $\mathfrak{B}\Phi(L_2) = e_*^B\big(\mathfrak{B}\varphi(L_2)\big)$, where $\varphi = (\varphi, 0) : (M_1, \varphi^* H_2) \to (M_2, H_2)$, and for each $x \in M_1$, $\mathfrak{B}\varphi(L_2)_x$ is nothing but the backward image of $(L_2)_{\varphi(x)}$ along the map $\varphi_* : T_x M_1 \to T_{\varphi(x)} M_2$ in the sense of (1.31). It thus follows from Lemma 1.2.15 that $\mathfrak{B}\Phi(L_2)$ defines a Lagrangian subspace of $\mathbb{T}_x M_1$ over each point $x \in M_1$. Unfortunately though, it is not always smooth.

**Example 1.2.20.** Let $L_2 = \Delta \oplus \mathrm{Ann}(\Delta)$ be the Dirac structure associated to a foliation $\Delta$ on $M_2$. Then by Lemma 1.2.15, $\mathfrak{B}\varphi(L_2) = \varphi_*^{-1}\Delta \oplus \mathrm{Ann}(\varphi_*^{-1}\Delta)$. This is smooth if and only if $\varphi_*^{-1}\Delta$ has constant rank on $M_1$, which is not always the case.

We will now describe a sufficient condition that guarantees the smoothness of $\mathfrak{B}\Phi(L_2)$. From the definition of $\mathfrak{B}\Phi(L_2)$ it follows that there is a short exact sequence

$$0 \to \ker(\varphi^*) \cap \varphi^* L_2 \to \Gamma_\Phi \cap \big(\mathbb{T}M_1 \oplus \varphi^* L_2\big) \xrightarrow{p_1} \mathfrak{B}\Phi(L_2) \to 0, \tag{1.34}$$

where the graph $\Gamma_\Phi$ was defined in (1.15), and $p_1 : \mathbb{T}M_1 \oplus \varphi^*\mathbb{T}M_2 \to \mathbb{T}M_1$ denotes the projection. Note again the double usage of the symbol $\varphi^*$; it denotes both the pull-back of forms as well as the pull-back of vector bundles. The first two objects in this sequence are intersections of smooth vector bundles, hence they are smooth precisely when their ranks are constant. In that case we are guaranteed that $\mathfrak{B}\Phi(L_2)$ is smooth, being the quotient of a bundle map of constant rank. Since the sequence is exact, the alternating sum of the ranks is zero. Moreover, $\mathfrak{B}\Phi(L_2)$ has constant rank because it is Lagrangian, even when it is singular. Hence, we obtain the following sufficient criterion for smoothness:

$$\text{If } \ker(\varphi^*) \cap \varphi^* L_2 \text{ has constant rank then } \mathfrak{B}\Phi(L_2) \text{ is smooth.} \tag{1.35}$$

**Example 1.2.21.** Consider again the case of a foliation $L_2 = \Delta \oplus \mathrm{Ann}(\Delta)$. In that case

$$\ker(\varphi^*) \cap \varphi^* L_2 = \ker(\varphi^*) \cap \varphi^* \mathrm{Ann}(\Delta) = \ker(\varphi^*|_{\mathrm{Ann}(\Delta)}) = \mathrm{Ann}\big(\Delta + \mathrm{Im}(\varphi_*)\big).$$

Since $\mathrm{rank}\big(\Delta + \mathrm{Im}(\varphi_*)\big) = \dim(M_1) + \mathrm{rank}(\Delta) - \mathrm{rank}(\varphi_*^{-1}\Delta)$, $\ker(\varphi^*) \cap \varphi^* L_2$ is of constant rank if and only if $\varphi_*^{-1}\Delta$ is of constant rank on $M_1$.

Contrarily to what this example seems to suggest, (1.35) is not a necessary condition for smoothness in general. Example 1.3.24 below will demonstrate this.

**Lemma 1.2.22** ([8, Proposition 5.6])**.** If $\mathfrak{B}\Phi(L_2)$ is smooth then it is automatically Dirac.

*Proof.* Since $\mathfrak{B}\Phi(L_2) = e_*^B\big(\mathfrak{B}\varphi(L_2)\big)$ we may assume that $B = 0$. By Remark 1.2.5 we know that integrability is equivalent to the vanishing of the Nijenhuis tensor, and it suffices to check this on some open dense set $U \subset M_1$. By Lemma 1.1.7 it suffices to show that for each $x_0 \in U$ and each element $u_{x_0} \in \mathfrak{B}\varphi(L_2)_{x_0}$, there is a local section $u$ of $\mathfrak{B}\varphi(L_2)$ around $x_0$ which is $\varphi$-related to a local section $v$ of $L_2$ around $\varphi(x_0)$. Let

$$U := \{x \in M_1 |\ \varphi_* \text{ and } \ker(\varphi^*) \cap \varphi^* L_2 \text{ have constant rank around } x\}.$$

Then $U$ is open and dense and for $x_0 \in U$ the constant rank theorem for smooth maps gives us a neighborhood of $x_0$ of the form $\mathbb{R}^a \times Y$, together with a factorization

$$\mathbb{R}^a \times Y \to Y \hookrightarrow Y \times \mathbb{R}^b \subset M_2$$

of $\varphi|_{\mathbb{R}^a \times Y}$ into a submersion followed by an immersion. For $u_{x_0} \in \mathfrak{B}\varphi(L_2)_{x_0}$ we first pick an extension $u \in \Gamma(\mathfrak{B}\varphi(L_2)|_Y)$. Since $\mathfrak{B}\varphi(L_2)$ is smooth and $\ker(\varphi^*) \cap \varphi^* L_2$ has constant rank, (1.34) is a sequence of vector bundles and we can pick a local smooth pre-image for $u$ under $p_1$. In particular, this gives a $v \in \Gamma(L_2|_Y)$ which is $\varphi$-related to $u$. We extend $u$ in a constant way to a section defined over $\mathbb{R}^a \times Y$, which is then still $\varphi$-related to $v$. Extending $v$ in an arbitrary way to a section of $L_2$ over $Y \times \mathbb{R}^b$, we obtain the desired $\varphi$-related sections. □

If $\rho_2$ is a local spinor for $L_2$ then $e^B(\varphi^*\rho_2)$ is annihilated by $\mathfrak{B}\Phi(L_2)$ and so forms a local spinor for the latter, provided it is nonzero. The latter can be verified as follows. If we write $\rho_2 = e^{B+i\omega} \wedge \Omega$ at a particular point, then $\varphi^*\rho_2 \neq 0$ if and only if $\varphi^*\Omega \neq 0$. Since $\Omega \in \Lambda^{\text{top}}(T^*M_2 \cap L_2)$, this amounts to the restriction of $\varphi^*$ to $T^*M_2 \cap L_2$ being injective, which is equivalent to

$$\ker(\varphi^*) \cap \varphi^* L_2 = 0. \tag{1.36}$$

Hence, (1.36) not only guarantees smoothness of $\mathfrak{B}\Phi(L_2)$, but also that pull-backs of spinors for $L_2$ are spinors for the backward image. The converse is true as well; if the pull-back of a spinor for $L_2$ is nonzero, then the backward image is smooth and defined by that spinor.

In a similar fashion there is the *forward image* of a Dirac structure $L_1$ on $(M_1, H_1)$, defined by

$$\begin{aligned} \mathfrak{F}\Phi(L_1) :&= \{v \in \varphi^*\mathbb{T}M_2 | u \sim_\Phi v \text{ for some } u \in L_1\} \\ &= \{\varphi_* X + \xi |\ X + \varphi^*\xi - \iota_X B \in L_1\} \subset \varphi^*\mathbb{T}M_2. \end{aligned} \tag{1.37}$$

Again we have $\mathfrak{F}\Phi(L_1) = \mathfrak{F}\varphi(e_*^{-B} L_1)$, and there is a short exact sequence

$$0 \to L_1 \cap e_*^B\big(\ker(\varphi_*)\big) \to \Gamma_\Phi \cap \big(L_1 \oplus \varphi^*\mathbb{T}M_2\big) \overset{p_2}{\to} \mathfrak{F}\Phi(L_1) \to 0,$$

where now $p_2$ denotes the projection to $\varphi^*\mathbb{T}M_2$. By the same argument as for backward images, we obtain the following sufficient criterion for smoothness:

$$\text{If } \ker(\varphi_*) \cap e_*^{-B} L_1 \text{ has constant rank then } \mathfrak{F}\Phi(L_1) \text{ is smooth.} \tag{1.38}$$

Still, even if $\mathfrak{F}\Phi(L_1) \subset \varphi^*\mathbb{T}M_2$ is smooth, it is not guaranteed that it defines a subbundle of $\mathbb{T}M_2$. Let us address this issue only in the case where $\varphi$ is a surjective submersion, as that is the only case in which we shall need the forward image. Then, we need $\mathfrak{F}\Phi(L_1)$ to be invariant on the fibers of $\varphi$. More precisely, for fixed $y \in M_2$, we need $\mathfrak{F}\Phi(L_1)_x = \mathfrak{F}\Phi(L_1)_{x'}$ for all $x, x' \in \varphi^{-1}(y)$. Note that this is an equality between Lagrangian subspaces of $\mathbb{T}_y M_2$. If this is the case we can define, unambiguously, the forward image on $M_2$ by $\mathfrak{F}\Phi(L_1)_y := \mathfrak{F}\Phi(L_1)_x$ for any $x \in \varphi^{-1}(y)$. To see that this is smooth on $M_2$, we pick a local section $s$ of $\varphi$ and observe that $\mathfrak{F}\Phi(L_1) = s^*\mathfrak{F}\Phi(L_1)$ inside[9] $s^*\varphi^*\mathbb{T}M_2 = \mathbb{T}M_2$. Similar to Lemma 1.2.22, the result is then automatically integrable on $M_2$.

**Lemma 1.2.23.** Let $(\varphi, B) : (M_1, H_1) \to (M_2, H_2)$ be a map such that $\varphi : M_1 \to M_2$ is a surjective submersion with connected fibers. Suppose that $\ker(\varphi_*) \subset e_*^{-B} L_1$ and that $\iota_X(H + dB) = 0$ for all $X \in \ker(\varphi_*)$. Then $\mathfrak{F}\Phi(L_1) \subset \varphi^*\mathbb{T}M_2$ is smooth and constant on the fibers of $\varphi$. If $\rho_2$ is a form on $M_2$ such that $e^B\varphi^*\rho_2$ is a spinor for $L_1$, then $\rho_2$ is a spinor for $\mathfrak{F}\Phi(L_1)$.

*Proof.* Since $\ker(\varphi_*) \cap e_*^{-B} L_1 = \ker(\varphi_*)$ is of constant rank, $\mathfrak{F}\Phi(L_1)$ is smooth by (1.38). To show that it is constant on fibers it suffices, by connectivity of the fibers, to check this locally. Let $X$ be a local vector field on $M_1$ which is tangent to the fibers, i.e. $\varphi_* X = 0$. By assumption, $X \in \Gamma(e_*^{-B} L_1)$, so by integrability of the latter (with respect to $H + dB$), we deduce from Lemma 1.1.5 that $e_*^{-B} L_1$ is invariant under the flow $\psi_t$ of $X$. Since by assumption $\iota_X(H + dB) = 0$, we have $\psi_t = \varphi_{t*}$ (see (1.14)) where $\varphi_t$ is the flow of $X$. In particular,

$$\mathfrak{F}\varphi((e_*^{-B} L_1)_{\varphi_t(x)}) = \mathfrak{F}\varphi\mathfrak{F}\varphi_t((e_*^{-B} L_1)_x) = \mathfrak{F}(\varphi \circ \varphi_t)((e_*^{-B} L_1)_x) = \mathfrak{F}\varphi((e_*^{-B} L_1)_x),$$

which shows that the push-forward is indeed constant along the fibers of $\varphi$. The last statement follows from immediate computations, using that $\varphi^*$ is injective on forms. □

*Remark* 1.2.24. For maps $\Phi_1$ and $\Phi_2$ we have $\mathfrak{B}\Phi_2 \circ \mathfrak{B}\Phi_1 = \mathfrak{B}(\Phi_1 \circ \Phi_2)$ and $\mathfrak{F}\Phi_1 \circ \mathfrak{F}\Phi_2 = \mathfrak{F}(\Phi_1 \circ \Phi_2)$. Moreover, if $\varphi$ is invertible, then $\mathfrak{B}\Phi = \mathfrak{F}(\Phi^{-1})$ and $\mathfrak{F}\Phi(L_1) = \varphi_*(e_*^{-B}(L_1))$ if $\Phi = (\varphi, B)$.

As in Section 1.2.1 we can use the diagonal map $\Delta : (M, H_1 + H_2) \to (M, H_1) \times (M, H_2)$ to define the Baer sum of Dirac structures

$$L_1 \boxtimes L_2 := \mathfrak{B}\Delta(L_1 \times L_2) = \{X + \xi + \eta |\ X + \xi \in L_1,\ X + \eta \in L_2\}.$$

From (1.35) we see that it is smooth whenever $T^*M \cap L_1 \cap L_2$ has constant rank. Note that there is a natural map $L_1 \times_{TM} L_2 \to L_1 \boxtimes L_2$ given by $(X + \xi, X + \eta) \mapsto X + \xi + \eta$,

[9] To be precise, the left-hand $\mathfrak{F}\Phi(L_1)$ refers to the one we just defined on $M_2$, while the right-hand $\mathfrak{F}\Phi(L_1)$ refers to the subbundle of $\varphi^*\mathbb{T}M_2$.

which is an isomorphism if and only if $L_1 \cap L_2 \cap T^*M = 0$. The latter condition also ensures that for (local) spinors $\rho_1$, $\rho_2$ for $L_1$, $L_2$, the product $\rho_1 \wedge \rho_2$ does not vanish and forms a spinor for $L_1 \boxtimes L_2$ (see the discussion above (1.36)).

## 1.3 Generalized complex structures

Recall that a complex structure on a manifold $M$ can be defined as a complex structure on $TM$, i.e. an endomorphism $I : TM \to TM$ with $I^2 = -1$, for which the $(+i)$-eigenbundle $T^{1,0}M$ is involutive for the Lie bracket of vector fields, i.e.

$$[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subset \Gamma(T^{1,0}M).$$

Replacing $TM$ by $\mathbb{T}M$ and the Lie bracket by the Courant bracket, we obtain

**Definition 1.3.1.** A *generalized complex structure* on $(M, H)$ is a complex structure $\mathcal{J}$ on $\mathbb{T}M$ such that its $(+i)$-eigenbundle $L \subset \mathbb{T}M_{\mathbb{C}}$ is involutive, i.e. $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$.

The bundle $L$ is involutive and satisfies $L \cap \overline{L} = 0$, but for $L$ to be Dirac we need it to be isotropic. This is equivalent to $\mathcal{J}$ being orthogonal with respect to the natural pairing. Somewhat surprisingly, this is already implied by the integrability condition.

**Lemma 1.3.2** ([24, Proposition 2.8])**.** Let $L \subset \mathbb{T}M_{\mathbb{C}}$ be an involutive subbundle. Then $L$ is either isotropic or of the form $\pi^{-1}(\Delta)$ for some integrable distribution $\Delta \subset TM_{\mathbb{C}}$.

*Proof.* Suppose that $L$ is not isotropic. Then there is a point $x \in M$ and a $u \in L_x$ with $\langle u, u \rangle \neq 0$. Extend $u$ to a local section of $L$ and let $f$ be any function on $M$. Then, by Lemma 1.1.1,

$$[\![fu, u]\!] = f[\![u, u]\!] - (\pi(u) \cdot f)u + 2\langle u, u \rangle df.$$

Since $L$ is involutive and $\langle u, u \rangle$ is nonzero at $x$ we obtain $d_x f \in L_x$. Hence, $T_x^*M \subset L_x$ and so $L_x = \pi^{-1}(\Delta_x)$ where $\Delta_x := \pi(L_x) \neq 0$. But then the rank of $L$ is bigger than $m = \dim(M)$, implying that it can not be isotropic at any point of $M$. Consequently, the above trick applies at every point and we obtain $T^*M \subset L$ and so $L = \pi^{-1}(\Delta)$ where $\Delta = \pi(L)$. □

If $\mathcal{J}$ is generalized complex and $L$ the associated $(+i)$-eigenbundle, the rank of $L$ is $m$ as a consequence of the equation $L \oplus \overline{L} = \mathbb{T}M_{\mathbb{C}}$. Therefore, by the lemma, $L$ is isotropic and hence Dirac. In particular, Definitions 1.3.1 and 1.2.7 are equivalent, i.e. there is a one-to-one correspondence between generalized complex structures $\mathcal{J}$ and complex Dirac structures $L$ satisfying $L \cap \overline{L} = 0$. The latter condition on $L$ is also referred to as *non-degeneracy*. Occasionally we will talk about *almost generalized complex structures*, which by definition are complex structures on $\mathbb{T}M$ which are orthogonal with respect to $\langle \cdot, \cdot \rangle$, but with no integrability condition. These correspond to non-degenerate complex almost Dirac structures. In section 1.2.2 we saw that Dirac structures $L$ can also be described by certain line bundles $K \subset \Lambda^\bullet T^*M_{\mathbb{C}}$. For convenience we collect the three equivalent descriptions of generalized complex structures in the following lemma.

**Lemma 1.3.3.** There is a one-to-one correspondence between

- Generalized complex structures $\mathcal{J}$.
- Complex Dirac structures $L$ which are non-degenerate, i.e. $L \cap \overline{L} = 0$.
- Complex line bundles $K \subset \Lambda^\bullet T^* M_{\mathbb{C}}$ which satisfy the following three conditions:
  i) $K$ is pointwise spanned by spinors of the form
  $$e^{B+i\omega} \wedge \Omega, \tag{1.39}$$
  where $B, \omega$ are real two-forms and $\Omega$ is a complex decomposable[10] form.
  ii) If $\rho$ is a local section of $K$ then $d^H \rho = u \cdot \rho$ for some $u \in \Gamma(\mathbb{T}M_{\mathbb{C}})$.
  iii) For every $0 \neq \rho_x \in K_x$ we have $(\rho_x, \overline{\rho_x})_{Ch} = (\rho_x \wedge \overline{\rho_x}^T)_{\text{top}} \neq 0$.

The relation between $\mathcal{J}$ and $L$ was explained above, while $L$ and $K$ are related via

$$L = \{u \in \mathbb{T}M_{\mathbb{C}} |\, u \cdot K = 0\}.$$

The line bundle $K$ associated to a generalized complex structure is called its *canonical line bundle*. The degree of the form $\Omega$ appearing in (1.39) coincides with the type of the Dirac structure $L$, and we will call it the type of $\mathcal{J}$ at $x$. We can give another description of it in terms of $\mathcal{J}$ alone. Consider the map $\pi_{\mathcal{J}}$ given by

$$\pi_{\mathcal{J}} : T^*M \overset{\pi^*}{\to} \mathbb{T}M \overset{\mathcal{J}}{\longrightarrow} \mathbb{T}M \overset{\pi}{\to} TM. \tag{1.40}$$

**Lemma 1.3.4.** $\pi_J$ is a Poisson structure on $M$.

*Proof.* Since $\beta(\pi_J(\alpha)) = 2\langle \beta, J\alpha \rangle$ is skew in $\alpha$ and $\beta$, $\pi_{\mathcal{J}}$ is skew and therefore given by a bivector on $M$. Let $L$ denote the $(+i)$-eigenbundle for $\mathcal{J}$, and $L^T$ its transpose defined in (1.32). Then

$$L \boxtimes \overline{L}^T = \{X + \xi + \eta |\, X + \xi \in L,\ X - \eta \in \overline{L}\}.$$

For such elements we have $\mathcal{J}(X + \xi) = i(X + \xi)$ and $\mathcal{J}(X - \eta) = -i(X - \eta)$. Subtracting these equations from each other yields $\mathcal{J}(\xi + \eta) = i(2X + \xi - \eta)$, so in particular $2iX = \pi_{\mathcal{J}}(\xi + \eta)$. Hence $L \boxtimes \overline{L}^T \subset e_*^{-i\pi_{\mathcal{J}}/2} T^* M_{\mathbb{C}}$, and since both sides are Lagrangian they must be equal. As $L \boxtimes \overline{L}^T$ is integrable with respect to the zero three-form, we deduce from Example 1.2.3 that $\pi_{\mathcal{J}}$ is Poisson. $\square$

Note that $\pi_{\mathcal{J}}$ is an invariant of $\mathcal{J}$ in the sense that it is unchanged when we conjugate $\mathcal{J}$ by a $B$-field transformation. The Poisson structure $\pi_{\mathcal{J}}$ induces a (possibly singular) foliation on the manifold, with leaves that carry symplectic structures. The conormal bundle to such a symplectic leaf agrees with the kernel of $\pi_{\mathcal{J}}$ and is given by

$$\nu_{\mathcal{J}} := T^*M \cap \mathcal{J} T^* M.$$

[10] A differential form is called *decomposable* if it can be written as a product of one-forms.

In general it might be singular, as its rank may vary from one point to the next. Since $\mathcal{J}$ induces a complex structure on $\nu_{\mathcal{J}}$, we can picture a generalized complex structures as a possibly singular foliation of $M$, with symplectic leaves and with transverse complex structures, integrable in a suitable sense. The type of $\mathcal{J}$ then coincides with the number of transverse complex directions;

$$\text{type}_x(\mathcal{J}) = \dim_{\mathbb{C}}(\nu_{\mathcal{J}})_x. \tag{1.41}$$

This description coincides with the one given in Remark 1.2.14 at every point.

Next we give some examples of generalized complex structures.

**Example 1.3.5.** Let $I$ be an almost complex structure on $M$. It defines an almost generalized complex structure

$$\mathcal{J}_I := \begin{pmatrix} -I & 0 \\ 0 & I^* \end{pmatrix}. \tag{1.42}$$

The corresponding almost Dirac structure is given by $L_I = T^{0,1}M \oplus T^{*1,0}M$. It is integrable with respect to $[\![\cdot,\cdot]\!]_H$ if and only if $I$ itself is integrable and $H$ is of type $(2,1)+(1,2)$ with respect to $I$. The spinor line corresponding to $L_I$ is given by $\Lambda^{n,0}T^*M$, and the type is $n = \frac{1}{2}\dim_{\mathbb{R}}(M)$ at all points.

**Example 1.3.6.** Let $\omega$ be a non-degenerate two-form on $M$. It defines an almost generalized complex structure

$$\mathcal{J}_\omega := \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \tag{1.43}$$

The associated almost Dirac structure is given by $L_\omega = e_*^{i\omega}(TM_{\mathbb{C}}) = \{X - i\omega(X) |\, X \in TM_{\mathbb{C}}\}$. It is integrable for $[\![\cdot,\cdot]\!]_H$ if and only if $\omega$ is closed and $H = 0$ (see also Example 1.2.2). A spinor for $\mathcal{J}$ is given by $e^{i\omega}$, and the type is 0 everywhere.

**Example 1.3.7.** Let $I$ be an almost complex structure and $Q$ a bivector that satisfies $QI^* = IQ$, viewed as maps $T^*M \to TM$. The bivector $\sigma := Q - iIQ$ is then of type $(2,0)$ with respect to $I$ and

$$\mathcal{J}_{(I,\sigma)} := \begin{pmatrix} -I & 4IQ \\ 0 & I^* \end{pmatrix} \tag{1.44}$$

defines an almost generalized complex structure. The corresponding almost Dirac structure is given by

$$L_{(I,\sigma)} := T^{0,1}M \oplus e_*^{\sigma}(T^{*1,0}M) = \{X + \sigma(\xi) + \xi |\, X \in T^{0,1}M,\ \xi \in T^{*1,0}M\}$$

and the spinor line by $e^\sigma \cdot \Lambda^{n,0}T^*M$. The type coincides with $\frac{1}{2}\text{corank}(Q)$. Let us investigate when $L_{(I,\sigma)}$ is integrable with respect to the zero three-form. For $X, Y \in$

$T^{0,1}M$ we need $[X,Y] \in L_{(I,\sigma)}$, which can only happen if $T^{0,1}M$ is involutive, i.e. if $I$ is integrable. Next, consider

$$[\![X, \sigma(\xi) + \xi]\!] = \mathcal{L}_X(\sigma(\xi) + \xi) = (\mathcal{L}_X\sigma)(\xi) + (\sigma(\mathcal{L}_X\xi) + \mathcal{L}_X\xi).$$

Since $X \in T^{0,1}M$, $\xi \in T^{*1,0}M$ and $I$ is integrable we have $\mathcal{L}_X\xi \in T^{*1,0}M$, so the second term above is automatically in $L_{(I,\sigma)}$. Hence, what remains is the condition $(\mathcal{L}_X\sigma)(\xi) \in T^{0,1}M$. This is equivalent to $\sigma$ being holomorphic, as can be seen e.g. in local coordinates. Finally, we have

$$[\![\sigma(\xi) + \xi, \sigma(\eta) + \eta]\!] = [\sigma(\xi), \sigma(\eta)] + \mathcal{L}_{\sigma(\xi)}\eta - \iota_{\sigma(\eta)}d\xi.$$

As $[\sigma(\xi), \sigma(\eta)]$ lies in $T^{1,0}M$, the above expression lies in $L_{(I,\sigma)}$ if and only if it lies in $e_*^{\sigma}(T^{*1,0}M)$. Just as in Example 1.2.3, this is equivalent to $\sigma$ being Poisson, i.e. the Poisson bracket, defined now on holomorphic functions, must satisfy the Jacobi identity. To summarize, (1.44) defines a generalized complex structure if and only if $(M, I, \sigma)$ is a holomorphic Poisson manifold.

**Example 1.3.8.** Here is a less concrete example that is interesting from a topological point of view. If $M$ and $N$ are smooth $k$-dimensional manifolds we denote by $M\#N$ their *connected sum*, obtained by removing a small $k$-dimensional ball from both $M$ and $N$ and glue the resulting manifolds with boundary to each other via an orientation reversing diffeomorphism of $S^{k-1}$. We will abbreviate by $mM := M\#M\#\ldots\#M$ the connected sum of $m$ copies of $M$. The manifolds $M_{m,n} := m\mathbb{CP}^2\#n\overline{\mathbb{CP}}^2$, where $\overline{\mathbb{CP}^2}$ denotes the manifold $\mathbb{CP}^2$ with the opposite orientation, are known to carry complex or symplectic structures only for $m = 1$. However, it was shown in [15] that they carry a generalized complex structure if and only if $m$ is odd, which is precisely the condition for $M_{m,n}$ to admit an almost complex structure[11].

Using the notions discussed in Section 1.1.2 we can talk about generalized holomorphic maps.

**Definition 1.3.9.** A map $\Phi : (M_1, H_1, \mathcal{J}_1) \to (M_2, H_2, \mathcal{J}_2)$ is called *generalized holomorphic* if for every $u \in \mathbb{T}M_1$ and $v \in \mathbb{T}M_2$ with $u \sim_\Phi v$, we have $\mathcal{J}_1(u) \sim_\Phi \mathcal{J}_2(v)$.

As before, we will often drop the prefix "generalized" and simply say holomorphic. Let us try to get some feeling for this notion. Using the splittings $\mathbb{T}M_i = TM_i \oplus T^*M_i$ we can write $\mathcal{J}_i$ as

$$\mathcal{J}_i = \begin{pmatrix} A_i & \pi_{\mathcal{J}_i} \\ \sigma_i & -A_i^* \end{pmatrix}.$$

In general this decomposition is not very useful because it is only $\pi_{\mathcal{J}_i}$ that is invariant under $B$-field transformations. Nevertheless, the condition for a map $\Phi = (\varphi, B)$ to be

[11] The topological obstruction for having an almost generalized complex structure coincides with that for an almost complex structure, see Lemma 1.3.42.

holomorphic becomes very explicit:

$$\begin{aligned} i)\ &\varphi_* \circ (A_1 - \pi_{\mathcal{J}_1} B) = A_2 \circ \varphi_*, \\ ii)\ &\varphi_*(\pi_{\mathcal{J}_1}) = \pi_{\mathcal{J}_2}, \\ iii)\ &\sigma_1 + A_1^* B + B(A_1 - \pi_{\mathcal{J}_1} B) = \varphi^*(\sigma_2). \end{aligned} \tag{1.45}$$

Note that the second equation tells us that $\varphi$ has to be a Poisson map. Here are some concrete examples.

**Example 1.3.10.** Let $\mathcal{J}_1 = \mathcal{J}_{I_1}$ and $\mathcal{J}_2 = \mathcal{J}_{I_2}$, for complex structures $I_1$ and $I_2$. Then (1.45) is equivalent to $\varphi$ being holomorphic and $B$ being of type $(1,1)$ with respect to $I_1$.

*Remark* 1.3.11. As this example shows, generalized holomorphic maps need not satisfy either $\mathfrak{B}\Phi(L_2) = L_1$ or $\mathfrak{F}(L_1) = L_2$. Indeed, if $\varphi : (M_1, I_1) \to (M_2, I_2)$ is a constant map between two complex manifolds then $\varphi$ is generalized holomorphic, but $\mathfrak{B}\varphi(L_2) = (TM_1)_{\mathbb{C}}$ while $\mathfrak{F}\varphi(L_1) = (T^*M_2)_{\mathbb{C}}$.

**Example 1.3.12.** Let $\mathcal{J}_1 = \mathcal{J}_{I_1}$ and $\mathcal{J}_2 = \mathcal{J}_{\omega_2}$, with $I_1$ complex and $\omega_2$ symplectic. Since $\pi_{\mathcal{J}_1} = 0$ while $\pi_{\mathcal{J}_2}$ is invertible, there does not exist any holomorphic map in this case.

**Example 1.3.13.** Let $\mathcal{J}_1 = \mathcal{J}_{\omega_1}$ and $\mathcal{J}_2 = \mathcal{J}_{I_2}$. The three equations in (1.45) reduce to

$$\varphi_* \circ \omega_1^{-1} B = -I_2 \circ \varphi_*, \qquad \varphi_*(\omega_1^{-1}) = 0, \qquad \omega_1 + B\omega_1^{-1} B = 0.$$

The third equation says that $I_1 := -\omega_1^{-1} B$ is an almost complex structure, while the first equation says that $\varphi$ should be pseudo-holomorphic, i.e. $\varphi_* \circ I_1 = I_2 \circ \varphi_*$. The second equation can be rephrased by saying that the fibers of $\varphi$ should be coisotropic, meaning that $\omega_1^{-1}\big(\mathrm{Ann}(\ker(\varphi_*))\big) \subset \ker(\varphi_*)$. Since $\varphi$ is pseudo-holomorphic, this is equivalent to demanding coisotropicity with respect to $B$. Here is an example of this situation. Let $(M_1, I_1)$ be a complex manifold and let $\kappa$ be a *holomorphic symplectic form*, i.e. a closed, non-degenerate $(2,0)$-form on $M_1$. Then $B := \mathrm{Re}(\kappa)$ is closed and non-degenerate as well, as is $\omega_1 := BI_1$. Any holomorphic map $\varphi : (M_1, I_1) \to (M_2, I_2)$ whose fibers are coisotropic with respect to $\kappa$ satisfies the above equations.

**Example 1.3.14.** Let $\mathcal{J}_1 = \mathcal{J}_{\omega_1}$ and $\mathcal{J}_2 = \mathcal{J}_{\omega_2}$. Then (1.45) reduces to

$$\varphi_* \circ \omega_1^{-1} B = 0, \qquad \varphi_*(\omega_1^{-1}) = \omega_2^{-1}, \qquad \omega_1 + B\omega_1^{-1} B = \varphi^* \omega_2.$$

The Poisson condition, written as $\omega_2^{-1} = \varphi_* \circ \omega_1^{-1} \circ \varphi^*$, shows that $\varphi_*$ has to be surjective, i.e. $\varphi$ must be a submersion. Let $V := \ker(\varphi_*)$ denote the vertical tangent bundle and $V^{\omega_1}$ its symplectic orthogonal. Since $V^{\omega_1} = \omega_1^{-1}(\mathrm{Ann}(V)) = \omega_1^{-1}(\mathrm{Im}(\varphi^*))$ and $\varphi_*(\omega_1^{-1}\varphi^*\alpha) = \omega_2^{-1}\alpha$, it follows that $V \cap V^{\omega_1} = 0$, i.e. $V$ is symplectic. Moreover, the induced symplectic structure on $V^{\omega_1}$ coincides with $\varphi^*\omega_2$. The remaining two conditions can be understood as follows. The first says that $B(TM_1) \subset \omega_1(V)$, which is equivalent to $\iota_u B = 0$ for all $u \in V^{\omega_1}$. Finally, the third equation is equivalent to $\omega_1^{-1} B|_V$ defining an almost complex structure on $V$. So if e.g. $B = 0$, the fibers have to be zero dimensional and $\varphi$ has to be a symplectic local diffeomorphism.

As these examples show, the notion of generalized holomorphic maps in generalized complex geometry behaves rather differently in different contexts.

When $\varphi$ is a diffeomorphism we can give a more concrete description in terms of spinors. If $K_i$ is the canonical bundle for $\mathcal{J}_i$, $\Phi$ being an isomorphism amounts to

$$K_1 = e^B \wedge \varphi^* K_2.$$

Here $\varphi^* K_2$ refers to the pull-back applied to forms, so $\varphi^* K_2 \subset \Lambda^\bullet T^* M_1$. The following lemma justifies calling structures of type 0 symplectic and those of type $n = \frac{1}{2}\mathrm{dim}_{\mathbb{R}}(M)$ complex.

**Lemma 1.3.15.** Let $\mathcal{J}$ have type 0 (type $n$ respectively). Then there is a two-form $B$ such that $e_*^B \mathcal{J} e_*^{-B}$ equals $\mathcal{J}_\omega$ for some symplectic structure $\omega$ (respectively $\mathcal{J}_I$ for some complex structure).

*Proof.* If $\mathcal{J}$ has type 0 then $L \cap T^*M_{\mathbb{C}} = 0$. From the complex version of Example 1.2.2 we deduce that $L = e_*^{B+i\omega}(TM_{\mathbb{C}})$, where $H = -dB$ and $d\omega = 0$. In particular, $L$ is the $B$-field transform of a symplectic structure (see Example 1.3.6). If $\mathcal{J}$ has type $n$, then $\pi(L) \subset TM_{\mathbb{C}}$ is a complex subspace of dimension $n$. Since we always have $TM_{\mathbb{C}} = \pi(L) + \pi(\overline{L})$, we see that $\pi(L) \cap \pi(\overline{L}) = 0$. This equips $M$ with an integrable complex structure, by defining $T^{0,1}M := \pi(L)$. Note that $L \cap T^*M_{\mathbb{C}}$ is also $n$-dimensional and annihilates $\pi(L)$, hence $L \cap T^*M_{\mathbb{C}} = T^{*1,0}M$. Consequently, $L = e_*^B(T^{0,1}M) \oplus T^{*1,0}M$ for some complex two-form $B$. The $(2,0)$ and $(1,1)$ components of $B$ are irrelevant in this expression, so we may replace $B$ by a real two-form without changing $L$. In particular, $L$ is the $B$-field transform of a complex structure. □

In other words, structures of type 0 and $n$ are isomorphic to symplectic and complex structures respectively. We now state the analogue of the Newlander-Nirenberg and Darboux theorems in generalized complex geometry.

**Theorem 1.3.16** ([6])**.** *Let $(M, H, \mathcal{J})$ be a generalized complex manifold. If $x \in M$ is a point where $\mathcal{J}$ has type $k$, then a neighborhood of $x$ is isomorphic to a neighborhood of $(0,0)$ in*

$$(\mathbb{R}^{2n-2k}, \omega_{st}) \times (\mathbb{C}^k, \sigma) := (\mathbb{R}^{2n-2k} \times \mathbb{C}^k, \mathcal{J}_{\omega_{st}} \times \mathcal{J}_{(i,\sigma)}) \tag{1.46}$$

*where $\omega_{st}$ is the standard symplectic form, $\sigma$ is a holomorphic Poisson structure which vanishes at 0, and $\mathcal{J}_{\omega_{st}}$ and $\mathcal{J}_{(i,\sigma)}$ are defined in Examples 1.3.5 and 1.3.6.*

## 1.3.1 Generalized complex submanifolds

Our main objective in this thesis is to investigate which submanifolds of generalized complex manifolds admit a blow-up. The starting point for that is to define the appropriate notion of generalized complex submanifold. The one that we will use generalizes the usual definition of complex and symplectic submanifolds, as those are the submanifolds that are known to admit a blow-up in their respective categories (there is, for instance, no blow-up theory for (co)isotropic submanifolds of a symplectic manifold). In particular, the definition that we will use should not be confused with so-called *branes* (see below), which sometimes are also called generalized complex submanifolds. These branes are generalizations of Lagrangian submanifolds in symplectic geometry.

Roughly speaking, a generalized complex submanifold for us will be one that inherits a generalized complex structure from the ambient space. To state this precisely we use the backward image operation on Dirac structures that was introduced in Section 1.2.3.

**Definition 1.3.17.** A *generalized complex submanifold* of $(M, H, \mathcal{J})$ is a submanifold $i : Y \hookrightarrow M$ with the property that $\mathfrak{B}i(L)$ is generalized complex, i.e. is smooth and satisfies $\mathfrak{B}i(L) \cap \overline{\mathfrak{B}i(L)} = 0$.

*Remark* 1.3.18. Note that we are identifying the inclusion $i$ with the generalized map $(i, 0)$, so the relevant three-form on $Y$ is given by $i^*H$. It seems natural to also incorporate a two-form $B \in \Omega^2(Y)$ in the above definition. However, this does not make a difference since $\mathfrak{B}(i, B)(L) = e^B_* \mathfrak{B}i(L)$, which is generalized complex if and only if $\mathfrak{B}i(L)$ is.

The notion of being a generalized complex submanifold entails two conditions of rather different natures. The smoothness condition is something that depends on how the bundle $L$ is varying along the submanifold, while the non-degeneracy condition is a linear algebraic condition that we can check at each given point of $Y$. The following lemma will help us to recognize when these two conditions are satisfied. Below, the superscript $\perp$ refers to the orthogonal complement in $\mathbb{T}M$ with respect to the natural pairing.

