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Transitions toward a Semantics for Real Possibility

Transities naar een Semantiek voor Werkelijke Mogelijkheid

(met een samenvatting in het Nederlands)

Proefschrift

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door

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Voor Greet

Langs de takken van de bomen
Blös de wind zien sloffe lied
Over alles wat veurbij gung
Over wat der kommen giet
Ik stao der naor te lustern
Zunder weemoed, zunder spiet
Want wat west is dat is west
En wat komp dat weet ie nie

"Alles kan ja"
Daniël Lohues

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Antje Rumberg
Fall 2016

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Contents

Introduction: Welcome to a World of Possibilities

Welcome to a world of possibilities! At the heart of this thesis, there is the idea that the world is full of possibilities, real possibilities for what the future may bring. At the moment, I am sitting here in Constance at my desk typing these sentences, and it is possible that I will continue writing for another few hours. However, there are alternate possibilities. I can take a break from writing and go for a swim in the Lake of Constance, and I can just as well go to the ice cream parlor around the corner and have a hot fudge sundae. The future is open. There are alternative possibilities for the future to evolve. The picture that underlies our research is the one that is famously invoked by Borges's "The garden of forking paths". When facing the future, we are facing a maze of forking paths, and actuality makes its way through that maze. At each juncture, actuality traces one of the alternative possible paths toward the future, and all paths are equally viable. At each point in time, one of the immediate alternative future possibilities is actualized, and the remainder fades away. The alternative paths leading toward the future are thereby not to be understood as mere epistemic possibilities that represent our epistemic uncertainty with respect to what the future will bring. Rather, at the core of the picture we will be working with, there is the idea of objective indeterminism. Each of the alternative paths leading toward the future represents a genuine possibility for the future in virtue of what the world is like. Real possibilities are alternative possibilities for the future in an indeterministic world.

This thesis centers around the question "What is really possible?". Our research is not of an empirical nature, however. We do not aim to establish the truth of indeterminism. We leave that to the physicists. Certainly, we make ample use of concrete examples throughout this thesis. We consider, for instance, the jumping behavior of frogs and the rolling behavior of balls. But we have not conducted experiments in a laboratory in order to find out how frogs and balls behave in a certain experimental setup. You can be assured that no frogs have been harmed during the research of this thesis. This thesis is a philosophical one, and our method is thinking rather than empirical investigation. Thinking is what philosophers do. In particular, our method consists in conceptual modeling, and our investigations are altogether of a formal nature. We use the tools of logic and formal metaphysics. Our endeavor is not to establish by empirical means what is really possible. And the real possibilities that we are after do not have to be possibilities in our world, the world we actually live in. Rather, our endeavor is to develop a rigorous formal conceptual framework that allows us to reason about what is really possible under the assumption of indeterminism.

As the title suggests, the overall aim of this thesis is to develop a semantics for the notion of real possibility. In other words, we aim at devising a modal-temporal language that is suited for reasoning about what is really possible and providing an appropriate interpretation of that language. In our investigations, we depart from the theory of branching time, which provides an adequate formal basis for modeling the notion of real possibility. Branching time structures allow for a perspicuous representation of alternative future possibilities all of which share some common past. The task of providing a semantics for the notion of real possibility is then twofold: there is a logical and a metaphysical challenge. From a logical point of view, the challenge consists in setting up a formal branching time semantics that makes use of parameters of truth that are suited for modeling the temporal dynamics of real possibilities. The metaphysical challenge, on the other hand, consists in providing a systematic explanation of branching time models that properly accommodates the worldly aspect of real possibilities.

In chapter 1, we set the stage: we introduce the notion of real possibility and the theory of branching time, and we point out the adequacy of the branching time framework for the formal representation of real possibilities. Real possibility is one among many different kinds of possibilities, and it stands out in several respects. We discuss the peculiarities of real possibilities and locate the notion of real possibility in the vast landscape of possibilities. We then provide the definition of a branching time structure and discuss the two most popular semantic approaches based on the framework of branching time, viz. Peirceanism and Ockhamism. We finally motivate our choice of the branching time framework over the popular possible worlds framework. We highlight crucial differences between the two frameworks and show why the possible worlds framework is ruled out as a suitable alternative to the branching time framework in the formal representation of real possibilities.

In chapter 2, we present a novel propositional semantics based on the framework of branching time, viz. the so-called transition semantics. The basic idea is to replace the moment-history pairs employed as parameters of truth in the Ockhamist semantics by pairs consisting of a moment and a set of transitions. Whereas histories represent complete possible courses of events, sets of transitions can represent incomplete parts thereof as well. Every transition selects one of the immediate future possibilities open at a moment. The transition semantics exploits the structural resources a branching time structure has to offer and provides a fine-grained picture of the interrelation of modality and time. In addition to temporal and modal operators, a so-called stability operator becomes interpretable as a universal quantifier over the possible future extensions of a given transition set. The stability operator enables us to adequately capture the dynamics of real possibilities: their transition from contingency to settledness. We show that the semantics developed along those lines allows for great generality: both Peirceanism and Ockhamism can be viewed as limiting cases of the transition approach that build on restricted resources only. The relevant restrictions reveal their limitations: on both accounts, stability collapses into truth.

In chapter 3, we make a transition toward completeness. The transition semantics is not a genuine Kripke-style semantics. Sentences are not only evaluated at moments, which form the basic constituents of a branching time structure. Rather, the semantics makes use of a second, defined parameter of truth next to the moment parameter, and the language is equipped with intensional operators that are interpreted as quantifiers over that second parameter. What is more, the sets of transitions that are employed as a second parameter of truth are set-theoretically rather complex. We first show that sets of transitions correspond one-to-one to certain substructures of branching time structures, which are first-order definable. On the basis of that result, we then provide a general characterization of a class of genuine Kripke structures that is shown to preserve validity. Those Kripke structures have the following properties: the relevant structures comprise a primitive accessibility relation for each of the three different kinds of intensional operators of the transition language, the set of indices of evaluations equals the set of points of the structure, and quantification over the second parameter of truth dissolves into quantification over that set. The result we establish in this chapter constitutes a first and significant step toward a future completeness proof for the transition framework, which can then proceed by a chronicle construction on the defined structures.

In chapter 4, we turn to the worldly aspect of real possibilities. For a branching time model to constitute a model for real possibility, each possible course of events must conform to the prevailing laws of nature. In particular, the model has to fit the underlying branching time structure: if, for example, the situation obtaining at a moment is such that the laws of nature allow for three possible future continuations, the local branching structure must be a tripartition. Singling out those branching time models that are in fact models for real possibility requires a transition from logical to metaphysical considerations. We present a dynamic, modal explanation of branching time models for real possibility in terms of potentialities. The core idea is this: by manifesting their potentialities, objects become causally efficacious and jointly give direction to the possible future courses of events. A rigorous formal characterization of potentialities is provided, and it is shown how a branching time model for real possibility together with its underlying structure can be lifted in a dynamic fashion from a single momentary circumstance on the basis of the potentialities of objects. By grounding real possibilities in potentialities, we provide a systematic account of branching time models for real possibility that elucidates why all possible courses of events really are in accordance with the prevailing laws of nature.

* * *

Chapter 1

Real Possibilities and Branching Time

1.1 Introduction

This thesis centers around the notion of *real possibility*. Real possibility is one among many different kinds of possibility, and it stands out in several respects. In a nutshell, real possibilities are alternative possibilities for the future in an indeterministic world. They are indexically anchored in concrete situations, and they bear an intimate relation to time and to the world.

As the title suggests, the overall aim of this thesis consists in developing a semantics for the notion of real possibility. To this end, we are in need of a formal framework that allows for an adequate representation of real possibilities. We can then devise semantic models for a language that is suited for reasoning about what is really possible. The framework that will constitute the basis of our investigations throughout this thesis is the so-called framework of *branching time*. In that framework, the modal-temporal structure of the world is represented as a tree of moments that branches toward the future.

In this chapter, we introduce the notion of real possibility and the theory of branching time and point out the adequacy of the branching time framework for the formal representation of real possibilities. In section 1.2, we characterize the notion of real possibility by relating it to other kinds of possibilities. In section 1.3, we present the framework of branching time and discuss the two most popular semantic approaches based on that framework, which, after Prior, are referred to as Peirceanism and Ockhamism. In section 1.4, we motivate our choice of the branching time framework over the popular possible worlds framework.

1.2 Real possibilities

Today is Sunday, August 16, 2015. It is half past two in the afternoon. I am in Constance, sitting at my desk typing these sentences. Outside the sun is shining. It is a warm summer day. At this very moment, in the stadium Galgenwaard in Utrecht, there is the kick-off of the match between FC Utrecht and SC Heerenveen. It is the first home match that FC Utrecht is playing in this season.

There is a sense in which it is possible that I am in Utrecht now, standing in the stadium Galgenwaard cheering for FC Utrecht. In the very same sense, it is also possible that I am now enjoying a swim in the North Sea at the Hoek van Holland, a small port in the neighborhood of Rotterdam. Both scenarios are possible in a certain respect, while, actually, I am in Constance now. Neither of them constitutes a real possibility, however. What is really possible, on the other hand, is that I now take a break from writing and follow the soccer match on the radio. And it is likewise really possible that I go for a swim in the Lake of Constance in the late afternoon, and I may just as well stay at home.

The notion of possibility comes in a great variety. Things can be possible in several different ways. What is common to all kinds of possibility is that they are closely tied up with the notions of necessity and contingency. What cannot possibly be different is necessary, and that which is neither necessary nor impossible is contingent. At the same time, all kinds of possibility are tightly interwoven with the notions of actuality and reality, and that is where the differences arise. The variety of different kinds of possibility is essentially due to the fact that possibilities may differ with respect to how they relate to those latter notions.

In this section, we map the landscape of possibilities and locate the notion of real possibility within that landscape. In section 1.2.1, we review several notions of possibility standardly discussed in the literature.[1] In section 1.2.2,

[1] An overview of different kinds of possibility is provided, for example, in Fine (2002), Kment (2012) and Müller (2012). The latter also contains a discussion of the notion of real possibility.

we introduce the notion of real possibility and carve out the peculiarities of that notion by carefully distinguishing it from other kinds of possibilities.

1.2.1 Varieties of possibility

In this section, we briefly discuss the most prominent kinds of possibility to be found in the literature. There is, first of all, the notion of epistemic possibility. Next to epistemic possibility, it is common to distinguish, among others, between logical, metaphysical and physical possibility. The latter group is usually referred to as alethic possibilities. To be sure, the list of epistemic, logical, metaphysical and physical possibility does by no means exhaust the vast landscape of possibilities.[2] However, it is those kinds of possibility that surface in the literature. And it is those kinds that will become important throughout this thesis against the background of our question as to what is really possible. We will therefore focus on them in what follows. In section 1.2.1.1, we introduce the notion of epistemic possibility, and we turn to the threefold variety of alethic possibility in section 1.2.1.2.

A brief disclaimer is in order here. Our discussion of the standard notions of possibility is an opinionated one. We take a step back from the existing literature and characterize different kinds of possibilities in a rather abstract way, viz. in terms of the relations they bear to the notions of actuality and reality. We do not discuss in detail any specific account of possibility as proposed by any particular author. Also, we refrain from referring to any technicalities or formal apparatus. Engaging in a discussion of the subtleties of specific accounts and technical intricacies would lead us astray. What we aim at in this section is carving out the main ideas behind different kinds of possibility in an intuitive and non-technical way.

1.2.1.1 Epistemic possibility

Epistemic possibility is a subjective kind of possibility that pertains to an agent's epistemic state. Epistemic possibilities mirror an agent's state of knowledge. What is epistemically possible is what might actually be the case for all the agent knows. On the basis of his knowledge, the agent cannot tell apart those possibilities from actuality. Given what he knows, either of them could

[2] There is another major kind of possibility that we do not discuss here, viz. deontic possibility. Deontic possibility pertains to the notions of obligation and permission. What is deontically possible is what one is allowed to do against the background of certain rules or norms. Formally, deontic possibility has first been studied by von Wright (1951, 1968, 1971) and Hintikka (1957).

equally be actual. Epistemic possibilities represent modal alternatives to actuality in virtue of an agent's subjective reality, i.e. in virtue of what the agent knows, or rather, does not know.

At the moment, I am sitting here in Constance at my desk, typing in my laptop. And I have actually spent the entire weekend doing so. But suppose that you are ignorant of my weekend's activities. In particular, suppose that you do not know that I am in Constance now working on my thesis. In that case, for all you know, it is epistemically possible that I went to Utrecht this weekend and am now standing in the stadium Galgenwaard watching the match between FC Utrecht and SC Heerenveen. And your imperfect state of knowledge also renders it epistemically possible that at this very moment I am enjoying a swim in the North Sea at the Hoek van Holland. Due to your lack of knowledge, both scenarios constitute epistemic possibilities, as does the possibility of me sitting in Constance at my desk working on my thesis, as I actually do.

Actuality itself is always part of an agent's epistemic possibilities, but an agent's epistemic possibilities may comprise much more than what actually is the case. They may comprise numerous alternatives to actuality. The multiplicity of epistemic possibilities reflects the agent's epistemic uncertainty with respect to what actually is the case. To be sure, objectively, one and only one of the agent's epistemic possibilities accords with actuality, while the others constitute mere possibilities. The agent himself, however, cannot discriminate between his epistemic possibilities. From his epistemic perspective, all those possibilities are on a par. His knowledge does not allow him to rule out either of them. Once you learn that I am presently in Constance, me being in the stadium Galgenwaard in Utrecht now does no longer constitute an epistemic possibility for you. Nor does the possibility of me having a swim in the North Sea at the Hoek van Holland at this very moment. In the light of new information, the range of epistemic possibilities decreases. In the extreme case of perfect knowledge, there is but a single possibility left, viz. actuality itself.

The notion of epistemic possibility is a rather weak notion. For it to be epistemically possible that I am in the stadium Galgenwaard in Utrecht now, it is not required that you consider that possibility very likely, let alone that you believe that possibility to be the case. Everything that is compatible with an agent's state of knowledge is epistemically possible, no matter how implausible the agent takes those possibilities to be. The notion of belief brings in another kind of possibility that is sometimes subsumed under the heading 'epistemic possibility', namely, doxastic possibility. Just as epistemic possibilities, doxastic possibilities represent alternatives to actuality in virtue of an agent's subjective reality. Yet, in the case of doxastic possibility, an agent's subjective reality is identified with his state of belief rather than with his state of knowledge. What is doxastically possible is what is compatible with what the agent, rightly or wrongly, believes to actually be the case.

Unlike in the case of epistemic possibility, actuality itself does not have to be among an agent's doxastic possibilities. An agent may entertain false beliefs. Someone who knows that I am a big FC Utrecht fan, may, for example, falsely believe that I am in Utrecht now for the kick-off of the season's first home match. And the sunny weather and my long-standing plans of heading to the Hoek van Holland this summer may mislead someone else to believe that I am floundering about in the North Sea right now. In either case, me being in Constance at this very moment typing these sentences is ruled out as a doxastic possibility.

The notion of doxastic possibility is stronger than the notion of epistemic possibility. While it is commonly assumed that an agent's doxastic possibilities, at the same time, also constitute epistemic possibilities for that agent,[3] an agent's epistemic possibilities may very well outrange his doxastic possibilities. I believe that the world is indeterministic. Yet, I do not know whether it really is indeterministic or whether it is deterministic after all. Both the world being indeterministic and the world being deterministic are epistemically possible, given what I know. Yet, only the former is also possible doxastically. The world being deterministic does not constitute a doxastic possibility of mine. It is incompatible with what I believe to actually be the case.[4]

1.2.1.2 Alethic possibility

Logical, metaphysical and physical possibility are often referred to as alethic possibilities, in order to contrast them with epistemic and doxastic possibility. Whereas epistemic and doxastic possibility pertain to an agent's subjective state of knowledge or belief, alethic possibilities concern possibility in an objective, ontological sense.[5] One may put it this way: epistemic and doxastic

[3] This is simply due to the traditional assumption that knowledge is true belief (plus x).

[4] The interaction of knowledge and belief has famously been studied in Stalnaker (2006). For an overview of epistemic logic, see van Ditmarsch *et al.* (2008).

[5] One may wish to say that, in contrast to epistemic and doxastic possibility, alethic possibilities are linked up with the notion of truth. Here, we consider possibilities in their own right, independently of their linguistic renderings. We disregard discussions as to whether

possibilities represent modal alternatives to actuality in virtue of an agent's subjective reality, viz. his state of knowledge or belief. Alethic possibilities, in contrast, represent modal alternatives to actuality in virtue of some objective reality, some real aspect of the world. They rest on the idea that, for all there is to the fundamental ontology of the world, what actually is the case could have been different in various ways. Actuality is deemed to be but one of many possibilities of how things could have been.

What distinguishes logical, metaphysical and physical possibility is which aspect of the world is at the core of those notions. The notion of reality at play is a different one in each case. The relevant notion of reality bears on the question as to which entities may possibly exist as part of the fundamental ontology that makes up the furniture of the world as well as on the question as to which configurations of those entities constitute modal alternatives to actuality.[6] We will see that the transition from logical to metaphysical to physical possibility comes with a strengthening of the notion of reality. When moving from logical to metaphysical to physical possibility, the link with the world becomes stronger and stronger. The resulting picture is the following: the class of logical possibilities comprises the class of metaphysical possibilities, which in turn includes the class of physical possibilities. In this section, we will discuss the three different kinds of alethic possibility one by one. In section 1.2.1.2.1, we start out with the weakest notion, viz. logical possibility, in section 1.2.1.2.2, we then turn to the notion of metaphysical possibility, and in section 1.2.1.2.3, we review the notion of physical possibility.

1.2.1.2.1 Logical possibility Logical possibility is alethic possibility in the broadest possible sense. It is a rather abstract notion, and it is only very loosely tied up with the world. The link with the world only comes in via the notion of logical form. Logical possibilities represent modal alternatives to actuality in virtue of the logical form of things.

The notion of logical possibility does not bear on our assumptions as to which entities may possibly exists as part of our fundamental ontology—over

linguistic expressions are rigid designators, as defended by Kripke (1980), or whether whether they can possibly refer to other entities than they actually do. Questions such as, for example, whether it is necessary in virtue of the meaning associated with the term 'water' that water is H_2O, which arise in the context of Putnam's famous Twin Earth thought experiment, are not our focus. (Cf. Putnam 1973).

[6] What we call 'possible configurations' here is often referred to as 'possible worlds'. It seems that the term 'possible world' has established itself as a synonym for 'possibility'. We reserve the term 'possible world' for technical use.

and above the actually existing ones. It only presupposes that the various entities that are part of the fundamental ontology have some logical form, which determines their most general form of combination with other entities. To speak less abstractly, the notion of logical possibility presupposes a classification of the totality of fundamental entities into objects, properties and relations, according to their logical form. The logical form of those entities, viz. their being objects, properties or relations, determines which entities can be combined with each other in order to form a state of affairs and how different states of affairs can again be combined to form larger and larger configurations. Every configuration that respects the logical form of the fundamental entities constitutes a logical possibility, and actuality is one of them.[7]

It is certainly logically possible that I am in the stadium Galgenwaard in Utrecht now, watching the kick-off of the season's first home match of FC Utrecht, even though I am actually in Constance at present. And it is also logically possible that I am taking a swim in the North Sea at the Hoek van Holland right now. Neither of those scenarios is in conflict with the logical form of the entities involved. Either of them constitutes a logical possibility, a logically salient modal alternative to actuality.

It is even logically possible that I am simultaneously both in the stadium Galgenwaard in Utrecht and at the Hoek van Holland, which is located at a distance of more than 60 kilometers from Utrecht. The contradiction that arises from me being co-located at different places at the same time is not a logical one, i.e. none that arises in virtue of the logical form. Me being co-located at different places at the same time is logically possible, just as it is logically possible that my office door is at once both open and closed, or that the very same thing is simultaneously both red all over and green all over. The notion of logical possibility rests solely on the logical form of the fundamental entities, abstracting away from their very nature.

What is logically impossible, however, is that the very same object at the very same time both has a given property and lacks it, or similarly, both stands in a certain relation to other objects and does not. Me being in the stadium Galgenwaard and not being in the stadium Galgenwaard in Utrecht now is a logical contradiction, just as it is logically impossible that my office door is

[7] The account of logical possibility that we outline here is, of course, strongly inspired by the account of possibility that Wittenstein develops in his *Tractatus* (cf. Wittgenstein 1922). For an inspiring discussion of Wittgenstein's notion of logical form and the logically possible configurations it gives rise to, see Visser (2012).

at once both open and not open, or that the very same thing is simultaneously both red all over and not red all over. The logical form rules out such combinations.

1.2.1.2.2 Metaphysical possibility The notion of metaphysical possibility is much more strongly tied up with the world than the notion of logical possibility is. Whereas in the case of logical possibility, all that matters is the logical form of things, in the case of metaphysical possibility, in addition to the logical form, the very nature of the entities that make up the furniture of our world is taken into account as well. Metaphysical possibilities represent modal alternatives to actuality in virtue of the nature of things.

What is metaphysically possible must also be logically possible. However, the converse does not hold. The class of metaphysical possibilities forms a proper subclass of the class of logical possibilities. We said that it is logically possible that I am both in the stadium Galgenwaard in Utrecht and at the Hoek van Holland at the very same time. Yet, this is arguably not a metaphysical possibility. For, it challenges my very identity. And while it is logically possible that my office door is at once both open and closed or that the very same thing is simultaneously both red all over and green all over, configurations like those are likewise excluded on metaphysical grounds. They are in conflict with the very nature of the properties of being red all over and being green all over or, respectively, the properties of being open and closed. Still, me being in the stadium Galgenwaard now attending the soccer match between FC Utrecht and SC Heerenveen is metaphysically possible. And it is also metaphysically possible that I am now enjoying a swim in the North Sea at the Hoek van Holland. Neither of those scenarios is ruled out by the nature of the entities involved. Both of them constitute metaphysically possible alternatives to actuality.

The notion of metaphysical possibility is certainly one of the most employed modal concepts in contemporary analytic philosophy. And it is probably the most vague one. The point is simply this: there is no absolute notion of metaphysical possibility. What we consider metaphysically possible is inextricably interwoven with our metaphysical assumptions as to what the world is like. Are there genuine modal properties such as dispositions? Can an electron be positively charged? Our metaphysical assumptions bear on the question which entities may possibly exist as part of the fundamental ontology and, at the same time, constrain the range of configurations that are possible in virtue of the nature of those entities. In order to not get caught up in a circle, it seems, there are basically two viable options: first, one can start out with a primitive stock of fundamental entities that have a modally robust nature and derive the metaphysically possible configurations from there. Second, one can stipulate a totality of metaphysically possible configurations of modally-flat entities as primitive and trace facts about the existence and nature of things back to those possible configurations. Let us have a look at the two possible options one by one.

Suppose that our starting point is a primitive stock of objects, properties and relations. Assume, moreover, that each of those entities has a primitive modal nature. We can then simply construct the metaphysically possible configurations from the entities of our fundamental ontology by taking into account their very nature, just as we have constructed the logically possible configurations from some given stock of fundamental entities on the basis of their logical form. By their very nature, objects have some properties necessarily, i.e. in every possible configuration, others contingently, i.e. in some possible configuration but not in all. Likewise, there are necessary relations among objects and contingent ones. Moreover, some properties and relations necessarily co-occur, simply in virtue of being the properties and relations they are; others necessarily exclude each other. On that option, the range of metaphysically possible configurations is reduced to the primitive modal nature of the entities of our fundamental ontology. And among the various metaphysically possible configurations, there is actuality itself.[8]

Let us now consider the second option. That option amounts to taking the totality of possible configurations that we deem to be metaphysically possible as primitive. In doing so, we advance a principle of plenitude: we presuppose that our totality of metaphysically possible configurations covers everything that is metaphysically possible, including actuality itself. There are no metaphysical possibilities beyond that totality. Provided with the entire range of metaphysically possible configurations, we can then draw conclusions as to which entities

[8] An account of metaphysical possibility in terms of essence is advanced in Fine (1994). However, it seems that every account that builds on the idea that the range of metaphysically possible configurations can be derived from some given fundamental ontology has to presuppose a primitive notion of modality that in one way or other relates back to the idea that the fundamental entities come along with a modally robust nature. For otherwise, the notion of metaphysical possibility collapses into the notion of logical possibility. Primitive modal assumptions may either be incorporated directly into the recombination principle, enter via a notion of consistency or come under the disguise of a notion of conceivability.

may possibly exist, namely all those that are part of at least one metaphysically possible configuration. The totality of metaphysically possible configurations also sheds light on the nature of those fundamental entities. Objects have some properties in all possible configurations, others only in some. Some properties and relations always co-occur, others never. If an object has the very same property in every possible configuration, we may conclude that it is in the nature of that object to have that property. If certain properties or relations co-occur in every possible configuration, we may say that such co-occurrence is due to the very nature of properties and relations.[9]

On both options, metaphysical possibility in the end comes down to compatibility with the nature of things. However, the order of explanation is a different one in each case. According to the first option, the metaphysically possible configurations are accounted for in terms of the primitive modal nature of things. According to the second option, on the other hand, the nature of otherwise modally-flat entities is reduced to the primitive totality of metaphysically possible configurations. Both accounts rest on a primitive notion of modality that is either built into the fundamental entities themselves or enters with the range of metaphysically possible configurations. The kind of primitive modality is thereby the same in each case, even though it comes under a different disguise: it is metaphysical modality itself.

1.2.1.2.3 Physical possibility The notion of physical possibility is even more strongly tied up with the world than the notion of metaphysical possibility is. When it comes to the notion of physical possibility, in addition to the logical form and the nature of things, the laws of nature enter the picture. Physical possibilities represent modal alternatives to actuality with respect to the prevailing laws of nature.[10]

It is commonly assumed that the class of physical possibilities forms a subclass of the class of metaphysical possibilities: the laws of nature must be compatible with the nature of things. Whether it is a proper subclass or whether the notion of physical possibility collapses into the notion of metaphysical pos-

[9] The account of metaphysical possibility that we are alluding to here is, of course, the one advocated in D. Lewis (1986a); although we are here ignoring the question whether there is 'transworld identity' or, as Lewis thinks, a relation of counterparthood.

[10] Not all laws of nature need to be physical laws, of course. The term physical possibility might therefore be misleading. What we call physical possibility here is sometimes also referred to as nomological or natural possibility.

sibility depends on one's conception of metaphysical possibility as well as on one's conception of the laws of nature.

However, no matter the precise conception of what the laws of nature are, everyone will admit, that it is physically possible that I am in the stadium Galgenwaard in Utrecht now cheering for FC Utrecht, even though I am actually sitting in Constance at my desk. And it is also physically possible that I am taking a swim in the North Sea at the Hoek van Holland at this very moment. Neither of those scenarios is incompatible with the prevailing laws of nature. Our laws of nature do not rule out a course of events in which I have taken a train to the Netherlands yesterday and went to the stadium Galgenwaard this afternoon or else made a trip to the Hoek van Holland. Both scenarios constitute physically possible alternatives to the actual course of events.

What is physically possible depends on what the laws of nature are. And since there is no absolute notion of the laws of nature, there is no absolute notion of physical possibility either. Let us briefly consider what the options are. First of all, there are two options that parallel the two approaches that we have discussed in the context of metaphysical possibility in the previous section. We will discuss them in turn and then briefly turn to a third option.

First, suppose that we have started out with a fundamental stock of objects, properties and relations that have a primitive modal nature. We may then derive the laws of nature from the very nature of the entities that are part of our fundamental ontology. The laws of nature may be conceived of as generalizations that describe the nature of our fundamental entities. Suppose, for example, that among our fundamental properties, there is the property of being charged. Assume moreover that it is in the very nature of that property that like charges repel each other. We may then conclude that it is a law of nature that all objects that are like charged repel each other, rendering a configuration in which two electrons, both of which are negatively charged, attract each other physically impossible. If we ground the laws of nature in the nature of the entities of our fundamental ontology, the laws are metaphysically necessary. They hold in every metaphysically possible configuration. For, each such configuration is compatible with the nature of things. Physical possibility then coincides with metaphysical possibility.[11]

[11] An account of laws of nature, according to which the laws are grounded in the nature of dispositional properties, is, for example, defended in Bird (2007).

Let us consider the second option. Assume that we have stipulated a primitive totality of metaphysically possible configurations of entities that are themselves modally-flat. We may then read off the laws of nature from the actual configuration, just as we have read off the nature of the fundamental entities from what is the case across the metaphysically possible configurations. The laws of nature may be considered systematizations of the regularities found in actuality. Suppose that from the regularities found in actuality, we conclude that it is a law of nature that like charges repel each other. That like charges attract each other is accordingly physically impossible. This does not exclude, however, that in some metaphysically possible configuration, like charges attract each other after all, in which case, in that metaphysically possible configuration, different laws of nature hold. On such a view, the laws of nature are no longer metaphysically necessary, and metaphysical possibility and physical possibility come apart. Physical possibility becomes a relative notion: what is physically possible depends on what the laws of nature are and the latter in turn depends on what is the case in the configuration at hand.[12]

There is a third option. Rather than deriving the laws of nature from the nature of things or from what is the case in a metaphysically possible configuration, one could simply stipulate primitive laws of nature. If we add primitive laws as an extra ingredient into our fundamental ontology, every configuration that contradicts those laws of nature is rendered physically impossible. It may be claimed that on such an account, the laws of nature gain explanatory power: they govern what is actually the case and explain what is physically possible. On the first option, in contrast, it is the nature of things that does the explanatory work and the laws of nature basically become redundant. On the second option, actuality explains the laws, and not the other way around. For example, on the first account, the fact that uranium spheres of more than a mile in diameter are physically impossible can be explained by the very nature of uranium. On the second account, it is explained by reference to the regularities found in actuality. Yet, those regularities cannot account for the contrast with the claim that gold spheres of more than a mile in diameter are physically possible.[13] On the third option, the laws of nature take over the explanatory role. However, what we are looking for in vain is an explanation of what the laws of nature have to do with, e.g., uranium and gold. We are missing any

[12] Accounts along those lines are discussed under the heading 'Neo-Humeanism' and go back to D. Lewis (1986a). We will come back to those accounts in chapter 4 below.

[13] The example is borrowed from van Fraassen (1989, p.27).

explanation of how the laws of nature relate to the nature of the fundamental entities that make up the furniture of our world.[14]

1.2.2 What is really possible?

In the previous section, we have briefly reviewed the various different kinds of possibility that are most prominent in the literature. In section 1.2.1.1, we have introduced the notion of epistemic possibility, and in section 1.2.1.2, we have discussed the prevalent group of alethic possibilities: logical, metaphysical and physical possibility. In this section, we will eventually get to the bottom of the question 'What is really possible?'.

We have seen that it is epistemically possible that I am in the stadium Galgenwaard in Utrecht now watching the soccer match between FC Utrecht and SC Heerenveen, while, actually, I am sitting here in Constance at my desk typing these sentences. In fact, me standing in the stadium Galgenwaard right now, cheering for FC Utrecht is even a logical, a metaphysical and also a physical possibility. In the same vein, it not only epistemically but also logically, metaphysically as well as physically possible that I am now enjoying a swim in the North Sea at the Hoek van Holland. However, as said, neither me being in the stadium Galgenwaard in Utrecht right now attending the season's first home match of FC Utrecht, nor me drifting with the waves in the North Sea at the Hoek van Holland at this very moment is really possible, given that I am actually dwelling in Constance at present. But what is really possible then?

In this section, we will carve out the peculiarities of the notion of real possibility and point out how it distinguishes itself from those other kinds of possibility that we have introduced so far. In our discussion of the standard notions of possibility in section 1.2.1, we have primarily focused on two aspects, namely the relation that those kinds of possibility bear to the notion of actuality, on the one hand, and the notion of reality, on the other. We have illustrated that both epistemic possibilities as well as the various different kinds of alethic possibility represent modal alternatives to actuality in virtue of some reality. Differences between those different kinds of possibility have been shown to be essentially due to the fact that the relevant notion of reality is a different one in each case. Epistemic possibilities constitute modal alternatives to actuality in virtue of an agent's subjective reality, viz. his state of knowledge. Alethic pos-

[14] An account of primitive laws of nature is proposed, for example, in Maudlin (2007).

sibilities, on the other hand, represent modal alternatives to actuality in virtue of some objective reality, some real aspect of the world—the logical form, the nature of things or the laws of nature, respectively. The transition from logical to metaphysical to physical possibility involves thereby a strengthening of the link with the world.

We will see that the notion of real possibility requires a fundamental shift in perspective. The relation between possibility and actuality changes radically, and the relation between possibility and the world is further sharpened. Real possibilities are alternative possibilities for the future in an indeterministic world. They are indexically anchored in concrete situations, and they bear an intimate relation to time and to the world. The notion of actuality gains dynamic traits and the relation between possibility and actuality is a temporal rather than a modal one: unlike epistemic and alethic possibilities, real possibilities do not constitute modal alternatives to some given actuality but rather constitute temporal alternatives for actuality: they are open possibilities for the future. Similar to logical, metaphysical and physical possibilities, real possibilities are alethic possibilities: they are possibilities in an ontological sense. However, the objective notion of reality that is at play here does not exhaust itself in the logical form, the nature of things or the laws of nature. What is really possible depends in addition on the concrete momentary circumstances at hand. In section 1.2.2.1, we will first devote ourselves to the temporal aspect of the notion of real possibility, and we will turn to the worldly aspect in section 1.2.2.2 below.

1.2.2.1 The temporal aspect of real possibilities

So let us have a closer look at what the temporal aspect of real possibilities involves. That is, let us consider how the notion of real possibility relates to the notion of time and how its interrelation with time affects the relation between actuality and possibility.

In our discussion of the standard notions of alethic possibility in section 1.2.1.2, time has played virtually no role. As we have seen, what is common to all those different kinds of possibility—logical, metaphysical and physical—is that they represent modal alternatives to some given actuality. Time may enter the picture only in as so far as we may conceive of actuality as being extended in time. We may understand actuality as not only involving me sitting here in Constance at my desk writing on my thesis but as embracing also the entire past and future course of events, of which the present situation forms just a part. If actuality is extended in time, possibilities that represent modal alternatives to that actuality are, as a matter of course, extended in time as well. This is not to say, however, that what is logically, metaphysically or physically possible then is time-dependent. The range of modal alternatives remains the same at all times. Logical, metaphysical and physical possibility are essentially atemporal notions.

When it comes to the notion of real possibility, the picture changes drastically. In the context of real possibilities, time plays a pivotal role, and the way in which time enters the picture crucially differs from the way in which time may be incorporated in the case of logical, metaphysical and physical possibilities. Real possibilities are future possibilities. And their being future possibilities requires them to be indexically anchored in time. This is simply due to the fact that there is no future without there being a present: the future conceptually depends on the present. By reason of being possibilities for the future, real possibilities can only be understood with respect to some local standpoint in time. In contrast to logical, metaphysical and physical possibilities, real possibilities are thus time-dependent. What is really possible may accordingly vary from time to time.

With the local standpoint in time, there also enters the notion of the past. The notion of real possibility presupposes a fundamental asymmetry between the past and the future. It rests on the idea that the future is open while the past is fixed. Real possibilities are alternative possibilities for the future given the unique past course of events. The relativization to the past renders real possibilities historical possibilities. What is characteristic of those kinds of possibilities is that they may diminish but never increase as time progresses. The passage of time constrains the range of possibilities, so to say. What is possible at one point in time may not be possible at some later time anymore. Yesterday in the early afternoon, me attending the kick-off of the season's first home match of FC Utrecht was still a real possibility. I could have taken a train to Utrecht yesterday and gone to the stadium today. Then, there was also the real possibility of me taking a swim in the North of Sea at the Hoek van Holland at this very moment. However, time has passed, and those real possibilities have vanished. As it happens, I have not taken a train to the Netherlands yesterday, and now that I am sitting here in Constance at my desk, neither of those scenarios is really possible anymore.

At the heart of the notion of real possibility, there is the idea that the future is open in a genuine sense. Given the unique past course of events, there are real open alternatives for the future. Either of those alternatives is equally possible. Modally, they are all on a par.[15] But as time progresses, only one of them will be actualized, and the remainder disappears. The notion of actuality at play is a temporal one. It comprises but the present and the past, and it is dynamic. The future is not actual yet. The future is still to come, and real possibilities represent genuine alternatives for the future to unfold. Either of the alternative future possibilities can be actualized, but, as of now, none of them is actual yet. Unlike logical, metaphysical and physical possibilities then, real possibilities do not constitute mere modal alternatives to some given actuality. Rather, they constitute temporal alternatives for the dynamic actuality to evolve. The contrast becomes most visible when we focus on the asymmetry between the future and the past. From the local standpoint in time, the past and the present are already actual. There certainly may be modal alternatives to the actual course of events up to the present—logical, metaphysical or physical ones. But there is no real possibility for the past and the present to be different from what they actually are. What has happened has happened, and what is the case is the case. There are only open alternatives for the future, temporal alternatives for actuality to evolve. And there is no fact of the matter yet, which of the alternative possibilities for the future will eventually be actualized. I can now take a break from writing and follow the soccer match on the radio, and I can also skip the match and continue writing for a little while. In either case it is really possible that I take a swim in the Lake of Constance in the late afternoon, and I may just as well stay at home. If you want to know what happens, you have to wait and see.

1.2.2.2 The worldly aspect of real possibilities

In the previous section, we have been concerned with the temporal aspect of real possibilities, and we said that real possibilities represent genuine temporal alternatives for the future to evolve. Given a concrete situation in time, however, surely not everything whatsoever can happen in the future. Real possibilities are worldly possibilities. They build on a limited kind of indeterminism rather than on absolute randomness. In this section, we illustrate how the notion of real possibility relates to the world.

[15] This is not to say that all alternatives for the future need to be on a par probabilistically. They do not have to be equally probable.

Just as logical, metaphysical and physical possibilities, real possibilities are alethic possibilities. What is really possible in a concrete situation is so in an ontological rather than in a merely epistemic sense. Like the standard notions of alethic possibilities, real possibilities then are possibilities in virtue of some objective reality, some real aspect of the world. Yet, while logical, metaphysical and physical possibilities are mere modal alternatives to a given actuality, real possibilities are temporal alternatives for a dynamic actuality, and that difference has an impact on the notion of reality that is at the core of the notion of real possibility. Real possibilities do not only have to conform to the logical form, the nature of things and the laws of nature but also to the concrete momentary circumstances at hand. In light of being anchored in concrete situations, they are more strongly tied up with the world than any of the various different kinds of alethic possibilities that we have discussed so far. What is really possible in a situation is what can temporally evolve from that concrete situation against the background of what the world is like, logically, metaphysically and physically. And, in accordance with the idea of indeterminism, there may be more than one such possible future continuation.

In order to see what the worldly aspect of real possibilities involves, let us consider a concrete example. We said that given that I am actually in Constance at this very moment, it is not really possible that I am in the stadium Galgenwaard right now for the kick-off of the first home match of FC Utrecht. While I will certainly have to miss out on the opening of that match, we may wonder whether it is not really possible after all to make it to the stadium Galgenwaard before the onset of the second half. The second half starts only in one hour, and Utrecht is just about 800 kilometers away.

Being in the stadium Galgenwaard in one nanosecond from now would require to travel faster than the speed of light and is hence excluded by the laws of nature governing our world. Note that it is not physically impossible in general though to be in the stadium Galgenwaard in Utrecht on Sunday, August 16, 2015, one nanosecond after half past two in the afternoon. It is only physically impossible given my present situation, according to which at half past two in the afternoon, I am still located in Constance. In the case of real possibility, the local configuration matters; while in the case of physical possibility, the initial conditions may be varied.

Given my current location, by the speed of light, Utrecht is reachable in less than a second. However, there is another caveat. There is nothing that would allow me to travel by the speed of light. While me being in the stadium Galgenwaard in Utrecht in a second from now is in accordance with the prevailing laws of nature and my present location, it is not in accordance with the available technological means: it is a physical possibility, in the present situation, but not a technological one.

The fastest airplane of the world can cover a distance of almost 3600 kilometers in one hour. If we took off right now, I could be in Utrecht in less than 15 minutes from now and hence make it to the stadium Galgenwaard on time for the second half of the soccer match. That possibility, while technically sound, does by no means constitute a real possibility, however. As it happens, the plane is not parking next to my desk in Constance but is based at the NASA headquarters in Washington. For the plane to come to Constance to pick me up, it would take more than one and a half hours. By the time we would reach Utrecht then, the match would already be finished.

By the usual public means, we would not do any better. There is no real possibility to make it to the stadium Galgenwaard before the onset of the second half of the ongoing match between FC Utrecht and SC Heerenveen by taking a train from Constance or a scheduled flight from Zurich airport. In principle, it is possible to travel from Constance to Utrecht by train in less than eight hours. But not even that is really possible right now. The last train that takes less than eight hours has already left an hour ago. If I hurry, I could manage the train leaving Constance at 14:38, arriving in Utrecht briefly before midnight, and the next and last train for today runs only one hour later. Taking a plane from Zurich airport is not a viable option either. There is only one flight scheduled for this evening, at half past five, precisely. Although by taking the next train from Constance to Zurich, I will manage to be on the airport before boarding time, I will not make it on the plane because the flight is already fully booked.

To be sure, the options we have discussed do not exhaust the entire range of possibilities of getting from Constance to Utrecht now. However, they should nevertheless suffice to make clear how tight the link between the notion of real possibility and the world is. Unlike in the case of logical, metaphysical and physical possibilities, what matters in the case of real possibilities is not only the logical form, the nature of things and the laws of nature, but real possibilities depend in addition on the concrete momentary circumstances at hand from which they evolve.

1.3 The theory of branching time

The interaction of modality and time became a topic of formal investigation early on in the development of tense logic by Prior (1957). The initial suggestion was to understand temporal modalities in terms of quantification over moments in a linear temporal structure—the so-called Diodorean modality. In a letter to Prior in 1958, Kripke suggested a branching representation of historical modalities instead.[16] Formal languages based on branching time structures were studied in Prior (1967), and Thomason (1970) contains the first detailed overview of a temporal logic based on those structures. Then as now, the main appeal of the theory of branching time is that it provides a perspicuous representation of alternative possibilities for the future, all of which share some common past.

In this section, we introduce the theory of branching time. In section 1.3.1, we provide the formal definition of a branching time structure. In section 1.3.2, we then discuss the two most popular semantic approaches based on the framework of branching time, which, following Prior (1967), are labeled Peirceanism and Ockhamism.

1.3.1 Branching time structures

In the Prior-Thomason theory of branching time, the interaction of modality and time is represented in a tree-like fashion. The modal-temporal structure of the world is depicted as a tree of moments that branches toward the future. We state the definition of a branching time structure outright and then discuss its implications.

A *BT structure* is a Kripke frame $\mathcal{M} = \langle M, < \rangle$ consisting of a non-empty set of moments M and a strict partial order $<$ on that set that fulfills the following conditions: (BT1) the temporal ordering $<$ on M is linear toward the past, (BT2) any two moments in M have a greatest common lower $<$-bound in M, and (BT3) M has no $<$-maximal elements. We use $m \leq m'$ to stand for ($m < m'$ or $m = m'$). An example of a BT structure is provided in Fig. 1.1.

Definition 1.1 (BT structure). *A* BT structure $\mathcal{M} = \langle M, < \rangle$ *is a non-empty strict partial order* (*i.e. a set* $M \neq \emptyset$ *together with a relation* $<$ *that is irreflexive, asymmetric and transitive*) *such that*

[16] See Ploug and Øhrstrøm (2012) for the Kripke-Prior letters of 1958 and Øhrstrøm and Hasle (1995, 2011) for a broader overview of the history of temporal logic and the treatment of future possibilities.

(BT1) *for all* $m, m', m'' \in M$*, if* $m' < m$ *and* $m'' < m$*, then* $m' \leq m''$ *or* $m'' \leq m'$*;*

(BT2) *for all* $m, m' \in M$*, there is some* $m'' \in M$ *s.t.* $m'' \leq m$*,* $m'' \leq m'$ *and for all* $m''' \in M$*, if* $m''' \leq m$ *and* $m''' \leq m'$*, then* $m''' \leq m''$*;*

(BT3) *for all* $m \in M$*, there is some* $m' \in M$ *s.t.* $m < m'$*.*

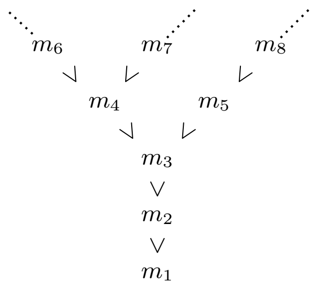

Figure 1.1: A BT structure.

Condition (BT1) captures the idea that the past is fixed while the future may be open. It requires the strict partial order $<$ on the set of moments M to be *left-linear*: there is no backward branching. Every moment has a unique past, while, at the same time, there may be $<$-incomparable moments in its future. The left-linearity of the temporal relation ensures that the resulting structure is tree-like.