**Proposition 1.3.19.** *Let $Y$ be a submanifold of a generalized complex manifold $(M, \mathcal{J})$.*

$$i)\ \textit{If } N^*Y \cap \mathcal{J}N^*Y \textit{ is of constant rank, then } \mathfrak{B}i(L) \textit{ is smooth.} \tag{1.47}$$

$$ii)\ \mathfrak{B}i(L) \cap \mathfrak{B}i(\overline{L}) = 0 \textit{ if and only if } \mathcal{J}N^*Y \cap (N^*Y)^\perp \subset N^*Y. \tag{1.48}$$

*Proof.* i): The smoothness criterion (1.35) tells us that if $\ker(i^*)_\mathbb{C} \cap i^*L$ has constant rank, then the backward image $\mathfrak{B}i(L)$ is smooth. In this situation $\ker(i^*) = N^*Y$, the conormal bundle of $Y$ in $M$, and $\ker(i^*)_\mathbb{C} \cap i^*L$ is precisely the $(+i)$-eigenspace of the restriction of $\mathcal{J}$ to the invariant subspace $N^*Y \cap \mathcal{J}N^*Y$ of $N^*Y$. Hence, $\ker(i^*)_\mathbb{C} \cap i^*L$ has constant rank if and only if $N^*Y \cap \mathcal{J}N^*Y$ has constant rank, and this proves i).

ii): From the definition of the backward image it follows that

$$\mathfrak{B}i(L) \cap \mathfrak{B}i(\overline{L}) = \{X + i^*\xi = X + i^*\eta |\ X \in TY_\mathbb{C},\ X + \xi \in L,\ X + \eta \in \overline{L}\}.$$

For such elements we have $\xi - \eta \in N^*Y_\mathbb{C}$ and $\mathcal{J}(\xi - \eta) = i(2X + \xi + \eta)$. In this last equation the left-hand side lies in $\mathcal{J}N^*Y_\mathbb{C}$, while the right-hand side lies in $\pi^{-1}(TY_\mathbb{C}) = (N^*Y_\mathbb{C})^\perp$. On the other hand, given $\alpha \in N^*Y_\mathbb{C}$ and $X + \beta \in \pi^{-1}(TY_\mathbb{C})$ with $\mathcal{J}\alpha = X + \beta$, then $X + \beta + i\alpha \in L$ and $X + \beta - i\alpha \in \overline{L}$ and so $X + i^*(\beta + i\alpha) = X + i^*(\beta - i\alpha) \in \mathfrak{B}i(L) \cap \mathfrak{B}i(\overline{L})$. Consequently, $\mathfrak{B}i(L) \cap \mathfrak{B}i(\overline{L}) = 0$ precisely if the equation $\mathcal{J}\alpha = X + \beta$ above forces $X = 0$ and $\beta \in N^*Y_\mathbb{C}$, which amounts to the condition $\mathcal{J}N^*Y \cap (N^*Y)^\perp \subset N^*Y$. □

*Remark* 1.3.20. a) The intersection $N^*Y \cap \mathcal{J}N^*Y$ inherits a complex structure given by $\mathcal{J}$ itself, and its complex rank can be thought of as the type of $\mathcal{J}$ in transverse directions (see also (1.41)). Hence, (1.47) can be rephrased by saying that if $\mathcal{J}$ has constant type in directions transverse to $Y$, then the backward image is smooth. This condition is not strictly necessary, as Example 1.3.24 below will demonstrate.

b) We can give a more geometric interpretation of (1.48) as follows. We have

$$\mathcal{J}N^*Y \cap (N^*Y)^\perp = \{\mathcal{J}\alpha |\ \alpha \in N^*Y \cap \pi_{\mathcal{J}}^{-1}(TY)\}.$$

For this to lie in $N^*Y$, we first of all need the tangential part of such a $\mathcal{J}\alpha$ to vanish. This is nothing but $\pi_{\mathcal{J}}(\alpha)$, so one condition we obtain is that $\pi_{\mathcal{J}}\big(N^*Y \cap \pi_{\mathcal{J}}^{-1}(TY)\big) = 0$, or equivalently

$$TY \cap \pi_{\mathcal{J}}(N^*Y) = 0. \tag{1.49}$$

If this condition holds we have $N^*Y \cap \pi_{\mathcal{J}}^{-1}(TY) = \ker(\pi_{\mathcal{J}}) \cap N^*Y$, and so for (1.48) to hold we need in addition that $\mathcal{J}(\ker(\pi_{\mathcal{J}}) \cap N^*Y) \subset N^*Y$. Since $\pi_{\mathcal{J}}(\mathcal{J}\alpha) = 0$ for any $\alpha \in T^*M$ with $\mathcal{J}\alpha \in T^*M$, this is equivalent to demanding

$$\mathcal{J}(\ker(\pi_{\mathcal{J}}) \cap N^*Y) \subset \ker(\pi_{\mathcal{J}}) \cap N^*Y. \tag{1.50}$$

Conditions (1.49) and (1.50) can be visualized as follows. Firstly, the symplectic distribution $\pi_{\mathcal{J}}(T^*M)$ on $M$ induces the distribution $\Delta := TY \cap \pi_{\mathcal{J}}(T^*M)$ on $Y$. Then $\Delta \subset \pi_{\mathcal{J}}(T^*M)$ is a symplectic subspace precisely when (1.49) holds. Hence, (1.49) means that $Y$ intersects all the leaves in symplectic subspaces. Furthermore, the annihilator $\mathrm{Ann}(\Delta) \subset T^*Y$ fits into the exact sequence

$$0 \to \ker(\pi_{\mathcal{J}}) \cap N^*Y \to \ker(\pi_{\mathcal{J}}) \to \mathrm{Ann}(\Delta) \to 0.$$

The space $\ker(\pi_{\mathcal{J}}) = T^*M \cap \mathcal{J}T^*M$ has a canonical complex structure induced by $\mathcal{J}$, so (1.50) amounts to $\mathrm{Ann}(\Delta)$ being the quotient of $\ker(\pi_{\mathcal{J}})$ by a complex subspace. In particular, $\mathrm{Ann}(\Delta)$ itself inherits a complex structure. Summarizing, (1.48) can be rephrased by saying that $Y$ intersects the symplectic leaves of $M$ in symplectic subspaces with an induced complex structure on the conormal bundles.

Now let us give some examples of generalized complex submanifolds.

**Example 1.3.21.** A point $x \in M$ is always a generalized complex submanifold. Indeed, (1.47) holds trivially in this case, while (1.48) is satisfied because $N^*\{x\} = T_x^*M = (T_x^*M)^\perp$.

**Example 1.3.22.** Suppose that $\mathcal{J} = \mathcal{J}_I$ for a complex structure $I$. Since $\mathcal{J}N^*Y = I^*N^*Y \subset (N^*Y)^\perp$, (1.48) holds if and only if $I^*N^*Y = N^*Y$, i.e. $Y$ is a complex submanifold. Condition (1.47) is then automatically satisfied because $N^*Y \cap \mathcal{J}N^*Y = N^*Y$.

**Example 1.3.23.** Let $\mathcal{J} = \mathcal{J}_\omega$ for a symplectic structure $\omega$. Since $\mathcal{J}N^*Y = \omega^{-1}N^*Y \subset TM|_Y$, (1.48) is satisfied if and only if $\omega^{-1}(N^*Y) \cap TY = 0$. This is equivalent to $i^*\omega$ being non-degenerate on $Y$, i.e. $Y$ needs to be a symplectic submanifold. In that case $N^*Y \cap \mathcal{J}N^*Y = 0$, so (1.47) holds as well.

Note that if $M$ is symplectic, $i$ is holomorphic in the sense of Definition 1.3.9 if and only if $Y$ is an open subset (Example 1.3.14). Hence holomorphicity is not the right notion to define submanifolds, for it would exclude symplectic submanifolds.

We now give an example to illustrate how (1.47), although sufficient, is not strictly necessary.

**Example 1.3.24.** Consider $(M = \mathbb{C}^3, H = 0)$ with coordinates $(u, z, w)$, and consider the spinor $\rho = du \wedge (z + dz \wedge dw)$. On $\{z \neq 0\}$ we can write it as $\rho = ze^{d \log z \wedge dw} du$, while on $\{z = 0\}$ we have $\rho = du \wedge dz \wedge dw$. In particular, $\rho$ is a pure spinor at each point in $M$. Moreover,

$$(\rho, \overline{\rho})_{\text{Ch}} = -du \wedge dz \wedge dw \wedge d\overline{u} \wedge d\overline{z} \wedge d\overline{w}$$

is nowhere vanishing, so we obtain an almost generalized complex structure $\mathcal{J}$ on $M$. Since $d\rho = -du \wedge dz = -\partial_w \cdot \rho$, the structure is integrable by Lemma 1.2.19.

On the open dense set $\{z \neq 0\}$, $\mathcal{J}$ is of type 1, with symplectic leaves given by $u =$ constant. On $\{z = 0\}$, $\mathcal{J}$ is of type 3, given by the standard complex structure. Let $Y = \{w = 0, u - z = 0\}$. Then $\mathfrak{B}i(L)$ is given by the complex structure on $Y$ viewed as a complex submanifold of $\mathbb{C}^3$, so $Y$ is a generalized complex submanifold of $M$ according to Definition 1.3.17. Indeed, on $z \neq 0$ we have $i^*\rho = zdu$, which defines the standard complex structure on $Y$, while at $z = 0$ $\mathcal{J}$ is complex and $Y$ is given by a complex line through the origin. Nevertheless, at $z = 0$ we have $N^*Y \cap \mathcal{J}N^*Y = N^*Y$, while on $z \neq 0$ we have $N^*Y \cap \mathcal{J}N^*Y = 0$. So this is an example where (1.47) is not satisfied, even though the submanifold is generalized complex.

Since there is no smoothness issue for submanifolds in the symplectic and complex categories, one may wonder whether it really appears in generalized complex geometry. Could the smoothness of $\mathfrak{B}i(L)$ perhaps be deduced from the non-degeneracy condition alone? Unfortunately it does not.

**Example 1.3.25.** Consider $(M = \mathbb{C}^2, H = 0)$ with coordinates $(z, w)$, and consider the spinor $\rho = z + dz \wedge dw + zdz \wedge d\bar{z}$. As in the previous example one verifies that $\rho$ is pure everywhere and that $(\rho, \overline{\rho})_{\text{Ch}}$ is nowhere zero. Furthermore, $d\rho = dz = -\partial_w \cdot \rho$, so $\rho$ defines a generalized complex structure $\mathcal{J}$. On $\{z = 0\}$ it is of complex type, while on $\{z \neq 0\}$ it is of symplectic type. Consider $Y := \{w = 0\}$. On $Y \cap \{z \neq 0\}$ the backward image is induced by $i^*\rho = z(1 + dz \wedge d\bar{z})$, hence is generalized complex of symplectic type. At $z = 0$ however, $\mathfrak{B}i(L)$ is given by the standard complex structure on $Y$. Hence, $\mathfrak{B}i(L)$ is generalized complex at every point, but is not smooth. The latter follows from the fact that for smooth generalized complex structures the type varies in even steps, so in particular it will be constant on a manifold of real dimension 2 , where it will be either 0 or 1 throughout. This is thus an example of a submanifold which is pointwise generalized complex, i.e. (1.48) holds, but not in a smooth manner.

For the purpose of blowing up, the class of all generalized complex submanifolds is too large. For instance, in Example 1.3.21 we saw that any point $x \in (M, \mathcal{J})$ forms a generalized complex submanifold. If $0 < \text{type}_x(\mathcal{J}) < n$, there will be normal directions that are symplectic or complex, but there will also be normal directions which are neither. It seems difficult to say anything reasonable about a blow-up in such a situation, and in fact in Section 3.3 we will give concrete situations where these kind of points cannot be blown-up in a generalized complex fashion.

We will therefore restrict our attention to a further subclass of submanifolds, namely those which are purely complex or purely symplectic in transverse directions. Following nomenclature from Poisson geometry we call these *generalized Poisson submanifolds* and *generalized Poisson transversals* respectively, simply because they are Poisson submanifolds and Poisson transversals for the underlying Poisson structure. The precise definitions are as follows.

**Definition 1.3.26.** Let $i : Y \hookrightarrow M$ be a submanifold.

- $Y$ is a *generalized Poisson submanifold* if $\mathcal{J}N^*Y = N^*Y$.
- $Y$ is a *generalized Poisson transversal* if $\mathcal{J}N^*Y \cap (N^*Y)^\perp = 0$.

If $Y$ is a generalized Poisson submanifold then $\mathcal{J}N^*Y \cap N^*Y = \mathcal{J}N^*Y \cap (N^*Y)^\perp = N^*Y$, hence both (1.47) and (1.48) are satisfied and so $Y$ is a generalized complex submanifold. Since $\pi_{\mathcal{J}}(N^*Y) = 0$, $Y$ is a Poisson submanifold for $\pi_{\mathcal{J}}$.

Similarly, if $Y$ is a generalized Poisson transversal, $\mathcal{J}N^*Y \cap N^*Y \subset \mathcal{J}N^*Y \cap (N^*Y)^\perp = 0$ and both (1.47) and (1.48) are again satisfied, so that $Y$ is a generalized complex submanifold. Since $(N^*Y)^\perp = \pi^{-1}(TY)$, the condition $\mathcal{J}N^*Y \cap (N^*Y)^\perp = 0$ is equivalent to $TY + \pi_{\mathcal{J}}(N^*Y) = TM|_Y$. This is precisely the transversality condition in Poisson geometry (see [18]), and is equivalent to $Y$ intersecting the leaves not just symplectically but also transversally. Note that generalized Poisson submanifolds inherit a complex structure on $N^*Y$ given by the restriction of $\mathcal{J}$, while for generalized Poisson transversals $N^*Y$ comes equipped with a symplectic structure given by

$$(\alpha, \beta) \mapsto \langle \mathcal{J}\alpha, \beta \rangle.$$

The fact that this form is non-degenerate is precisely the condition for $Y$ to be a generalized Poisson transversal. Hence, generalized Poisson submanifolds and generalized Poisson transversals really look complex, respectively, symplectic in transverse directions.

Looking at Example 1.3.21, we see that $\{x\} \subset M$ is a generalized Poisson submanifold precisely if $\text{type}_x(\mathcal{J}) = n$, while it is a generalized Poisson transversal precisely if $\text{type}_x(\mathcal{J}) = 0$. Regarding Examples 1.3.22 and 1.3.23, we see that complex submanifolds are always generalized Poisson submanifolds, while symplectic submanifolds are always generalized Poisson transversals.

We already remarked that the inclusion of a generalized complex submanifold is rarely generalized holomorphic. In fact, we can say precisely when this happens.

**Lemma 1.3.27.** Let $i : Y \hookrightarrow (M, \mathcal{J})$ be a generalized complex submanifold. Then $i$ is generalized holomorphic if and only if $Y$ is a generalized Poisson submanifold.

*Proof.* Let us recall how the induced generalized complex structure $\mathcal{J}_Y$ on $Y$ is defined. Given $X+\xi \in \mathbb{T}Y$, we write it as $X + i^*\xi'$ for some $\xi' \in T^*M$. Then we decompose $X+\xi' = (X_1+\xi_1') + (\overline{X_1}+\overline{\xi_1'})$ into $L$ and $\overline{L}$ components so that, by definition, $X+\xi = (X_1 + i^*\xi_1') + (\overline{X_1} + i^*\overline{\xi_1'})$ is the decomposition into $L_Y$ components. In particular,

$$\mathcal{J}_Y(X+\xi) = i\big((X_1-\overline{X_1}) + i^*(\xi_1' - \overline{\xi_1'})\big).$$

Now suppose that $X+\xi$ is $i$-related to $Y+\eta \in \mathbb{T}M$, i.e. $Y = i_*X$ and $\xi = i^*\eta$. Then $\eta = \xi_1' + \overline{\xi_1'} + \alpha$, where $\alpha \in N^*Y$. In particular,

$$\mathcal{J}(Y+\eta) = i\big((X_1-\overline{X_1}) + (\xi_1'-\overline{\xi_1'})\big) + \mathcal{J}\alpha.$$

This is $i$-related to $\mathcal{J}_Y(X+\xi)$ if and only if $\mathcal{J}\alpha \in N^*Y$, i.e. if and only if $Y$ is a generalized Poisson submanifold. □

**Branes**

We will see in Section 3.1 that when we blow up a point of complex type, the exceptional divisor is not always a generalized Poisson submanifold. Nevertheless, it does carry the structure of a so-called generalized complex brane, a concept which we will now explain.

Let $Y \subset (M,H)$ be a submanifold, and denote by $K := N^*Y$ its conormal bundle. We have $K^\perp = \pi^{-1}(TY)$, where the orthogonal complement is taken with respect to the natural pairing and $\pi : \mathbb{T}M \to TM$ denotes the anchor. The bundle $K^\perp/K$ inherits a canonical bracket from $\mathbb{T}M$ as follows. Given $u, v \in \Gamma(K^\perp)$, we can choose extensions $\tilde{u}, \tilde{v} \in \Gamma(\mathbb{T}M)$ and consider the element $[\![\tilde{u},\tilde{v}]\!]|_Y$. Since $K^\perp = \pi^{-1}(TY)$, $\pi$ is a morphism of brackets, and $TY \subset TM$ is involutive, it follows that $[\![\tilde{u},\tilde{v}]\!]|_Y \in \Gamma(K^\perp)$. We can see how this expression depends on the choice of extensions by looking at what happens if we add, to either $\widetilde{u}$ or $\widetilde{v}$, something of the form $fw$, where $w \in \Gamma(\mathbb{T}M)$ and $f \in C^\infty(M)$ with $f|_Y = 0$. By the Leibniz rule (Lemma 1.1.1 iii)) we have

$$[\![\tilde{u}, fw]\!]|_Y = (f[\![\tilde{u},w]\!] + (\pi(\tilde{u})\cdot f)w)|_Y = 0,$$

because $\pi(\tilde{u})$ is tangent to $Y$, along which $f$ vanishes. Hence, $[\![\tilde{u},\tilde{v}]\!]|_Y$ is independent of the choice of $\widetilde{v}$. Next, using Lemma 1.1.1 iii) and iv), we obtain

$$[\![fw, \tilde{v}]\!]|_Y = (f[\![w,\tilde{v}]\!] - (\pi(\tilde{v})\cdot f)w + 2\langle u, v\rangle \cdot df)|_Y.$$

The first two terms vanish for the same reason as before, but the third term will in general be nonzero. Hence, $[\![\tilde{u},\tilde{v}]\!]|_Y$ does depend on the choice of $\widetilde{u}$, but since $df|_Y \in \Gamma(K)$ we do get a well-defined operation $\Gamma(K^\perp)\times\Gamma(K^\perp) \to \Gamma(K^\perp/K)$. Finally, for $v \in \Gamma(K)$ and $u, w \in \Gamma(K^\perp)$, we have

$$\langle [\![\widetilde{u},\widetilde{v}]\!]|_Y, w\rangle = \pi(u)\cdot\langle v, w\rangle - \langle v, [\![\widetilde{u},\widetilde{w}]\!]|_Y\rangle = 0,$$

because $v$ is perpendicular to $K^\perp$ and $[\![\widetilde{u}, \widetilde{w}]\!]|_Y \in \Gamma(K^\perp)$. Hence, $[\![\widetilde{u}, \widetilde{v}]\!]|_Y \in \Gamma((K^\perp)^\perp) = \Gamma(K)$. A similar argument holds when $u \in \Gamma(K)$, and so we obtain a bracket

$$[\![\cdot, \cdot]\!] : \Gamma(K^\perp/K) \times \Gamma(K^\perp/K) \to \Gamma(K^\perp/K).$$

The natural pairing induces a non-degenerate pairing on $K^\perp/K$ and there is a short exact sequence

$$0 \to T^*Y \to K^\perp/K \to TY \to 0, \tag{1.51}$$

expressing $K^\perp/K$ as an exact Courant algebroid over $Y$ (see Remark 1.1.4).

**Definition 1.3.28.** A *trivialization of* $\mathbb{T}M$ *along* $Y$ is a splitting $s : TY \to K^\perp/K$ of (1.51) whose image is isotropic and involutive.

Since maximal isotropic subspaces of $K^\perp/K$ correspond bijectively to maximal isotropic subspaces $\tau \subset K^\perp$ that contain $K$, isotropic splittings of (1.51) are the same as isotropic extensions of $TY$ by $K$, i.e. short exact sequences

$$0 \to K \to \tau \to TY \to 0$$

with $\tau \subset \mathbb{T}M|_Y$ isotropic (note that $K \subset \tau$ together with $\tau$ being isotropic implies that $\tau \subset K^\perp$). The splitting that corresponds to $\tau$ is a trivialization precisely when $\tau/K \subset K^\perp/K$ is involutive. From now on, a trivialization will refer to the subspace $\tau \subset K^\perp$.

**Definition 1.3.29.** A *generalized complex brane* in $(M, H, \mathcal{J})$ is a submanifold $Y$ together with a trivialization $\tau$ on $Y$ such that $\mathcal{J}\tau = \tau$.

*Remark* 1.3.30. The definition of a brane given in [24] requires in addition the input of a certain vector bundle over $Y$ (a representation of the complex Lie algebroid $\tau^{1,0}$, to be precise). For us the current definition will suffice.

*Remark* 1.3.31. Since $\pi_{\mathcal{J}}(N^*Y) = \pi(\mathcal{J}K) \subset \pi(\mathcal{J}\tau) = \pi(\tau) = TY$, we see that generalized complex branes are always coisotropic submanifolds for $\pi_J$.

We can make things more concrete by choosing an isotropic splitting $s : TM \to \mathbb{T}M$, with corresponding three-form[12] $H$. This induces an isotropic splitting of (1.51) with three-form $i^*H$, where $i : Y \hookrightarrow M$ is the inclusion. In particular, a trivialization along $Y$ exists if and only if $i^*H$ is exact. In the given splitting, $K^\perp = TY \oplus T^*M|_Y$, and any trivialization $\tau$ can be written as

$$\tau = \tau(B) := \{X + \xi \in TY \oplus T^*M|_Y \mid \xi|_{TY} = -\iota_X B\} \tag{1.52}$$

for a uniquely determined[13] $B \in \Omega^2(Y)$. Then $\tau(B)/K$ is involutive if and only if $i^*H = -dB$. Note that any symmetry $e_*^{\widetilde{B}}$ of $\mathbb{T}M$ maps a generalized complex brane $\tau$ for $(M, H, \mathcal{J})$ to $e_*^{\widetilde{B}}\tau$, which is a generalized complex brane for $(M, H - d\widetilde{B}, e_*^{\widetilde{B}}\mathcal{J}e_*^{-\widetilde{B}})$. Moreover, we have $e_*^{\widetilde{B}}\tau(B) = \tau(B + i^*\widetilde{B})$.

[12]Recall that this is defined by $H(X, Y, Z) = -2\langle[\![s(X), s(Y)]\!], s(Z)\rangle$.

[13]The proof of this fact is identical to the one we gave for writing a Dirac structure $L$ in terms of a subspace $E \subset TM$ and a two-form $\varepsilon \in \Lambda^2 E^*$.

**Example 1.3.32.** Let $(M, I)$ be a complex manifold and $H$ a closed three-form of type $(2,1)+(1,2)$, considered as a generalized complex manifold (see Example 1.3.5). Let $Y$ be a real submanifold and $B$ a two-form on $Y$. From (1.52) we see that $\tau(B)$ is a generalized complex brane if and only if $i^*H = -dB$, $Y$ is a complex submanifold, and $B$ is of type $(1,1)$. In particular, putting $H = 0$, we see that generalized complex branes on complex manifolds correspond to complex submanifolds equipped with closed $(1,1)$-forms.

Before giving more examples we explain how to take the preimage of a generalized complex brane under a generalized holomorphic map. Let $\Phi = (\varphi, B) : (M_1, H_1, \mathcal{J}_1) \to (M_2, H_2, \mathcal{J}_2)$ be a generalized holomorphic map and $(Y_2, \tau_2)$ a generalized complex brane in $(M_2, H_2, \mathcal{J}_2)$. Suppose that $Y_1 = \varphi^{-1}(Y_2)$ is a submanifold of $M_1$ for which $\varphi_* : NY_1 \to \varphi^* NY_2$ is injective. This is the case for instance if $\varphi$ is transverse to $Y_2$, but the application we have in mind is when $Y_1$ is the exceptional divisor of the blow-up of $Y_2$ in $M$, and then the blow-down map is never transverse to $Y_2$. Define

$$\tau_1 := \{u \in \mathbb{T}M_1|_{Y_1} \mid u \sim_\Phi v \text{ for some } v \in \tau_2\}.$$

**Lemma 1.3.33.** The pair $(Y_1, \tau_1)$ forms a generalized complex brane in $(M_1, H_1, \mathcal{J}_1)$.

*Proof.* It is clear that for each $y \in Y_1$ the subspace $(\tau_1)_y \subset \mathbb{T}_y M_1$ is obtained from $(\tau_2)_{\varphi(y)}$ by the backward image construction (see Section 1.2.3). In particular, it is a maximal isotropic subspace of $\mathbb{T}_y M_1$. Since $N^*Y_2 \subset \tau_{Y_2}$ and $\varphi^* : \varphi^* N^*Y_2 \to N^*Y_1$ is surjective, it follows that $N^*Y_1 \subset \tau_1$ and therefore $\tau_1 \subset (N^*Y_1)^\perp$. The analogue of the short exact sequence (1.34) for $\tau_1$ reads

$$0 \to \ker(\varphi^*) \cap \varphi^*\tau_2 \to \Gamma_\Phi|_{Y_1} \cap (\mathbb{T}M_1|_{Y_1} \oplus \varphi^*\tau_2) \to \tau_1 \to 0.$$

Now $\ker(\varphi^*) \cap \varphi^*\tau_2 = \ker(\varphi^*) \cap \varphi^* N^*Y_2$, which is of constant rank since $\varphi^* : \varphi^* N^*Y_2 \to N^*Y_1$ is surjective. Consequently, $\tau_1$ is smooth. By arguments similar to those used in Lemma 1.2.22 it follows that $\tau_1/N^*Y_1$ is involutive (alternatively, use Remark 1.3.34 below) and since $\Phi$ is generalized holomorphic it follows that $\mathcal{J}_1\tau_1 = \tau_1$. Hence, $(Y_1, \tau_1)$ is a generalized complex brane. □

*Remark* 1.3.34. If $\tau_2 = \tau(B_2)$ for $B_2 \in \Omega^2(Y_2)$, then $\tau_1 = \tau(B_1)$, where $B_1 := \varphi^* B_2 + i_1^* B$ and $i_1 : Y_1 \hookrightarrow M_1$ denotes the inclusion.

*Remark* 1.3.35. Note that the analogous statement for generalized complex submanifolds is not true, i.e. the pre-image of a generalized complex submanifold by a generalized holomorphic map need not be generalized complex again (see Remark 3.1.15).

**Example 1.3.36.** Let $(M, \omega)$ be a symplectic manifold endowed with the zero three-form[14], considered as a generalized complex manifold (Example 1.3.6). Let us see when $(Y, \tau(B))$ is a generalized complex brane. If $\widetilde{B} \in \Omega^2(M)$ denotes a local extension of $B$ we can write $\tau(B) = e_*^{\widetilde{B}}(TY) \oplus N^*Y$. For $\xi \in N^*Y$ we have

$$\mathcal{J}(\xi) = -\omega^{-1}(\xi) \in \tau(B) \text{ if and only if } \omega^{-1}(\xi) \in TY \text{ and } \widetilde{B}(\omega^{-1}(\xi)) \in N^*Y.$$

[14] If $H$ is nonzero we can always apply a symmetry to reduce to this case.

In other words, $Y$ has to be coisotropic and $\iota_X B = 0$ for all $X \in TY^\omega := \omega^{-1}(N^*Y) \subset TY$. The subbundle $TY^\omega \subset TY$ is involutive and the associated foliation is called the *characteristic foliation* of the coisotropic submanifold $Y$. The condition on $B$ implies that it descends to a two-form on the normal bundle $TY/TY^\omega$ of this foliation. Continuing, for $X \in TY$ we have

$$\mathcal{J}(e_*^{\widetilde{B}} X) = \mathcal{J}(X - \widetilde{B}(X)) = \omega(X) + \omega^{-1}\widetilde{B}(X),$$

which lies in $\tau(B)$ if and only if

$$\omega^{-1}\widetilde{B}(X) \in TY \text{ and } \omega(X) + \widetilde{B}\omega^{-1}\widetilde{B}(X) \in N^*Y.$$

The first condition amounts to $\omega^{-1}\widetilde{B}$ mapping $TY$ to itself and is dual to the condition that $\widetilde{B}\omega^{-1}$ maps $N^*Y$ to itself, which is precisely the condition we already imposed on $B$. The second can be reformulated by saying that $1 + (\omega^{-1}\widetilde{B})^2$ maps $TY$ to $TY^\omega$. Note that $\omega^{-1}\widetilde{B} : TY \to TY$ preserves the subspace $TY^\omega$, so it induces a map from $TY/TY^\omega$ to itself. The last condition is then equivalent to this map being a complex structure. Now $\omega$ induces a symplectic form on $TY/TY^\omega$ while $B$ induces a two-form on $TY/TY^\omega$, and we have $\omega^{-1}\widetilde{B} = \omega^{-1}B$ on this space. Note that the fact that $\omega^{-1}B$ is complex implies that $B + i\omega$ is non-degenerate of type $(2,0)$ on[15] $TY/TY^\omega$.

Summarizing, $(Y, \tau(B))$ is a generalized complex brane if and only if $Y$ is coisotropic, $B$ annihilates vectors tangent to the characteristic foliation $TY^\omega$, and $\omega^{-1}B$ induces a complex structure on the normal bundle $TY/TY^\omega$. Since $Y$ is coisotropic we have $\dim_{\mathbb{R}}(Y) \geq n = \frac{1}{2}\dim_{\mathbb{R}}(M)$, with equality if and only if $Y$ is Lagrangian. In that case $B$ is forced to be zero, so Lagrangians are examples of generalized complex branes. In general, since $TY/TY^\omega$ carries a non-degenerate complex two-form it has even complex dimension, so $\dim_{\mathbb{R}}(Y) = \mathrm{rank}_{\mathbb{R}}(TY^\omega) + 4k$ for some $k \in \mathbb{Z}_{\geq 0}$. Since $\mathrm{rank}_{\mathbb{R}}(TY^\omega) = \mathrm{codim}_{\mathbb{R}}(Y) = 2n - \dim_{\mathbb{R}}(Y)$, we have $\dim_{\mathbb{R}}(Y) = n + 2k$. These higher dimensional generalized complex branes first appeared in the work of Kapustin and Orlov [31].

One way to obtain these kind of branes would be by combining Example 1.3.32 and Lemma 1.3.33: if $(\varphi, B) : (M, \mathcal{J}_\omega) \to (N, \mathcal{J}_I)$ is a generalized holomorphic map as in Example 1.3.13, then any regular fiber $Y \subset M$, together with the restriction of $B$ to $Y$, forms a generalized complex brane in $(M, \mathcal{J}_\omega)$.

**Example 1.3.37.** Let $(M, I, \sigma)$ be a holomorphic Poisson manifold, considered as generalized complex manifold with respect to the zero three-form (Example 1.3.7). If $Y \subset M$ is a complex submanifold which is coisotropic for $\sigma$, then $TY \oplus N^*Y = \tau(0)$ equips $Y$ with the structure of a generalized complex brane. The particular case of $Y = M$ is usually called a *space filling brane*, because it is supported on the entire manifold.

We have talked about generalized complex submanifolds and about generalized complex branes, so at this point it seems natural to ask what the intersection is between these two definitions.

[15] In general, if $\alpha, \beta$ are non-degenerate two-forms on a vector space $V$ such that $\alpha^{-1}\beta$ is a complex structure, then $\beta + i\alpha$ is a non-degenerate $(2,0)$-form on $V$.

**Lemma 1.3.38.** Let $Y$ be a submanifold of a generalized complex manifold $(M, H, \mathcal{J})$. Then $Y$ is both a generalized complex submanifold and a generalized complex brane if and only if $Y$ is a generalized Poisson submanifold for which the induced generalized complex structure $\mathcal{J}_Y$ is $B$-field equivalent to a holomorphic Poisson structure (with zero three-form).

*Proof.* We continue to abbreviate $K = N^*Y$ and $K^\perp = \pi^{-1}(TY)$. If $Y$ is a brane, then $\mathcal{J}K \subset \tau \subset K^\perp$, so that $\mathcal{J}K \cap K^\perp = \mathcal{J}K$. Condition (1.48) for $Y$ being a generalized complex submanifold then reduces to $\mathcal{J}K \subset K$, which is equivalent to the generalized Poisson submanifold condition $\mathcal{J}K = K$. In that case, the induced structure $\mathcal{J}_Y$ on $\mathbb{T}Y$ can be described explicitly via the isomorphism $\mathbb{T}Y = K^\perp/K$. A generalized complex brane in $Y$ is nothing but an isotropic, involutive subbundle $\tau/K \subset K^\perp/K$, such that $\mathcal{J}\tau = \tau$. Since $\mathcal{J}K = K$, we then also get $\mathcal{J}(\tau/K) = \tau/K$. We can view $\tau/K$ as giving a splitting $K^\perp/K = TY \oplus T^*Y$ with zero three-form, with the property that $\mathcal{J}(TY) = TY$. By Example 1.3.7, this must be a holomorphic Poisson structure. □

*Remark* 1.3.39. Applying this lemma to the case of a point (which is always a generalized complex submanifold), we see that a point in $(M, H, \mathcal{J})$ supports the structure of a brane if and only if the point lies in the complex locus. Similarly, since the entire manifold is always a generalized Poisson submanifold, it carries the structure of a brane if and only if $\mathcal{J}$ is equivalent to a holomorphic Poisson structure. In other words, space filling branes are given by holomorphic Poisson manifolds.

### 1.3.2 Generalized Hermitian structures

Given a complex manifold $(M, I)$ it is often useful to choose a Hermitan metric, i.e. a metric $g$ for which $I$ is orthogonal. This induces a non-degenerate two-form $\omega := gI$, which need not be closed in general. Moreover, the space of all metrics that are compatible with $I$ forms a closed convex cone in the space of all metrics, hence is contractible.

Similarly, if $(M, \omega)$ is symplectic one can pick a compatible almost complex structure $I$, i.e. one for which $g := -\omega I$ defines a metric. Again, the auxiliary structure $I$ is not necessarily integrable. It is a well-known fact that the space of compatible almost complex structures is non-empty and contractible.

A similar construction is possible on a generalized complex manifold $(M, \mathcal{J})$, using the notion of a generalized metric (see Section 1.1.4)

**Definition 1.3.40.** A generalized metric $\mathcal{G}$ is *compatible* with $\mathcal{J}$ if $\mathcal{G}\mathcal{J} = \mathcal{J}\mathcal{G}$. The pair $(\mathcal{G}, \mathcal{J})$ is then called a *generalized Hermitian structure*.

It is not immediately clear from this definition that a compatible generalized metric $\mathcal{G}$ exists.

**Lemma 1.3.41.** Let $(M, \mathcal{J})$ be an almost generalized complex manifold. Then there is a generalized metric $\mathcal{G}$ compatible with $\mathcal{J}$, and the space of all compatible generalized metrics is contractible.

*Proof.* Since $(\mathbb{T}M, \mathcal{J})$ is a complex vector bundle in the ordinary sense, we know that the space of associated Hermitian metrics for it is non-empty and contractible. If $(\cdot,\cdot)$ is one, we have $\langle u, v\rangle = (\tilde{\mathcal{G}}u, v)$ for some isomorphism $\tilde{\mathcal{G}}$ of $\mathbb{T}M$. Since both pairings are symmetric and compatible with $\mathcal{J}$, it follows that $\tilde{\mathcal{G}}$ is symmetric with respect to $(\cdot,\cdot)$ and commutes with $\mathcal{J}$. Consequently, $\tilde{\mathcal{G}}^2$ is positive and has a positive square root, which we denote by $|\tilde{\mathcal{G}}|$, and we define $\mathcal{G} := |\tilde{\mathcal{G}}|^{-1}\tilde{\mathcal{G}}$. It commutes with $\mathcal{J}$ and satisfies $\mathcal{G}^2 = 1$. Moreover, since $\langle \mathcal{G}u, v\rangle = (|\tilde{\mathcal{G}}|u, v)$ is positive definite and symmetric, it follows that $\mathcal{G}$ is a generalized metric. The above construction realizes the space of compatible generalized metrics as a retract of the contractible space of Hermitian metrics, and as such is itself contractible. □

Recall that $\mathcal{G}$ induces a splitting $\mathbb{T}M = V_+ \oplus V_-$ into $(\pm 1)$-eigenspaces, where $V_\pm$ is the graph of $\pm g + b$ for some metric $g$ and two-form $b$. Since $\mathcal{G}$ commutes with $\mathcal{J}$, we get induced complex structures on $V_\pm$. Using the isomorphism $\pi : V_\pm \to TM$ we can transfer these to $TM$, giving two almost complex structures $I_\pm$ on $M$ that are compatible with $g$. Thus $\mathcal{J}$, as an almost generalized complex structure, is completely determined by the data $(g, b, I_+, I_-)$, and conversely any such tuple $(g, b, I_+, I_-)$ gives rise to an almost generalized complex structure. This correspondence is not one-to-one, as there is a whole contractible space of tuples giving rise to the same $\mathcal{J}$. It is therefore of limited use for studying $\mathcal{J}$, but it will play a prominent role later in the context of generalized Kähler geometry. One thing that is trivial to see in this picture, but which is not immediately obvious from the definition alone, is the following.

**Lemma 1.3.42.** There exists an almost generalized complex structure on $M$ if and only if there exists an ordinary almost complex structure on $M$.

Hence the topological obstructions for having a generalized complex structure are the same as those for having an ordinary complex structure.

In Section 1.2.1 we saw that, given a generalized metric $\mathcal{G}$ on $M$, there is a one-to-one correspondence between complex almost Dirac structures on $M$ and complex structures on the bundle $\mathbb{T}M$ that are compatible with the metric $\mathbb{G}$ that is induced by $\mathcal{G}$. To see what this gives if the Dirac structure $L$ corresponds to a generalized complex structure $\mathcal{J}$, we choose $\mathcal{G}$ so that $(\mathcal{G}, \mathcal{J})$ is Hermitian. It is not difficult to see from (1.27) that the complex structure on $\mathbb{T}M$ that induces $L$ is precisely $\mathcal{J}$ itself. From Corollary 1.2.10 we deduce the following.

**Lemma 1.3.43.** Let $(\mathcal{G}, \mathcal{J})$ be a generalized Hermitian structure and $(g, b, I_\pm)$ the associated bi-Hermitian structure on $M$. If $\dim(M) = 2n$, then $\text{type}(\mathcal{J}) = n \bmod 2$ if and only if $I_+$ and $I_-$ induce the same orientation on $M$.

*Proof.* By Corollary 1.2.10 we know that the orientation induced by $\mathcal{J}$ agrees with the canonical orientation on $\mathbb{T}M$ if and only if $\text{type}(\mathcal{J}) \equiv n \bmod 2$. Moreover, if we choose any orientation on $TM$ and transport it to $V_\pm$ via the isomorphisms $\pi : V_\pm \to TM$, then the canonical orientation on $\mathbb{T}M = V_+ \oplus V_-$ coincides with the product orientation (in general they differ by the sign $(-1)^{\dim(M)}$). Since $\mathcal{J}$ preserves this decomposition, its induced orientation equals the product of the induced orientations on $V_+$ and $V_-$. Since these are precisely the orientations induced by $I_+$ and $I_-$, the lemma follows. □

The study of generalized complex structures via pairs of Hermitian structures $(g, I_+, I_-)$ is made difficult by the fact that the two complex structures do not commute in general. We will see in the next section that the commutator $[I_+, I_-]$ plays an important role in generalized Kähler geometry, for now we give an elementary but useful lemma regarding this commutator.

**Lemma 1.3.44.** $\ker([I_+, I_-]) = \ker(I_+ + I_-) \oplus \ker(I_+ - I_-)$.

*Proof.* Clearly $\ker(I_+ + I_-) \cap \ker(I_+ - I_-) = 0$, and using

$$[I_+, I_-] = (I_+ - I_-)(I_+ + I_-) = -(I_+ + I_-)(I_+ - I_-)$$

we see that $\ker([I_+, I_-]) \supset \ker(I_+ + I_-) \oplus \ker(I_+ - I_-)$. As both $I_\pm$ anti-commute with $[I_+, I_-]$, they preserve $\ker([I_+, I_-])$. On this subspace $I_+$ and $I_-$ commute with each other, so they admit a simultaneous eigenspace decomposition, all of whose eigenvalues are $\pm i$. The result follows. □

## 1.4 Generalized Kähler geometry

Generalized complex geometry provides a framework that incorporates both complex and symplectic geometry. On a Kähler manifold we have both a complex and a symplectic structure which are compatible with each other. Here is the generalized version.

**Definition 1.4.1.** A *generalized Kähler structure* on $M$ is a pair of commuting generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ such that $\mathcal{G} := -\mathcal{J}_1\mathcal{J}_2$ defines a generalized metric.

Put differently, a generalized Kähler structure is a generalized Hermitian structure $(\mathcal{G}, \mathcal{J}_1)$ for which the induced almost generalized complex structure $\mathcal{J}_2 = \mathcal{G}\mathcal{J}_1$ is integrable as well.

**Example 1.4.2.** The natural example is given by an ordinary Kähler manifold $(M, I, \omega)$. Define $\mathcal{J}_1 := \mathcal{J}_I$ and $\mathcal{J}_2 := -\mathcal{J}_\omega$, which commute because $I$ is compatible with $\omega$, and

$$\mathcal{G} = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}$$

is indeed positive, where $g := -\omega I$ is the associated Kähler metric.

As discussed in Section 1.3.2, the Hermitian structure $(\mathcal{G}, \mathcal{J}_1)$ is equivalently described by a tuple $(g, b, I_+, I_-)$, where $\text{graph}(\pm g + b) = V_\pm$ equals the $(\pm 1)$-eigenspace of $\mathcal{G}$, and $I_\pm$ are the almost complex structure on $TM$ that correspond to $\mathcal{J}_1|_{V_\pm}$. As $\mathcal{J}_2$ equals $\pm\mathcal{J}_1$ on $V_\pm$, we obtain

$$L_1 = V_+^{1,0} \oplus V_-^{1,0}, \qquad L_2 = V_+^{1,0} \oplus V_-^{0,1}. \tag{1.53}$$

Here $(V_\pm)_\mathbb{C} = V_\pm^{1,0} \oplus V_\pm^{0,1}$ is the decomposition associated to $\mathcal{J}_1|_{V_\pm}$. Since now both $\mathcal{J}_1$ and $\mathcal{J}_2$ are part of the data, there is no ambiguity in the tuple $(g, b, I_\pm)$ anymore. We will refer to such a tuple as an almost[16] *bi-Hermitian structure*. The above construction can be reversed, giving a bijection between almost bi-Hermitian structures $(g, b, I_+, I_-)$ and almost generalized Kähler structures $(\mathcal{J}_1, \mathcal{J}_2)$. As explained in Section 1.1.4 there is a unique splitting of $\mathbb{T}M$, the metric splitting, in which $b = 0$. It can be obtained from the given splitting by applying $e_*^b$, which transforms the data $(g, b, I_\pm, H)$ to $(g, 0, I_\pm, H - db)$. In the sequel we will always position ourselves in this metric splitting, and refer to $(g, I_\pm, H)$ as the bi-Hermitian structure associated to $(\mathcal{G}, \mathcal{J}_1, \mathcal{J}_2)$.