The meaning of condition (BT2) is twofold. The condition, first of all, secures the unity of the structure by demanding the temporal ordering $<$ on the set of moments M to be *connected*: any two moments must have a common lower bound in M. Besides, we have added an amendment to the effect that there be a *greatest* common lower bound in each case. That requirement is nonstandard. It is inserted here as it affords a perspicuous definition of branching points, which is crucial to our account of transitions, as we shall see (cf. section 2.3). Structures that are not only connected but also fulfill the stronger requirement will be said to be *jointed.*[17]

Condition (BT3) expresses the idea that time does not end, i.e., there is no last moment. The earlier-later relation $<$ is *serial.* In chapter 4, we will come across BT structures in which that condition is violated, and we will call a BT structure in which every branch eventually comes to an end a *finite* BT

[17] The terminology is due to Reynolds, and it is employed in Zanardo *et al.* (1999).

structure. Note that we do not include an analogous requirement for the past: our definition of a BT structure does not rule out branching time structures that contain a first moment.

Irreflexivity, asymmetry and transitivity are very natural properties to impose on a temporal relation. They provide a temporal direction. We do not place any further conditions on the earlier-later relation $<$ that represents the temporal ordering of moments. We allow it to be discrete, dense or continuous. While discrete structures are important for applications in computer science, in a more general setting triggered by philosophical considerations, dense and continuous structures should be allowed for as well.

Given a BT structure $\mathcal{M} = \langle M, < \rangle$, we can identify maximal $<$-chains in the temporal ordering of moments. Each such chain represents a complete possible course of events. Moments that are incomparable by the earlier-later relation $<$ can never be part of the same possible course of events. Every maximal $<$-chain of moments in M is called a *history.* That is, a history is a $<$-linear subset h of M that is such that none of its proper supersets $h' \supsetneq h$ in M is linearly ordered via $<$ as well. We denote the set of histories in the BT structure $\mathcal{M}$ by $\mathsf{hist}(\mathcal{M})$, and given a moment $m \in M$, we use H_m to stand for the set of all histories in $\mathsf{hist}(\mathcal{M})$ that contain m. In other words, $\mathsf{H}_m := \{h \in \mathsf{hist}(\mathcal{M}) \mid m \in h\}$. In Fig. 1.2, the histories in the given BT structure are indicated.

Definition 1.2 (History). *Given a BT structure $\mathcal{M} = \langle M, < \rangle$, a set $h \subseteq M$ is a* history *iff h is a maximal $<$-linear subset of M (i.e. a subset that is linearly ordered via $<$ (for all $m, m' \in h$, $m \leq m'$ or $m' \leq m$) and such that there is no proper superset $h' \supsetneq h$ in M that is linearly ordered via $<$ as well). The* set of histories in $\mathcal{M}$ *is denoted by* $\mathsf{hist}(\mathcal{M})$. *For a moment $m \in M$, the* set of histories containing *m is denoted by H_m; so $\mathsf{H}_m := \{h \in \mathsf{hist}(\mathcal{M}) \mid m \in h\}$.*

Due to the absence of backward branching, histories are *downward closed*: a history h contains for every moment $m \in h$, all moments $m' \in M$ in that moment's past, i.e., if $m \in h$ and $m' < m$, then $m' \in h$. The reason is simply this: by condition (BT1) of Def. 1.1, the past of any given moment is a $<$-linear subset of M, and histories are defined as maximal linear $<$-chains in M. In particular, we thus have that if $m' \leq m$, then $\mathsf{H}_m \subseteq \mathsf{H}_{m'}$.

Lemma 1.3. *Given a BT structure $\mathcal{M} = \langle M, < \rangle$, every history $h \in \mathsf{hist}(\mathcal{M})$ is* downward closed, *i.e. for all $m, m' \in M$ the following holds: if $m' < m$ and $m \in h$, then $m' \in h$.*

Proof. Follows from condition (BT1) of Def. 1.1 and Def. 1.2. □

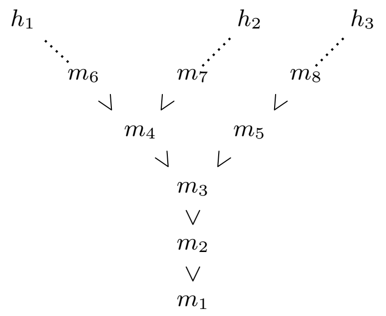

Figure 1.2: Histories in a BT structure.

Recall that in the definition of a BT structure, we have not only required that the earlier-later relation $<$ on the set of moments M be connected but also added a requirement to the effect that it be jointed: any two moments must have a greatest common lower bound. In the light of Lem. 1.3, our condition (BT2) guarantees that the intersection of any two histories in $\mathsf{hist}(\mathcal{M})$ is non-empty and contains a $<$-greatest element.[18] This greatest element constitutes a *branching point*. Whenever a moment $m \in M$ is the $<$-maximal element in the intersection $h \cap h'$ of any two histories $h, h' \in \mathsf{hist}(\mathcal{M})$, we say that the histories h and h' branch at m, and we call the moment m a branching point.

Definition 1.4 (Branching). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, two histories $h, h' \in \mathsf{hist}(\mathcal{M})$* branch at m, *in symbols: $h \perp_m h'$, iff $m \in h \cap h'$ and for all $m' \in M$ such that $m' \in h \cap h'$, we have $m' \leq m$. If for a moment $m \in M$, there are histories $h, h' \in \mathsf{hist}(\mathcal{M})$ s.t. $h \perp_m h'$, we call the moment m a* branching point.

Conditions (BT1) and (BT2) of Def. 1.1 thus jointly ensure that any two histories branch at some moment. There is always 'a last moment of indetermination'.[19]

[18]That the intersection of any two histories in a branching time structure $\mathcal{M} = \langle M, < \rangle$ is non-empty is a straightforward consequence from the fact that the earlier-later relation $<$ on M is connected. Note that the condition does not imply that the intersection of all histories in $\mathsf{hist}(\mathcal{M})$ is non-empty. For concreteness, consider a BT structure $\mathcal{M} = \langle M, < \rangle$ that contains a history h isomorphic to the integers $\mathbb{Z}$ such that at every moment $m_i \in h$ with $i \in \mathbb{Z}$, there is a history $h_i \in \mathsf{hist}(\mathcal{M})$ such that $h \perp_{m_i} h_i$ (cf. Def. 1.4). Assume that $\mathsf{hist}(\mathcal{M}) = \{h\} \cup \{h_i \mid i \in \mathbb{Z}\}$. Then any two histories in $\mathsf{hist}(\mathcal{M})$ have a non-empty intersection, but $\bigcap \mathsf{hist}(\mathcal{M}) = \emptyset$.

[19]Lem. 1.5 is the perfect analog of the *Prior Choice Principle* that is an integral part of the theory of branching space-time put forth in Belnap (1992). In Belnap, Perloff, and Xu (2001, ch.7A.3), the possibility of adopting the principle to the case of branching time is discussed, and the idea of assuming a 'last moment of indetermination' in the theory of branching time

Lemma 1.5. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and let $h, h' \in \mathsf{hist}(\mathcal{M})$. Then there is some moment $m \in M$ such that $h \perp_m h'$.*

Proof. Let $m, m' \in M$ be moments s.t. $m \in h \setminus h'$ and $m' \in h' \setminus h$. Let $m'' \in M$ be the greatest common lower bound of m and m' in M, which exists by condition (BT2) of Def. 1.1. By Lem. 1.3, it follows that $m'' \in h \cap h'$ and for all $m''' \in h \cap h'$, we have $m''' \leq m''$. Hence, $h \perp_{m''} h'$. □

The basic building blocks of a branching time structure are moments. They are structural elements that are to be carefully distinguished from *times*. Different moments in different histories may correspond to the same clock time. In many cases in which we are provided with a BT structure $\mathcal{M} = \langle M, < \rangle$ which is such that all histories in $\mathsf{hist}(\mathcal{M})$ are order isomorphic, we can project a linear series of times onto the branching tree of moments. There may then exist a $<$-linearly ordered set of times T that is order isomorphic to either history and such that we can define a function $time : M \to T$ that maps every moment $m \in M$ onto a time $t \in T$ in such a way that for every history $h \in \mathsf{hist}(\mathcal{M})$, the restriction $time|_h$ is an order isomorphism from h onto T. We call the function *time* a *time function*. The time function induces a horizontal partition of the set of moments M. It identifies moments that correspond to the same time across histories. Different moments contained in the same history are associated with different times and the temporal order on M is preserved, i.e. for all $m, m' \in M$ such that $m < m'$, we have $time(m) < time(m')$.[20] An illustration of a BT structure with an associated series of times is provided in Fig. 1.3.

Definition 1.6 (Time function). *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let T be a $<$-linearly ordered set of times.* *A* time function *is a function $time : M \to T$ such that for every $h \in \mathsf{hist}(\mathcal{M})$, the restriction $time|_h$ is an order isomorphism from h onto T.*

One final remark on the notion of a moment is in order here. Moments, while temporally flat, are conceived of as spanning all of space. The physical picture underlying the theory of branching time is a Newtonian one: space is

as well is approved. In the case of branching space-time, the *Prior Choice Principle* has lately been put into question against the background of considerations concerning the compatibility of the framework with general relativity. In that context, a topology that presupposes first moments of determination rather than a last moment of indetermination seems preferable. Cf. Müller (2013) and Placek (2014).

[20] Our notion of a time function is equivalent to the theory of instants provided in Belnap, Perloff, and Xu (2001, ch.7A.5). In Belnap, Perloff, and Xu (2001), it is admitted that the requirement that every history be order isomorphic to the linear series of times may be too strong.

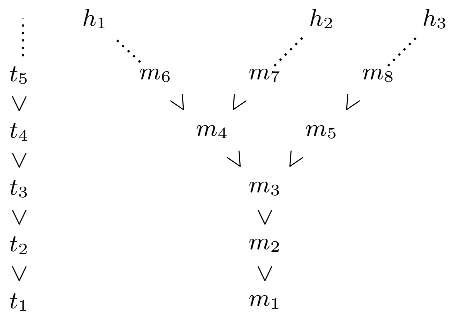

Figure 1.3: Times in a BT structure.

absolute and persists through time. The same is, of course, true for linear time. Branching time generalizes linear time by combining the idea of Newtonian physics with the idea of an open future. The theory of branching space-time provided in Belnap (1992) constitutes a refinement of the theory of branching time. Branching space-time relates to space-time just as branching time relates to linear time. Whereas the theory of branching time builds on moments, which span all of space simultaneously, in the theory of branching space-time, the fundamental structural elements are space-time points. Space becomes relative and spatial and space-like relations become expressible as well. In this thesis, we confine ourselves to the theory of branching time. In that context, 'locality' is always to be understood as temporal locality.[21]

1.3.2 Branching time semantics

BT structures can be employed in the semantics of formal languages containing temporal and modal operators—in fact, the branching time framework has been developed for precisely those semantic purposes in the context of Prior's tense logic. There are two popular semantic approaches based on the framework of branching time. Prior (1967) refers to them as *Peirceanism* and *Ockhamism*, and we will now briefly discuss them in turn.[22] In section 1.3.2.1, we deal with the Peircean account, and we address the Ockhamist account in section 1.3.2.2.

[21] For an illuminating discussion of the relation between the theory of branching time and the theory of branching space-time, see Belnap (2007, 2012). From a semantic point of view, the theory of branching space-times poses an interesting challenge: what are the indices of evaluation in a branching space-time universe? It seems that space-time points are too small to replace moments in the semantic evaluation.

[22] For Peirceanism and Ockhamism, see Prior (1967, pp.122-134) and Thomason (1970, pp.266-271).

In chapter 2, we will provide a novel propositional semantics based on the framework of branching time, viz. the so-called transition semantics. Peirceanism, Ockhamism and the transition semantics differ with respect to which structural elements—over and above a moment of evaluation—are employed as parameters of truth in the recursive semantic machinery, and the corresponding languages are equipped with different kinds of intensional operators. All languages—the Peircean, the Ockhamist and the transition language—are extensions of the basic propositional language $\mathcal{L}$, whose sentences are built up from a given stock of propositional variables and the usual Boolean connectives. We denote the set of propositional variables of the basic propositional language $\mathcal{L}$ by At and make use of negation $\neg$ and conjunction $\wedge$ as the only primitive truth-functional connectives. Disjunction $\vee$, implication $\rightarrow$ and the biconditional $\leftrightarrow$ can be defined in terms of negation $\neg$ and conjunction $\wedge$ in the usual way.

DEFINITION 1.7 (The basic propositional language $\mathcal{L}$). *The alphabet of the basic propositional language $\mathcal{L}$ consists of the set of propositional variables* At *and the following list of primitive Boolean connectives:* $\neg$, $\wedge$. *The syntax of the basic propositional language $\mathcal{L}$ is specified by the following* BNF:[23]

$$\phi \ ::= \ p \mid \neg\phi \mid \phi \wedge \phi.$$

1.3.2.1 Peirceanism

In the Peircean semantics, sentences are evaluated from the local standpoint of a moment in a BT structure, and the Peircean language contains but one kind of intensional operators, viz. temporal ones. The Peircean language $\mathcal{L}_{\mathsf{p}}$ extends the propositional language $\mathcal{L}$ by a past operator P_{p} and two operators for the future, a weak one f_{p} and a strong one F_{p}.

DEFINITION 1.8 (The Peircean language $\mathcal{L}_{\mathsf{p}}$). *The alphabet of the Peircean language $\mathcal{L}_{\mathsf{p}}$ consists of the set of propositional variables* At *and the following list of primitive operators:* $\neg$, $\wedge$, P_{p}, f_{p}, F_{p}. *The syntax of the Peircean language $\mathcal{L}_{\mathsf{p}}$ is specified by the following* BNF:

$$\phi \ ::= \ p \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{P}_{\mathsf{p}}\phi \mid \mathsf{f}_{\mathsf{p}}\phi \mid \mathsf{F}_{\mathsf{p}}\phi.$$

[23] BNF stands for 'Backus-Naur Form', which allows for a straightforward specification of the syntax of a formal language.

On the Peircean account, the recursive semantic machinery is relativized to a single parameter of truth: the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$ depends solely on a moment of evaluation $m \in M$. A *Peircean model* is a triple $\mathfrak{M} = \langle M, <, v_{\mathsf{p}} \rangle$, where $\mathcal{M} = \langle M, < \rangle$ is a BT structure and v_{p} a valuation function that assigns truth values to the propositional variables $p \in \mathsf{At}$ relative to a moment $m \in M$.

Definition 1.9 (Peircean model). *A* Peircean model *is an ordered triple* $\mathfrak{M} = \langle M, <, v_{\mathsf{p}} \rangle$ *consisting of a BT structure* $\mathcal{M} = \langle M, < \rangle$ *and a valuation function* $v_{\mathsf{p}} : \mathsf{At} \times M \to \{0, 1\}$.

The truth of an arbitrary sentence $\phi \in \mathcal{L}_{\mathsf{p}}$ in a Peircean model $\mathfrak{M} = \langle M, <, v_{\mathsf{p}} \rangle$ at a moment $m \in M$ can now be defined recursively. We use $\mathfrak{M}, m \vDash_{\mathsf{p}} \phi$ in order to indicate that a sentence $\phi \in \mathcal{L}_{\mathsf{p}}$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_{\mathsf{p}} \rangle$ on a BT structure $\mathcal{M} = \langle M, < \rangle$ according to the Peircean semantics. The expressions $\mathfrak{M} \vDash_{\mathsf{p}} \phi$ for validity in a Peircean model, $\mathcal{M} \vDash_{\mathsf{p}} \phi$ for Peircean validity in a BT structure and $\vDash_{\mathsf{p}} \phi$ for general Peircean validity are defined in the obvious way by means of the usual generalizations over indices of evaluations, models and structures, respectively.

The recursive semantic clauses for the propositional variables $p \in \mathsf{At}$ and the truth-functional connectives $\neg$ and $\wedge$ are straightforward:

(At) $\mathfrak{M}, m \vDash_{\mathsf{p}} p$ iff $v_{\mathsf{p}}(p, m) = 1$;

($\neg$) $\mathfrak{M}, m \vDash_{\mathsf{p}} \neg\phi$ iff $\mathfrak{M}, m \nvDash_{\mathsf{p}} \phi$;

($\wedge$) $\mathfrak{M}, m \vDash_{\mathsf{p}} \phi \wedge \psi$ iff $\mathfrak{M}, m \vDash_{\mathsf{p}} \phi$ and $\mathfrak{M}, m \vDash_{\mathsf{p}} \psi$.

The semantic clauses for the future operators, f_{p} and F_{p}, and the past operator P_{p} are discussed separately.

Let us start with the weak future operator f_{p}. The interpretation of the weak future operator f_{p} is analogous to the interpretation of the future operator in linear time. The weak future operator f_{p} is an existential quantifier over the various moments in the future of the moment of evaluation. A sentence of the form $\mathsf{f}_{\mathsf{p}}\phi$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_{\mathsf{p}} \rangle$ if and only if there is some later moment $m' > m$ at which ϕ is true.

(f_{p}) $\mathfrak{M}, m \vDash_{\mathsf{p}} \mathsf{f}_{\mathsf{p}}\phi$ iff there is some $m' > m$ such that $\mathfrak{M}, m' \vDash_{\mathsf{p}} \phi$.

Unlike in linear time, however, in a branching time structure, there may be more than one possible future continuation of the moment of evaluation, and the weak future operator requires a future witness in only a single one of them. The condition provided in the semantic clause for $\mathsf{f_p}$ is equivalent to the requirement that there be at least one history $h \in \mathsf{H}_m$ that contains a future witness $m' > m$. More interesting than the weak future operator itself is its dual $\mathsf{G_p} := \neg\mathsf{f_p}\neg$. A sentence of the form $\mathsf{G_p}\phi$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_\mathsf{p}\rangle$ if and only if the sentence ϕ is true at every future moment $m' > m$. On an intuitive reading, the meaning the operator $\mathsf{G_p}$ is "always in the future".

Whereas the weak future operator $\mathsf{f_p}$ requires a future witness in only one possible future continuation of the moment of evaluation, the strong future operator $\mathsf{F_p}$ universally quantifies over all histories containing the moment of evaluation, demanding a future witness in every single one of them. That is, a sentence of the form $\mathsf{F_p}\phi$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_\mathsf{p}\rangle$ if and only every history passing through m contains some later moment $m' > m$ at which ϕ is true.

($\mathsf{F_p}$) $\mathfrak{M}, m \vDash_\mathsf{p} \mathsf{F_p}\phi$ iff for all $h \in \mathsf{H}_m$, there is some $m' \in h$ s.t. $m' > m$ and $\mathfrak{M}, m' \vDash_\mathsf{p} \phi$.

Note that in the case of the strong future operator $\mathsf{F_p}$, the quantification over histories cannot be dispensed with. The condition provided in the semantic clause for $\mathsf{F_p}$ does not have a first-order counterpart. Following Burgess (1980), we set $\mathsf{g_p} := \neg\mathsf{F_p}\neg$ for the dual of $\mathsf{F_p}$. A sentence of the form $\mathsf{g_p}\phi$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_\mathsf{p}\rangle$ if and only if there is at least one history $h \in \mathsf{H}_m$ in which ϕ is true at every future moment $m' > m$.

The semantic clause for the past operator $\mathsf{P_p}$ is straightforward: a sentence of the form $\mathsf{P_p}\phi$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_\mathsf{p}\rangle$ if and only if ϕ is true at some earlier moment $m' < m$. Since every moment has a unique past, no quantification over histories is needed in that case.

($\mathsf{P_p}$) $\mathfrak{M}, m \vDash_\mathsf{p} \mathsf{P_p}\phi$ iff there is some $m' < m$ s.t. $\mathfrak{M}, m' \vDash_\mathsf{p} \phi$.

The past operator is an existential quantifier over the set of moments in the unique past of the moment of evaluation. Its dual $\mathsf{H_p} := \neg\mathsf{P_p}\neg$ then amounts to a universal quantifier over that set. A sentence of the form $\mathsf{H_p}\phi$ is true at a moment $m \in M$ in a Peircean model $\mathfrak{M} = \langle M, <, v_\mathsf{p}\rangle$ if and only if ϕ is true at every earlier moment $m' < m$. Whereas the past operator $\mathsf{P_p}$ intuitively reads "at some point in the past", the intuitive meaning of its dual $\mathsf{H_p}$ is "always in the past". The mnemonic for $\mathsf{H_p}$ is "it has always been the case".

Both the weak and the strong future operator are modalized, and neither of them is apt to provide a notion of plain future truth. The weak future operator $\mathsf{f_p}$ conflates future truth with possibility and is too weak, the strong future operator $\mathsf{F_p}$, on the other hand, conflates future truth with inevitability and is too strong. The Peircean model $\mathfrak{M} = \langle M, <, v_{\mathsf{p}} \rangle$ provided in Fig. 1.4 below satisfies the conjunction $\mathsf{f_p} p \wedge \mathsf{f_p} \neg p$ and falsifies the disjunction $\mathsf{F_p} p \vee \mathsf{F_p} \neg p$. The model contains a branching point m with one possible future continuation in which p always is the case and another in which p never is the case. For all $m' > m$, we have $v_{\mathsf{p}}(p, m') = 1$ if $m' \in h_1 \cup h_2$, and $v_{\mathsf{p}}(p, m') = 0$ if $m' \in h_3$. The conjunction $\mathsf{f_p} p \wedge \mathsf{f_p} \neg p$ is true at the branching point m, while the disjunction $\mathsf{F_p} p \vee \mathsf{F_p} \neg p$ is false at that moment. Intuitively, however, either p or $\neg p$ will be the case but not both.

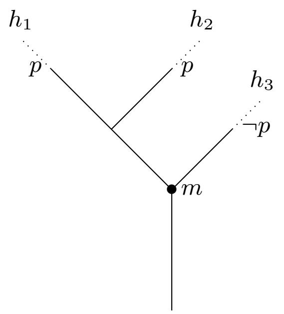

Figure 1.4: Truth in a Peircean model.

1.3.2.2 Ockhamism

The Ockhamist semantics is not a genuine Kripke-style semantics. The indices of evaluation are not identical to moments, which form the building blocks of a BT structure. Ockhamism makes use of a history as a second parameter of truth next to the moment parameter, and the Ockhamist language contains intentional operators that are interpreted as quantifiers over that second parameter. In addition to temporal operators, modal operators enter the picture. The Ockhamist language $\mathcal{L}_{\mathsf{o}}$ enriches the propositional language $\mathcal{L}$ by a past operator $\mathsf{P_o}$ and a future operator $\mathsf{F_o}$, as well as by an operator for inevitability, or settledness, $\Box_{\mathsf{o}}$.

Definition 1.10 (The Ockhamist language $\mathcal{L}_{\mathsf{o}}$). *The alphabet of the Ockhamist language $\mathcal{L}_{\mathsf{o}}$ consists of the set of propositional variables* At *and the following list of primitive operators:* $\neg$, $\wedge$, P_{o}, F_{o}, $\Box_{\mathsf{o}}$. *The syntax of the Ockhamist language $\mathcal{L}_{\mathsf{o}}$ is specified by the following* BNF:

$$\phi \ ::= \ p \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{P}_{\mathsf{o}}\phi \mid \mathsf{F}_{\mathsf{o}}\phi \mid \Box_{\mathsf{o}}\phi.$$

On the Ockhamist account, the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$ depends on a history as a second parameter of truth next to the moment parameter. That is, truth at a moment of evaluation is relativized to a complete possible course of events. Sentences are evaluated at moment-history pairs. In order for a pair consisting of a moment $m \in M$ and a history $h \in \mathsf{hist}(\mathcal{M})$ to constitute a suitable index of evaluation it is thereby required that the moment m be contained in the history h, i.e. $h \in \mathsf{H}_m$. The notation 'm/h' is used in order to indicate that the condition is met.

The employment of a second, defined parameter of truth raises a question as to the status of propositional variables. Are they to be treated as arbitrary sentences, or do they have a special status in that they lay down the fundamental facts holding at a moment? On the first conception, it seems natural to have the truth values of propositional variables depend on both parameters of truth. On the second conception, on the other hand, the treatment of the propositional variables depends on whether the propositional variables that are supposed to represent fundamental facts are allowed to contain a 'trace of futurity' (cf. Prior 1967, p.124). If one denies that propositional variables may contain traces of futurity, the truth values of the propositional variables may be assumed to be solely moment-dependent; otherwise, a general treatment seems preferable after all. Different authors take a different stance toward the status of propositional variables.[24] In the present context, we allow the truth values of propositional variables to vary with both parameters of truth in order to preserve the substitution property of the resulting logic. When we make a transition from logical to metaphysical considerations in chapter 4, we will adopt a different attitude toward propositional variables. But for now, let us define an *Ockhamist model* $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$ as a BT structure $\mathcal{M} = \langle M, < \rangle$

[24] For example, Zanardo (1996, 1998) and Burgess (1978, 1979) treat propositional variables as arbitrary sentences, Reynolds (2002, 2003) and Thomason (1970, 1984) have the truth values of propositional variables depend only on the moment parameter, and Prior (1967) entertains the idea to have a two-sorted language that contains two different kinds of propositional variables that each require a different treatment.

together with a valuation function v_{o} that assigns truth values to the propositional variables $p \in \mathsf{At}$ relative to moment-history pairs m/h with $m \in M$ and $h \in \mathsf{H}_m$. A moment-dependent valuation of propositional variables can be obtained by placing the following additional constraint on the valuation function v_{o}: $v_{\mathsf{o}}(p, m/h) = 1$ iff for all $h' \in \mathsf{H}_m$, $v_{\mathsf{o}}(p, m/h') = 1$.

Definition 1.11 (Ockhamist model). *An* Ockhamist model *is a triple* $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$ *consisting of a BT structure* $\mathcal{M} = \langle M, < \rangle$ *and a valuation function* $v_{\mathsf{o}} : \mathsf{At} \times \{m/h \mid m \in M \text{ and } h \in \mathsf{H}_m\} \to \{0, 1\}$.

Given an Ockhamist model $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$, we can extend the valuation v_{o} on the set At of propositional variables to any arbitrary sentence $\phi \in \mathcal{L}_{\mathsf{o}}$ by means of recursive semantic clauses. We use $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \phi$ in order to indicate that a sentence $\phi \in \mathcal{L}_{\mathsf{o}}$ is true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$ on a BT structure $\mathcal{M} = \langle M, < \rangle$ according to the Ockhamist semantics. Validity in an Ockhamist model $\mathfrak{M} \vDash_{\mathsf{o}} \phi$, Ockhamist validity in a BT structure $\mathcal{M} \vDash_{\mathsf{o}} \phi$ and general Ockhamist validity $\vDash_{\mathsf{o}} \phi$ are defined as usual.

The recursive semantic clauses for the propositional variables $p \in \mathsf{At}$ and the Boolean connectives $\neg$ and $\wedge$ are straightforward:

(At) $\mathfrak{M}, m/h \vDash_{\mathsf{o}} p$ iff $v_{\mathsf{o}}(p, m/h) = 1$;

($\neg$) $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \neg\phi$ iff $\mathfrak{M}, m/h \nvDash_{\mathsf{o}} \phi$;

($\wedge$) $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \phi \wedge \psi$ iff $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \phi$ and $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \psi$.

In the case of the temporal operators F_{o} and P_{o}, the history parameter is kept fixed, and the moment of evaluation is shifted forward or backward, respectively, on the given history, just as in linear tense logic. A sentence of the form $\mathsf{F}_{\mathsf{o}}\phi$ is true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$ if and only if the history h contains a later moment $m' > m$ at which ϕ is true with respect to h.

(F_{o}) $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \mathsf{F}_{\mathsf{o}}\phi$ iff there is some $m' \in h$ s.t. $m' > m$ and $\mathfrak{M}, m'/h \vDash_{\mathsf{o}} \phi$.

The past operator is interpreted analogously: a sentence of the form $\mathsf{P}_{\mathsf{o}}\phi$ is true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$ if and only if ϕ is true with respect to h at some earlier moment $m' < m$. Note that in the case of the past operator the requirement $m' \in h$ can be dropped since, by Lem. 1.3, histories are downward closed.

(P_{o}) $\mathfrak{M}, m/h \vDash_{\mathsf{o}} \mathsf{P}_{\mathsf{o}}\phi$ iff there is some $m' < m$ s.t. $\mathfrak{M}, m'/h \vDash_{\mathsf{o}} \phi$.

Both the future operator $\mathsf{F}_\circ$ and the past operator $\mathsf{P}_\circ$ are existential quantifiers. Their respective duals $\mathsf{G}_\circ := \neg\mathsf{F}_\circ\neg$ and $\mathsf{H}_\circ := \neg\mathsf{P}_\circ\neg$ accordingly involve universal quantification. A sentence of the form $\mathsf{G}_\circ\phi$ is true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_\circ \rangle$ if and only if ϕ is true with respect to the history h at every later moment $m' > m$ contained in h. Analogously, a sentence of the form $\mathsf{H}_\circ\phi$ is true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_\circ \rangle$ if and only if ϕ is true with respect to h at any earlier moment $m' < m$.

Since the semantic evaluation is relativized to a history as a second parameter of truth, modal operators become interpretable as quantifiers over the set of histories containing the moment of evaluation. The inevitability operator $\Box_\circ$ amounts to universal quantification. A modal operator for possibility, $\Diamond_\circ$, can be defined as its dual, i.e., $\Diamond_\circ := \neg\Box_\circ\neg$. The modality involved is S5. A sentence of the form $\Box_\circ\phi$ is true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_\circ \rangle$ if and only if ϕ is true at the moment m with respect to every history that passes through m.

($\Box_\circ$) $\mathfrak{M}, m/h \vDash_\circ \Box_\circ\phi$ iff for all $h' \in \mathsf{H}_m, \mathfrak{M}, m/h' \vDash_\circ \phi$.

A sentence of the form $\Diamond_\circ\phi$ is accordingly true at a moment-history pair m/h in an Ockhamist model $\mathfrak{M} = \langle M, <, v_\circ \rangle$ if and only if there is at least one history in H_m with respect to which ϕ is true at the moment m. Inevitability $\Box_\circ$ and possibility $\Diamond_\circ$ are historical modalities: what is inevitable or possible depends on how far time has yet unfolded. In the course of time, some possibilities may disappear, and what has not yet been settled becomes settled.

By making use of a history as a second parameter of truth next to the moment parameter, Ockhamism generalizes Peirceanism. Modal operators become interpretable, and truth and inevitability come apart while they coincide on the Peircean account. In the Ockhamist semantics, a sentence can be assigned different truth values at the very same moment with respect to different histories. Assume that in the Ockhamist model provided in Fig. 1.5, for all $m' \in M$, we have $v_\circ(p, m'/h_1) = 1$ if $m' \in h_1$, $v_\circ(p, m'/h_2) = 1$ if $m' \in h_2$, and $v_\circ(p, m'/h_3) = 0$ if $m' \in h_3$. Then the conjunction $\Diamond_\circ\mathsf{F}_\circ p \wedge \Diamond_\circ\mathsf{F}_\circ\neg p$ is true at the moment m with respect to any history in H_m. Moreover, with respect to none of the histories in H_m does it hold that the conjunction $\mathsf{F}_\circ p \wedge \mathsf{F}_\circ\neg p$ is true at the moment m, while the disjunction $\mathsf{F}_\circ p \vee \mathsf{F}_\circ\neg p$ is true at the moment m with respect to each such history. In fact, under the assumption that time does not end, the disjunction $\mathsf{F}_\circ p \vee \mathsf{F}_\circ \neg p$ is an Ockhamist validity, and condition (BT3) of Def. 1.1 ensures that this is here the case.

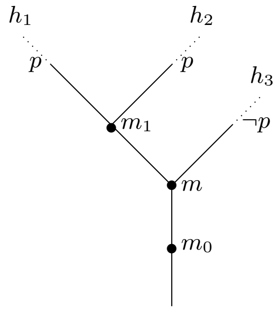

Figure 1.5: Truth in an Ockhamist model.

When introducing the definition of a BT structure in section 1.3.1 above, we said the condition (BT1) in Def. 1.1 captures the idea that the past is fixed while the future is open. The idea underlying that condition is not to be confounded with the idea that the past is necessary while the future is contingent. The former idea pertains to the properties of a branching time structure whereas the latter idea pertains to the properties of the semantic models on those structures. Obviously, there may be models that contain moments whose future—albeit open—is not contingent. In the model in Fig. 1.5, the sentence $\Box_\circ \mathsf{F}_\circ p$ is true at the moment m_1 with respect to the history h_1, even though there is a second history passing through that moment. And, what is more, since we have taken a logical stance toward propositional variables and have their truth values depend on both parameters of truth, the implication $\mathsf{P}_\circ p \to \Box_\circ \mathsf{P}_\circ p$, which expresses that the past is necessary, is falsifiable. In our model, the implication $\mathsf{P}_\circ p \to \Box_\circ \mathsf{P}_\circ p$ comes out false at the moment-history pair m/h_1 because $\mathsf{P}_\circ p$ is false with respect to m_0/h_3. A moment-dependent valuation on the set At of propositional variables renders the implication $\mathsf{P}_\circ p \to \Box_\circ \mathsf{P}_\circ p$ valid for every $p \in \mathsf{At}$. However, the structural idea that the past is fixed while the future is open finds its expression in the validity of the implication $\mathsf{P}_\circ \Box_\circ p \to \Box_\circ \mathsf{P}_\circ p$ and the invalidity of the implication $\mathsf{F}_\circ \Box_\circ p \to \Box_\circ \mathsf{F}_\circ p$.[25] In our model in Fig. 1.5, the latter implication is false at the moment m with respect to the history h_1: the sentence $\Box_\circ p$ is true at the moment $m_1 > m$ with respect to h_1, but $\mathsf{F}_\circ p$ is false at the moment m with respect to the history h_3.

[25] For a detailed discussion of those issues, see Burgess (1978).

1.4 Possibilities without possible worlds

In the previous sections, we have introduced the notion of real possibility and the theory of branching time, which builds the basis of our investigations throughout this thesis. In this section, we will discuss the appeal of the branching time framework for the formal representation of real possibilities. In particular, we motivate our choice of the branching time framework over the popular possible worlds framework, which seems to have established itself as the standard for modeling all kinds of notions of possibility.

In section 1.4.1, we consider how possibilities are represented in the branching time framework and emphasize the adequacy of the theory of branching time for the notion of real possibility we aim to model. In section 1.4.2, we introduce the possible worlds framework and discuss the representation of future possibilities within that framework. In section 1.4.3, we highlight crucial differences between the branching time framework and the possible worlds approach. On the basis of those differences it becomes apparent why the possible worlds framework is ruled out as a suitable alternative to the branching time framework in the formal representation of real possibilities.

In our discussion of the two frameworks, we confine ourselves mainly to aspects of the respective structures, setting aside questions concerning the semantic models definable on those structures, which will be discussed at length in chapter 4. While the contrastive juxtaposition of the branching time framework and the possible worlds approach serves, first of all, the purpose of making a case for the theory of branching time, it also affords the opportunity to discuss several cognate notions along the way that will become important throughout this thesis. We introduce the definition of a $T \times W$ frame, the notion of a Kamp frame as well as the idea of a bundled tree, and we address the theory of the so-called Thin Red Line.

1.4.1 Possibilities in branching time

The framework of branching time is, first of all, a formal device, a technical tool to be employed in the semantics of formal languages containing modal and temporal operators. At the same time, the very idea underlying the theory of branching time is inextricably tied up with certain metaphysical assumptions. In this section, we aim at making explicit those underlying metaphysical assumptions. In section 1.4.1.1, we address the question of what exactly is represented by a branching time structure, and we illustrate how possibilities fit into the picture. In section 1.4.1.2, we spell out why the framework of branching time is especially well suited for modeling the notion of real possibility.

1.4.1.1 The representation of possibilities

The theory of branching time provides some metaphysical picture, but what exactly is a branching time structure a picture of? And how does the precise nature of the structure bear on the representation of possibilities? In this section, we show that a branching time structure allows for two different perspectives, a global and a local one, and we argue that what is represented by a branching time structure depends, first of all, on which perspective one takes. We will see that the representations corresponding to the two different perspectives are not unrelated and in the end add up to form a uniform whole. In section 1.4.1.1.1, we first focus on the global perspective, and we discuss the local perspective subsequently in section 1.4.1.1.2.

1.4.1.1.1 A global perspective Let us start out with the global perspective. The global perspective is an external perspective, a perspective from outside a given branching time structure. It is the perspective that resonates with the definition of a branching time structure provided in Def. 1.1, and the corresponding picture is the one depicted in Fig. 1.6.

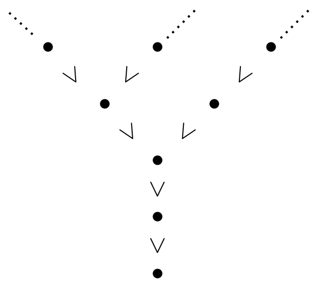

Figure 1.6: A global perspective on a BT structure.

In section 1.3.1, we have characterized a branching time structure as a pair $\langle M, < \rangle$ consisting of a non-empty set of moments M and a strict partial order $<$ on that set which is required to be left-linear, jointed and serial. Each moment in the tree-like ordering is thought of as spanning all of space simultaneously, and the relation $<$ is interpreted temporally as an earlier-later relation. If we take a standpoint outside a given branching time structure and look down on the structure from a God's eye view perspective, the entire structure is spread out before us. The complete tree of moments is lying there in its totality. The picture is an entirely static one. All moments are on a par. There is no marked actuality. There is no distinguished present, no past and no future. All there is is the ordered set of moments, which is connected in a tree-like fashion by the earlier-later relation $<$. In McTaggart's terminology, we may want to say that the picture is a purely B-theoretic one.[26]

It is tempting to mistake the static tree of moments spread out in front of us for an eternalist picture of the world. A branching time structure, however, is not supposed to represent our world, nor does it represent time. Neither our world nor time branches. Both evolve linearly.[27] That the temporal earlier-later relation among moments does not represent the structure of time should already be evident from the fact that we can add a linear series of times as an additional layer on a branching structure. Referring to the set of moments as "Our World" is misleading. It is a misnomer, just as the label "branching time" itself is.[28]

What a branching time structure depicts, when viewed from an outside perspective, is the modal-temporal structure of some indeterministic world—ideally, of course, that of ours. The idea of indeterminism is at the heart of the theory of branching time, and it finds its expression in the tree-like ordering of moments by the earlier-later relation. The ordering is, strictly speaking, not a purely temporal one, but it is essentially modal in nature, for, it admits branching points. Those branching points are the loci of indeterminism.[29]

[26] Cf. McTaggart (1908).

[27] It seems that several criticisms of the theory of branching time rest on a false understanding of what a branching time structure is supposed to represent. It is frequently assumed that a branching time structure provides a metaphysical picture of our world or time itself. Building on a misunderstanding, the criticisms lead to unwarranted conclusions. The most peculiar example is surely the claim that the theory of branching time is incompatible with indeterminism, which is defended, for example, in Rosenkranz (2013). This claim seems especially striking against the background that the branching time framework has been developed precisely for the purpose to give expression to the idea of indeterminism. Recent contributions to the metaphysics of branching time are included in Correia and Iacona (2013).

[28] In Belnap, Perloff, and Xu (2001), the set of moments of a branching time structure is called "Our World" in order to emphasize that what a branching time structure represents are inner-worldly possibilities, possibilities in the modal-temporal structure of our world. In the same work, moments are referred to as 'concrete events'. We take this to mean that moments are concretely anchored in the modal-temporal structure of the world rather than that they are constitutive parts of the latter. That the label "branching time" is misleading is often stressed by Belnap himself. He often speaks of "branching histories" instead, and in Belnap, Perloff, and Xu (2001, p.29, fn.1) we read: "We never, ever mean to suggest that time itself—which is presumably best thought of as linear—ever, ever 'branches'."

[29] On a robust understanding of indeterminism, it seems natural to look for indeterminism on the level of semantic or metaphysical models rather than on the sheer structural level. A

A branching time structure is full of possibilities. Due to the existence of branching points, there are countless possibilities to be found in the modal-temporal structure of the world. Each branching point allows for several immediate possible future continuations, and each such immediate possible future continuation represents a local future possibility. Besides, a branching time structure allows us to carve out complete possible courses of events, viz. histories. In contrast to the local future possibilities arising at a branching point, histories may be said to represent global possibilities. Every history captures a complete possible temporal development of the world, and any two histories branch at some moment. What both kinds of possibilities, local and global ones, have in common is that they constitute temporal alternatives: they are possibilities in the modal-temporal structure of the world and represent open alternatives for the future.

1.4.1.1.2 A local perspective Let us now turn to the local perspective. Whereas the global perspective is an external perspective on a branching time structure, the local perspective is an internal one. It mirrors the perspective of language users and is thus closely related to the intended purpose of the branching time framework as a semantic means to be employed in the interpretation of languages. We will see that if we abandon our outside perspective on a branching time structure and instead take a standpoint within the tree of moments, the purely static character of the structure disappears and the tree gains dynamic traits.

As soon as we take an internal perspective, the various moments of the branching time structure are no longer on a par, and the B-theoretic picture transforms into an A-theoretic one. By locating ourselves at a moment in the structure, we mark that moment as present. At the same time, our local standpoint determines a unique past, viz. the set of all moments preceding the present moment. The present moment and its past jointly provide a notion of actuality. They capture what has actually happened so far. But then, there may be various branches lying ahead of us. Those branches represent alternative possibilities for the future. There may be immediate and remote, viz. local and global ones. And all of them represent temporal alternatives. Any

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robust notion of indeterminism seems to presuppose a conception of laws of nature as well as a world that is not devoid of any content. In the theory of branching time, the idea of indeterminism is already present on the structural level. In chapter 4, we will make the idea more robust: we will provide branching time models in which each possible course of events is compatible with the prevailing laws of nature.

of them can be actualized, but none of them is actual yet. In addition, there are also branches that branch off from the past course of events. Those branches represent mere counterfactual possibilities, possibilities that once could have been actualized but actually were not. The partition of the tree of moments into actuality, possibility and counterfactuality is depicted in Fig. 1.7. The thick line indicates actuality, the thin lines represent possibilities, and the dashed line marks a counterfactuality.

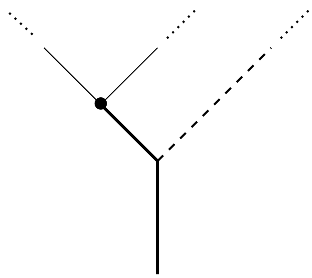


Figure 1.7: A local perspective on a BT structure.

The notion of actuality that enters the picture once we take a local standpoint within the tree of moments is a dynamic one, and so is the corresponding notion of possibility. Actuality comprises the present and the past. The various branches that are spread out before us are not part of actuality. The multiplicity of those branches reflects the openness of the future. That the future is open is thereby not supposed to mean that there are multiple futures lying ahead of us. Objections such as the one by Lewis, who holds that "branching [...] conflicts with our ordinary presupposition that we have a single future" (D. Lewis 1986a, p.207) rest on a false understanding of what the branching time structure represents. Strictly speaking, there is no future yet. The future is yet to come. And the various branches that are lying before us represent possibilities for that single future to unfold. From the perspective of the present moment, all those possibilities for the future are equally possible. Any of them can be actualized. Yet, as time progresses, only one of them will be actualized, and the remainder disappears.

In the dynamic nature of the notion of actuality, the idea of the passage of time shines through. When time passes, the present moment becomes past, moments that have not been actual yet become actual, and possibilities that once have been open, drop out of existence.[30] Actuality evolves by and by, and the range of open possibilities for the future diminishes accordingly. The passage of time enters the picture once we take an inside perspective and take seriously the claim that only one of the alternative possibilities for the future will be actualized in the end. It is incorporated in our understanding of a branching time structure, not in the branching time structure itself.

1.4.1.2 The adequacy for the notion of real possibility

In the previous section, we have discussed in detail what a branching time structure represents, as viewed from two different perspectives. In this section, we link our discussion back to the notion of real possibility. That is, we point out the adequacy of the framework of branching time for the formal representation of real possibilities.

We said that real possibilities are alternative possibilities for the future in an indeterministic world. They are locally anchored in a concrete situation in time, and they crucially depend on what has actually happened so far. In other words, they are historical possibilities. What is really possible at one point in time may not be really possible at some later time anymore. Some real possibilities disappear as time progresses. What is really possible at a given point in time, though, is so in a very genuine sense. Real possibilities are open possibilities for the future. They are equal alternatives for actuality to unfold. Either of them can in fact be actualized. Branching time structures allow for a perspicuous representation of the temporal aspect of real possibilities. Both the idea that real possibilities are historical possibilities as well as the idea that they are genuinely open alternatives for the future can be adequately captured.

For concreteness, let us consider how the example that we have discussed in section 1.2.2 is represented in the theory of branching time. The corresponding picture is provided in Fig. 1.8.[31]

The very moment in the middle of the picture corresponds to my actual situation. It depicts me sitting in Constance at my desk typing into my laptop. There is a branch branching off in the past. In light of what has actually happened so far, that branch does not represent a real possibility. Rather, it captures a course of events that once has been really possible but has gone

[30] On McCall's account, unactualized possibilities really drop out of existence, which is referred to as branch attrition. See, McCall (1984, 1994). Our notion of a pruning rests on a similar idea; cf. section 3.2.1.

[31] Icons are taken from www.icons8.com.