**Proposition 1.4.3** ([23, Theorem 6.28]). *Let $(\mathcal{J}_1, \mathcal{J}_2)$ be an almost generalized Kähler structure and $(g, I_+, I_-, H)$ the associated almost bi-Hermitian structure. Then the following are equivalent:*

*i)* $(\mathcal{J}_1, \mathcal{J}_2)$ *is generalized Kähler.*

*ii)* $V_\pm^{1,0}$ *are both involutive.*

*iii)* $I_\pm$ *are both integrable complex structures and* $\pm d_\pm^c \omega_\pm = H$, *where*[17] $\omega_\pm = gI_\pm$ *and* $d_\pm^c = i(\bar{\partial}_\pm - \partial_\pm)$.

*iv)* $I_\pm$ *are both integrable and* $\nabla^\pm I_\pm = 0$, *where* $\nabla^\pm := \nabla \mp \frac{1}{2} g^{-1} H$ *and* $\nabla$ *is the Levi-Cevita connection associated to g.*

*Proof.* i) ⇔ ii): From (1.53) we see that $L_1$ and $L_2$ are involutive if and only if both $V_\pm^{1,0}$ are involutive and[18]

$$[\![V_+^{1,0}, V_-^{1,0}]\!] \subset V_+^{1,0} \oplus V_-^{1,0}, \qquad [\![V_+^{1,0}, V_-^{0,1}]\!] \subset V_+^{1,0} \oplus V_-^{0,1}. \tag{1.54}$$

However, (1.54) is in fact a consequence of the involutivity of $V_\pm^{1,0}$. For instance, to verify the first inclusion it suffices, since $V_+^{1,0} \oplus V_-^{1,0}$ is Lagrangian, to check that $\langle [\![u, v]\!], w\rangle = 0$ for all $u \in V_+^{1,0}$, $v \in V_-^{1,0}$ and $w \in V_\pm^{1,0}$. Using Lemma 1.1.1, we compute

$$\langle [\![u, v]\!], w\rangle = \pi(u) \cdot \langle v, w\rangle - \langle v, [\![u, w]\!]\rangle = -\pi(v) \cdot \langle u, w\rangle + \langle u, [\![v, w]\!]\rangle.$$

If $w \in V_+^{1,0}$ then this vanishes by the first equality, while for $w \in V_-^{1,0}$ this follows from the second equality. Hence, $\mathcal{J}_1$ and $\mathcal{J}_2$ are integrable if and only if $V_\pm^{1,0}$ are both involutive.
ii)⇔ iii): By definition we have

$$V_\pm^{1,0} = \{X \mp i\omega_\pm(X) |\ X \in T_\pm^{1,0} M\} = e_*^{\pm i\omega_\pm}(T_\pm^{1,0} M),$$

where $T_\pm^{1,0} M$ denotes the $(+i)$-eigenbundle of $I_\pm$. Using (1.9) we compute

$$[\![e_*^{\pm i\omega_\pm}(X), e_*^{\pm i\omega_\pm}(Y)]\!]_H = e_*^{\pm i\omega_\pm}([X, Y]) - \iota_Y \iota_X (H \pm i d\omega_\pm).$$

[16] The adjective "almost" refers to a structure without assuming any integrability conditions. The appropriate integrability conditions in this case are given by Proposition 1.4.3 *ii*).

[17] Here and in the rest of this text, an equation of the form $\pm d_\pm^c \omega_\pm = H$ refers to two separate equations, one for which the signs are the overlying ones, and the other for which the signs are the underlying ones.

[18] For notational convenience we identify below all bundles with their spaces of sections.

This lies in $V_{\pm}^{1,0}$ if and only if $I_{\pm}$ are integrable and $\iota_Y \iota_X(H \pm i d\omega_{\pm}) = 0$ for all $X, Y \in T_{\pm}^{1,0}M$. This last condition can be rewritten as $H^{(3,0)+(2,1)} = \mp i\partial_{\pm}\omega_{\pm}$, where the type decomposition on the left is taken with respect to $I_{\pm}$. Since $H$ is real, this is equivalent to $H = \pm d^c_{\pm}\omega_{\pm}$.

ii)$\Leftrightarrow$ iv): We will show that $V_+^{1,0}$ is involutive if and only if $I_+$ is integrable and $\nabla^+ I_+ = 0$, the case of $V_-^{1,0}$ being similar. Since $(V_+^{1,0})^{\perp} = V_+^{1,0} \oplus (V_-)_{\mathbb{C}}$, involutivity of $V_+^{1,0}$ is equivalent to $\langle [\![V_+^{1,0}, V_+^{1,0}]\!], V_+^{1,0} \oplus (V_-)_{\mathbb{C}}\rangle = 0$. Now for $u, v \in V_+^{1,0}$ and $w \in (V_-)_{\mathbb{C}}$ we have

$$\langle [\![u, v]\!], w\rangle = -\langle v, [\![u, w]\!]\rangle = \langle v, \pi^+[\![w, u]\!]\rangle.$$

This vanishes if and only if $\pi^+[\![w, u]\!] \in V_+^{1,0}$, which by Lemma 1.4.4 below is equivalent to $\nabla^+ I_+ = 0$. In that case $[\![V_+^{1,0}, V_+^{1,0}]\!] \subset (V_+)_{\mathbb{C}}$. Since $\pi[\![u, v]\!] = [\pi(u), \pi(v)]$, we have $[\![V_+^{1,0}, V_+^{1,0}]\!] \subset V_+^{1,0}$ if and only if $I_+$ is integrable. □

**Lemma 1.4.4.** For $X, Y \in \Gamma(TM)$ we have

$$[\![X - gX, Y + gY]\!] = \big(\nabla^+_X Y + g\nabla^+_X Y\big) + \big(\nabla^-_Y X - g\nabla^-_Y X\big). \tag{1.55}$$

*Proof.* We will show that $\widetilde{\nabla}^+_X Y := \pi\pi^+[\![X - gX, Y + gY]\!]$ coincides with $\nabla^+_X Y$, the case of $\nabla^-$ being similar. Here $\pi^+$ denotes the projection to $V_+$ and $\pi$ the projection to $TM$. Now $\nabla^+$ is characterized as the unique connection that preserves $g$ and has skew-symmetric torsion given by $-g^{-1}H$, so we need to show the same for $\widetilde{\nabla}^+$. By Lemma 1.1.1 we see that

$$\widetilde{\nabla}^+_{fX} Y = f\widetilde{\nabla}^+_X Y, \qquad \widetilde{\nabla}^+_X (fY) = f\widetilde{\nabla}^+_X Y + X(f)Y,$$

so $\widetilde{\nabla}^+$ indeed defines a connection on $TM$. Since

$$\begin{aligned} g(\widetilde{\nabla}^+_X Y, Z) + g(Y, \widetilde{\nabla}^+_X Z) =& \langle \widetilde{\nabla}^+_X Y + g\widetilde{\nabla}^+_X Y, Z + gZ\rangle + \langle Y + gY, \widetilde{\nabla}^+_X Z + g\widetilde{\nabla}^+_X Z\rangle \\ =& \langle [\![X - gX, Y + gY]\!], Z + gZ\rangle \\ &+ \langle Y + gY, [\![X - gX, Z + gZ]\!]\rangle \\ =& X \cdot \langle Y + gY, Z + gZ\rangle \\ =& X \cdot g(Y, Z), \end{aligned}$$

we see that $\widetilde{\nabla}^+$ preserves $g$. Finally, the torsion of $\widetilde{\nabla}^+$ is given by

$$\begin{aligned} g(\widetilde{\nabla}^+_X Y - \widetilde{\nabla}^+_Y X - [X, Y], Z) =& \langle [\![X - gX, Y + gY]\!] - [\![Y - gY, X + gX]\!] \\ &- 2[X, Y], Z + gZ\rangle \\ =& - H(X, Y, Z). \end{aligned}$$

Hence $\widetilde{\nabla}^+ = \nabla^+$. □

It is useful to have an explicit relation between $(\mathcal{J}_1, \mathcal{J}_2)$ and $(g, I_\pm)$. From the decomposition

$$X + \xi = \frac{1}{2}\big((X + g^{-1}\xi) + g(X + g^{-1}\xi)\big) + \frac{1}{2}\big((X - g^{-1}\xi) - g(X - g^{-1}\xi)\big)$$

and the definition of $I_+$ and $I_-$ we deduce that

$$\begin{aligned}\mathcal{J}_1(X + \xi) =& \frac{1}{2}\big(I_+(X + g^{-1}\xi) + gI_+(X + g^{-1}\xi)\big)\\ &+ \frac{1}{2}\big(I_-(X - g^{-1}\xi) - gI_-(X - g^{-1}\xi)\big),\end{aligned}$$

and similarly for $\mathcal{J}_2$ with $I_-$ replaced by $-I_-$. Consequently,

$$\mathcal{J}_1 = \frac{1}{2}\begin{pmatrix} I_+ + I_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(I_+^* + I_-^*) \end{pmatrix}, \qquad \mathcal{J}_2 = \frac{1}{2}\begin{pmatrix} I_+ - I_- & -(\omega_+^{-1} + \omega_-^{-1}) \\ \omega_+ + \omega_- & -(I_+^* - I_-^*) \end{pmatrix}. \tag{1.56}$$

From this we see that

$$\pi_{\mathcal{J}_1} = -\frac{1}{2}(\omega_+^{-1} - \omega_-^{-1}), \qquad \pi_{\mathcal{J}_2} = -\frac{1}{2}(\omega_+^{-1} + \omega_-^{-1}). \tag{1.57}$$

In particular, $\pi_{\mathcal{J}_1} + \pi_{\mathcal{J}_2} = -\omega_+^{-1}$ is invertible, hence $TM = \mathrm{Im}(\pi_{\mathcal{J}_1}) + \mathrm{Im}(\pi_{\mathcal{J}_2})$. So the symplectic leaves of $\pi_{\mathcal{J}_1}$ and $\pi_{\mathcal{J}_2}$ are transverse to each other, and consequently

$$\begin{aligned}\mathrm{type}(\mathcal{J}_1) + \mathrm{type}(\mathcal{J}_2) &= \frac{1}{2}\mathrm{codim}\big(\mathrm{Im}(\pi_{\mathcal{J}_1})\big) + \frac{1}{2}\mathrm{codim}\big(\mathrm{Im}(\pi_{\mathcal{J}_2})\big)\\ &= \frac{1}{2}\mathrm{codim}\big(\mathrm{Im}(\pi_{\mathcal{J}_1}) \cap \mathrm{Im}(\pi_{\mathcal{J}_2})\big) \le n,\end{aligned}$$

where $2n = \dim(M)$. From Lemma 1.3.43 we see that $\mathrm{type}(\mathcal{J}_1) = n$ mod 2 if and only if $I_+$ and $I_-$ induce the same orientation, while $\mathrm{type}(\mathcal{J}_2) = n$ mod 2 if and only if $I_+$ and $-I_-$ induce the same orientations. For example, in four dimensions this leaves three possibilities. If $I_+$ and $I_-$ induce opposite orientations then both $\mathcal{J}_1$ and $\mathcal{J}_2$ are everywhere of type 1. If $I_+$ and $I_-$ induce the same orientation then both $\mathcal{J}_1$ and $\mathcal{J}_2$ are of even type, so either they are both generically of type 0 with disjoint type change loci[19], or one is symplectic and the other is complex (with no type change).

**Example 1.4.5.** Let $(M, g, I, J, K)$ be a hyper-Kähler manifold, i.e. $I, J, K$ are integrable complex structures compatible with $g$, such that $IJ = -JI = K$ and $d\omega_I = d\omega_J = d\omega_K = 0$. Here $\omega_I, \omega_J$ and $\omega_K$ are the two-forms associated to the Hermitian structures $(g, I)$, $(g, J)$ and $(g, K)$ respectively. Then $(g, I, J)$ defines a bi-Hermitian structure (with respect to the zero three-form), hence a generalized Kähler structure on $M$. Since $(I - J)(I + J) = 2K$, both $I \pm J$ are invertible and therefore so are $\omega_I^{-1} \pm \omega_J^{-1}$. From (1.57) we deduce that both $\mathcal{J}_1$ and $\mathcal{J}_2$ are everywhere of type 0 .

[19] The type change locus of a generalized complex structure is the subset of points where the type does not assume its global minimal value. If the manifold is connected then this is a nowhere dense set, by Theorem 1.3.16.

**Example 1.4.6.** Here we present an example that we will more thoroughly investigate in Section 4.4, where we will also give proofs of the following statements. Let $G$ be a compact, connected, even dimensional Lie group, equipped with a bi-invariant metric $\langle\cdot,\cdot\rangle$. There exists a complex structure $I$ on the Lie algebra $\mathfrak{g}$ which is compatible with $\langle\cdot,\cdot\rangle$, and whose $(+i)$-eigenspace $\mathfrak{g}^{1,0}\subset\mathfrak{g}_{\mathbb{C}}$ is closed under the Lie bracket. It follows that its left and right invariant extensions over $G$ give integrable complex structures $I_+$ and $I_-$. There are two connections $\nabla^+$ and $\nabla^-$ on $G$, characterized by the property that left-, respectively, right-invariant vector fields are parallel, and we have $\nabla^{\pm}I_{\pm}=0$. The torsion of $\nabla^{\pm}$ is given by the Cartan three-form on $G$, which is a closed, bi-invariant three-form $H$ that on $\mathfrak{g}$ is given by $H(\xi,\eta,\zeta)=\langle[\xi,\eta],\zeta\rangle$. By Proposition 1.4.3 it follows that the tuple $(\langle\cdot,\cdot\rangle,I_{\pm},H)$ defines a generalized Kähler structure on $G$. Furthermore, if $T$ is a maximal torus in $G$, then the above generalized Kähler structure can be constructed in such a way that $T$ acts by symmetries of the structure. Since $I_+=I_-$ at the identity of $G$, $I_+=I_-$ at all points of $T$, which by (1.57) implies that $T$ is a submanifold of the complex locus. In fact, $T$ turns out to be the connected component of the complex locus containing the identity, hence $T$ is a generalized Poisson submanifold with respect to $\mathcal{J}_1$.

*Remark* 1.4.7. If $(\mathcal{J}_1,\mathcal{J}_2)$ is generalized Kähler then so is $(\mathcal{J}_2,\mathcal{J}_1)$, with the same generalized metric $\mathcal{G}$. So when considering e.g. a generalized Poisson submanifold for one of the two structures, we may as well assume this to be $\mathcal{J}_1$.

The difficult feature of bi-Hermitian geometry lies in the fact that $I_+$ and $I_-$ do not commute in general. Therefore, standard techniques in complex geometry such as the decomposition of forms into types, become difficult as they can be performed only for one of the two complex structures at a time. This failure of commutativity suggests that important information about the generalized Kähler structure is contained in the tensor

$$Q:=-\frac{1}{2}[I_+,I_-]g^{-1}:T^*M\to TM. \tag{1.58}$$

A quick calculation shows that $Q$ is skew-symmetric so we can regard it as a bivector, and it was observed in [2] in the four-dimensional case and in [29] in the general case, that $Q$ is Poisson. In fact, it turns out to be the real part of two holomorphic Poisson structures

$$\sigma_{\pm}:=Q-iI_{\pm}Q. \tag{1.59}$$

One can prove this directly in local coordinates using the integrability conditions (see [29]), or in the following more abstract way (see [25]). In the proof of Lemma 1.3.4 we saw that

$$L\boxtimes\bar{L}^T=\{-\frac{i}{2}\pi_{\mathcal{J}}(\xi)+\xi|\,\xi\in T^*M_{\mathbb{C}}\}=e_*^{-\frac{i}{2}\pi_{\mathcal{J}}}(T^*M_{\mathbb{C}}).$$

This allowed us to conclude that $\pi_{\mathcal{J}}$ is integrable. In a similar spirit we have the following proposition, which follows from the results in [25].

**Proposition 1.4.8.** *Let $(\mathcal{J}_1,\mathcal{J}_2)$ be a generalized Kähler structure with associated Dirac structures $L_1$ and $L_2$, and let $(g,I_{\pm},H)$ be the corresponding bi-Hermitian structure.*

*Then*

$$\bar{L}_1^T \boxtimes \overline{L}_2 = L_{(I_+, -\frac{1}{8}\sigma_+)}, \qquad \bar{L}_1^T \boxtimes L_2 = L_{(I_-, -\frac{1}{8}\sigma_-)},$$

*where $L_{(I_\pm, -\frac{1}{8}\sigma_\pm)}$ was defined in Example 1.3.7 and the Baer sum was introduced in Section 1.2.3. In particular, $\sigma_\pm$ are both holomorphic Poisson.*

*Proof.* We will show that $L_{(I_+, -\frac{1}{8}\sigma_+)} \subset \bar{L}_1^T \boxtimes \overline{L}_2$; equality then follows from dimensional reasons, and the case of $\sigma_-$ is similar. The fact that $\sigma_\pm$ are holomorphic Poisson then follows from Example 1.3.7. We have

$$L_{(I_+, -\frac{1}{8}\sigma_+)} = \{X + \sigma_+(\xi) - 8\xi |\ X \in T_+^{0,1}M,\ \xi \in T_+^{*1,0}M\},$$

where $T_+^{1,0}M$ denotes $(+i)$-eigenspace for $I_+$. For $X \in T_+^{0,1}M$ we write $X = X - g(X) + g(X)$, and since $X + g(X) \in V_+^{0,1} = \overline{L}_1 \cap \overline{L}_2$ we see that $X \in \bar{L}_1^T \boxtimes \overline{L}_2$. Next, let us denote by $P_\pm := \frac{1}{2}(1 - iI_\pm^*)$ the projections onto $T_\pm^{*1,0}M$. A quick calculation yields

$$\sigma_+ = 4g^{-1}\overline{P}_+\overline{P}_-P_+.$$

For $\xi \in T_+^{*1,0}M$, using $\xi = P_+\xi$ and $1 = P_\pm + \overline{P}_\pm$, we obtain

$$\sigma_+(\xi) = 4g^{-1}(\xi - P_+\overline{P}_-\xi) - 4g^{-1}(P_-\xi) \tag{1.60}$$

$$= -4g^{-1}(P_+\overline{P}_-\xi) + 4g^{-1}(\overline{P}_-\xi). \tag{1.61}$$

Note that (1.60) and (1.61) are decompositions of $\sigma_+(\xi)$ in $T_+^{0,1}M + T_-^{0,1}M$ and $T_+^{0,1}M + T_-^{1,0}M$ respectively. Writing $\zeta := 4(\xi - P_+\overline{P}_-\xi + P_-\xi)$ and $\eta := -4P_+\overline{P}_-\xi - 4\overline{P}_-\xi$, we have

$$\sigma_+(\xi) - 8\xi = \sigma_+(\xi) - \zeta + \eta.$$

Equation (1.60) implies that $\sigma_+(\xi) + \zeta \in \overline{L}_1$ while (1.61) implies that $\sigma_+(\xi) + \eta \in \overline{L}_2$. In particular $\sigma_+(\xi) - 8\xi \in \bar{L}_1^T \boxtimes \overline{L}_2$, so indeed $L_{(I_+, -\frac{1}{8}\sigma_+)} \subset \bar{L}_1^T \boxtimes \overline{L}_2$. □

The fact that $\bar{L}_1^T \boxtimes \overline{L}_2$ is smooth can also be seen directly from $\bar{L}_1^T \cap \overline{L}_2 \cap T^*M_{\mathbb{C}} = V_+^{0,1} \cap T^*M_{\mathbb{C}} = 0$, which in addition shows that

$$\bar{\rho}_1^T \wedge \overline{\rho}_2 = e^{-\frac{1}{8}\sigma_+}\Omega_+, \tag{1.62}$$

where $\Omega_+$ is a suitably scaled $(n, 0)$-form for $I_+$ and $\rho_1$ and $\rho_2$ are spinors for $L_1$ and $L_2$. Similarly,

$$\bar{\rho}_1^T \wedge \rho_2 = e^{-\frac{1}{8}\sigma_-}\Omega_-. \tag{1.63}$$

# Chapter 2

# Blowing up Submanifolds

In this chapter we introduce the concept of blowing up in the category of smooth manifolds. The definition that we will give resembles very much the algebro-geometric one. Specifically, for a given submanifold we introduce the notion of a holomorphic ideal for it, and use that to define the blow-up of the submanifold by means of a universal property. The advantage of this approach is that it is canonical, and from the point of view of the submanifold the only choice that has been made is that of a holomorphic ideal. In Section 2.2 we give a normal form that describes all the holomorphic ideals for a given submanifold, and use it to study the topology of the blow-up. This includes a calculation of the fundamental group and its cohomology. Then, in Sections 2.3 and 2.4 we discuss how to endow blow-ups with complex and symplectic structures. This part is well-known, and serves as both a review as well as a warm-up for the constructions in Chapters 3 and 4.

## 2.1 Blow-ups in differential geometry

Blowing up a submanifold consists of replacing it by all complex directions normal to it. As such it as a surgery procedure, where a subset of the manifold is removed and a new piece is glued back in. When performing surgeries on differentiable manifolds one often needs to make specific choices, such as a tubular embedding of the submanifold. In the following, we present a way to blow up submanifolds in which this choice takes the form of an ideal of functions, that has the submanifold as its zero set.

**Definition 2.1.1.** Let $M$ be a smooth manifold and $C^\infty(M;\mathbb{C})$ the sheaf of complex valued smooth functions. Let $Y \subset M$ be a closed[1] submanifold of real codimension $2l$, with $l \geq 1$. A *holomorphic ideal* for $Y$ is an ideal sheaf $I_Y \subset C^\infty(M;\mathbb{C})$ with the following properties:

i) $I_Y|_{M\setminus Y} = C^\infty(M;\mathbb{C})|_{M\setminus Y}$.

[1]Closed in the sense of topological subspace, we do not necessarily assume that $Y$ is compact. Our submanifolds will always be embedded.

ii) Each $y \in Y$ has a neighborhood $U$ together with $z^1, \dots, z^l \in I_Y(U)$, such that $z := (z^1, \dots, z^l) : U \to \mathbb{C}^l$ is a submersion with $Y \cap U = z^{-1}(0)$, and $I_Y|_U = \langle z^1, \dots, z^l \rangle$.

*Remark* 2.1.2. It follows from properties i) and ii) that $Z(I_Y) := \{x | \ f(x) = 0 \ \forall f \in (I_Y)_x\} = Y$, where $(I_Y)_x$ denotes the stalk of $I_Y$ at $x$. So basically $I_Y$ is an ideal which has $Y$ as its zero set, but makes it look complex in transverse directions. In particular, a holomorphic ideal turns $NY$ into a complex vector bundle via the decomposition $N^*Y_{\mathbb{C}} = N^{*1,0}Y \oplus N^{*0,1}Y$, where $N_y^{*1,0}Y := \langle d_y z | \ z \in (I_Y)_y \rangle$. In terms of sheaves, $N^{*1,0}Y = (I_Y/I_Y^2)|_Y$. Note that if $I_Y$ and $I'_Y$ are holomorphic ideals for $Y$ with $I_Y \subset I'_Y$, then $I_Y = I'_Y$.

For any smooth map $f : M_1 \to M_2$ and ideal sheaf $I \subset C^\infty(M_2; \mathbb{C})$ we can form the pull-back $f^*I$, which by definition is the ideal generated by $\{f^*g | \ g \in I\}$. We have $Z(f^*I) = f^{-1}Z(I)$ and if $f$ is transverse to a submanifold $Y_2$ then it pulls back any holomorphic ideal for $Y_2$ to a holomorphic ideal for $Y_1 := f^{-1}Y_2$. Note that $f^*I_{Y_2} = I_{Y_1}$ implies that the induced map $df : NY_1 \to f^*NY_2$ is complex linear and injective, but the converse is not true in general.

**Example 2.1.3.** Let $Y = \{0\} \subset \mathbb{C}$. One possible holomorphic ideal for $Y$ is given by the ideal $I_Y := \langle z \rangle$, where $z$ is the holomorphic coordinate on $\mathbb{C}$. However, for each $k \in \mathbb{Z}_{\geq 2}$ the ideal $I_Y(k)$ generated by the function $z + (\overline{z})^k$ is also a holomorphic ideal for $Y$. Note that the ideals $I_Y$ and $I_Y(k)$ are all mutually distinct holomorphic ideals for $Y$ that induce the same complex structure on $N^*Y$.

We will mainly be interested in holomorphic ideals for smooth submanifolds, but in order to state the definition of the blow-up we also want to consider singular submanifolds in complex codimension 1.

**Definition 2.1.4.** A *divisor* on $M$ is an ideal sheaf $I_Y \subset C^\infty(M; \mathbb{C})$ which locally can be generated by a single function, and whose zero set $Y$ is nowhere dense in $M$.

As in complex geometry, there is an alternative description of divisors in terms of line bundles. Let $L$ be a complex line bundle and $s$ a section whose zero set $Y := s^{-1}(0)$ is nowhere dense. Then there is a map $\Gamma(L^*) \to C^\infty(M; \mathbb{C})$ given by evaluation on $s$, and the image will be a divisor in the above sense. Conversely, given a divisor $I_Y$, choose an open cover $\{U_\alpha\}$ of $M$ together with generators $f_\alpha \in I_Y(U_\alpha)$. On $U_\alpha \cap U_\beta$ both $f_\alpha$ and $f_\beta$ generate the same ideal, so we have $f_\alpha = g_{\alpha\beta} f_\beta$ for some function $g_{\alpha\beta}$. In particular $f_\alpha = g_{\alpha\beta} g_{\beta\alpha} f_\alpha$ and $f_\alpha = g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} f_\alpha$. Since the zero set of $f_\alpha$ is nowhere dense, we see that $(U_{\alpha\beta}, g_{\alpha\beta})$ defines a line bundle on $M$, which is trivial over each $U_\alpha$. The functions $f_\alpha$ then glue together to form a section of this bundle, giving a pair $(L, s)$ as above. Note that in this language, a divisor $(L, s)$ is a holomorphic ideal in the sense of Definition 2.1.1, i.e. is smooth, if and only if $s$ intersects the zero-section of $L$ transversally.

Equipped with this language we can define the blow-up of a holomorphic ideal $I_Y$ in $M$.

**Definition 2.1.5.** Let $Y \subset M$ be a closed submanifold and $I_Y$ a holomorphic ideal for $Y$. The *blow-up* of $I_Y$ in $M$ is a pair $(\widetilde{M}, p)$, consisting of a smooth manifold $\widetilde{M}$ and a smooth *blow-down map* $p : \widetilde{M} \to M$ such that $p^* I_Y$ is a divisor and which satisfies the following universal property: For any smooth map $f : X \to M$ such that $f^* I_Y$ is a divisor, there is a unique $\widetilde{f} : X \to \widetilde{M}$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\widetilde{f}} & \widetilde{M} \\ & \searrow^{f} & \downarrow p \\ & & M \end{array}$$

Given $(\widetilde{M}, p)$, we define $E := p^{-1}(Y)$ and $I_E := p^* I_Y$. This $E$ is called the *exceptional divisor* of the blow-up.

**Theorem 2.1.6.** *The blow-up $(\widetilde{M}, p)$ exists and is unique up to unique isomorphism.*

*Proof.* The uniqueness part follows immediately from the universal property. Since $Y$ is closed in $M$, we can cover $M$ by charts which are either disjoint from $Y$, or of the form $\mathbb{C}^l \times \mathbb{R}^m$ with coordinates $(z^1, \ldots, z^l, x^1, \ldots, x^m)$, where the $z^i$ are as in Definition 2.1.1 ii) and $x^i$ are real coordinates on $Y$. We will construct the blow-up on each individual chart, so that the universal property implies that all the local constructions can be glued together uniquely to form the desired manifold $\widetilde{M}$. On a chart not intersecting $Y$ we do nothing, i.e. take $p$ to be the identity, as $I_Y$ is already (trivially) a divisor there. On a chart $U = \mathbb{C}^l \times \mathbb{R}^m$ as above with $Y \cap U = \{0\} \times \mathbb{R}^m$, we define $\widetilde{U} := \widetilde{\mathbb{C}}^l \times \mathbb{R}^m$ and $p = (p', \mathrm{Id}) : \widetilde{U} \to U$, where $\widetilde{\mathbb{C}}^l$ is defined by

$$\widetilde{\mathbb{C}}^l := \{(z, \tau) |\ z \in \tau\} \subset \mathbb{C}^l \times \mathbb{CP}^{l-1} \tag{2.1}$$

and $p' : \widetilde{\mathbb{C}}^l \to \mathbb{C}^l$ is given by the first projection. The manifold $\widetilde{\mathbb{C}}^l$ has a cover by $l$ different charts, given by

$$(v^1, \ldots, v^{i-1}, z^i, v^{i+1}, \ldots, v^l) \leftrightarrow \big(z^i \cdot (v^1, \ldots, v^{i-1}, 1, v^{i+1}, \ldots, v^l), \\ [v^1 : \ldots : v^{i-1} : 1 : v^{i+1} : \ldots : v^l]\big)$$

for $1 \leq i \leq l$. On these charts $p'$ is given by

$$p' : (v^1, \ldots, v^{i-1}, z^i, v^{i+1}, \ldots, v^l) \mapsto (z^i v^1, \ldots, z^i v^{i-1}, z^i, z^i v^{i+1}, \ldots, z^i v^l). \tag{2.2}$$

Now suppose that $f : X \to U$ is a map such that $f^*(I_Y|_U)$ is a divisor, with corresponding nowhere dense zero set $D = f^{-1}(Y \cap U)$. The desired lift $\widetilde{f} : X \to U$ is already uniquely determined and smooth on $X \backslash D$, because $p'$ is an isomorphism over $\mathbb{C}^l \backslash \{0\}$, so we only have to show that $\widetilde{f}$ extends smoothly over $D$. To that end, write

$f = (f^1, \dots, f^l, f'^1, \dots, f'^m)$, so that $f^*(I_Y|_U) = \langle f^1, \dots, f^l \rangle$. By definition of being a divisor there exists, on a neighborhood $V$ of any $x_0 \in D$, a function $g \in C^\infty(V;\mathbb{C})$ with $\langle g \rangle = \langle f^1, \dots, f^l \rangle$. Therefore there exist $a^i, b_i \in C^\infty(V;\mathbb{C})$ with $f^i = a^i g$ and $g = \sum_i b_i f^i$ and so, since $g \neq 0$ on a dense set, we obtain $\sum_i a^i b_i = 1$. In particular there is an index $i_0$ such that, after possibly shrinking $V$, $a^{i_0}$ is nowhere zero. The map $\widetilde{f} : V \backslash (D \cap V) \to \widetilde{U}$ has its image contained in the chart (2.2) for $i = i_0$, where it is given by

$$\widetilde{f} : x \mapsto \Big(\frac{f^1(x)}{f^{i_0}(x)}, \dots, f^{i_0}(x), \dots, \frac{f^l(x)}{f^{i_0}(x)}, f'^1(x), \dots, f'^m(x)\Big).$$

Since $f^i/f^{i_0} = a^i/a^{i_0}$ and $a^{i_0}$ is nonzero on $V$, we see that $\widetilde{f}$ indeed extends smoothly over the whole of $V$, and therefore over the whole of $D$. Hence, the blow-up exists over each chart, and from the above discussion we conclude that the blow-up $p : \widetilde{M} \to M$ indeed exists and is unique. □

*Remark* 2.1.7. It follows from the universal property that the blow-up construction is functorial in the sense that for any map $f : (M_1, I_{Y_1}) \to (M_2, I_{Y_2})$ with $f^* I_{Y_2} = I_{Y_1}$, there is a unique map $\widetilde{f} : \widetilde{M_1} \to \widetilde{M_2}$ making the obvious diagram commute.

## 2.2 Topology

In Remark 2.1.2 we saw that a holomorphic ideal $I_Y$ for a submanifold $Y \subset M$ turns the normal bundle $NY$ into a complex vector bundle. It is then natural to ask whether any complex structure on $NY$ is induced by such an ideal. In this section we will show that this is the case, and moreover prove that any two ideals that induce the same complex structure on $NY$ are related to each other by a diffeomorphism around $Y$. We then use this to describe the topology of the blow-up, including a computation of the fundamental group and the cohomology groups.

Suppose $NY$ has the structure of a complex vector bundle, so that $NY_{\mathbb{C}} = N^{1,0}Y \oplus N^{0,1}Y$. Viewing $Y$ as a submanifold of the total space of $NY$ given by the zero section, there is a natural holomorphic ideal $I_Y^{\text{lin}}$ for $Y$ generated by $\Gamma(N^{*1,0}Y)$, considered as fiberwise linear complex functions on $NY$. We call this the *linear ideal* associated to the complex structure on $NY$. Explicitly, if $q : N^{1,0}Y \to Y$ denotes the projection[2] and $U \subset Y$ is an open set over which $q^{-1}U \cong U \times \mathbb{C}^k$, then $I_Y^{\text{lin}}|_{q^{-1}U} = \langle z^1, \dots, z^k \rangle$, where the latter are the coordinates on $\mathbb{C}^k$. If $\iota : NY \to M$ is any tubular embedding, we obtain a holomorphic ideal for $Y$ on the image of $\iota$ by pushing forward $I_Y^{\text{lin}}$. As this ideal coincides with the trivial one on the complement of $Y$, we can extend it to the rest of $M$, obtaining a holomorphic ideal for $Y$ on $M$ that induces the given complex structure on $NY$. We now show that in fact all holomorphic ideals for $Y$ arise in this way.

[2] $NY$ and $N^{1,0}Y$ are isomorphic as complex vector bundles, so we will often implicitly identify them with each other.

**Proposition 2.2.1.** *Let $I_Y$ be a holomorphic ideal for $Y$, inducing a complex structure on $NY$. Then there exists a tubular embedding $\iota : NY \to M$ such that $\iota^* I_Y = I_Y^{lin}$.*

*Proof.* Let $\kappa : NY \to M$ be any tubular embedding and consider $\kappa^* I_Y$. This is a holomorphic ideal for $Y$ on $NY$ that induces the same complex structure on $NY$ as $I_Y^{\text{lin}}$, and it suffices to show that there exists a diffeomorphism $\varphi$ of $NY$ defined in a neighborhood of $Y$, that fixes $Y$ and satisfies $\varphi^* \kappa^* I_Y = I_Y^{\text{lin}}$. Then $\iota := \kappa \circ \varphi$ will be the desired tubular embedding. We will construct $\varphi$ on $N^{1,0}Y$, which is isomorphic as a complex vector bundle to $NY$.

Pick an open cover $\{U_\alpha\}$ of $Y$ together with trivializing frames $e^\alpha = (e_1^\alpha, \dots, e_k^\alpha)$ for $N^{1,0}Y$ over $U_\alpha$, with $k = \text{codim}_{\mathbb{C}}(Y)$. Such a local frame induces an identification $q^{-1}U_\alpha \cong U_\alpha \times \mathbb{C}^k$, by letting $(x, z_\alpha) \in U_\alpha \times \mathbb{C}^k$ correspond to $\sum_i z_\alpha^i e_i^\alpha(x)$. Here $q : N^{1,0}Y \to Y$ denotes the projection. Let $e_i^\alpha = (g_{\alpha\beta})_i^{\ j} e_j^\beta$ be[3] the transition on a double overlap $U_\alpha \cap U_\beta$, so that in particular $z_\beta^i = (g_{\alpha\beta})_j^{\ i} z_\alpha^j$. By taking the $U_\alpha$ sufficiently small we may assume that $\kappa^* I_Y$ is generated, on a neighborhood of $U_\alpha$ in $q^{-1}U_\alpha$, by functions $w_\alpha^1, \dots, w_\alpha^k$. By assumption, we know that

$$dw_\alpha^i|_{U_\alpha} = (h_\alpha)^i_{\ j} dz_\alpha^j|_{U_\alpha} \tag{2.3}$$

for some family of invertible matrices $h_\alpha$ on $U_\alpha$. We may absorb $h_\alpha$ in the local frame $e_\alpha$ and assume without loss of generality that $h_\alpha = \text{Id}$. Let $\{\rho_\alpha\}$ be a partition of unity subordinated to $\{U_\alpha\}$. The expression $w_\alpha q^*(\rho_\alpha e^\alpha)$ defines a map from a neighborhood of $Y$ in $N^{1,0}Y$ to $N^{1,0}Y$, given by

$$v \mapsto \sum_i w_\alpha^i(v) \rho_\alpha(q(v)) e_i^\alpha(q(v)).$$

It sends fibers to fibers and restricts to the identity on $Y$, and outside of $U_\alpha$ it maps all fibers to zero. Note that the same expression without the $\rho_\alpha$ would only be defined over $U_\alpha$. Define

$$\psi := \sum_\alpha w_\alpha q^*(\rho_\alpha e^\alpha) : N^{1,0}Y \supset\!\!\to N^{1,0}Y. \tag{2.4}$$

Again, this map is fiber preserving and restricts to the identity on $Y$. We claim that its derivative along $Y$ is the identity, so that it induces a diffeomorphism of neighborhoods of $Y$ in $NY$. Indeed, to check this we look in a particular coordinate chart $q^{-1}U_\alpha \cong U_\alpha \times \mathbb{C}^k$. There, $\psi$ is given by

$$\psi : (x, z_\alpha^i) \mapsto \big(x, \sum_\beta \rho_\beta(x) (g_{\beta\alpha}(x))_j^{\ i} w_\beta^j(x, z_\alpha)\big). \tag{2.5}$$

[3] In an expression with a repeated index an implicit summation over that index is understood.

Using (2.3) with $h_\alpha = \mathrm{Id}$ we see that

$$d\Big(\sum_\beta \rho_\beta(x)(g_{\beta\alpha}(x))^i_j w^j_\beta(x,z_\alpha)\Big)|_Y = \sum_\beta \rho_\beta(x)(g_{\beta\alpha}(x))^i_j dw^j_\beta(x,z_\alpha)|_Y$$
$$= \sum_\beta \rho_\beta(x) dz^i_\alpha|_Y = dz^i_\alpha|_Y$$

which implies that $d\psi|_Y = \mathrm{Id}$, hence $\psi$ is a local diffeomorphism around $Y$. From the local expression (2.5) it is clear that $\psi^* I^{\mathrm{lin}}_Y = \kappa^* I_Y$, so $\varphi := \psi^{-1}$ is the desired diffeomorphism. □

The main point in the above proof is the construction of the map $\psi$ in (2.4) which relates any holomorphic ideal for $Y$ on $NY$ to its linearization. Since the functions $w_\alpha(t) := (1-t)z_\alpha + t w_\alpha$ all satisfy $dw_\alpha(t)|_Y = dz_\alpha|_Y$, the family $\psi_t$, defined by the same equation as $\psi$ but with $w_\alpha$ replaced by $w_\alpha(t)$, defines an isotopy from the identity to $\psi = \psi_1$ on a small enough neighborhood of $Y$ in $NY$. Consequently, if $Y$ is compact we may use the isotopy extension theorem to obtain the following corollary.

**Corollary 2.2.2.** If $Y$ is compact and if $I_Y$ and $I'_Y$ are two holomorphic ideals for $Y$ that induce the same complex structure on $NY$, then there is a diffeomorphism $\varphi$ of $M$ with $\varphi^* I_Y = I'_Y$.

*Remark* 2.2.3. In particular, the two associated blow-ups are diffeomorphic. The diffeomorphism itself is however not unique, so the differentiable blow-up is not canonical.

This last remark admits a generalization.

**Corollary 2.2.4.** If $Y$ is compact and $I_Y$ and $I'_Y$ are holomorphic ideals whose induced complex structures on $NY$ are isotopic, then the blow-ups are diffeomorphic.

*Proof.* Let $J_t$ be a family of complex structures on $NY$ such that $J_0$ corresponds to the complex structure induced by $I_Y$ and $J_1$ to that of $I'_Y$. By Proposition 2.2.1 there exist tubular embeddings $\iota$ and $\iota'$ which pull-back $I_Y$ and $I'_Y$ to their respective linearizations. We can view $\{J_t\}_t$ as a complex structure on the normal bundle of $Y \times [0,1] \subset M \times [0,1]$, and there exists a tubular embedding $\kappa$ of the latter into $M \times [0,1]$, which restricts to $\iota$ and $\iota'$ at times 0 and 1. This induces a holomorphic ideal for $Y \times [0,1]$ on $M \times [0,1]$, denoted by $I_{Y\times[0,1]}$, and by construction we have $i_0^*(I_{Y\times[0,1]}) = I_Y$ and $i_1^*(I_{Y\times[0,1]}) = I'_Y$, where $i_0, i_1 : M \to M \times [0,1]$ denote the two inclusions. Let $X$ denote the blow-up of $I_{Y\times[0,1]}$ in $M \times [0,1]$. The composition

$$X \xrightarrow{p} M \times [0,1] \xrightarrow{\pi_2} [0,1]$$

is a submersion, whose fibers over 0 and 1 correspond to the blow-ups of $I_Y$ and $I'_Y$ in $M$. Since the restriction of $\pi_2 \circ p$ to $X \backslash p^{-1}(Y \times [0,1])$ is a trivial fiber bundle over $[0,1]$ and $p^{-1}(Y \times [0,1])$ is compact, the statement of the corollary follows from the following lemma, which is a slight variant of the theorem of Ehresmann. It is in that lemma that we need that $Y$ is compact. □

**Lemma 2.2.5.** Let $p : M \to N$ be a surjective submersion and let $K \subset M$ be a compact subset such that $p|_{M\backslash K} : M\backslash K \to N$ is a locally trivial fiber bundle. Then $p$ itself is a locally trivial fiber bundle.