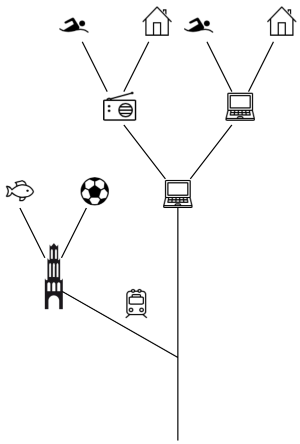


Figure 1.8: Real possibilities in branching time.

unactualized. Even though I did not, I could have taken a train to Utrecht yesterday. And if I had done so, I could be in the stadium Galgenwaard now watching the season's first home match of FC Utrecht, and I could just as well be at the Hoek van Holland at this very instant swimming with the fish in the North Sea. Both me standing in the stadium Galgenwaard cheering for FC Utrecht as well as me drifting in the waves in the North Sea at the Hoek van Holland constitute alternatives to my present situation. In fact, the branching time structure contains moments in alternative histories in which I am doing so right now. Yet, those histories represent mere counterfactual possibilities. Possibilities that once could have been actualized but actually were not. For, as it happens, I did not take the train to Utrecht yesterday but stayed in Constance instead, and so the real possibility of me being in the stadium Galgenwaard in Utrecht or at the Hoek van Holland right now has vanished. Yet, there are still open alternatives for the future. There are various branches spread before me, and each such branch represents a real possibility, a genuine alternative for actuality to unfold. I can take a break from writing and follow the soccer match on the radio, or I can skip the soccer match and keep on working. In either case, there is the possibility of taking a swim in the Lake of Constance in the late afternoon, and there is also the real possibility of simply staying at home.

It is worthwhile to note that branching time structures, as Kripke frames, can, first of all, capture only the temporal aspect of real possibilities, viz.

the interrelation of modality and time. We have seen that in addition to the temporal aspect, there is also a worldly aspect to real possibilities: what is really possible in a concrete situation that is locally anchored in time is what can temporally evolve from that situation against the background of the prevailing laws of nature. That worldly aspect of real possibilities cannot be warranted on purely structural grounds. Rather, it needs to be incorporated into the corresponding branching time models. In chapter 4, we deal with worldly models for real possibility. We discuss the question of how we can provide an appropriate valuation of a suitable language on a branching time structure that respects the worldly aspect of real possibilities, and we offer a dynamic explanation of branching time models in terms of potentialities.

Even though the branching time framework is still in the shadow of the more popular possible worlds framework, it is nowadays famously employed in many different areas in philosophy and computer science where real possibilities play a role. In computer science, the branching time framework finds application in model checking and artificial intelligence. There are computational variants of Peirceanism and Ockhamism. The computational tree logic CTL is the discrete counterpart of Peirceanism. The logics called CTL* and PCTL*, on the other hand, correspond to Ockhamism—the latter extends the former by a past operator.[32] The stit-framework provides a logic for agency. The expression 'stit' stands short for 'seeing to it that'. The logic contains a stit-operator that captures how an agent's choices bear on the possible courses of events.[33] Furthermore, in the philosophical literature, various accounts of indeterministic causation have been proposed on the basis of the framework of branching time, and the framework is also extensively discussed in the context of the free will debate, as it promises a sound basis for a libertarian account of free action.[34]

1.4.2 Possibilities in a possible worlds framework

Having stressed the merits of the branching time framework for the formal representation of real possibilities, we now turn to its rival, viz. the possible worlds

[32] For an overview of computational variants of Peirceanism and Ockhamism, CTL and CTL*, see Emerson (1990).

[33] For stit accounts of agency, see, for example, Belnap and Perloff (1990), Horty and Belnap (1995), Belnap, Perloff, and Xu (2001) and Broersen (2011).

[34] Accounts of causation based on the framework of branching time are provided, for example, in von Kutschera (1993), Xu (1997) and Belnap (2005). Recent contributions to the application of the theory of branching time in theories of agency and indeterminism are included in Müller (2014b).

framework. The possible worlds framework is enjoying great popularity, both in a logical and in a metaphysical context, and it is also predominant in linguistics. It is certainly right to say that it has established itself as the standard framework for modeling all kinds of notions of possibility and necessity.

In the possible worlds framework, possibilities are represented by so-called possible worlds. The idea of a possible world is often traced back to Leibniz's dictum that ours is "the best of all possible worlds" (Leibniz 1985, p.228). Attempts to analyze possibility and necessity in terms of quantification over possible worlds are already present in Carnap's work.[35] Kripke devised a rigorous formal framework based on the idea of possible worlds and thereby laid the foundation for a semantic approach to modal logic, which, so far, had been studied only from a syntactic point of view.[36] The Kripke-semantics named after him allows for relative notions of possibility and necessity and constitutes the model-theoretic basis of modern modal logic. The Kripkean idea has also made its way into linguistics, where it is famously employed in the interpretation of modals, such as "must", "can" or "might".[37] That the possible worlds framework has gained such popularity in metaphysics is doubtlessly not least due to Lewis's metaphysical account of possible worlds, which he himself describes as "a philosopher's paradise" (D. Lewis 1986a, ch.1). In contemporary metaphysics, possible worlds are not only employed in analyses of possibility and necessity but figure prominently in various accounts of all kinds of modal notions, as, for instance, in theories of causation, the debate on dispositions, the definition of (in)determinism, etc.[38]

In this section, we introduce the possible worlds framework in completely general terms. To be sure, the precise conception of possible worlds crucially differs across the logical, the linguistic and the metaphysical setup. Yet, what is common to all those different approaches is the underlying structure of the possible worlds pluriverse, and it is that common structure that we will focus on in what follows. In particular, we are interested in how time can be incorporated into the picture. In section 1.4.2.1, we consider possible combinations of modality and time in a possible worlds framework, and we discuss the impact of the resulting structures for the representation of possibilities in section 1.4.2.2.

[35] Cf. Carnap (1947).

[36] Cf. Kripke (1959, 1963a,b). Syntactic systems for modal logic had already been extensively studied in C. Lewis and Langford (1932).

[37] Cf. Kratzer (1977, 1981, 1991).

[38] An illuminating overview of the emergence and conception of possible worlds in logic and metaphysics is provided in Menzel (2016).

1.4.2.1 Possible worlds and time

From a purely structural point of view, a possible worlds framework, in its most basic version, is but a simple Kripke frame $\langle W, \sim \rangle$, that is, an ordered pair consisting of a non-empty set of possible worlds W and an accessibility relation $\sim$ on that set. Each possible world stands in for some possibility, and the accessibility relation reflects which possible worlds are considered alternatives of a given world. In a Kripke frame, possibility and necessity are then analyzed in terms of restricted quantification over the set of possible worlds in accordance with the accessibility relation.

Kripke frames that comprise but a set of possible worlds and an accessibility relation between those worlds yield, first of all, a theory of pure modality. Even though one might conceive of the various possible worlds as being extended in time, such temporality is not reflected on the structural level. While those frames are perfectly suited in order to model atemporal notions of possibility and necessity, such as logical, metaphysical and physical possibility, they are inadequate for modeling historical modalities. If we are to assess the adequacy of the possible worlds framework as a means to model the notion of real possibility, which is an essentially temporal kind of possibility, we need to consider systems in which time is explicitly incorporated into the structure. In this section, we discuss the most promising possible worlds candidates that accommodate a notion of time. In section 1.4.2.1.1, we introduce the so-called $T \times W$ framework, and in section 1.4.2.1.2, we turn to a generalization of that approach, viz. to the notion of a Kamp frame.

1.4.2.1.1 T × W frames The most straightforward way to combine modality and time in a possible worlds framework is to simply add time as a further dimension. The result is what is called a $T \times W$ *frame*.[39] An example of such a frame is provided in Fig. 1.9.

Formally, a $T \times W$ frame is an ordered quadruple $\langle W, T, <, \sim \rangle$. It comprises next to a non-empty set of possible worlds W, a non-empty set of times T, which is linearly ordered by an earlier-later relation $<$. The earlier-later relation $<$ is assumed to be irreflexive and transitive and is thus also asymmetric. The $<$-linear series of times is then projected onto the various possible worlds, providing each possible world with a temporal structure. The modal accessibility relation $\sim$ between possible worlds is accordingly relativized to times,

[39] The definition of a $T \times W$ frame goes back to Thomason (1984).

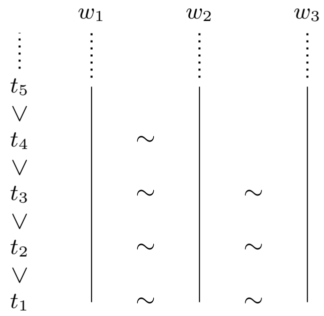


Figure 1.9: A $T \times W$ frame.

and required to be an equivalence relation in each case. Besides, one further constraint is added: any two possible worlds that are accessible at a given time, must also be accessible at all prior times. This latter condition gives expression to the idea that the past is fixed while the future is open. It renders possible worlds that are accessible at a certain time historical alternatives. What is possible depends on how far time has yet unfolded. In the course of time, the range of possibilities may diminish, but it never increases. We state the formal definition of a $T \times W$ frame in Def. 1.12.

Definition 1.12 ($T \times W$ frame). *A* $T \times W$ frame *is an ordered quadruple* $\langle W, T, <, \sim \rangle$ *where*

(i) W *is a non-empty set of worlds;*

(ii) T *is a non-empty set of times;*

(iii) $<$ *is a strict linear order on* T *(i.e. a relation that is irreflexive, asymmetric, transitive and linear);*

(iv) $\sim \subseteq (T \times W \times W)$ *such that*

  (a) *for all* $t \in T$*, the relation* $\sim_t := \{\langle w', w'' \rangle \mid \langle t, w', w'' \rangle \in \sim\}$ *is an equivalence relation (i.e. a relation that is reflexive, symmetric and transitive);*

  (b) *for all* $w', w'' \in W$ *and for all* $t, t' \in T$*, if* $w' \sim_t w''$ *and* $t' < t$*, then* $w' \sim_{t'} w''$*.*

In a $T \times W$ frame $\langle W, T, <, \sim \rangle$, the semantic evaluation is not only relativized to a world parameter. Rather, the indices of evaluation are pairs $\langle t, w \rangle$

consisting of a time $t \in T$ and a possible world $w \in W$. The $T \times W$ framework is tailored to the interpretation of languages such as the Ockhamist one, which extends the basic propositional language $\mathcal{L}$ by a past operator, a future operator and an operator for inevitability. The semantic clauses of those operators are analogous to the Ockhamist ones: just replace moments by times and histories by worlds. In the case of the temporal operators, the world parameter is held fixed, and the time parameter is shifted forward or backward, respectively, along the temporal structure of the given world. In the case of the inevitability operator, on the other hand, the time parameter is kept constant, and the modal operator is interpreted as a universal quantifier over the set of possible worlds accessible from the given one at that time.

If we assume, as it is commonly done, that whenever two possible worlds, say w and w', are accessible at a given time t, the corresponding pairs $\langle t, w\rangle$ and $\langle t, w'\rangle$ make exactly the same propositional variables true, the resulting models are close semantic counterparts of Lewis's divergence models.[40] Since, by condition (iv.b), the time-relative accessibility is preserved toward the past, in that case, any two possible worlds that are accessible at a time share their entire past. That is, they perfectly coincide on some initial segment and differ only with respect to the future—at least in so far as the truth values of the propositional variables are concerned. We will come back to Lewis's divergence models in chapter 4. The additional semantic constraint on the valuation of the propositional variables captures the idea that the past is necessary. As already noted in the case of Ockhamism in section 1.3.2.2, that idea must be carefully distinguished from the idea that the past is fixed, which is catered for on purely structural grounds. And while the additional semantic constraint seems metaphysically sound, it is undesirable from a logical point of view: in the presence of that constraint, the substitution property of the resulting logic in its unrestricted form is lost.

1.4.2.1.2 Kamp frames In a $T \times W$ frame, time is added as an extra ingredient. A $T \times W$ frame rests on an absolute notion of time that is projected onto each of the possible worlds. As a result, in a $T \times W$ frame, all possible worlds are isomorphic with respect to their temporal structure. This certainly seems to be apt from a metaphysical point of view and also from the point of

[40] We say close semantic counterparts, for, in order for such a model to be a divergence model in the strict sense, there must be some time at which all possible worlds are mutually accessible. Note that by the definition of a $T \times W$ frame, it is not required that there be some time such that $\sim_t = W \times W$.

view of Newtonian physics. From a logical point of view, however, the assumption of an absolute notion of time does not constitute a necessary requirement. The notion of a *Kamp frame* generalizes the notion of a $T \times W$ frame. As in the case of $T \times W$ frames, a primitive notion of time is added to the set of possible worlds. But now, the notion of time is no longer an absolute one. Unlike in the case of a $T \times W$ frame, in a Kamp frame, worlds may differ with respect to their temporal structure.[41]

A Kamp frame is an ordered triple $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$. Just as a $T \times W$ frame, a Kamp frame comprises, first of all, a non-empty set of possible worlds W. Then, instead of building on a unique linearly ordered set of times, a Kamp frame comes equipped with a function $\mathcal{T}$ that assigns to each possible world $w \in W$, a strict linear ordering $\langle T_w, <_w \rangle$ of times. The linear series of times associated with different possible worlds are thereby in general not completely disjoint but rather overlap. The modal accessibility relation $\sim$ between possible worlds is again relativized to times and required to be an equivalence relation in each case. For two possible worlds to be accessible at a given time, that time must, as a matter of course, be part of the temporal structure of both worlds. The idea that two possible worlds that are accessible at a given time constitute historical alternatives is accommodated by requiring that those worlds have the same temporal structure with respect to the past, viz. they share the same times toward the past, and are accessible at all those prior times. Here again the idea that the past is fixed while the future is open shines through. Note that the requirement is again a purely structural one. It only guarantees that the past is 'unique', which is not to say that the past is necessary.

Definition 1.13 (Kamp frame). *A* Kamp frame *is an ordered triple* $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$ *where*

(i) W *is a non-empty set of worlds;*

(ii) $\mathcal{T} : w \mapsto \langle T_w, <_w \rangle$ *is a function that maps every world* $w \in W$ *onto a pair* $\langle T_w, <_w \rangle$ *consisting of a non-empty set of times* T_w *and a strict linear order* $<_w$ *on that set (i.e. a relation that is irreflexive, asymmetric, transitive and linear);*

[41] The definition of a Kamp frame has apparently been put forth by Kamp in an unpublished manuscript (cf. Kamp 1979), and it is restated in Thomason (1984). In Thomason (1984), a Kamp frame is defined as an ordered triple $\langle \mathcal{T}, W, \sim \rangle$, whose first entry equals the function $\mathcal{T}$ and whose second entry equals the set of possible worlds. We have changed the order of the entries for systematic reasons.

(iii) $\sim \subseteq (\bigcup_{w \in W} T_w \times W \times W)$ *such that*

  (a) *for all* $t \in \bigcup_{w \in W} T_w$*, the relation* $\sim_t := \{\langle w', w'' \rangle \mid \langle t, w', w'' \rangle \in \sim\}$ *is an equivalence relation (i.e. a relation that is reflexive, symmetric and transitive);*

  (b) *for all* $w', w'' \in W$ *and for all* $t \in \bigcup_{w \in W} T_w$*, if* $w' \sim_t w''$*, then* $t \in T_{w'} \cap T_{w''}$*;*

  (c) *for all* $w', w'' \in W$ *and for all* $t \in \bigcup_{w \in W} T_w$*, if* $w' \sim_t w''$*, then* $\{t' \in \mathcal{T}(w') \mid t' <_{w'} t\} = \{t'' \in \mathcal{T}(w'') \mid t'' <_{w''} t\}$*;*

  (d) *for all* $w', w'' \in W$ *and for all* $t, t' \in \bigcup_{w \in W} T_w$*, if* $w' \sim_t w''$ *and* $t' < t$*, then* $w' \sim_{t'} w''$*.*

Obviously, every $T \times W$ frame is also a Kamp frame, viz. a Kamp frame in which every world $w \in W$ is associated with exactly the same linear series of times $\langle T, < \rangle$. As in the case of $T \times W$ frames, in a Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$, the indices of evaluation are pairs $\langle t, w \rangle$ consisting of a time and a world. Since the series of times may vary from world to world, it is, however, no longer the case that every time is part of the temporal structure of every world. For a pair $\langle t, w \rangle$ to constitute a suitable index of evaluation, it has therefore in addition to be required that $t \in T_w$.

In the branching time framework, complete possible courses of events are represented by histories; in Kamp frames and $T \times W$ frames, respectively, they are represented by possible worlds. Whereas histories are built up from moments, possible worlds are, first of all, unstructured entities. Possible worlds receive an internal structure only via the combination with the notion of time. By projecting linear series of times onto the various possible worlds, the latter are provided with an internal temporal structure. They are divided into temporal parts, so to say. The times themselves, however, are not identical to those temporal parts. Times are not constitutive of possible worlds. That the very same time can be part of the temporal structure of two different possible worlds does not mean that those possible worlds overlap. In fact, they do not even have to be accessible at that time. We have seen that we can also project a linear series of times onto the various histories in a branching time structure, identifying moments that correspond to the same clock time across histories. Yet, unlike possible worlds, histories have an internal structure irrespective of whether they are associated with a linear series of times.

From a logical point of view, the ontological difference between times and moments as well as the question of how complete possible courses of events relate to their temporal parts is of no importance.[42] All that matters from a logical point of view is the structure among the indices of evaluation. And that structure is the same independently of whether we take the possible worlds themselves or their temporal parts as ontologically prior, and hence independently of whether the internal temporal structure of possible worlds is provided by times or by the temporal parts themselves. In other words, rather than projecting onto each possible world a linear series of times, we could just as well conceive of possible worlds as linearly ordered sets of temporal parts. That is, we could define Kamp frames as disjoint unions of strict $<$-linear orders of temporal parts together with a suitable accessibility relation $\sim$. From the abstract point of view of logic, the $T \times W$ frame provided in Fig. 1.9 above is equivalent to the index structure provided in Fig. 1.10, and sometimes structures of the latter sort are referred to as Kamp frames as well.[43] In section 1.4.3.2, we will see that what matters though is that possible worlds with their internal linear temporal structure are ontologically prior to the modal universe as a whole: the pluriverse is essentially made up from possible worlds.

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
\langle t_5, w_1\rangle & & \langle t_5, w_2\rangle & & \langle t_5, w_3\rangle \\
\vee & & \vee & & \vee \\
\langle t_4, w_1\rangle & \sim & \langle t_4, w_2\rangle & & \langle t_4, w_3\rangle \\
\vee & & \vee & & \vee \\
\langle t_3, w_1\rangle & \sim & \langle t_3, w_2\rangle & \sim & \langle t_3, w_3\rangle \\
\vee & & \vee & & \vee \\
\langle t_2, w_1\rangle & \sim & \langle t_2, w_2\rangle & \sim & \langle t_2, w_3\rangle \\
\vee & & \vee & & \vee \\
\langle t_1, w_1\rangle & \sim & \langle t_1, w_2\rangle & \sim & \langle t_1, w_3\rangle
\end{array}$$

Figure 1.10: The index structure of a Kamp frame.

1.4.2.2 The representation of possibilities

In the previous section, we have illustrated how time can be accommodated in a possible worlds framework. We have introduced the notion of a $T \times W$ frame and its generalization, the notion of a Kamp frame, which bring out the basic idea very clearly. Both $T \times W$ frames and Kamp frames rest on a primitive

[42] This becomes especially vivid if one considers a further generalization of Kamp frames, viz. so-called neutral frames, where the temporal series of times associates with different worlds are considered to be disjoint. For the definition of a neutral frame, see Thomason (1984, p.149).

[43] See, for example, Reynolds (2002, 2003), where also a precise definition of those kinds of structures is provided.

set of possible worlds and a primitive notion of time. By combining the two notions, each possible world is bestowed with a temporal structure.

In this section, we discuss the representation of possibilities in a possible worlds framework. Whether the structure under consideration is a $T \times W$ frame or a Kamp frame does thereby not matter. All that matters is the general idea that is common to both kinds of structures, and we therefore speak more broadly of possible worlds frameworks. As in our discussion of the representation of possibilities in the branching time framework in section 1.4.1.1, we differentiate again between a global and a local perspective. In section 1.4.2.2.1, we start out with the global perspective, and we turn to the local perspective in section 1.4.2.2.2.

1.4.2.2.1 A global perspective So let us take a standpoint outside a given possible worlds framework. From that global, external, God's eye view perspective, we overlook the entire pluriverse, which is laid out in front of us. We behold multiple possible worlds, which are lying there, one next to the other. Time boils down to the earlier-later relation that is built into the possible worlds themselves. Each possible world is maximally extended in time and exhibits a linear temporal structure. What unifies the pluriverse as a whole are modal relations between possible worlds at various times. The corresponding picture is provided in Fig. 1.11

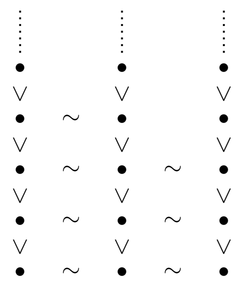


Figure 1.11: A global perspective on a possible worlds framework.

The picture is an entirely static one. All possible worlds are on a par, and so are all their temporal stages. There is no actuality, no distinguished present, no past and no future. All there is, is the side by side of possible worlds, each endowed with an inner-worldly temporal structure and connected among each other by inter-worldly modal relations. Unlike in the case of branching time, there is no risk to mistake the static picture of the pluriverse for an eternalist picture of the world. Nobody would ever want to hold that the totality of temporally unrelated modal chunks represents our world or that it depicts time.

Clearly, what a possible worlds framework portrays are possibilities. Every possible world of the pluriverse stands in for one possibility. The possibilities represented by those possible worlds correspond to complete possible evolutions. Every possible world spans an entire possible temporal development and hence represents a global possibility. The various possible worlds are thereby inclusive in time. There are no temporal relations across different possible worlds, only modal ones. Possible worlds represent possibilities for other possible worlds, which are temporally closed as well. The modal accessibility relation between possible worlds specifies, for each possible world, which worlds are considered modal alternatives of that world. The modal alternativeness-relation is not an absolute one, however, but it is relativized to times. Which possible worlds constitute modal alternatives of a given world may vary from time to time. As time progresses, fewer and fewer possible worlds are accessible—or, at least, their range never increases. The relativization to times renders possible worlds that are accessible at a certain time historical alternatives. They are modal alternatives given the past course of events. Even though which possible worlds are accessible from a given world is time-dependent, the accessible worlds nevertheless constitute global, modal alternatives to the complete possible course of events represented by the given world.

1.4.2.2.2 A local perspective As we have seen, when viewed from an outside perspective, a possible worlds framework depicts in completely general terms all possible possibilities. This is to say, it pins down, for each possible world, which possible worlds constitute modal alternatives of that world at which times. Let us now consider what happens if we take a local standpoint within a possible worlds framework. The local perspective is the perspective that guides a semantic approach. For, it is the perspective from which language is used.

In the case of a possible worlds framework, taking an inside perspective requires that we locate ourselves in the linear temporal structure of some possible world. By positioning ourselves within a possible world, we thereby mark that world as the actual world. It is the world we live in. The actual world comprises everything that actually happens in the course of time. It spans the entire actual temporal development. The notion of actuality involved here is, first of all, a purely modal one that is atemporal in nature. However, since worlds are temporally structured, yet another notion of actuality enters the picture. Our position in the actual world is a standpoint at a certain point in time, and that time is thereby marked as present. The present, in turn, fixes a unique past, and together, present and past provide a temporal notion of actuality. They capture what has happened so far. To be sure, since worlds are linearly extended in time, the present also determines a unique future. That future is part of the actual world and hence of modal actuality. It is not, however, part of the local temporal actuality. Our local standpoint in the temporal structure of the actual world determines a range of possible worlds that are presently accessible. Those possible worlds represent historical possibilities for the actual world. Worlds that are presently not accessible from the actual world might be dubbed couterfactualities. They may either represent what once has been possible from the perspective of the actual world or what would have been possible, had another world been actual. The partition of the structure in modal and temporal actuality, possibility and counterfactuality is presented in Fig. 1.12. The thick line represents the modal actuality, the very thick one marks the temporal actuality, and the thin line depicts a possibility, whereas the dashed line corresponds to a counterfactuality.

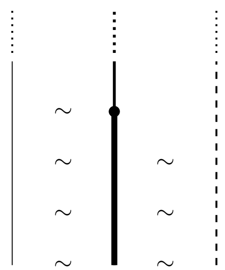


Figure 1.12: A local perspective on a possible worlds framework.

Let us consider in a bit more detail what kind of notion of possibility is represented here and how the two different notions of actuality interrelate. Given our local standpoint in the temporal structure of the actual world, there may be various possible worlds that are modally accessible at that time, and each such possible world represents a historical possibility for the actual world. The multiplicity of those possible worlds is supposed to give expression to the idea that the future is open. In a possible worlds framework, the idea of an open future is not, and cannot be, tantamount to the claim that our world lacks a unique future. After all, all possible worlds are linearly extended in time hence come with a unique past as well as with a unique future. Since possible worlds are maximally inclusive in time, as of now, it is fixed what actually will be the case. Alternatives for the future can only enter the picture via the various possible worlds that are presently considered historical alternatives to the actual world. They are represented by complete possible courses of events that, as of the present moment, constitute historical alternatives to the unique actual temporal development of the world. Possibilities for the future, thus understood, are modal rather than temporal alternatives, and they are necessarily global. Our local standpoint in time only determines which possible worlds are considered possibilities, but those possibilities are possibilities for the complete actual course of events. They are not alternatives for the temporal actuality to evolve but rather constitute alternatives to the overarching modal actuality with respect to the temporal one. The temporal actuality merely fixes the range of modal alternatives. While the modal actuality is static, the temporal actuality is dynamic. When time progresses, the temporal actuality evolves and the range of possible worlds that are considered alternatives to the static modal actuality diminishes accordingly. And there is but one possibility for the temporal actuality to evolve: the one dictated by the modal actuality.

1.4.3 Branching time vs. possible worlds

In section 1.4.1, we have emphasized the adequacy of the branching time framework for the formal representation of real possibilities, and in section 1.4.2, we have introduced the possible worlds framework and discussed how future possibilities are represented within that framework. In this section, we point out crucial differences between the theory of branching time and the possible worlds approach. The differences we discuss challenge the adequacy of the possible worlds framework as a basis for the formal representation of real possibilities. This is certainly not to say that the possible worlds framework is of no avail. The possible worlds framework surely provides a powerful tool for modeling various notions of possibility and necessity. Our argument is only to the effect that the combination of modality and time in that framework is not suited for the particular notion of real possibility that we aim to model.

The framework of branching time and the possible worlds framework crucially differ in their representation of the interrelation of modality and time.

That disparity leads to both conceptual and logical differences between the two frameworks. We show that there are certain modifications of the theory of branching time that in fact render the two frameworks conceptually and logically equivalent. Yet, we argue that they do so only at the expense of the advantages the branching time framework has over the possible worlds framework in the first place. In section 1.4.3.1, we focus on conceptual differences between the theory of branching time and the possible worlds framework. We introduce the theory of the so-called Thin Red Line and demonstrate that while that theory does away with any conceptual differences concerning the representation of future possibilities, it sacrifices the idea that the future is genuinely open. In section 1.4.3.2, we then turn to logical differences between the theory of branching time and the possible worlds approach. We introduce the notion of a so-called bundled tree, which provides a logical equivalent to the notion of a Kamp frame. The trade-off is the negligence of the rich structural resources a branching time structure has to offer.

1.4.3.1 Conceptual differences

Both the framework of branching time and the possible worlds framework allow for a formal representation of historical possibilities and accommodate the idea that the future is open. However, the two frameworks crucially differ with respect to how the idea of an open future is made precise in each case. There are fundamental differences in the conceptual representation of future possibilities between the two frameworks. In section 1.4.3.1.1, we make those differences explicit. In section 1.4.3.1.2, we then show that once a so-called Thin Red Line is introduced into the framework of branching time, the differences disappear. However, as we shall see, with the differences, we also lose what makes the theory of branching time so appealing for the representation of real possibilities.

1.4.3.1.1 Histories, worlds and the open future We have seen that in the case of branching time, our local standpoint in the tree of moments provides us with a notion of temporal actuality that comprises but the present and the past and is essentially dynamic in nature. The various branches that are spread out before us represent possibilities for the future. They are temporal alternatives: alternatives for the temporal actuality to unfold. And all of them are equally possible.

In the case of a possible worlds framework, on the other hand, our local standpoint in the structure does not only determine a notion of temporal actuality but, at the same time, also provides us with a notion of modal actuality. Future possibilities are represented by possible worlds that constitute historical alternatives to the actual world. They are modal alternatives: alternatives to the modal actuality given the temporal one. And there is but one possibility for the temporal actuality to unfold.

The idea of an open future is a different one in each case. In the theory of branching time, where future possibilities are temporal alternatives, the future is genuinely open. There is no actual future yet, and each of the various branches that are lying before us can equally well be actualized. There is no fact of the matter yet what the future will bring. In a possible worlds framework, in contrast, where future possibilities are modal alternatives, the future is not genuinely open in an objective, ontological sense. As of the present moment, it is settled what actually will be the case. There is an actual future even though that actual future is not part of temporal actuality yet. The actual future is nevertheless part of modal actuality, which uniquely constrains the development of the dynamic, temporal actuality. Rather than constituting genuine alternatives for the future to unfold, alternative future possibilities represent ways the future could have been, given the past course of events, but actually is not.

We have characterized real possibilities as possibilities for the future that are genuinely open and can equally be actualized. As should be clear by now, the representation of future possibilities in a possible worlds framework runs counter to that idea. In the possible worlds framework, from the local perspective, it is already settled which of the alternative future possibilities will be actualized in the end. While the possible worlds framework provides a rigorous notion of historical possibility, is does not do justice to the idea that there are genuinely open alternatives for the future. The idea of modal alternatives that is at the bottom of the possible worlds approach is certainly fruitful for many different kinds of possibility. However, for modeling real possibilities as genuine open future possibilities, the framework is of no avail. Merely providing possible worlds with a temporal structure and relativizing the modal accessibility relation to times does not suffice.

A proponent of the possible worlds approach may object to our above verdict and claim that the various possibilities for the future really are alternatives for actuality rather than mere modal alternatives to a given actuality. Such a claim, however, makes perfect sense if, and only if, the possibilities for the future are understood as epistemic rather than ontological possibilities. That is, if there is epistemic uncertainty about which of the various possible worlds represents the actual world. In the case of a possible worlds framework, epistemic uncertainty comes down to a problem of self-location in the possible worlds pluriverse. And, of course, if the agent does not know which world he is presently located in, then he may also be uncertain about what the future will bring. From an objective point of view, however, his standpoint in the pluriverse is uniquely fixed and with it also the unique actual future course of events. While the alternative possibilities for the future may be equally possible from an agent's subjective point of view, viz. given his state of knowledge, they are not equally possible from an ontological point of view. In the branching time framework, on the contrary, it is not only epistemically but also ontologically indeterminate what the future will bring. There simply is no fact of the matter which of the various histories passing through the present moment will be actualized in the end. All of them are equally possible, and ontologically so. This is exactly the so-called initialization problem that the Ockhamist semantics is facing: a local context does not suffice to provide an initial value for the history parameter. We will come back to that 'problem' in section 2.5.3. In fact the problem is not a problem but a mark of objective indeterminism.

1.4.3.1.2 The Thin Red Line At first glance, it might seem that the difference between the branching time framework and the possible worlds framework with regard to the idea of the open future is essentially due to the fact that in branching time, future possibilities are possibilities in the temporal structure of the world whereas possible worlds are temporally unrelated modal chunks. In this section, we illustrate that it is rather the presence of a temporally global, modal actuality that makes the difference. We show that once a so-called Thin Red Line is inserted into the branching time picture, we are conceptually back to the possible worlds approach.

The theory of the Thin Red Line rests on the assumption that from all the various histories depicted in a branching time structure, one can be singled out as 'the actual history'. That unique history, which is supposed to represent the complete actual temporal development of the world, is dubbed the Thin Red Line. We can define a branching time structure with a Thin Red Line as a triple $\mathcal{M}_{\mathsf{TRL}} = \langle \mathcal{M}, <, \mathsf{TRL} \rangle$ where $\mathcal{M} = \langle M, < \rangle$ is a BT structure and TRL is a unique history, viz. the history that captures the complete actual course of events.

Definition 1.14 (BT+TRL structure). *A* BT+TRL structure *is a triple* $\mathcal{M}_{\mathsf{TRL}} = \langle M, <, \mathsf{TRL}\rangle$ *where* $\mathcal{M} = \langle M, <\rangle$ *is a BT structure and* $\mathsf{TRL} \in \mathsf{hist}(\mathcal{M})$ *is a history.*

It is plausible, though, to assume that the notion of a Thin Red Line is not an absolute one but depends on our internal standpoint in a branching time structure, just as the notion of the actual world in a possible worlds framework depends on our position in the pluriverse.[44] The crucial difference with the possible worlds framework, of course, consists in the fact that our local standpoint at a moment still leaves open several possibilities for the Thin Red Line, whereas once we position ourselves in the possible worlds pluriverse, the actual world is uniquely determined. Marking one of the histories passing through the present moment as the 'the actual history' requires a perspective from the end of time where it is settled which of the various histories is eventually actualized.

Let us see how the introduction of a Thin Red Line into a branching time framework bears on the representation of future possibilities and hence on the idea that the future is open. As before, our local standpoint in the tree of moments marks a temporal kind of actuality. But now the multiple branches lying ahead of us are no longer on a par. One of them has a distinguished status and is marked as 'the actual future', viz. the one that is part of the Thin Red Line. As a consequence, the remaining branches we are facing can no longer constitute temporal alternatives for the dynamic actuality to evolve. To be sure, those remaining branches still represent future possibilities, but they are future possibilities of a fundamentally different kind: they are no longer genuine alternatives for the future. There is but one possibility for the temporal actuality to unfold.

The introduction of a Thin Red Line assimilates the representation of future possibilities in a branching time framework to their respective representation in the possible worlds framework. By inserting a Thin Red Line into a branching time structure, in addition to the temporal actuality, a modal actuality enters

[44]There are various versions of the Thin Red Line theory to be found in the literature. First of all, there is the theory of an absolute Thin Red Line. That theory suffers from the defect that it cannot deal with utterances about the future made at contexts that lie off the Thin Red Line. Then, there is the idea of a context-relative Thin Red Line, which, however, runs into similar problems: it cannot deal with nested sentences about the future. Finally, there is the idea of a moment-relative Thin Red Line. Here, the idea is that at every moment, one of the immediate possible future continuations has a distinguished status. For an overview of the theory of the Thin Red Line and its variants, see Belnap, Perloff, and Xu (2001, ch.II.6). Versions of the Thin Red Line theory are defended, for example, in Øhrstrøm (2009), Malpass and Wawer (2012) and Wawer (2014). For a critical discussion of the Thin Red Line, see Belnap and Green (1994) and Belnap, Perloff, and Xu (2001, ch.II.6).

the picture. Whereas the temporal actuality comprises only what has happened so far, the modal actuality captures the entire actual temporal development of the world. In the presence of an overarching modal actuality such as provided by the Thin Red Line, future possibilities degenerate to modal alternatives. Possibilities for the future are represented by histories that constitute modal alternatives to the Thin Red Line. Once the Thin Red Line has entered the picture, there are no temporal alternatives for temporal actuality to be had. The temporal actuality merely determines which histories presently count as modal alternatives to the Thin Red Line, namely all those that overlap with the Thin Red Line at the present moment. In other words, the various branches unfolding before us represent possibilities for the future only in virtue of their being part of histories that, as of the present, are historical alternatives to the Thin Red Line. Histories that no longer constitute historical alternatives may again be considered counterfactualities. The partition of the branching time structure into modal and temporal actuality, possibility and counterfactuality is provided in Fig. 1.13. The thick line represents the modal actuality, viz. the Thin Red Line, the very thick line marks the temporal actuality, and the thin line depicts a possibility, while the dashed line corresponds to a counterfactuality.

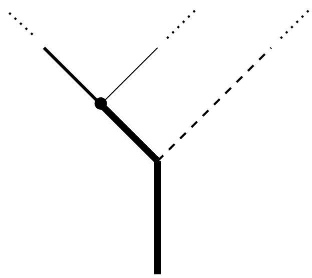


Figure 1.13: The Thin Red Line.

The analogy with the representation of future possibilities in the possible worlds framework is obvious. As in the case of the possible worlds framework, the meaning of actuality is twofold. A static, modal actuality is juxtaposed to the dynamic, temporal actuality, which is determined by the local standpoint in time, and the sole function of the temporal actuality remains to determine which complete possible courses of events represent historical alternatives to the modal actuality. Again, future possibilities are mere modal alternatives to the overarching modal actuality relative to the temporal one. The difference that of course remains is that in the case of branching time, the Thin Red Line has to be added artificially from a perspective from the end of time, whereas in the possible worlds framework, the modal actuality flows directly from the internal perspective. At any rate, as soon as the Thin Red Line is there, we lose the very motivation of employing the branching time framework in the formal representation of real possibilities in the first place: there is no room anymore for genuine temporal alternatives for the future.

1.4.3.2 Logical differences

One might think that the difference between the branching time framework and the possible worlds framework is a purely conceptual one and that from a logical point of view, the two frameworks come down to essentially the same. After all, there is a close resemblance between histories and their constitutive moments, on the one hand, and possible worlds with their internal temporal structure, on the other. Whether we evaluate sentences at moment-history pairs in a branching time structure as in the standard Ockhamist semantics or with respect to pairs consisting of a time and a world in a possible worlds framework should not make too much of a difference.

In this section, we illustrate that the difference between the branching time framework and the possible worlds framework is not a purely conceptual one, but that the two frameworks in fact differ also logically. To wit, there are Ockhamist validities that are not valid with respect to the class of Kamp frames. It is a well-established result that validity with respect to the class of Kamp frames coincides with validity with respect to the class of so-called bundled trees rather than with Ockhamist validity. The restriction to bundled trees smoothes the logical differences between the branching time framework and the possible worlds framework, but it also makes apparent how much of the rich structural resources of a branching time structure we have to give away in order to assimilate the two frameworks. We point out the logical differences between the branching time framework and the possible worlds approach in section 1.4.3.2.1 and we turn to the correspondence with bundled trees in section 1.4.3.2.2.

1.4.3.2.1 Histories, worlds and validity In section 1.4.2.1.2, we have briefly commented on the index structure of a Kamp frame, and we said that from a logical point of view it does not matter whether we conceive of possible worlds as unstructured entities that are provided an internal temporal structural only by the combination with times or whether we view possible worlds as strict linear sets of temporal parts, in analogy with histories. From a logical point of view, the ontological difference between times and moments is of no significance. What is of importance, however, is that in the possible worlds framework, the temporally structured possible worlds are considered primitive elements in the modal structure whereas in a branching time framework, the modal structure is carved out from the temporal one. The basic constituents of a branching time structure are moments, and histories are defined elements in the tree-like ordering of moments. In the following, we will illustrate how that difference bears on the notion of validity.

Consider the Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$ and the branching time structure $\mathcal{M} = \langle M, < \rangle$ provided in Fig. 1.14. Obviously, those two structures validate exactly the same Ockhamist sentences. There is a natural one-to-one correspondence between the possible worlds $w \in W$ in the given Kamp frame $\mathcal{K}$ and the histories $h \in \mathsf{hist}(\mathcal{M})$ in the given branching time structure $\mathcal{M}$ such that all relevant relations are preserved. We can define a surjective mapping $\delta : \langle t, w \rangle \mapsto m$ between the set of indices in $\mathcal{K}$, say $\mathsf{Ind}(\mathcal{K})$, and the set of moments M in $\mathcal{M}$ such that for all $w', w'' \in W$ and $t', t'' \in T_{w'}$ (i) $t' < t''$ iff $\delta(\langle t', w' \rangle) < \delta(\langle t'', w' \rangle)$ and (ii) $\langle w', w'' \rangle \in \sim_{t'}$ iff $\delta(\langle t', w' \rangle) = \delta(\langle t', w'' \rangle)$. The function δ maps every possible world w in $\mathcal{K}$, i.e. every $<$-linearly ordered set $\{\langle w, t \rangle \mid t \in T_w\}$, onto an order isomorphic history h in $\mathcal{M}$ in such a way that whenever two possible worlds are accessible at a given time, the corresponding histories overlap at the respective moment. The function δ thus ensures that the structure of the image histories in $\mathcal{M}$ perfectly matches the structure of possible worlds in $\mathcal{K}$. Whenever, for two structures $\mathcal{K}$ and $\mathcal{M}$, there exists such a function δ, we say that the structures $\mathcal{K}$ and $\mathcal{M}$ are *δ-related*.

Definition 1.15 (δ-related). *Let $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$ be a Kamp frame, and let $\mathcal{M} = \langle M, < \rangle$ be a BT structure. We say that $\mathcal{K}$ and $\mathcal{M}$ are* δ-related *if there is a surjective function*

$$\begin{array}{rccc} \delta : & \mathsf{Ind}(\mathcal{K}) & \to & M \\ & \langle t, w \rangle & \mapsto & m \end{array}$$

such that for all $w', w'' \in W$ and $t', t'' \in T_{w'}$ the following holds:

(i) $t' < t''$ *iff* $\delta(\langle t', w' \rangle) < \delta(\langle t'', w' \rangle)$;

(ii) $\langle w', w'' \rangle \in \sim_{t'}$ *iff* $\delta(\langle t', w' \rangle) = \delta(\langle t', w'' \rangle)$.

In our example, the possible world w_1 is mapped onto the history h_1, w_2 onto h_2 and w_3 onto h_3. That is, every history $h \in \mathsf{hist}(\mathcal{M})$ in the given branching time structure $\mathcal{M}$ is the δ-image of exactly one possible world $w \in W$ in the Kamp frame $\mathcal{K}$. Consequently, the function δ induces a bijection between the indices of evaluation in $\mathcal{K}$ and the moment-history pairs in $\mathcal{M}$ that preserves satisfiability: every sentence $\phi \in \mathcal{L}_\circ$ that is satisfiable in the one structure is also satisfiable in the other. However, the result does not generalize, as we shall see. There are pairs of Kamp structures and BT structures that are δ-related without there being a one-to-one correspondence between worlds and histories so that preservation of satisfiability fails.

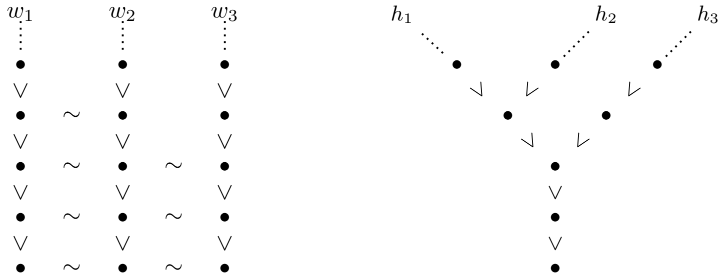


Figure 1.14: Histories and possible worlds.

To be sure, for every branching time structure $\mathcal{M} = \langle M, < \rangle$, there is a δ-related Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$ such that there is a one-to-one correspondence between the set of histories $\mathsf{hist}(\mathcal{M})$ in $\mathcal{M}$ and the set of possible worlds W in $\mathcal{K}$. Just let W be $\mathsf{hist}(\mathcal{M})$, extract $\mathcal{T}$ from the temporal order $<$ on M and define $\sim$ in terms of overlap of histories, i.e. for all $h, h' \in \mathsf{hist}(\mathcal{M})$, we set $\mathcal{T}(h) = \langle h, < \rangle$ and $\langle h, h' \rangle \in \sim_m$ iff $m \in h \cap h'$. However, the converse does not hold. That is to say, it is not the case that for every Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$, there is a δ-related BT structure $\mathcal{M} = \langle M, < \rangle$ such that every history $h \in \mathsf{hist}(\mathcal{M})$ is the δ-image of exactly one possible world $w \in W$. The transition from a Kamp frame to a δ-related branching time structure involves merging all time-world pairs whose respective possible worlds are $\sim$-related at a given time into a single moment. In order to end up with a single tree, we, first of all, have to ensure that any two possible worlds are $\sim$-related at some time. Otherwise, the resulting branching time structure is not connected. Moreover, if the resulting branching time structure is supposed to be jointed, we even have to require that there be a maximal such time in each case. In order to rule out that the resulting branching time structure contains a last moment, it must furthermore be guaranteed that the strict linear orders $\langle T_w, <\rangle$ associated with the various possible worlds $w \in W$ are serial. What is more, we have to exclude a case in which there are two possible worlds $w, w' \in W$ that are $\sim$-related at every single time $t \in T_w \cap T_{w'}$. For, those two possible worlds would obviously collapse into the very same history. More importantly, however, the resulting branching time structure can also contain histories that are not identical to the δ-image of any possible world of the Kamp frame. In what follows, we restrict our considerations to Kamp frames that fulfill the above provisos of jointedness, seriality and distinctness, and we show that by merging possible worlds into a tree, additional histories may emerge.

For concreteness, consider a Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim\rangle$ such that the set W contains for every $n \in \mathbb{N} \setminus \{0\}$, a possible world w_n and assume that the linear series of times $\mathcal{T}(w_n)$ associated with each such possible world is order isomorphic to the natural numbers $\mathbb{N}$. Assume moreover that the time-relative accessibility relation $\sim$ on $W \times W$ is such that at every time $m \in \mathbb{N}$, it groups together all worlds w_n such that $n \geq m$ (where $n \geq 1$ and $m \geq 0$); so $\sim_m = \{\langle w_k, w_l\rangle \mid k, l \geq m\}$. If we merge all time-world pairs whose respective possible worlds are $\sim$-related at a given time into a single moment, in the limit, a new history h_ω emerges, for which there was no corresponding possible world. Even though the two structures depicted in Fig. 1.15 are δ-related, there is no one-to-one correspondence between the possible worlds and histories in those two structures. The phenomenon of emergent histories is due to the fact that, unlike possible worlds, histories are defined elements in the modal-temporal structure. They are maximal chains in the tree-like ordering of moments, whereas possible worlds and their linear temporal structure are considered primitive modal elements.

The two structures provided in Fig. 1.15 do not validate the same Ockhamist sentences. Consider the following counterexample, which we take from Reynolds (2002). Assume that in the Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim\rangle$, as specified above, the sentence p is true at all and only those time-world pairs $\langle t_m, w_n\rangle$ such that $n \geq m$ (where $n \geq 1$ and $m \geq 0$). Assume an analogous valuation on the δ-related branching time structure $\mathcal{M} = \langle M, <\rangle$. That is, assume that in that structure, the sentence p is true at the moment $\delta(\langle t_m, w_n\rangle)$ iff p is true at $\langle t_m, w_n\rangle$. Given our constraints on the function δ and the time-relative accessibility relation $\sim$ in $\mathcal{K}$, that condition yields a moment-dependent val-

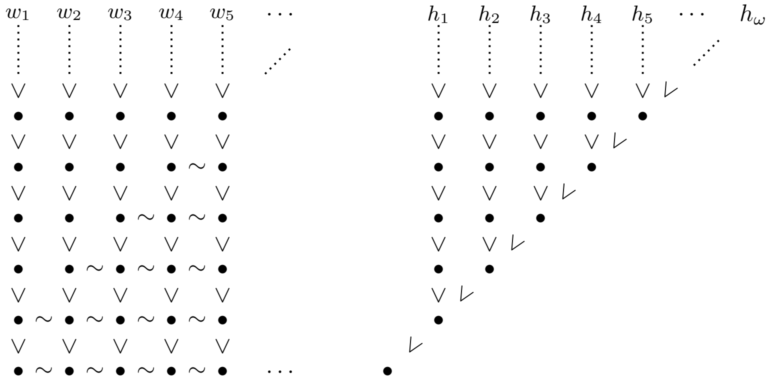


Figure 1.15: Emergent histories.

uation on the branching time structure $\mathcal{M}$: p is true at all and only those moments $m \in M$ that are contained in the emerging history h_ω. The corresponding models are as depicted in Fig. 1.16 below. In Reynolds (2002), the formula $\Box_\circ \mathsf{G}_\circ(p \to \Diamond_\circ \mathsf{F}_\circ p) \to \Diamond_\circ \mathsf{G}_\circ(p \to \mathsf{F}_\circ p)$ is provided. That formula is an Ockhamist validity and is thus valid in the BT structure $\mathcal{M} = \langle M, < \rangle$. But it is not valid in the given model on our Kamp frame. The sentence $\Box_\circ \mathsf{G}_\circ(p \to \Diamond_\circ \mathsf{F}_\circ p) \to \Diamond_\circ \mathsf{G}_\circ(p \to \mathsf{F}_\circ p)$ is false at the time-world pair $\langle t_0, w_1 \rangle$. Hence, Ockhamist validity is distinct from validity with respect to the class of Kamp frames.[45]

1.4.3.2.2 Bundled trees The notion of a bundled tree provides the logical analogue to a Kamp frame. Just as a Kamp frame builds on a primitive set of possible worlds, a bundled tree rests on a primitive set of histories. Bundled trees provide a remedy for the problem of emergent histories: validity with respect to the class of Kamp frames is equivalent to bundled tree validity.

A *bundled tree* is defined as a triple $\mathcal{B} = \langle M, <, B \rangle$ consisting of a BT structure $\mathcal{M} = \langle M, < \rangle$ and a non-empty set of histories $B \subseteq \mathsf{hist}(\mathcal{M})$ that is required to be such that it covers the entire BT structure. That is, every moment $m \in M$ must be contained in at least one history $h \in B$. A set of

[45] In Burgess (1978, 1979), the formula $\Box_\circ \mathsf{G}_\circ \Diamond_\circ \mathsf{F}_\circ \Box_\circ p \to \Diamond_\circ \mathsf{G}_\circ \mathsf{F}_\circ p$ is discussed. That formula is also Ockhamist valid but falsifiable in a Kamp frame. Yet, contrary to what is said in Thomason (1984, pp.151f.), a counterexample to that formula requires a model on a structure more complex than the one we are considering here. The structure that is considered in Thomason (1984), is, however, even simpler than the structure we consider: the $<$-minimal row of times or moments, respectively, is omitted. Note that in a model on such a structure not even Reynold's formula can be shown to be falsifiable.

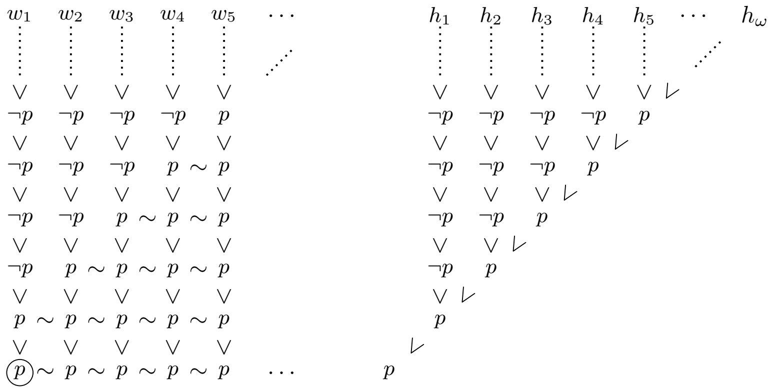


Figure 1.16: Counterexample: Kamp validity vs. Ockhamist validity.

histories in $\mathsf{hist}(\mathcal{M})$ that fulfills that condition is called a *bundle*. What is important is that the bundle B does not have to be identical to the set of histories $\mathsf{hist}(\mathcal{M})$ in the underlying BT structure. Whenever $B = \mathsf{hist}(\mathcal{M})$, the bundled tree $\mathcal{B}$ is said to be *complete*. We denote the class of all bundled trees by $\mathscr{B}$.

Definition 1.16 (Bundled tree). *A* bundled tree *is an ordered triple* $\mathcal{B} = \langle M, <, B\rangle$ *where* $\mathcal{M} = \langle M, <\rangle$ *is a BT structure and* $B \subseteq \mathsf{hist}(\mathcal{M})$ *is a non-empty set of histories such that for every* $m \in M$*, there is some history* $h \in B$ *such that* $m \in h$*. A bundled tree* $\mathcal{B} = \langle M, <, B\rangle$ *is* complete *iff* $B = \mathsf{hist}(\mathcal{M})$*. Let* $\mathscr{B}$ *be the* class of all bundled trees.

Consider the branching time structure $\mathcal{M} = \langle M, <\rangle$ provided in Fig. 1.15 above. That branching time structure contains a history h_ω isomorphic to the natural numbers $\mathbb{N}$, and for every moment $m_i \in h_\omega$ with $i \in \mathbb{N} \setminus \{0\}$, there is a history h_i such that $h_\omega \perp_{m_i} h_i$. We have $\mathsf{hist}(\mathcal{M}) = \{h_\omega\} \cup \{h_i \mid i \in \mathbb{N} \setminus \{0\}\}$. Note that every moment $m \in M$ is contained in at least one of the histories h_i. The history h_ω is not needed in order to span the entire tree of moments and can be 'omitted'. The BT structure $\mathcal{M} = \langle M, <\rangle$ together with the bundle $\mathsf{hist}(\mathcal{M}) \setminus \{h_\omega\}$ constitutes a bundled tree $\mathcal{B} = \langle M, <, \mathsf{hist}(\mathcal{M}) \setminus \{h_\omega\}\rangle$ according to Def. 1.16.

The bundle B does of course not really 'add' anything to the BT structure $\mathcal{M} = \langle M, <\rangle$. It merely constrains the range of histories that are employed as a second parameter of truth in the semantic evaluation. In a bundled tree $\mathcal{B} = \langle M, <, B\rangle$, sentences of the Ockhamist language $\mathcal{L}_{\mathsf{o}}$ are evaluated at moment history-pairs, just as in the standard Ockhamist semantics, but now only histories $h \in B$ are taken into account: the indices of evaluation are restricted to pairs m/h with $m \in M$ and $h \in B$ such that $m \in h$. A *bundled tree model* $\mathfrak{B} = \langle M, <, B, v_{\mathsf{o}} \rangle$ is a bundled tree $\mathcal{B} = \langle M, <, B \rangle$ together with a valuation function v_{o} that assigns truth values to the propositional variables $p \in \mathsf{At}$ relative to a moment $m \in M$ and a history $h \in B$ such that $m \in h$.

Definition 1.17 (Bundled tree model). *A* bundled tree model *is an ordered quadruple* $\mathfrak{B} = \langle M, <, B, v_{\mathsf{o}} \rangle$ *consisting of a bundled tree* $\mathcal{B} = \langle M, <, B \rangle$ *and a valuation function* $v_{\mathsf{o}} : \mathsf{At} \times \{m/h \mid m \in M \text{ and } h \in B \text{ s.t. } m \in h\} \to \{0, 1\}$.

The following semantic clauses extend the valuation v_{o} on the set of propositional variables At to all sentences $\phi \in \mathcal{L}_{\mathsf{o}}$. We use $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \phi$ in order to indicate that a sentence $\phi \in \mathcal{L}_{\mathsf{o}}$ is true in a bundled tree model $\mathfrak{B} = \langle M, <, B, v_{\mathsf{o}} \rangle$ on a bundled tree $\mathcal{B} = \langle M, <, B \rangle$ at the index pair m/h according to the Ockhamist semantics. The notation extends to $\mathfrak{B} \vDash_{\mathsf{o}} \phi$, $\mathcal{B} \vDash_{\mathsf{o}} \phi$ and $\mathscr{B} \vDash_{\mathsf{o}} \phi$ in the obvious way.

(At) $\mathfrak{B}, m/h \vDash_{\mathsf{o}} p$ iff $v_{\mathsf{o}}(p, m/h) = 1$;

($\neg$) $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \neg\phi$ iff $\mathfrak{B}, m/h \nvDash_{\mathsf{o}} \phi$;

($\wedge$) $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \phi \wedge \psi$ iff $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \phi$ and $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \psi$;

(F_{o}) $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \mathsf{F}_{\mathsf{o}}\phi$ iff there is some $m' \in h$ s.t. $m' > m$ and $\mathfrak{B}, m'/h \vDash_{\mathsf{o}} \phi$;

(P_{o}) $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \mathsf{P}_{\mathsf{o}}\phi$ iff there is some $m' < m$ s.t. $\mathfrak{B}, m'/h \vDash_{\mathsf{o}} \phi$;

($\Box_{\mathsf{o}}$) $\mathfrak{B}, m/h \vDash_{\mathsf{o}} \Box_{\mathsf{o}}\phi$ iff for all $h' \in B$ s.t. $m \in h'$, $\mathfrak{B}, m/h' \vDash_{\mathsf{o}} \phi$.

In a bundled tree, the domain of the inevitability operator $\Box_{\mathsf{o}}$ is restricted to the bundle B. The semantic clauses of the truth-functional connectives and the temporal operators remain intact as those do not involve a shift of the history parameter.[46] While the formula $\Box_{\mathsf{o}}\mathsf{G}_{\mathsf{o}}(p \to \Diamond_{\mathsf{o}}\mathsf{F}_{\mathsf{o}}p) \to \Diamond_{\mathsf{o}}\mathsf{G}_{\mathsf{o}}(p \to \mathsf{F}_{\mathsf{p}}p)$ is an Ockhamist validity and thus valid in the BT structure $\mathcal{M} = \langle M, < \rangle$ provided in Fig. 1.16, it is not valid in the model on the corresponding bundled tree $\mathcal{B} = \langle M, <, \mathsf{hist}(\mathcal{M}) \setminus \{h_\omega\} \rangle$ with the missing history h_ω. It is false in that model at the <-minimal moment m_0, just as it is falsifiable in the corresponding model on the corresponding Kamp frame $\mathcal{K}$.

[46] There are philosophical discussions as to the appropriateness of bundled trees, i.e. trees with missing histories. See, for example, Nishimura (1979) and Belnap, Perloff, and Xu (2001, ch.7A.6).

Ockhamist validity is equivalent to validity with respect to the class of complete bundled trees. Kamp validity, on the other hand, is equivalent to validity with respect to the class of all bundled trees.[47] First, for every bundled tree $\mathcal{B} = \langle M, <, B \rangle$, there is a δ-related Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$ that validates exactly the same Ockhamist sentences. Just let $W = B$, extract $\mathcal{T}$ from the temporal order $<$ on M and define $\sim$ in terms of overlap of histories. And now, also the converse holds—at least under the provisos of jointedness, seriality and distinctness discussed above: for every Kamp frame $\mathcal{K} = \langle W, \mathcal{T}, \sim \rangle$ that fulfills our provisos, there is a δ-related bundled tree $\mathcal{B} = \langle M, <, B \rangle$ such that every history of the bundle is the δ-image of exactly one possible world in W. Hence, there is a bijection between the indices of evaluation in $\mathcal{K}$ and the indices of evaluation in $\mathcal{B}$ that preserves satisfiability, and so validity on the two structures coincides.

We have seen that the logical differences between Kamp frames and branching time structures are due to the fact that the histories that are employed as a second parameter of truth in the Ockhamist semantics are defined elements in a branching time structure whereas in a Kamp frame, one starts out with a primitive set of possible worlds. The transition to bundled trees dispels that difference. A bundled tree rests on a primitive set of histories, just as a Kamp frame rests on a primitive set of possible worlds. However, the modal-temporal structure of the world that is represented by a branching time structure is much richer than that it would just allow us to carve out histories. A branching time structure does not only allow for histories as global possibilities but also allows for local future possibilities in the tree-like ordering of moments. And the second parameter of truth employed in the semantic evaluation does not have to be a history, which represents a complete possible course of events. That Kamp validity is equivalent to bundled tree validity makes apparent how poor a possible worlds framework is in the first place: the expense of assimilating the possible worlds framework and the theory of branching time is the loss of the rich structural resources a branching time structure has to offer. And as soon as one abandons the idea that the second parameter of truth must be a history, the overall similarity between the possible worlds approach and the branching time framework disappears altogether.

[47]For a thorough proof of that claim, see Reynolds (2002). We restrict ourself here to a mere sketch of the proof.

1.5 Concluding remarks

In this chapter, we have provided the general background for this thesis. We have introduced the notion of real possibility and the framework of branching time, and we have pointed out the appeal of the branching time framework for the representation of real possibilities. In particular, we have motivated our choice of the branching time framework over the popular possible worlds framework.

We have seen that real possibilities are genuinely open possibilities for the future. They are temporal alternatives for a dynamic actuality to evolve. While the branching time framework does justice to that idea, the possible worlds framework fails in that respect. In the possible worlds framework, future possibilities boil down to mere modal alternatives to an overarching actuality. While the idea of modal alternatives that is at the core of the possible worlds framework is useful for modeling various different kinds of possibilities, such as epistemic, logical, metaphysical and physical possibility, it is not suited for the particular notion of real possibility we aim to model. Just adding a temporal dimension to possible worlds is not enough in order to capture the idea of an open future, and a branching time structure with a Thin Red Line does not do any better.

There are also logical differences between the branching time framework and the possible worlds framework. Kamp validity coincides with bundled tree validity rather than with Ockhamist validity. Smoothing the logical differences is only possible at the cost of giving away the rich structural resources a branching time structure has to offer. It is important to bear in mind that histories are defined elements in a branching time structure, and that there is no need to make use of histories as a second parameter of truth in the semantic evaluation. The basic constituents of a branching time structure are moments, and the ordering of moments is full of possibilities other than histories. The tree of moments harbors countless local future possibilities. In the next chapter, we present a novel branching time semantics that, unlike Ockhamism, exploits the structural resources a branching time structure has to offer. In the transition semantics we are proposing, the semantic evaluation can be relativized to incomplete possible courses of events as well, and in chapter 3, the corresponding 'Kamp frames' will be shown to be trees of branching time substructures that mirror the passage of time rather than pluriverses of parallel linear possible worlds.

As a Kripke frame, a branching time structure can first of all only account for the temporal aspect of the notion of real possibility. Real possibilities, however, are worldly possibilities. They are alternative possibilities for the future in an indeterministic world, and they are even closer tied up with the world than standard notions of alethic possibility are, to wit, logical, metaphysical and physical possibility. We will come back to the worldly aspect of real possibilities in chapter 4, when we provide a metaphysical explanation of branching time models for real possibility that accommodates the worldly aspect. We will see that, in accordance with the inherent temporal dynamics of real possibilities, the explanation of the corresponding models must be dynamic well. Branching time models for real possibility will be lifted in a dynamic fashion from a single momentary circumstance on the basis of the potentialities of objects.

* * *

Chapter 2

Transition Semantics*

2.1 Introduction

In the previous chapter, we have introduced the framework of branching time, which will build the basis of our investigations throughout this thesis. We have illustrated how possibilities are presented in that framework, and we have discussed the two most popular semantic approaches based on that framework, labeled Peirceanism and Ockhamism. In this chapter, we put forth the transition semantics we are proposing.

We have seen that Peirceanism and Ockhamism differ with respect to which structural elements are employed as parameters of truth in the semantic evaluation. In Peirceanism, the semantic evaluation is relativized to a moment parameter only, whereas Ockhamism makes use of a history as a second parameter of truth next to the moment parameter. The basic idea behind the transition semantics is to replace the history parameter employed in the Ockhamist semantics by a set of so-called transitions in order to allow for a more local perspective. Transitions are structural elements that capture the local change at a moment: every transition specifies one of the immediate possible future continuations of a moment in a branching time structure. Whereas histories represent complete possible courses of events, sets of transitions can represent incomplete parts thereof as well.

*This chapter is based on Rumberg (2016).

That sets of transitions can serve as a local alternative to histories in the semantic evaluation on a branching time structure has been suggested in Müller (2014a). The transition semantics we are proposing here goes significantly beyond that suggestion: by enriching the purely temporal language considered in Müller (2014a) by modal operators and a so-called stability operator, we obtain a language that is able to reflect the richness of the semantic machinery. The stability operator is interpreted as a universal quantifier over the possible future extensions of a given transition set. In light of the stability operator, the transition parameter becomes dynamic: it provides a second temporal perspective that can be shifted independently of the moment parameter and enables us to capture the behavior of the truth value of a sentence at a moment in the course of time.

The transition semantics exploits the structural resources a branching time structure has to offer and provides a fine-grained, dynamic picture of the interrelation of modality and time. The merits are twofold: on the one hand, the transition semantics enables a perspicuous treatment of real possibilities, on the other hand, it allows for great generality. Firstly, the dynamics inherent to the transition approach adequately reflects the dynamics of real possibilities. As said, real possibilities are genuine alternatives for the future. Which possibility will be realized depends on how the future unfolds. The interest of the transition semantics is this: with its stability operator, it allows us to specify exactly how and how far the future has to unfold for the contingency to dissolve. It tells us precisely what needs to happen for it to be settled that a given future possibility will be realized: it allows for local witnesses for future possibilities. Secondly, both Peirceanism and Ockhamism can be viewed as limiting cases of the transition approach that are obtained by restricting the range of transition sets that are taken into account in the semantic evaluation. Both accounts fall short of the transition semantics in terms of expressive strength. The restrictions reveal their limitations: on both accounts, the dynamic character is lost stability collapses into truth.

The chapter is structured as follows. In section 2.2, we discuss, by means of a concrete example, how the choice of parameters of truth affects the interpretation of real possibilities and motivate the prospects of a dynamic transition parameter. In section 2.3, we introduce the notion of a transition and establish the basis of our transition semantics by showing how the various possible courses of events in a BT structure can be represented by means of transition sets. In section 2.4, we put forth the semantic clauses of the transition semantics we are proposing and illustrate how sentences about the future are treated in that semantic framework. In section 2.5, we point out the generality of the transition framework. We show that the transition semantic unifies and properly extends Peirceanism and Ockhamism: both accounts are limiting cases of the transition approach that make use of restricted resources only, and on both accounts stability collapses into truth. We end with a brief remark on how our approach relates to the accounts put forward in Zanardo (1998) and MacFarlane (2003, 2014): the transition semantics goes beyond both Zanardo's idea of indistinguishability and MacFarlane's assessment sensitive postsemantics.

2.2 Motivation: An example

A branching time structure allows for different parameters of truth in the semantic evaluation, and different parameters of truth allow for different witnesses for future possibilities. In this section, we will consider a concrete example in order to illustrate how the choice of parameters of truth bears on the interpretation of real possibilities. The discussion sheds light on the advantages of a dynamic framework and motivates our approach.

In section 1.2.2, we said that real possibilities are genuinely open alternatives for the future, alternative possibilities for actuality to evolve. Either of them can be actualized while none of them is actual yet. Time will settle the matter. Branching time structures adequately capture that idea. At a given moment, there may be more than one possibility for the future, with nothing yet deciding between them. For concreteness, assume that you have purchased a ticket for tomorrow's lottery. Given that the outcome of the drawing is in fact objectively indeterminate, it is both possible that your ticket will win and that your ticket will lose, with nothing yet to tip the balance. Whether your ticket will win or lose, depends on how the future unfolds, or more precisely, on the outcome of tomorrow's drawing.

On the Peircean account, the semantic evaluation is relativized to a moment parameter only. Sentences are evaluated from the perspective of a local standpoint in time, which provides a notion of temporal actuality. On the Peircean account, there is no room for plain modalities, nor is there room for plain future truth. Future truth and modality are conflated. Both Peircean future operators are modalized: the strong future operator combines future truth with inevitability, the weak operator, on the other hand, combines future truth with possibility. On a strong reading, the Peircean semantics renders both the sentence "Your ticket will win" and its contrary prediction "Your ticket will lose" false at any moment before the drawing, whereas, on a weak reading, both sentences come out true. The strong future operator is certainly inapt for an account of real possibility. The weak future operator fares better in this respect; however, it fails to relate possibility to actuality in any interesting way. There is no way to make explicit what needs to happen for either possibility to come about. If we are to make the link between actuality and possibility explicit, a second parameter of truth next to the moment parameter is needed: the semantic evaluation has to relativized—one way or another—to a possible course of events so as to make room for a notion of plain future truth.

The Ockhamist account makes use of a history as a second parameter of truth next to the moment parameter and thereby generalizes Peirceanism. Sentences are evaluated at moment-history pairs. While the moment parameter pins down a local standpoint in time and thereby provides a notion of temporal actuality, the history parameter introduces a hypothetical modal actuality. Future truth and modality come apart. Modal operators become interpretable as quantifiers over histories, and owing to the fact that the semantic evaluation at a moment is relativized to a history parameter, we are provided with a notion of plain future truth. In the case of the future operator, the moment of evaluation is simply shifted forward on the given history, just as in linear tense logic. Sentences about the contingent future can be assigned different truth values at the very same moment with respect to different histories. In our example, at any moment before the drawing, the sentence "Your ticket will win" is true with respect to one history, while its contrary prediction "Your ticket will lose" is true with respect to another history, rendering both outcomes possible. By relativizing truth at a moment to some hypothetical modal actuality, Ockhamism makes explicit the idea that which of the alternative future possibilities will eventually be realized depends on how the future unfolds. It correctly tells us that both alternatives are possibilities for the future because each of the respective future claims is plainly true with respect to some history. Histories, however, represent complete possible courses of events, whereas, in most cases, an incomplete course of events suffices to dissolve the contingency. In the lottery example, it is the drawing that constitutes the relevant tipping point. With respect to any incomplete course of events that encompasses one of the possible outcomes of tomorrow's drawing, it is settled which possibility will be realized.

The transition semantics we are proposing generalizes and extends both Peirceanism and Ockhamism. The basic idea is to replace the moment-history pairs employed as parameters of truth in the Ockhamist semantics by pairs consisting of a moment and a transition set so as to allow for the relativization to incomplete possible courses of events as well. That is to say, just as in Ockhamism, the transition semantics introduces, in addition to the temporal actuality provided by the moment parameter, a hypothetical modal actuality. However, whereas the relevant notion of a hypothetical modal actuality employed in Ockhamism is a global and static one, the transition semantics entertains a local and dynamic notion of hypothetical modal actuality. It introduces a second local perspective in time that can be shifted independently of the moment parameter. Not only do future truth and modality come apart, but an additional notion of stability enters the picture. In addition to temporal and modal operators, we adopt a so-called stability operator into our language. Modal operators are interpreted as quantifiers over transition sets. Our future operator provides a notion of plain future truth with respect to a possibly incomplete hypothetical course of events: it demands a future witness in every possible future continuation of the moment of evaluation that is an extension of the given, possibly incomplete course of events. And the stability operator allows us to capture the behavior of the truth value of a sentence about the future at a moment in the course of time: its changing from contingent to stably-true or stably-false. It is a universal quantifier over the possible extensions of a given transition set.

With the stability operator at our disposal, we cannot only express that, at any moment prior to the drawing, it is both possible that your ticket will win and that your ticket will lose, but we can also specify exactly how and how far the future has to unfold for things to become settled either way. At any moment before the drawing, the sentences "Your ticket will win" and "Your ticket will lose" are neither stably-true nor stably-false but contingent with respect to the partial course of events up to the drawing. Their truth values only stabilize as the future unfolds. The drawing dissolves the contingency. With respect to any incomplete possible course of events that encompasses one of the possible outcomes of tomorrow's drawing, each of our sentences is either stably-true or stably-false at any moment before the drawing. Any incomplete possible course of events that encompasses one of the possible outcomes of the drawing suffices to settle the matter, even though there might be several histories that contain the very same outcome of the drawing and which differ only with respect to what will happen hundreds or thousands years hence. Unlike Ockhamism, the transition semantics allows for local witnesses for future possibilities.

2.3 Transitions and transition sets

As said, the basic idea behind the transition semantics is to replace the history parameter employed in the Ockhamist semantics by a set of transitions so as to allow the semantic evaluation to be relativized to incomplete possible courses of events as well. In this section, we pave the way for the transition semantics. We provide the definition of a transition and characterize the class of transition sets that will be employed in the semantic evaluation.

The notion of a transition is closely connected with the notion of change. It is prominent in the work of von Wright, and Belnap has adopted the idea to branching theories. Roughly speaking, transitions are ordered pairs consisting of an initial and a subsequent outcome, which capture the change or transformation from the initial to the corresponding outcome. In von Wright (1963), transitions are construed as pairs whose initial is a state of affairs and whose outcome is either a later state of affairs or a process.[48] The language studied by von Wright in the context of his logic of change and agency contains expressions that make reference to such transitions: the expression 'pTq' is used to stand for the transition from the generic state of affairs described by p to the generic state of affairs described by q. Belnap (1999) picks up on von Wright's notion of a transition but suggests to make transitions more concrete. On von Wright's conception of a transition, the very same transition may occur at different points in time and may even be part of different possible courses of events. Belnap, in contrast, proposes to anchor transitions concretely in the modal-temporal structure of the world. Rather than conceiving of transitions as pairs consisting of two successive states of affairs (or a state of affairs and a process, respectively), Belnap defines transitions as pairs whose respective initials and outcomes are successive chains of moments in a branching time structure. Transitions become structural elements.[49]

[48]For von Wright's definition of a transition, see von Wright (1963, ch.II)

[49]For Belnap's definition of a transition, see Belnap (1999, 2005) and Belnap, Perloff, and Xu (2001). Belnap (1999, 2005) deals with the notion of a transition in the framework of branching space-time. In Belnap, Perloff, and Xu (2001, ch.7A.4, cf. also ch.3A, 6A.3), the

We follow Belnap in anchoring transitions concretely in the tree-like ordering of moments in a branching time structure. Our definition of a transition is a special case of the definition advanced by Belnap, and it is the notion that is also to be found in Müller (2014a). Transitions are conceived of as structural elements that capture the local change at a moment. A transition specifies one of the immediate possible future continuations of a moment. The notion of a transition provides an excellent tool to model possible courses of events in a branching time structure. Chains of transitions allow us to adequately mirror the forking paths in the tree of moments. Unlike histories, sets of transitions can represent complete or incomplete possible courses of events that stretch linearly all the way from the past toward a possibly open future. In section 2.3.1, we put forth our definition of a transition, and in section 2.3.2, we illustrate how possible courses of events can be represented by means of transition sets.

2.3.1 Transitions

Transitions provide a local alternative to histories. Whereas a history represents a complete possible course of events, a transition captures one of the immediate possible future continuations of a moment in a branching time structure. One may put it this way: while histories represent global possibilities, transitions represent local ones.

Our definition of a transition is based on the notion of undividedness, which captures the local branching behavior of histories at a given moment. In section 2.3.1.1, we first introduce the relation of undividedness. In section 2.3.1.2, we then spell out the definition of a transition. In section 2.3.1.3, we show that we can define a natural partial ordering among the various transitions in a branching time structure.

2.3.1.1 The relation of undividedness

When introducing the Prior-Thomason theory of branching time in section 1.3.1 above, we said that conditions (BT1) and (BT2) of the definition of a BT structure (Def. 1.1) jointly ensure that any two histories branch at some moment. Given a BT structure $\mathcal{M} = \langle M, < \rangle$ and a branching point $m \in M$, not all pairs of histories in $\mathsf{hist}(\mathcal{M})$ that contain the moment m need to branch at m, however. Some may still continue to overlap for a certain while after m, and in

definition is adopted to the case of branching time. Given a BT structure $\mathcal{M} = \langle M, < \rangle$, a transition is defined as a pair $\langle I, O \rangle$ where $I \subseteq M$ is a non-empty upper-bounded chain of moments and $O \subseteq M$ is a non-empty lower-bounded chain of moments such that every moment in I precedes every moment in O.

that case, they are said to be undivided at m. At each moment $m \in M$, the relation of undividedness partitions the set of histories H_m into sets of undivided histories according to their local branching behavior, and it thereby provides a local notion of possibility, as we shall see.[50]

Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let $m \in M$ be a moment. Assume that there are two histories $h, h' \in \mathsf{H}_m$. If the moment m is the greatest element in the intersection $h \cap h'$ of the two histories h and h', then, according to Def. 1.4, the two histories branch at m, in symbols: $h \perp_m h'$. If, on the other hand, their intersection $h \cap h'$ contains a moment later than m, the histories h and h' are *undivided at m*. Note that, since histories are downward closed, whenever two histories in $\mathsf{hist}(\mathcal{M})$ share some moment later than m, they also share m. For two histories to be undivided at a moment m, it is therefore not only necessary but, at the same time, also sufficient that they overlap at some moment in the future of m.

Definition 2.1 (Undividedness-at-m). *Given a BT structure $\mathcal{M} = \langle M, < \rangle$ and a moment $m \in M$, two histories $h, h' \in \mathsf{hist}(\mathcal{M})$ are* undivided at m, *in symbols: $h \equiv_m h'$, iff there is some moment $m' \in M$ such that $m' > m$ and $m' \in h \cap h'$.*

Obviously, any two histories h and h' that overlap at some moment m are undivided at that moment m if and only if they do not branch at m. To wit, if the moment m is maximal in $h \cap h'$, they branch; otherwise they are undivided.

Lemma 2.2. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, $m \in M$ a moment, and let $h, h' \in \mathsf{H}_m$. Then:*

$$h \equiv_m h' \quad \textit{iff} \quad h \not\perp_m h'.$$

Proof. Follows from Def. 1.4 and Def. 2.1. □

The relation undividedness-at-m is an equivalence relation on the set H_m of histories containing the moment m and thus yields a partition of that set. It is straightforward to verify that the relation $\equiv_m$ on H_m is reflexive, symmetric and transitive.[51]

[50] The notion of undividedness plays a crucial role in stit accounts of agency, as provided, for example, in Belnap and Perloff (1990), Horty and Belnap (1995) and Belnap, Perloff, and Xu (2001), and it is also prominent in von Kutschera (1993) and Zanardo (1998). The notion of choice that is at the core of the stit framework closely relates to the notion of undividedness. An agent's choices at a moment have to respect undividedness at that moment: there is no choice between undivided histories.

[51] Note that the reflexivity of $\equiv_m$ flows directly from Def. 2.1 since we have required from the outset that the earlier-later relation $<$ in a BT structure be serial. In the absence of condition (BT3) of Def. 1.1, some additional constraint is needed in order to ensure the reflexivity of $\equiv_m$.

LEMMA 2.3. *Given a BT structure $\mathcal{M} = \langle M, < \rangle$ and a moment $m \in M$, the relation $\equiv_m$ is an equivalence relation on the set H_m (i.e. a relation that is reflexive, symmetric and transitive).*

Proof. The reflexivity of the relation $\equiv_m$ is guaranteed by condition (BT3) of Def. 1.1. Its symmetry is straightforward. We show that the relation is transitive as well. Assume that there are histories $h, h', h'' \in \mathsf{H}_m$ s.t. $h \equiv_m h'$ and $h' \equiv_m h''$. By Def. 2.1 it follows that there is some $m' \in M$ s.t. $m' > m$ and $m' \in h \cap h'$ and, likewise, that there is some $m'' \in M$ s.t. $m'' > m$ and $m'' \in h' \cap h''$. Since both $m', m'' \in h'$, we have either $m' \leq m''$ or $m'' \leq m'$. If $m' \leq m''$, then $m' \in h''$ and hence $m' \in h \cap h''$. If, on the other hand, $m'' \leq m'$, then $m'' \in h$ and so $m'' \in h \cap h''$. Consequently, $h \equiv_m h''$. □

The equivalence class of a history $h \in \mathsf{H}_m$ with respect to the relation $\equiv_m$ is denoted by $[h]_m$; so $[h]_m = \{h' \in \mathsf{H}_m \mid h \equiv_m h'\}$. We use Π_m to stand for the partition $\mathsf{H}_m / \equiv_m$ of H_m modulo undividedness-at-m. As a matter of course, the equivalence classes in Π_m are pairwise disjoint and jointly exhaustive: for all $h, h' \in \mathsf{H}_m$, we have either $[h]_m = [h']_m$ or $[h]_m \cap [h']_m = \emptyset$, and $\mathsf{H}_m = \bigcup \Pi_m$.

DEFINITION 2.4 (The equivalence class $[h]_m$). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, $m \in M$ a moment and $h \in \mathsf{H}_m$ a history, let*

$$[h]_m := \{h' \in \mathsf{H}_m \mid h \equiv_m h'\}.$$

The partition of H_m with respect to the equivalence relation $\equiv_m$ is denoted by Π_m.

The cells of the partition Π_m induced by the relation of undividedness-at-m represent local possibilities. Each equivalence class is a set of undivided histories that corresponds to one immediate possible future continuation of the moment m. In case no histories branch at m, there is but a single immediate possibility for the future: the partition is trivial, i.e. $\Pi_m = \{\mathsf{H}_m\}$. If, on the other hand, the moment m is a branching point, the partition Π_m contains at least two elements, each of which captures one of the alternative immediate future possibilities open at m. The partition reflects the local branching structure. At each branching point, some histories branch, while others may be undivided.

From the definition of undividedness it immediately follows that all histories that share some moment m are undivided at any moment m' in that moment's past: if $m' < m$ and $h, h' \in \mathsf{H}_m$, then $h \equiv_{m'} h'$, and hence $\mathsf{H}_m \subseteq [h]_{m'} = [h']_{m'}$. In particular, it holds that any two histories that are undivided at some moment

m are also undivided at any earlier moment m': if $m' < m$ and $h \equiv_m h'$, then $h \equiv_{m'} h'$, and consequently $[h]_m = [h']_m \subseteq [h]_{m'} = [h']_{m'}$. We summarize those results in the following lemma.

LEMMA 2.5. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and let $m, m' \in M$ be moments such that $m' < m$. Then for all $h \in \mathsf{H}_m$ the following holds:*

(*i*) $\mathsf{H}_m \subseteq [h]_{m'}$,

(*ii*) $[h]_m \subseteq [h]_{m'}$.

Proof.

(i) Let $h' \in \mathsf{H}_m$. Then $m \in h \cap h'$, and since $m' < m$, by Lem. 1.3, we also have $m' \in h \cap h'$. By Def. 2.1 it follows that $h \equiv_{m'} h'$ and hence $h' \in [h]_{m'}$.

(ii) Let $h' \in [h]_m$. Then $h' \in \mathsf{H}_m$, from which it follows by (i) that $h' \in [h]_{m'}$. □

2.3.1.2 The definition of a transition

The relation of undividedness introduced in the previous section builds the basis of our definition of a transition. At any moment m, the relation of undividedness-at-m partitions the set of histories H_m into immediate future possibilities. A transition links the moment m to one such local future possibility provided by the relation of undividedness. It selects one of the immediate future possibilities open at m and thereby specifies one immediate possible future continuation of that moment.

Given a BT structure $\mathcal{M} = \langle M, < \rangle$, we define a *transition* as a pair $\langle m, H \rangle$, also written $\langle m \rightarrowtail H \rangle$, consisting of moment $m \in M$ and a set of locally undivided histories $H \in \Pi_m$. The moment m is called the *initial* of the transition $\langle m \rightarrowtail H \rangle$ and the set of histories H its *outcome*. In case the moment m is a branching point, there are at least two transitions that share the initial m but have pairwise disjoint outcomes. If, on the other hand, no histories branch at m, there is but a single transition with initial m, whose outcome is identical to H_m. In the former case, we say that the transitions are *indeterministic*, whereas transitions of the latter type are called *trivial*. We use $\mathsf{TRANS}(\mathcal{M})$ to stand for the set of all transitions in $\mathcal{M}$, and we denote the set of indeterministic transition in $\mathcal{M}$ by $\mathsf{trans}(\mathcal{M})$.

Definition 2.6 (Transition). *For* $\mathcal{M} = \langle M, < \rangle$ *a BT structure, a* transition *is a pair* $\langle m, H \rangle$*, also written* $\langle m \rightarrowtail H \rangle$*, with* $m \in M$ *and* $H \in \Pi_m$*. The moment* m *is the* initial *and the set of histories* H *the* outcome *of the transition* $\langle m \rightarrowtail H \rangle$*. A transition* $\langle m \rightarrowtail H \rangle$ *is* trivial *if* $H = \mathsf{H}_m$*; otherwise,* indeterministic*. Let* $\mathsf{TRANS}(\mathcal{M})$ *be the* set of all transitions in $\mathcal{M}$*, and let* $\mathsf{trans}(\mathcal{M})$ *be the* set of all indeterministic transitions in $\mathcal{M}$*.*

A fact that deserves explicit mention, no matter how trivial, is the following one, which concerns the interrelation between transitions, moments and histories: consider some transition $\langle m \rightarrowtail H \rangle \in \mathsf{TRANS}(\mathcal{M})$ and let $h \in H$. Then the initial m of the transition $\langle m \rightarrowtail H \rangle$ is obviously contained in the history h, i.e. $m \in h$, and its outcome H equals the equivalence class of that history with respect to the relation $\equiv_m$ of undividedness-at-m, i.e. $H = [h]_m$.

Transitions are set theoretically rather complex. The complexity of the definition of a transition is due to the fact that the definition is intended to be general enough to accommodate the case of dense and continuous BT structures. In case $\mathcal{M} = \langle M, < \rangle$ is discrete, so that for every moment $m \in M$, there is a set $\mathsf{succ}(m)$ of immediate successors of m, a transition can be defined as a pair of moments $\langle m, m' \rangle$ with $m \in M$ and $m' \in \mathsf{succ}(m)$. This is the notion of a transition that is prevalent in computer science, where primarily discrete structures are studied. If the BT structure under consideration is not discrete, however, transitions cannot generally be defined as pairs of moments. In order to see this, consider a structure that contains a history h isomorphic to the real numbers $\mathbb{R}$ such that at any moment $m_i \in h$ with $i \in \mathbb{R}$, the partition Π_{m_i} is non-trivial, i.e. $\Pi_{m_i} \neq \{H_{m_i}\}$. For every moment $m_i \in h$, we then have a transition $\langle m_i \rightarrowtail H \rangle$ with $h \in H$ that is not reducible to a pair of moments.

2.3.1.3 The transition ordering

Given a BT structure $\mathcal{M} = \langle M, < \rangle$, we can define a left-linear strict partial order $\prec$ on the set $\mathsf{TRANS}(\mathcal{M})$ of all transitions in $\mathcal{M}$. In light of the order relation $\prec$ on $\mathsf{TRANS}(\mathcal{M})$, sets of transitions then become structured entities.

The ordering $\prec$ among the various transitions in a BT structure $\mathcal{M} = \langle M, < \rangle$ is defined in terms of the earlier-later relation $<$ between the initials and a reverse inclusion relation $\subseteq$ on the outcomes. Given two transitions $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \in \mathsf{TRANS}(\mathcal{M})$, we say that $\langle m \rightarrowtail H \rangle$ *precedes* $\langle m' \rightarrowtail H' \rangle$, in symbols: $\langle m \rightarrowtail H \rangle \prec \langle m' \rightarrowtail H' \rangle$, iff (i) the initial of $\langle m \rightarrowtail H \rangle$ is prior to the initial of $\langle m' \rightarrowtail H' \rangle$, i.e. $m < m'$, and (ii) the outcome of $\langle m' \rightarrowtail H' \rangle$ is included in the outcome of $\langle m \rightarrowtail H \rangle$, i.e. $H' \subseteq H$.

DEFINITION 2.7 (Transition ordering). *Given a BT structure* $\mathcal{M} = \langle M, < \rangle$ *and transitions* $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \in \mathsf{TRANS}(\mathcal{M})$, *we say that* $\langle m \rightarrowtail H \rangle$ precedes $\langle m' \rightarrowtail H' \rangle$, *in symbols:* $\langle m \rightarrowtail H \rangle \prec \langle m' \rightarrowtail H' \rangle$, *iff* (i) $m < m'$ *and* (ii) $H' \subseteq H$.

Just as the temporal earlier-later relation $<$ on M, the transition ordering $\prec$ on $\mathsf{TRANS}(\mathcal{M})$ is a left-linear strict partial order. We use $t \preceq t'$ to stand for ($t \prec t'$ or $t = t'$).

LEMMA 2.8. *Given a BT structure* $\mathcal{M} = \langle M, < \rangle$, *the transition ordering* $\prec$ *on* $\mathsf{TRANS}(\mathcal{M})$ *is a strict partial order* (*i.e. a relation that is irreflexive, asymmetric and transitive*) *that is left-linear.*

Proof. Being defined in terms of the earlier-later relation $<$ on M and set inclusion, the relation $\prec$ on $\mathsf{TRANS}(\mathcal{M})$ is obviously irreflexive and transitive and hence also asymmetric. We show that the relation is left-linear as well. Assume that there are $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \langle m'' \rightarrowtail H'' \rangle \in \mathsf{TRANS}(\mathcal{M})$ s.t. $\langle m' \rightarrowtail H' \rangle \preceq \langle m \rightarrowtail H \rangle$ and $\langle m'' \rightarrowtail H'' \rangle \preceq \langle m \rightarrowtail H \rangle$. By Def. 2.7 it follows that $m' \leq m$ and $m'' \leq m$ and $H \subseteq H' \cap H''$. Since the relation $<$ on M is left-linear (condition (BT1) of Def. 1.1), we either have $m' = m''$, $m' < m''$ or $m'' < m'$. If $m' = m''$, then $H' = H''$, since $H' \cap H'' \neq \emptyset$. Now assume that $m' \neq m''$. By Lem. 2.5 (ii), $m' < m''$ implies $H'' \subseteq H'$ and $m'' < m'$ implies $H' \subseteq H''$. Consequently, we have $\langle m' \rightarrowtail H' \rangle \preceq \langle m'' \rightarrowtail H'' \rangle$ or $\langle m'' \rightarrowtail H'' \rangle \preceq \langle m' \rightarrowtail H' \rangle$. □

If we restrict the order relation $\prec$ on the set $\mathsf{TRANS}(\mathcal{M})$ to the subset $\mathsf{trans}(\mathcal{M})$ of indeterministic transitions, the two conditions imposed on the transition ordering $\prec$ are no longer independent. Take two indeterministic transitions $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \in \mathsf{trans}(\mathcal{M})$ and assume that $m < m'$. Then, by Lem. 2.5 (ii), it follows that $H' \subsetneq H$. Note that the converse does not hold in general: only under the presupposition that $H \cap H' \neq \emptyset$, can we infer from $H' \subsetneq H$ that $m' < m$ holds as well.