*Proof.* Let $M_y$ denote the fiber of a point $y \in N$. By assumption there exists a neighborhood $V$ of $y$ in $N$ such that $p^{-1}(V)\backslash K$ is diffeomorphic to $M_y\backslash K \times V$, and we can think of this as a tubular neighborhood of $M_y\backslash K$ in $M$. On the other hand, since $M_y$ is a submanifold of $M$, it has itself a tubular neighborhood $U$. Then over $M_y\backslash K$ we have two tubular neighborhoods, and after interpolating between them we obtain a tubular neighborhood $W$ of $M_y$ in $M$, which near $M_y \cap K$ is given by $U$ and away from $M_y \cap K$ is given by $p^{-1}(V)\backslash K$. Since $K \cap M_y$ is compact there exists an open neighborhood $V'$ of $y$ in $N$ such that $p^{-1}(V') \subset W$. Let $r : p^{-1}(V') \to M_y$ denote the retraction coming from the tubular neighborhood $W$, and consider the map $(p, r) : p^{-1}(V') \to V' \times M_y$. It is invertible away from $r^{-1}(K \cap M_y)$, and since its derivative is invertible at points in $M_y$ and $K \cap M_y$ is compact, we can shrink $V'$ to ensure that $(r, p)$ becomes invertible, exhibiting $p^{-1}(V')$ as a product $V' \times M_y$. □

Let us now focus on the topology of the blow-up of one fixed ideal. First we do this for the linear ideal on $NY$, associated to some given complex structure on the bundle $NY$. Define

$$\widetilde{NY} = \{(z, l) \in NY \times \mathbb{P}(NY) |\ z \in l\} \subset NY \times \mathbb{P}(NY), \tag{2.6}$$

where $\mathbb{P}(NY)$ denotes the complex projectivization of $NY$. Let $p : \widetilde{NY} \to NY$ be the restriction to $\widetilde{NY}$ of the projection $NY \times \mathbb{P}(NY) \to NY$. The pair $(\widetilde{NY}, p)$ gives the blow-up of $Y$ in $NY$, and the exceptional divisor is given by $E := p^{-1}(0) = \mathbb{P}(NY)$.

Now suppose that $I_Y$ is a holomorphic ideal for $Y$ in $M$, and let $\iota : NY \xrightarrow{\sim} U \subset M$ be a tubular embedding with $\iota^* I_Y = I_Y^{\text{lin}}$, as provided by Proposition 2.2.1. Here $U$ is an open neighborhood of $Y$ in $M$. We can write $M$ as a glueing of two manifolds along open subsets,

$$M = M\backslash Y \cup_\iota NY,$$

where the glueing is given by $\iota : NY\backslash Y \xrightarrow{\sim} U\backslash Y$. Since $\iota^* I_Y = I_Y^{\text{lin}}$, it follows by naturality of the blow-up that $\widetilde{M}$ is given by

$$\widetilde{M} = M\backslash Y \cup_{\widetilde{\iota}} \widetilde{NY} \tag{2.7}$$

where $\widetilde{\iota} : \widetilde{NY}\backslash E \xrightarrow{p} NY\backslash Y \xrightarrow{\iota} U\backslash Y$.

**Corollary 2.2.6.** The restriction $p : \widetilde{M}\backslash E \to M\backslash Y$ is a diffeomorphism, $I_E = p^* I_Y$ is smooth and $p : E \to Y$ is isomorphic to $\mathbb{P}(NY) \to Y$.

The first statement actually follows from the universal property, while the other statements could also be deduced from the construction of the blow-up itself. Nevertheless, (2.7) gives a nice way to see these things directly. In some cases we can be more explicit about the topology of the blow-up.

**Lemma 2.2.7.** If $Y$ is a point and $\dim_{\mathbb{R}}(M) = 2n$, then[4] $\widetilde{M} \cong M \# \overline{\mathbb{CP}}^n$.

*Proof.* Consider the map $\pi : \mathbb{CP}^n \backslash [0 : \ldots : 0 : 1] \to \mathbb{CP}^{n-1}$ given by

$$\pi([z_0 : \ldots : z_n]) = [z_0 : \ldots : z_{n-1}].$$

The fiber over any point naturally forms a complex line, and over an affine chart $U_i := \{z_i \neq 0\}$ in $\mathbb{CP}^{n-1}$ we have a nonzero section $e_i$ of $\pi$ given by $e_i([z_0 : \ldots : z_{n-1}]) := [z_0 : \ldots : z_{n-1} : z_i]$. On an overlap $U_i \cap U_j$ we have $e_i = \frac{z_i}{z_j} \cdot e_j$. On the other hand, the tautological bundle $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$ (see 2.1) also has sections over the opens $U_i$, given by

$$e_i'([z_0 : \ldots : z_{n-1}]) := ((\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, 1, \frac{z_{i+1}}{z_i}, \ldots, \frac{z_{n-1}}{z_i}), [z_0 : \ldots : z_{n-1}]).$$

This time we have $e_i' = \frac{z_j}{z_i} \cdot e_j'$ on $U_i \cap U_j$, hence we see that $\mathbb{CP}^n$ minus a point is identified with the dual of the tautological line bundle. In general, the dual of a complex line bundle $L$ is equal to its complex conjugate. This can be seen by choosing an Hermitian metric, which gives an isomorphism $\overline{L} \to L^*$. Consider the manifold $\overline{\mathbb{CP}}^n \backslash [0 : \ldots : 0 : 1]$, which is the same as $\mathbb{CP}^n \backslash [0 : \ldots : 0 : 1]$ but with the opposite orientation, and let $\pi$ denote the same map onto $\mathbb{CP}^{n-1}$. The change of orientation has, from the point of view of $\mathbb{CP}^{n-1}$, the effect of conjugation in the fibers. Hence, $\overline{\mathbb{CP}}^n \backslash [0 : \ldots : 0 : 1]$ is diffeomorphic to $\widetilde{\mathbb{C}}^n$. Since the blow-up of a point consists of glueing the complement of a disc in $M$ to $\widetilde{\mathbb{C}}^n$, which we now know to be equal to the complement of a disc in $\overline{\mathbb{CP}}^n$, we see that the blow-up is given by the connected sum. □

Let us return to the general case. We can use (2.7) to compute the fundamental group.

**Lemma 2.2.8.** $\pi_1(\widetilde{M}) = \pi_1(M)$.

*Proof.* Since $\mathrm{codim}_{\mathbb{R}}(Y) \geq 4$, we have $\pi_1(M \backslash Y) = \pi_1(M)$. Moreover, $\widetilde{NY}$ deformation retracts onto $E$, which is a bundle over $Y$ with fiber given by $\mathbb{CP}^{k-1}$ for $k = \mathrm{codim}_{\mathbb{C}}(Y)$. Since this is simply connected, we have $\pi_1(E) = \pi_1(Y)$. Similarly, $\widetilde{NY} \backslash E \cong NY \backslash Y$ is homotopy equivalent to a sphere bundle over $Y$, with fiber given by $S^{2k-1}$. Again, this implies that $\pi_1(\widetilde{NY} \backslash E) = \pi_1(NY \backslash Y) = \pi_1(Y)$. Using Seifert-Van Kampen we obtain

$$\pi_1(\widetilde{M}) = \pi_1(M \backslash Y) \star_{\pi_1(\widetilde{NY} \backslash E)} \pi_1(\widetilde{NY}) = \pi_1(M) \star_{\pi_1(Y)} \pi_1(Y) = \pi_1(M). \quad \square$$

As far as cohomology is concerned we have the following lemma.

**Lemma 2.2.9.** There is an exact sequence $0 \to H^*(M) \stackrel{p^*}{\to} H^*(\widetilde{M}) \to A^* \to 0$, where $A^* \cong H^*(\mathbb{P}(NY))/H^*(Y)$ is a free module over $H^*(Y)$ generated by $\alpha, \alpha^2, \ldots, \alpha^{k-1}$ for a certain class $\alpha$ of degree 2. Here $k = \mathrm{codim}_{\mathbb{C}}(Y)$, and cohomology is taken over $\mathbb{Z}$.

[4] The symbol $\#$ stands for connected sum, see also Example 1.3.8.

*Proof.* The map of pairs $p : (\widetilde{M}, E) \to (M, Y)$ induces the commutative diagram

$$\begin{array}{ccccccccccc} \dots \to & H^{i-1}(E) & \to & H^i(\widetilde{M}, E) & \to & H^i(\widetilde{M}) & \to & H^i(E) & \to & H^{i+1}(\widetilde{M}, E) & \to \dots \\ & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \dots \to & H^{i-1}(Y) & \to & H^i(M, Y) & \to & H^i(M) & \to & H^i(Y) & \to & H^{i+1}(M, Y) & \to \dots \end{array} \tag{2.8}$$

whose rows are exact and all vertical maps are given by $p^*$. By homotopy invariance we have $H^*(M, Y) \cong H^*(M, U)$, where $U$ is a neighborhood of $Y$ in $M$ that deformation retracts onto $Y$, and similarly $H^*(\widetilde{M}, E) \cong H^*(\widetilde{M}, p^{-1}U)$. Using excision and the fact that $p : (\widetilde{M} \backslash E, p^{-1}U \backslash E) \to (M \backslash Y, U \backslash Y)$ is a diffeomorphism, we deduce that $p^* : H^*(M, Y) \to H^*(\widetilde{M}, E)$ is an isomorphism.

Since $E = \mathbb{P}(NY)$, there is a tautological line bundle $L$ over $E$, whose fiber over a point $l \in NY$ is the line $l$ itself. In fact the total space of $L$ is precisely $\widetilde{NY}$, see (2.6). Denote by $\alpha := c_1(L) \in H^2(E)$. Since the fiber of $p : E \to Y$ equals $\mathbb{CP}^{k-1}$ and the restriction of $L$ to such a fiber is precisely the tautological bundle over $\mathbb{CP}^{k-1}$, the restriction of the classes $1, \alpha, \dots, \alpha^{k-1}$ form a $\mathbb{Z}$-basis of the cohomology of the fiber. By the Leray-Hirsch theorem we deduce that $1, \alpha, \dots, \alpha^{k-1}$ form a basis for $H^*(E)$ as a module over $H^*(Y)$. In particular, $p^* : H^*(Y) \to H^*(E)$ is injective.

We would like to conclude, via some version of the five-lemma, that $p^* : H^*(M) \to H^*(\widetilde{M})$ is injective. In general, given a diagram like (2.8), a sufficient condition for the third vertical arrow to be injective is that the first vertical arrow is surjective and that the second and fourth vertical arrows are injective. We just saw that the second and fourth arrows are injective, but the first vertical arrow is not surjective, its cokernel being generated by the classes $\alpha, \dots, \alpha^{k-1}$ over $H^*(Y)$. However, the tautological bundle over $E$ is actually the restriction of a line bundle on $\widetilde{M}$, given by the line bundle corresponding to the divisor $E$ itself. In particular the class $\alpha$ is the restriction of a class[5] in $H^2(\widetilde{M})$, which we will continue to denote by $\alpha$. Consequently, the map $H^*(E) \to H^{*+1}(\widetilde{M}, E)$ kills the classes $\alpha^i$ for $i > 0$, hence its image coincides with that of the composition $H^*(Y) \to H^*(E) \to H^{*+1}(\widetilde{M}, E)$. This implies that the version of the five-lemma described above still holds, and we conclude that $p^* : H^*(M) \to H^*(\widetilde{M})$ is injective. Given this fact, a quick diagram chase along (2.8) shows that the cokernel of $p^* : H^*(M) \to H^*(\widetilde{M})$ coincides with that of $p^* : H^*(Y) \to H^*(E)$, which proves the lemma. □

Let $b_i(M) = \dim_{\mathbb{R}}(H^i(M; \mathbb{R}))$ denote the $i$-th Betti number, understood to be zero whenever $i$ is negative.

**Corollary 2.2.10.** $b_i(\widetilde{M}) = b_i(M) + \sum_{j=1}^{k-1} b_{i-2j}(Y)$.

[5] Another interpretation of this class is as the Poincaré dual of $E$ in $\widetilde{M}$. Note that since $E$ is co-oriented it has a Poincaré dual, even if $M$ is not oriented.

## 2.3 Complex blow-up

We have seen how to construct a blow-up differentiably, now it is time to endow all manifolds with geometric structures. We first discuss the complex setting.

The complex blow-up fits nicely in the language introduced in Section 2.1. If $(M, I)$ is a complex manifold and $Y \subset M$ is a complex submanifold, there is a canonical holomorphic ideal for $Y$, given by the holomorphic functions that vanish on $Y$. Consequently, there is a canonical differentiable blow-up $\widetilde{M}$.

**Proposition 2.3.1.** *The blow-up $\widetilde{M}$ carries a unique complex structure such that the blow-down map is holomorphic.*

*Proof.* Recall that the construction of $\widetilde{M}$ was performed locally, using the universal property to glue all local constructions together. On a local holomorphic chart $\mathbb{C}^l \times \mathbb{C}^m$, where $Y$ is given by $\{0\} \times \mathbb{C}^m$, the blow-up is given by $\widetilde{\mathbb{C}}^l \times \mathbb{C}^m$, where $\widetilde{\mathbb{C}}^l$ is the blow-up of the origin in $\mathbb{C}^l$ (see (2.1)). This clearly carries a unique complex structure for which the blow-down map is holomorphic. The glueing maps over intersections of charts, as provided by the universal property, will be holomorphic on a dense set, hence everywhere. In particular, $\widetilde{M}$ is a complex manifold. $\square$

*Remark* 2.3.2. What we observed in this proof is that the differentiable blow-up $\widetilde{M}$ satisfies not only the universal property of Definition 2.1.5, but also its analogue in the category of complex manifolds and holomorphic maps. Note that in this category the condition $f^* I_{Y_2} = I_{Y_1}$, for a holomorphic map $f : (M_1, Y_1) \to (M_2, Y_2)$, is in fact equivalent to $f^{-1}(Y_2) = Y_1$ and injectivity of the bundle map $df : NY_1 \to NY_2$.

The most important application of the complex blow-up is to resolutions of singularities. Here is a concrete example that shows the intuition behind this.

**Example 2.3.3.** Consider the curve $C := \{y^2 - x^2(x+1) = 0\}$ in $\mathbb{C}^2$, which has a singularity at $(0, 0)$. Let $p : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ denote the blow-up of the origin with exceptional divisor $E \cong \mathbb{CP}^1$. Then $\widetilde{\mathbb{C}}^2$ is covered by two charts with coordinates $(u_1, v_1)$ and $(u_2, v_2)$, where $p(u_1, v_1) = (u_1, u_1 v_1)$ and $p(u_2, v_2) = (u_2 v_2, v_2)$. There are two analytic subspaces of $\widetilde{\mathbb{C}}^2$ associated to $C$, the *total transform* given by $p^{-1}(C)$, and the *proper transform* $\widetilde{C}$ which is the closure of $p^{-1}(C) \backslash (E \cap p^{-1}(C))$ in $\widetilde{\mathbb{C}}^2$. The total transform is given by the equation $p^*(y^2 - x^2(x+1)) = 0$, which on the two charts looks like $u_1^2(v_1^2 - (u_1 + 1)) = 0$ and $v_2^2(1 - u_2^2(u_2 v_2 + 1)) = 0$ respectively. It follows that the proper transforms are given by $v_1^2 - (u_1 + 1) = 0$ and $1 - u_2^2(u_2 v_2 + 1) = 0$. In particular, $\widetilde{C}$ is smooth. Note that the intersection of the proper transform with the exceptional divisor is given by the two points $(u_1, v_1) = (0, \pm 1)$ (or $(u_2, v_2) = (\pm 1, 0)$ in the other chart), which on $\mathbb{CP}^1$ correspond precisely to the two tangents of the curve $C$ at $(0, 0)$.

This example can be generalized to the case of any complex curve $C$ in a smooth complex surface $S$. Each point in $C$ comes with a multiplicity, which by definition is the lowest order nonzero homogeneous component of an equation for the curve in local coordinates on

$S$ centered at the point in question. The singular points are those for which the multiplicity is bigger than one. If $x \in C$ is a singular point, let $p : \widetilde{S} \to S$ denote the blow-up of $S$ in $x$. The total transform $p^{-1}C$ and the proper transform $\widetilde{C}$ are defined as in the example above, and the intersection $\widetilde{C} \cap E$ is the set of points that correspond to the lines in $T_xS$ that are tangent to $C$ at $x$. By a careful analysis of what happens to the local equation for the curve under the blow-up, similar to the example above, one can show that after a finite number of blow-ups the multiplicity of a singular point must decrease. In particular, a finite number of blow-ups will transform the singular point into a set of smooth points on a new curve $\widetilde{C}$. Therefore, a locally finite number of blow-ups in $S$ transforms the curve $C$ into a smooth curve. For details we refer to Theorem 7.1 in [7].

In much greater generality, Hironaka [27] proved that any complex analytic space admits a desingularization. The proof is very hard, but the notion of local multiplicity (now intrinsically defined on the singular space itself) and the theory of blow-ups still play a major role in the construction.

## 2.4 Symplectic blow-up

The construction of the symplectic blow-up is different from the complex one, the reason being that symplectic submanifolds do not carry a canonical holomorphic ideal. In particular, we can not define the blow-up immediately via a universal property. Another difference is that in this context we need to require the submanifold to be compact, for reasons that will become clear later in Section 3.2. In this section we will only describe the symplectic blow-up of a point in $(M, \omega)$, since that already describes the main ideas. The fine details of the construction, as well as the symplectic blow-up in general, are left to Section 3.2 where we discuss blow-ups of generalized Poisson transversals. This includes symplectic submanifolds as a special case.

The construction we present here is due to [36]. Let $Y$ be given by a point $y_0 \in M$, and pick a Darboux chart $\mathbb{C}^n$ around $y_0$ on which $\omega = \frac{i}{2}\sum_i dz^i d\bar{z}^i$ (in the case of a general submanifold $Y$ we need to pick a tubular embedding that puts the symplectic structure in a normal form around $Y$, which is provided by the symplectic neighborhood theorem). The blow-up of the origin in $\mathbb{C}^n$ is given by $\widetilde{\mathbb{C}}^n = \{(z, l) \in \mathbb{C}^n \times \mathbb{CP}^{n-1} | \ z \in l\}$, and comes equipped with maps $p : \widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ and $\pi : \widetilde{\mathbb{C}}^n \to \mathbb{CP}^{n-1}$ induced by the two projections. The latter realizes $\widetilde{\mathbb{C}}^n$ as the tautological line bundle over $\mathbb{CP}^{n-1}$, with zero section given by $E$. The pullback $p^*\omega$ is symplectic on $\widetilde{\mathbb{C}}^n \backslash E$, but degenerates on $E$ itself. To remedy this, consider the Fubini-Study form $\omega_{\text{FS}}$ on $\mathbb{CP}^{n-1}$. Then $\pi^*\omega_{\text{FS}}$ is positive on $E$, and $\omega_\varepsilon := p^*\omega + \varepsilon\pi^*\omega_{\text{FS}}$ is positive on $\widetilde{\mathbb{C}}^n$ for all $\varepsilon > 0$. We saw in Section 2.2 that the blow-up $\widetilde{M}$ is given by the glueing of $M\backslash\{y_0\}$ to $\widetilde{\mathbb{C}}^n$ via the isomorphism $p : \widetilde{\mathbb{C}}^n \backslash E \xrightarrow{\sim} \mathbb{C}^n \backslash \{0\} \subset M\backslash\{y_0\}$. This glueing map does not pull-back $\omega$ to $\omega_\varepsilon$ because of the extra Fubini-Study term in $\omega_\varepsilon$. Fortunately, $\pi^*\omega_{\text{FS}}$ has the convenient property that its restriction to $\widetilde{\mathbb{C}}^n \backslash E$ is exact, i.e. $\pi^*\omega|_{\widetilde{\mathbb{C}}^n \backslash E} = d\alpha$ for a specific one-form $\alpha$, whose precise expression does not concern us at the moment. If $\tau$ is a function on $\mathbb{C}^n$

with compact support and which equals 1 on a neighborhood of 0, the form

$$\widetilde{\omega}_\varepsilon := \begin{cases} p^*\omega + \varepsilon\pi^*\omega_{\text{FS}} & \text{on } E \\ p^*\omega + \varepsilon d((p^*\tau)\alpha) & \text{on } \widetilde{\mathbb{C}}^n \backslash E \end{cases} \tag{2.9}$$

is smooth and symplectic on $\widetilde{\mathbb{C}}^n$ for $\varepsilon$ small enough. It agrees with $p^*\omega$ outside a neighborhood of $E$. Specifically, let $U$ be a neighborhood of 0 in $\mathbb{C}^n$ which contains the support of $\tau$, so that $p^{-1}U$ is a neighborhood of $E$ in $\widetilde{\mathbb{C}}^n$, outside of which $\widetilde{\omega}_\varepsilon$ equals $p^*\omega$. Then we can write $\widetilde{M}$ as the result of glueing $M\backslash\overline{U}$ to $\widetilde{\mathbb{C}}^n$ via the same glueing map, but now restricted to $\widetilde{\mathbb{C}}^n\backslash p^{-1}\overline{U} \to \mathbb{C}^n\backslash\overline{U}$. This glueing map is symplectic, equipping $\widetilde{M}$ with a symplectic structure $\widetilde{\omega}_\varepsilon$. The blow-down map is symplectic on $\widetilde{M}\backslash p^{-1}U$, but not near the exceptional divisor $E$. Note that the parameter $\varepsilon$ is related to the symplectic volume of $E$: if $i : E \hookrightarrow Y$ denotes the inclusion, then

$$\int_E i^*\widetilde{\omega}_\varepsilon = \varepsilon \int_{\mathbb{CP}^{n-1}} \omega_{\text{FS}}.$$

**Example 2.4.1.** In [36] the symplectic blow-up was used to construct new examples of non-Kählerian symplectic manifolds. At the time such examples, especially simply connected ones, were rather scarce. The idea is as follows. Given any symplectic manifold $(M,\omega)$ of dimension $2n$, there exists a symplectic embedding of $M$ into $\mathbb{CP}^{2n+1}$ (see [22, Section 3.4] or [39]). From Section 2.2 we know that the blow-up of $M$ in $\mathbb{CP}^n$, call it $X$, is simply connected, and $b_3(X) = b_1(M)$. Hence if $b_1(M)$ is odd, $X$ is a simply connected symplectic manifold which can not be Kähler. For instance, $M$ can be taken to be Thurston's manifold, which is a compact symplectic four-manifold with $b_1(M) = 3$.

### Symplectic cuts

An elegant alternative approach to symplectic blow-ups uses symplectic cuts, and is due to [34]. This approach will be useful for us when we discuss blow-ups of generalized Poisson transversals.

We first discuss symplectic cuts in general. Let $(M,\omega)$ be a symplectic manifold, endowed with a Hamiltonian $S^1$-action with moment map $\mu : M \to \mathbb{R}$. Recall that this means that $\iota_X\omega = d\mu$, for $X$ the action vector field on $M$ given by

$$X(x) = \left.\frac{d}{dt}\right|_{t=0} e^{it}\cdot x. \tag{2.10}$$

If $\varepsilon$ is a regular value for $\mu$, it separates $M$ into the regions $\{\mu \geq \varepsilon\}$ and $\{\mu \leq \varepsilon\}$, which are manifolds with boundaries given by $\mu^{-1}(\varepsilon)$. Intuitively, the symplectic cut disconnects these two regions, and collapses their boundaries via the $S^1$-action to form two new manifolds without boundaries, denoted by $M_{\geq\varepsilon}$ and $M_{\leq\varepsilon}$. Here is the precise construction. Consider the symplectic manifold $(N,\widetilde{\omega}) := (M,\omega)\times(\mathbb{C},\omega_{\text{st}})$, where $\omega_{\text{st}} = \frac{i}{2}dwd\bar{w}$, and consider the $S^1$-action on $N$ given by $e^{i\theta}\cdot(x,w) = (e^{i\theta}\cdot x, e^{i\theta}w)$.

This action also carries a moment map $\nu : N \to \mathbb{R}$, given by $\nu(x, w) = \mu(x) - \frac{1}{2}|w|^2$. A regular value $\varepsilon$ for $\mu$ is also a regular value for $\nu$, and

$$\nu^{-1}(\varepsilon) = \{(x, w)|\ \mu(x) = \varepsilon + \frac{1}{2}|w|^2\}.$$

This is a union of two subsets which are $S^1$-invariant, namely $\nu^{-1}(\varepsilon) \cap \{w = 0\}$ and $\nu^{-1}(\varepsilon) \cap \{w \neq 0\}$. The former coincides with $\mu^{-1}(\varepsilon)$, while on the latter we can write down a slice[6] for the $S^1$-action:

$$\varphi : \{\mu > \varepsilon\} \to \nu^{-1}(\varepsilon) \cap \{w \neq 0\}, \qquad x \mapsto (x, \sqrt{2(\mu(x) - \varepsilon)}). \tag{2.11}$$

Clearly $\varphi^*\tilde{\omega} = \omega$, hence the induced diffeomorphism $\{\mu > \varepsilon\} \to (\nu^{-1}(\varepsilon) \cap \{w \neq 0\})/S^1$ is a symplectomorphism, with respect to the restricted symplectic structure on $\{\mu > \varepsilon\}$ and the reduced symplectic structure on the quotient. Consequently, we see that the reduced symplectic manifold $M_{\geq\varepsilon} := \nu^{-1}(\varepsilon)/S^1$ (which has no boundary), has an open subset which is symplectomorphic to $\{\mu > \varepsilon\}$, whose complement is given by the codimension 2 symplectic submanifold $\mu^{-1}(\varepsilon)/S^1$. This confirms the above described intuitive picture of the cut.
Similarly, we can also consider the $S^1$-action on $N$ given by $e^{i\theta}\cdot(x, w) = (e^{i\theta}\cdot x, e^{-i\theta}w)$, with moment map $\tilde{\nu}(x, w) = \mu(x) + \frac{1}{2}|w|^2$. As above, the quotient $M_{\leq\varepsilon} := \tilde{\nu}^{-1}(\varepsilon)/S^1$ has an open set which is symplectomorphic to $\{\mu < \varepsilon\}$, whose complement is given by $\mu^{-1}(\varepsilon)/S^1$.

Now let us apply this to blow-ups. Consider the $S^1$-action on $M := \mathbb{C}^n$ given by $e^{i\theta} \cdot z := e^{-i\theta}z$. A moment map for this action is given by $\mu(z) = \frac{1}{2}|z|^2$, and every $\varepsilon > 0$ is a regular value with level set given by the sphere of radius $\sqrt{2\varepsilon}$.

**Lemma 2.4.2.** $M_{\geq\varepsilon/2} = \widetilde{\mathbb{C}}^n$, the blow-up of $\mathbb{C}^n$ at the origin, with symplectic structure given by $p^*\omega_{\text{st}} + \varepsilon\pi^*\omega_{\text{FS}}$. Here $p$ and $\pi$ are the two projections to $\mathbb{C}^n$ and $\mathbb{CP}^{n-1}$ respectively.

*Proof.* Let $\nu : \mathbb{C}^n \times \mathbb{C} \to \mathbb{R}$ be the moment map $\nu(z, w) = \frac{1}{2}|z|^2 - \frac{1}{2}|w|^2$ used for the symplectic cut. Consider the map $\kappa : \nu^{-1}(\varepsilon/2) \to \mathbb{C}^n \times \mathbb{CP}^{n-1}$ given by

$$\kappa : (z, w) \mapsto \big(\frac{wz}{|z|}, [z]\big). \tag{2.12}$$

It is $S^1$-invariant with image $\widetilde{\mathbb{C}}^n = \{(x, l)|\ x \in l\}$, and induces a diffeomorphism $\nu^{-1}(\varepsilon/2)/S^1 \cong \widetilde{\mathbb{C}}^n$. Since $\pi \circ \kappa : \nu^{-1}(\varepsilon/2) \to \mathbb{CP}^{n-1}$ factors through the quotient map $\mathbb{C}^n\backslash\{0\} \to \mathbb{CP}^{n-1}$, we have $\kappa^*\pi^*\omega_{\text{FS}} = \frac{i}{2}\partial\bar{\partial}\log(|z|^2)$, basically by definition of $\omega_{\text{FS}}$. Consequently,

$$\begin{aligned}\kappa^*(p^*\omega_{\text{st}} + \varepsilon\pi^*\omega_{\text{FS}}) &= (p \circ \kappa)^*(\frac{i}{2}du^i d\bar{u}^i) + \frac{i\varepsilon}{2}\partial\bar{\partial}\log(|z|^2)\\ &= \frac{i}{2}d(\frac{wz^i}{|z|})d(\frac{\bar{w}\bar{z}^i}{|z|}) + \frac{i\varepsilon}{2}\partial\bar{\partial}\log(|z|^2).\end{aligned}$$

[6] A *slice* for a $G$-action on a manifold $M$ is a submanifold $S \hookrightarrow M$ such that the action map $G \times S \to M$ is a diffeomorphism onto an open neighborhood of $S$ in $M$. It implies that $S \to M/G$ is an open embedding.

As usual, a double index implicitly implies a summation over that index. We have

$$\begin{aligned}
d\big(\frac{wz^i}{|z|}\big)d\big(\frac{\bar{w}\bar{z}^i}{|z|}\big) =&\big(\frac{w}{|z|}dz^i + \frac{z^i}{|z|}dw - \frac{wz^i}{2|z|^3}(z^j d\bar{z}^j + \bar{z}^j dz^j)\big) \wedge \big(\text{complex conjugate}\big)\\
=&\frac{|w|^2}{|z|^2}dz^i d\bar{z}^i + dwd\bar{w} + \frac{w}{|z|^2}(\bar{z}^i dz^i)d\bar{w} + \frac{\bar{w}}{|z|^2}dw(z^i d\bar{z}^i)\\
&- \frac{|w|^2}{|z|^4}(\bar{z}^i dz^i)(z^j d\bar{z}^j) - \frac{\bar{w}}{2|z|^2}dw(z^j d\bar{z}^j + \bar{z}^j dz^j)\\
&- \frac{w}{2|z|^2}(z^j d\bar{z}^j + \bar{z}^j dz^j)d\bar{w}\\
=&\frac{|w|^2}{|z|^2}dz^i d\bar{z}^i + dwd\bar{w} + \frac{1}{2|z|^2}(wd\bar{w} + \bar{w}dw)(z^i d\bar{z}^i - \bar{z}^i dz^i)\\
&- \frac{|w|^2}{|z|^4}(\bar{z}^i dz^i)(z^j d\bar{z}^j)\\
=&dz^i d\bar{z}^i + dwd\bar{w} - \varepsilon\big(\frac{1}{|z|^2}dz^i d\bar{z}^i - \frac{1}{|z|^4}(\bar{z}^i dz^i)(z^j d\bar{z}^j)\big)\\
=&dz^i d\bar{z}^i + dwd\bar{w} - \varepsilon\partial\bar{\partial}\log(|z|^2).
\end{aligned}$$

In the fourth step we used that $\nu^{-1}(\varepsilon/2) = \{(z,w)|\ |z|^2 = \varepsilon + |w|^2\}$, so that in particular $wd\bar{w} + \bar{w}dw = z^i d\bar{z}^i + \bar{z}^i dz^i$. We obtain $\kappa^*(p^*\omega_{\text{st}} + \varepsilon\pi^*\omega_{\text{FS}}) = \omega_{\text{st}}$ on $\mathbb{C}^n \times \mathbb{C}$, which proves the lemma. □

Besides providing an alternative construction of the symplectic manifold $\widetilde{\mathbb{C}}^n$, the nice feature of the cut is the slice given by (2.11), which in this context gives rise to a symplectomorphism

$$\varphi : \big(\mathbb{C}^n\backslash\overline{B_\varepsilon}, \omega_{\text{st}}\big) \to \big(\widetilde{\mathbb{C}}^n\backslash E, p^*\omega_{\text{st}} + \varepsilon\pi^*\omega_{\text{FS}}\big).$$

Here $B_\varepsilon$ denotes the ball of radius $\sqrt{\varepsilon}$ in $\mathbb{C}^n$. We can use this to give an alternative description of the blow-up. Consider again a point $y_0 \in M$ and choose a Darboux chart $\mathbb{C}^n$ around it. Via the chart we consider $B_\varepsilon$, for $\varepsilon$ small enough, as a subset of $M$, and we define the blow-up of $y_0$ in $M$ by

$$\widetilde{M} := (M\backslash\overline{B_\varepsilon}) \cup_\varphi \widetilde{\mathbb{C}}^n. \tag{2.13}$$

This is a symplectic manifold again because the glueing map $\varphi$ is symplectic. As compared to the construction given before, the advantage of this approach is that we do not need the choice of a bump function to create the symplectic form. The only choices involved are that of the symplectic coordinate chart and the parameter $\varepsilon$. Note that the latter again has to be chosen sufficiently small, this time in order for the ball of radius $\sqrt{\varepsilon}$ to fit into the Darboux chart.

The disadvantage of this approach is the fact that now there is no canonically defined blow-down map. Indeed, using (2.11) and (2.12), we see that the blow-down map $p$ :

$\widetilde{\mathbb{C}}^n \to \mathbb{C}^n$ corresponds to the map $p \circ \varphi : \mathbb{C}^n \backslash \overline{B_\varepsilon} \to \mathbb{C}^n$ given by

$$p \circ \varphi : z \mapsto \left(1 - \frac{\varepsilon}{|z|^2}\right)^{\frac{1}{2}} \cdot z.$$

Since this is not the identity map, we can not canonically extend the blow-down map on $\widetilde{\mathbb{C}}^n$ to all of $\widetilde{M}$. However, for $|z| \to \infty$ we do have that $p \circ \varphi$ converges to the identity map, so it is possible to interpolate between $p \circ \varphi$ and the identity, which can then be extended to the rest of $\widetilde{M}$. This interpolation step basically consists of choosing a bump function again. No matter the construction, the blow-down map is never going to be symplectic in a neighborhood of the exceptional divisor.

*Remark* 2.4.3. The symplectic blow-up is, in contrast to the complex one, not unique. Even in the above described case of a point, the problem is that different Darboux charts around it are not necessarily symplectically isotopic.

# Chapter 3

# Blow-ups of Generalized Complex Structures

Our objective in this chapter is to study which kind of generalized complex submanifolds admit a blow-up that is again generalized complex. We will give three answers to this question. First, we consider generalized complex submanifolds which look complex in transverse directions, i.e. generalized Poisson submanifolds. We show that these inherit a canonical holomorphic ideal, and hence a canonical blow-up. We then give a necessary and sufficient condition for the blow-up to admit a generalized complex structure for which the blow-down map is generalized holomorphic. Secondly, we consider generalized complex submanifolds which look symplectic in transverse directions, i.e. generalized Poisson transversals. For these we construct a global neighborhood theorem, which we then use to endow the blow-up with a generalized complex structure, if the submanifold is in addition compact. This structure agrees with the original structure outside of a neighborhood of the exceptional divisor. There is no obstruction to blow-up in this case, but the blow-up is not canonical. Finally, we describe explicitly a class of generalized complex submanifolds which are not of the above mentioned types, and show that they do not admit a generalized complex blow-up. The results in this chapter are based on joint work with Bailey and Cavalcanti [4].

## 3.1 Generalized Poisson submanifolds

In this section we will look at generalized Poisson submanifolds, which were introduced in Section 1.3.1. For convenience we recall the definition here.

**Definition 3.1.1.** Let $(M, \mathcal{J})$ be a generalized complex manifold. A *generalized Poisson submanifold* is a submanifold $Y \subset M$ such that

$$\mathcal{J} N^*Y = N^*Y. \tag{3.1}$$

We saw in Section 1.3.1 that these are generalized complex submanifolds and that they are Poisson submanifolds for the underlying Poisson structure $\pi_{\mathcal{J}}$, for which the inclusion map is generalized holomorphic. The key fact in the blow-up theory of generalized Poisson submanifolds is the following.

**Proposition 3.1.2.** *Let $Y \subset (M, \mathcal{J})$ be a closed generalized Poisson submanifold. Then there is a canonical holomorphic ideal $I_Y$ for $Y$, whose induced complex structure on $N^*Y$ is given by $\mathcal{J}$.*

*Proof.* Consider a generalized complex chart $U = (\mathbb{R}^{2n-2k}, \omega_{\mathrm{st}}) \times (\mathbb{C}^k, \sigma)$ around a point in $Y$, as provided by Theorem 1.3.16. Since $Y$ is a union of symplectic leaves we have $Y \cap U = W \times Z$, where $W \subset \mathbb{R}^{2n-2k}$ is open and $Z \subset \mathbb{C}^k$ is a complex submanifold which is Poisson for $\sigma$. By choosing appropriate holomorphic coordinates $z^i$ on $\mathbb{C}^k$, we may assume that $Z = \{z^1, \ldots, z^l = 0\}$, and a natural choice of holomorphic ideal for $Y$ in $U$ is then given by $\langle z^1, \ldots, z^l \rangle$. To patch these local ideals together into a global one we need to show that on the overlap of two charts the corresponding ideals match. So suppose $(\mathbb{R}^{2n-2k}, \omega_i) \times (\mathbb{C}^k, \sigma_i)$, $i = 1, 2$, are two local charts[1], and suppose that $(\varphi, B)$ is a generalized complex isomorphism between them which maps $Y$ to itself. Let $(x, z)$ and $(y, w)$ be coordinates on the two charts, where $x, y$ and $z, w$ denote the symplectic and complex directions respectively, and such that $I_Y$ is given by $\langle z^1, \ldots, z^l \rangle$, respectively $\langle w^1, \ldots, w^l \rangle$. By symmetry, it suffices to show that $\varphi^* w^i \in \langle z^1, \ldots, z^l \rangle$ for all $i \leq l$. As is shown in [35, Ch.VI], this condition may be verified on the level of Taylor series, and since we already know that $\varphi^* w^i \in \langle z^1, \ldots, z^l, \bar{z}^1, \ldots, \bar{z}^l \rangle$ because $\varphi(Y) = Y$, we only need to verify that

$$\left. \frac{\partial^r w^i}{\partial \bar{z}^{i_1} \ldots \partial \bar{z}^{i_r}} \right|_Y = 0, \quad \forall r \geq 0, \ \forall i, i_1, \ldots, i_r \in \{1, \ldots, l\}. \tag{3.2}$$

Here we are abusing notation slightly in writing $w^i$ instead of $\varphi^* w^i$. The case $r = 0$ reads $w^i|_Y = 0$, which is satisfied since $\varphi(Y) = Y$. To verify (3.2), we first write out what it means for $(\varphi, B)$ to be an isomorphism on the level of spinors:

$$e^{i\omega_1} e^{\sigma_1} (dz^1 \ldots dz^k) = e^{f + B + i\omega_2} e^{\sigma_2} (dw^1 \ldots dw^k). \tag{3.3}$$

The scaling factor $e^f$ is there because representatives for spinor lines are unique only up to scaling. At $Y$, using that $Y$ is Poisson, (3.3) becomes

$$e^{i\omega_1} (dz^1 \ldots dz^l) e^{\sigma_1} (dz^{l+1} \ldots dz^k) = e^{f + B + i\omega_2} (dw^1 \ldots dw^l) e^{\sigma_2} (dw^{l+1} \ldots dw^k).$$

Now apply $dw^i \wedge \iota_{\partial_{\bar{z}^{i_1}}}$, with $i, i_1 \leq l$, to both sides. The left-hand side vanishes, while the only survivor on the right is given by

$$\frac{\partial w^i}{\partial \bar{z}^{i_1}} e^{f + B + i\omega_2} (dw^1 \ldots dw^l) e^{\sigma_2} (dw^{l+1} \ldots dw^k),$$

[1] Strictly speaking we should look at open neighborhoods of 0 but for sake of notation we suppress this. Also note that we can assume that the "$k$" in both charts is the same, as the type can only jump in even steps and $(\mathbb{R}^{4s}, \omega_{\mathrm{st}})$ is isomorphic to $(\mathbb{C}^{2s}, \sigma_0)$ for $\sigma_0$ an invertible holomorphic Poisson structure.

so (3.2) holds for $r \leq 1$. This implies in particular that the forms $dz^1 \wedge \ldots \wedge dz^l$ and $dw^1 \wedge \ldots \wedge dw^l$ are proportional along $Y$, where we think of $w^i$ as a function of $(x, z)$.