LEMMA 2.9. *Let* $\mathcal{M} = \langle M, < \rangle$ *be a BT structure, and let* $\langle m \rightarrowtail H \rangle$, $\langle m' \rightarrowtail H' \rangle \in \mathsf{trans}(\mathcal{M})$. *The following holds*

$$(m' < m \textit{ and } H \cap H' \neq \emptyset) \quad \textit{iff} \quad H \subsetneq H'$$

Proof. "⇒": Assume that $m' < m$ and $H \cap H' \neq \emptyset$. Let $h \in H \cap H'$. By Lem. 2.5 (i) it follows that $H_m \subseteq H' = [h]_{m'}$. Since $\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M})$ is an indeterministic transition, we have $H \subsetneq \mathsf{H}_m$, and hence $H \subsetneq H'$.
"⇐": Assume that $H \subsetneq H'$. Then $H \cap H' \neq \emptyset$ since, by Lem. 2.3, $H \neq \emptyset$.

Let $h \in H \cap H'$. It follows that $m, m' \in h$. Three cases can be considered: we either have (a) $m = m'$, (b) $m' < m$ or (c) $m < m'$. If (a) $m = m'$, then $H = H'$, which contradicts our assumption that $H \subsetneq H'$. If (b) $m < m'$, then, by "⇒", it follows that $H' \subsetneq H$, which likewise contradicts our assumption. Therefore, (c) $m' < m$. □

Lem. 2.9 shows that on the set $\mathsf{trans}(\mathcal{M})$ of indeterministic transitions, we can reconstruct the transition ordering $\prec$ solely in terms of proper set inclusion $\subsetneq$ between the outcomes. Note that the proper set inclusion reverses the transition ordering. Given two indeterministic transitions $\langle m \rightarrowtail H\rangle$, $\langle m' \rightarrowtail H'\rangle \in \mathsf{trans}(\mathcal{M})$, we have $\langle m \rightarrowtail H\rangle \prec \langle m' \rightarrowtail H'\rangle$ if and only if the outcome of $\langle m' \rightarrowtail H'\rangle$ is properly included in the outcome of $\langle m \rightarrowtail H\rangle$, i.e. $H' \subsetneq H$.

Corollary 2.10. *Let $\mathcal{M} = \langle M, <\rangle$ be a BT structure, and let $\langle m \rightarrowtail H\rangle$, $\langle m' \rightarrowtail H'\rangle \in \mathsf{trans}(\mathcal{M})$ be indeterministic transitions in $\mathcal{M}$. Then:*

$$\langle m \rightarrowtail H\rangle \prec \langle m' \rightarrowtail H'\rangle \textit{ iff } H' \subsetneq H.$$

Proof. Follows from Lem. 2.9. □

2.3.2 Transition sets

In the previous section, we have introduced the notion of a transition. Transitions are structural elements that capture the local change at a given moment in a branching time structure by specifying one immediate possible future continuation of that moment. In this section, we show how we can construct from those local elements, complete and incomplete possible courses of events.

Sets of transitions are, first of all, structured entities: they are made up from local transitions and they receive an ordering by the relation $\prec$. From a more abstract point of view, however, viz. the point of view of logic, sets of transitions are tantamount to certain sets of histories: a set of transitions captures a possible course of events by allowing certain histories to occur and excluding others. It specifies a path in the tree of moments that stretches all the way from the past toward a possibly open future. Whether a set of transitions represents an incomplete possible course of events, depends solely on whether it admits more than one history and thus allows for alternative possible future continuations. From the abstract point of view of logic, sets of transitions are nothing over and above the set of histories they admit.

The transition semantics is based on sets of transitions: they are our substitute for the Ockhamist history parameter. From all possible sets of transitions definable in a BT structure, only a distinguished subclass will thereby be employed in the semantic evaluation. In our formal semantics, we make use of indeterministic transitions only, and only transition sets that are consistent and downward closed in the transition ordering $\prec$ are taken into account. The set of consistent, downward closed sets of indeterministic transitions exhausts the range of possible courses of events in a branching time structure when those are understood abstractly: each consistent, downward closed set of indeterministic transitions uniquely captures one possible course of events.

In this section, we characterize the set of all consistent, downward closed sets of indeterministic transitions. That is, we specify the range of transition sets on which our transition semantics is based. In section 2.3.2.1, we first spell out what it means for a set of transitions to be consistent. In section 2.3.2.2, we turn to the idea that transition sets are to be downward closed and motivate our restriction to indeterministic transitions. In section 2.3.2.3, we consider the resulting set of all consistent, downward closed sets of indeterministic transitions and show that each member of that set corresponds one-to-one to a possible course of events.

2.3.2.1 Consistency

In this section, we make concrete the idea of a close correspondence between transition sets and sets of histories alluded to above and introduce the notion of consistency. Consistency is a modal notion that is intimately linked up with the idea of a possible course of events. The notion of consistency governs which transitions can co-occur in a single possible course of events and thereby lays down which sets of transitions represent possible courses of events in the first place.

Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and consider some set of transitions $T \subseteq \mathsf{TRANS}(\mathcal{M})$. Each transition $\langle m \rightarrowtail H \rangle$ contained in the set T is anchored at some moment $m \in M$ and selects one of the immediate future possibilities open at that moment, ruling out the remainder. In other words, it allows certain histories in H_m to occur, namely those that are part of its outcome H, but may exclude others. On the whole then, there may or may not be histories that are allowed by all transitions in T. In case there is at least one such history that is in accordance with all the transitions in the set T, the set T can be said to be consistent.

The above considerations disclose a natural correspondence between sets of transitions and sets of histories. To every transition set, there corresponds a possibly empty set of histories, viz. the set of histories allowed by the transition set. Given a transition set $T \subseteq \mathsf{TRANS}(\mathcal{M})$, the *set of histories allowed by* T, denoted by $\mathsf{H}(T)$, is the intersection of the outcomes of those transitions.

Definition 2.11 (The set of histories allowed by T). *Given a BT structure* $\mathcal{M} = \langle M, < \rangle$ *and a set of transitions* $T \subseteq \mathsf{TRANS}(\mathcal{M})$*, the* set of histories allowed by T*, in symbols:* $\mathsf{H}(T)$*, is given by*

$$\mathsf{H}(T) := \bigcap_{\langle m \rightarrowtail H \rangle \in T} H.$$

The set of histories allowed by a transition set fully determines the course of events depicted by that set: from the abstract point of view of logic, sets of transitions that allow exactly the same histories depict the very same course of events.

Given the correspondence between transition sets and sets of histories provided in Def. 2.11, we can define a set of transitions to be *consistent* iff the set $\mathsf{H}(T)$ of histories allowed by T is non-empty. In that case, there is a possible course of events in which all transitions in T co-occur.

Definition 2.12 (Consistency). *For* $\mathcal{M} = \langle M, < \rangle$ *a BT structure, a transition set* $T \subseteq \mathsf{TRANS}(\mathcal{M})$ *is* consistent *iff* $\mathsf{H}(T) \neq \emptyset$.

We show that the consistency of a set of transitions comes down to the requirement that all the transitions in the set have to lie within one chain. That is, we show that a transition set $T \subseteq \mathsf{TRANS}(\mathcal{M})$ is consistent if and only if it is linearly ordered by the transition ordering $\prec$. A consistent set of transitions can in particular not contain two different transitions with the same initial.

Proposition 2.13. *For* $\mathcal{M} = \langle M, < \rangle$ *a BT structure and* $T \subseteq \mathsf{TRANS}(\mathcal{M})$ *a set of transitions, we have that*

$$\mathsf{H}(T) \neq \emptyset \quad \textit{iff} \quad T \textit{ is linearly ordered via } \prec .$$

Proof. "⇒": Assume that $\mathsf{H}(T) \neq \emptyset$. Then there is some history $h \in \mathsf{H}(T)$, and we have that $\{m \in M \mid \langle m \rightarrowtail H \rangle \in T\} \subseteq h$. By Def. 1.2 it follows that the set $\{m \in M \mid \langle m \rightarrowtail H \rangle \in T\}$ is linearly ordered by the relation $<$. Let $\langle m' \rightarrowtail H' \rangle, \langle m'' \rightarrowtail H'' \rangle \in T$. Then $m', m'' \in \{m \in M \mid \langle m \rightarrowtail H \rangle \in T\}$.

Three cases can be considered: (a) if $m' = m''$, then $H' = H''$, since otherwise $\mathsf{H}(T) = \emptyset$; (b) if $m' < m''$, then, by Lem. 2.5 (ii), it follows that $H'' = [h]_{m''} \subseteq [h]_{m'} = H'$, which implies that $\langle m' \rightarrowtail H' \rangle \prec \langle m'' \rightarrowtail H'' \rangle$; (c) if $m'' < m'$, we accordingly have $H' = [h]_{m'} \subseteq [h]_{m''} = H''$ and thus $\langle m'' \rightarrowtail H'' \rangle \prec \langle m' \rightarrowtail H' \rangle$.
"$\Leftarrow$": Assume that T is linearly ordered via $\prec$. If T contains a maximal element $\langle m \rightarrowtail H \rangle$, then by Def. 2.7 we have that $H \subseteq \mathsf{H}(T)$ and thus $\mathsf{H}(T) \neq \emptyset$. Now assume that T does not contain a maximal element. Since T is linearly ordered via $\prec$, it follows by Def. 2.7 that $\{m \in M \mid \langle m \rightarrowtail H \rangle \in T\}$ is linearly ordered via $<$. Hence, by the Axiom of Choice, there is some history $h \supseteq \{m \in M \mid \langle m \rightarrowtail H \rangle \in T\}$. Assume for reductio that $h \notin \mathsf{H}(T)$. Then there must be some transition $\langle m' \rightarrowtail H' \rangle \in T$ s.t. $h \notin H'$. Since, by assumption, T does not contain a maximal element, there is some transition $\langle m'' \rightarrowtail H'' \rangle \in T$ s.t. $\langle m' \rightarrowtail H' \rangle \prec \langle m'' \rightarrowtail H'' \rangle$. This implies that $m' < m''$ and $H'' \subseteq H'$, from which it follows by Lem. 2.5 (i) that $\mathsf{H}_{m''} \subseteq H'$. Since $m'' \in \{m \in M \mid \langle m \rightarrowtail H \rangle \in T\} \subseteq h$, we have that $h \in \mathsf{H}_{m''} \subseteq H'$, which contradicts our assumption that $h \notin H'$. □

2.3.2.2 Downward closedness

In the previous section, we have dealt with the requirement of consistency. We have defined a consistent transition set as one that allows at least one history, and we have shown that a consistent transition set amounts to a chain in the transition ordering $\prec$. In this section, we turn to the requirement of downward closedness, i.e. we close transition sets downward in the transition ordering $\prec$. And henceforth, we restrict our considerations to indeterministic transitions. As will become apparent, once transition sets are downward closed, trivial transition sets become redundant from a logical point of view: they have no bearing on the set of histories admitted by a downward closed transition set.

Closing a transition set downwards in the transition ordering $\prec$ captures the idea that the past is fixed, in accordance with the absence of backwards branching. A set of transitions $T \subseteq \mathsf{trans}(\mathcal{M})$ is *downward closed* if it contains all indeterministic transitions preceding any transition occurring in it as well. We call the set that results from closing a given transition set $T \subseteq \mathsf{trans}(\mathcal{M})$ downwards the *downward completion of* T.

Definition 2.14 (Downward closed). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, a transition set $T \subseteq \mathsf{trans}(\mathcal{M})$ is* downward closed *iff for all $t, t' \in \mathsf{trans}(\mathcal{M})$, if $t \in T$ and $t' \prec t$, then $t' \in T$.*

DEFINITION 2.15 (Downward completion). *For* $\mathcal{M} = \langle M, < \rangle$ *a BT structure and* $T \subseteq \mathsf{trans}(\mathcal{M})$ *a set of transitions, the* downward completion of T, *in symbols:* $\mathsf{dc}(T)$*, is defined as follows:*

$$\mathsf{dc}(T) := \{t \in \mathsf{trans}(\mathcal{M}) \mid \textit{there is some } t' \in T \textit{ s.t. } t \preceq t'\}.$$

The set of histories allowed by the downward completion $\mathsf{dc}(T)$ of a transition set $T \subseteq \mathsf{trans}(\mathcal{M})$ is identical to the set of histories allowed by the set T itself, i.e. $\mathsf{H}(\mathsf{dc}(T)) = \mathsf{H}(T)$. Since the transition ordering reverses the inclusion relation on the outcomes, closing a transition set downwards does not exclude any histories. From this it follows immediately that the downward completion $\mathsf{dc}(T)$ of a transition set $T \subseteq \mathsf{trans}(\mathcal{M})$ is consistent if and only if T is consistent.

LEMMA 2.16. *For* $\mathcal{M} = \langle M, < \rangle$ *a BT structure and* $T \subseteq \mathsf{trans}(\mathcal{M})$ *a set of transitions, we have that* $\mathsf{H}(T) = \mathsf{H}(\mathsf{dc}(T))$.

Proof. Since $T \subseteq \mathsf{dc}(T)$, we obviously have that $\mathsf{H}(\mathsf{dc}(T)) \subseteq \mathsf{H}(T)$. We show that $\mathsf{H}(T) \subseteq \mathsf{H}(\mathsf{dc}(T))$ holds as well. Assume for reductio that there is some history $h \in \mathsf{H}(T) \setminus \mathsf{H}(\mathsf{dc}(T))$. Then there is some transition $\langle m \rightarrowtail H \rangle \in \mathsf{dc}(T)$ s.t. $h \notin H$. By Def. 2.15 it follows that there is some $\langle m' \rightarrowtail H' \rangle \in T$ s.t. $\langle m \rightarrowtail H \rangle \prec \langle m' \rightarrowtail H' \rangle$. This implies that $H' \subseteq H$. Since $h \in \mathsf{H}(T) \subseteq H'$, it follows that $h \in H$, which contradicts our assumption. □

Due to the absence of backward branching, closing a transition set downward has no bearing on the set of histories allowed by the transition set and thus on the possible course of events represented by that transition set from an abstract point of view. And once transition sets are downward closed, trivial transition do not have any effect in that respect either. They become redundant. To be sure, it is not the case that for every transition set $T \subseteq \mathsf{TRANS}(\mathcal{M})$, it holds that $\mathsf{H}(T) = \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M}))$, i.e., not every transition set admits precisely the same histories as its restriction to indeterministic transitions does. In general, trivial transitions may very well rule out histories. They may, as it were, witness a previous indeterministic transition without being identical to that transition. However, whenever the set T is closed toward the past in the transition ordering $\prec$, we have $\mathsf{H}(T) = \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M}))$ because every trivial transition that excludes at least one history is preceded by an indeterministic transition with the very same outcome.[52]

[52]Note that in case $\mathsf{H}(T) = \mathsf{hist}(\mathcal{M})$, the intersection $T \cap \mathsf{trans}(\mathcal{M})$ equals the empty transition set $\emptyset_{\mathsf{Tr}}$, and hence $\mathsf{H}(T) = \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M}))$ is trivially fulfilled.

PROPOSITION 2.17. *Let* $\mathcal{M} = \langle M, < \rangle$ *be a BT structure, and let* T *be a set of transitions in* $\mathsf{TRANS}(\mathcal{M})$ *such that for all* $t, t' \in \mathsf{TRANS}(\mathcal{M})$*, the following holds: if* $t \in T$ *and* $t' \prec t$*, then* $t' \in T$*. Then* $\mathsf{H}(T) = \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M}))$*.*

Proof. Obviously $\mathsf{H}(T) \subseteq \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M}))$, since $(T \cap \mathsf{trans}(\mathcal{M})) \subseteq T$. We show that $\mathsf{H}(T \cap \mathsf{trans}(\mathcal{M})) \subseteq \mathsf{H}(T)$ holds as well. Assume for reductio that there is some $h \in \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M})) \setminus \mathsf{H}(T)$. Then there is some transition $\langle m \rightarrowtail H \rangle \in T$ s.t. $h \notin H$. This transition cannot be a member of $\mathsf{trans}(\mathcal{M})$ and must therefore be trivial, i.e. $H = \mathsf{H}_m$. Consider any history $h' \in H$. There exists some moment $m' < m$ s.t. $h \perp_{m'} h'$. Since $m' < m$, it follows by Lem. 2.5 (i) that $H = \mathsf{H}_m \subseteq [h']_{m'}$. This implies that $\langle m' \rightarrowtail [h']_{m'} \rangle \prec \langle m \rightarrowtail H \rangle$. Then $\langle m' \rightarrowtail [h']_{m'} \rangle \in T \cap \mathsf{trans}(\mathcal{M})$. Consequently, $\mathsf{H}(T \cap \mathsf{trans}(\mathcal{M})) \subseteq [h']_{m'}$, which contradicts our assumption that $h \in \mathsf{H}(T \cap \mathsf{trans}(\mathcal{M}))$. □

Trivial transitions are certainly interesting when filled with content, as will become apparent in chapter 4. However, in a purely structural setting, they entirely lose their significance. As purely structural elements, they become impotent when part of a downward closed transition set: for every trivial transition that excludes at least one history, there is always an indeterministic transition that fulfills precisely the same structural role, i.e. that excludes exactly the same histories.

One related remark is in order here. If we restrict our considerations to downward closed transition sets, we are provided with a local notion of consistency. Obviously, a consistent set of transitions cannot contain two different transitions with the same initial as those represent mutually exclusive alternatives for the future. In the context of downward closed transition sets, that condition already constitutes a sufficient condition for consistency. We call a transition set that contains two different transitions with the same initial *locally inconsistent*, and we show that a downward closed transition set is consistent according to Def. 2.12 if and only if it is not locally inconsistent.[53]

DEFINITION 2.18 (Local inconsistency). *Let* $\mathcal{M} = \langle M, < \rangle$ *be a BT structure. A transition set* $T \subseteq \mathsf{TRANS}(\mathcal{M})$ *is* locally inconsistent *iff there are transitions* $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \in T$ *s.t.* $\langle m \rightarrowtail H \rangle \neq \langle m' \rightarrowtail H' \rangle$ *and* $m = m'$*.*

[53] While, in the case of branching time, the two notions of consistency coincide, they come apart in the case of branching space-times, as put forth in Belnap (1992), due to what is called modal correlations or 'funny business'. See Müller *et al.* (2008) and Müller (2014a).

Proposition 2.19. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let T be a set of transitions in* $\mathsf{TRANS}(\mathcal{M})$ *such that for all $t, t' \in \mathsf{TRANS}(\mathcal{M})$, the following holds: if $t \in T$ and $t' \prec t$, then $t' \in T$. Then:*

$$\mathsf{H}(T) \neq \emptyset \quad \textit{iff} \quad T \textit{ is locally consistent.}$$

Proof. "⇒": Assume that T is locally inconsistent. Then there are $\langle m \rightarrowtail H \rangle$, $\langle m' \rightarrowtail H' \rangle \in T$ s.t. $\langle m \rightarrowtail H \rangle \neq \langle m' \rightarrowtail H' \rangle$ and $m = m'$. It follows that $H \cap H' = \emptyset$ and hence $\mathsf{H}(T) = \emptyset$.
"⇐": Assume that $\mathsf{H}(T) = \emptyset$. By Prop. 2.13 it follows that there are transitions $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \in T$ s.t. $\langle m \rightarrowtail H \rangle \not\preceq \langle m' \rightarrowtail H' \rangle$ and $\langle m' \rightarrowtail H' \rangle \not\preceq \langle m \rightarrowtail H \rangle$. Consequently, we have $H \not\subseteq H'$ and $H' \not\subseteq H$. Let $h \in H \setminus H'$, and let $h' \in H' \setminus H$. Then there exists some $m'' \in M$ s.t. $m \geq m'' \leq m'$ and $h \perp_{m''} h'$. Consequently, there are two transitions $\langle m'' \rightarrowtail [h]_{m''} \rangle$ and $\langle m'' \rightarrowtail [h']_{m''} \rangle$, and we have that $\langle m'' \rightarrowtail [h]_{m''} \rangle \preceq \langle m \rightarrowtail H \rangle$ and $\langle m'' \rightarrowtail [h']_{m''} \rangle \preceq \langle m' \rightarrowtail H' \rangle$. Since $\langle m \rightarrowtail H \rangle, \langle m' \rightarrowtail H' \rangle \in T$ and T is downward closed, both $\langle m'' \rightarrowtail [h]_{m''} \rangle$, $\langle m'' \rightarrowtail [h']_{m''} \rangle \in T$, and so T is locally inconsistent. □

A consistent set of transitions $T \subseteq \mathsf{trans}(\mathcal{M})$ in a BT structure $\mathcal{M} = \langle M, < \rangle$ is a chain in the transition ordering $\prec$ on $\mathsf{trans}(\mathcal{M})$. Accordingly, a consistent, downward closed transition set $T \subseteq \mathsf{trans}(\mathcal{M})$ amounts to a $\prec$-chain of indeterministic transitions that is complete with respect to the past. Every possible consistent extension $T' \supsetneq T$ of such a consistent, downward closed transition set is a future extension, i.e. a $\prec$-chain of transitions that stretches further into the future. Due to the restriction to indeterministic transitions, each such possible future extension rules out at least one additional history and thus represents a different possible course of events. It is worthwhile to note that not every consistent, downward closed transition set in $\mathsf{trans}(\mathcal{M})$ is identical to the downward completion of the singleton of some indeterministic transition. A consistent, downward closed set of transitions does not have to contain a greatest element.

Given a BT structure $\mathcal{M} = \langle M, < \rangle$, we can define for every moment $m \in M$, some downward closed transition set that contains all transitions in the past of m. For $m \in M$, the *set of transitions preceding* m, $\mathsf{Tr}(m)$, is the set of all transitions in $\mathsf{trans}(\mathcal{M})$ whose outcome includes the set of histories H_m.

Definition 2.20 (The set of transitions preceding m). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure and $m \in M$ a moment, the* set of transitions preceding m, *in symbols:* $\mathsf{Tr}(m)$, *is defined as follows:*

$$\mathsf{Tr}(m) := \{\langle m' \rightarrowtail H' \rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{H}_m \subseteq H'\}.$$

The set $\mathsf{Tr}(m)$ captures the past course of events up to the moment m. Note that since we are dealing with indeterministic transitions only, whenever $\langle m' \rightarrowtail H' \rangle \in \mathsf{Tr}(m)$, we have $m' < m$. Obviously, $\mathsf{Tr}(m)$ is consistent according to Def. 2.12 because $\mathsf{H}_m \subseteq \mathsf{H}(\mathsf{Tr}(m))$. We show that we even have $\mathsf{H}(\mathsf{Tr}(m)) = \mathsf{H}_m$: the set of transitions preceding m allows exactly those histories that contain the moment m.

LEMMA 2.21. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and $m \in M$ a moment. Then $\mathsf{H}(\mathsf{Tr}(m)) = \mathsf{H}_m$.*

Proof. Obviously, $\mathsf{H}_m \subseteq \mathsf{H}(\mathsf{Tr}(m))$. We show that $\mathsf{H}(\mathsf{Tr}(m)) \subseteq \mathsf{H}_m$ holds as well. Assume for reductio that there is some history $h \in \mathsf{H}(\mathsf{Tr}(m)) \setminus \mathsf{H}_m$, and consider any history $h' \in \mathsf{H}_m$. It follows that there exists some $m' < m$ such that $h \perp_{m'} h'$. Then $\mathsf{H}_m \subseteq [h']_{m'}$ and hence $\mathsf{H}_m \cap [h]_{m'} = \emptyset$. Consequently, we have that $\langle m' \rightarrowtail [h']_{m'} \rangle \in \mathsf{Tr}(m)$ and $\langle m' \rightarrowtail [h]_{m'} \rangle \notin \mathsf{Tr}(m)$. Since $h \notin [h']_{m'}$ this implies that $h \notin \mathsf{H}(\mathsf{Tr}(m))$, which contradicts our assumption. □

Consistent, downward closed transition sets allow for more generality than individual transitions do. As we said above, not every consistent, downward closed transition set can be captured in terms of the downward completion of the singleton of an individual transition. For concreteness, assume we are given a BT structure $\mathcal{M} = \langle M, < \rangle$ that contains a history h isomorphic to the real numbers $\mathbb{R}$ such that at every moment $m_i \in h$ with $i \in \mathbb{R}$, the partition Π_{m_i} is non-trivial, i.e. $\Pi_{m_i} \neq \{H_{m_i}\}$. Take some moment $m_j \in h$ and consider the set $\mathsf{Tr}(m_j)$ of transitions preceding that moment. The set $\mathsf{Tr}(m_j)$ is an ascending $\prec$-chain of transitions in $\mathsf{trans}(\mathcal{M})$ that does not have a maximum: it is not identical to the downward completion of any singleton transition set.

2.3.2.3 The set of consistent, downward closed transition sets

In section 2.3.2.1, we have spelled out what it means for a transition set to be consistent, and in section 2.3.2.2, we have closed transition sets downwards in the transition ordering $\prec$, restricting our considerations to sets of indeterministic transitions only. In this section, we consider the properties of the set of all consistent, downward closed sets of indeterministic transitions, which will build the basis of our semantics. In particular, we show that each member of that set uniquely pins down one possible course of events.

For $\mathcal{M} = \langle M, < \rangle$ a BT structure, let $\mathsf{dcts}(\mathcal{M})$ be the *set of all consistent, downward closed sets of indeterministic transitions in $\mathcal{M}$*. That is, the set $\mathsf{dcts}(\mathcal{M})$ contains, for every set $T \subseteq \mathsf{trans}(\mathcal{M})$ of indeterministic transitions that is consistent, its downward completion $\mathsf{dc}(T)$; so $\mathsf{dcts}(\mathcal{M}) = \{\mathsf{dc}(T) \mid T \subseteq \mathsf{trans}(\mathcal{M}) \text{ and } \mathsf{H}(T) \neq \emptyset\}$.

Definition 2.22 (The set $\mathsf{dcts}(\mathcal{M})$). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, the* set of consistent, downward closed sets of indeterministic transitions *in $\mathcal{M}$, $\mathsf{dcts}(\mathcal{M})$, is given by:*

$$\mathsf{dcts}(\mathcal{M}) := \{\mathsf{dc}(T) \mid T \subseteq \mathsf{trans}(\mathcal{M}) \textit{ and } \mathsf{H}(T) \neq \emptyset\}.$$

The set $\mathsf{dcts}(\mathcal{M})$ specifies the full range of transition sets that will be employed in the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$. The consistency requirement guarantees that every transition set in $\mathsf{dcts}(\mathcal{M})$ represents a possible course of events in the first place. The downward completion reflects the idea that the past is fixed and ensures that every possible consistent extension of a given transition set is a future extension. The restriction to indeterministic transitions is due to the fact that from the abstract point of view of logic, a transition set is nothing over and above the set of histories it admits: transition sets that allow precisely the same histories represent the very same course of events. In case the branching time structure under consideration is a linear one, i.e. one that does not contain any indeterministic transitions, the set $\mathsf{dcts}(\mathcal{M})$ comprises but the empty transition set, which is denoted by $\emptyset_{\mathsf{Tr}}$.

The set $\mathsf{dcts}(\mathcal{M})$ receives an ordering by set inclusion $\subseteq$. The ordering on set $\mathsf{dcts}(\mathcal{M})$ induced by the inclusion relation $\subseteq$ is a partial order that is, above all, left-linear and jointed. The left-linearity is a straightforward consequence of the fact that all transition sets in $\mathsf{dcts}(\mathcal{M})$ are downward closed chains in the transition ordering $\prec$. The jointedness of the inclusion relation $\subseteq$ on $\mathsf{dcts}(\mathcal{M})$, on the other hand, follows from the fact that for any two transition sets $T, T' \in \mathsf{dcts}(\mathcal{M})$, their intersection $T \cap T'$ is contained in $\mathsf{dcts}(\mathcal{M})$ as well. Endowed with the inclusion relation, the set $\mathsf{dcts}(\mathcal{M})$ then forms a tree. This tree represents the modal space for time to pass: the multiple possible future extensions of a given transition set represent alternative future continuations of the course of events depicted by the transition set. Whenever time passes, some possibility is realized while others fade away.[54]

[54] As Pooley (2013, p.340) rightly observes, in a branching time structure, the passage of time cannot be represented by a single chain of transition sets in the inclusion ordering, as has been suggested, for example, by McCall (1984, 1994): in a branching time structure, there is no distinguished direction, and time passes linearly. What can be represented are possibilities for time to pass, and those are best represented by a tree of transition sets. Every maximal chain of transition sets in $\mathsf{dcts}(\mathcal{M})$ depict a possibility for the passage of time. To

To every transition set $T \in \mathsf{dcts}(\mathcal{M})$, there corresponds a set of histories $\mathsf{H}(T) \in \mathsf{hist}(\mathcal{M})$, which captures the course of events depicted by the transition set. We show that the correspondence between transition sets $T \in \mathsf{dcts}(\mathcal{M})$ and sets of histories $\mathsf{H}(T) \in \mathsf{hist}(\mathcal{M})$ is injective and order-reversing. Different transition sets in $\mathsf{dcts}(\mathcal{M})$ allow for different histories, i.e. $T = T'$ iff $\mathsf{H}(T) = \mathsf{H}(T')$, and the smaller the transition set, the more histories it admits, and *vice versa*, i.e. $T \subsetneq T'$ iff $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$.

Lemma 2.23. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure. For all $T, T' \in \mathsf{dcts}(\mathcal{M})$, we have $T = T'$ iff $\mathsf{H}(T) = \mathsf{H}(T')$. In particular, it holds that $T \subsetneq T'$ iff $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$.*

Proof. Let $T, T' \in \mathsf{dcts}(\mathcal{M})$. We first show that $T = T'$ iff $\mathsf{H}(T) = \mathsf{H}(T')$. Obviously, $T = T'$ implies $\mathsf{H}(T) = \mathsf{H}(T')$. We prove that the converse holds as well. Assume that $T \neq T'$ with $\langle m' \rightarrowtail H' \rangle \notin T$ and $\langle m' \rightarrowtail H' \rangle \in T'$. Two cases can be considered. Case (i): Assume that $T \subsetneq T'$. Then for all $\langle m \rightarrowtail H \rangle \in T$, it holds that $\langle m \rightarrowtail H \rangle \prec \langle m' \rightarrowtail H' \rangle$, which implies that $\mathsf{H}_{m'} \subseteq \mathsf{H}(T)$. Since $\mathsf{H}(T') \subseteq H' \subsetneq \mathsf{H}_{m'}$, it follows that $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$. Case (ii): Assume that $T \not\subseteq T'$. Then there is some $\langle m \rightarrowtail H \rangle \in T$ s.t. $\langle m \rightarrowtail H \rangle \notin T'$, and we have $H \not\subseteq H'$ and $H' \not\subseteq H$. Let $h \in H \setminus H'$ and $h' \in H' \setminus H$. Then there is some $m'' \in M$ s.t. $m \geq m'' \leq m'$ and $h \perp_{m''} h'$. Consequently, there are two transitions $\langle m'' \rightarrowtail [h]_{m''} \rangle$ and $\langle m'' \rightarrowtail [h']_{m''} \rangle$ in $\mathsf{trans}(\mathcal{M})$ with $\langle m'' \rightarrowtail [h]_{m''} \rangle \preceq \langle m \rightarrowtail H \rangle$ and $\langle m'' \rightarrowtail [h']_{m''} \rangle \preceq \langle m' \rightarrowtail H' \rangle$. Since T and T' are downward closed in $\mathsf{trans}(\mathcal{M})$ via $\prec$, this implies that $\langle m'' \rightarrowtail [h]_{m''} \rangle \in T$ and $\langle m'' \rightarrowtail [h']_{m''} \rangle \in T'$, from which it follows that $\mathsf{H}(T) \cap \mathsf{H}(T') \neq \emptyset$ and hence $\mathsf{H}(T) \neq \mathsf{H}(T')$.
We finally show that $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$ implies $T \subsetneq T'$, having established the converse implication in case (i) above. Assume that $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$. Then there is some $h \in \mathsf{H}(T) \setminus \mathsf{H}(T')$. Let $h' \in \mathsf{H}(T')$. It follows there is some $m \in M$ s.t. $h \perp_m h'$, and we moreover have that there is some $\langle m' \rightarrowtail H' \rangle \in T'$ s.t. $h \notin H'$ and $h' \in H'$. The latter requires that $\langle m \rightarrowtail [h']_m \rangle \preceq \langle m' \rightarrowtail H' \rangle$. Since T' is downward closed in $\mathsf{trans}(\mathcal{M})$ via $\prec$, this implies that $\mathsf{dc}(\langle m \rightarrowtail [h']_m \rangle) \subseteq T'$. Since $\mathsf{H}(T) \supseteq \{h, h'\}$, for every transition $\langle m'' \rightarrowtail H'' \rangle \in T$, it holds that $m'' < m$. Consequently, $T \subseteq \mathsf{dc}(\langle m \rightarrowtail [h']_m \rangle) \setminus \{\langle m \rightarrowtail [h']_m \rangle\}$ and hence $T \subsetneq T'$. □

be sure, Pooley himself does not speak about transition sets. What he refers to are the BT substructures that correspond to transition sets, viz. our so-called prunings (cf. section 3.2).

By Prop. 2.13, every transition set $T \in \mathsf{dcts}(\mathcal{M})$ amounts to a chain in the transition ordering $\prec$ on $\mathsf{trans}(\mathcal{M})$ that is complete toward the past. There are two extreme cases: the $\prec$-chain may either be empty or maximal. Accordingly, the inclusion ordering $\subseteq$ on $\mathsf{dcts}(\mathcal{M})$ has both a minimum and maxima. The minimum consists in the empty transition set $\emptyset_{\mathsf{Tr}}$, which allows all histories in the BT structure $\mathcal{M}$, i.e. $\mathsf{H}(\emptyset_{\mathsf{Tr}}) = \mathsf{hist}(\mathcal{M})$. The maxima in the inclusion ordering $\subseteq$ on $\mathsf{trans}(\mathcal{M})$, on the other hand, are given by maximal consistent transition sets in $\mathsf{dcts}(\mathcal{M})$. In the remainder of this section, we show that each such maximal consistent transition set in $\mathsf{dcts}(\mathcal{M})$ corresponds one-to-one to a history in $\mathsf{hist}(\mathcal{M})$ via the notion of the set of histories admitted by a transition set. Of course, there are lots of intermediate cases. In chapter 3, we generalize our result: we show that every transition set in $\mathsf{dcts}(\mathcal{M})$ corresponds one-to-one to a substructure of $\mathcal{M}$ with domain $\bigcup \mathsf{H}(T)$, a so-called pruning, which comprises at least one history from $\mathsf{hist}(\mathcal{M})$ and is such that if it contains a branching point, it contains all moments above that branching point as well.

A set of transitions $T \subseteq \mathsf{trans}(\mathcal{M})$ is maximal consistent if and only if it is consistent but none of its proper supersets $T' \supsetneq T$ in $\mathsf{trans}(\mathcal{M})$ is. The notion of a maximal consistent set of transitions provides the analog to the notion of a history. Just as a history is a maximal $<$-chain in M, a maximal consistent transition set is a maximal $\prec$-chain in $\mathsf{trans}(\mathcal{M})$. We show that every maximal consistent transition in $\mathsf{dcts}(\mathcal{M})$ set allows exactly one history and thus represents a complete possible course of events. Non-maximal $\prec$-chains in $\mathsf{dcts}(\mathcal{M})$, on the other hand, allow more than one history, and the possible courses of events they represent are incomplete ones, viz. ones that allow for alternative possible future continuations. This is a straightforward consequence from Prop. 2.24 given Lem. 2.23.

Proposition 2.24. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and $T \subseteq \mathsf{trans}(\mathcal{M})$ a maximal consistent set of transitions. Then $\mathsf{H}(T) = \{h\}$ for some $h \in \mathsf{hist}(\mathcal{M})$.*

Proof. Since $T \subseteq \mathsf{trans}(\mathcal{M})$ is maximal consistent, it holds that $\mathsf{H}(T) \neq \emptyset$. Then there is at least one history $h \in \mathsf{H}(T)$. Assume for reductio that there is another history $h' \in \mathsf{hist}(\mathcal{M})$ s.t. $h' \neq h$ and $\mathsf{H}(T) \supseteq \{h, h'\}$. Then there is some moment $m \in M$ s.t. $h \perp_m h'$ and so $\Pi_m \supseteq \{[h]_m, [h']_m\}$. We show that $\langle m \rightarrowtail [h]_m \rangle \notin T$ while $\mathsf{H}(T \cup \{\langle m \rightarrowtail [h]_m \rangle\}) \neq \emptyset$. Assume that $\langle m \rightarrowtail [h]_m \rangle \in T$. Then $\mathsf{H}(T) \subseteq [h]_m$. Since $h' \notin [h]_m$, it follows that $h' \notin \mathsf{H}(T)$, which contradicts our assumption that $\mathsf{H}(T) \supseteq \{h, h'\}$. Therefore, $\langle m \rightarrowtail [h]_m \rangle \notin T$. Since $h \in \mathsf{H}(T) \cap [h]_m$, it nevertheless holds that $\mathsf{H}(T \cup \{\langle m \rightarrowtail [h]_m \rangle\}) \neq \emptyset$. This contradicts the maximal consistency of T. □

For any history $h \in \mathsf{hist}(\mathcal{M})$, we can define the *set of transitions characterizing* h, $\mathsf{Tr}(h)$, as the subset of $\mathsf{trans}(\mathcal{M})$ that contains all and only those transitions that allow h to occur.

Definition 2.25 (The set of transitions characterizing h). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure and $h \in \mathsf{hist}(\mathcal{M})$ a history, the* set of transitions characterizing h*, in symbols:* $\mathsf{Tr}(h)$*, is given by:*

$$\mathsf{Tr}(h) := \{\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M}) \mid h \in H\}.$$

Obviously, the set $\mathsf{Tr}(h)$ is consistent according to Def. 2.12, since $h \in \mathsf{H}(\mathsf{Tr}(h))$. We show that the set $\mathsf{Tr}(h)$ is even maximal consistent.

Proposition 2.26. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and $h \in \mathsf{hist}(\mathcal{M})$ a history. The set $\mathsf{Tr}(h)$ of transitions characterizing the history h is maximal consistent.*

Proof. Obviously, $\mathsf{Tr}(h)$ is consistent, since $h \in \mathsf{H}(\mathsf{Tr}(h))$. We show that $\mathsf{Tr}(h)$ is maximal consistent. Let $\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M})$ be a transition s.t. $\langle m \rightarrowtail H \rangle \notin \mathsf{Tr}(h)$. Then $h \notin H$. Consider some history $h' \in H$. We have that $h \not\equiv_m h'$. It follows that there is some moment $m' \in M$ s.t. $m' \leq m$ and $h \perp_{m'} h'$. Since $H = [h']_m \subseteq [h']_{m'}$ and $\langle m' \rightarrowtail [h]_{m'} \rangle \in \mathsf{Tr}(h)$, it holds that $\mathsf{H}(\mathsf{Tr}(h) \cup \{\langle m \rightarrowtail H \rangle\}) = \emptyset$. □

From Prop. 2.24 and Prop. 2.26 it then immediately follows that for every history $h \in \mathsf{hist}(\mathcal{M})$, the set of histories characterizing that history allows but a single history, viz. the history h itself. Moreover, on the basis of Lem. 2.23, we can conclude that every maximal consistent set of transitions is identical to a set $\mathsf{Tr}(h)$ for some history $h \in \mathsf{hist}(\mathcal{M})$. There is thus a natural one-to-one correspondence between histories and maximal consistent sets of transitions: every maximal consistent set of transitions allows exactly one history $h \in \mathsf{hist}(\mathcal{M})$ and is of the form $\mathsf{Tr}(h)$; and for every history $h \in \mathsf{hist}(\mathcal{M})$, there is a maximal consistent set of transitions $\mathsf{Tr}(h)$ that characterizes that history.

2.4 BT semantics with sets of transitions

In the previous section, we have introduced the notion of a transition, and we have defined the set $\mathsf{dcts}(\mathcal{M})$ of all consistent, downward closed sets of indeterministic transitions in a BT structure $\mathcal{M} = \langle M, < \rangle$. We have thereby provided the basis of the transition semantics we are proposing. In this section, we present the transition framework and point out the adequacy of the novel semantics for the representation of real possibilities.

The basic idea behind the transition semantics is to replace the Ockhamist history parameter by sets of transitions: sentences are evaluated at a moment in a BT structure $\mathcal{M} = \langle M, < \rangle$ relative to a consistent, downward closed set of indeterministic transitions $T \in \mathsf{dcts}(\mathcal{M})$. Whereas histories represent complete possible courses of events, sets of transitions can represent incomplete parts thereof as well. Every consistent, downward closed set of indeterministic transitions corresponds one-to-one to a possible course of events that stretches linearly from the past toward a possibly open future. Due to the relativization of the semantic evaluation to both complete and incomplete possible courses of events, a new kind of intensional operator becomes interpretable. In addition to temporal and modal operators, we introduce a so-called stability operator into our language, which is interpreted as a universal quantifier over the possible future extensions of a given transition set. In the light of the stability operator, the transition parameter becomes dynamic. Not only does it provide a second local perspective in time, but that second perspective can also be varied independently of the moment parameter.

The semantics developed along those lines exploits the structural resources a branching time structure has to offer and provides a fine-grained picture of the interrelation of modality and time that adequately captures the dynamics of real possibilities: the stability operator enables us to specify exactly how and how far the future has to unfold for the contingency at a moment to dissolve, and local witnesses for future possibilities become available. In section 2.4.1, we introduce the language of the transition semantics and define the notion of a transition model. In section 2.4.2, we then discuss, one by one, the semantic clauses for the various kinds of intensional operators of the transition language. In section 2.4.3, we put the semantics to work and illustrate how sentences about the contingent future are treated within that semantic framework.

2.4.1 The transition language $\mathcal{L}_\mathrm{t}$ and its models

The richness of a BT language does not least depend on the choice of parameters of truth. In Peirceanism, where the semantic evaluation is relativized to a moment parameter only, only temporal operators are interpretable. On the Ockhamist account, which makes use of a history as a second parameter of truth next to the moment parameter, in addition to temporal operators, modal operators are interpretable. And in the transition semantics, where the Ockhamist history parameter is replaced by a set of transitions, over and above temporal operators and modal operators, a stability operator becomes interpretable as well. The inclusion of a stability operator into the language is enabled by the fact in the transition semantics, the semantic evaluation at a moment depends on a set of transitions, which can represent a complete or incomplete possible course of events.

The language of the transition semantics then contains three different kinds of intensional operators: temporal, modal and stability operators. To be concrete, the transition language $\mathcal{L}_{\mathsf{t}}$ extends the propositional language $\mathcal{L}$ by the following primitive operators: we have a past operator P, two operators f and F for the future, a weak and a strong one, an operator for inevitability $\Box$ and a stability operator S.

Definition 2.27 (The transition language $\mathcal{L}_{\mathsf{t}}$). *The alphabet of the transition language $\mathcal{L}_{\mathsf{t}}$ consists of the set of propositional variables* At *and the following list of primitive operators:* $\neg$, $\wedge$, P, f, F, $\Box$ *and* S. *The syntax of the transition language $\mathcal{L}_{\mathsf{t}}$ is specified by the following* BNF:

$$\phi \ ::= \ p \mid \neg\phi \mid \phi \wedge \phi \mid \mathsf{P}\phi \mid \mathsf{f}\phi \mid \mathsf{F}\phi \mid \Box\phi \mid \mathsf{S}\phi.$$

In the transition semantics, sentences are evaluated on a BT structure $\mathcal{M} = \langle M, < \rangle$ with respect to a moment $m \in M$ and a consistent, downward closed set of indeterministic transitions in $\mathsf{dcts}(\mathcal{M})$. The set $\mathsf{dcts}(\mathcal{M})$ provides the full range of transition sets that can serve as a second parameter of truth.[55] The semantics makes use of indeterministic transitions only, and only transition sets that are consistent and downward closed are taken into account. The requirement of consistency is a natural prerequisite: for, an inconsistent set of transitions does not represent a possible course of events at all. The restriction to downward closed transition sets reflects the idea that the past is fixed and ensures that every proper superset of a transition set is a future exten-

[55] Of course, there are alternative ways of spelling out the semantics without restricting the semantic evaluation to downward closed sets of indeterministic transitions from the outset. Yet, generalizing the transition parameter is only possible at high costs. One possibility would be to take into account all consistent transition sets definable in a BT structure and treat transition sets that allow exactly the same histories as equivalent. This, however, is only possible at the cost of either sacrificing the substitution property of the resulting logic or increasing the complexity of the semantic clauses in virtue of having the semantics operate on equivalence classes of transition sets. Another possibility would be to replace transition sets by prunings, i.e. by the substructures that are spanned by the corresponding sets of histories (cf. Def. 3.3). In this case, however, the semantics would get rather abstract. Rather than considering chains of local transitions, we would be working with global objects, which are much harder to grasp. Transitions would lose their relevance and the internal structure of transition sets would be entirely lost.

sion. Confining the semantic evaluation to sets of indeterministic transitions is motivated by the fact that, from a logical point of view, a transition set is equivalent to the set of histories it admits. Trivial transitions become negligible: they do not contribute to the course of events represented by a downward closed transition set.

The parameters of truth are then pairs consisting of a moment $m \in M$ and a transition set $T \in \mathsf{dcts}(\mathcal{M})$. In order for a pair consisting of a moment $m \in M$ and a set of transitions $T \in \mathsf{dcts}(\mathcal{M})$ to constitute a suitable index of evaluation, the transition set T must allow at least one history that contains the moment of evaluation m, i.e. $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. In other words, the moment $m \in M$ must be *compatible* with the transition set $T \in \mathsf{dcts}(\mathcal{M})$. Compatibility is the counterpart of the Ockhamist requirement that the moment of evaluation must always be contained in the respective history of evaluation. In analogy with the Ockhamist case, we employ the notation "m/T" in order to indicate that the compatibility condition $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$ is met.