Suppose inductively that for some $m \geq 1$ (3.2) is satisfied for all $r \leq m$. Apply $dw^i \wedge \mathcal{L}_{\partial_{\bar{z}^{i_1}}} \ldots \mathcal{L}_{\partial_{\bar{z}^{i_m}}}$, for any $i, i_1, \ldots, i_m \leq l$, to both sides of (3.3) and evaluate the resulting expression at $Y$. The left-hand side will vanish again because $\omega_1$ is independent of $z$ and $\sigma_1$ is holomorphic. Using multi-index notation, the Leibniz rule gives

$$0 = dw^i \sum_{\substack{I \sqcup J \sqcup K \sqcup L \\ = \{i_1, \ldots, i_m\}}} \mathcal{L}_{\partial_{\bar{z}^I}}(e^{f+B+i\omega_2}) \mathcal{L}_{\partial_{\bar{z}^J}}(e^{\sigma_2}) \mathcal{L}_{\partial_{\bar{z}^K}}(dw^1 \ldots dw^l) \mathcal{L}_{\partial_{\bar{z}^L}}(dw^{l+1} \ldots dw^k). \tag{3.4}$$

**Claim:** We have $\mathcal{L}_{\partial_{\bar{z}^J}} \sigma_2(dw^j)|_Y = 0$ for all $J \subset \{i_1, \ldots, i_m\}$ and $j \leq l$.

Let us accept this claim for the moment and continue with the proof. We compute

$$\mathcal{L}_{\partial_{\bar{z}^K}} dw^j = \sum_{1 \leq a \leq k} \frac{\partial^{|K|+1} w^j}{\partial z^a \partial \bar{z}^K} dz^a + \sum_{1 \leq a \leq k} \frac{\partial^{|K|+1} w^j}{\partial \bar{z}^a \partial \bar{z}^K} d\bar{z}^a + \sum_{1 \leq b \leq 2n-2k} \frac{\partial^{|K|+1} w^j}{\partial x^b \partial \bar{z}^K} dx^b. \tag{3.5}$$

If $j \leq l$, the function $\partial^{|K|} w^j / \partial \bar{z}^{|K|}$ vanishes along $Y$ by the induction hypothesis. Hence, at $Y$, the first and second terms above with $a > l$ together with the entire third term vanish, because we differentiate in directions tangent to $Y$. If in addition $|K| < m$, the second term vanishes by the induction hypothesis. It follows that for $K \subsetneq \{i_1, \ldots, i_m\}$, $(\mathcal{L}_{\partial_{\bar{z}^K}}(dw^1 \ldots dw^l))|_Y$ is proportional to $(dw^1 \ldots dw^l)|_Y$. Using the Claim, these terms all disappear from (3.4) because we wedge everything with $dw^i$. Hence, (3.4) reduces to

$$0 = e^{f+B+i\omega_1} e^{\sigma_2} \sum_{1 \leq i_{m+1} \leq l} \frac{\partial^{m+1} w^i}{\partial \bar{z}^{i_1} \ldots \partial \bar{z}^{i_{m+1}}} d\bar{z}^{i_{m+1}} dw^1 \ldots dw^k$$

at $Y$. Since $d\bar{z}^{i_{m+1}}$ is a linear combination of $d\bar{w}^1, \ldots, d\bar{w}^l$, the terms in this summation are all linear independent. So (3.2) holds for $r = m + 1$ as well, and therefore for all $r$ by induction. □

*Proof of the Claim.* If we write $\sigma_2 = \sigma_2^{ab} \partial_{w^a} \partial_{w^b}$, the Poisson condition implies that $\sigma_2^{ab}$ vanishes at $Y$ for $a \leq l$ or $b \leq l$. A repeated Lie derivative on $\sigma_2$ will be a sum of terms of the form

$$\frac{\partial^r \sigma_2^{ab}}{\partial \bar{z}^{i_1} \ldots \partial \bar{z}^{i_r}} (\mathcal{L}_{\partial_{\bar{z}^{j_1}}} \ldots \mathcal{L}_{\partial_{\bar{z}^{j_s}}} \partial_{w^a})(\mathcal{L}_{\partial_{\bar{z}^{k_1}}} \ldots \mathcal{L}_{\partial_{\bar{z}^{k_t}}} \partial_{w^b}). \tag{3.6}$$

Using the chain rule and the fact that $\sigma_2$ is holomorphic we can rewrite the first term in terms of $w$-derivatives. By the induction hypothesis there are no derivatives in the $w^i$-directions for $i \leq l$, because these come together with a term of the form $\partial w^i / \partial \bar{z}^{i_j}$ or a further derivative thereof (note that $i_1, \ldots, i_r \leq l$). Moreover, if either $a \leq l$ or $b \leq l$ there are also no $w^i$-derivatives for $i > l$ because these are tangent to $Y$ along which $\sigma^{ab}$ is constantly equal to zero. Hence (3.6) will only be nonzero at $Y$ for $a, b > l$, and so to prove the Claim it suffices to show that $(\mathcal{L}_{\partial_{\bar{z}^{j_1}}} \dots \mathcal{L}_{\partial_{\bar{z}^{j_s}}} \partial_{w^a})(dw^j)|_Y = 0$ for $a > l, j \leq l$. Abbreviating $J = \{j_1, \dots, j_s\}$, we have

$$0 = \mathcal{L}_{\bar{z}^J}(dw^j(\partial_{w^a})) = \sum_{J_1 \sqcup J_2 = J} (\mathcal{L}_{\bar{z}^{J_1}} dw^j)(\mathcal{L}_{\bar{z}^{J_2}} \partial_{w^a}).$$

From (3.5) and the comments below that we see that $\mathcal{L}_{\bar{z}^{J_1}} dw^j$ is either a linear combination of $dw^1, \dots, dw^l$, or it is proportional to $d\bar{w}^b$ for some $b$. The latter can only happen if $J_1 = J = \{i_1, \dots, i_m\}$, but then obviously $\mathcal{L}_{\bar{z}^J} dw^j(\partial_{w^a}) = 0$. The result now follows by induction over $s$. $\square$

Having a canonical holomorphic ideal for $Y$, we obtain a canonical blow-up $\widetilde{M}$ as a smooth manifold. We now investigate whether $\widetilde{M}$ carries a generalized complex structure for which the blow-down map $p$ is holomorphic. Clearly this structure exists and is unique on $\widetilde{M} \backslash E$, where $E$ is the exceptional divisor, and we need to verify whether it extends over $E$. From the definition of the ideal $I_Y$ and the construction of the blow-up given in the proof of Theorem 2.1.6, the blow-down map $p$ is locally given by

$$\mathbb{R}^{2n-2k} \times \mathrm{Bl}_Z \mathbb{C}^k \to \mathbb{R}^{2n-2k} \times \mathbb{C}^k,$$

where $Y$ is locally given by $\mathbb{R}^{2n-2k} \times Z \subset \mathbb{R}^{2n-2k} \times \mathbb{C}^k$, and $\mathrm{Bl}_Z \mathbb{C}^k$ denotes the complex blow-up of $Z \subset \mathbb{C}^k$. The target is equipped with the generalized complex structure determined by the standard symplectic form on $\mathbb{R}^{2n-2k}$ and a holomorphic Poisson structure $\sigma$ on $\mathbb{C}^k$. Clearly, this structure lifts if and only if $\sigma$ lifts to $\mathrm{Bl}_Z \mathbb{C}^k$. So we are led to the following question: When does a holomorphic Poisson structure lift to a complex blow-up? This was addressed by Polishchuk in [37], and for completeness we review the results here. Let $(X, \sigma)$ be a holomorphic Poisson manifold, and $Z \subset X$ a holomorphic Poisson submanifold. Recall that this means that $Z$ is a complex submanifold, and that $\sigma(N^{*1,0}Z) = 0$. The latter is equivalent to the ideal $I_Z$ of $Z$ being a Poisson ideal in the ring of holomorphic functions on $X$. In particular, $N^{*1,0}Z$ inherits a fiberwise Lie algebra structure, given by the Poisson bracket under the natural isomorphism $N^{*1,0}Z \cong I_Z/I_Z^2$. Explicitly, if $f$ and $g$ are two local holomorphic functions that vanish on $Z$, then $[df, dg] = d\{f, g\}$.

To state the blow-up conditions on $Z$, we need the following terminology.

**Definition 3.1.3.** A Lie algebra $\mathfrak{g}$ is *degenerate* if the map $\Lambda^3 \mathfrak{g} \to \mathrm{Sym}^2(\mathfrak{g})$ given by

$$x \wedge y \wedge z \mapsto [x, y]z + [y, z]x + [z, x]y$$

vanishes.

This is a rather abstract condition, so we first give an alternative characterization which is more geometric.

**Lemma 3.1.4.** A Lie algebra $\mathfrak{g}$ is degenerate if and only if $[x, y] \in k \cdot x + k \cdot y$ for all $x, y \in \mathfrak{g}$, where $k$ is the field over which $\mathfrak{g}$ is defined.

*Proof.* If $\dim(\mathfrak{g}) \leq 2$ then both conditions are always satisfied, so assume that $\dim(\mathfrak{g}) \geq 3$. If $\mathfrak{g}$ is degenerate, then for $x, y \in \mathfrak{g}$ we choose a $z$ which is not in the plane spanned by $x$, $y$, and from

$$[x, y]z = -[y, z]x - [z, x]y$$

we see that $[x, y]$ must be a linear combination of $x$ and $y$. Conversely, suppose that the bracket of any two elements lies in their linear span. Then, for $x$, $y$ and $z$ linearly independent we have $[x, y] = ax + by$, $[y, z] = cy + dz$ and $[z, x] = ez + fx$ for some $a, b, c, d, e, f \in k$. Moreover, we also have $[y, x + z] = gy + h(x + z)$ for some $g, h \in k$. Since $x, y, z$ are linearly independent it follows that $a = -d$, and similarly we have $b = -e$ and $c = -f$. Consequently,

$$[x, y]z + [y, z]x + [z, x]y = (c + f)xy + (e + b)yz + (a + d)zx = 0,$$

showing that $\mathfrak{g}$ is degenerate. □

*Remark* 3.1.5. The notion of degeneracy depends on the base field over which $\mathfrak{g}$ is defined. For instance, the complexification of a degenerate Lie algebra over $\mathbb{R}$ is degenerate over $\mathbb{C}$, but a degenerate Lie algebra over $\mathbb{C}$ need not be degenerate over $\mathbb{R}$ when we restrict scalars. It is shown in [37] that degeneracy is equivalent to being either Abelian, or isomorphic to the algebra generated by $e_1, \ldots, e_{n-1}, f$, with relations $[e_i, e_j] = 0$ and $[f, e_i] = e_i$. Note that 2-dimensional Lie algebras are always degenerate.

If $Z \subset (X, \sigma)$ is a holomorphic Poisson submanifold, we call $N^{*1,0}Z$ degenerate if its fiberwise Lie algebra structure is degenerate over $\mathbb{C}$. This is equivalent to the condition

$$\{f, g\}h + \{g, h\}f + \{h, f\}g \in I_Z^3 \quad \forall f, g, h \in I_Z. \tag{3.7}$$

Now let $p : \widetilde{X} \to X$ denote the complex blow-up along a complex submanifold $Z$, which for the moment we do not assume to be a Poisson submanifold, and let $E$ denote the exceptional divisor. We say that $\sigma$ can be lifted if there exists a holomorphic Poisson structure $\widetilde{\sigma}$ on $\widetilde{X}$ for which $p$ is a Poisson map. Such a lift is necessarily unique, because $p$ is an isomorphism almost everywhere.

**Proposition 3.1.6** ([37])**.** *There exists a lift $\widetilde{\sigma}$ on $\widetilde{X}$ if and only if $Z$ is a Poisson submanifold and $N^{*1,0}Z$ is degenerate. The exceptional divisor is a Poisson submanifold with respect to $\widetilde{\sigma}$ if and only if $N^{*1,0}Z$ is Abelian.*

*Proof.* Let $(z^1, \ldots, z^k)$ be local coordinates on $X$ with $Z = \{z^1, \ldots, z^l = 0\}$ for some $l \leq k$. This chart is covered by $l$ charts on the blow-up $\widetilde{X}$ on which the projection has the form (see (2.2))

$$p : (v^1, \ldots, z^a, \ldots, v^l, z^{l+1}, \ldots, z^k) \mapsto (z^a v^1, \ldots, z^a, \ldots, z^a v^l, z^{l+1}, \ldots, z^k) \tag{3.8}$$

for $a \leq l$. Then $p$ is an isomorphism on the open dense set $\{z^a \neq 0\}$, where we have $v^j = z^j/z^a$. We have to verify when the Poisson brackets extend smoothly over the exceptional divisor $\{z^a = 0\}$. There are two types of brackets that cause trouble. Firstly, we have

$$\{z^i, v^j\} = \{z^i, \frac{z^j}{z^a}\} = \frac{1}{(z^a)^2}\big(z^a\{z^i, z^j\} - z^j\{z^i, z^a\}\big), \tag{3.9}$$

for $i = a$ or $i > l$, and $j \leq l$ with $j \neq a$. On the blow-up we know that something of the form $f/z^a$, where $f \in I_Z = \langle z^1, \dots, z^l\rangle$, defines a non-singular function. In particular, (3.9) is smooth on the blow-up if and only if $z^a\{z^i, z^j\} - z^j\{z^i, z^a\}$ lies in $I_Z^2$. Since $j \neq a$, this can only happen if both $\{z^i, z^j\}$ and $\{z^i, z^a\}$ lie in $I_Z$. Since this must hold for all $a \leq l$, we need that $I_Z$ is a Poisson ideal, i.e. $Z$ has to be a Poisson submanifold. The second type of brackets that may become singular are

$$\{v^i, v^j\} = \{\frac{z^i}{z^a}, \frac{z^j}{z^a}\} = \frac{1}{(z^a)^3}\big(z^a\{z^i, z^j\} + z^i\{z^j, z^a\} + z^j\{z^a, z^i\}\big), \tag{3.10}$$

for $1 \leq i, j \leq l$, $i \neq a \neq j$. As above, this is smooth on the blow-up if and only if the term between brackets lies in $I_Z^3$, which is precisely condition (3.7) for degeneracy of $I_Z$. Finally, to check whether $E$ is Poisson we need to check whether its ideal, which on the above chart is generated by $z^a$, is a Poisson ideal. In other words, we need to check when $z^a$ divides the brackets $\{z^i, z^a\}$ for $i > l$ and $\{v^j, z^a\}$ for $1 \leq j \leq l$, $j \neq a$. The Poisson condition on $Z$ implies that $\{z^i, z^a\} \in I_Z$, which means that it is divisable by $z^a$ on the blow-up. However, for the second one we have

$$\{v^j, z^a\} = \frac{1}{z^a}\{z^j, z^a\}.$$

For this to be divisable by $z^a$, we need $\{z^j, z^a\} \in I_Z^2$. Since this must hold for all the local charts, this means that $\{I_Z, I_Z\} \subset I_Z^2$, i.e. $N^{*1,0}Z$ needs to be Abelian. □

If $Y \subset (M, \mathcal{J})$ is a generalized Poisson submanifold then $Y$ is in particular a Poisson submanifold for $\pi_{\mathcal{J}}$ and so $N^*Y$ inherits a fiberwise real Lie algebra structure $[\cdot,\cdot]_{\pi_{\mathcal{J}}}$ in the same manner as discussed above in the holomorphic Poisson context. Specifically, for $\alpha, \beta \in N_y^*Y$, we extend them to one-forms $\tilde{\alpha}, \tilde{\beta}$ and set

$$[\alpha, \beta]_{\pi_{\mathcal{J}}} := d_y(\pi_{\mathcal{J}}(\tilde{\alpha}, \tilde{\beta})).$$

This is independent of the choice of extensions since $\pi_{\mathcal{J}}(N^*Y) = 0$. Note that for functions $f$ and $g$ with $f|_Y = g|_Y = 0$ we have $[d_y f, d_y g]_{\pi_{\mathcal{J}}} = d_y\{f, g\}$. Hence, the bundle $N^*Y$ has a fiberwise Lie algebra structure, as well as a complex structure induced by $\mathcal{J}$. These two are compatible, in the sense that $[\cdot,\cdot]_{\pi_{\mathcal{J}}}$ is complex linear. This can for instance be seen in a local chart, see also the proof of the theorem below. We call $N^*Y$ degenerate if $[\cdot,\cdot]_{\pi_{\mathcal{J}}}$ is degenerate over $\mathbb{C}$.

**Theorem 3.1.7.** *Let $Y \subset (M, \mathcal{J})$ be a generalized Poisson submanifold and let $p : \widetilde{M} \to M$ denote the blow-up with respect to the canonical holomorphic ideal $I_Y$ from Proposition 3.1.2. Then $\widetilde{M}$ has a generalized complex structure for which $p$ is generalized holomorphic if and only if $N^*Y$ is degenerate. The exceptional divisor is a generalized Poisson submanifold if and only if $N^*Y$ is Abelian.*

*Proof.* Pick a local chart where $Y = W \times Z \subset (\mathbb{R}^{2n-2k}, \omega_{\text{st}}) \times (\mathbb{C}^k, \sigma)$, with $W$ open and $Z$ a holomorphic Poisson submanifold (see the proof of Proposition 3.1.2). As explained in the discussion above Definition 3.1.3, the generalized complex structure lifts to the blow-up if and only if $\sigma$ lifts to the blow-up of $Z$ in $\mathbb{C}^k$, which we now know to be equivalent to $N^{*1,0}Z$ being degenerate. It remains to relate degeneracy of $N^{*1,0}Z$ to that of $N^*Y$. The latter coincides with $N^*Z$, the normal bundle of $Z$ considered as a real submanifold of $\mathbb{C}^k$, which inherits a complex structure because $Z$ is a complex submanifold. If $Q = \text{Re}(\sigma)$, we have

$$\begin{aligned}[\alpha, \beta]_Q = d\big(Q(\widetilde{\alpha}, \widetilde{\beta})\big) =& d\big(\frac{1}{2}\sigma(\widetilde{\alpha}^{1,0}, \widetilde{\beta}^{1,0}) + \frac{1}{2}\bar{\sigma}(\widetilde{\alpha}^{0,1}, \widetilde{\beta}^{0,1})\big) \\ =& \frac{1}{2}[\alpha^{1,0}, \beta^{1,0}]_\sigma + \frac{1}{2}[\alpha^{0,1}, \beta^{0,1}]_{\bar{\sigma}},\end{aligned}$$

for $\alpha, \beta \in N^*Z$ and $\widetilde{\alpha}, \widetilde{\beta} \in \Gamma(T^*M)$ smooth extensions. Consequently, the complex linear isomorphism $N^*Z \to N^{*1,0}Z$ given by $\alpha \mapsto \alpha^{1,0}$ carries $[\cdot,\cdot]_Q$ over to $\frac{1}{2}[\cdot,\cdot]_\sigma$ and so $[\cdot,\cdot]_Q$ is complex linear. In particular, $N^{*1,0}Z$ is degenerate if and only if $(N^*Z, [\cdot,\cdot]_Q)$ is degenerate as a complex Lie algebra. Now in the local chart, $\pi_{\mathcal{J}} = -\omega_{\text{st}}^{-1} \oplus 4IQ$, so that $[\cdot,\cdot]_Q$ and $[\cdot,\cdot]_{\pi_{\mathcal{J}}}$ agree up to a complex multiple. In particular, $[\cdot,\cdot]_{\pi_{\mathcal{J}}}$ is complex linear and degenerate over $\mathbb{C}$ if and only if $[\cdot,\cdot]_Q$ is. Finally, the statement about the exceptional divisor follows from Proposition 3.1.6. □

Since degeneracy is automatic for Lie algebras of dimension 2 , we obtain

**Corollary 3.1.8.** Let $Y \subset (M, \mathcal{J})$ be a generalized Poisson submanifold of complex codimension 2 . Then it can be blown up in a generalized complex way.

**Example 3.1.9.** Let $(M, \mathcal{J})$ be a generalized complex manifold. In [5] it is shown that the complex locus, i.e. the points of type 0, carries canonically the structure of a complex analytic space. In a local generalized complex chart, this structure is induced by regarding the complex locus as the vanishing set of the corresponding holomorphic Poisson structure[2]. As such, any complex submanifold of the complex locus is automatically a generalized Poisson submanifold. In particular, such a submanifold can be blown up as soon as its conormal bundle is degenerate. For example, any point in the complex locus on a generalized complex four-manifold can be blown up. This generalizes the corresponding result from [15], where it was assumed that the point lies in the smooth part of the complex locus.

**Example 3.1.10.** An example where the submanifold has positive dimension is the maximal torus $S^1 \times S^1 \subset S^3 \times S^3$. Here we view $S^3 = SU(2)$ as a compact Lie group, so that $S^3 \times S^3$ is a compact, connected, even dimensional Lie group. From Example 1.4.6 we know that it carries a generalized complex structure (even a generalized Kähler structure) for which the maximal torus $S^1 \times S^1$ is a generalized Poisson submanifold, which is automatically degenerate because it has complex codimension 2 . More details on this example, as well as a study of other Lie groups will be given in Section 4.4.

[2]Note that both the holomorphic structure as well as the holomorphic Poisson tensor on the chart are not unique, but, and this is the crucial point proved in [5], they do induce the same structure on the complex locus.

**Example 3.1.11.** Let $(M, \mathcal{J})$ be a four-dimensional generalized complex manifold which is generically symplectic, with non-empty complex locus $Z$. Since $Z$ is locally described by the vanishing of a holomorphic Poisson tensor in two complex dimensions, it looks locally like a complex curve on a complex surface. By example 3.1.9 we can blow up $M$ at any point in $Z$, and use this to "desingularize" $Z$. Indeed, as is proven for example in [7] (see also Section 2.3), if $C \subset X$ is any complex curve on a smooth complex surface $X$, one can perform a locally finite number of blow-ups on $X$ so that the underlying analytic set of the total transform of the curve $C$ has only ordinary double points. In particular, the total transform[3] itself will be a normal crossing divisor with possible multiplicities (so in local coordinates $(z_1, z_2)$, it will be given by $z_1^a z_2^b = 0$ for some $a, b \in \mathbb{Z}_{\geq 0}$). Now we do not have a global complex structure available, but this desingularization procedure is of a local nature, so we conclude that after a (locally finite) number of blow-ups, we get a generalized complex manifold whose complex locus, as a complex analytic space, has only normal crossing singularities.

In general there is not so much we can say about the structure of the exceptional divisor $E$, except for the fact that it is *coisotropic* for $\pi_{\mathcal{J}}$, i.e. $\pi_{\mathcal{J}}(N^*E) \subset TE$. This is equivalent to the ideal of $E$ being a Poisson subalgebra, which is indeed the case because it is the pull-back of the ideal of $Y$ by the Poisson map $p$, and $Y$ is a Poisson submanifold (so in particular coisotropic). From Lemma 1.3.33 we obtain the following corollary.

**Corollary 3.1.12.** Let $Y$ be a generalized Poisson submanifold with degenerate conormal bundle, and let $E$ denote the exceptional divisor of the blow-up. If $Y$ carries the structure of a generalized complex brane then so does $E$.

In the special case that $Y$ is given by a point $x \in M$ we can say more.

**Proposition 3.1.13.** *Let $x \in (M, \mathcal{J})$ be a point of complex type such that $T_x^*M$ is degenerate. Then the exceptional divisor $E$ of the generalized complex blow-up is a generalized complex brane. Moreover, $E$ is a generalized Poisson submanifold if and only if $T_x^*M$ is Abelian, which is the case if either $\mathcal{J}$ is everywhere of complex type, or if $x$ is a singular point of the complex locus. If $T_x^*M$ is not Abelian, then the complex locus of $M$ is a smooth complex curve around $x$, and $E$ intersects the complex locus of $\widetilde{M}$ transversally at the point that corresponds to the tangent of this curve.*

*Proof.* The statement about $E$ being a brane follows from the previous corollary, since any point of complex type carries the structure of a generalized complex brane. Let $(\mathbb{C}^n, \sigma)$ be a holomorphic Poisson structure serving as a local model for $(M, \mathcal{J})$ around $x$. We already know that $E$ is a generalized Poisson submanifold if and only if $T_x^*M$ is Abelian. If we write $\sigma = \sum_{i,j} \sigma^{ij} \partial_{z^i} \partial_{z^j}$, then the complex locus is given by the common vanishing locus of the holomorphic functions $\sigma^{ij}$. The induced Lie algebra on $T_0^*\mathbb{C}^n$ is given by

$$[dz^i, dz^j] = \sum_k (\partial_{z^k} \sigma^{ij}(0)) dz^k.$$

[3]The *total transform* of a subset $C$ under a blow-up equals $p^{-1}(C)$ where $p$ is the blow-down map, while the *proper transform* is given by $\overline{p^{-1}(C) \backslash E}$.

If this is Abelian then either the $\sigma^{ij}$ are identically 0 on $\mathbb{C}^n$ (the complex case), or $x$ is a singular point[4]. From Remark 3.1.5 we know that if $T_x^*M$ is not Abelian, then there exists a basis $\alpha^1, \ldots, \alpha^{n-1}, \beta$ of $\mathbb{C}^n$ such that $[\alpha^i, \beta] = \alpha^i$, with all other brackets vanishing. By a linear change of coordinates we may assume that $\alpha^i = dz^i$ and $\beta = dw$ for coordinates $(z^1, \ldots, z^{n-1}, w)$ on $\mathbb{C}^n$. If we write $\sigma = \sum_{i,j} \sigma^{ij}\partial_i\partial_j + \sum_i \sigma^i \partial_i \partial_w$ where $\partial_i := \partial_{z^i}$, we obtain $\sigma^{ij}(0) = \sigma^i(0) = 0$, $d_0\sigma^{ij} = 0$ and $d_0\sigma^i = dz^i$. Consequently, $C := \{\sigma^1 = 0, \ldots, \sigma^{n-1} = 0\}$ is a smooth curve in a neighborhood of $0 \in \mathbb{C}^n$, and we claim that it equals the complex locus, i.e. the set of points where $\sigma = 0$. For that we need to show that the $\sigma^{ij}$ vanish whenever the $\sigma^i$ do. By the Jacobi identity we have $0 = \{w, \{z^i, z^j\}\} + \{z^j, \{w, z^i\}\} + \{z^i, \{z^j, w\}\}$, which translates into

$$0 = \sigma^i \partial_w \sigma^j - \sigma^j \partial_w \sigma^i + 2 \sum_k \left( \sigma^{kj} \partial_k \sigma^i + \sigma^{ik} \partial_k \sigma^j - \sigma^k \partial_k \sigma^{ij} \right).$$

On $C$, where all the $\sigma^i$ vanish, we obtain the equation

$$\sigma^{ij}(\partial_i \sigma^i + \partial_j \sigma^j) = - \sum_{k \neq i,j} (\sigma^{kj} \partial_k \sigma^i + \sigma^{ik} \partial_k \sigma^j).$$

Since $\partial_i \sigma^j = \delta_i^j$ at 0, we see that $(\partial_i \sigma^i + \partial_j \sigma^j)$ is invertible around 0, while the right-hand side of the equation vanishes to third order at 0. But this implies that $\sigma^{ij}$ vanishes to third order at 0, which implies that the right-hand side actually vanishes to fourth order at 0. Continuing in this fashion we deduce that $\sigma^{ij}|_C$ vanishes to infinite order at 0, hence $\sigma^{ij}|_C = 0$. For the final statement, consider the coordinate charts on the blow-up $\widetilde{\mathbb{C}}^n$ given by $(v^1, \ldots, z^i, \ldots, v^n)$ for $i < n$ and $(v^1, \ldots, v^{n-1}, w)$ (see (2.2)). On the first $n-1$ charts we have

$$\{z^i, v^n\} = \frac{1}{z^i}\{z^i, w\} = \frac{\sigma^i}{z^i}.$$

Since $\sigma^i = z^i$ modulo terms of order at least two, $\{z^i, v^n\}|_E = 1$, showing that in these charts the intersection of $E$ with the type change locus is empty. In the final chart we have

$$\{v^i, w\} = \frac{1}{w}\sigma^i, \qquad \{v^i, v^j\} = \frac{\sigma^{ij}}{w^2} + v^i \frac{\sigma^j}{w^2} - v^j \frac{\sigma^i}{w^2}.$$

Note that $\sigma^i = z^i$ modulo terms of higher order, so the first bracket vanishes at 0. Furthermore, the singular part of $v^i\sigma^j/w^2$ cancels agains that from $v^j\sigma^i/w^2$. Finally, $\sigma^{ij}$ is of order two at 0, and by the computation above we know that $\frac{\partial^2 \sigma^{ij}}{\partial w^2}(0) = 0$. In particular, the second bracket also vanishes at 0. □

*Remark* 3.1.14. If $T_x^*M$ is Abelian, $E$ may or may not be contained in the complex locus of $\widetilde{M}$. In a holomorphic Poisson chart $(\mathbb{C}^n, \sigma)$ around $x$ as above, the intersection of

[4]The notion of singularity here is in the context of complex analytic geometry. Specifically, even when 0 is an isolated zero of the $\sigma^{ij}$, we still consider it singular if the Lie algebra is Abelian, because the Jacobian of the defining equations has zero rank at 0.

$E \cong \mathbb{P}^{n-1}$ with the complex locus is given by the common zero set on $\mathbb{P}^{n-1}$ of the homogeneous polynomials

$$F^{ijk} := \sum_{1\le r,s\le n} \left(\sigma^{ij}_{rs} z^k z^r z^s + \sigma^{jk}_{rs} z^i z^r z^s + \sigma^{ki}_{rs} z^j z^r z^s\right),$$

where $\sum_{r,s} \sigma^{ij}_{rs} z^r z^s$ is the homogeneous term of degree two of the holomorphic function $\sigma^{ij}$. This statement follows from (3.10). In particular, if $\dim_{\mathbb{R}}(M) = 4$ or if the order of vanishing of the $\sigma^{ij}$ at $x$ is bigger or equal than three, then $E$ is fully contained in the type change locus.

*Remark* 3.1.15. In contrast to the blow-up in complex geometry, the exceptional divisor of a generalized complex blow-up need not be a generalized complex submanifold again. Indeed, consider the holomorphic Poisson manifold $(\mathbb{C}^2, \sigma = z^1\partial_{z^1}\partial_{z^2})$. It is of complex type on $\{z^1 = 0\}$ and of symplectic type on $\{z^1 \neq 0\}$. The generalized complex blow-up of $0 \in \mathbb{C}^2$ is given by $(\widetilde{\mathbb{C}}^2, \widetilde{\sigma})$ where $\widetilde{\sigma}$ is the lift of $\sigma$. It is of complex type along the proper transform of $\{z_1 = 0\}$, and of symplectic type on the complement. On this symplectic piece we have $\widetilde{\sigma} = \kappa^{-1}$, where $\kappa$ is a holomorphic symplectic form. Since $E$ has complex dimension 1 it is Lagrangian for $\kappa$, and so $E$ is not a generalized complex submanifold of $(\widetilde{\mathbb{C}}^2, \widetilde{\sigma})$.

## 3.2 Generalized Poisson transversals

We now turn our attention to submanifolds which are symplectic in transverse directions, i.e. generalized Poisson transversals (see Section 1.3.1). We recall the definition here for convenience.

**Definition 3.2.1.** Let $(M, \mathcal{J})$ be a generalized complex manifold. A *generalized Poisson transversal* is a submanifold $Y \subset M$ with

$$\mathcal{J}(N^*Y) \cap (N^*Y)^\perp = 0. \tag{3.11}$$

As explained in Section 1.3.1, a generalized Poisson transversal is automatically a generalized complex submanifold, and condition (3.11) is equivalent to

$$\pi_{\mathcal{J}}(N^*Y) + TY = TM|_Y,$$

which means that $Y$ intersects the symplectic leaves of $(M, \pi_{\mathcal{J}})$ symplectically and transversally. We will show that any compact generalized Poisson transversal can be blown-up. In order to do so we will first give a normal form, i.e. a tubular embedding of the conormal bundle that puts the generalized complex neighborhood of $Y$ into a standard form. We then only have to construct a blow-up for the latter.

### 3.2.1 A normal form

Let $Y \hookrightarrow (M, J)$ be a generalized Poisson transversal. Since $Y$ is a generalized complex submanifold it has its own generalized complex structure $\mathcal{J}_Y$. Moreover, the splitting

$TM|_Y = TY \oplus NY$, with $NY := \pi_{\mathcal{J}}(N^*Y)$, induces a decomposition $(\pi_{\mathcal{J}})|_Y = \pi_{\mathcal{J}_Y} + \omega_Y$, where $\pi_{\mathcal{J}_Y}$ equals the Poisson structure on $Y$ induced by $\mathcal{J}_Y$ and $\omega_Y \in \Gamma(\bigwedge^2 NY)$ is non-degenerate. The suggestive notation for the latter indicates that we will consider $\omega_Y$ as a symplectic form on the bundle $N^*Y$. In what follows, we will identify $Y$ with the zero section in $N^*Y$, and denote by $p : N^*Y \to Y$ the projection. Before we can state a neighborhood theorem we first need to give a normal form associated to the data above. For that we need two lemmas.

Recall that $T(N^*Y)$ has a canonical decomposition along $Y$ given by

$$T(N^*Y)|_Y = N^*Y \oplus TY. \tag{3.12}$$

**Lemma 3.2.2.** There exists a closed two-form $\sigma$ on the total space of $N^*Y$, which along $Y$ is given by $\omega_Y \oplus 0$.

*Proof.* Choose an Hermitian structure $(g, I)$ on $N^*Y$ compatible with $\omega_Y$. Let $e_j$ be a local unitary frame on $N^*Y$ with dual frame $e^j$, such that $\omega_Y = \sum_j \frac{i}{2} e^j \wedge \bar{e}^j$. We obtain local coordinates $(x, z)$ on $N^*Y$ by identifying $(x, z)$ with the point $\sum_j z^j e_j(x)$. Note that the $z$-coordinates are complex. If $\rho_\alpha$ is a partition of unity on $Y$ and $e_j^\alpha$ are local frames as above, define

$$\lambda := \sum_{\alpha, j} p^*(\rho_\alpha) \frac{i}{2} z_\alpha^j d\bar{z}_\alpha^j. \tag{3.13}$$

Then $\sigma := d\lambda$ restricts to $\omega_Y$ on $Y$ and its restriction to each fiber of $N^*Y$ is the translation invariant extension of $\omega_Y$. In addition, this particular choice of $\lambda$ is also $U(1)$-invariant. □

If $\sigma$ is such a closed extension of $\omega_Y$, we define a Dirac structure $L_\sigma$ on $N^*Y$ by[5]

$$L_\sigma := e_*^{i\sigma}(\mathfrak{B}p(L_Y)). \tag{3.14}$$

Since $p$ is a submersion, $\ker(p^*) = 0$ and hence (1.36) is satisfied. In particular $\mathfrak{B}p(L_Y)$ is a smooth Dirac structure, integrable with respect to the three-form $\widetilde{H} := p^* H_Y$ where $H_Y := i^* H$. Along the zero section $Y$ we have

$$L_\sigma|_Y = \{X + \xi + e - i\omega_Y(e) | X + \xi \in L_Y, e \in N^*Y\},$$

where we used the decomposition (3.12). In particular, $L_\sigma \cap \overline{L_\sigma} = 0$ at $Y$, hence also in a neighborhood of $Y$ in $N^*Y$. We will denote the resulting generalized complex structure by $\mathcal{J}_\sigma$. This will be our candidate for the normal form. In order to show that it does not depend on the choice of $\sigma$ we need the following.

**Lemma 3.2.3.** Let $\sigma_t$ be a smooth family of closed two-forms extending $\omega_Y$. Then there exists a family $\Phi_t = (\varphi_t, B_t)$ of generalized diffeomorphisms around $Y$ with $\Phi_0 = (\mathrm{Id}, 0)$, that satisfies $\mathfrak{F}\Phi_t(L_{\sigma_0}) = L_{\sigma_t}$ and which fixes $Y$ up to first order, i.e. $\varphi_t|_Y = \mathrm{Id}$, $d\varphi_t|_Y = \mathrm{Id}$ and $B_t|_Y = 0$.

[5] For the definition of the backward image see Section 1.2.3.

*Proof.* Since $\sigma_t - \sigma_0$ vanishes on $Y$, Lemma 3.2.4 below provides a family $\eta_t \in \Omega^1(N^*Y)$ with $\sigma_t - \sigma_0 = d\eta_t$, and such that $\eta_t$ and its first partial derivatives vanish along $Y$. By definition,

$$L_t := L_{\sigma_t} = e_*^{i\sigma_t}(\mathfrak{B}p(L_Y)) = e_*^{id\eta_t}(L_{\sigma_0}).$$

Since $\eta_t$ and $d\eta_t$ vanish along $Y$, $L_t$ defines a family of generalized complex structures $\mathcal{J}_t$ in a neighborhood of $Y$, integrable with respect to the (fixed) three-form $\widetilde{H}$. Consider the time-dependent generalized vector field $\mathcal{J}_t\dot{\eta}_t =: X_t + \xi_t$ and let $\psi_{t,s}$ be its flow, given by

$$\psi_{t,s} = (\varphi_{t,s})_* \circ e_*^{-\int_s^t \varphi_{r,s}^*(d\xi_r + \iota_{X_r}\widetilde{H})dr}, \tag{3.15}$$

where $\varphi_{t,s}$ is the flow of the time-dependent vector field $X_t$. It satisfies $\psi_{s,s} = \mathrm{Id}$ and

$$\frac{d}{dt}\psi_{t,s}(u) = -[\![\mathcal{J}_t\dot{\eta}_t, \psi_{t,s}(u)]\!].$$

Since $\eta_t$ together with its first derivatives vanish along $Y$, $\varphi_{t,s}$ is well defined in a neighborhood of $Y$ and fixes $Y$ to first order. We claim that

$$L_t = \psi_{t,0}L_0. \tag{3.16}$$

From the formula for $L_t$ this amounts to showing that $e_*^{-id\eta_t}\psi_{t,0}L_0 = L_0$. We have

$$\begin{aligned}\frac{d}{dt}e_*^{-id\eta_t}\psi_{t,0}(u) &= -i[\![\dot{\eta}_t, e_*^{-id\eta_t}\psi_{t,0}(u)]\!] - e_*^{-id\eta_t}[\![\mathcal{J}_t\dot{\eta}_t, \psi_{t,0}(u)]\!]\\ &= [\![-i\dot{\eta}_t - \mathcal{J}_0\dot{\eta}_t, e_*^{-id\eta_t}\psi_{t,0}(u)]\!].\end{aligned} \tag{3.17}$$

Here we used (1.9) and the definition of $\mathcal{J}_t$. This shows that $e_*^{-id\eta_t}\psi_{t,0}$ integrates the adjoint action of $-i\dot{\eta}_t - \mathcal{J}_0\dot{\eta}_t \in \Gamma(L_0)$. Since $\Gamma(L_0)$ is involutive, (3.16) indeed holds on the account of Lemma 1.1.5. The desired family of diffeomorphisms is then given by

$$\Phi_t = (\varphi_t, B_t) := (\varphi_{t,0}, \int_0^t \varphi_{r,0}^*(d\xi_r + \iota_{X_r}\tilde{H})dr). \qquad \square$$

In the proof above we used the following parametrized version of the Poincaré lemma.

**Lemma 3.2.4.** Let $\alpha_t \in \Omega^k_{\mathrm{cl}}(E)$ be a smooth family of closed forms on a vector bundle $E$ over $M$ which vanish along $M$. Then there exists a smooth family $\eta_t \in \Omega^{k-1}(E)$ with $d\eta_t = \alpha_t$, such that for each $t$ the form $\eta_t$ together with its first partial derivatives vanishes along $M$.

*Proof.* Let $V$ denote the Euler vector field on $E$, i.e. $V_\xi = \xi$ for $\xi \in E$. Its flow is given by $\varphi_s(\xi) = e^s\xi$, and we have

$$\alpha_t = \lim_{s\to-\infty}\left(\varphi_0^*\alpha_t - \varphi_s^*\alpha_t\right) = \int_{-\infty}^0 \frac{d}{ds}\varphi_s^*\alpha_t ds = d\left(\iota_V \int_{-\infty}^0 \varphi_s^*\alpha_t ds\right) =: d\eta_t.$$

Another formula for $\eta_t$ is given by $\eta_t = \iota_V \int_0^1 \frac{1}{s}L_s^*\alpha_t ds$, where $L_s$ denotes multiplication by $s$ on $E$. The forms $\eta_t$ satisfy all the desired properties. $\square$

With the help of the previous results we can now properly construct a normal form around a generalized Poisson transversal.