DEFINITION 2.28 (Compatibility). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, $m \in M$ a moment and $T \in \mathsf{dcts}(\mathcal{M})$ a set of transitions, we say that m is* compatible *with T iff $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$.*

With those preliminaries in place, we can now provide the definition of a transition model. A *transition model* is defined as a triple $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$, where $\mathcal{M} = \langle M, < \rangle$ is a BT structure and v_{t} a valuation function that assigns truth values to the propositional variables $p \in \mathsf{At}$ relative to a moment $m \in M$ and a transition set $T \in \mathsf{dcts}(\mathcal{M})$ compatible with that moment.[56]

DEFINITION 2.29 (Transition model). *A* transition model *is an ordered triple $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ consisting of a BT structure $\mathcal{M} = \langle M, < \rangle$ and a valuation function $v_{\mathsf{t}} : \mathsf{At} \times \{m/T \mid m \in M,\ T \in \mathsf{dcts}(\mathcal{M})$ and $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset\} \to \{0, 1\}$.*

Note that in order to preserve the substitution property of the resulting logic, we have the truth values of the propositional variables depend on both parameters of truth. A solely moment-dependent evaluation of the propositional variables can be obtained within that more general framework by imposing the following additional constraint on the valuation function v_{t}: $v_{\mathsf{t}}(p, m/T) = 1$ iff for all $T' \in \mathsf{dcts}(\mathcal{M})$ such that $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$, $v_{\mathsf{t}}(p, m/T') = 1$. In light of the meaning of the inevitability operator $\Box$, that additional constraint amounts to

[56] In section 2.5.1, we generalize the definition of a transition model by invoking the notion of a transition structure. See Def. 2.33 for the notion of a transition structure and Def. 2.34 for the general definition of a model on such a structure.

the claim that propositional variables are either inevitably true or inevitable false.[57]

2.4.2 The semantic clauses

Given a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$, we can extend the valuation v_{t} on the set of propositional variables At to any arbitrary sentences of the language. We use $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \phi$ in order to indicate that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true at an index of evaluation m/T in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ on a BT structure $\mathcal{M} = \langle M, < \rangle$ according to the transition semantics. The expressions $\mathfrak{M} \vDash_{\mathsf{t}} \phi$ for validity in a transition model, $\mathcal{M} \vDash_{\mathsf{t}} \phi$ for validity in a BT structure and $\vDash_{\mathsf{t}} \phi$ for general validity are defined in the obvious way by means of the usual generalizations over indices of evaluation, models and structures, respectively.

The truth of an arbitrary sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ of the transition language in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ at an index pair m/T is defined recursively. The recursive semantic clauses for the propositional variables $p \in \mathsf{At}$ and the truth-functional connectives $\neg$ and $\wedge$ are straightforward and are stated below without further comment.

(At) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} p$ iff $v_{\mathsf{t}}(p, m/T) = 1$;

($\neg$) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \neg\phi$ iff $\mathfrak{M}, m/T \nvDash_{\mathsf{t}} \phi$;

($\wedge$) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \phi \wedge \psi$ iff $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \phi$ and $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \psi$.

In the following, we discuss the semantic clauses for the three different kinds of intensional operators one by one. In section 2.4.2.1, we provide the semantic clauses for the temporal operators, in section 2.4.2.2, we turn to the modal operators, and we discuss the stability operators in section 2.4.2.3.

2.4.2.1 Temporal operators

The transition language $\mathcal{L}_{\mathsf{t}}$ contains three primitive temporal operators: one for the past and two for the future. We discuss those different operators separately below. We start with the strong future operator F (section 2.4.2.1.1), we then discuss the weak future operator f (section 2.4.2.1.2) and finally the past operator P (section 2.4.2.1.3).

In the case of the temporal operators, the transition set is kept fixed and the moment of evaluation is shifted in a way compatible with the given transition set. The transition set places restrictions on the domain of quantification of the

[57] Cf. our discussion of the treatment of propositional variables in section 1.3.2.2.

temporal operators. It constrains the range of moments that are to be taken into account in the semantic evaluation. While such a restriction is insignificant in the case of the past operators, it is of great significance in the case of the future operators, as we shall see.

2.4.2.1.1 The strong future operator The crucial point in developing a semantics on a BT structure consists in providing an appropriate notion of future truth. The strong future operator F of the transition semantics captures what it is for a sentence about the future to be plainly true with respect to a set of transitions. That is, it captures future truth with respect to complete or incomplete possible courses of events.

Our strong future operator F has both pristine Peircean and Ockhamist traits: it combines the Peircean idea of universally quantifying over future possibilities with the Ockhamist idea of relativizing truth at a moment to a possible course of events. Unlike in Ockhamism, the possible course of events needs not to be a complete one, however; and whereas the strong Peircean future operator requires a future witness in every possible future continuation of the moment of evaluation, our future operator demands a witness only in those future continuations of the moment of evaluation that are possible extensions of the given transition set. Along those lines, we say that a sentence of the form $\mathsf{F}\phi$ is true in a transition model $\mathfrak{M} = \langle M, < \rangle$ at an index index of evaluation m/T if and only if for every future extension $T' \supseteq T$ that is compatible with the moment of evaluation m, there is a compatible future moment $m' > m$ at which ϕ is true with respect to the original transition set T.

($\mathsf{F}^\sharp$) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{F}\phi$ iff for all $T' \supseteq T$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$, there is some $m' > m$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_{m'} \neq \emptyset$ and $\mathfrak{M}, m'/T \vDash_{\mathsf{t}} \phi$.

Note that the sole function of the universal quantification over the possible future extensions T' of the given transition set T is to specify the range of possible future continuations of m that are required to contain a witness for the future claim. The semantic clause ($\mathsf{F}^\sharp$) can be shown to be equivalent to the following condition (F), in which the universal quantification over future possibilities is spelled out by reference to the set of histories allowed by the given transition set rather than in terms of its possible future extensions:[58]

(F) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{F}\phi$ iff for all $h \in \mathsf{H}(T) \cap \mathsf{H}_m$, there is some $m' \in h$ s.t. $m' > m$ and $\mathfrak{M}, m'/T \vDash_{\mathsf{t}} \phi$.

[58]This is the semantic clause for the future operator provided in Müller (2014a).

According to condition (F), a sentence of the form $\mathsf{F}\phi$ is true in a transition model $\mathfrak{M}$ at an index pair m/T if and only if every history in H_m that is allowed by T contains some later moment $m' > m$ at which ϕ is true with respect to T.

Lemma 2.30. *The semantic clauses* $(\mathsf{F}^\sharp)$ *and* (F) *are equivalent.*

Proof. "$\Rightarrow$": Assume that the $(\mathsf{F}^\sharp)$-condition holds, i.e., suppose that for every $T' \supseteq T$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$, there is a future moment $m' > m$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_{m'} \neq \emptyset$ and $\mathfrak{M}, m'/T \vDash_{\mathsf{t}} \phi$. Then this holds, in particular, for every maximal consistent extension T' of T that is compatible with m. Since, by Prop. 2.26, for every $h \in \mathsf{H}(T) \cap \mathsf{H}_m$, there is a maximal consistent extension $\mathsf{Tr}(h) \supseteq T$ with $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$, condition $(\mathsf{F}^\sharp)$ implies the existence of a future witness $m' > m$ in every history $h \in \mathsf{H}(T) \cap \mathsf{H}_m$.
"$\Leftarrow$": Assume that the (F)-condition holds, i.e., suppose that for all histories $h \in \mathsf{H}(T) \cap \mathsf{H}_m$, there is some $m' \in h$ s.t. $m' > m$ and $\mathfrak{M}, m'/T \vDash_{\mathsf{t}} \phi$. Let $T' \supseteq T$ be an extension of T s.t. $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$. Then $\mathsf{H}(T') \subseteq \mathsf{H}(T)$. Consider some $h \in \mathsf{H}(T') \cap \mathsf{H}_m$. By our assumption, it follows there is some future witness $m' > m$ in h, so that $\mathsf{H}(T') \cap \mathsf{H}_{m'} \neq \emptyset$. □

The semantic clause (F) is much easier to grasp, and it is also more general in that it does not depend on which transition sets are taken into account in the semantic evaluation, which will become important in section 2.5.1. We will therefore make use of that condition in what follows.

In analogy with the Peircean case, we set $\mathsf{g} := \neg\mathsf{F}\neg$ for the dual of the strong future operator F. A sentence of the form $\mathsf{g}\phi$ is true at an index of evaluation m/T in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}}\rangle$ if and only if there is some history in H_m that is allowed by T in which ϕ is true at every future moment.

2.4.2.1.2 The weak future operator Just as the Peircean language, the transition language contains, next to the strong future operator F, a weak future operator f. Whereas the strong future operator F requires a future witness in every possible future continuation of the moment of evaluation that is admitted by the transition set, the weak future f operator requires a future witness in only one such possible future continuation. A sentence of the form $\mathsf{f}\phi$ is true in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}}\rangle$ at an index of evaluation m/T if and only if there is some future moment $m' > m$ compatible with the transition set T at which ϕ is true with respect to T.

(f) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{f}\phi$ iff there is some $m' > m$ s.t. $\mathsf{H}(T) \cap \mathsf{H}_{m'} \neq \emptyset$ and $\mathfrak{M}, m'/T \vDash_{\mathsf{t}} \phi$.

The condition provided in the semantic clause for f is equivalent to the requirement that there be at least one history $h \in \mathsf{H}(T) \cap \mathsf{H}_m$ that contains a future witness $m' > m$. Unlike in the case of its strong counterpart, however, there is no need to quantify over histories. Just as the semantic clause for the weak Peircean future operator $\mathsf{f_p}$, the semantic clause for our weak future operator f only involves quantification over future moments. The crucial difference with the Peircean $\mathsf{f_p}$ operator consists in the role of the transition set, which places restrictions on the set of moments that are to be taken into consideration.

The weak future operator is too weak to provide a proper notion of future truth. It is more interesting from a technical than from a philosophical point of view, and it becomes important in the completeness proof. Its dual $\mathsf{G} := \neg\mathsf{f}\neg$ has some intuitive plausibility though. A sentence of the form $\mathsf{G}\phi$ is true in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ at an index of evaluation m/T if and only if ϕ is true with respect to T at every future moment $m' > m$ compatible with T. Whereas, intuitively, the meaning of the weak future operator f is "at some point in the common future of m and T", the meaning of its dual G is "always in the common future of m and T".

2.4.2.1.3 The past operator The semantic clause for the past operator P is straightforward: a sentence of the form $\mathsf{P}\phi$ is true in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ at an index of evaluation m/T if and only if there is some earlier moment $m' < m$ at which ϕ is true with respect to T. Due to the absence of backward branching, every moment has a unique past so that no further specification is needed in that case. The transition set adds no restrictions: given that the moment of evaluation m is compatible with the transition set T, all moments in its unique past are so as well.

(P) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{P}\phi$ iff there is some $m' < m$ s.t. $\mathfrak{M}, m'/T \vDash_{\mathsf{t}} \phi$.

The past operator P is an existential quantifier over the set of moments in the unique past of the moment of evaluation. Its dual $\mathsf{H} := \neg\mathsf{P}\neg$, accordingly, amounts to a universal quantifier over that set. A sentence of the form $\mathsf{H}\phi$ is true at an index of evaluation m/T in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ if and only if ϕ is true with respect to T at every earlier moment $m' < m$. Whereas, the past operator P intuitively reads as "at some point of the past", the intuitive meaning of its dual H is "always in the past".

2.4.2.2 Modal operators

Since—like in Ockhamism but unlike in Peirceanism—the semantic evaluation at a moment is relativized to a second parameter of truth, which specifies some possible course of events, modal operators are interpretable, and truth and inevitability come apart. The transition language contains a primitive operator $\Box$ for inevitability. An operator $\Diamond$ for possibility can be defined as its dual, i.e. $\Diamond := \neg\Box\neg$.

In the case of the modal operators, the moment of evaluation is kept constant and the modalities are interpreted as quantifiers over all transition sets in $\mathsf{dcts}(\mathcal{M})$ that are compatible with the moment of evaluation. The inevitability $\Box$ operator amounts to universal quantification, the possibility operator $\Diamond$ to existential quantification, respectively. As in standard Ockhamism, the modality involved is $\mathsf{S5}$. The relevant accessibility relation is an equivalence relation (cf. Prop. 3.17). The truth of a sentence of the form $\Box\phi$ in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}}\rangle$ at an index of evaluation m/T can be understood along the following lines: relative to any—complete or incomplete—possible course of events compatible with m, the sentence ϕ is true at m.

($\Box$) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \Box\phi$ iff for all $T' \in \mathsf{dcts}(\mathcal{M})$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$,
$\mathfrak{M}, m/T' \vDash_{\mathsf{t}} \phi$.

Note that, unlike in Ockhamism, an incomplete possible course of events suffices as a witness for a possibility claim. A sentence of the form $\Diamond\phi$ is true in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}}\rangle$ at an index of evaluation m/T if and only if there is some transition set compatible with m with respect to which ϕ is true at m. What is possible is thereby moment-dependent. Inevitability $\Box$ and possibility $\Diamond$ are historical modalities: the range of admissible transition sets may decrease toward the future but never toward the past. That is to say, the implication $\mathsf{P}\Box p \rightarrow \Box\mathsf{P}p$ is generally valid while the implication $\mathsf{F}\Box p \rightarrow \Box\mathsf{F}p$ is not.

2.4.2.3 Stability operators

Evaluating sentences with respect to possibly non-maximal chains of transitions rather than with respect to entire histories brings in a new phenomenon that is specific to the transition approach. The truth value of a sentence at a moment can change if the transition set is extended so that it stretches further into the future. What has been contingent eventually becomes settled.

Next to temporal and modal operators, the transition semantics allows for a stability operator S and its dual $\mathsf{Ƨ}$. As in the case of the modal operators, in the case of the stability operators, the moment of evaluation is kept fixed, and the transition set is shifted in a way compatible with the moment of evaluation. Unlike in the case of the modal operators, the domain of quantification is, however, restricted to supersets of the given transition set. That is, the stability operator S is interpreted as a universal quantifier over the possible future extensions of the given transition set that are compatible with the moment of evaluation. Its dual $\mathsf{Ƨ}$ then amounts to an existential quantifier over that set. The modality involved is $\mathsf{S4}$. The relevant accessibility relation is reflexive, antisymmetric and transitive (cf. Prop. 3.17).

The stability operator S enables us to specify how and how far the future has to unfold for the truth value of a sentence at a moment to become settled, or stable, as we will say. A sentence of the form $\mathsf{S}\phi$ is true in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ at an index of evaluation m/T if and only if ϕ is true at m with respect to every possible extension T' of T that is compatible with m. Given that $\mathsf{S}\phi$ is true with respect to m/T, whatever else will happen, i.e., no matter how we extend the transition set, ϕ remains true at m. Relative to the course of events specified by T, it is already settled that ϕ is true at m.

(S) $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{S}\phi$ iff for all $T' \supseteq T$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$, $\mathfrak{M}, m/T' \vDash_{\mathsf{t}} \phi$.

Truth and stability come apart. If a sentence of the form $\mathsf{S}\phi$ is true at an index of evaluation m/T in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$, i.e. $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{S}\phi$, we say that ϕ is *stably-true* relative to that index. Accordingly, we say that ϕ is *stably-false* relative to an index of evaluation m/T if its negation $\neg\phi$ is stably-true at that index, i.e. $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{S}\neg\phi$.

DEFINITION 2.31 (Stable truth and falsity). *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ a transition model on that structure. Given a moment $m \in M$ and a set of transitions $T \in \mathsf{dcts}(\mathcal{M})$ such that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$, we say that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is* stably-true *with respect to m/T iff $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{S}\phi$. We say that ϕ is* stably-false *with respect to m/T iff $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \mathsf{S}\neg\phi$.*

There are sentences that classify as neither stably-true nor stably-false in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ with respect to a given index of evaluation m/T. Those sentences are said to be *contingent* with respect to m/T. They are true at the moment m with respect to one possible extension of T that is compatible with m but false with respect to another. To put it in concrete terms, given a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$, a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is contingent relative to an index of evaluation m/T if and only if $\text{Ƨ}\phi \wedge \text{Ƨ}\neg\phi$ is true at m/T, i.e. $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \text{Ƨ}\phi \wedge \text{Ƨ}\neg\phi$, or equivalently, $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \neg\mathsf{S}\phi \wedge \neg\mathsf{S}\neg\phi$.

Definition 2.32 (Contingency). *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ a transition model on that structure. Given a moment $m \in M$ and a set of transitions $T \in \mathsf{dcts}(\mathcal{M})$ such that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$, we say that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is* contingent *with respect to m/T iff $\mathfrak{M}, m/T \vDash_{\mathsf{t}} \text{Ƨ}\phi \wedge \text{Ƨ}\neg\phi$.*

A sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ that is contingent in a transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ relative to some index of evaluation m/T can become stably-true or stably-false with respect to some extension T' of the transition set T. Once the truth value of the sentence has stabilized with respect to some index, however, it remains stable under all extensions of the transition set in question. In particular, a sentence that is stably-true with respect to some index m/T is stably-true at m with respect to all future extensions $T' \supseteq T$ compatible with m.

As said, the domain of quantification of the stability operator S is only a subset of the domain of quantification of the inevitability operator $\Box$. Whereas the inevitability operator quantifies over all transition sets compatible with the moment of evaluation, the stability operator quantifies over only the possible future extensions of the given transition set. Inevitability implies stability, but not *vice versa*. That the truth of a sentence ϕ is inevitable at an index m/T means that the sentence ϕ is true at the moment m, no matter what happens. That a sentence ϕ is stably-true relative to an index m/T, on the other hand, expresses that the course of events captured by the transition that T is sufficient to settle the truth of ϕ at m: with respect to that course of events, the sentence ϕ is true at the moment m, no matter what will happen later on. The stability operator allows for a perspicuous treatment of sentences about the future whose truth value at a moment only stabilizes as the future unfolds. It captures the behavior of the truth value of a sentence about the open future at a moment in the course of time, viz. its changing from contingent to stably-true or stably-false, and thus adequately reflects the dynamics of real possibilities.

2.4.3 Evaluating sentences about the future

In the transition semantics, sentences are evaluated at a moment with respect to a possible course of events compatible with that moment, just as in Ockhamism. Both frameworks are tailored to the idea that the truth value of a sentence about the future depends on how the future unfolds. The crucial difference between the transition semantics and Ockhamism consists in the fact that whereas the histories employed in the Ockhamist semantics represent complete possible courses of events, the transition semantics allows for the relativization to incomplete possible courses of events as well. The static, global Ockhamist history parameter is replaced by a dynamic, local transition parameter. In order to illustrate what that difference amounts to, let us have a look at how sentences about the future are treated in the transition semantics.

For that purpose, consider a transition model on a BT structure $\mathcal{M} = \langle M, < \rangle$ that contains a branching point $m \in M$. Assume that there is one possible future continuation of m in which p always is the case and another in which p never is the case, where the truth value of p is only moment-dependent. In the transition model $\mathfrak{M} = \langle M, <, v_{\mathsf{t}} \rangle$ provided in Fig. 2.1, for all moments $m' \in M$ such that $m' > m$ and for all transition sets $T \in \mathsf{dcts}(\mathcal{M})$ such that $\mathsf{H}(T) \cap \mathsf{H}_{m'} \neq \emptyset$, we have $v_{\mathsf{t}}(p, m'/T) = 1$ if $m' \in h_2 \cup h_3$, and $v_{\mathsf{t}}(p, m'/T) = 0$ if $m' \in h_4$. We will now investigate the behavior of the truth value of the sentence $\mathsf{F}p$ and its contrary prediction $\mathsf{F}\neg p$ in the model $\mathfrak{M}$ at the moment m with respect to different transition sets, as indicated in Fig. 2.1.

Let us start with the empty transition set, $\emptyset_{\mathsf{Tr}}$, the smallest transition set possible. Since $\mathsf{H}(\emptyset_{\mathsf{Tr}}) = \mathsf{hist}(\mathcal{M}) = \{h_1, h_2, h_3, h_4\}$, the transition set allows all histories in H_m: it excludes none of the histories passing through our moment of evaluation m. We have $\mathsf{H}(\emptyset_{\mathsf{Tr}}) \cap \mathsf{H}_m = \mathsf{H}_m = \{h_2, h_3, h_4\}$. If we evaluate the sentences $\mathsf{F}p$ and $\mathsf{F}\neg p$ at the index $m/\emptyset_{\mathsf{Tr}}$, both sentences turn out false, i.e. $\mathfrak{M}, m/\emptyset_{\mathsf{Tr}} \not\models_{\mathsf{t}} \mathsf{F}p$ and $\mathfrak{M}, m/\emptyset_{\mathsf{Tr}} \not\models_{\mathsf{t}} \mathsf{F}\neg p$. The sentence $\mathsf{F}p$ is false at $m/\emptyset_{\mathsf{Tr}}$ since the empty transition set $\emptyset_{\mathsf{Tr}}$ admits the history $h_4 \in \mathsf{H}_m$, in which p is false at every moment later than m; and $\mathsf{F}\neg p$ is likewise false at $m/\emptyset_{\mathsf{Tr}}$ since the empty transition set $\emptyset_{\mathsf{Tr}}$ also allows the history $h_2 \in \mathsf{H}_m$, in which p is the case at every moment later than m.

The same is true if we evaluate our sentence pair at the moment m with respect to the set of transitions preceding m, $\mathsf{Tr}(m)$. The transition set $\mathsf{Tr}(m)$ is a proper extension of the empty transition set $\emptyset_{\mathsf{Tr}}$. It captures the past course of events up to the moment of evaluation m. Even though, as a proper extension of $\emptyset_{\mathsf{Tr}}$, the transition set $\mathsf{Tr}(m)$ rules out at least one history in $\mathsf{H}(\emptyset_{\mathsf{Tr}})$, viz. the history h_1, the set of histories compatible with m is nevertheless the same in each case. We have $\mathsf{H}(\emptyset_{\mathsf{Tr}}) \cap \mathsf{H}_m = \mathsf{H}(\mathsf{Tr}(m)) \cap \mathsf{H}_m$ despite the fact that $\mathsf{H}(\mathsf{Tr}(m)) = \mathsf{H}(\emptyset_{\mathsf{Tr}}) \setminus \{h_1\} = \{h_2, h_3, h_4\}$. Consequently, at the index $m/\mathsf{Tr}(m)$,

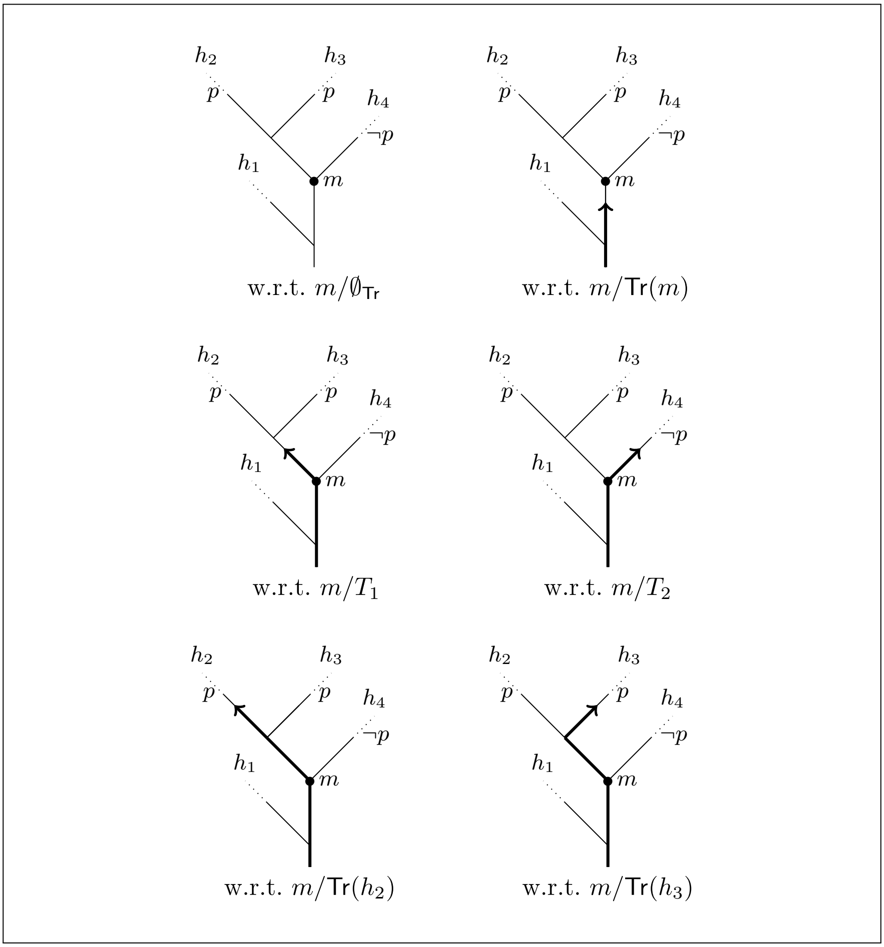
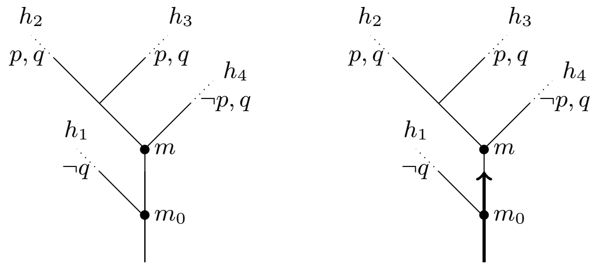


Figure 2.1: Evaluation w.r.t. different transition sets.

again both the sentence $\mathsf{F}p$ and its contrary prediction $\mathsf{F}\neg p$ turn out false, i.e. $\mathfrak{M}, m/\mathsf{Tr}(m) \not\models_{\mathsf{t}} \mathsf{F}p$ and $\mathfrak{M}, m/\mathsf{Tr}(m) \not\models_{\mathsf{t}} \mathsf{F}\neg p$. With both the sentences $\mathsf{F}p$ and $\mathsf{F}\neg p$ being false at $m/\emptyset_{\mathsf{Tr}}$ and $m/\mathsf{Tr}(m)$, respectively, their disjunction is likewise false at those indices, i.e. $\mathfrak{M}, m/\emptyset_{\mathsf{Tr}} \not\models_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$ and $\mathfrak{M}, m/\mathsf{Tr}(m) \not\models_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$. Yet, the disjunction is false at those indices only contingently, as we shall see.

The two transition sets $\emptyset_{\mathsf{Tr}}$ and $\mathsf{Tr}(m)$ then do not differ as to which truth values are assigned on their basis to our sentences $\mathsf{F}p$ and $\mathsf{F}\neg p$ at the moment m. This is, however, not to say that the two transition sets always make precisely the same sentences true. Assume that in the model $\mathfrak{M}$, it holds that q is true at all and only those moments that are later than m, as illustrated in Fig. 2.2.

That is, suppose that for all $m' \in M$ and for all transition sets $T \in \mathsf{dcts}(\mathcal{M})$ such that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$, we have $v_{\mathsf{t}}(q, m'/T) = 1$ iff $m' > m$. Then the sentence $\mathsf{HF}q$ is true at the moment m with respect to the transition set $\mathsf{Tr}(m)$ but false with respect to the empty transition set $\emptyset_{\mathsf{Tr}}$, i.e. $\mathfrak{M}, m/\mathsf{Tr}(m) \models_{\mathsf{t}} \mathsf{HF}q$ and $\mathfrak{M}, m/\emptyset_{\mathsf{Tr}} \not\models_{\mathsf{t}} \mathsf{HF}q$. Since the transition set $\mathsf{Tr}(m)$ allows only those histories in which q eventually is the case, at any moment in the past of m, the sentence $\mathsf{F}q$ is true with respect to $\mathsf{Tr}(m)$. With respect to the empty transition set $\emptyset_{\mathsf{Tr}}$, however, the sentence $\mathsf{F}q$ is false at at least one moment in the past of m: to wit, the sentence $\mathsf{F}q$ is false with respect to $\emptyset_{\mathsf{Tr}}$ at the past moment m_0 at which h_1 and h_2 branch, i.e. $\mathfrak{M}, m_0/\emptyset_{\mathsf{Tr}} \not\models_{\mathsf{t}} \mathsf{F}q$. Since the empty transition set $\emptyset_{\mathsf{Tr}}$ admits, at every moment of evaluation, the entire set of histories passing through that moment, at the moment m_0 it also allows the history h_1, which lacks a witness for the future claim $\mathsf{F}q$.



w.r.t. $m/\emptyset_{\mathsf{Tr}}$ w.r.t. $m/\mathsf{Tr}(m)$

Figure 2.2: Evaluation w.r.t. $\emptyset_{\mathsf{Tr}}$ and $\mathsf{Tr}(m)$.

Let us now return to our original example. At the moment m, there are two immediate possible future continuations, corresponding to the cells of the partition $\{\{h_2, h_3\}, \{h_4\}\}$ induced by the relation of undividedness-at-m. Those immediate possible future continuations are captured by the transition sets T_1 and T_2, respectively, both of which are proper extensions of the transition set $\mathsf{Tr}(m)$. We have $T_1 = \mathsf{Tr}(m) \cup \{m \rightarrowtail \{h_2, h_3\}\}$ and $T_2 = \mathsf{Tr}(m) \cup \{m \rightarrowtail \{h_4\}\}$ and, consequently, $\mathsf{H}(T_1) = \{h_2, h_3\}$ and $\mathsf{H}(T_2) = \{h_4\}$.

With respect to the transition set T_1, which specifies the immediate possible future continuation of m in which p always is the case, the sentence $\mathsf{F}p$ is true at the moment m, while its contrary prediction $\mathsf{F}\neg p$ is false at that index, i.e. $\mathfrak{M}, m/T_1 \models_{\mathsf{t}} \mathsf{F}p$ and $\mathfrak{M}, m/T_1 \not\models_{\mathsf{t}} \mathsf{F}\neg p$. The transition set T_1 rules out any history in H_m that lacks a moment later than m at which p is true. What is more, it only allows histories in which p is always true in the future of m.

With respect to the transition set T_2, on the other hand, which captures the immediate possible future continuation of m in which p never is the case, precisely the opposite holds. The sentence $\mathsf{F}p$ is false at the moment m with respect to the transition set T_2, and its contrary prediction $\mathsf{F}\neg p$ is true at that index, i.e. $\mathfrak{M}, m/T_2 \not\vDash_{\mathsf{t}} \mathsf{F}p$ and $\mathfrak{M}, m/T_2 \vDash_{\mathsf{t}} \mathsf{F}\neg p$. The transition set T_2 only admits the history h_4, in which p is false at any moment in the future m.

Since there is a transition set compatible with m with respect to which $\mathsf{F}p$ is true and $\mathsf{F}\neg p$ is false at m, viz. the transition set T_1, and another with respect to which $\mathsf{F}\neg p$ is true and $\mathsf{F}p$ is false at m, viz. the transition set T_2, each of the possibility claims $\Diamond\mathsf{F}p$, $\Diamond\neg\mathsf{F}\neg p$, $\Diamond\mathsf{F}\neg p$ and $\Diamond\neg\mathsf{F}p$ is true at the moment m with respect to any transition set compatible with that moment. In particular, we have that at the index m/T_1 both $\mathsf{F}p$ and $\Diamond\neg\mathsf{F}p$ are true, i.e. $\mathfrak{M}, m/T_1 \vDash_{\mathsf{t}} \mathsf{F}p \wedge \Diamond\neg\mathsf{F}p$, which shows that future truth and inevitability come apart, as they should.

The transition set T_2 is maximal consistent: it allows but a single history, viz. the history h_4, and it is thus identical to the set $\mathsf{Tr}(h_4)$ that characterizes that history. The transition set T_1, on the other hand, can still be further extended. Two proper future extensions are possible, namely the maximal consistent extensions $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$. Each of those transition sets admits exactly one history, and in each of those histories, the sentence p is true at any moment later than m. With respect to both transition sets, $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$, the sentence $\mathsf{F}p$ is thus true at the moment m, while the contrary prediction $\mathsf{F}\neg p$ is false, respectively, i.e. $\mathfrak{M}, m/\mathsf{Tr}(h_2) \vDash_{\mathsf{t}} \mathsf{F}p$ and $\mathfrak{M}, m/\mathsf{Tr}(h_2) \not\vDash_{\mathsf{t}} \mathsf{F}\neg p$, and accordingly for the transition set $\mathsf{Tr}(h_3)$.

Let us focus on the sentence $\mathsf{F}p$ for a moment and see how its truth value at the moment m changes with respect to different transition sets. We said that the sentence $\mathsf{F}p$ is false at the moment m with respect to both the empty transition set $\emptyset_{\mathsf{Tr}}$ and the set $\mathsf{Tr}(m)$ of transitions preceding m but true with respect to the future extensions T_1, $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$, while relative to $T_2 \supsetneq \mathsf{Tr}(m)$, it remains false m. The transition from $\mathsf{Tr}(m)$ to T_1 comes along with a change in truth value. Since the sentence $\mathsf{F}p$ is false at m with respect to $\mathsf{Tr}(m)$ and, relatedly, T_2, but true with respect to $T_1 \supsetneq \mathsf{Tr}(m)$, we have that $\mathfrak{M}, m/\mathsf{Tr}(m) \vDash_{\mathsf{t}} \mathsf{Ƨ}\mathsf{F}p \wedge \mathsf{Ƨ}\neg\mathsf{F}p$ or, equivalently, $\mathfrak{M}, m/\mathsf{Tr}(m) \vDash_{\mathsf{t}} \neg\mathsf{S}\mathsf{F}p \wedge \neg\mathsf{S}\neg\mathsf{F}p$. That is, the sentence $\mathsf{F}p$ is contingent with respect to $m/\mathsf{Tr}(m)$: it is neither stably-true nor stably-false relative to that index. Its truth value only stabilizes as the future unfolds. The contingency of $\mathsf{F}p$ at m already dissolves with respect to the immediate extension T_1. With respect to m/T_1, the sentence $\mathsf{F}p$ is not contingent anymore. Relative to that index, it is stably-true, i.e. $\mathfrak{M}, m/T_1 \vDash_{\mathsf{t}} \mathsf{SF}p$. The sentence $\mathsf{F}p$ is true at the index m/T_1 and it remains true at m with respect to any proper extensions of T_1, viz. $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$.[59] This is, however, not to say that at m/T_1 it is inevitable that $\mathsf{F}p$ be true. The sentence $\Box\mathsf{F}p$ is false at m/T_1, i.e. $\mathfrak{M}, m/T_1 \nvDash_{\mathsf{t}} \Box\mathsf{F}p$, because $\mathsf{F}p$ is false at m/T_2. A sentence can be stably-true relative to an index of evaluation without its truth at that index being inevitable.

The behavior of the truth value of the sentence $\mathsf{F}p$ at the moment m in the course of time, its transition from contingency to stability, reflects the dynamics of the future possibility $\Diamond\mathsf{F}p$. To be sure, the maximal consistent transition sets $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$ constitute witnesses for the possibility $\Diamond\mathsf{F}p$ at the moment m, just as the transition set T_1 does. Yet, in contrast to the maximal consistent transition sets $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$, the transition set T_1 provides a local witness for the possibility $\Diamond\mathsf{F}p$: the transition set T_1 suffices to dissolve the contingency at m and guarantees the truth of $\mathsf{F}p$. With respect to T_1, the sentence $\mathsf{F}p$ is stably-true with respect to m/T_1, as we have seen. For concreteness, assume that in the model $\mathfrak{M}$, the histories h_2 and h_3 differ as to whether r is true in the future of their respective branching point, as illustrated in Fig. 2.3. That is, suppose that all moments $m' \in M$ and for all transition sets $T \in \mathsf{dcts}(\mathcal{M})$ such that $\mathsf{H}(T) \cap \mathsf{H}_{m'} \neq \emptyset$, we have $v_{\mathsf{t}}(r, m'/T) = 1$ if $m' > m$ and $m' \in h_2 \setminus h_3$, and $v_{\mathsf{t}}(r, m'/T) = 0$ if $m' > m$ and $m' \in h_3 \setminus h_2$. While the histories h_2 and h_3 still agree as to the truth of p, they now disagree as to the future truth of r. Whether the future possibility $\Diamond\mathsf{F}p$ at m will eventually be realized depends on how the future unfolds. For p to come about, it is thereby, however, completely irrelevant whether r will be the case or not. The maximal consistent transition sets $\mathsf{Tr}(h_2)$ and $\mathsf{Tr}(h_3)$ provide more information than needed in order to provide a witness for the future possibility $\Diamond\mathsf{F}p$. It is the local transition set T_1 that settles the matter, even though that set allows two histories that differ as to the truth of r. With respect to T_1, it is already settled that p will be the case.

[59] At first glance, it might seem that the truth value of a sentence about the future can only change from 'false' to 'true' under extensions of a given transition set. This is, however, not true in general. In order to see why this is so, consider the behavior of the truth value of the sentence $\mathsf{F}\neg\mathsf{F}q$ in the model $\mathfrak{M}$ provided in Fig. 2.2. Let m_1 be any moment in the past of m_0, i.e. $m_1 < m_0$. Then $\mathfrak{M}, m_1/\emptyset_{\mathsf{Tr}} \vDash_{\mathsf{t}} \mathsf{F}\neg\mathsf{F}q$ since $\mathfrak{M}, m_0/\mathsf{Tr}(m) \nvDash_{\mathsf{t}} \mathsf{F}q$, but $\mathfrak{M}, m_1/T_1 \nvDash_{\mathsf{t}} \mathsf{F}\neg\mathsf{F}q$.

The transition set T_1 specifies exactly how and how far the future has to unfold for the possibility to realized.

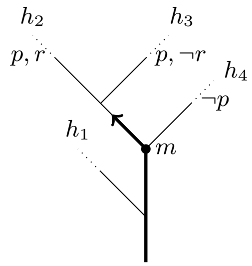


Figure 2.3: Local witnesses for future possibilities.

We said that the sentence $\mathsf{F}p$ is contingent with respect to $m/\mathsf{Tr}(m)$ but stably-true with respect to m/T_1. Its truth value at the moment m depends on how the future unfolds and stabilizes only as time progresses. The same is true for the contrary prediction $\mathsf{F}\neg p$: the sentence $\mathsf{F}\neg p$ is likewise contingent with respect to $m/\mathsf{Tr}(m)$, i.e. $\mathfrak{M}, m/\mathsf{Tr}(m) \vDash_{\mathsf{t}} \text{Ƨ}\mathsf{F}\neg p \wedge \text{Ƨ}\neg\mathsf{F}\neg p$, but stabilizes with respect to T_2, i.e. $\mathfrak{M}, m/T_2 \vDash_{\mathsf{t}} \mathsf{SF}\neg p$. With both the sentences $\mathsf{F}p$ and $\mathsf{F}\neg p$ being contingent with respect to $m/\mathsf{Tr}(m)$, their disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ is likewise contingent relative to that index, i.e. $\mathfrak{M}, m/\mathsf{Tr}(m) \vDash_{\mathsf{t}} \text{Ƨ}(\mathsf{F}p \vee \mathsf{F}\neg p) \wedge \text{Ƨ}\neg(\mathsf{F}p \vee \mathsf{F}\neg p)$. The disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ is false at the index of evaluation $m/\mathsf{Tr}(m)$, i.e. $\mathfrak{M}, m/\mathsf{Tr}(m) \nvDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$, and it is true with respect to either of the immediate possible future extensions of $\mathsf{Tr}(m)$. We have $\mathfrak{M}, m/T_1 \vDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$ since $\mathfrak{M}, m/T_1 \vDash_{\mathsf{t}} \mathsf{F}p$, and $\mathfrak{M}, m/T_2 \vDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$ because $\mathfrak{M}, m/T_2 \vDash_{\mathsf{t}} \mathsf{F}\neg p$. Relative to m/T_1 and m/T_2, respectively, the disjunction is even stably-true, i.e. $\mathfrak{M}, m/T_1 \vDash_{\mathsf{t}} \mathsf{S}(\mathsf{F}p \vee \mathsf{F}\neg p)$ and $\mathfrak{M}, m/T_2 \vDash_{\mathsf{t}} \mathsf{S}(\mathsf{F}p \vee \mathsf{F}\neg p)$. That the disjunction is false at the index pair $m/\mathsf{Tr}(m)$ is a sign of contingency: the past course of events up to m does not yet suffice to settle the matter. Only as the future unfolds, the truth value of either disjunct eventually stabilizes—in a maximal consistent extension, if not before—rendering the disjunction stably-true at the moment m as well. The intuition that the disjunction has the force of a tautology, as expressed by Thomason (1970), is reflected in the transition semantics by the validity $\neg\mathsf{S}\neg(\mathsf{F}p \vee \mathsf{F}\neg p)$ or, equivalently $\text{Ƨ}(\mathsf{F}p \vee \mathsf{F}\neg p)$. The disjunction is never stably-false. As time progresses, sooner or later, things become settled one way or the other.

2.5 The Generality of the transition semantics

In the transition semantics, sentences are assigned truth values in a model on a BT structure $\mathcal{M} = \langle M, < \rangle$ with respect to a moment $m \in M$ and a consistent, downward closed transition set $T \in \mathsf{dcts}(\mathcal{M})$ that is compatible with that moment, i.e. $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. Building on sets of transitions, the transition semantics exploits the structural resources a BT structure has to offer and allows for a fine-grained, dynamic picture of the interrelation of modality and time.

In this section, we point out the generality of the transition framework. In section 2.5.1, we show that the transition semantics generalizes both Peirceanism and Ockhamism. Both accounts can be obtained by restricting the range of transition sets that are taken into account in the semantic evaluation. The relevant restrictions reveal the limitations of the Peircean and the Ockhamist account: the dynamic picture collapses into a static one.

Zanardo (1998) likewise provides a framework that generalizes both Peirceanism and Ockhamism. We briefly review that framework in section 2.5.2 and show that it falls short of the transition framework: Zanardo's framework cannot capture the dynamics of time either. It is essentially the dynamicity of the second parameter of truth that distinguishes the transition semantics from extant accounts. In virtue of that feature, the transition semantics gains expressive means that are not available on extant accounts and allows for a perspicuous treatment of sentences about the contingent future. We conclude the section by relating our account to MacFarlane's assessment-sensitive postsemantics. In section 2.5.3, we show that MacFarlane's assessment-sensitive account of future contingents can be captured within the transition framework as well, at lower costs.

2.5.1 Unifying Peirceanism and Ockhamism

Every BT structure $\mathcal{M} = \langle M, < \rangle$ provides a set of moments M, a set of histories $\mathsf{hist}(\mathcal{M})$ and a set of transitions $\mathsf{trans}(\mathcal{M})$. Peirceanism, Ockhamism and the transition account differ with respect to which of those structural elements are employed as parameters of truth in the semantic evaluation. In this section, we show that the transition semantics unifies the Peircean and the Ockhamist approach and properly extends both of them. Both Peirceanism and Ockhamism can be viewed as limitations of the transition framework.

The transition semantics makes use of sets of transitions as a second parameter of truth next to the moment parameter in the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$. A set of transitions $T \in \mathsf{dcts}(\mathcal{M})$ can represent a complete or partial course of events that stretches linearly all the way from the past toward a possibly open future. It does so by excluding certain histories and allowing others to still occur. Among the consistent, downward closed transition sets definable in a BT structure, there are two extreme cases. The empty transition set $\emptyset_{\mathsf{Tr}}$ excludes no histories whatsoever: the set of histories allowed by the empty transition set $\emptyset_{\mathsf{Tr}}$ is the set of all histories in $\mathcal{M}$, i.e. $\mathsf{H}(\emptyset_{\mathsf{Tr}}) = \mathsf{hist}(\mathcal{M})$. Maximal consistent transition sets, on the other hand, exclude all but one history: every maximal consistent set of transitions is identical to a set $\mathsf{Tr}(h)$ that characterizes exactly one history $h \in \mathsf{hist}(\mathcal{M})$ and allows only that history to occur, i.e. $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$ (cf. Prop. 2.24 and Prop. 2.26).

We show that if only the empty transition set is taken into account in the semantic evaluation on a BT structure, we are back to Peirceanism, while a restriction to all maximal consistent transition sets yields Ockhamism. In order to be able to capture the relevant restrictions in a uniform way, in section 2.5.1.1, we introduce the notion of a transition structure. Restrictions on the admissible transition sets can then be viewed as restrictions on the class of transition structures. Correspondence results are provided, and Peircean and Ockhamist validity are shown to be definable in the transition semantics. In section 2.5.1.2, we deal with the Peircean case, and we treat the Ockhamist case in section 2.5.1.3. In section 2.5.1.4, we point out the consequences of the restrictions: on both the Peircean and the Ockhamist account, stability collapses into truth.