**Theorem 3.2.5.** *Let $Y \subset (M, \mathcal{J})$ be a generalized Poisson transversal. For any $\sigma \in \Omega^2_{cl}(N^*Y)$ which extends $\omega_Y$ there is a generalized complex structure $\mathcal{J}_\sigma$ on a neighborhood of $Y$ in $N^*Y$ given by (3.14). If $\sigma'$ is another extension of $\omega_Y$ then there is a family of generalized diffeomorphisms fixing $Y$ up to first order and taking $\mathcal{J}_\sigma$ to $\mathcal{J}_{\sigma'}$.*

*Proof.* We already know that $\mathcal{J}_\sigma$ is a generalized complex structure on a neighborhood of $Y$ in $N^*Y$. If $\sigma'$ is another closed extension of $\omega_Y$, we can apply Lemma 3.2.3 to $\sigma_t := (1-t)\sigma + t\sigma'$ to produce the desired family of generalized diffeomorphisms. □

Theorem 3.2.5 shows that any symplectic vector bundle over a generalized complex manifold has a generalized complex structure for which the base is a generalized Poisson transversal, and which up to isomorphism depends only on the symplectic vector bundle and the generalized complex manifold. The following theorem shows that all generalized Poisson transversals locally arise from this construction.

**Theorem 3.2.6.** *Let $Y \subset (M, \mathcal{J})$ be a generalized Poisson transversal. Then a neighborhood of $Y$ in $(M, \mathcal{J})$ is isomorphic to a neighborhood of $Y$ in $(N^*Y, \mathcal{J}_\sigma)$, where $\mathcal{J}_\sigma$ is one of the generalized complex structures of Theorem 3.2.5.*

*Remark* 3.2.7. In particular, this result tells us that on a neighborhood of $Y$, $\mathcal{J}$ is completely determined by the induced generalized complex structure $\mathcal{J}_Y$ on $Y$ and the induced symplectic structure on the vector bundle $N^*Y$.

*Proof.* We will prove this theorem by constructing an embedding of $N^*Y$ into $M$ that pulls back $\mathcal{J}$ to one of the structures of Theorem 3.2.5. This embedding will only depend on the choice of a connection on $TM$, and all such embeddings will turn out to be isotopic to each other.

Let $p : T^*M \to M$ be the cotangent bundle, and choose any connection $\nabla$ on $TM$, whose dual connection on $T^*M$ we also denote by $\nabla$. Using the Poisson structure $\pi_\mathcal{J}$ we obtain a vector field $V$ on $T^*M$ whose value at $\xi \in T^*M$ is given by $V_\xi := \pi_\mathcal{J}(\xi)^h_\xi$, where the superscript $h$ denotes the horizontal lift. We denote by $\varphi_t : T^*M \supset\!\to T^*M$ the flow of $V$.

**Lemma 3.2.8.** The map $exp := p \circ \varphi_1|_{N^*Y} : N^*Y \supset\!\to M$ gives a diffeomorphism from a neighborhood of $Y$ in $N^*Y$ onto an open neighborhood of $Y$ in $M$. If $\nabla'$ is a different connection then $exp'$ is isotopic to $exp$ via maps which are constant on $Y$ up to first order.

*Proof.* By definition of $V$ we have $L_s^*V = sV$ for $s \in \mathbb{R}$, where $L_s$ denotes multiplication by $s$ on the fibers of $T^*M$. It follows that[6] $\varphi_t(L_s\xi) = L_s(\varphi_{st}(\xi))$ for $\xi \in T^*M$. Hence,

$$d_y\varphi_t(\xi) = \frac{d}{ds}\bigg|_{s=0} \varphi_t(L_s\xi) = \xi + t\pi_\mathcal{J}(\xi),$$

[6] This equality is analogous to the more familiar equality $\gamma_{sX}(t) = \gamma_X(st)$ for geodesics.

for $y \in Y \subset N^*Y$. Since $V$ vanishes at $Y$, we have $exp|_Y = \text{Id}$, and so

$$d_y\varphi_t(\xi, v) = (\xi, v + t\pi_{\mathcal{J}}(\xi)) \tag{3.18}$$

in terms of the decomposition (3.12). Composing with $p$ gives $d_y exp(\xi, v) = v + \pi_{\mathcal{J}}(\xi)$, hence by transversality of $Y$ we see that $exp$ is a local diffeomorphism. Since $exp|_Y = \text{Id}$ and $Y$ is closed in $M$, $exp$ is a diffeomorphism around $Y$. If $\nabla'$ is a different connection, there is a path of connections $\nabla^t$ from $\nabla$ to $\nabla'$, whose exponentials $exp^t$ give the desired isotopy. Since (3.18) is independent of $\nabla^t$, the $exp^t$ all agree up to first order along $Y$. □

We will now construct explicitly one of the generalized complex structures $\mathcal{J}_\sigma$ from Theorem 3.2.5 together with a two-form $B$ on $N^*Y$, such that $(exp, B)$ is holomorphic with respect to $\mathcal{J}$ on $M$ and $\mathcal{J}_\sigma$ on $N^*Y$. For the proof of the following lemma, recall that if $\omega_{\text{can}}$ denotes the canonical symplectic form on $T^*M$ and if $X, Y \in TM$, $\alpha, \beta \in T^*M$, we have

$$(\omega_{\text{can}})_y(\alpha + X, \beta + Y) = \alpha(Y) - \beta(X), \tag{3.19}$$

in terms of $T(T^*M)|_M = T^*M \oplus TM$.

**Lemma 3.2.9.** Define

$$\widetilde{\sigma}_t := -\int_0^t (\varphi_s)^*\omega_{\text{can}} ds \ \in \ \Omega^2_{\text{cl}}(T^*M), \tag{3.20}$$

where $\varphi_s$ is the flow of the vector field $V$. Then $\sigma := i^*\widetilde{\sigma}_1$ is a closed extension of $\omega_Y$, where $i : N^*Y \hookrightarrow T^*M$ denotes the inclusion.

*Proof.* Using (3.18) and (3.19), we see that

$$\begin{aligned}\sigma_y(\alpha + X, \beta + Y) &= -\int_0^1 (\omega_{\text{can}})_y\big(\alpha + (X + s\pi_{\mathcal{J}}(\alpha)), \beta + (Y + s\pi_{\mathcal{J}}(\beta))\big) ds \\ &= -\int_0^1 2s\alpha(\pi_{\mathcal{J}}(\beta)) ds = \omega_Y(\alpha, \beta)\end{aligned}$$

for all $\alpha, \beta \in N^*Y$ and $X, Y \in TY$, proving the lemma. □

Observe that $V$ is the vector part of the generalized vector field $\mathcal{V} \in \Gamma(\mathbb{T}(T^*M))$, where $\mathcal{V}_\xi := (\mathcal{J}\xi)^h_\xi$. If $\psi_t$ denotes the flow of $\mathcal{V}$ on $\mathbb{T}(T^*M)$, then $\psi_t e_*^{i\widetilde{\sigma}_t}$ equals the flow of $\mathcal{V} + i\lambda_{\text{can}}$ where $\lambda_{\text{can}}$ denotes the canonical one-form on $T^*M$ (compare (1.14) and (3.20)). In particular, $\psi_t e_*^{i\widetilde{\sigma}_t}$ integrates the adjoint action of $-i\lambda_{\text{can}} - \mathcal{V}$. Since $(-i\lambda_{\text{can}} - \mathcal{V})_\xi = (-i\xi - \mathcal{J}\xi)^h_\xi \in \mathfrak{B}p(L)$ and $\mathfrak{B}p(L)$ is involutive, Lemma 1.1.5 implies that $\psi_t e_*^{i\widetilde{\sigma}_t}$ preserves $\mathfrak{B}p(L)$. Consequently,

$$e_*^{i\widetilde{\sigma}_t}\mathfrak{B}p(L) = \psi_{-t}\mathfrak{B}p(L), \tag{3.21}$$

as Dirac structures on the total space of $T^*M$. Here is an overview of all the maps involved:

$$\begin{array}{ccccc} N^*Y & \xrightarrow{\ i\ } & T^*M & \xrightarrow{\ \varphi_t\ } & T^*M \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle p} & \\ Y & \longrightarrow & M & & \end{array}$$

The left square is commutative but the right triangle is not. Now if we apply $\mathfrak{B}i$ to (3.21) at $t = 1$, the left-hand side becomes $e_*^{i\sigma}\mathfrak{B}i\mathfrak{B}p(L) = e_*^{i\sigma}\mathfrak{B}p(L_Y)$ where $\sigma = i^*\widetilde{\sigma}_1$. This is precisely one of the structures from Theorem 3.2.5. If we write $\psi_t = (\varphi_t)_* e_*^{-B_t}$ (see (1.14)), the right-hand side becomes

$$\mathfrak{B}i(\psi_{-1}\mathfrak{B}p(L)) = \mathfrak{B}i\mathfrak{B}\Phi_1\mathfrak{B}p(L) = \mathfrak{B}(p \circ \Phi_1 \circ i)(L),$$

where $\Phi_t := (\varphi_t, B_t)$. Now $p \circ \Phi_1 \circ i = (exp, i^*B_1)$, so if we define $B := i^*B_1$, then $(exp, B)$ is indeed holomorphic. This completes the proof of Theorem 3.2.6. □

### 3.2.2 Blowing up

In this section we will use the normal form (Theorems 3.2.5 and 3.2.6) to construct the blow-up of generalized Poisson transversals. The construction we are about to give is inspired by the symplectic blow-up using symplectic cuts, as outlined in Section 2.3. To use that in our setting, we need a reduction procedure for generalized complex structures. General reduction methods have been introduced in [9], but we will only need a very special case of this. In what follows, an $S^1$-action on a generalized complex manifold $(Z, H, \mathcal{J})$ is understood to be an $S^1$-action on the manifold $Z$ that preserves $\mathcal{J}$ and for which $\iota_X H = 0$, where $X$ is the associated action vector field (see (2.10)). In particular, this implies that $H$ is $S^1$-invariant. In analogy with symplectic geometry we call
$\mu : Z \to \mathbb{R}$ a *moment map* if $\mathcal{J}X = d\mu$. Since $d\mu(X) = 2\langle \mathcal{J}X, X\rangle = 0$ because $\mathcal{J}$ is skew-symmetric, the level sets of $\mu$ are $S^1$-invariant.

**Proposition 3.2.10.** *Suppose we are given a free $S^1$-action on $(Z, H, \mathcal{J})$ with moment map $\mu$. If $i : \mu^{-1}(c) \hookrightarrow Z$ is a regular level set with quotient map $q : \mu^{-1}(c) \to \mu^{-1}(c)/S^1$, then $\mathfrak{F}q(\mathfrak{B}i(L))$ gives a generalized complex structure $\mathcal{J}'$ on $\mu^{-1}(c)/S^1$. If there is a spinor $\rho$ for $\mathcal{J}$ which is $S^1$-invariant, then $i^*\rho = q^*\rho'$ for a unique form $\rho'$ on the quotient, forming a spinor for $\mathcal{J}'$.*

*Proof.* Let us abbreviate by $Z_c := \mu^{-1}(c)$ the regular level set and by $i : Z_c \hookrightarrow M$ the inclusion. Since $Z_c$ has real codimension 1 we have $N^*Z_c \cap \mathcal{J}N^*Z_c = 0$, so $\mathfrak{B}i(L)$ is smooth by (1.47). A quick computation gives

$$\mathfrak{B}i(L) \cap \mathfrak{B}i(\overline{L}) = \mathbb{C} \cdot X. \tag{3.22}$$

Since $dH = 0$ and $\iota_X H = 0$ by assumption, we can write $H = q^*H'$ for a (unique) three-form $H'$ on the quotient, so we can regard $q$ as a generalized map. It satisfies $\ker(q_*) \cap$

$\mathfrak{B}i(L) = \mathbb{C} \cdot X$, which is of constant rank 1, so that the forward image $\mathfrak{F}q(\mathfrak{B}i(L))$ is smooth by (1.38). By Lemma 1.2.23 it projects down to $Z_c/S^1$ and it is generalized complex because of (3.22) and the fact that $X$ spans the kernel of $q_*$.

Suppose that $\rho$ is a local spinor for $L$ which is $S^1$-invariant. Then, as $\mathcal{J}N^*Z_c \cap N^*Z_c = 0$, (1.36) holds and so $i^*\rho$ is nonzero on $Z_c$ and gives an $S^1$-invariant spinor for $\mathfrak{B}i(L)$. Moreover,

$$0 = (X - i\mathcal{J}X) \cdot \rho = (X - id\mu) \cdot \rho$$

implies that $\iota_X i^*\rho = 0$. Hence $i^*\rho$ comes from a unique differential form on $Z_c/S^1$ which, by Lemma 1.2.23, forms a spinor for the induced generalized complex structure on the quotient. □

Consider now a generalized Poisson transversal $Y \subset (M, \mathcal{J})$, with $\omega_Y$ the induced symplectic structure on $N^*Y$. As in the proof of Lemma 3.2.2, we choose a compatible Hermitian structure $(g, I)$ on the bundle $N^*Y$ and use it to construct an $S^1$-invariant one-form $\lambda$ on the manifold $N^*Y$, of the form (3.13). In particular, its differential $\sigma = d\lambda$ is a closed extension of $\omega_Y$ which is $S^1$-invariant, and whose restriction to the fibers is translation invariant. Consider the $S^1$-action on $Z := N^*Y \times \mathbb{C}$ given by

$$e^{i\theta} \cdot (z, w) = (e^{-i\theta}z, e^{i\theta}w),$$

and denote by $X$ the induced action vector field on $Z$. We equip $Z$ with the three-form $p^*H_Y$, where $p : N^*Y \times \mathbb{C} \to Y$ is the obvious map, and the generalized complex structure $\mathcal{J}_Z$ which is the product of the standard symplectic structure on $\mathbb{C}$ and $\mathcal{J}_\sigma$ on $N^*Y$ as defined by (3.14). We then have an $S^1$-action on $(Z, p^*H_Y, \mathcal{J}_Z)$.

**Lemma 3.2.11.** The map $\mu : Z \to \mathbb{R}$ given by $\mu(z, w) := \frac{1}{2}g(z, z) - \frac{1}{2}|w|^2$ is a moment map.

*Proof.* We can write $X = (X_1, X_2)$ on $N^*Y \times \mathbb{C}$, with $X_i$ the corresponding action vector field on the separate factors. In particular, $X_1$ is vertical on $N^*Y$, and by definition of $\mathcal{J}_\sigma$ we have $\mathcal{J}_Z(X_1, X_2) = (\sigma(X_1), \omega_{\text{st}}(X_2))$. Since $\sigma + \omega_{\text{st}} = d(\lambda + \lambda_{\text{st}})$, where both $\lambda$ and $\lambda_{\text{st}}$ are $S^1$-invariant, we get $\mathcal{J}_Z X = -d\iota_X(\lambda + \lambda_{\text{st}})$. Hence, it suffices to show that $\mu = -\iota_X(\lambda + \lambda_{\text{st}}) = -\iota_{X_1}\lambda - \iota_{X_2}\lambda_{\text{st}}$. This is a fiberwise equality and can readily be verified, e.g. by using a unitary frame. □

*Remark* 3.2.12. If one starts with an arbitrary extension $\sigma = d\lambda$ of $\omega_Y$, one can average it over $S^1$ to make it invariant, and the map $-\iota_X(\lambda_{\text{st}} + \lambda)$ is again a moment map. The advantage of our choice above is that the moment map has an explicit description in terms of a metric.

For $\varepsilon > 0$, Proposition 3.2.10 implies that $\widetilde{N^*Y}_\varepsilon := \mu^{-1}(\varepsilon^2/2)/S^1$ has a generalized complex structure, obtained from $\mathcal{J}_\sigma$ by taking a backward and forward image successively. As a manifold, $\widetilde{N^*Y}_\varepsilon$ is given by the blow-up of $Y$ in $N^*Y$. Indeed, as in Section 2.4, define

$$\widetilde{N^*Y} := \{(z, l) | z \in l\} \subset N^*Y \times \mathbb{P}(N^*Y).$$

This is the blow-up of $Y$ inside $N^*Y$, and we have a diffeomorphism

$$\kappa : \widetilde{N^*Y}_\varepsilon \to \widetilde{N^*Y}, \qquad (w,z) \mapsto (\frac{wz}{|z|}, [z]).$$

Here we abbreviate $|z| := \sqrt{g(z,z)}$. It remains to be shown that this blow-up can be glued back into the original manifold $M$ to produce the blow-up of $Y$ in $M$. For that, we consider the slice for the $S^1$-action given by

$$\widetilde{\varphi} : N^*Y \backslash \overline{B_\varepsilon} \hookrightarrow \mu^{-1}(\varepsilon^2/2) \subset Z, \qquad z \mapsto (z, \sqrt{|z|^2 - \varepsilon^2}). \tag{3.23}$$

Here $\overline{B_\varepsilon}$ is the disc bundle of radius $\varepsilon$. If $q$ denotes the quotient map of the $S^1$-action, we obtain a diffeomorphism

$$\varphi := q \circ \widetilde{\varphi} : N^*Y \backslash \overline{B_\varepsilon} \longrightarrow \widetilde{N^*Y}_\varepsilon \backslash E. \tag{3.24}$$

Here $E$ is the exceptional divisor, given by the image under $q$ of $\mu^{-1}(\varepsilon^2/2) \cap \{w = 0\}$. To see whether $\varphi$ is holomorphic, we look at what happens if we pull back a spinor. Since $\rho = e^{i\omega_{\mathrm{st}} + i\sigma} \wedge p^*\rho_Y$ is an $S^1$-invariant spinor on $Z$, Proposition 3.2.10 implies that we can choose a spinor $\rho'$ on $\widetilde{N^*Y}_\varepsilon \backslash E$ so that $q^*\rho' = i^*\rho$. In particular,

$$\varphi^*\rho' = \widetilde{\varphi}^* i^* \rho = e^{i\sigma} \wedge p^*\rho_Y,$$

which is a spinor for $J_\sigma$ on $N^*Y \backslash \overline{B_\varepsilon}$. So indeed, $\varphi$ is holomorphic.

**Theorem 3.2.13.** *Let $Y \subset (M, \mathcal{J})$ be a compact generalized Poisson transversal. For any tubular embedding as in Theorem 3.2.6, the associated blow-up carries a generalized complex structure which is, outside of a neighborhood of the exceptional divisor, isomorphic to the original structure.*

*Proof.* Let $\iota : N^*Y \rightarrowtail M$ be a tubular embedding as provided by Theorem 3.2.6. By Theorem 3.2.5 we may precompose $\iota$ by an isotopy to ensure that $\iota$ is holomorphic with respect to $\mathcal{J}$ on $M$, and a $\mathcal{J}_\sigma$ of our choosing on $N^*Y$. Note that this isotopy does not change the differentiable structure of the blow-up. Hence, we may assume that $\iota$ is holomorphic with respect to the $\mathcal{J}_\sigma$ on $N^*Y$ that we have been using above to construct $\widetilde{N^*Y}_\varepsilon$. The rest of the argument now proceeds as in Section 2.4. Let $U$ be a neighborhood of $Y$ in $N^*Y$ on which $\mathcal{J}_\sigma$ and $\iota$ are defined, and pick $\varepsilon > 0$ such that $\overline{B_\varepsilon} \subset U$. Here we use that $Y$ is compact. We can write

$$M = M \backslash \iota(\overline{B_\varepsilon}) \cup_\iota U,$$

using the holomorphic glueing map $\iota : U \backslash \overline{B_\varepsilon} \to \iota(U) \backslash \iota(\overline{B_\varepsilon})$. Then we define the blow-up by

$$\widetilde{M} := M \backslash \iota(\overline{B_\varepsilon}) \cup_{\iota \circ \varphi^{-1}} \widetilde{U}, \tag{3.25}$$

where $\widetilde{U} := \varphi(U\backslash\overline{B_\varepsilon}) \cup E$, and the glueing is performed by the holomorphic map

$$\iota \circ \varphi^{-1} : \widetilde{U}\backslash E \to \iota(U)\backslash\iota(\overline{B_\varepsilon}).$$

Here $\varphi$ is the slice that we defined in (3.23) and (3.24). Note that the map $\widetilde{M}\backslash\widetilde{U} \to M\backslash U$ is an isomorphism. □

*Remark* 3.2.14. i) As in Section 2.4, there is no natural blow-down map from $\widetilde{M}$ to $M$. It is possible to create one, but it requires an additional choice. See Section 2.4 for more details.

ii) The blow-up for generalized Poisson submanifolds, even though it does not always exist, is uniquely defined. In contrast, we say nothing about the uniqueness of the blow-up for generalized Poisson transversals. In fact, even in the symplectic setting it is unknown whether two symplectic blow-ups of the same point are always symplectomorphic. One of the main problems is that it is unknown whether any two symplectic embeddings of a ball into a symplectic manifold are always symplectically isotopic to each other.

**Example 3.2.15.** Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold and $Y \hookrightarrow M$ a generalized Poisson submanifold for $\mathcal{J}_1$, i.e. $\mathcal{J}_1 N^*Y = N^*Y$. By Lemma 4.0.1, $Y$ is a generalized Poisson transversal with respect to $\mathcal{J}_2$. In Example 3.1.10 we discussed how the maximal torus in a compact even-dimensional Lie group is a generalized Poisson submanifold for $\mathcal{J}_1$ which, because of the degeneracy condition, can almost never be blown up. With respect to $\mathcal{J}_2$ however, there are no restrictions, so all maximal tori can be blown up for $\mathcal{J}_2$. In Section 4.4 we will give a more thorough investigation of these examples, and show that if the maximal torus can be blown up for $\mathcal{J}_1$ and $\mathcal{J}_2$, then the blow-up is again generalized Kähler.

## 3.3 Other types of submanifolds

Our definition of a generalized complex submanifold is, besides a smoothness criterion, characterized by

$$JN^*Y \cap (N^*Y)^\perp \subset N^*Y. \tag{3.26}$$

In the previous sections we investigated the blow-up theory of the two extreme cases, namely those for which the above inclusion is either an equality (the generalized Poisson case) or when the intersection is zero (the generalized Poisson transversals). An obvious question at this point is whether the "intermediate" cases admit a blow-up theory as well. The techniques we used for generalized Poisson submanifolds and generalized Poisson transversals are so different from each other that it does not seem likely that we can use either of them when the type in the normal direction is mixed. We will now give an example where we can explicitly prove that there does not exist a blow-up. For that we will use the following fact that was proven by Atiyah [3], for which we give an elementary proof.

**Proposition 3.3.1** ([3]). *Let $M$ be a compact four-dimensional generalized complex manifold of type 1. Then the Euler characteristic $\chi(M)$ is even.*

*Proof.* Since the type can only change in even amounts, a structure which is of type 1 somewhere on a four-manifold is of type 1 everywhere. This gives rise to a decomposition $TM = L_1 \oplus L_2$, where $L_1$ is the distribution tangent to the symplectic foliation and $L_2$ is a choice of normal bundle. In particular $L_1$ and $L_2$ are orientable and we can think of them as complex line bundles[7], giving an almost complex structure on $TM$. By Wu's formula, using that $c_1(TM) \equiv w_2(M) \bmod 2$ and $c_1(TM) = c_1(L_1) + c_1(L_2)$, we obtain

$$\alpha^2 \equiv \alpha \cup c_1(L_1) + \alpha \cup c_1(L_2) \bmod 2 \quad \forall \alpha \in H^2(M, \mathbb{Z}).$$

Applying this to $\alpha = c_1(L_1)$, we see indeed that $\chi(M) = c_1(L_1)c_1(L_2)$ is even. □

Now let $M$ be a compact four-dimensional generalized complex manifold of type 1. From Lemma 2.2.7 we know that the blow-up of a point in $M$ is differentiably given by $\widetilde{M} = M \# \overline{\mathbb{CP}}^2$, which has Euler characteristic $\chi(\widetilde{M}) = \chi(M) + 1$. This is odd, which means that $\widetilde{M}$ does not admit a type 1 structure. In particular, if $\widetilde{M}$ admits a generalized complex structure, then it must be of type 0 or 2, or a mixture of these. In particular, such a structure would nowhere be related to the original structure on $M$, so it can not come from a blow-up construction.

In the example above, Equation (3.26) is neither zero nor an equality at any point. However, as Example 1.3.24 illustrates, it can happen that (3.26) is zero at some points, and an equality at others. There seems to be no easy argument to rule out a blow-up theory for these kind of submanifolds, hence further study is needed to see what can be said about them.

[7]In fact, $L_2$ inherits a canonical almost complex structure, being the normal to the symplectic foliation in a generalized complex manifold.

# Chapter 4

# Blow-ups of Generalized Kähler Manifolds

In the previous chapter we learned that there are two natural classes of submanifolds in generalized complex geometry that, under suitable conditions, admit a blow-up: those that look complex and those that look symplectic in transverse directions. On a Kähler manifold $(M, I, \omega)$, we know that a submanifold $Y$ which is complex for $I$ is automatically symplectic for $\omega$ (but not the other way around). Moreover, the complex blow-up of $Y$ in $M$ carries again a Kähler structure, at least when $Y$ is compact. The first of these two facts admits a straightforward generalization.

**Lemma 4.0.1.** Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold. A submanifold $Y \subset M$ which is a generalized Poisson submanifold for $\mathcal{J}_1$ is a generalized Poisson transversal for $\mathcal{J}_2$.

*Proof.* By the generalized Kähler condition we have

$$\langle \mathcal{J}_1\alpha, \mathcal{J}_2\alpha \rangle > 0 \qquad \forall \alpha \in N^*Y.$$

So if $\mathcal{J}_1 N^*Y = N^*Y$, then necessarily $\mathcal{J}_2 N^*Y \cap (N^*Y)^\perp = 0$. □

In light of this we are led to the following question:

If $(M, \mathcal{J}_1, \mathcal{J}_2)$ is a generalized Kähler manifold and $Y \subset M$ a compact generalized Poisson submanifold for $\mathcal{J}_1$ with degenerate conormal bundle, is the blow-up again generalized Kähler?

Here the degeneracy condition is to ensure that the blow-up with respect to $\mathcal{J}_1$ exists. Note that there is no loss of generality to assume that $Y$ is generalized Poisson for $\mathcal{J}_1$, by symmetry of the pair $(\mathcal{J}_1, \mathcal{J}_2)$. In the present chapter we will give a partial answer to this question. Specifically, we will give sufficient conditions on $(M, \mathcal{J}_1, \mathcal{J}_2)$ that ensure that the answer to the above question is positive. We will also give a concrete example, on compact Lie groups, to illustrate.

To understand and appreciate the steps that we will take in the coming sections, it is instructive to sketch the analogy with blow-ups in ordinary Kähler geometry. The way one constructs a new Kähler form on the blow-up of a complex submanifold is by pulling back the original Kähler form, and then adding to it a two-form which is positive on all the fibers of the blow-down map. This two-form can be written as $dd^c f$, where $f$ is a function on $\widetilde{M}$ with a singularity along $E$. We will refer to $f$ as a "potential". Here is another way to look at this. We first pull back the Kähler structure to obtain a "singular" Kähler structure, whose metric is no longer non-degenerate. Then, we deform the structure to become non-degenerate. Of course this last sentence is quite an exaggeration in this context; we are merely adding a closed two-form to the Kähler form. However, it is a useful point of view in the generalized Kähler context. If $p : \widetilde{M} \to M$ denotes the blow-down map, we can consider the backward image $\mathfrak{B}p(L_2)$ on $\widetilde{M}$. It is smooth, but no longer non-degenerate along the exceptional divisor. In analogy with the story above, we could try to remedy this by deforming $\mathfrak{B}p(L_2)$ to a genuine generalized complex structure, in such a way that it remains compatible with $\mathcal{J}_1$. Even though this would be an elegant way to proceed, it is not clear how to write down such a deformation for $\mathfrak{B}p(L_2)$. Instead, we will operate on the level of the underlying bi-Hermitian structure $(g, I_+, I_-, H)$, and proceed as follows. We first show that the bi-Hermitian data can be lifted to the blow-up of $M$ with respect to $\mathcal{J}_1$ to form a degenerate bi-Hermitian structure (see Definition 4.1.3) on $\widetilde{M}$. Then, we set up a deformation process that allows one to flow from such a degenerate structure towards a non-degenerate one. The end result then defines the desired generalized Kähler structure. As in the Kähler case, this deformation also needs the input of a potential $f$. This time however, it is not just $dd^c f$ which plays a role, but also the Hamiltonian vector field $Q(df)$, where $Q$ is defined in (1.58). Once we have a good control on the singular behavior of $Q(df)$, which requires additional geometrical conditions on $M$, the flow works and we can blow up. As a special case we recover the ordinary Kähler blow-up, for which $Q = 0$. Nevertheless, in this generalized approach, *both* $\mathcal{J}_1$ and $\mathcal{J}_2$ are being deformed, in contrast to the ordinary Kähler setting.

The idea behind flowing a generalized Kähler manifold by potentials already appears in [30], where it was used to describe new examples of generalized Kähler structures. It was subsequently used in [16] to blow up a non-degenerate point on a four-dimensional manifold, and the methods that we introduce here extend these ideas. The content of this chapter appeared in [33].

## 4.1 Lifting the bi-Hermitian structure

Let $Y \subset (M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Poisson submanifold for $\mathcal{J}_1$. Associated to $(\mathcal{J}_1, \mathcal{J}_2)$ we have the pair of holomorphic Poisson structures $(I_\pm, \sigma_\pm)$ defined in (1.59), whose real parts coincide and are equal to $Q$. In this section we will show that $(I_\pm, \sigma_\pm)$ both lift to the generalized complex blow-up of $Y$ with respect to $\mathcal{J}_1$. To that end, we first prove that $\sigma_\pm$ lift to the blow-up of $Y$ with respect to $I_\pm$, and then show that the three different blow-ups all coincide.

From (1.56) it follows that $Y$ is generalized Poisson for $\mathcal{J}_1$ if and only if $I_\pm^* N^*Y = N^*Y$, and $I_+^*|_{N^*Y} = I_-^*|_{N^*Y}$. Consequently, $Y$ is a holomorphic Poisson submanifold for both $(I_\pm, \sigma_\pm)$.

**Lemma 4.1.1.** *$N^*Y$ is degenerate for $\pi_{\mathcal{J}_1}$ if and only if it is degenerate for $Q$.*

*Proof.* From Equations (1.58) and (1.57) we obtain

$$Q = \pi_{\mathcal{J}_1} \circ (I_+ + I_-)^*. \tag{4.1}$$

The endomorphism $A := (I_+ + I_-)^*$ restricts to an automorphism of $N^*Y$, and we have

$$[\alpha, \beta]_Q = d_y\big(Q(\tilde{\alpha}, \tilde{\beta})\big) = d_y\big(\pi_{\mathcal{J}_1}(A\tilde{\alpha}, \tilde{\beta})\big) = [A\alpha, \beta]_{\pi_{\mathcal{J}_1}} \tag{4.2}$$

for $\alpha, \beta \in N_y^*Y$, where $\tilde{\alpha}, \tilde{\beta}$ are smooth local extensions of $\alpha$ and $\beta$. In the last equality we used that $A\tilde{\alpha}$ is a smooth local extension of $A\alpha$. Abstractly, if $(\mathfrak{g}, [\cdot,\cdot])$ is a Lie algebra and $A : \mathfrak{g} \to \mathfrak{g}$ a linear map such that $[u, v]_A := [Au, v]$ is again a Lie bracket[1], then degeneracy of $[\cdot,\cdot]$ implies that for $[\cdot,\cdot]_A$ as well. To prove this, we need to show that $[x, y]_A \in \mathbb{C}x + \mathbb{C}y$ for all $x, y \in \mathfrak{g}$, and it suffices to verify this for $x, y$ that are linearly independent. Since $[\cdot,\cdot]$ is degenerate, there are $\lambda, \mu \in \mathbb{C}$ with $[x, y]_A = [Ax, y] = \lambda Ax + \mu y$. Since $[\cdot,\cdot]_A$ is skew, we have $[Ax, y] = [Ax, x + y]$, and so there are $\lambda', \mu' \in \mathbb{C}$ with $[x, y]_A = \lambda' Ax + \mu'(x + y)$. Comparing both equations and using that $x$ and $y$ are linearly independent, we see that either $Ax \in \mathbb{C}x + \mathbb{C}y$, hence also $[Ax, y] \in \mathbb{C}x + \mathbb{C}y$, or $[Ax, y] = \lambda Ax$. Running the same argument with $x$ and $y$ interchanged, we see that either $[Ax, y] \in \mathbb{C}x + \mathbb{C}y$, or $[Ax, y] = \mu Ay$ for some $\mu \in \mathbb{C}$. In conclusion, if $[Ax, y]$ does not lie in $\mathbb{C}x + \mathbb{C}y$, then $Ax$ is proportional to $Ay$ and so $[Ax, y] = 0$ by skew symmetry of $[\cdot,\cdot]_A$, a contradiction. Hence, $[x, y]_A \in \mathbb{C}x + \mathbb{C}y$ for all $x, y \in \mathfrak{g}$, and so $[\cdot,\cdot]_A$ is degenerate. □

From Proposition 3.1.6 it follows that if $Y$ can be blown up for $\mathcal{J}_1$, then $\sigma_\pm$ lift to the blow-ups of $Y$ with respect to $I_\pm$. Let us denote by $\widetilde{M}$ the blow-up for $\mathcal{J}_1$ and by $\widetilde{M}_\pm$ the blow-ups for $I_\pm$.

**Lemma 4.1.2.** *The blow-ups $\widetilde{M}$, $\widetilde{M}_+$ and $\widetilde{M}_-$ all coincide.*

*Proof.* As explained in Chapter 2, the blow-up $\widetilde{M}$ is constructed from a holomorphic ideal $I_{Y,\mathcal{J}_1}$ that $Y$ inherits from $\mathcal{J}_1$, while the blow-ups $\widetilde{M}_\pm$ use the natural holomorphic ideals $I_{Y,I_\pm}$ that $Y$ inherits from being a complex submanifold in $(M, I_\pm)$. It thus suffices to show that these three ideals coincide, which turns out to be true up to a conjugation, i.e. $\overline{I_{Y,\mathcal{J}_1}} = I_{Y,I_\pm}$. This is not a problem, for the blow-up of a conjugate ideal is given by the same manifold but with conjugate divisor. Pick a local chart $(\mathbb{R}^{2n-2k}, \omega_{\text{st}}) \times (\mathbb{C}^k, \sigma)$ for $\mathcal{J}_1$ as provided by Theorem 1.3.16, in which $Y$ necessarily looks like $W \times Z$, where $W \subset$

[1] In fact as the argument shows, the Jacobi identity plays no role here. Hence this is really a statement about skew-symmetric brackets.

$\mathbb{R}^{2n-2k}$ is open and $Z \subset \mathbb{C}^k$ a complex Poisson submanifold. If $(x, z)$ are coordinates on this chart in which $Y = \{z^1, \ldots, z^l = 0\}$, then the ideal $I_{Y,\mathcal{J}_1}$ is on this chart given by $\langle z^1, \ldots, z^l\rangle$ (see the proof of Proposition 3.1.2). Let us verify that $\overline{I_{Y,\mathcal{J}_1}} = I_{Y,I_+}$, the case of $I_-$ being similar. In general, if $I_Y$ and $I'_Y$ are holomorphic ideals for $Y$, then $I_Y \subset I'_Y$ implies that $I_Y = I'_Y$. This follows easily from property ii) of Definition 2.1.1. In particular, we only need to show that, say, $\overline{I_{Y,\mathcal{J}_1}} \subset I_{Y,I_+}$. Pick a holomorphic chart for $I_+$ with coordinates $u^i$ so that $Y$ is given by $\{u^1, \ldots, u^l = 0\}$, so that $I_{Y,I_+} = \langle u^1, \ldots, u^l\rangle$. Using the same criterion from [35] that we used in proving Proposition 3.1.2, we can verify $\overline{I_{Y,\mathcal{J}_1}} \subset I_{Y,I_+}$ merely by looking at Taylor series, i.e. we need to show that

$$\left.\frac{\partial^m \bar{z}^i}{\partial \bar{u}^{i_1} \ldots \partial \bar{u}^{i_m}}\right|_Y = 0 \quad \forall m \geq 0,\ \forall i, i_1, \ldots, i_m \in \{1, \ldots, l\}. \tag{4.3}$$

For this we use (1.62), which now explicitly becomes

$$(-1)^{\frac{1}{2}k(k-1)} e^{i\omega_{\mathrm{st}}} e^{-\bar{\sigma}} (d\bar{z}^1 \ldots d\bar{z}^k) \wedge \bar{\rho}_2 = e^f e^{-\frac{1}{8}\sigma_+} (du^1 \ldots du^n), \tag{4.4}$$

where $e^f$ is some rescaling. One can now prove (4.3) by induction on $m$, and by applying appropriate Lie derivatives to (4.4). This part of the argument is very similar to the one used in the proof of Proposition 3.1.2, and so we omit further details. □

From Lemma 4.1.2 it follows that if $Y \subset (M, \mathcal{J}_1, \mathcal{J}_2)$ is a compact generalized Poisson submanifold for $\mathcal{J}_1$ with degenerate conormal bundle, then both complex structures $I_+$ and $I_-$ lift to the blow-up for $\mathcal{J}_1$. We will continue to denote these by $I_\pm$, and it should be clear from the context on which manifold we are considering them. In particular, we can look at the tuple $(p^*g, I_+, I_-, p^*H)$ on the blow-up. This tuple still satisfies the integrability conditions, in the sense that $I_\pm$ are integrable, $p^*H$ is closed and $\pm d^c_\pm p^*\omega_\pm = p^*H$. However, $p^*g$ is no longer non-degenerate and so we do not have a generalized Kähler structure on the blow-up. Nevertheless, Lemma 4.2 tells us that $Q$ can be lifted to $\widetilde{M}$, even though (1.58) no longer makes sense. Let us formalize the current situation.

**Definition 4.1.3.** A *degenerate bi-Hermitian structure* on $M$ is a tuple $(g, I_+, I_-, H)$, where $g$ is a symmetric bilinear form, $I_\pm$ are complex structures and $H$ is a closed three-form, such that

i) $g$ is positive on $M\backslash E$, where $E \subset M$ is a closed and nowhere dense submanifold, on which $TM^\perp := \{v \in TM \mid g(v, w) = 0\ \forall w \in TM\}$ has constant rank.

ii) $I_\pm$ are compatible with $g$, and satisfy the integrability condition $\pm d^c_\pm \omega_\pm = H$.

iii) The bivector $Q := -\frac{1}{2}[I_+, I_-]g^{-1}$, defined on $M\backslash E$, extends smoothly over $M$.

We summarize the results of this section in the following proposition.

**Proposition 4.1.4.** *Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold with bi-Hermitian data given by $(g, I_\pm, H)$. Let $Y \subset M$ be a compact generalized Poisson submanifold for $\mathcal{J}_1$ with degenerate conormal bundle, and let $p : \widetilde{M} \to M$ denote the corresponding blow-up. Then there are complex structures $I_\pm$ on $\widetilde{M}$ for which $p$ is holomorphic, and the tuple $(p^*g, I_\pm, p^*H)$ defines a degenerate bi-Hermitian structure on $\widetilde{M}$.*

## 4.2 A flow of bi-Hermitian structures

To deal with the degeneracy of the metric on the blow-up we introduce a deformation procedure to flow a degenerate structure into a non-degenerate one. Let $(g, I_\pm, H)$ be a degenerate bi-Hermitian structure with degeneracy set $E$. Since $E$ is nowhere dense, the relation $Qg = -\frac{1}{2}[I_+, I_-]$ holds everywhere on $M$, and $\sigma_\pm := Q - iI_\pm Q$ are holomorphic Poisson with respect to $I_\pm$. To define the flow, we need the following extra ingredient.

**Definition 4.2.1.** A *potential* for $I_+$ (respectively $I_-$) is a closed one-form $\alpha$ defined on an open dense set, such that $X_\alpha := Q(\alpha)$ and $d_+^c\alpha$ (respectively $d_-^c\alpha$) extend smoothly over $M$.

*Remark* 4.2.2. The terminology originates from the situation where $\alpha = -df$ for a densely defined function $f$, which is usually referred to as the potential. Although this is the situation in which we are interested, we state the results in this section for general closed one-forms for ease of notation.