2.5.1.1 Transition structures

As said, our aim is to show that both Peirceanism and Ockhamism are limiting cases of the transition approach that result from restricting the range of transition sets that are taken into account in the semantic evaluation. In order to be able to capture the restriction of the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$ to subsets of $\mathsf{dcts}(\mathcal{M})$ in a uniform way, we generalize the notion of a transition model by introducing the notion of a *transition structure*.

A transition structure is the counterpart of an Ockhamist bundled tree (cf. Def. 1.16). Just as a bundled tree involves a primitive set of histories, a transition structure involves a primitive set of transition sets. In either case, the respective 'bundle' is thereby required to be such that it covers the entire structure. To be concrete, we define a transition structure as a triple $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ consisting of a BT structure $\mathcal{M} = \langle M, < \rangle$ and a non-empty set of transition sets $ts \subseteq \mathsf{dcts}(\mathcal{M})$ such that every moment $m \in M$ is compatible with at least one transition set $T \in ts$. The compatibility requirement ensures that every moment of the underlying structure $\mathcal{M} = \langle M, < \rangle$ is contained in a history that is admitted by some transition of the bundle ts. Note that this is not to say that every history of the underlying structure is to be allowed by some transition set in ts. The compatibility requirement only concerns the basic constituents of a branching time structure, which are moments rather than histories.[60] We denote the class of all transition structures by $\mathscr{C}$.

DEFINITION 2.33 (Transition structure). *A* transition structure *is an ordered triple* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$*, where* $\mathcal{M} = \langle M, < \rangle$ *is a BT structure and* $ts \subseteq \mathsf{dcts}(\mathcal{M})$ *a non-empty set of transition sets such that for every moment* $m \in M$*, there is some transition set* $T \in ts$ *such that* $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$*. Let* $\mathscr{C}$ *be the* class of all transition structures.

The additional set ts of transition sets does of course not really 'add' anything to the BT structure $\mathcal{M} = \langle M, < \rangle$: the BT structure itself already determines all possible sets of transitions in $\mathsf{dcts}(\mathcal{M})$. The set ts merely indicates which of those transition sets are employed in the semantic evaluation. A transition model $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ is a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ together with a valuation function v_{t} that assigns truth values to the propositional variables of the language $\mathcal{L}_{\mathsf{t}}$ at a moment $m \in M$ relative to a transition set $T \in ts$ such that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$.

DEFINITION 2.34 (Transition model). *A* transition model *is an ordered quadruple* $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$*, where* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ *is a transition structure and* $v_{\mathsf{t}} : \mathsf{At} \times \{m/T \mid m \in M, T \in ts \text{ and } \mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset\} \to \{0, 1\}$ *a valuation function.*

[60] Rather than requiring that every moment $m \in M$ has to be compatible with some transition set of the bundle ts, we could instead require that every history $h \in \mathsf{hist}(\mathcal{M})$ has to be allowed by some such transition set, which would amount to imposing the following condition on transition structures instead: for every $h \in \mathsf{hist}(\mathcal{M})$, there is some $T \in ts$ such that $h \in \mathsf{H}(T)$. Such a constraint would rule out incomplete Ockhamist transition structures, i.e. transition structures where $ts \subsetneq \{\mathsf{Tr}(h) \mid h \in \mathsf{hist}(\mathcal{M})\}$, from the outset and hence facilitate the proof of the correspondence result for Ockhamism (cf. section 2.5.1.3.2). Our reason for preferring the more general definition over the stronger alternative is that BT structures are defined in terms of moments rather than in terms of histories. Histories are defined elements in a BT structure.

The following semantic clauses extend the valuation v_{t} on the set of propositional variables At in a transition model $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ to all sentences $\phi \in \mathcal{L}_{\mathsf{t}}$. As usual, we use $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \phi$ in order to indicate that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true in a transition model $\mathfrak{M}^{ts}$ at a moment $m \in M$ with respect to a transition set $T \in ts$ such that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. The expressions $\mathfrak{M}^{ts} \models_{\mathsf{t}} \phi$ for validity in a transition model, $\mathcal{M}^{ts} \models_{\mathsf{t}} \phi$ for validity in a transition structure and $\models_{\mathsf{t}} \phi$ for general validity are defined in the obvious way. Moreover, for $\mathscr{C}' \subsetneq \mathscr{C}$, we use $\mathscr{C}' \models_{\mathsf{t}} \phi$ to stand for validity with respect to the class of transition structures $\mathscr{C}'$.

(At) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} p$ iff $v_{\mathsf{t}}(p, m/T) = 1$;

($\neg$) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \neg\phi$ iff $\mathfrak{M}^{ts}, m/T \not\models_{\mathsf{t}} \phi$;

($\wedge$) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \phi \wedge \psi$ iff $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \phi$ and $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \psi$;

(P) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \mathsf{P}\phi$ iff there is some $m' < m$ s.t. $\mathfrak{M}^{ts}, m'/T \models_{\mathsf{t}} \phi$;

(f) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \mathsf{f}\phi$ iff there is some $m' > m$ s.t.$\mathsf{H}(T) \cap \mathsf{H}_{m'} \neq \emptyset$ and $\mathfrak{M}^{ts}, m'/T \models_{\mathsf{t}} \phi$;

(F) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \mathsf{F}\phi$ iff for all $h \in \mathsf{H}(T) \cap \mathsf{H}_m$, there is some $m' \in h$ s.t. $m' > m$ and $\mathfrak{M}^{ts}, m'/T \models_{\mathsf{t}} \phi$;

($\Box$) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \Box\phi$ iff for all $T' \in ts$ s.t. $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$, $\mathfrak{M}^{ts}, m/T' \models_{\mathsf{t}} \phi$;

(S) $\mathfrak{M}^{ts}, m/T \models_{\mathsf{t}} \mathsf{S}\phi$ iff for all $T' \in ts$ s.t. $T' \supseteq T$ and $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$, $\mathfrak{M}^{ts}, m/T' \models_{\mathsf{t}} \phi$.

In a transition model $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$, the inevitability operator $\Box$ and the stability operator S quantify over transition sets in ts only. The semantic clauses for the past operator P and the future operators f and F remain intact, as those operators do not involve a shift of the transition parameter. In the case of the temporal operators, the transition parameter merely places restrictions on the respective domain of quantification. Note that in the case of the strong future operator F, we make use of the formulation in which the quantification over future possibilities is spelled out in terms of the set of histories allowed by the given transition set rather than in terms of its possible future extensions. While the two formulations are equivalent whenever $ts = \mathsf{dcts}(\mathcal{M})$, for $ts \subsetneq \mathsf{dcts}(\mathcal{M})$ this is no generally the case.[61]

[61] Consider a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ where $ts = \{\emptyset_{\mathsf{Tr}}\}$. If in the semantic clause for the strong future operator F, the quantification over future possibilities were spelled

In section 2.4, we have focused on transition models on transition structures $\mathcal{M}^{\mathsf{dcts}(\mathcal{M})} = \langle M, <, \mathsf{dcts}(\mathcal{M})\rangle$, and the semantic clauses were formulated for only that class of models. Transition models on a transition structure $\mathcal{M}^{\mathsf{dcts}(\mathcal{M})}$ make use of the entire range of consistent, downward closed sets of indeterministic transitions the BT structure $\mathcal{M}$ has to offer, whereas all other transition models rest upon limited resources only. We will now consider two such possible limitations, namely those that correspond to Peirceanism and Ockhamism, respectively.[62]

2.5.1.2 Generalizing Peirceanism

Let us have a look at the Peircean account first. We show that Peirceanism can be captured in the transition framework by restricting the semantic evaluation on a BT structure to the empty transition set. In section 2.5.1.2.1, we prove that Peircean validity is equivalent to validity with respect to the class of all transition structures whose bundle comprises but the empty transition set, and in section 2.5.1.2.2, we show that class to be characterizable in the transition language $\mathcal{L}_{\mathsf{t}}$. Not only then does the transition semantics generalize Peirceanism, but Peircean validity can even be defined in the transition semantics.

2.5.1.2.1 The class of Peircean transition structures On the Peircean account, the semantic evaluation is relativized to a moment parameter only: sentences $\phi \in \mathcal{L}_{\mathsf{p}}$ are assigned truth values in a Peircean model $\mathfrak{M} = \langle M, <, v_{\mathsf{p}}\rangle$ on a BT structure $\mathcal{M} = \langle M, <\rangle$ relative to a moment $m \in M$. The Peircean language $\mathcal{L}_{\mathsf{p}}$ is equipped with two future operators, F_{p} and f_{p}, whose semantic clauses—explicitly or implicitly—involve quantification over the set of histories containing the moment of evaluation. Whereas the strong future operator F_{p}

out in terms of the possible future extensions of the given transition set in ts (in parallel with condition ($\mathsf{F}^\sharp$) on p. 99) rather than in terms of the set of histories allowed by that set, the equivalence $\mathsf{f}p \leftrightarrow \mathsf{F}p$ would be valid in $\mathcal{M}^{ts}$. That is, the strong future operator would simply collapse into the weak one. For a sentence of the form $\mathsf{F}\phi$ to be true at an index $m/\emptyset_{\mathsf{Tr}}$, it would be sufficient that there is a single moment later than m at which ϕ is true. On our account, on the contrary, it is necessary that every history in H_m contains a future witness. The model in Fig. 2.1 provides a counterexample to the above equivalence: in that model, the sentence $\mathsf{f}p$ is true at $m/\emptyset_{\mathsf{Tr}}$ while $\mathsf{F}p$ is false at that index.

[62]Whether there are other transition structures that are interesting from a logical point of view remains to be seen and is subject to future research. From a philosophical point of view, one might wonder to what extent transition structures with missing transition sets are appropriate at all, a discussion that we will not go into here. Similar considerations arise as those brought forth as arguments against incomplete bundled trees in the context of Ockhamism. For a discussion of the problems surrounding bundled trees with missing histories, see Belnap, Perloff, and Xu (2001, ch. 7A.6).

requires a future witness in each such history, the weak future operator $\mathsf{f_p}$ requires a witness in only a single one of them.

We show that we can capture Peirceanism with its sole dependence on a moment parameter in the transition framework by restricting the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$ to the empty transition set $\emptyset_{\mathsf{Tr}}$. Due to the restriction to the empty transition set $\emptyset_{\mathsf{Tr}}$, the possible indices of evaluation are restricted to pairs $m/\emptyset_{\mathsf{Tr}}$ consisting of a moment $m \in M$ and the empty transition set $\emptyset_{\mathsf{Tr}}$, for which it holds that $\mathsf{H}(\emptyset_{\mathsf{Tr}}) \cap \mathsf{H}_m = \mathsf{H}_m$. At any moment of evaluation $m \in M$, all histories containing m are admitted by the empty transition set $\emptyset_{\mathsf{Tr}}$, and this is exactly what is needed in order to mimic the Peircean future operators.

For every BT structure $\mathcal{M} = \langle M, < \rangle$, the structure $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\} \rangle$ is a transition structure according to Def. 2.33, since $\mathsf{H}(\emptyset_{\mathsf{Tr}}) = \mathsf{hist}(\mathcal{M})$: every moment $m \in M$ is compatible with the empty transition set $\emptyset_{\mathsf{Tr}}$. We call a transition structure $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\} \rangle$ a *Peircean transition structure* and a model $\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}, v_{\mathsf{t}} \rangle$ on such a structure a *Peircean transition model.* The *class of all Peircean transition structures* is denoted by $\mathscr{C}_{\mathsf{p}}$.

DEFINITION 2.35 (Peircean transition structure and model). *A transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ is called a* Peircean transition structure *iff $ts = \{\emptyset_{\mathsf{Tr}}\}$. A transition model $\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}, v_{\mathsf{t}} \rangle$ on a Peircean transition structure $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\} \rangle$ is called a* Peircean transition model. *Let $\mathscr{C}_{\mathsf{p}}$ be the* class of all Peircean transition structures.

There is a natural one-to-one correspondence Peircean transition structures and BT structures. Let us denote the class of all BT structures by $\mathscr{T}$. We define a function $\eta : \mathscr{C}_{\mathsf{p}} \to \mathscr{T}$ that maps every Peircean transition structure $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\} \rangle$ onto the BT structure $\mathcal{M} = \langle M, < \rangle$. Obviously, the function η is a bijection.

LEMMA 2.36. *The function*

$$\begin{array}{rccc} \eta : & \mathscr{C}_{\mathsf{p}} & \to & \mathscr{T} \\ & \langle M, <, \{\emptyset_{\mathsf{Tr}}\} \rangle & \mapsto & \langle M, < \rangle \end{array}$$

is a bijection.

Proof. Straightforward. □

Let $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}\rangle$ be a Peircean transition structure, and let $\eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}})$ be the corresponding BT structure. Since every moment $m \in M$ is compatible with the empty transition set, the mapping $m/\emptyset_{\mathsf{Tr}} \mapsto m$ is a bijection between the set of indices of evaluation in $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}$ and the set of Peircean indices in $\eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}})$. We can then extend the correspondence η between Peircean transition structures and BT structures provided in Lem. 2.36 to models. Given a Peircean transition structure $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}\rangle$, the function $\eta' : \langle \mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}, v_{\mathsf{t}}\rangle \mapsto \langle \eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}), v_{\mathsf{p}}\rangle$ with $v_{\mathsf{p}}(p, m) = v_{\mathsf{t}}(p, m/\emptyset_{\mathsf{Tr}})$ for all $p \in \mathsf{At}$ and $m \in M$ is a bijection between Peircean transition models on $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}$ and Peircean BT models on $\eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}})$. We show that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true in a Peircean transition model $\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}, v_{\mathsf{t}}\rangle$ on $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}$ at an index of evaluation $m/\emptyset_{\mathsf{Tr}}$ if and only if its respective translation $\phi^* \in \mathcal{L}_{\mathsf{p}}$ is true in the Peircean model $\eta'(\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}})$ on the corresponding BT structure $\eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}})$ at the moment m.

For any sentence $\phi \in \mathcal{L}_{\mathsf{t}}$, its translation ϕ^* into the Peircean language $\mathcal{L}_{\mathsf{p}}$ can be defined recursively, and every sentence of the Peircean language is a translation of some sentence in the transition language. As both $\mathcal{L}_{\mathsf{t}}$ and $\mathcal{L}_{\mathsf{p}}$ are extension of the propositional language $\mathcal{L}$, suffice it to say that $(\mathsf{F}\phi)^* = \mathsf{F}_{\mathsf{p}}\phi^*$, $(\mathsf{f}\phi)^* = \mathsf{f}_{\mathsf{p}}\phi^*$, $(\mathsf{P}\phi)^* = \mathsf{P}_{\mathsf{p}}\phi^*$, $(\Box\phi)^* = \phi^*$ and $(\mathsf{S}\phi)^* = \phi^*$. Note that due to the restriction to a single set of transitions, viz. the empty transition set, the equivalences $\Box p \leftrightarrow p$ and $\mathsf{S}p \leftrightarrow p$ are valid in every Peircean transition structure, in symbols: $\mathscr{C}_{\mathsf{p}} \vDash_{\mathsf{t}} \Box p \leftrightarrow p$ and $\mathscr{C}_{\mathsf{p}} \vDash_{\mathsf{t}} \mathsf{S}p \leftrightarrow p$.

Proposition 2.37. *Let $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}\rangle$ be a Peircean transition structure. The mapping $\eta' : \langle \mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}, v_{\mathsf{t}}\rangle \mapsto \langle \eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}), v_{\mathsf{p}}\rangle$ with $v_{\mathsf{p}}(p, m) = v_{\mathsf{t}}(p, m/\emptyset_{\mathsf{Tr}})$ for all $p \in \mathsf{At}$ and $m \in M$ is a bijection between Peircean transition models on $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}}$ and Peircean BT models on $\eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}})$. The following holds: given a Peircean transition model $\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}, v_{\mathsf{t}}\rangle$, for every $\phi \in \mathcal{L}_{\mathsf{t}}$ and every $m \in M$:*

$$\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}}, m/\emptyset_{\mathsf{Tr}} \vDash_{\mathsf{t}} \phi \quad \textit{iff} \quad \eta'(\mathfrak{M}^{\{\emptyset_{\mathsf{Tr}}\}}), m \vDash_{\mathsf{p}} \phi^*.$$

Proof. The proof runs by induction on the structure of a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ and rests on the induction hypothesis that the claim holds for every proper subformula ψ of ϕ. Given the correspondence η', the base clause for atomic ϕ is straightforward. The cases for the truth-functional connectives $\neg$ and $\wedge$ are banal, the case for P is trivial, and the cases for $\Box$ and S dissolve due to the validity of the equivalences $\Box p \leftrightarrow p$ and $\mathsf{S}p \leftrightarrow p$. In the case of the future operators f and F, the proof makes use of the fact that $\mathsf{H}(\emptyset_{\mathsf{Tr}}) \cap \mathsf{H}_m = \mathsf{H}_m$. □

From Prop. 2.37 it follows that a sentence $\phi \in \mathcal{L}_\mathsf{t}$ is valid in a Peircean transition structure $\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}} = \langle M, <, \{\emptyset_{\mathsf{Tr}}\}\rangle$ if and only if its Peircean translation $\phi^* \in \mathcal{L}_\mathsf{p}$ is valid in the corresponding BT structure $\eta(\mathcal{M}^{\{\emptyset_{\mathsf{Tr}}\}})$ according to the Peircean semantics. By Lem. 2.36, this implies that a sentence $\phi \in \mathcal{L}_\mathsf{t}$ is valid with respect to the class $\mathscr{C}_\mathsf{p}$ of all Peircean transition structures if and only if its translation $\phi^* \in \mathcal{L}_\mathsf{p}$ is generally valid in the Peircean semantics.

Corollary 2.38. *For every $\phi \in \mathcal{L}_\mathsf{t}$:*

$$\mathscr{C}_\mathsf{p} \vDash_\mathsf{t} \phi \quad \textit{iff} \quad \vDash_\mathsf{p} \phi^*.$$

Proof. Follows from Lem. 2.36 and Prop. 2.37. □

Because every sentence of the Peircean language $\mathcal{L}_\mathsf{p}$ is a translation of some sentence in the transition language $\mathcal{L}_\mathsf{t}$, by Cor. 2.38, Peircean validity is equivalent to validity with respect to the class $\mathscr{C}_\mathsf{p}$ of all Peircean transition structures.

2.5.1.2.2 A correspondence result for Peirceanism We now show that the class $\mathscr{C}_\mathsf{p}$ of all Peircean transition structures is characterizable in the transition language $\mathcal{L}_\mathsf{t}$. On the basis of the result established in the previous section, it then follows that Peircean validity can be defined in the transition semantics.

We prove that the following correspondence holds: a transition structure $\mathcal{M}^{ts}$ is a Peircean transition structure if and only if the equivalence $\Box p \leftrightarrow p$ is valid in $\mathcal{M}^{ts}$.

Proposition 2.39. *For every transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$:*

$$\mathcal{M}^{ts} \in \mathscr{C}_\mathsf{p} \quad \textit{iff} \quad \mathcal{M}^{ts} \vDash_\mathsf{t} \Box p \leftrightarrow p.$$

Proof. "⇒": Let $\mathcal{M}^{ts} \in \mathscr{C}_\mathsf{p}$. As said, we have $\mathscr{C}_\mathsf{p} \vDash_\mathsf{t} \Box p \leftrightarrow p$, and hence $\mathcal{M}^{ts} \vDash_\mathsf{t} \Box p \leftrightarrow p$.
"⇐": Assume that $\mathcal{M}^{ts} \notin \mathscr{C}_\mathsf{p}$. Then $ts \neq \{\emptyset_{\mathsf{Tr}}\}$. We first show that this implies that ts comprises at least two transition sets, i.e. $ts \subseteq \{T, T'\}$. Assume for reductio that $ts = \{T\}$ for some $T \in \mathsf{dcts}(\mathcal{M})$ s.t. $T \neq \emptyset_{\mathsf{Tr}}$. It follows that there is some transition $\langle m \rightarrowtail H\rangle \in T$. Then there is some $h \in \mathsf{H}_m \backslash H$ and some $m' \in h$ s.t. $H \cap \mathsf{H}_{m'} = \emptyset$. Since $\mathsf{H}(T) \subseteq H$, this implies that $\mathsf{H}(T) \cap \mathsf{H}_{m'} = \emptyset$, which shows that for $ts = \{T\} \neq \{\emptyset_{\mathsf{Tr}}\}$, the structure $\mathcal{M}^{ts}$ is not a transition structure according to Def. 2.33. Hence, $ts \subseteq \{T, T'\}$.
Now take some moment $m \in M$ s.t. $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$ and $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$. The existence of m is guaranteed by Lem. 1.5. Consider a model $\mathfrak{M}^{ts} = \langle M, <, ts, v_\mathsf{t}\rangle$ s.t. $v_\mathsf{t}(p, m/T) = 1$ and $v_\mathsf{t}(p, m/T') = 0$. We then have that $\mathfrak{M}^{ts}, m/T \vDash_\mathsf{t} p$ but

$\mathfrak{M}^{ts}, m/T \not\models_{\mathsf{t}} \Box p$, because $\mathfrak{M}^{ts}, m/T' \not\models_{\mathsf{t}} p$. Consequently, $\mathfrak{M}^{ts}, m/T \not\models_{\mathsf{t}} \Box p \leftrightarrow p$, and hence $\mathcal{M}^{ts} \not\models_{\mathsf{t}} \Box p \leftrightarrow p$. □

It is worthwhile to note that rather than characterizing the class of Peircean transition structures $\mathscr{C}_{\mathsf{p}}$ by means of the equivalence $\Box p \leftrightarrow p$, we could likewise have characterized that class by the equivalence $\Diamond p \leftrightarrow p$ or $\Box p \leftrightarrow \Diamond p$, respectively. Since in a Peircean transition structure, the bundle of transition sets comprises but a single transition set, namely, the empty one, inevitability, possibility and truth coincide. We do not only have $\mathscr{C}_{\mathsf{p}} \models_{\mathsf{t}} \Box p \leftrightarrow p$ but also $\mathscr{C}_{\mathsf{p}} \models_{\mathsf{t}} \Diamond p \leftrightarrow p$ and $\mathscr{C}_{\mathsf{p}} \models_{\mathsf{t}} \Box p \leftrightarrow \Diamond p$, and the model constructed in the second part of the proof of Prop. 2.39 constitutes a counterexample to either of those equivalences.

Let us eventually bring together the results established so far. By Cor. 2.38, Peircean validity is equivalent to validity with respect to the class $\mathscr{C}_{\mathsf{p}}$ of all Peircean transition structures, and, by Prop. 2.39, that class can be characterized by the equivalence $\Box p \leftrightarrow p$. Peircean validity is thus definable in the transition semantics: a sentence of the Peircean language $\mathcal{L}_{\mathsf{p}}$ is valid in the Peircean semantics if and only if its $\mathcal{L}_{\mathsf{t}}$-correspondent is valid in all transition structures that validate the equivalence $\Box p \leftrightarrow p$.

Corollary 2.40. *For every $\phi \in \mathcal{L}_{\mathsf{t}}$:*

$$(\textit{for all } \mathcal{M}^{ts} \textit{ s.t. } \mathcal{M}^{ts} \models_{\mathsf{t}} \Box p \leftrightarrow p,\ \mathcal{M}^{ts} \models_{\mathsf{t}} \phi) \quad \textit{iff} \quad \models_{\mathsf{p}} \phi^*.$$

Proof. Follows from Cor. 2.38 and Prop. 2.39. □

2.5.1.3 Generalizing Ockhamism

Let us now turn to the Ockhamist account. We show that if we restrict the semantic evaluation on a BT structure to maximal consistent transition sets, we are back to Ockhamism. Our approach is parallel to the Peircean case. In section 2.5.1.3.1, we show that Ockhamist validity is equivalent to validity with respect to the class of all transition structures whose respective bundle equals the set of all maximal consistent transitions sets. In section 2.5.1.3.2, we prove, in two steps, that this class of structures can be characterized in the transition language $\mathcal{L}_{\mathsf{t}}$, which then allows us to conclude that Ockhamist validity is definable in the transition semantics.

2.5.1.3.1 The class of Ockhamist transition structures On the Ockhamist account, the semantic evaluation is relativized to a history as a second parameter of truth next to the moment parameter: sentences $\phi \in \mathcal{L}_{\mathsf{o}}$ are assigned truth values in an Ockhamist model $\mathfrak{M} = \langle M, <, v_{\mathsf{o}} \rangle$ on a BT structure $\mathcal{M} = \langle M, < \rangle$ relative to a moment-history pair m/h, where $m \in M$ and $h \in \mathsf{H}_m$. The Ockhamist future operator F_{o} simply shifts the moment of evaluation forward on the given history, just as in linear tense logic. The history parameter also becomes relevant in the case of the Ockhamist modal operators. The inevitability operator $\Box_{\mathsf{o}}$ and its dual $\Diamond_{\mathsf{o}}$ are interpreted as quantifiers over the set of histories containing the moment of evaluation.

In section 2.3.2, we have established a one-to-one correspondence between histories and maximal consistent transition sets. Making use of that result, we now show that we can capture Ockhamism with its dependence on a history parameter in the transition semantics by restricting the semantic evaluation on a BT structure to maximal consistent transition sets. For $\mathcal{M} = \langle M, < \rangle$ a BT structure, let $\mathsf{mcts}(\mathcal{M}) \subseteq \mathsf{dcts}(\mathcal{M})$ be the *set of all maximal consistent sets of indeterministic transitions in* $\mathcal{M}$. By Prop. 2.24 and Prop. 2.26, for every history $h \in \mathsf{hist}(\mathcal{M})$, there is a maximal consistent transition set $\mathsf{Tr}(h) \in \mathsf{dcts}(\mathcal{M})$ with $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$, and every maximal consistent set of transitions in $\mathsf{dcts}(\mathcal{M})$ is identical to some such set. Hence, we have that $\mathsf{mcts}(\mathcal{M}) = \{\mathsf{Tr}(h) \mid h \in \mathsf{hist}(\mathcal{M})\}$.

Definition 2.41 (The set $\mathsf{mcts}(\mathcal{M})$). *For* $\mathcal{M} = \langle M, < \rangle$ *a BT structure, the* set of all maximal consistent transition sets in $\mathcal{M}$ *is given by*

$$\mathsf{mcts}(\mathcal{M}) := \{\mathsf{Tr}(h) \mid h \in \mathsf{hist}(\mathcal{M})\}.$$

By restricting the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$ to the set $\mathsf{mcts}(\mathcal{M})$ of maximal consistent transition sets, we ensure that the possible indices of evaluation are restricted to pairs $m/\mathsf{Tr}(h)$ consisting of a moment $m \in M$ and a maximal consistent set of transitions $\mathsf{Tr}(h) \in \mathsf{mcts}(\mathcal{M})$ such that $m \in h$. At any index of evaluation $m/\mathsf{Tr}(h)$, exactly one history containing the moment m is admitted by the maximal consistent transition set $\mathsf{Tr}(h)$, viz. the corresponding history h. This allows us to mimic the Ockhamist future operator. Moreover, the restriction guarantees that the modal operators likewise quantify over maximal consistent transition sets $\mathsf{Tr}(h) \in \mathsf{mcts}(\mathcal{M})$ only, which is in accordance with their interpretation in the Ockhamist semantics.

Due to the one-to-one correspondence between histories and maximal consistent transition sets, for every BT structure $\mathcal{M} = \langle M, < \rangle$, the structure $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M}) \rangle$ qualifies as a transition structure according to

Def. 2.33. We call a transition structure $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$ an *Ockhamist transition structure* and a model $\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M}), v_{\mathsf{t}}\rangle$ on such a structure an *Ockhamist transition model.* The *class of all Ockhamist transition structures* is denoted by $\mathscr{C}_{\mathsf{o}}$.

Definition 2.42 (Ockhamist transition structure and model). *A transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ is called an* Ockhamist transition structure *iff $ts = \mathsf{mcts}(\mathcal{M})$. A transition model $\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M}), v_{\mathsf{t}}\rangle$ on an Ockhamist transition structure $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$ is called an* Ockhamist transition model*. Let $\mathscr{C}_{\mathsf{o}}$ be the* class of all Ockhamist transition structures*.*

Obviously, there is again a natural one-to-one correspondence between the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures and the class of BT structures $\mathscr{T}$, which can be described by the function $\kappa : \mathscr{C}_{\mathsf{o}} \to \mathscr{T}$ that maps every Ockhamist transition structure $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$ onto the BT structure $\mathcal{M} = \langle M, <\rangle$.

Lemma 2.43. *The function*

$$\begin{array}{rccc} \kappa : & \mathscr{C}_{\mathsf{o}} & \to & \mathscr{T} \\ & \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle & \mapsto & \langle M, <\rangle \end{array}$$

is a bijection.

Proof. Straightforward. □

Let $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$ be an Ockhamist transition structure, and let $\kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})})$ be the corresponding BT structure. Since $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$ for all $h \in \mathsf{hist}(\mathcal{M})$, the mapping $m/\mathsf{Tr}(h) \mapsto m/h$ induced by the correspondence κ is a bijection between the set of indices of evaluation in $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})}$ and the set of Ockhamist index pairs in $\kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})})$. We can then extend the correspondence κ to models. Given an Ockhamist transition structure $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$, the function $\kappa' : \langle \mathcal{M}^{\mathsf{mcts}(\mathcal{M})}, v_{\mathsf{t}}\rangle \mapsto \langle \kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})}), v_{\mathsf{o}}\rangle$ with $v_{\mathsf{o}}(p, m/h) = v_{\mathsf{t}}(p, m/\mathsf{Tr}(h))$ for all $p \in \mathsf{At}$, $m \in M$ and $h \in \mathsf{H}_m$ is a bijection between Ockhamist transition models on $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})}$ and Ockhamist BT models on $\kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})})$. We show that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true in an Ockhamist transition model $\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M}), v_{\mathsf{t}}\rangle$ on $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})}$ at an index of evaluation $m/\mathsf{Tr}(h)$ if and only if its respective translation $\phi^* \in \mathcal{L}_{\mathsf{o}}$ is true in the Ockhamist model $\kappa'(\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})})$ on the corresponding BT structure $\kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})})$ at the moment-history pair m/h.

For any sentence $\phi \in \mathcal{L}_\mathsf{t}$, its translation ϕ^* into the the Ockhamist language can be defined recursively, and every sentence of the Ockhamist language is a translation of some sentence in the transition language. As both $\mathcal{L}_\mathsf{t}$ and $\mathcal{L}_\mathsf{o}$ are extensions of the propositional language $\mathcal{L}$, suffice it to say that $(\mathsf{F}\phi)^* = \mathsf{F}_\mathsf{o}\phi^*$, $(\mathsf{f}\phi)^* = \mathsf{F}_\mathsf{o}\phi^*$, $(\mathsf{P}\phi)^* = \mathsf{P}_\mathsf{o}\phi^*$, $(\Box\phi)^* = \Box_\mathsf{o}\phi^*$ and $(\mathsf{S}\phi)^* = \phi^*$. Note that due to the restriction to maximal consistent transition sets, which single out exactly one history, the weak and the strong future operator coincide. That is, the equivalence $\mathsf{f}p \leftrightarrow \mathsf{F}p$ is valid with respect to the class $\mathscr{C}_\mathsf{o}$ of Ockhamist transition structures, in symbols: $\mathscr{C}_\mathsf{o} \vDash_\mathsf{t} \mathsf{f}p \leftrightarrow \mathsf{F}p$. And since maximal consistent transition sets cannot be further extended, the equivalence $\mathsf{S}p \leftrightarrow p$ is valid in every Ockhamist transition structure as well, in symbols: $\mathscr{C}_\mathsf{o} \vDash_\mathsf{t} \mathsf{S}p \leftrightarrow p$.

Proposition 2.44. *Let $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$ be an Ockhamist transition structure. The mapping $\kappa' : \langle \mathcal{M}^{\mathsf{mcts}(\mathcal{M})}, v_\mathsf{t}\rangle \mapsto \langle \kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})}), v_\mathsf{o}\rangle$ with $v_\mathsf{o}(p, m/h) = v_\mathsf{t}(p, m/\mathsf{Tr}(h))$ for all $p \in \mathsf{At}$, $m \in M$ and $h \in \mathsf{H}_m$ is a bijection between Ockhamist transition models on $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})}$ and Ockhamist BT models on $\kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})})$. The following holds: given an Ockhamist transition model $\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M}), v_\mathsf{t}\rangle$, for every $\phi \in \mathcal{L}_\mathsf{t}$, every $m \in M$ and every $h \in \mathsf{H}_m$:*

$$\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})}, m/\mathsf{Tr}(h) \vDash_\mathsf{t} \phi \quad \textit{iff} \quad \kappa'(\mathfrak{M}^{\mathsf{mcts}(\mathcal{M})}), m/h \vDash_\mathsf{o} \phi^*.$$

Proof. The proof runs by induction on the structure of a sentence $\phi \in \mathcal{L}_\mathsf{t}$ and rests on the induction hypothesis that the claim holds for every proper subformula ψ of ϕ. Given the correspondence κ', the base clause is straightforward. The cases for the truth-functional connectives $\neg$ and $\wedge$ are banal, the case for P is trivial, and the cases for f and S dissolve due to the validity of the equivalences $\mathsf{f}\phi \leftrightarrow \mathsf{F}\phi$ and $\mathsf{S}p \leftrightarrow p$. In the case of the future operator F and the inevitability operator $\Box$, the proof makes use of the fact that $\mathsf{mcts}(\mathcal{M}) = \{\mathsf{Tr}(h) \mid h \in \mathsf{hist}(\mathcal{M})\}$ and $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$ for all $h \in \mathsf{hist}(\mathcal{M})$. □

By Prop. 2.44, a sentence $\phi \in \mathcal{L}_\mathsf{t}$ is satisfiable in an Ockhamist transition structure $\mathcal{M}^{\mathsf{mcts}(\mathcal{M})} = \langle M, <, \mathsf{mcts}(\mathcal{M})\rangle$ if and only if its Ockhamist translation is satisfiable in the corresponding BT structure $\kappa(\mathcal{M}^{\mathsf{mcts}(\mathcal{M})})$ according to the Ockhamist semantics. Lem. 2.43 and Prop. 2.44 then jointly imply that a sentence $\phi \in \mathcal{L}_\mathsf{t}$ is valid in every Ockhamist transition structure if and only if its translation $\phi^* \in \mathcal{L}_\mathsf{o}$ is valid in the Ockhamist semantics. Since every Ockhamist sentence is the translation of some sentence in the transition language, Ockhamist validity coincides with validity with respect to the class $\mathscr{C}_\mathsf{o}$ of Ockhamist transition structures.

Corollary 2.45. *For every $\phi \in \mathcal{L}_{\mathsf{t}}$:*

$$\mathscr{C}_{\mathsf{o}} \vDash_{\mathsf{t}} \phi \quad \textit{iff} \quad \vDash_{\mathsf{o}} \phi^{*}.$$

Proof. Follows from Lem. 2.43 and Prop. 2.44. □

2.5.1.3.2 A correspondence result for Ockhamism We finally show that the class $\mathscr{C}_{\mathsf{o}}$ of all Ockhamist transition structures can be characterized in the transition language $\mathcal{L}_{\mathsf{t}}$. Given the results established in the previous section, this then allows us to define Ockhamist validity in the transition semantics.

Proving that the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures is characterizable in $\mathcal{L}_{\mathsf{t}}$ is slightly more complicated than proving the corresponding result in the case of Peirceanism. In the definition of a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$, we have merely required that every moment $m \in M$ must be compatible with at least one transition set $T \in ts$. We have not required that every history $h \in \mathsf{hist}(\mathcal{M})$ must be allowed by some such transition set. Accordingly, there are transition structures such that $ts \subsetneq \mathsf{mcts}(\mathcal{M})$. The phenomenon is the one familiar to us from bundled tress (cf. section 1.4.3.2).

In order to show that the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures can be characterized in the transition language, we proceed in two steps: in section 2.5.1.3.2.1, we generalize the results that we have established in the preceding section to bundled trees. We prove that the class of all transition structures $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ in which ts comprises maximal consistent transition sets only, i.e. $ts \subseteq \mathsf{mcts}(\mathcal{M})$, is definable in $\mathcal{L}_{\mathsf{t}}$, and that validity with respect to that class is equivalent to bundled tree validity. In section 2.5.1.3.2.2, we then show, by making recourse to a result in Zanardo *et al.* (1999), that from that class we can single out the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures, viz. the class of those transition structures $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ for which $ts = \mathsf{mcts}(\mathcal{M})$.

2.5.1.3.2.1 Bundled Ockhamism In this section, we show, in a first step, that the class of all transition structures $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ in which ts does not contain any transition set that is not maximal consistent is definable in the transition language $\mathcal{L}_{\mathsf{t}}$ and that validity with respect to that class coincides with bundled tree validity.

We call a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ where $ts \subseteq \mathsf{mcts}(\mathcal{M})$ a *bundled Ockhamist transition structure* and a model based on such a structure *a bundled Ockhamist transition model.* The *class of all bundled Ockhamist transition structures* is denoted by $\mathscr{C}_{\mathsf{b}}$.

Definition 2.46 (Bundled Ockhamist transition structure and model). *A transition structure* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ *is called a* bundled Ockhamist transition structure *iff* $ts \subseteq \mathsf{mcts}(\mathcal{M})$. *A transition model* $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ *on a bundled Ockhamist transition structure* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ *with* $ts \subseteq \mathsf{mcts}(\mathcal{M})$ *is called a* bundled Ockhamist transition model. *Let* $\mathscr{C}_{\mathsf{b}}$ *be the* class of all bundled Ockhamist transition structures.

We now show that the class $\mathscr{C}_{\mathsf{b}}$ of all bundled Ockhamist transition structures is definable in the transition language $\mathcal{L}_{\mathsf{t}}$. More precisely, we prove that a transition structure $\mathcal{M}^{ts}$ is a bundled Ockhamist transition structure if and only if the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ is valid in $\mathcal{M}^{ts}$.

Proposition 2.47. *For every transition structure* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$*:*

$$\mathcal{M}^{ts} \in \mathscr{C}_{\mathsf{b}} \quad \textit{iff} \quad \mathcal{M}^{ts} \vDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p.$$

Proof. "⇒": Let $\mathcal{M}^{ts} \in \mathscr{C}_{\mathsf{b}}$. Then $ts \subseteq \mathsf{mcts}(\mathcal{M})$. Let $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ be a transition model on $\mathcal{M}^{ts}$. Let $m \in M$ and $\mathsf{Tr}(h) \in ts$ s.t. $\mathsf{H}(\mathsf{Tr}(h)) \cap \mathsf{H}_m \neq \emptyset$, and assume that $\mathfrak{M}^{ts}, m/\mathsf{Tr}(h) \nvDash_{\mathsf{t}} \mathsf{F}p$. By the semantic clause for F, it follows that for all $m' \in M$ s.t. $m' \in h$ and $m' > m$, we have $\mathfrak{M}^{ts}, m'/\mathsf{Tr}(h) \nvDash_{\mathsf{t}} p$. This implies that $\mathfrak{M}^{ts}, m/\mathsf{Tr}(h) \vDash_{\mathsf{t}} \mathsf{F}\neg p$; for, by condition (BT3) of Def. 1.1, there is at least one $m' \in h$ s.t. $m' > m$. Hence, $\mathfrak{M}^{ts}, m/\mathsf{Tr}(h) \vDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$. Since $\mathfrak{M}^{ts}$ on $\mathcal{M}^{ts}$, $m \in M$ and $\mathsf{Tr}(h) \in ts$ were arbitrarily chosen, it follows that $\mathcal{M}^{ts} \vDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$.
"⇐": Assume that $\mathcal{M}^{ts} \notin \mathscr{C}_{\mathsf{b}}$. Then $ts \nsubseteq \mathsf{mcts}(\mathcal{M})$. It follows that there is some $T \in ts$ s.t. T is not maximal consistent. Accordingly, there is some transition $\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M})$ s.t. $\langle m \rightarrowtail H \rangle \notin T$ and $\mathsf{H}(T \cup \{\langle m \rightarrowtail H \rangle\}) \neq \emptyset$. Since the set T is downward closed, we have that for all $\langle m' \rightarrowtail H' \rangle \in T$, $\langle m' \rightarrowtail H' \rangle \prec \langle m \rightarrowtail H \rangle$. This implies that $\mathsf{H}_m \subseteq \mathsf{H}(T)$. Because $\langle m \rightarrowtail H \rangle$ is an indeterministic transition, there must be histories $h, h' \in \mathsf{H}_m$ s.t. $h \perp_m h'$. Consider some transition model $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ on $\mathcal{M}^{ts}$ s.t. for all $m' \in M$ s.t. $\mathsf{H}(T) \cap \mathsf{H}_{m'} \neq \emptyset$, we have $v_{\mathsf{t}}(p, m'/T) = 1$ if $m' > m$ and $m' \in h$, and $v_{\mathsf{t}}(p, m'/T) = 0$ if $m' > m$ and $m' \in h'$. Then $\mathfrak{M}^{ts}, m/T \nvDash_{\mathsf{t}} \mathsf{F}p$ and $\mathfrak{M}^{ts}, m/T \nvDash_{\mathsf{t}} \mathsf{F}\neg p$. Consequently, $\mathfrak{M}^{ts}, m/T \nvDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$, and hence we have $\mathcal{M}^{ts} \nvDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p$. □

A brief remark is in order here. Rather than characterizing the class $\mathscr{C}_{\mathsf{b}}$ of all bundled Ockhamist transition structures by the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$, we could alternatively have characterized that class by means of the equivalence $\mathsf{f}p \leftrightarrow \mathsf{F}p$. Since maximal consistent transition sets allow exactly one history, that equivalence is valid in every bundled Ockhamist transition structure, i.e.

$\mathscr{C}_{\mathsf{b}} \vDash_{\mathsf{t}} \mathsf{f}p \leftrightarrow \mathsf{F}p$, and the model constructed in the second part of the proof of Prop. 2.47 also constitutes a counterexample to that equivalence. The equivalence $\mathsf{f}p \leftrightarrow \mathsf{F}p$ is equivalent to the equivalence $\neg\mathsf{G}\neg p \leftrightarrow \mathsf{F}p$, which captures the Ockhamist idea that G is the dual of F. Considering the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ is more interesting, however. That the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ characterizes the class $\mathscr{C}_{\mathsf{b}}$ of all transition structures $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ in which ts comprises maximal consistent transition sets only reveals that we can only make the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ valid at the cost of giving away the rich structural resources BT structures have to offer. If, and only if, one neglects the local aspect of the modal-temporal structure and instead focuses on maximal consistent transition sets only, which require a standpoint from the end of time, does the disjunction come out valid.

Now consider the class $\mathscr{C}_{\mathsf{b}}$ of all bundled Ockhamist transition structures. Obviously, the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures is a subset of that class, i.e. $\mathscr{C}_{\mathsf{o}} \subseteq \mathscr{C}_{\mathsf{b}}$. However, the converse does not hold. As said, the definition of a transition structure (Def. 2.33) does not rule out transition structures $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ for which $ts \subsetneq \mathsf{mcts}(\mathcal{M})$. The following example illustrates the case. Consider a BT structure $\mathcal{M}$ that contains a history h^* isomorphic to the natural numbers $\mathbb{N}$ such that at any moment $m_i \in h^*$ with $i \in \mathbb{N}$, there is a history h_i such that $h^* \perp_{m_i} h_i$. Assume that $\mathsf{hist}(\mathcal{M}) = \{h^*\} \cup \{h_i \mid i \in \mathbb{N}\}$. The structure $\mathcal{M}^{ts}$ with $ts = \{\mathsf{Tr}(h_i) \mid i \in \mathbb{N}\}$ is a transition structure according to Def. 2.33: every moment is contained in some history in ts. It is not a complete Ockhamist transition structure, however, but rather a bundled one, for $ts = \mathsf{mcts}(\mathcal{M}) \backslash \{\mathsf{Tr}(h^*)\}$. Given the example, the parallel with bundled trees becomes apparent. We have considered a very similar example in our discussion of bundled trees in section 1.4.3.2.2 below. Recall that a bundled tree is a BT structure $\mathcal{M} = \langle M, <\rangle$ together with a non-empty subset $B \subseteq \mathsf{hist}(\mathcal{M})$ such that for every moment $m \in M$, there is a history $h \in B$ such that $m \in h$ (cf. Def. 1.16). A bundled tree is said to be complete if and only if $B = \mathsf{hist}(\mathcal{M})$. We use $\mathscr{B}$ to stand for the *class of all bundled trees.*

In the previous section, we have shown that there is a one-to-one correspondence κ between the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures and BT structures $\mathscr{T}$ that preserves satisfiability. We now generalize that result. That is, we extend the correspondence to the class $\mathscr{C}_{\mathsf{b}}$ of bundled Ockhamist transition structures and the class $\mathscr{B}$ of bundled trees, respectively. Validity with respect to the class $\mathscr{C}_{\mathsf{b}}$ of bundled Ockhamist transition structures is shown to coincide with bundled tree validity.