Let $\alpha$ be a potential for $I_+$. Denote by $\varphi_t$ the flow of $X_\alpha$ and define closed two-forms

$$G_t^\pm := \varphi_{t*}(d_\pm^c\alpha), \qquad F_t^\pm := \int_0^t G_s^\pm ds.$$

We will have to be careful with $G_t^-$ and $F_t^-$, since $d_-^c\alpha$ is not assumed to be smooth everywhere. The aim of this section is to prove

**Theorem 4.2.3.** *Let $(g, I_+, I_-, H)$ be a degenerate bi-Hermitian structure with compact degeneracy submanifold $E$. Let $\alpha$ be a potential for $I_+$, such that $d_+^c\alpha$ has compact support and $I_-^*(-d_+^c\alpha)_{I_-}^{1,1}$ is positive on $TM^\perp$. Then the tuple $(g_t, I_{+,t}, I_{-,t}, H_t)$, defined by*[2]

$$\begin{aligned} g_t :=& g - I_-^*\big(F_t^+\big)_{I_-}^{1,1}, \qquad I_{+,t} := \varphi_{t*}(I_+), \qquad I_{-,t} := I_-, \\ H_t :=& H + id\Big(\big(F_t^+\big)_{I_-}^{2,0} - \big(F_t^+\big)_{I_-}^{0,2}\Big), \end{aligned} \tag{4.5}$$

*forms a bi-Hermitian structure for sufficiently small $t > 0$.*

*Proof.* It is clear that $g_t$ is symmetric, $I_{\pm,t}$ are integrable and that $H_t$ is closed for all $t$. By construction, $g_t$ is compatible with $I_{-,t} = I_-$. Let us show that $g_t$ is a metric for sufficiently small $t > 0$. Choose a relatively compact open neighborhood $V$ of $E$ in $M$ with $\mathrm{supp}(d_+^c\alpha) \subset V$ and pick a $\delta_1 > 0$ and a relatively compact open set $W$ with $\overline{\varphi_t(V)} \subset W$ for all $t \leq \delta_1$. By construction, $F_t^+ = 0$ on $M \backslash W$ for $t \leq \delta_1$ and so $g_t = g$ is non-degenerate there. Writing $g_t = g + h_t$, we have

$$\lim_{t \to 0^+} \frac{h_t}{t} = \dot{h}_0 = -I_-^*(d_+^c\alpha)_{I_-}^{1,1},$$

[2]Here and in the remainder of this section, an expression of the form $\alpha_I^{p,q}$ denotes the $(p, q)$-component of a form $\alpha$ with respect to the complex structure $I$.

which by assumption is positive on $TM^\perp$. For small $\varepsilon > 0$, $g + \varepsilon\dot{h}_0$ is positive on $TM|_E$ and therefore also on $TM|_{\overline{U}}$ for $U \subset W$ a small enough neighborhood of $E$. Hence the same is true for $g + \varepsilon h_t/t$, and therefore also for $tg/\varepsilon + h_t$, provided $t > 0$ is close to zero. If in addition $t \leq \varepsilon$ then $tg/\varepsilon + h_t \leq g + h_t = g_t$ since $g \geq 0$. In conclusion, there exist a neighborhood $U$ of $E$ in $W$ and a $\delta_2 > 0$ such that $g_t$ is positive on $U$ for all $0 < t \leq \delta_2$. Since $\overline{W}\backslash U$ is compact and $g_0 = g$ is non-degenerate on $M\backslash E$, there is a $\delta_3 > 0$ such that $g_t$ is positive on $\overline{W}\backslash U$ for $0 \leq t \leq \delta_3$. Consequently, $g_t$ is a metric for $0 < t \leq \min(\delta_1, \delta_2, \delta_3)$.

As already observed above, $g_t$ is compatible with $I_{-,t}$ for all $t$. Moreover, since any closed two-form $F$ on a complex manifold satisfies

$$d^c F^{1,1} = id(F^{2,0} - F^{0,2}),$$

we obtain

$$-d^c_{-,t}\omega_{-,t} = -d^c_-(\omega_- - (F_t^+)^{1,1}_{I_-}) = H + id\big((F_t^+)^{2,0}_{I_-} - (F_t^+)^{0,2}_{I_-}\big) = H_t.$$

Hence all that is left to verify is that $I_{+,t}$ is also compatible with $g_t$ and that $d^c_{+,t}\omega_{+,t} = H_t$. In contrast with $I_{-,t}$ this is not immediately obvious, the reason being that the flow seems to treat $I_+$ and $I_-$ on an unequal footing. However, we will now show that if we pull back the entire flow (4.5) by $\varphi_t$, we obtain a similar flow but with the roles of $I_+$ and $I_-$ interchanged. We begin by giving an alternative formula for $I_{+,t}$.

**Lemma 4.2.4.** $\varphi_{t*}(I_\pm) = I_\pm - QF_t^\pm$.

*Proof.* Consider the generalized complex structure

$$\mathcal{J}_+ = \begin{pmatrix} I_+ & Q \\ 0 & -I_+^* \end{pmatrix},$$

integrable with respect to the zero three-form (see Example 1.3.7). As $\alpha$ is closed, the generalized vector field $\mathcal{J}_+\alpha = X_\alpha - I_+^*\alpha$ is a symmetry of $\mathcal{J}_+$, which means that $\mathcal{J}_+$ is preserved by its flow[3]

$$\psi_t^+ = e_*^{F_t^+} \circ \varphi_{t*}.$$

Hence $e_*^{-F_t^+} \circ \mathcal{J}_+ \circ e_*^{F_t^+} = \varphi_{t*}(\mathcal{J}_+)$, and so in particular $\varphi_{t*}(I_+) = I_+ - QF_t^+$. For $I_-$ we have to be careful since we do not know whether $F_t^-$ is smooth. We can apply the above argument at a given point in the open dense set where $\alpha$ is smooth, if we keep the time parameter small enough. Differentiating at $t = 0$, we can at least conclude that $\mathcal{L}_{X_\alpha} I_- = Qd^c_-\alpha$ holds on the dense set where $\alpha$ is defined. Since $X_\alpha$ is smooth, we learn that $Qd^c_-\alpha$, and therefore also $QG_t^-$ and $QF_t^-$, are smooth. The equation $\varphi_{t*}(I_-) = I_- - QF_t^-$ then holds because it does so at $t = 0$, and because both sides have equal time derivatives. □

[3] Although $\mathcal{J}_+\alpha$ is only densely defined, its associated adjoint action on $\Gamma(\mathbb{T}M)$ depends only on $X_\alpha$ and $d^c_+\alpha$ and is therefore defined everywhere. In particular, the flow is also defined everywhere.

In the proof of the lemma below we denote by $\nabla$ the Levi-Cevita connection and by $\nabla^{\pm}$ the metric connections whose torsions are given by $\mp g^{-1}H$ (see also Proposition 1.4.3). They are defined on the open dense set $M\backslash E$ where $g$ is non-degenerate.

**Lemma 4.2.5.**

$$\mathcal{L}_{X_\alpha} g = I_-^*(d_+^c\alpha)_{I_-}^{1,1} - I_+^*(d_-^c\alpha)_{I_+}^{1,1} \tag{4.6}$$

$$\iota_{X_\alpha} H = \frac{1}{2}\big(d_+^c(I_-^*\alpha) + d_-^c(I_+^*\alpha)\big) \tag{4.7}$$

*Proof.* We will verify these expressions on the intersection of $M\backslash E$ with the open dense set where $\alpha$ is defined. Since the left-hand sides of both equations are smooth, this shows that the right-hand sides have smooth extensions to all of $M$.

As both sides of (4.6) are symmetric, it suffices to evaluate them on a pair $(Y, Y)$. Using the well-known formula $(\mathcal{L}_{g^{-1}\alpha} g)(Y, Y) = 2\nabla_Y\alpha(Y)$ together with the definition $X_\alpha = g^{-1}(\frac{1}{2}[I_+, I_-]^*\alpha)$, we obtain

$$\begin{aligned}(\mathcal{L}_{X_\alpha} g)(Y, Y) =& \nabla_Y\big([I_+, I_-]^*\alpha\big)(Y)\\ =& \nabla_Y\alpha\big([I_+, I_-]Y\big) + \alpha\big((\nabla_Y[I_+, I_-])Y\big).\end{aligned}$$

Using the identities[4] $d_\pm^c\beta = [d, I_\pm^*\cdot]\beta$ and $d\beta = (\nabla\beta)^{\text{skew}}$ for $\beta \in \Omega^*(M)$, together with the fact that $\alpha$ is closed, the right-hand side of (4.6) evaluates to[5]

$$\begin{aligned}d(I_+^*\alpha)(Y, I_-Y) - (+\leftrightarrow -) =& \Big(\nabla_Y\alpha(I_+I_-Y) + \alpha((\nabla_Y I_+)I_-Y) - \nabla_{I_-Y}\alpha(I_+Y)\\ &- \alpha((\nabla_{I_-Y}I_+)Y)\Big) - (+\leftrightarrow -)\\ =& \nabla_Y\alpha([I_+, I_-]Y) + \alpha\big((\nabla_Y[I_+, I_-])Y\big)\\ &+ \alpha\Big(\big(I_-(\nabla_Y I_+)Y - (\nabla_{I_-Y}I_+)Y\big) - (+\leftrightarrow -)\Big).\end{aligned} \tag{4.8}$$

Using $\nabla_Y I_\pm = \pm\frac{1}{2}[g^{-1}\iota_Y H, I_\pm]$, a tedious but straightforward calculation shows that

$$\begin{aligned}(I_\mp\nabla_Y I_\pm - \nabla_{I_\mp Y}I_\pm)Z = \mp\frac{1}{2}g^{-1}\big(&I_\mp^*\iota_{I_\pm Z}\iota_Y + I_\pm^*\iota_Z\iota_{I_\mp Y}\\ &+ I_\mp^*I_\pm^*\iota_Z\iota_Y + \iota_{I_\pm Z}\iota_{I_\mp Y}\big)H.\end{aligned} \tag{4.9}$$

From this we see that the last term in (4.8) vanishes, proving (4.6).

For (4.7), we use $d_\pm^c = [d, I_\pm^*\cdot]$ and $d\alpha = (\nabla\alpha)^{\text{skew}}$ to compute

$$\begin{aligned}(d_+^c(I_-^*\alpha))(Y, Z) =& (d(I_+^*I_-^*\alpha))(Y, Z) - (d(I_-^*\alpha))(I_+Y, Z) - (d(I_-^*\alpha))(Y, I_+Z)\\ =& -\nabla_{I_+Y}\alpha(I_-Z) + \alpha\big(I_-(\nabla_Y I_+)Z - (\nabla_{I_+Y}I_-)Z\big) - (Y\leftrightarrow Z).\end{aligned}$$

[4] Here $I_\pm^*\cdot$ acts on a form of degree $(p, q)$ by $i(p-q)$. In particular, for a one-form $\alpha$ we have $I_\pm^*\cdot\alpha = I_\pm^*\alpha$, while for a two-form $\beta$ we have $I_\pm^*\cdot\beta = I_\pm^*\beta + \beta I_\pm$.

[5] Here we use the notation $(+\leftrightarrow -)$ to denote the same term that precedes it but with $\pm$ symbols interchanged.

The same equation with $\pm$ interchanged also holds. Using again (4.9) and the fact that $\nabla\alpha$ is symmetric, we get

$$\begin{aligned}\big(d^c_+(I^*_-\alpha)+d^c_-(I^*_+\alpha)\big)(Y,Z) =&\alpha\Big(\big(I_-\nabla_Y I_+ - \nabla_{I_-Y}I_+\big)Z + (+\leftrightarrow -)\Big) - (Y\leftrightarrow Z)\\ =&\alpha\big(g^{-1}(I^*_+I^*_- - I^*_-I^*_+)\iota_Z\iota_Y H\big)\\ =&2(\iota_{X_\alpha}H)(Y,Z),\end{aligned}$$

proving (4.7). □

From the proof of Lemma 4.2.4 we learned that $Qd^c_-\alpha$ is smooth, and therefore also $d^c_-\alpha Q$ by taking adjoints. Since $d^c_+\alpha$ is smooth by assumption, (4.6) gives us in addition smoothness of $(d^c_-\alpha)^{1,1}_{I_+}$. Consequently, using Lemma 4.2.4 we see that

$$\begin{aligned}I^*_+F^-_t - F^-_t I_+ =&\int_0^t \varphi_{s*}\big((\varphi^*_s I_+)^* d^c_-\alpha - d^c_-\alpha(\varphi^*_s I_+)\big)ds\\ =&\int_0^t \varphi_{s*}\big(I^*_+d^c_-\alpha - F^+_{-s}Qd^c_-\alpha - d^c_-\alpha I_+ + d^c_-\alpha Q F^+_{-s}\big)ds \qquad (4.10)\end{aligned}$$

is smooth as well. Moreover, if we apply $d$ to (4.7), we obtain

$$\begin{aligned}\mathcal{L}_{X_\alpha}H =&-\frac{1}{2}d\big(I^*_+\cdot d(I^*_-\cdot\alpha) + I^*_-\cdot d(I^*_+\cdot\alpha)\big)\\ =&-\frac{1}{2}d(I^*_+d^c_-\alpha + d^c_-\alpha I_+ + I^*_-d^c_+\alpha + d^c_+\alpha I_-), \qquad (4.11)\end{aligned}$$

which implies that $d(I^*_+d^c_-\alpha + d^c_-\alpha I_+)$ is smooth. Similar to (4.10), it follows that $d(I^*_+F^-_t + F^-_t I_+)$ is smooth. We now have the right ingredients to make sense of the following lemma.

**Lemma 4.2.6.** Let $(g_t, I_{+,t}, I_-, H_t)$ be as in (4.5). Then

$$\varphi^*_t g_t = g + \frac{1}{2}\big(I^*_+F^-_{-t} - F^-_{-t}I_+\big), \qquad (4.12)$$

$$\varphi^*_t H_t = H + \frac{1}{2}d\big(I^*_+F^-_{-t} + F^-_{-t}I_+\big). \qquad (4.13)$$

*Proof.* Both equations hold at $t=0$, so it suffices to show that both sides have the same time derivative. Since $\varphi^*_t F^\pm_t = -F^\pm_{-t}$, we have $\varphi^*_t g_t = \varphi^*_t g + \frac{1}{2}(I^*_{-,-t}F^+_{-t} - F^+_{-t}I_{-,-t})$, where $I_{-,-t} = \varphi^*_t(I_-)$. Using Lemma 4.2.4 and (4.6), we obtain

$$\begin{aligned}\frac{d}{dt}(\varphi^*_t g_t) =&\varphi^*_t(\mathcal{L}_{X_\alpha}g) + \frac{1}{2}\big(G^-_{-t}QF^+_{-t} - I^*_{-,-t}G^+_{-t} - F^+_{-t}QG^-_{-t} + G^+_{-t}I_{-,-t}\big)\\ =&\varphi^*_t\Big(\mathcal{L}_{X_\alpha}g + \frac{1}{2}\big(-d^c_-\alpha QF^+_t - I^*_-d^c_+\alpha + F^+_tQd^c_-\alpha + d^c_+\alpha I_-\big)\Big)\\ =&\frac{1}{2}\varphi^*_t\big(-I^*_{+,t}d^c_-\alpha + d^c_-\alpha I_{+,t}\big)\\ =&\frac{1}{2}(-I^*_+G^-_{-t} + G^-_{-t}I_+).\end{aligned}$$

This equals the time derivative of the right-hand side of (4.12), thereby proving it.

For (4.13), using $\varphi_t^* H_t = \varphi_t^* H - \frac{1}{2} d(I_{-,-t}^* F_{-t}^+ + F_{-t}^+ I_{-,-t})$ and (4.11), we have

$$\begin{aligned}\frac{d}{dt}(\varphi_t^* H_t) =& \varphi_t^*(\mathcal{L}_{X_\alpha} H) - \frac{1}{2} d\big(G_{-t}^- Q F_{-t}^+ - I_{-,-t}^* G_{-t}^+ + F_{-t}^+ Q G_{-t}^- - G_{-t}^+ I_{-,-t}\big)\\ =& \varphi_t^*(\mathcal{L}_{X_\alpha} H) + \frac{1}{2}\varphi_t^* d\big(d_-^c \alpha Q F_t^+ + I_-^* d_+^c \alpha + F_t^+ Q d_-^c \alpha + d_+^c \alpha I_-\big)\\ =& \frac{1}{2}\varphi_t^* d\big(-I_{+,t}^* d_-^c \alpha - d_-^c \alpha I_{+,t}\big)\\ =& -\frac{1}{2} d(I_+^* G_{-t}^- + G_{-t}^- I_+),\end{aligned}$$

which equals the time derivative of the right-hand side of (4.13), thereby proving it. □

From Lemma 4.2.6, together with the arguments applied before to $I_{-,t}$, it follows that $\varphi_t^* I_{+,t} = I_+$ is compatible with $\varphi_t^* g_t$, and that

$$\begin{aligned}\varphi_t^*(d_{+,t}^c \omega_{+,t}) =& d_+^c((\varphi_t^* g_t) I_+) = d_+^c(\omega_+ + (F_{-t}^-)_{I_+}^{1,1}) = H + id\big((F_{-t}^-)_{I_+}^{2,0} - (F_{-t}^-)_{I_+}^{0,2}\big)\\ =& \varphi_t^* H_t.\end{aligned}$$

Pushing everything forward again by $\varphi_t$ we obtain the desired compatibility of $I_{+,t}$ with $g_t$ and $H_t$, finishing the proof of Theorem 4.2.3. □

*Remark* 4.2.7. We stated the theorem for potentials for $I_+$ but of course a similar result is true for potentials for $I_-$. In that case we need $I_+^*(-d_-^c \alpha)_{I_+}^{1,1}$ to be positive on $TM^\perp$.

*Remark* 4.2.8 ([30]). Theorem 4.2.3 is stated for metrics which are almost everywhere non-degenerate. It can however, also be applied to the following situation, where $g$ is identically zero. Suppose $(M, I)$ is a compact complex manifold and $\sigma$ a holomorphic Poisson structure with real part $Q$. Setting $g = 0$, $I_\pm = I$ and $H = 0$, this is a degenerate bi-Hermitian structure in the sense of Definition 4.1.3, except for the fact that the degeneracy set $E = M$ is no longer nowhere dense. Still, Lemma 4.2.4 only used that $Q$ is the real part of holomorphic Poisson structures $\sigma_\pm$, while Lemma 4.2.5 is trivially true in this case (both sides of both equations are zero). Hence Theorem 4.2.3 still applies in this case, and all we need is a potential $\alpha$ such that $-d^c\alpha$ is positive on $M$. To that end, suppose that $D \subset M$ is a divisor which is Poisson for $\sigma$ and which in addition is positive, i.e. the line bundle $\mathcal{O}_X(D)$ is positive. Let $s \in \Gamma(\mathcal{O}_X(D))$ be a holomorphic section which vanishes to first order along $D$, and choose a Hermitian metric $h$ on $\mathcal{O}_X(D)$ such that $iR_h$ is positive on $M$, where $R_h$ is the curvature of the unitary connection induced by $h$. Then $\alpha := -d \log |s|$ is a potential with the desired properties, for $-d^c\alpha = iR_h$ is smooth and positive, while $Q(\alpha)$ is smooth because $D$ is Poisson (c.f. the proof of Theorem 4.3.2 i)). The deformation procedure then gives us a bi-Hermitian structure where $I_+$ and $I_-$ are no longer equal. In [30] this was applied to find examples of generalized Kähler structures on Del Pezzo surfaces, which are complex surfaces whose anticanonical bundle is positive.

## 4.3 Flowing towards a non-degenerate structure

Let $Y \subset (M, \mathcal{J}_1, \mathcal{J}_2)$ be a compact generalized Poisson submanifold for $\mathcal{J}_1$ with degenerate normal bundle, and denote by $p : \widetilde{M} \to M$ the corresponding blow-up. By Proposition 4.1.4, the tuple $(p^*g, I_\pm, p^*H)$ defines a degenerate bi-Hermitian structure, whose degeneracy set $E$ is given by the exceptional divisor. Moreover, $T\widetilde{M}^\perp$ equals the vertical tangent bundle of the fibration $p : E \cong \mathbb{P}(NY) \to Y$. In order to apply the deformation procedure from Section 4.2, we need a suitable potential[6] $f$. This will be based on the following idea (see also [16]). Consider $M$ and $\widetilde{M}$ as complex manifolds with respect to either $I_+$ or $I_-$, so that $E$ is a divisor on $\widetilde{M}$, and consider the holomorphic line bundle $\mathcal{O}_{\widetilde{M}}(-E)$. Recall that a Hermitian metric on a holomorphic line bundle induces a unitary connection, whose curvature $R_h$ is of type $(1,1)$.

**Lemma 4.3.1.** If $U$ is any neighborhood of $E$ in $\widetilde{M}$, there exists a metric $h$ on $\mathcal{O}_{\widetilde{M}}(-E)$ such that $iR_h$ is supported in $U$ and restricts to a positive $(1,1)$-form on $T\widetilde{M}^\perp$.

*Proof.* On $E$ we have the tautological line bundle $\mathcal{O}_E(-1) \subset p^*NY$, whose fiber over a point $l \in E = \mathbb{P}(NY)$ is the corresponding line in $NY$. If we equip $NY$ with a Hermitian metric, then this induces one on $\mathcal{O}_E(-1)$ and therefore also on $\mathcal{O}_E(1) := \mathcal{O}_E(-1)^*$. Denote the latter by $h'$ and its curvature by $R_{h'}$. If we set $E_y := p^{-1}(y) = \mathbb{P}(N_yY)$ for $y \in Y$, then $iR_{h'}|_{E_y}$ equals (a multiple of) the Fubini-Study form[7] on $\mathbb{P}(N_yY)$. Now $\mathcal{O}_{\widetilde{M}}(-E)|_E \cong N^*E$, the conormal bundle of $E$ in $\widetilde{M}$, which equals $\mathcal{O}_E(1)$. We can extend the metric $h'$ on $\mathcal{O}_E(1)$ to a metric on $\mathcal{O}_{\widetilde{M}}(-E)$ as follows. Forgetting about the holomorphic structure for a moment, pick a tubular neighborhood $V \supset E$ in $\widetilde{M}$ such that $\overline{V} \subset U$. Denoting by $p : V \to E$ the corresponding retraction, we identify $\mathcal{O}_{\widetilde{M}}(-E)|_V \cong p^*\mathcal{O}_E(1)$. Equip $\mathcal{O}_{\widetilde{M}}(-E)|_V$ with the metric $p^*h'$ with curvature $p^*R_{h'}$, which has the same restriction to all of the $E_y$'s as $R_{h'}$ does. On the complement of $E$, the bundle $\mathcal{O}_{\widetilde{M}}(-E)$ is trivial so can be given a flat metric $h''$. We let $h$ be equal to $h''$ on $\widetilde{M}\backslash U$, $p^*h'$ on a neighborhood of $E$ in $V$, and a suitable interpolation in between. Clearly, $R_h$ is compactly supported in $U$, and $iR_h|_{E_y}$ is positive for all $y \in Y$. □

It is a well-known fact that if $s$ is any meromorphic section of $\mathcal{O}_{\widetilde{M}}(-E)$ which is not identically zero, then $iR_h = -dd^c \log|s|$. So, $f := -\log|s|$ is a good candidate for a potential. Unfortunately, we can not always guarantee smoothness of the Hamiltonian vector field $Q(df)$. The theorem below gives two situations where $Q(df)$ can be controlled.

**Theorem 4.3.2.** *Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be a generalized Kähler manifold and $Y$ a compact generalized Poisson submanifold for $\mathcal{J}_1$ whose conormal bundle is degenerate. Then if one of the following conditions holds, the blow-up $\widetilde{M}$ with respect to $\mathcal{J}_1$ has a generalized Kähler structure.*

*i) $(N^*Y, [\cdot,\cdot]_{\pi_{\mathcal{J}_1}})$ is Abelian.*

[6] We will consider potentials $\alpha = df$ and also refer to $f$ as the potential, slightly abusing terminology from Section 4.2.

[7] In fact this is one of the standard ways to define the Fubini-Study form on projective space.

*ii) $Y \subset D$, where $D$ is a compact Poisson divisor in $M$ with respect to $(I_+, \sigma_+)$ or $(I_-, \sigma_-)$.*

*Moreover, in situation i) the generalized Kähler structure on the blow-up agrees with the original structure on the complement of a neighborhood of the exceptional divisor. In situation ii), the same is true if we make the additional assumption that $\mathcal{O}_M(D)|_Y$ is trivial. In that case it is also not necessary to assume that $D$ is compact.*

*Proof.* i): Consider $M$ as a complex manifold with respect to, say, $I_+$. From (4.2) we see that $[\cdot,\cdot]_Q$ is Abelian on $N^*Y$ and therefore $E \subset \widetilde{M}$ is a Poisson submanifold for the lift of $Q$. Consider the potential $f = -\log|s|$, where $s$ is a meromorphic section of $\mathcal{O}_{\widetilde{M}}(-E)$ with a simple pole along $E$, and the norm is taken with respect to a metric as in Lemma 4.3.1. We claim that $Q(df)$ extends smoothly to the whole of $\widetilde{M}$. To see this, let $x \in E$ and let $e$ be a local holomorphic section of $\mathcal{O}_{\widetilde{M}}(-E)$ with $e(x) \neq 0$. Then $s = \frac{1}{z}e$, where $z$ is a local equation for $E$, hence

$$Q(df) = \frac{1}{4}\sigma_+(\frac{dz}{z}) + \frac{1}{4}\overline{\sigma_+}(\frac{d\bar{z}}{\bar{z}}) - Q(d\log|e|).$$

Since $\sigma_+(dz)$ is holomorphic and vanishes on $E$, it is divisible by $z$ and we see that $Q(df)$ is indeed smooth. Now we already know that $dd^c_+f$ is smooth and that $dd^c_+f|_{T\widetilde{M}^\perp}$ is positive, but in order to apply Theorem 4.2.3 we need that $(I_-^*(dd^c_+f)^{1,1}_{I_-})|_{T\widetilde{M}^\perp}$ is positive. However, the complex structure on $E_y$ is induced from $N_yY$ under the isomorphism $E_y = \mathbb{P}(N_yY)$. Since $Y$ is generalized Poisson, both $I_+$ and $I_-$ coincide on $NY$ and preserve it. So $E_y$ is a complex submanifold of $\widetilde{M}$ with respect to both $I_+$ and $I_-$, with the same induced complex structure. In particular $(I_-^*(dd^c_+f)^{1,1}_{I_-})|_{E_y} = dd^c_+f|_{E_y}$ is positive. So Theorem 4.2.3 applies and we obtain a generalized Kähler structure by perturbing the structure in a neighborhood of $E$, whose size is controlled by the choice of metric in Lemma 4.3.1 (so in particular can be arbitrarily small).

ii): Let us assume that $D$ is a Poisson divisor for $(I_+, \sigma_+)$, the other case being similar. Let $\widetilde{D}$ denote the proper transform[8] of $D$ on the blow-up $\widetilde{M}$. In terms of divisors, $\widetilde{D} = p^*D - kE$ for some $k \in \mathbb{Z}_{>0}$ and so $\mathcal{O}_{\widetilde{M}}(\widetilde{D}) = \mathcal{O}_{\widetilde{M}}(-kE) \otimes p^*\mathcal{O}_M(D)$. Equip $\mathcal{O}_{\widetilde{M}}(-kE) = \mathcal{O}_{\widetilde{M}}(-E)^{\otimes k}$ with the metric $h^{\otimes k}$, where $h$ is a metric on $\mathcal{O}_{\widetilde{M}}(-E)$ as in Lemma 4.3.1. If $h'$ is any metric on $\mathcal{O}_M(D)$, the metric $h^{\otimes k} \otimes p^*h'$ on $\mathcal{O}_{\widetilde{M}}(\widetilde{D})$ satisfies $iR_{h^{\otimes k}\otimes p^*h'} = ikR_h + ip^*R_{h'}$, which is positive on $T\widetilde{M}^\perp$ since $p^*R_{h'}$ vanishes there. Let $s$ be a holomorphic section of $\mathcal{O}_{\widetilde{M}}(\widetilde{D})$ with a simple zero along $\widetilde{D}$, and define $f := -\log|s|$. Then $dd^cf = ikR_h + ip^*R_{h'}$ is smooth on $\widetilde{M}$ and positive on $T\widetilde{M}^\perp$, while the same argument as in i) shows that $Q(df)$ is smooth, using the fact that $\widetilde{D}$ is Poisson. So again Theorem 4.2.3 applies, but this time the structure is perturbed along $\widetilde{D}$ as well, so we can not contain the deformation to a neighborhood of $E$. If however we know that $\mathcal{O}_M(D)|_Y$ is trivial, then we can choose $h'$ above to be flat around $Y$ and so $dd^c_+f = ikR_h$ around $E$. If $s'$ is a section of $\mathcal{O}_{\widetilde{M}}(kE)$ with a zero of order $k$ along $E$,

[8] Recall that this is the closure of $p^{-1}(D)\backslash E$ in $\widetilde{M}$.

and $\mathcal{O}_{\widetilde{M}}(kE)$ is equipped with the metric dual to $h^{\otimes k}$, we can define $f' := -\rho \cdot \log |s'|$, where $\rho$ is a function which is 0 near $E$ and 1 outside of a neighborhood of $E$. Then $f + f'$ still has the property that $Q(df)$ is smooth, but in addition satisfies $dd^c(f + f') = 0$ on an annulus around $E$. We then apply the deformation procedure only on a neighborhood of $E$, keeping it fixed on an annulus around it, and then glue the result back to the original structure. □

*Remark* 4.3.3. i) Suppose that $M$ is a Kähler manifold, seen as a generalized Kähler manifold as in Example 1.4.2, and $Y$ is a complex submanifold regarded as a generalized Poisson submanifold for $\mathcal{J}_1$. Then, since $\pi_{\mathcal{J}_1} = 0$, $N^*Y$ is Abelian and we are in situation i) of the theorem. Equation (4.5) that defines the flow reduces in this equation to simply adding $dd^c f$ to the symplectic form, and this is how one usually produces a Kähler metric on the blow-up.

ii) Let us clarify why we need $\mathcal{O}_M(D)|_Y$ to be trivial if we want to contain the deformation to a neighborhood of $E$ in situation ii) of the theorem. In the first part of the proof we are flowing the structure by the two-form $ikR_h + ip^*R_{h'}$, and in the second part we want to cancel this on an annulus around $E$ by $dd^c f'$, where $f'$ is a smooth function. In particular we need $ikR_h + ip^*R_{h'}$ to be exact on the annulus, which is automatic for $ikR_h$ since $\mathcal{O}_{\widetilde{M}}(kE)|_{\widetilde{M}\backslash E}$ is trivial. For $p^*R_{h'}$ to be exact, we need $R_{h'}$ to be exact[9] around $Y$, which amounts to $\mathcal{O}_M(D)|_Y$ being trivial around $Y$.

Although condition ii) of the theorem is clear as it is stated, it is unclear whether it has any applications. For that reason we state the following

**Corollary 4.3.4.** Let $(M, \mathcal{J}_1, \mathcal{J}_2)$ be generalized Kähler with $\mathcal{J}_1$ generically of symplectic type, and $Y$ a compact generalized Poisson submanifold for $\mathcal{J}_1$ with degenerate conormal bundle and which is contained in the type change locus[10] of $\mathcal{J}_1$. Then the blow-up is generalized Kähler.

*Proof.* Let $X_1$ be the type change locus for $\mathcal{J}_1$. In a local chart of the form (1.46), $X_1$ is given by the vanishing of the holomorphic function $\sigma^{k/2}$ and as such is either empty or a codimension 1 analytic subset of $\mathbb{C}^n$. We assume $X_1 \neq \emptyset$, otherwise the statement is vacuous. Let $D' \subset M$ denote the Poisson subvariety of points where $Q$ does not assume its maximal rank on $M$. By Lemma 1.3.44 and (1.57) we have

$$\ker(Q) = \ker(I_+^* - I_-^*) \oplus \ker(I_+^* + I_-^*) = \ker(\pi_{\mathcal{J}_1}) \oplus \ker(\pi_{\mathcal{J}_2}).$$

Consequently, $D' = X_1 \cup X_2$, where $X_i$ is the set of points where $\pi_{\mathcal{J}_i}$ is not of maximal rank (or equivalently, where $\mathcal{J}_i$ does not assume its minimal type). Let $D$ be the union of the codimension 1 components of $D'$. Then $D$ is also a Poisson subvariety which a priori could be empty, but we claim that $X_1 \subset D$. Indeed, if $x \in X_1 \backslash D$, then a neighborhood $U$ of $x$ in $X_1$ is disjoint from $D$. However, since $U$ is given by the vanishing

[9] As $Y$ has complex codimension 2 or bigger (otherwise the blow-up is trivial), the Gysin sequence shows that the second degree cohomology of an annulus around $Y$ agrees with that of $Y$ itself.

[10] Since $\mathcal{J}_1$ is generically symplectic, for $Y$ to be generalized Poisson it has to be either an open set in the symplectic locus or fully contained in the type change locus. In the former case there is nothing to blow up.

of a holomorphic function (for a complex structure which need not coincide with either $I_\pm$), an open dense set in $U$ is a smooth submanifold of $M$ of real dimension $2n-2$. But $U \subset D'$ , and a real $2n-2$ dimensional submanifold of a complex manifold can not be contained in a finite union of analytic subsets of complex codimension bigger than 1. So indeed $X_1 \subset D$, and Theorem 4.3.2 ii) applies. □

A special case of this corollary is when $Y$ is a point and $M$ is four-dimensional. This situation was considered in [16], where it was assumed that the type change locus was smooth at the point in question.

*Remark* 4.3.5. In [21] Goto proved that every compact Kähler manifold with a holomorphic Poisson structure $\sigma$ has a generalized Kähler structure, where $\mathcal{J}_1$ is given by the Poisson deformation of the complex structure and $\mathcal{J}_2$ is a suitable deformation of the symplectic structure. If $Y$ is a complex Poisson submanifold for $\sigma$ whose conormal bundle is degenerate, then $\sigma$ lifts to the complex blow-up of $Y$, which has a Kähler metric of its own (see Remark 4.3.3 i)). Applying Goto's theorem again, we see that the blow-up is again generalized Kähler and the blow-down map is generalized holomorphic with respect to $\mathcal{J}_1$.

The proof of Goto's theorem relies on the use of Green's operators for finding the right deformation of $\mathcal{J}_2$, and as such is non-local. In fact, since the Kähler metric on the blow-up differs from the original metric on a neighborhood of the exceptional divisor, there is a-priori nothing we can say about the relation between $\mathcal{J}_2$ on the blow-up and on the original manifold, even arbitrarily far away from the exceptional divisor. As such it seems rather difficult to relate this to any blow-up procedure which is surgery-theoretic in nature.

## 4.4 An example: compact Lie groups

One source of examples of generalized Kähler manifolds is provided by Lie theory.

**Proposition 4.4.1** ([25]). *Let $G$ be an even-dimensional compact Lie group. Then $G$ has a generalized Kähler structure.*

In order to find suitable submanifolds to blow up, we need to understand these generalized Kähler structures in some detail. Let $G$ be a Lie group. An element $\xi \in \mathfrak{g}$ defines left- and right-invariant vector fields $\xi^L_g := d_eL_g(\xi)$ and $\xi^R_g := d_eR_g(\xi)$, and we have $[\xi^L, \eta^L] = [\xi, \eta]^L$, $[\xi^R, \eta^R] = -[\xi, \eta]^R$. These two trivializations of $TG$ define connections $\nabla^\pm$, characterized by $\nabla^+\xi^L = 0 = \nabla^-\xi^R$ for $\xi \in \mathfrak{g}$. Their torsion is given by

$$T^+(\xi^L, \eta^L) = -[\xi, \eta]^L, \qquad T^-(\xi^R, \eta^R) = [\xi, \eta]^R.$$

Suppose now that $G$ is compact, and let $\langle \cdot, \cdot \rangle$ be a metric on $\mathfrak{g}$ which is invariant under the adjoint action. In particular, its left- and right-invariant extensions over $G$ coincide and we denote this common extension by the same symbol $\langle \cdot, \cdot \rangle$. The three-form on $\mathfrak{g}$ defined by $H(\xi, \eta, \zeta) := \langle [\xi, \eta], \zeta \rangle$ is also invariant under the adjoint action, and so extends to a bi-invariant three-form on $G$. From the Jacobi identity it follows that $H$ is closed[11] and we have $\langle T^{\pm}(X,Y),Z\rangle = \mp H(X,Y,Z)$, hence $\nabla^{\pm}$ coincide with the connections defined in Proposition 1.4.3.

Since $G$ is compact it is automatically reductive, i.e. its Lie algebra splits as $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{g}'$, with $\mathfrak{a}$ Abelian and $\mathfrak{g}'$ semi-simple. Let $\mathfrak{t}$ be a maximal torus of $\mathfrak{g}'$ and $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}}\oplus\sum_{\alpha\in R}\mathfrak{g}'_\alpha$ the associated root space decomposition. Since $G$ is compact, the roots are contained in $i\mathfrak{t}^* \subset \mathfrak{t}^*_{\mathbb{C}}$, hence they come in pairs $\pm\alpha$ and we have $\overline{\mathfrak{g}'_\alpha} = \mathfrak{g}'_{-\alpha}$. Consequently, dim$(\mathfrak{g}')$ and dim$(\mathfrak{t})$ have the same parity, and since $\mathfrak{g}$ is even dimensional it follows that $\mathfrak{a}\oplus\mathfrak{t}$ is even dimensional. Now choose a decomposition $R = R^+ \cup R^-$ into positive and negative roots, so that in particular $-R_+ = R_-$. We define a complex structure $I$ on $\mathfrak{g}$ by the decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}^{1,0}\oplus\mathfrak{g}^{0,1}$, where

$$\mathfrak{g}^{1,0} = (\mathfrak{a}\oplus\mathfrak{t})^{1,0}\oplus\sum_{\alpha\in R^+}\mathfrak{g}'_\alpha$$

and $(\mathfrak{a}\oplus\mathfrak{t})^{1,0}$ is defined by an arbitrary but fixed complex structure on $\mathfrak{a}\oplus\mathfrak{t}$, compatible with $\langle\cdot,\cdot\rangle$ in the sense that the complex structure is an orthogonal transformation. This is equivalent to $(\mathfrak{a}\oplus\mathfrak{t})^{1,0}$ being isotropic for the complex linear extension of $\langle\cdot,\cdot\rangle$. By invariance of the metric, $\mathfrak{g}_\alpha$ is orthogonal to $\mathfrak{g}_\beta$ unless $\alpha = -\beta$, hence $I$ is compatible with $\langle\cdot,\cdot\rangle$. Since the sum of two positive roots is again positive, it follows that $[\mathfrak{g}^{1,0},\mathfrak{g}^{1,0}] \subset \mathfrak{g}^{1,0}$, so the complex structures $I_+$ and $I_-$, which are defined by the left-, respectively, right-invariant extensions of $I$ over $G$, are integrable. Since they are constant in the two respective trivializations, we have $\nabla^{\pm}I_{\pm} = 0$, so $(G,\langle\cdot,\cdot\rangle,I_{\pm},H)$ is generalized Kähler by Proposition 1.4.3.

Next we look for generalized Poisson submanifolds for $\mathcal{J}_1$. A natural candidate is given by the complex locus of $\mathcal{J}_1$, i.e. the set of points where $I_+ = I_-$ (see (1.4.3)). For $g\in G$ we have $(I_+)_g = (L_g)_*I$ and $(I_-)_g = (R_g)_*I$, so that $(I_+)_g = (I_-)_g$ if and only if Ad$(g)_*I = I$. This condition defines a subgroup of $G$, whose Lie algebra is given by

$$\{\xi\in\mathfrak{g}|\ [\mathrm{ad}(\xi),I] = 0 \Leftrightarrow [\xi,\mathfrak{g}^{1,0}]\subset\mathfrak{g}^{1,0}\}.$$

This algebra coincides with $\mathfrak{a}\oplus\mathfrak{t}$, hence the connected component of the complex locus of $\mathcal{J}_1$ that contains the identity equals the connected subgroup $T$ whose Lie algebra is $\mathfrak{a}\oplus\mathfrak{t}$. Thus $T$, or any complex submanifold $Y\subset T$ for that matter, is a generalized Poisson submanifold of $(G,\mathcal{J}_1)$. To blow up $Y$ in $G$ with respect to $\mathcal{J}_1$, we need to understand the induced Lie algebra structure on $N^*Y$. Since $Y\subset T$, we have an inclusion of Lie algebras

$$N^*T|_Y \subset N^*Y \subset T^*G|_Y. \tag{4.14}$$

The action of $T$ on $G$, either from the left or the right, is a symmetry of the whole generalized Kähler structure that preserves $T$. In particular, the Lie brackets on $T^*_yG$ and $N^*_yT$

[11] In fact, since $H$ is constant in both trivializations it is parallel with respect to both $\nabla^{\pm}$, and therefore also for their affine combination $\nabla := \frac{1}{2}(\nabla^+ + \nabla^-)$, which is nothing but the Levi-Cevita connection for $\langle\cdot,\cdot\rangle$.

are independent of $y \in Y$ and we can compute them at $e \in G$. From (1.57) we see that

$$(\pi_{\mathcal{J}_1})_g = -\frac{1}{2}(R_g)_*\big(\mathrm{Ad}(g)_*\omega^{-1} - \omega^{-1}\big) = -\frac{1}{2}(R_g)_*\Big(\mathrm{Ad}(g)\circ\omega^{-1}\circ\mathrm{Ad}(g)^* - \omega^{-1}\Big),$$

where $\omega(\xi,\eta) = \langle I\xi,\eta\rangle$ is the associated Hermitian two-form on $\mathfrak{g}$. Consequently, since for $\zeta \in \mathfrak{g}$ the flow of $\zeta_L$ is given by $\varphi_t = R_{\exp(t\zeta)}$, we have

$$(\mathcal{L}_{\zeta^L}\pi_{\mathcal{J}_1})_e = \frac{d}{dt}\bigg|_{t=0}\big(R^*_{\exp(t\zeta)}(\pi_{\mathcal{J}_1})\big)_e = -\frac{1}{2}\Big(\mathrm{ad}(\zeta)\circ\omega^{-1} + \omega^{-1}\circ\mathrm{ad}(\zeta)^*\Big).$$

Let $\xi,\eta \in \mathfrak{g}$ and denote by $\tilde{\xi},\tilde{\eta} \in \mathfrak{g}^*$ their images under the metric. We obtain

$$[\tilde{\xi},\tilde{\eta}]_{\pi_{\mathcal{J}_1}}(\zeta) = (\mathcal{L}_{\zeta^L}\pi_{\mathcal{J}_1})_e(\tilde{\xi},\tilde{\eta}) = \frac{1}{2}\langle [I\xi,\eta]+[\xi,I\eta],\zeta\rangle.$$

Hence, using the metric, the bracket $[\cdot,\cdot]_{\pi_{\mathcal{J}_1}}$ induces the following bracket on $\mathfrak{g}$:

$$[\xi,\eta]_{{}_1} := \frac{1}{2}\big([I\xi,\eta]+[\xi,I\eta]\big).$$

Write $\xi = \xi' + \sum_\alpha(\xi_\alpha + \overline{\xi_\alpha})$, where $\xi' \in \mathfrak{a}\oplus\mathfrak{t}$ and $\sum_\alpha(\xi_\alpha+\overline{\xi_\alpha}) \in \sum_\alpha(\mathfrak{g}_\alpha\oplus\overline{\mathfrak{g}_\alpha})_{\mathbb{R}}$, and similarly for $\eta$. Here and below, the summation on $\alpha$ is over all positive roots. Then

$$\begin{aligned}[\xi,\eta]_{{}_1} =& i\sum_{\alpha,\beta}\big([\xi_\alpha,\eta_\beta] - [\overline{\xi_\alpha},\overline{\eta_\beta}]\big)\\ &+ i\sum_\alpha\Big(\alpha^{1,0}(\xi')\eta_\alpha + \alpha^{0,1}(\xi')\overline{\eta_\alpha} - \alpha^{1,0}(\eta')\xi_\alpha - \alpha^{0,1}(\eta')\overline{\xi_\alpha}\Big).\end{aligned} \tag{4.15}$$

Here we are regarding the roots $\alpha,\beta \in (\mathfrak{a}_{\mathbb{C}}\oplus\mathfrak{t}_{\mathbb{C}})^*$ by extending them trivially over $\mathfrak{a}_{\mathbb{C}}$ and define their $(1,0)$- and $(0,1)$-components with respect to $I|_{\mathfrak{a}\oplus\mathfrak{t}}$. From (4.14) we see that if $N^*_yY$ is degenerate then so is $N^*_yT$, and therefore the whole of $N^*T$ by $T$-equivariance. So a necessary condition to blow up anything in $T$ is that $T$ itself can be blown up. This can be investigated by restricting (4.15) to $(T_eT)^\perp = \sum_\alpha(\mathfrak{g}_\alpha\oplus\overline{\mathfrak{g}_\alpha})_{\mathbb{R}} \subset \mathfrak{g}$. There, (4.15) reduces to

$$[\xi,\eta]_{{}_1} = i\sum_{\alpha,\beta}\big([\xi_\alpha,\eta_\beta] - [\overline{\xi_\alpha},\overline{\eta_\beta}]\big).$$

Recall that a Lie algebra is degenerate if and only if the bracket of any two elements lies in the plane spanned by them. In particular, as $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for root decompositions, $N^*_eT \cong (T_eT)^\perp$ is degenerate if and only if the sum of any two positive roots is not a root itself. The only root systems satisfying this are products of $A_1$, corresponding to the Lie group $SU(2)\cong S^3$. In conclusion, in order to blow up $T$ in $G$ with respect to $\mathcal{J}_1$, we need $G$ to be of the form $G = (S^1)^n\times(S^3)^m$ for $n+m$ even, with $T = (S^1)^n\times(S^1)^m$. We then still have some residual freedom in choosing the complex submanifold $Y\subset T$. Instead of determining precisely all $Y$ for which (4.15) becomes degenerate, let us give a concrete example. If $m$ is even, we can choose the complex structure $I$ on $\mathfrak{g}$ such that the induced complex structure on $(S^1)^n \times (S^1)^m$ is a product of two complex structures. This is because in this case both $\mathfrak{a}$ and $\mathfrak{t}$ are even-dimensional and we can choose the complex structure on $\mathfrak{a} \oplus \mathfrak{t}$ to be a direct sum of complex structures. Then if $Y$ is of the form $Y' \times (S^1)^m$, with $Y'$ a complex submanifold of $(S^1)^n$, (4.15) vanishes on $N^*Y$ because all roots vanish on $\mathfrak{a}_\mathbb{C}$. Alternatively, if $m$ is odd we can choose $I$ so that the induced complex structure on $(S^1)^{n-1} \times (S^1 \times (S^3)^m)$ is a product. Then if $Y = Y' \times (S^1)^{1+m}$, with $Y' \subset (S^1)^{n-1}$ a complex submanifold, (4.15) vanishes again on $N^*Y$. Note that since in all these cases $N^*Y$ is Abelian, Theorem 4.3.2 i) applies.

**Theorem 4.4.2.** *Let $G = (S^1)^n \times (S^3)^m$, where $n+m$ is even and let $T = (S^1)^n \times (S^1)^m \subset G$ be a maximal torus. Equip $G$ with a generalized Kähler structure as above for which $T$ is a generalized Poisson submanifold. Then in the following two cases*

*i) $m$ is even and $Y = Y' \times (S^1)^m$ with $Y'$ a complex submanifold of $(S^1)^n$,*

*ii) $m$ is odd and $Y = Y' \times (S^1)^{m+1}$ with $Y'$ a complex submanifold of $(S^1)^{n-1}$,*

*$Y$ can be blown up to a generalized Kähler manifold.*

*Remark* 4.4.3. Note that this example provides in particular generalized Poisson submanifolds of arbitrarily high codimension which can be blown-up. The topology of $Y$ can still be quite general because of the freedom we have in choosing the complex submanifold $Y'$ of the torus. For instance, any complex curve can be embedded in a torus, via its Jacobian embedding.

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# Nederlandse samenvatting

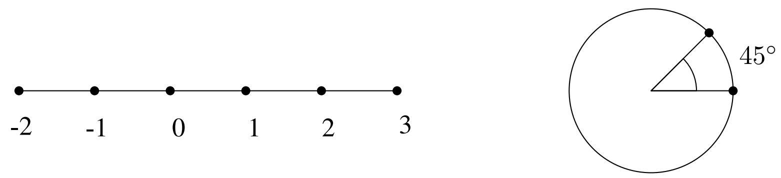
De titel van deze scriptie luidt: "Opblazen in Gegeneraliseerde Complexe Meetkunde". In dit hoofdstuk zullen we een toelichting geven bij deze titel en in het kort samenvatten wat er in deze scriptie bewezen wordt. De daadwerkelijke samenvatting van de resultaten bevindt zich op de laatste pagina van dit hoofdstuk, al het voorgaande is bedoeld ter inleiding voor diegene die niet thuis zijn in dit vakgebied.

### Variëteiten

Laten we beginnen door te verduidelijken wat we bedoelen met "meetkunde". Er zijn verschillende vakgebieden in de wiskunde die zich met meetkunde bezighouden en deze scriptie valt in het onderzoeksgebied "differentiaalmeetkunde". Dit is de studie van ruimtes die wiskundigen *variëteiten* noemen. Aan de hand van illustraties zullen we proberen hier een gevoel voor te geven. Het eerste voorbeeld van een variëteit is simpelweg een punt, zie Figuur 1(a). De *dimensie* van deze ruimte is nul, omdat iemand die zich in deze ruimte bevindt zich op geen enkele manier kan verplaatsen. Als volgend voorbeeld beschouwen we een lijn, zie Figuur 1(b). Deze variëteit is één-dimensionaal, aangezien iemand die zich op de lijn bevindt zich in één enkele richting kan verplaatsen. Ons laatste voorbeeld in dimensie één is de cirkel, zie Figuur 1(c).

**Figuur 1** (a) (b) (c)

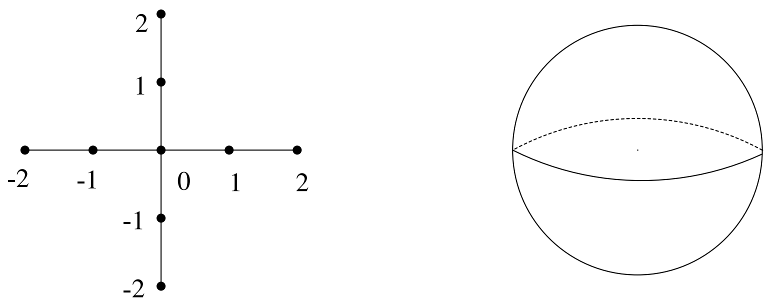
Een precieze manier om de dimensie te definiëren is met behulp van coördinaten. Als we de lijn voorzien van een assenstelsel, dan kunnen we de positie van een punt op de lijn vastleggen met één enkel getal, als in Figuur 2(a). Evenzo, als we een vast punt op de cirkel kiezen, dan zijn alle andere punten aan te duiden met de hoek die ze maken ten opzichte van het vaste punt, zie Figuur 2(b). We hebben dus één enkel getal nodig om punten op de cirkel te beschrijven en dus is de cirkel één-dimensionaal. Het enige verschil met de lijn is dat een gegeven punt met meerdere hoeken te beschrijven valt. Bijvoorbeeld, de hoeken $-360°, 0°$, $360°$, $720°$ en alle andere veelvouden van $360°$ beschrijven allemaal hetzelfde punt.




**Figuur 2** (a) (b)

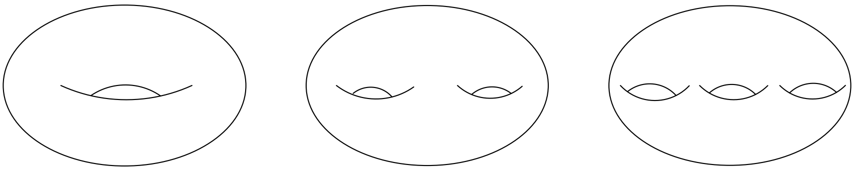
Nu, hoe zou iemand die in een één-dimensionale variëteit leeft kunnen weten of hij of zij zich op een cirkel of op een lijn bevindt? Als diegene maar een klein beetje zou mogen bewegen is er geen enkele manier om daar achter te komen. Pas als na een bepaalde tijd lopen (zonder om te keren) het punt van vertrek wordt bereikt is het duidelijk dat het om een cirkel gaat. Dit is de essentie van een één-dimensionale variëteit: het is een ruimte die er *lokaal* uitziet als een lijn. Anders gezegd: vanuit elk punt in de ruimte zien we een lijn, tenminste als we niet te ver kijken.

De lijn en de cirkel zijn de enige één-dimensionale variëteiten, maar in dimensie twee wordt de situatie iets interessanter. Als eerste voorbeeld beschouwen we het twee-dimensionale vlak, zie Figuur 3(a). We hebben hier al een assenstelsel op aangebracht, wat duidelijk maakt dat het vlak inderdaad twee-dimensionaal is; voor elk punt in het vlak hebben we twee getallen nodig om te beschrijven waar het zich bevindt. Net als met de lijn in één dimensie, kunnen we het twee-dimensionale vlak gebruiken voor de definitie van een twee-dimensionale variëteit: een ruimte die er lokaal uitziet als het twee-dimensionale vlak. Laten we kijken naar wat voorbeelden. Neem bijvoorbeeld een bol als in Figuur 3(b), waarbij het gaat om het oppervlak van de bol en dus niet om de inhoud.




**Figuur 3** (a) (b)

Het feit dat een bol er lokaal uitziet als een vlak beseffen we ons maar al te goed, aangezien het oppervlak van de aarde een twee-dimensionale bol vormt en we voor lange tijd dachten dat de aarde plat was. Een ander voorbeeld wordt gegeven door de twee-dimensionale torus, zie Figuur 4(a). We benadrukken weer dat het gaat om het getekende oppervlak, niet om de inhoud. Als we de torus interpreteren als een bol met één gat erin, dan krijgen we automatisch een hele familie van voorbeelden door het maken van meerdere gaten. Het resultaat wordt een *oppervlak van geslacht $g$* genoemd, waarbij $g$ het aantal gaten is. Zie Figuur 4(b),(c) voor de gevallen $g = 2$ en $g = 3$.



**Figuur 4** (a) (b) (c)

Er zijn nog meer voorbeelden van twee-dimensionale variëteiten maar laten we doorgaan naar hogere dimensies. Om te begrijpen wat een $n$-dimensionale variëteit is, waarbij $n$ een zeker natuurlijk getal is (dus $n = 0, 1, 2, 3, 4$ of groter), herhalen we dezelfde stappen die we in één en twee dimensies namen. We bouwen eerst het $n$-dimensionale analogon van de lijn en het vlak. Deze ruimte wordt ook wel *$n$-dimensionale Euclidische ruimte* genoemd, en een korte notatie hiervoor is $\mathbb{R}^n$. Per definitie is dit de ruimte bestaande uit alle mogelijke collecties van $n$ getallen, die men kan opvatten als de $n$ coördinaten van een punt. Het geval $n = 0$ is hierbij een beetje speciaal, we definiëren $\mathbb{R}^0$ simpelweg als een punt (als in Figuur 1(a)). Voor $n = 1$ bestaat $\mathbb{R}^1$ dus uit punten die beschreven worden door één getal en is dus voor te stellen als een lijn (Figuur 1(b)). Evenzo bestaat $\mathbb{R}^2$ uit punten die worden gegeven door twee getallen en dus is $\mathbb{R}^2$ te identificeren met het vlak (Figuur 3(a)). Het laatste geval waar we ons een voorstelling van kunnen maken is $n = 3$. Punten in $\mathbb{R}^3$ worden beschreven door drie getallen en we kunnen $\mathbb{R}^3$ dus identificeren met de ruimte die we om ons heen lijken te zien[12]. De ruimtes $\mathbb{R}^n$ voor $n = 4, 5, 6$ en hoger kunnen we niet voor ons zien, maar wiskundig gezien is dit geen probleem. Een punt in $\mathbb{R}^4$ is simpelweg een collectie van vier getallen, zoals bijvoorbeeld $(1, 5, \frac{1}{2}, 0)$ of $(0, \pi, 0, 3)$, en we hebben geen illustraties nodig om met zulke punten te kunnen werken. In feite is in de wiskunde een plaatje nooit voldoende om te gelden als een formeel bewijs. Zelfs als we iets over het vlak willen bewijzen moeten we het doen met de abstracte beschrijving als zijnde $\mathbb{R}^2$. Illustraties zijn er alleen voor de intuïtie en om bepaalde abstracte of technische argumenten te verhelderen. Met behulp van $\mathbb{R}^n$ kunnen we nu de definitie geven van een $n$-dimensionale variëteit: een ruimte die er lokaal uitziet als $\mathbb{R}^n$. We hebben hierboven dus al voorbeelden gezien voor $n = 0, 1, 2$. Zoals gezegd is het vanaf $n = 3$ lastig, zo niet onmogelijk, om directe illustraties te geven, maar wiskundigen hebben abstracte technieken ontwikkelt die het toelaten om zulke ruimtes te behandelen

[12] Aangezien we niet weten hoe het heelal er in zijn geheel uitziet, kan het best zijn dat de ruimte om ons heen niet zoals $\mathbb{R}^3$ is, maar, net zoals de cirkel, de eigenschap heeft dat als we maar genoeg in een bepaalde richting lopen we weer terugkeren op het punt van vertrek.

op dezelfde manier als bijvoorbeeld het één- en twee-dimensionale geval. Dit wil niet zeggen dat alle dimensies even moeilijk zijn, integendeel. Al in de eerder genoemde voorbeelden is het duidelijk dat we in hogere dimensies meer voorbeelden tegenkomen dan in lagere.

Voor iemand die de definitie van een variëteit hier voor het eerst ziet zal het waarschijnlijk niet direct duidelijk zijn wat het nut ervan is. Waarom zou men variëteiten überhaupt willen bestuderen? Het antwoord is simpelweg dat variëteiten op heel veel verschillende plekken tevoorschijn komen, met name in de natuurkunde. Dit geldt voor zowel complexe theoriën zoals de algemene relativiteitstheorie, als ook voor bijvoorbeeld de klassieke mechanica. Denk bijvoorbeeld aan een slinger zoals in een klok aan de muur, wiens ruimte van mogelijke posities gelijk is aan een cirkel. Een groot gedeelte van de klassieke mechanica bestaat uit het bestuderen van dynamische systemen op variëteiten, zoals hoe de slinger zich gedraagt op de cirkel naarmate de tijd verstrijkt.

## Meetkundige structuren

In de vorige sectie zagen we de definitie van een variëteit. Ondanks dat dit op zichzelf een interessant concept is, vereisen veel toepassingen de aanwezigheid van iets extra's op de ruimte zelf. Zoiets extra's noemen we een *meetkundige structuur*. Ons doel in deze sectie is het introduceren van een voorbeeld hiervan, namelijk *gegeneraliseerde complexe structuren*. Dit is een tamelijk abstract concept, dus we zullen dit in meerdere stappen uitleggen.

*Metrieken*

We beginnen met het beschrijven van een meetkundige structuur die het eenvoudigst is om te begrijpen en de basis vormt voor de daaropvolgende voorbeelden. Dit voorbeeld is dat van een *metriek*. Een metriek op een variëteit is een voorschrift dat ons in staat stelt om hoeken en afstanden op de variëteit te meten. Preciezer gezegd, een metriek vertelt ons wat de hoek is tussen twee lijnen die elkaar snijden, en wat de lengte is van een pad tussen twee gegeven punten. We benadrukken hierbij dat een metriek niet zomaar cadeau komt met een gegeven variëteit; het hebben van een metriek is extra informatie. Neem bijvoorbeeld een lijn als in Figuur 1(b). Aangezien in dimensie één er geen concept van hoek bestaat, is het geven van een metriek op de lijn hetzelfde als aangeven wat de afstand is tussen twee gegeven punten. Dit kan op een vrij willekeurige manier, het enige waaraan voldaan moet zijn is de eigenschap dat als we twee lijnstukken van lengtes $a$ en $b$ aan elkaar plakken, de totale lengte gelijk is aan de som $a + b$. Bijvoorbeeld, we zouden in Figuur 2(a) het lijnstuk tussen de punten 0 en 1 lengte 1 kunnen geven, maar ook elk ander positief getal naar keuze, en de lengte tussen punten 0 en 1 hoeft niet persé hetzelfde te zijn als dat tussen de punten 1 en 2. In dimensie twee en hoger geeft een metriek niet alleen lengtes maar ook hoeken, zoals we bijvoorbeeld ondervinden op het oppervlak van onze aardbol[13]. Als we de aarde twee keer zo groot zouden maken dan

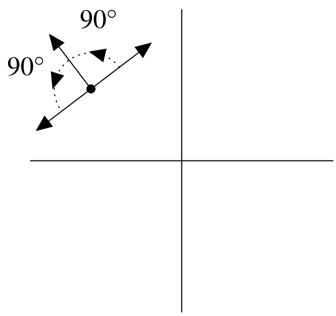
[13]De metriek op het aardoppervlak is vastgelegd doordat de aarde zich in het drie-dimensionale heelal bevindt.

zouden we daar, in de afwezigheid van een metriek, helemaal niks van merken. Een metriek stelt ons echter in staat om te meten hoe "gekromd" een variëteit is, en dus ook hoe groot de straal van een bol zoals de aarde is. Tenslotte merken we op dat we met een metriek niet alleen lengtes van paden in een variëteit kunnen meten, maar ook oppervlaktes van twee-dimensionale objecten en volumes van drie-dimensionale objecten. We kunnen zelfs "volumes" definiëren van objecten die een hogere dimensie hebben dan drie, vooropgesteld dat die dimensie kleiner of gelijk is aan dat van de variëteit zelf waarin ze leven. Het is gebruikelijk om al deze concepten aan te duiden met volume, zelfs in dimensies één en twee waar de terminologie lengte en oppervlakte gebruikelijker is.

*Symplectische structuren*
Het volgende voorbeeld is wat abstracter dan het voorgaande, maar brengt ons dicht bij gegeneraliseerde complexe structuren. We zagen hierboven dat een metriek ons in staat stelt om, behalve het meten van hoeken, ook volumes toe te kennen aan objecten van allerlei dimensies in de variëteit. Neem nu een variëteit die een zekere even dimensie heeft $(2, 4, 6$, etcetera). Een *symplectische structuur* is, per definitie, een voorschrift dat een getal toekent aan alle even-dimensionale objecten in de variëteit. Dus, in bijvoorbeeld een 6-dimensionale variëteit kent een symplectische structuur een getal toe aan objecten van dimensie 2, 4 en 6. We zouden dit getal kunnen opvatten als een soort van "volume", alhoewel dit getal best negatief kan zijn en aan de volgende opmerkelijke eigenschap voldoet. We weten dat het normale volume van een object verandert als we het object aanpassen; als we een ballon nemen en die opblazen zodat de diameter twee keer zo groot wordt, dan wordt de oppervlakte van de ballon vier keer zo groot. Echter, voor het "volume" wat gegeven is door een symplectische structuur eisen we juist dat als we een object zoals een ballon opblazen (of leeg laten lopen), het volume *gelijk* blijft. Dit vreemde aspect is moeilijk te illustreren, omdat het zich niet uit in ruimtes van dimensie twee, maar alleen in dimensies vier en hoger. De reden hiervoor kunnen we begrijpen door te kijken naar een twee-dimensionaal vlak. Een symplectische structuur op het vlak is simpelweg een voorschrift wat een getal toekent aan regio's in het vlak, zoals bijvoorbeeld een schijf dat omschreven wordt door de binnenkant van een cirkel. Voor dit soort regio's is het wel het geval dat het volume verandert als we de regio aanpassen. Het verschil met een schijf en een boloppervlak zoals de ballon is dat eerstgenoemde een *rand* heeft. Immers, iemand die op een schijf rondloopt zal uiteindelijk de rand tegenkomen en niet verder kunnen lopen. Op de bol daarentegen is dit niet het geval, diegene kan voor altijd rond blijven lopen zonder ooit ergens een obstructie tegen te komen. De precieze conditie die we dan op een symplectische structuur opleggen is dat het volume van een object *zonder rand* niet veranderd, als we dat object op een geleidelijke manier aanpassen (zoals het opblazen of leeg laten lopen van de ballon). In dimensie twee is dit niet te zien, omdat het enige twee-dimensionale object zonder rand gelijk is aan de hele ruimte zelf, en deze kunnen we niet geleidelijk groter of kleiner maken.

*Complexe structuren*
We hebben hierboven gezien dat een metriek ons zowel hoeken als volumes geeft, en dat een symplectische structuur ons alleen volumes geeft (al zij het een vreemd soort volume,

die bovendien alleen even-dimensionale objecten kan meten). Ons voorlaatste voorbeeld van een meetkundige structuur is iets dat zich alleen focust op het aspect van hoeken. We bekijken eerst een voorbeeld. Gegeven een metriek op het vlak, dus een notie van afstanden en hoeken, dan kunnen we tegen iemand die in het vlak staat en in een bepaalde richting kijkt, de opdracht geven om zichzelf over 90 graden te draaien met de klok mee[14]. Een belangrijk aspect van draaien over 90 graden is dat als we dit twee keer achter elkaar doen, we eindigen in de richting die tegenovergesteld is aan de oorspronkelijke richting, zie Figuur 5.




**Figuur 5**

Dit leidt ons tot de algemene definitie van een *complexe structuur*: het is een voorschrift die aan elk punt en elke richting in de variëteit een nieuwe richting toekent, zodat als we dit twee keer achter elkaar doen, we eindigen in de richting die tegenovergesteld is aan de oorspronkelijke. We zouden over deze nieuwe richting kunnen nadenken als een abstracte "rotatie over 90 graden met de klok mee", zij het echter dat er niet noodzakelijkerwijs een concept van hoeken aanwezig is op de variëteit. Het is duidelijk dat het onmogelijk is om zoiets te doen in een ruimte van dimensie één, en het blijkt zelfs dat de ruimte even-dimensionaal moet zijn om mogelijkerwijs een complexe structuur te kunnen dragen. Net als met symplectische structuren is de hierboven gegeven definitie niet volledig; er is nog een technische conditie die we helaas niet eenvoudig kunnen illustreren, zoals met het opblazen van een ballon, en die pas optreedt in dimensies vier en hoger.

*Gegeneraliseerde complexe structuren*

Eindelijk zijn we aangekomen bij de definitie van een *gegeneraliseerde complexe structuur*. Dit is een meetkundige structuur die, in de meest simpele bewoording, een hybride combinatie is van een symplectische structuur en een complexe structuur. Wat dit precies betekent is lastig om hier te omschrijven, maar intuïtief gezien betekent het dat voor elk

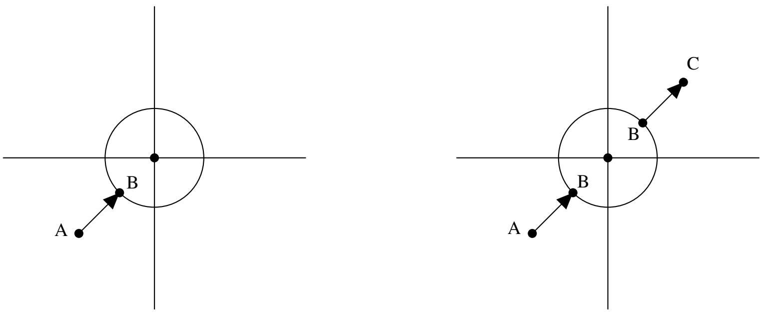
[14] Er is hier een subtiliteit waar we verder geen aandacht aan zullen schenken, en dat is hoe zo iemand zou kunnen weten wat de richting van de klok is. Het precieze antwoord is dat we eerst zelf een draairichting moeten kiezen die we "met de klok mee" noemen, dit is dus weer een extra meetkundige structuur die we op de variëteit aanbrengen! Er zijn variëteiten waar op geen enkele consistente manier een klokrichting kan worden gekozen, het beroemdste voorbeeld hiervan is de zogenaamde Möbiusband.

punt in de variëteit er richtingen zijn die uitgerust zijn met een symplectische structuur (dus in die richting kunnen we “volumes” meten), en andere richtingen die uitgerust zijn met een complexe structuur (in die richting kunnen we dus “over $90°$ draaien”). De hoeveelheid richtingen die symplectisch of complex zijn moeten allebei even zijn en bij elkaar optellen tot de dimensie van de variëteit. Bovendien mag deze verdeling van punt tot punt variëren, maar alleen in stappen van $4$. Dus, als we bijvoorbeeld een twee-dimensionale variëteit beschouwen, dan moet in elk punt ofwel alle richtingen symplectisch zijn of allemaal complex. Immers, we kunnen het getal $2$ maar op twee manier schrijven als de som van twee even getallen, namelijk $2 = 0 + 2$ en $2 = 2 + 0$. Aangezien we niet van de een naar de ander kunnen gaan in stappen van vier, is een gegeneraliseerde complexe structuur op een twee-dimensionale variëteit niets anders dan ofwel een symplectische structuur of een complexe structuur. Beschouw nu een variëteit van dimensie vier. Aangezien we $4$ kunnen schrijven als $4 = 4 + 0 = 0 + 4 = 2 + 2$ zijn er nu drie mogelijke combinaties; ofwel alle $4$ de richtingen zijn symplectisch, of alle $4$ zijn complex, of $2$ zijn symplectisch en $2$ zijn complex. Dit keer kan de verdeling wel veranderen van punt tot punt, want we kunnen bijvoorbeeld van een punt met $4$ symplectische richtingen $4$ richtingen omzetten naar $4$ complexe richtingen. In dit geval is de variëteit dus op te splitsen in twee gebieden, de één uitgerust met een symplectische- en de andere met een complexe structuur. In hogere dimensies wordt het aantal mogelijke verdelingen steeds groter. Tenslotte merken we op dat, net als bij symplectische en complexe structuren, er een technische conditie is waaraan een gegeneraliseerde complexe structuur aan moet voldoen; we zeggen dat een gegeneraliseerde complexe structuur *integreerbaar* moet zijn. We zullen hier niet verder op ingaan.

Men is vooral geïnteresseerd in gegeneraliseerde complexe meetkunde vanwege de rol die het speelt in de snaartheorie. Een van de fundamentele aspecten van deze natuurkundige theorie is dat de ruimtetijd die we om ons heen zien een hogere dimensie heeft dan $4$. De extra dimensies die we niet kunnen zien zijn, in zekere zin, opgerold en worden alleen zichtbaar op extreem kleine schaal. Voor een bepaald onderdeel van de snaartheorie, genaamd supersymmetrie, zijn extra meetkundige structuren nodig op die extra dimensies, en dit is waar gegeneraliseerde complexe structuren tevoorschijn komen.

## Opblazen

Er rest ons nog één laatste term in de titel van deze scriptie die we moeten toelichten, namelijk het concept van een variëteit *opblazen*. Dit is een operatie die bepaalde wijzigingen aanbrengt op een gegeven variëteit en aldus een nieuwe variëteit produceert. We zullen dit proces uitleggen aan de hand van enkele illustraties. Beschouw een punt in het vlak, en verwijder een kleine schijf die zich om dat punt heen bevindt als in Figuur 6(a). Het resultaat, dat wil zeggen het vlak minus de inhoud van de schijf, is een variëteit met *rand*. Zoals al eerder gezegd betekent dit dat iemand die zich in de ruimte buiten de schijf bevindt en besluit om in de richting van de schijf te lopen, zoals van punt $A$ naar punt $B$ in Figuur 6(a), op een gegeven moment niet meer verder kan. Diegene loopt bij het punt $B$ als het ware tegen een muur aan. De rand is in dit geval gelijk aan een cirkel en we kunnen nu een nieuwe variëteit zonder rand maken door tegengestelde punten op die cirkel met elkaar te identificeren. Ze beschrijven dan per definitie hetzelfde punt in de nieuwe ruimte. Concreet gezien betekent dit het volgende. Beschouw weer dezelfde persoon die van buitenaf naar de schijf toe loopt, zie Figuur 6(b). Ook nu komt diegene aan bij punt $B$, wat eerst deel van de rand was, maar nu in de nieuwe ruimte kan diegene wel doorlopen, hij of zij komt er aan de andere kant van de schijf weer uit en eindigt bij punt $C$. In het bijzonder is er nu geen rand meer, want er is geen enkele obstructie om door te lopen. We benadrukken hier dat het gebied binnen in de schijf geen deel uitmaakt van de nieuwe ruimte. In zekere zin wordt de persoon die van $A$ naar $C$ loopt op het tussenstuk "geteleporteerd" van het ene punt op de cirkel naar het andere punt.




**Figuur 6** (a) (b)

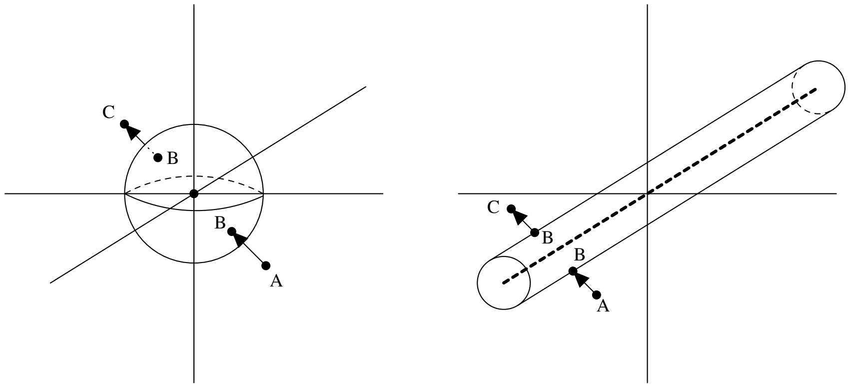
Deze nieuw verkregen ruimte noemen we het resultaat van het *opblazen* van het vlak in het punt. Het is een nieuwe ruimte die, behalve rond het opgeblazen punt, identiek is aan de oude ruimte. We kunnen hetzelfde ook doen in drie dimensies. Beschouw een punt in drie dimensies en verwijder de inhoud van een bol die zich om het punt heen bevindt, zie Figuur 7(a). Het resultaat is een ruimte met rand, waarbij dit keer de rand gelijk is aan de twee-dimensionale bol. Door tegengestelde punten met elkaar te identificeren krijgen we weer een ruimte zonder rand. In Figuur 7(a) is iemand weergeven die van punt $A$ naar punt $C$ loopt.

Behalve punten kunnen we ook andere deelvariëteiten[15] opblazen. Beschouw bijvoorbeeld een lijn in drie dimensies, zoals in Figuur 7(b) is aangegeven met de stippellijn. We verwijderen nu de inhoud van een cilinder die zich om de lijn heen bevindt en verkrijgen zo een ruimte met rand gegeven door de cilinder. Als we weer tegengestelde punten op de cilinder met elkaar identificeren krijgen we een nieuwe ruimte zonder rand. Deze is dus verkregen uit de oude ruimte door het opblazen van een lijn in de drie-dimensionale ruimte.

Bovenstaande beschrijft hoe men een variëteit kan opblazen. Strict genomen is dit echter niet datgene wat in deze scriptie onderzocht wordt. Het proces hierboven wordt ook wel opblazen over de reële getallen genoemd en in deze scriptie beschouwen we in plaats daarvan het opblazen over de complexe getallen. Ondanks dat beide operaties

[15]Een deelvariëteit is een deelverzameling van een variëteit die zelf ook weer een variëteit is.

daadwerkelijk verschillend zijn, geeft het bovenstaande vrij nauwkeurig het idee weer.




**Figuur 7** (a) (b)

**Resultaten in dit proefschrift**

We hebben hierboven gezien wat een gegeneraliseerde complexe structuur is en wat het inhoudt om een variëteit op te blazen. De hoofdvraag van dit proefschrift luidt als volgt:

*Stel we hebben een variëteit $M$ uitgerust met een gegeneraliseerde complexe structuur, en stel dat $N$ een deelvariëteit is van $M$. Kunnen we dan de variëteit die verkregen wordt door het opblazen van $N$ in $M$ opnieuw uitrusten met een gegeneraliseerde complexe structuur?*

In het algemeen is de vraag of een gegeven variëteit een gegeneraliseerde complexe structuur toelaat lastig om te beantwoorden. Er zijn een aantal eigenschappen bekend waaraan moet zijn voldaan, zoals bijvoorbeeld dat de dimensie van de variëteit even moet zijn, maar indien een variëteit aan deze bekende lijst van eigenschappen voldoet is het onbekend[16] of er daadwerkelijk een gegeneraliseerde complexe structuur op bestaat. Om een beter beeld te krijgen van welke variëteiten we kunnen uitrusten met een gegeneraliseerde complexe structuur is het dus belangrijk om veel expliciete voorbeelden te construeren en het proces van opblazen zou, indien het antwoord op bovenstaande vraag ja is, hier een belangrijke bijdrage aan kunnen leveren.

In Hoofdstuk 3 van dit proefschrift geven we een antwoord op de bovenstaande vraag in twee speciale gevallen. Zoals eerder beschreven bestaat een gegeneraliseerde complexe structuur, in elk punt op de variëteit $M$, uit een mix van symplectische en complexe

[16] Strikt genomen geldt dit alleen voor dimensies groter dan twee, want in twee dimensies is dit wel volledig bekend.

richtingen. We bekijken dan alleen deelvariëteiten $N$ in $M$ waarvoor geldt dat de normale richtingen ofwel geheel symplectisch ofwel geheel complex zijn. Hier verstaan we onder een normale richting een richting die niet in de richting van de deelvariëteit wijst. Bijvoorbeeld, als $M$ gegeven is door het vlak en $N$ is gegeven door de $x$-as, dan is de $y$-as een normale richting. In Stelling 3.1.7 bewijzen we dat als alle normale richtingen complex zijn en als aan een bepaalde technische conditie voldaan is, dat het antwoord op de bovenstaande vraag ja is. Dus, als we dit soort deelvariëteiten opblazen dan kan het resultaat weer uitgerust worden met een gegeneraliseerde complexe structuur. Bovendien is de structuur die we op de opgeblazen ruimte aanbrengen in zekere zin uniek.

Vervolgens bewijzen we in Stelling 3.2.13 dat dit ook het geval is voor deelvariëteiten waarvan alle normale richtingen symplectisch zijn. In dit geval is er geen technische conditie waaraan voldaan moet zijn, maar de structuur zelf die we op de opgeblazen ruimte aanbrengen is niet uniek, er zijn wat keuzes mee gemoeid. Nog een verschil met het complexe geval is dat in dit geval de deelvariëteit $N$ ook compact[17] moet zijn.

In Hoofdstuk 4 van dit proefschrift behandelen we dezelfde vraag als hierboven voor zogeheten *gegeneraliseerde Kähler structuren*. Een gegeneraliseerde Kähler structuur op een variëteit bestaat uit twee gegeneraliseerde complexe structuren die compatibel met elkaar zijn. Bijvoorbeeld, op een twee-dimensionaal oppervlak zoals in Figuren 3 en 4 kunnen we eerst een metriek construeren (dus een notie van hoeken, afstanden en volumes), en met behulp daarvan kunnen we zowel volumes meten van twee-dimensionale regio's, alsmede aangeven wat $90°$ draaien met de klok mee betekent. Dus, we hebben zowel een symplectische structuur als ook een complexe structuur, en ze zijn met elkaar compatibel omdat ze allebei bepaald worden door dezelfde metriek.

Stel nu dat we een gegeneraliseerde Kähler structuur hebben op de variëteit $M$, met $\mathcal{J}_1$ en $\mathcal{J}_2$ de twee bijbehorende gegeneraliseerde complexe structuren. Laat $N$ een deelvariëteit zijn waarvan de normale richtingen allemaal complex zijn voor $\mathcal{J}_1$, en symplectisch voor $\mathcal{J}_2$. Wegens bovenstaande behaalde resultaten weten we dan dat $M$ opgeblazen kan worden in $N$ en dat we het resultaat kunnen uitrusten met nieuwe gegeneraliseerde complexe structuren (er is nog wel die technische conditie voor $\mathcal{J}_1$ waar ook aan voldaan moet zijn), die we weer met $\mathcal{J}_1$ en $\mathcal{J}_2$ noteren. We kunnen ons dan afvragen of die twee nieuwe structuren met elkaar compatibel zijn en zo een nieuwe gegeneraliseerde Kähler structuur vormen. Dit is niet automatisch en het blijkt lastiger dan op het eerste gezicht lijkt om dit te bewerkstelligen. In Stelling 4.3.2 geven we enkele meetkundige voorwaarden waar de deelvariëteit $N$ aan moet voldoen, die garanderen dat de opgeblazen ruimte inderdaad weer een gegeneraliseerde Kähler structuur heeft. Met behulp daarvan construeren we in Sectie 4.4 enkele nieuwe voorbeelden van variëteiten met een gegeneraliseerde Kähler structuur.

[17] Compactheid is, in zekere zin, een synoniem voor begrensdheid. Zo is bijvoorbeeld de cirkel (Figuur 1(c)) compact, maar de (oneindige) lijn (Figuur 1(b)) niet.

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# Curriculum Vitae

Joey Leen van der Leer Durán was born on June 26, 1989 in Terrassa, in the province of Barcelona in Spain. He is son of Leo van der Leer and Dolores Durán Jiménez and has one younger sister, Meritxell van der Leer Durán. He lived in Spain until 1993, when he moved to the Netherlands.

In the year 2007 he graduated from the high school DevelsteinCollege in Zwijndrecht. Subsequently, he studied applied physics at the Technical University of Delft for one year, and then switched to Utrecht University to study mathematics and physics, for which in 2010 he received his bachelor degree in both subjects with cum laude distinction.

In the years 2010-2012 he attended the two master programs Mathematical Sciences and Theoretical Physics, also at Utrecht University. In 2012 he obtained both master degrees with cum laude distinction. His master thesis was entitled "Supersymmetric sigma models and generalized complex geometry", and was supervised by Prof. S. Vandoren and Dr. G. R. Cavalcanti.

In 2012 he started his PhD project under the supervision of Dr. G. R. Cavalcanti and defended his thesis, entitled "Blow-ups in generalized complex geometry", on November 30, 2016.

In 2016 Joey got married to Simone Tempelaar.

Starting from January 2017, he will be a Postdoctoral Fellow at the University of Toronto, in the research group of Prof. M. Gualtieri.