Let $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ with $ts \subseteq \mathsf{mcts}(\mathcal{M})$ be a bundled Ockhamist transition structure, and consider the bundled tree $\mathcal{B} = \langle M, <, B \rangle$ with $B = \{h \mid \mathsf{Tr}(h) \in ts\} \subseteq \mathsf{hist}(\mathcal{M})$. Note that the structure $\mathcal{B} = \langle M, <, B \rangle$ with $B = \{h \mid \mathsf{Tr}(h) \in ts\}$ is a bundled tree according to Def. 1.16 if and only if the structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ with $ts \subseteq \mathsf{mcts}(\mathcal{M})$ is a transition structure according to Def. 2.33: since for all $h \in \mathsf{hist}(\mathcal{M})$, we have that $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$, it holds that for every $m \in M$, there is some history $h \in B$ such that $m \in h$ if and only if for every $m \in M$, there is some maximal consistent transition set $\mathsf{Tr}(h) \in ts$ such that $\mathsf{H}(\mathsf{Tr}(h)) \cap \mathsf{H}_m \neq \emptyset$. Along those lines, we obtain a one-to-one correspondence between the class $\mathscr{C}_{\mathsf{b}}$ of bundled Ockhamist transition structures and the class $\mathscr{B}$ of bundled trees. Obviously, the function $\upsilon : \mathscr{C}_{\mathsf{b}} \to \mathscr{B}$ that maps every bundled Ockhamist transition structure $\langle M, <, ts \rangle$ with $ts \subseteq \mathsf{mcts}(\mathcal{M})$ onto the bundled tree $\langle M, <, B_{ts} \rangle$ with $B_{ts} := \{h \mid \mathsf{Tr}(h) \in ts\}$ is a bijection

Lemma 2.48. *The function*

$$\begin{array}{cccc} \upsilon : & \mathscr{C}_{\mathsf{b}} & \to & \mathscr{B} \\ & \langle M, <, ts \rangle & \mapsto & \langle M, <, B_{ts} \rangle \end{array}$$

is a bijection.

Proof. Straightforward. □

Let $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ with $ts \subseteq \mathsf{mcts}(\mathcal{M})$ be a bundled Ockhamist transition structure, and let $\upsilon(\mathcal{M}^{ts})$ be the corresponding bundled tree. The mapping $m/\mathsf{Tr}(h) \mapsto m/h$ is a bijection between the set of indices of evaluation in $\mathcal{M}^{ts}$ and the set of indices in $\upsilon(\mathcal{M}^{ts})$. Consequently, we can extend the correspondence υ to models: the function $\upsilon' : \langle \mathcal{M}^{ts}, v_{\mathsf{t}} \rangle \mapsto \langle \upsilon(\mathcal{M}^{ts}), v_{\mathsf{o}} \rangle$ with $v_{\mathsf{o}}(p, m/h) = v_{\mathsf{t}}(p, m/\mathsf{Tr}(h))$ for all $p \in \mathsf{At}$, $m \in M$ and $h \in B_{ts}$ with $m \in h$ is a bijection between bundled Ockhamist transition models on $\mathcal{M}^{ts}$ and bundled tree models on $\upsilon(\mathcal{M}^{ts})$. We show that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true in a bundled Ockhamist transition model $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ on $\mathcal{M}^{ts}$ at an index of evaluation $m/\mathsf{Tr}(h)$ if and only if its respective translation $\phi^* \in \mathcal{L}_{\mathsf{o}}$ is true in the bundled tree model $\upsilon'(\mathfrak{M}^{ts})$ on the corresponding structure $\upsilon(\mathcal{M}^{ts})$ at the moment-history pair m/h.

Proposition 2.49. *Let* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ *with* $ts \subseteq \mathsf{mcts}(\mathcal{M})$ *be a bundled Ockhamist transition structure. The mapping* $\upsilon' : \langle \mathcal{M}^{ts}, v_{\mathsf{t}} \rangle \mapsto \langle \upsilon(\mathcal{M}^{ts}), v_{\mathsf{o}} \rangle$ *with* $v_{\mathsf{o}}(p, m/h) = v_{\mathsf{t}}(p, m/\mathsf{Tr}(h))$ *for all* $p \in \mathsf{At}$, $m \in M$ *and* $h \in B_{ts}$ *such that* $m \in h$ *is a bijection between bundled Ockhamist transition models on* $\mathcal{M}^{ts}$ *and bundled tree models on* $\upsilon(\mathcal{M}^{ts})$*. The following holds: given a bundled Ockhamist transition model* $\mathfrak{M}^{ts} = \langle M, <, ts, v_{\mathsf{t}} \rangle$ *on* $\mathcal{M}^{ts}$*, for every* $\phi \in \mathcal{L}_{\mathsf{t}}$*, every* $m \in M$ *and every* $h \in B_{ts}$ *such that* $m \in h$*:*

$$\mathfrak{M}^{ts}, m/\mathsf{Tr}(h) \vDash_{\mathsf{t}} \phi \quad \textit{iff} \quad \upsilon'(\mathfrak{M}^{ts}), m/h \vDash_{\mathsf{o}} \phi^*.$$

Proof. The proof runs by induction on the structure of a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ and rests on the induction hypothesis that the claim holds for any proper subformula ψ of ϕ. Given the correspondence υ', the base clause is straightforward. The cases for the truth-functional connectives $\neg$ and $\wedge$ are banal, the case for P is trivial, and the cases for f and S dissolve due to the validity of the equivalences $\mathsf{f}p \leftrightarrow \mathsf{F}p$ and $\mathsf{S}p \leftrightarrow p$. In the case of the future operator F and the inevitability operator $\Box$, the proof makes use of the fact that $\mathsf{Tr}(h) \in ts$ iff $h \in B_{ts}$ and $\mathsf{H}(\mathsf{Tr}(h)) = \{h\}$ for all $h \in B_{ts}$. □

From Prop. 2.49 it follows that a bundled Ockhamist transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ with $ts \subseteq \mathsf{mcts}(\mathcal{M})$ validates a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ if and only if the corresponding bundled tree $\upsilon(\mathcal{M}^{ts})$ validates its Ockhamist translation $\phi^* \in \mathcal{L}_{\mathsf{o}}$. By the one-to-one correspondence υ between bundled Ockhamist transition structures and bundled trees established in Lem. 2.48, this implies that validity with respect to the class $\mathscr{C}_{\mathsf{b}}$ of all bundled Ockhamist transition structures is equivalent to bundled tree validity.

Corollary 2.50. *For every* $\phi \in \mathcal{L}_{\mathsf{t}}$*:*

$$\mathscr{C}_{\mathsf{b}} \vDash_{\mathsf{t}} \phi \quad \textit{iff} \quad \mathscr{B} \vDash_{\mathsf{o}} \phi^*.$$

Proof. Follows from Lem. 2.48 and Prop. 2.49. □

Since by the correspondence result established in Prop. 2.47, the class $\mathscr{C}_{\mathsf{b}}$ of bundled Ockhamist transition structures can be characterized by the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$, bundled tree validity is definable in the transition semantics:

Corollary 2.51. *For every* $\phi \in \mathcal{L}_{\mathsf{t}}$*:*

$$(\textit{for all } \mathcal{M}^{ts} \textit{ s.t. } \mathcal{M}^{ts} \vDash_{\mathsf{t}} \mathsf{F}p \vee \mathsf{F}\neg p,\ \mathcal{M}^{ts} \vDash_{\mathsf{t}} \phi) \quad \textit{iff} \quad \mathscr{B} \vDash_{\mathsf{o}} \phi^*.$$

Proof. Follows from Cor. 2.50 and Prop. 2.47. □

2.5.1.3.2.2 Full Ockhamism In the previous section, we have shown that the class of bundled Ockhamist transition structures is definable in the transition language $\mathcal{L}_{\mathsf{t}}$ and that validity with respect to that class is equivalent to bundled tree validity. As is well known, however, bundled tree validity is distinct from Ockhamist validity. In section 1.4.3.2.2, we have illustrated that the sentence $\Box_{\mathsf{o}}\mathsf{G}_{\mathsf{o}}(p \to \Diamond_{\mathsf{o}}\mathsf{F}_{\mathsf{o}}p) \to \Diamond_{\mathsf{o}}\mathsf{G}_{\mathsf{o}}(p \to \mathsf{F}_{\mathsf{o}}p)$, while Ockhamistically valid, is not a bundled tree validity: $\mathscr{B} \vDash_{\mathsf{o}} \phi$ implies $\vDash_{\mathsf{o}} \phi$, but the opposite direction does not hold. Ockhamist validity is validity with respect to the class of all complete bundled trees.

In this section, we single out from the class $\mathscr{C}_{\mathsf{b}}$ of all bundled Ockhamist transition structures those that are complete, viz. the class $\mathscr{C}_{\mathsf{o}}$ genuine Ockhamist transition structures. Due to the correspondence between Ockhamist transition structures and BT structures, on the one hand, and bundled Ockhamist transition structures and bundled trees, on the other, this amounts to characterizing the class of complete bundled trees within the class of bundled trees.

In Zanardo *et al.* 1999 it is shown that although in general it is not possible to define the class of complete bundled trees within the class of bundled trees, the class of complete bundled trees can be defined within the class of all bundled trees that are such that the intersection of any two histories contains a greatest element. Condition (BT2) of Def. 1.1 guarantees that this is here the case. In order to be able to define branching points, which figure as the initials of our transitions, in a perspicuous way, we have required from the outset that any two moments in a BT structure must have a greatest common lower bound. It is precisely that requirement that now allows us to apply the correspondence result provided in Zanardo *et al.* 1999 to our case.

Let $\delta := \Box\mathsf{F}\Box\mathsf{G}p \wedge \Diamond\mathsf{F}\neg p \to \Diamond\mathsf{F}(\Box\mathsf{G}p \wedge \mathsf{H}(\neg p \to \mathsf{F}\neg p))$ be the $\mathcal{L}_{\mathsf{t}}$-correspondent of the characterizing formula provided in Zanardo *et al.* 1999. Zanardo et al. show that a bundled tree $\mathcal{B}$ is complete if and only if δ^* is valid in $\mathcal{B}$.

Lemma 2.52 (Zanardo et al. 1999). *For every bundled tree $\mathcal{B} = \langle M, <, B\rangle$:*

$$\mathcal{B} \text{ is complete} \quad \text{iff} \quad \mathcal{B} \vDash_{\mathsf{o}} \delta^*$$

where $\delta^* := \Box_{\mathsf{o}}\mathsf{F}_{\mathsf{o}}\Box_{\mathsf{o}}\mathsf{G}_{\mathsf{o}}p \wedge \Diamond_{\mathsf{o}}\mathsf{F}_{\mathsf{o}}\neg p \to \Diamond_{\mathsf{o}}\mathsf{F}_{\mathsf{o}}(\Box_{\mathsf{o}}\mathsf{G}_{\mathsf{o}}p \wedge \mathsf{H}_{\mathsf{o}}(\neg p \to \mathsf{F}_{\mathsf{o}}\neg p))$.

Proof. See the proof of Theorem 2.3 in Zanardo *et al.* (1999, pp.127–129). □

Given the correspondences $\kappa : \mathscr{C}_\mathsf{o} \to \mathscr{T}$ and $\upsilon : \mathscr{C}_\mathsf{b} \to \mathscr{B}$, we can transfer the result cited in Lem. 2.52 to transition structures. On the basis of that result, we can define the class $\mathscr{C}_\mathsf{o}$ of genuine Ockhamist transition structures within the class $\mathscr{C}_\mathsf{b}$ of bundled Ockhamist transition structures.

Lemma 2.53. *For every transition structure $\mathcal{M}^{ts} \in \mathscr{C}_\mathsf{b}$:*

$$\mathcal{M}^{ts} \in \mathscr{C}_\mathsf{o} \quad \textit{iff} \quad \mathcal{M}^{ts} \vDash_\mathsf{t} \delta.$$

Proof. "⇒": Let $\mathcal{M}^{ts} \in \mathscr{C}_\mathsf{o}$. Since, by Lem. 2.52, δ^* is valid on all complete bundled trees, δ^* is an Ockhamist validity, i.e. $\vDash_\mathsf{o} \delta^*$. It then follows by Cor. 2.45 that $\mathscr{C}_\mathsf{o} \vDash_\mathsf{t} \delta$ and hence $\mathcal{M}^{ts} \vDash_\mathsf{t} \delta$.
"⇐": Assume that $\mathcal{M}^{ts} \vDash_\mathsf{t} \delta$. Let $\upsilon(\mathcal{M}^{ts})$ be the corresponding bundled tree. By Prop. 2.49 it follows that $\upsilon(\mathcal{M}^{ts}) \vDash_\mathsf{o} \delta^*$, which implies by Lem. 2.52 that $\upsilon(\mathcal{M}^{ts})$ is complete. Consequently, $ts = \mathsf{mcts}(\mathcal{M})$ and hence $\mathcal{M}^{ts} \in \mathscr{C}_\mathsf{o}$. □

By Lem. 2.53, the class $\mathscr{C}_\mathsf{o}$ of Ockhamist transition structures is definable in the class $\mathscr{C}_\mathsf{b}$ of bundled Ockhamist transition structures, and, by Prop. 2.47, the latter class is again definable in the overall class $\mathscr{C}$ of transition structures. Since by Cor. 2.45, Ockhamist validity with respect to our BT structures is equivalent to validity with respect to the class $\mathscr{C}_\mathsf{o}$ of Ockhamist transition structures, it then follows that Ockhamist validity is definable in the transition semantics: a sentence of the Ockhamist language is valid in the Ockhamist semantics if and only if its $\mathcal{L}_\mathsf{t}$-correspondent is valid in all transition structures in which both $\mathsf{F}p \vee \mathsf{F}\neg p$ and δ are valid.

Proposition 2.54. *For every $\phi \in \mathcal{L}_\mathsf{t}$:*

$$(\textit{for all } \mathcal{M}^{ts} \textit{ s.t. } \mathcal{M}^{ts} \vDash_\mathsf{t} \mathsf{F}p \vee \mathsf{F}\neg p \textit{ and } \mathcal{M}^{ts} \vDash_\mathsf{t} \delta,\ \mathcal{M}^{ts} \vDash_\mathsf{t} \phi) \quad \textit{iff} \quad \vDash_\mathsf{o} \phi^*.$$

Proof. Follows from Cor. 2.45, Prop. 2.47 and Lem. 2.53. □

2.5.1.4 Peirceanism and Ockhamism as limitations

The results provided in section 2.5.1.2 and section 2.5.1.3 above demonstrate that both Peirceanism and Ockhamism can be captured in the transition framework. With the notion of a transition structure at our disposal, we can show that the transition semantics unifies the Peircean and the Ockhamist account by generalizing both of them. In this section, we point out the impact of those results. The results do not only establish Peirceanism and Ockhamism as limiting cases of the transition approach but also exposes them as limitations.

In the previous sections, we have illustrated that both Peirceanism and Ockhamism can be captured in the transition framework by placing suitable restrictions on the transition sets employed in the semantic evaluation and, thus, by restricting the class of transition structures. Peircean validity coincides with validity with respect to the empty transition set, i.e. with respect to the class $\mathscr{C}_{\mathsf{p}}$ of all Peircean transition structures. Ockhamist validity, on the other hand, is equivalent to validity with respect to all maximal consistent transition sets, i.e. with respect to the class $\mathscr{C}_{\mathsf{o}}$ of Ockhamist transition structures. The relevant classes of transition structures have been shown to be characterizable in the transition language $\mathcal{L}_{\mathsf{t}}$, and, accordingly, both Peircean and Ockhamist validity are definable in the transition semantics. Whereas the transition semantics in its most general form, as spelled out in section 2.4, exploits the entire range of consistent, downward closed transition sets a BT structure has to offer, Peirceanism and Ockhamism are limiting cases that each rest upon only a proper subset thereof.

The transition semantics does, however, not only generalize both Peirceanism and Ockhamism but also exceeds both accounts in terms of expressive strength. Both Peirceanism and Ockhamism are limitations of the transition account. On the Peircean account, where the semantic evaluation depends solely on a moment parameter, inevitability and truth coincide: a sentence cannot be true without its truth being inevitable. The equivalence $\Box p \leftrightarrow p$ and hence also the equivalence $\mathsf{S}p \leftrightarrow p$ are valid in every Peircean transition structure. On the Ockhamist account, where the semantic evaluation at a moment is in addition relativized to a complete possible course of events, inevitability and truth come apart, but stability and truth still coincide: a sentence cannot be true without being stably-true. In every Ockhamist transition structure, the equivalence $\mathsf{S}p \leftrightarrow p$ is valid. Neither on the Peircean nor on the Ockhamist account is an instance of $\reflectbox{$\mathsf{S}$}p \wedge \reflectbox{$\mathsf{S}$}\neg p$ satisfiable.

Compared to the transition semantics, Peirceanism and Ockhamism build on limited resources only, and the limitations are such that the dynamic picture that is at the heart of the transition semantics dwindles away to a static one. All dynamicity is lost, and the stability operator loses its power: neither Peirceanism nor Ockhamism allow for quantification over proper future extensions of a transition set. On the Peircean account, we are provided with just a single transition set, and the maximal consistent transition sets that the Ockhamist account rests on cannot be further extended. Building on the entire range of consistent, downward closed transition sets, the transition semantics allows for a more fine-grained picture of the interaction of modality and time and thereby gains expressive means that are not available on either of those accounts.

2.5.2 A remark on Zanardo's idea of indistinguishability

Zanardo 1998 provides a branching time semantics, which aims at retrieving the resources provided by a BT structure and which generalizes both Peirceanism and Ockhamism, just as the transition approach does. In this section, we will briefly review that framework and show that the transition semantics with its inherent dynamicity also exceeds that framework in terms of expressive strength.

On Zanardo's account sentences are evaluated on so-called *I-trees*, which are BT structures $\mathcal{M} = \langle M, < \rangle$ with an indistinguishability function I that assigns to each moment $m \in M$ a partition of the set H_m of histories containing that moment. The indistinguishability function thereby has the following property: histories that are indistinguishable at a moment, are indistinguishable at all prior moments as well.

Definition 2.55 (I-tree). *An* I-tree *is an ordered triple* $\mathcal{I} = \langle M, <, I \rangle$, *where* $\mathcal{M} = \langle M, < \rangle$ *is a BT structure and* I *is a function with domain* M *such that for all* $m, m' \in M$ *and for all* $h, h' \in \mathsf{H}_m$, *(i)* $I(m)$ *is an equivalence relation on* H_m *and (ii) whenever* $m' < m$ *and* $\langle h, h' \rangle \in I(m)$, *then* $\langle h, h' \rangle \in I(m')$.

In a model $\mathcal{M}$ on an I-tree $\mathcal{I} = \langle M, <, I \rangle$, the semantic evaluation is relativized to pairs consisting of a moment $m \in M$ and an equivalence class modulo indistinguishability $[h]^I_m$ at that very moment. The language comes equipped with a past operator $\mathsf{P_z}$, two future operators $\mathsf{f_z}$ and $\mathsf{F_z}$ and an inevitability operator $\Box_\mathsf{z}$, and the following semantics clauses are provided:[63]

($\mathsf{P_z}$) $\mathfrak{M}, m/[h]^I_m \vDash_\mathsf{z} \mathsf{P_z}\phi$ iff there is some $m' < m$ s.t. $\mathfrak{M}, m'/[h]^I_{m'} \vDash_\mathsf{z} \phi$;

($\mathsf{f_z}$) $\mathfrak{M}, m/[h]^I_m \vDash_\mathsf{z} \mathsf{f_z}\phi$ iff there is some $h' \in [h]^I_m$ and some $m' \in h'$ s.t. $m' > m$ and $\mathfrak{M}, m'/[h']^I_{m'} \vDash_\mathsf{z} \phi$;

[63] We restrict ourselves here to the semantic clauses for the intensional operators. The semantic clauses for the truth-functional connectives are straightforward. Zanardo takes the operators $\mathsf{G_z}$ and $\Diamond_\mathsf{z}$ as primitive and defines the operators $\mathsf{f_z}$ and $\Box_\mathsf{z}$ as their duals. We stick with our usual set of operators. The particular choice as to which operators are considered primitive does not matter.

($\mathsf{F_z}$) $\mathfrak{M}, m/[h]^I_m \vDash_{\mathsf{z}} \mathsf{F_z}\phi$ iff for all $h' \in [h]^I_m$, there is some $m' \in h'$ s.t. $m' > m$ and $\mathfrak{M}, m'/[h']^I_{m'} \vDash_{\mathsf{z}} \phi$;

($\Box_{\mathsf{z}}$) $\mathfrak{M}, m/[h]^I_m \vDash_{\mathsf{z}} \Box_{\mathsf{z}}\phi$ iff for all $h' \in \mathsf{H}_m$, $\mathfrak{M}, m/[h']^I_m \vDash_{\mathsf{z}} \phi$.

The inevitability operator $\Box_{\mathsf{z}}$ is interpreted as a universal quantifier over the equivalence classes modulo undividedness at the given moment m. The temporal operators shift the moment of evaluation m in a way compatible with the given equivalence class $[h]^I_m$: the strong future operator $\mathsf{F_z}$ requires a future witness in every history in $[h]^I_m$, the weak future operator f in only a single one, and in the case of the past operator $\mathsf{P_z}$, the particular choice of a history in $[h]^I_m$ does not matter. What is important is that in either case, with the moment of evaluation m, the equivalence class $[h]^I_m$ is shifted as well.

In Zanardo (1998) it is shown that the I-tree framework generalizes both Peirceanism and Ockhamism. Both accounts can be obtained as limiting cases by imposing additional constraints on the indistinguishability function I. If the indistinguishability function is required to be such that for all $m \in M$ and $h \in \mathsf{H}_m$, we have $[h]^I_m = \{h\}$, we are back to Ockhamism, whereas the requirement $[h]^I_m = \mathsf{H}_m$ for all $m \in M$ and $h \in \mathsf{H}_m$ yields Peirceanism. The corresponding classes of structures are definable in the I-tree language.

The notion of undividedness, on which our definition of a transition rests, is a special kind of indistinguishability, and it is a very natural one, viz. one that flows immediately from the ordering of moments in a BT structure. In case the indistinguishability relation is the relation of undividedness, evaluating a sentence at an index $m/[h]^I_m$ in the I-tree framework amounts to evaluating the sentence at the moment m with respect to the downward completion of the singleton of the transition $\langle m \rightarrowtail [h]_m \rangle$ in the transition semantics.[64] Just as the transition semantics, the I-tree framework thus allows for the relativization of the semantic evaluation to incomplete possible courses of events as well.

However, unlike in the transition semantics, in the I-tree framework, indistinguishability classes cannot be shifted independently of the moment of evaluation: a sentence cannot be evaluated at a moment with respect to an indistinguishability class at another moment. Quantifying over extensions of a given transition set while keeping the moment of evaluation fixed, as in the case of the stability operator, is therefore not possible in the I-tree framework. In

[64] Note that the transition semantics allows for consistent, downward closed transition sets that do not contain a greatest element.

the I-tree framework, there is no room for contingency and stability. It is not only by employing a second parameter of truth that can capture an incomplete possible course of events, but also by allowing that second parameter to vary independently of the moment parameter that the transition semantics gains its expressive power.

2.5.3 A remark on MacFarlane's idea of assessment-sensitivity

Just as in the transition framework, in MacFarlane's assessment-sensitive post-semantics, sentences about the future are evaluated at a moment from a second local perspective in time. In this section, we will briefly introduce MacFarlane's framework and illustrate how it relates to the transition approach. We will see that we can capture MacFarlane's idea in the transition semantics, at lower costs.

It is important to note that moving to MacFarlane's framework requires, first of all, a fundamental shift in perspective. While we have so far been concerned with the logical notion of truth at an index of evaluation, we are now moving to the pragmatic notion of truth at a context. Let us briefly set the stage: sentences are used in contexts, and in order to determine whether a sentence is true in a context, both the notion of an index of evaluation and the notion of a context of use are needed. Just as an index of evaluation, a context of use is a sequence of parameters.[65] Yet, while the parameters of the index of evaluation are 'mobile', i.e. they can be shifted in the recursive semantics, the parameters of the context are 'immobile' parameters.[66] The context of use has basically two roles: it fixes the meaning of indexicals, such as 'now', and provides initial values for the parameters of the index of evaluation. In case the language under consideration does not contain indexicals, one may simply want to say that a sentence is true at a context of use if and only if it is true with respect to the index of evaluation that is initialized by the context of use. That analysis, however, is in need of modification when it comes to sentences about the future in a branching time setting.

[65] The notion of a context of use that we envisage here is the notion of context famously employed in Kaplan (1989), which was written in 1977. For an argument to the effect that the semantic evaluation needs to be relativized to both a context of use and an index of evaluation, see Kaplan (1989) and D. Lewis (1980).

[66] For the distinction between mobile and immobile parameters of truth, see Belnap, Perloff, and Xu (2001, ch.6B).

A future contingent is a sentence about the future used in a context in which its truth value is not yet settled. Whether the sentence is true or false in that context depends on how the future unfolds. It is true in one possible future but false in another, with nothing yet deciding between those possibilities. On the Peircean account, with its strong future operator, all future contingents are rendered false at the context of use: all of them are false at the moment initialized by the context of use according to the Peircean semantics. The recurring problem of Ockhamism is that the history parameter cannot be initialized by a context of use: while the context of use provides a unique moment of use, it cannot single out one of the histories passing through that moment.[67] And so the analysis for truth at a context that we have introduced above fails. Due to the mismatch between the parameters employed in the recursive semantics and the parameters provided by a context of use, Ockhamism is in need of a postsemantics that links the recursive semantic machinery to a context of use.[68]

There are two popular postsemantic accounts for Ockhamism: supervaluation, as put forth in Thomason (1970), and assessment-sensitivity, as provided in MacFarlane (2003, 2014).[69] On Thomason's supervaluationist account, a sentence is said to be true at a context of use if and only if the sentence is Ockhamistically true at the moment of use m_u with respect to all histories passing through that moment. A sentence is false at a context of use if and only if its negation is true at that context. Accordingly, future contingents are neither true nor false at their respective context of use. Their truth value at a context of use is undefined.

The assessment-sensitive postsemantics proposed by MacFarlane generalizes the supervaluationist account. On MacFarlane's assessment-sensitive account, in addition to the context of use, a so-called context of assessment enters the picture. The moment of assessment m_a provides a second perspective from which the truth of a future contingent at the moment of use m_u can be retrospectively assessed. A sentence is said to be true with respect to a context

[67] For a discussion of the problem that a context of use fails to provide an initial value for the Ockhamist history parameter, see Belnap, Perloff, and Xu (2001, ch.6B.5).

[68] The helpful term 'postsemantics' was coined by MacFarlane (2003).

[69] Interestingly enough, the structures defined in MacFarlane (2003, ch.9) are $T \times W$-frames (cf. Def. 1.12) rather than BT structures. The time-relative accessibility relation between worlds is merely required to be an equivalence relation; it is not defined as identity. Note that in a $T \times W$ framework the initialization problem does not arise. In that framework, the parameters employed in the recursive semantic machinery perfectly match the parameters provided by a context of use: both are time-world pairs.

of use and a context of assessment if and only if it is true at the moment of use m_u with respect to all histories containing the moment of assessment m_a according to the Ockhamist semantics. A sentence is false relative to a context of use and a context of assessment if and only if its negation is true with respect to those contexts.[70] Note that in order for a future contingent to be true at the context of use relative to a context of assessment in MacFarlane's sense, it is necessary that the moment of assessment m_a be later than the moment of use m_u. Supervaluation is just a special case of the assessment-sensitive postsemantics, viz. the case in which the moment of assessment m_a is identical to the moment of use m_u.

In the transition semantics, initialization is not a problem. Since in the transition semantics, sentences are assigned truth values at a moment with respect to a consistent, downward closed set of transitions rather than with respect to an entire history, it is possible to extract an initial value for the second parameter of truth from a context of use. A sentence can be evaluated at a moment with respect to the past course of events up to that moment: the context of use specifies the moment of use m_u, which in turn determines the set $\mathsf{Tr}(m_u)$ of transitions preceding that moment. The initialization problem does not arise.

What is more, the transition semantics allows us to express that a future contingent is contingent with respect to the parameters provided by the context of use, and the stability operator provides a means to specify how and how far the future has to unfold for its truth value at the moment of use to become settled. Future contingents are neither stably-true nor stably-false at the moment of use m_u with respect to the transition set $\mathsf{Tr}(m_u)$: they are contingent relative to that index. If a future contingent is stably-true relative to an index of evaluation $m_u/\mathsf{Tr}(m_a)$, the moment m_a constitutes a suitable moment of assessment in MacFarlane's sense. That is to say, if a sentence is stably-true in the transition semantics at an index of evaluation $m_u/\mathsf{Tr}(m_a)$, it is true in MacFarlane's assessment-sensitive postsemantics relative to the corresponding context of use and the corresponding context of assessment. Note that the converse does not hold, however. Since the disjunction $\mathsf{F}p \vee \mathsf{F}\neg p$ is an Ockhamist validity, it is true with respect to any context pair on MacFarlane's account, provided that the respective moments are order-related. In the transition semantics, on the other hand, the disjunction can be contingent.

[70]Belnaps's theory of double time references makes use of a similar idea. See Belnap 2002.

The transition semantics provides a second perspective from which the truth value of a future contingent at the moment of use can be retrospectively assessed within the semantics itself. No postsemantics and no additional parameters are needed. The transition semantics with its stability operator enables us to capture the behavior of the truth value of a future contingent in the course of time: its transition from contingency at $m_u/\mathsf{Tr}(m_u)$ to stability at $m_u/\mathsf{Tr}(m_a)$. There is even hope that a theory of conditionals allows us to give propositional content to the transition from $m_u/\mathsf{Tr}(m_u)$ to $m_u/\mathsf{Tr}(m_a)$ in the transition language. The basic idea is to specify conditionals for which the following holds: whenever the conditional is true at $m_u/\mathsf{Tr}(m_u)$, its consequent is stably-true at $m_u/\mathsf{Tr}(m_a)$, where the value of $\mathsf{Tr}(m_a)$ is determined by the antecedent. To develop a theory of conditionals for the transition framework is subject to future research.

2.6 Concluding remarks

In this chapter, we have presented a novel propositional semantics based on the framework of branching time. In our transition semantics, the parameters of truth are provided by a moment and a set of transitions, which can represent a partial or a complete possible course of events. In addition to temporal and modal operators, the transition language is equipped with a stability operator, which is interpreted as a universal quantifier over the possible extensions of a given transition set. The stability operator allows us to specify how and how far time has to unfold for the truth value of a sentence at a moment to become settled, and thereby enables a perspicuous treatment of real possibilities. The dynamics of the transition semantics adequately reflects the dynamics of real possibilities.

The transition semantics generalizes and extends both Peirceanism and Ockhamism. Both accounts are limiting cases of the transition approach that can be obtained by placing suitable restrictions on the transition sets employed in the semantic evaluation. Restricting the semantic evaluation to the empty transition set yields Peirceanism with its sole dependence on the moment parameter, while a restriction to maximal consistent transition sets yields Ockhamism with its dependence on moment-history pairs. On both accounts, stability collapses into truth. Operating on the entire range of consistent, downward close sets of indeterministic transitions a BT structure has to offer, the transition semantics with its stability operator gains expressive means that are not available on either of those accounts and provides a fine-grained, dynamic picture of the interrelation of modality and time.

* * *

Chapter 3

Transitions toward Completeness*

3.1 Introduction

In the previous chapter, we have presented a novel propositional branching time semantics, viz. the so-called transition semantics. The language of the transition semantics contains, in addition to temporal and modal operators, a stability operator, which is interpreted as a universal quantifier over possible future courses of events. The inclusion of a stability operator into the language is enabled by the fact that in the transition semantics, the semantic evaluation on a branching time structure is relativized to a set of transitions as a second parameter of truth, next to the moment parameter. Sentences are evaluated at pairs consisting of a moment and a consistent, downward closed set of indeterministic transitions compatible with that moment.

As the title suggests, the present chapter is concerned with transitions toward completeness. The meaning of that title is twofold. Of course, our overall aim is to provide an axiomatization as well as sound inference rules for the transition framework and prove the completeness of that deductive system. In this chapter, however, we confine ourselves to making a transition toward such

*This chapter draws on joint work with Alberto Zanardo, to whom I am truly grateful for many inspiring whiteboard discussions and numerous exchanges.

a completeness result. While we do not present a complete axiomatic system here, we do make a significant step toward completeness.

The employment of a second parameter of truth, which can even be quantified over, crucially distinguishes the transition semantics from a genuine Kripke-style semantics. The same is of course true for Ockhamism. In this chapter, we establish the basis for the completeness proof for the transition framework by providing a class of genuine Kripke structures that validate exactly those sentences that are validities in the transition semantics. We refer to the relevant structures as *index structures*. In those structures, the set of indices of evaluation equals set of points of the structure, and quantification over the second parameter of truth dissolves into restricted quantification over that set.

Completeness results for branching time logics are an intricate matter, as becomes clear when we briefly review extant results. A standard technique employed in the proof of completeness of a branching time logic consists in a so-called step-by-step chronicle construction, starting out with some arbitrary maximal consistent set of sentences that is to be shown to be satisfiable. The construction yields a Kripke structure with primitive accessibility relations for the various kinds of intensional operators of the language, in which each point is associated with a maximal consistent set of sentences. The assignment of maximal consistent sets of sentences to the points of the structure is dubbed a *chronicle*, and it can be shown to provide a valuation of the language on the underlying structure.

In the case of Peirceanism, the completeness construction yields a Kripke structure with a chronicle that comprises but a single accessibility relation, which mirrors the temporal earlier-later relation among the various points of the structure. A finite axiomatic system and corresponding completeness proof for Peirceanism is provided in Burgess (1980). The proof presented there makes use of the so-called Gabbay Irreflexivity Rule.[71] The Gabbay Irreflexivity Rule is a technical tool that proves useful in the completeness construction as it allows one to generate irreflexive points in the temporal order. A complete axiomatization of the Peircean logic that does not rest on the Gabbay Irreflexivity Rule is put forth in Zanardo (1990). The trade-off is an infinite set of axioms. Completeness results for the computational variant CTL of the Peircean logic are established in Emerson and Halpern (1985).

[71] The Gabbay Irreflexivity Rule is first presented in Gabbay (1981).

When it comes to Ockhamism, the situation is more complex. While in the Peircean semantics, the indices of evaluation are just moments, which form the basic constituents of a branching time structure, the Ockhamist semantics rests on a second, defined, parameter of truth, just as the transition semantics does. In Ockhamism, sentences are evaluated at moment-history pairs rather than at moments, and the Ockhamist language is equipped with modal operators that involve quantification over histories. In this case, the chronicle construction results in models on Kripke structures in which each point corresponds to a moment-history pair and which involve two kinds of accessibility relations, one for the temporal operators and another one for the modal operators to range over. This creates the need to provide a general characterization of the relevant class of Kripke structures and show that validity with respect to that class coincides with Ockhamist validity.

A definition of a class of Kripke structures that accurately captures bundled tree validity is given, for example, in Zanardo (1985, 1991, 1996) and Reynolds (2002, 2003), where the relevant Kripke structures are referred to as 'Ockhamist frames' or 'Kamp frames', respectively. Complete axiomatic systems for the bundled Ockhamist logic and its extension by 'since' and 'until' operators are provided in Zanardo (1985) and Zanardo (1991), respectively; and in Stirling (1992), a completeness result for the computational variant of the bundled version of Ockhamism is established. Finally, a complete axiomatization for the full Ockhamist computational tree logic CTL^* and its extension PCTL^* by a past operator is given in Reynolds (2001) and Reynolds (2000, 2005), respectively. Full versions of Ockhamism face the problem of emergent histories, which is triggered by the limit closure property of complete bundled trees. In Reynolds (2003), an axiomatization and a completeness proof for the full Ockhamist logic are sketched. The system presented there makes use of an additional axiom scheme that is supposed to rule out the emergence of additional histories in the limit step of the completeness construction. The full version of the proof, which is said to comprise more than one hundred pages, has unfortunately never appeared in print.

Just as the Ockhamist semantics, the transition semantics is not a genuine Kripke style semantics. For, the transition semantics makes use of a set of transitions as a second, defined, parameter of truth. Accordingly, as in the case of Ockhamism, the completeness construction does not yield models on a branching time structure. Rather, the resulting models are again index structures that are endowed with a chronicle. Yet, what is more, since the language of the transition semantics extends the Ockhamist language by a stability operator, the index structures generated in the completeness construction for the transition framework need to be Kripke structures with three, rather than merely two, primitive accessibility relations: one for the temporal, one for the modal and one for the stability operators.

In this chapter, we provide a first-order characterization of the relevant index structures and show that validity with respect to the class of index structures is equivalent to validity with respect to the class of transition structures (section 3.3). The proof of a correspondence between transition structures and genuine Kripke structures, viz. index structures, that preserves satisfiability is an interesting result on its own, and at the same time builds the basis for a future completeness proof. Before we turn to the correspondence between transition structures and index structures, however, we will first engage in a closer investigation of the notion of a transition set. We will show that transition sets correspond one-to-one to certain substructures of a branching time structure, which are first-order definable and which we will refer to as *prunings* (section 3.2). The correspondence between transition sets and prunings shows, first of all, that the transition sets, which figure as a second parameter of truth in the semantic evaluation, are theoretically less complex than they seem at first glance. At the same time, the correspondence allows us to treat transition sets as unstructured entities, viz., it allows us to identify transition sets without having to identify transitions first, which becomes important in the definition of index structures.

3.2 Transition Sets and Prunings

In the transition framework, the semantic evaluation on a BT structure $\mathcal{M} = \langle M, < \rangle$ is relativized to pairs m/T consisting of a moment $m \in M$ and a set of transitions $T \in \mathsf{dcts}(\mathcal{M})$ compatible with that moment. The semantics makes use of indeterministic transitions only, and only transition sets that are consistent and downward closed are taken into account. A consistent, downward closed transition set is a possibly non-maximal chain of transitions that is closed toward the past in the transition ordering. As such, a transition set specifies a possibly incomplete course of events that stretches linearly all the way from the past toward a possibly open future. It excludes certain histories and allows others to still occur. Each transition of the set selects one of the local future possibilities open at a branching point and rules out the remainder. The set of histories $\mathsf{H}(T) \subseteq \mathsf{hist}(\mathcal{M})$ admitted by a transition set $T \in \mathsf{dcts}(\mathcal{M})$ then is just the set of histories that are admitted by all transitions of the set, i.e. $\mathsf{H}(T) = \bigcap_{\langle m \rightarrowtail H \rangle \in T} H$.

Being made up from transitions, which are pairs $\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M})$ consisting of a branching point $m \in M$ and a set of histories $H \in \Pi_m$ that are undivided at that branching point, sets of transitions are set-theoretically rather complex. When employed in the semantic evaluation, however, the internal structure of transition sets, their being built up from particular transitions, does not play a role. What matters in the semantic evaluation is only the set of histories $\mathsf{H}(T)$ admitted by a given transition set $T \in \mathsf{dcts}(\mathcal{M})$. If we focus solely on this aspect of transition sets, we can provide perfect correspondents of transition sets in a BT structure that are set-theoretically less complex. Transition sets can be shown to correspond one-to-one to certain substructures of BT structures, which we will call *prunings*.

In chapter 2, we have established a one-to-one correspondence between maximal consistent transition sets and histories. We will now generalize that result to arbitrary transition sets in $\mathsf{dcts}(\mathcal{M})$. Every transition set $T \in \mathsf{dcts}(\mathcal{M})$ corresponds one-to-one to a certain set of histories, viz. the set of histories $\mathsf{H}(T) \subseteq \mathsf{hist}(\mathcal{M})$ admitted by the transition set. Given a transition set, the corresponding set of histories is uniquely determined, and different transition sets allow for different histories. For any two transition sets $T, T' \in \mathsf{dcts}(\mathcal{M})$, we have $T = T'$ if and only if $\mathsf{H}(T) = \mathsf{H}(T')$. The correspondence between transition sets and the corresponding sets of histories admitted by those transition sets is thereby order-reversing, i.e., $T \subsetneq T'$ if and only if $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$. What is more, that correspondence induces a bijection between transition sets and certain substructures of a BT structure, viz. the substructures spanned by the set of histories admitted by a transition set.

Obviously, not every set of histories is identical to the set of histories admitted by some transition set, nor is every possible substructure of a BT structure a substructure that is spanned by one such set. In this section, we provide a general, first-order, characterization of the class of substructures that can in fact be viewed as being spanned by the set of histories $\mathsf{H}(T)$ admitted by some transition set $T \in \mathsf{dcts}(\mathcal{M})$, viz. the so-called prunings. In section 3.2.1, we introduce the definition of a pruning in completely general terms. In section 3.2.2, we then establish the one-to-one correspondence between consistent, downward closed transition sets and prunings.

3.2.1 Prunings

In this section, we provide a general definition of the notion of a pruning, which in section 3.2.2 will be shown to correspond one-to-one to the notion of a consistent, downward closed set of transitions via the set of histories admitted by a transition set.

A pruning of a given BT structure is a certain kind of substructure of that BT structure. We put forth the definition of a pruning in section 3.2.1.2, and we elaborate on the form of a pruning in section 3.2.1.3, which will make clear where the name 'pruning' originates from. Furthermore, we will see that the notion of a pruning induces an ordering among BT structures, and we investigate the properties of the pruning relation in section 3.2.1.4. Before we turn to the definition of a pruning, however, some preliminary remark on our use of the notion of a substructure is in order (section 3.2.1.1).

3.2.1.1 Substructures

To begin with, it is important to note that when we refer to a class of substructures of a given BT structure, we always mean substructures in a strict sense. Given a BT structure $\mathcal{M} = \langle M, < \rangle$, for a structure $\mathcal{M}' = \langle M', <' \rangle$ to be a substructure of $\mathcal{M}$, it is necessary that the set M' be a non-empty subset of the set M in the first instance. Yet, it is not yet sufficient that the relation $<'$ be likewise included in the relation $<$. Rather, the relation $<'$ is required to be the proper restriction $< \cap (M' \times M')$ of the relation $<$ to the subset $M' \subseteq M$. We use $\mathcal{M}' \subseteq \mathcal{M}$ to indicate that $\mathcal{M}'$ is a *substructure* of $\mathcal{M}$. Every BT structure is obviously a substructure of itself. In case $M' \subsetneq M$, we say that $\mathcal{M}'$ is a *proper substructure* of $\mathcal{M}$, and we use $\mathcal{M}' \subsetneq \mathcal{M}$ to stand for ($\mathcal{M}' \subseteq \mathcal{M}$ and $\mathcal{M}' \neq \mathcal{M}$).

Definition 3.1 (Substructure). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, the structure $\mathcal{M}' = \langle M', <' \rangle$ is a* substructure *of $\mathcal{M}$, in symbols: $\mathcal{M}' \subseteq \mathcal{M}$, iff $M' \neq \emptyset$, $M' \subseteq M$ and $<' = < \cap (M' \times M')$.*

A brief note on notation: whenever R is a binary relation on a set X, and X' is a non-empty subset of X, we use $R|_{X'}$ as an abbreviation for $R \cap (X' \times X')$. Following this convention, a substructure of a BT structure $\mathcal{M} = \langle M, < \rangle$ is a structure $\mathcal{M}' = \langle M', <|_{M'} \rangle$ where $M' \subseteq M$.

3.2.1.2 The definition of a pruning

Let us now turn toward the notion of a pruning of a BT structure. The definition of a pruning rests on a certain auxiliary notion. We first introduce the relevant auxiliary notion, and then provide the definition of a pruning. Finally, we show that every pruning of a BT structure is again a BT structure.

The auxiliary notion that figures in the definition of a pruning is the following: given a BT structure $\mathcal{M} = \langle M, < \rangle$ and a moment $m \in M$, we denote the subset of M that contains, next to the moment m, all moments in the past or future of m by M_m; so $M_m := \{m' \in M \mid m' \leq m \text{ or } m' > m\}$.

Definition 3.2 (The set M_m). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure and $m \in M$ a moment, let $M_m := \{m' \in M \mid m' \leq m$ or $m' > m\}$.*

Since the set $M_m \subseteq M$ comprises all moments that are temporally related to the moment m, every history $h \in \mathsf{hist}(\mathcal{M})$ that passes through the moment $m \in M$ forms a subset of M_m. In fact, the set M_m is just the union of all histories that contain the moment m; i.e. $M_m = \bigcup \mathsf{H}_m$.

With that auxiliary notion in place, we can now state the definition of a pruning. A *pruning* of a BT structure $\mathcal{M} = \langle M, < \rangle$ is a substructure $\mathcal{M}' = \langle M', <|_{M'} \rangle$ of $\mathcal{M}$ that (i) shares at least one history with $\mathcal{M}$ and is such that (ii) if the set M' contains two moments m' and m'' that are incomparable by $<$, it contains all past and future moments of their greatest common lower $<$-bound m in M as well, which is to say that it includes the set M_m. An example of two prunings of a given BT structure is provided in Fig. 3.1 below. We use $\mathsf{prun}(\mathcal{M})$ to stand for the set of all prunings of a given BT structure $\mathcal{M}$.

Definition 3.3 (Pruning). *For $\mathcal{M} = \langle M, < \rangle$ a BT structure, a* pruning *of $\mathcal{M}$ is a substructure $\mathcal{M}' = \langle M', <|_{M'} \rangle$ of $\mathcal{M}$ such that*

(i) $\mathsf{hist}(\mathcal{M}) \cap \mathsf{hist}(\mathcal{M}') \neq \emptyset$*;*

(ii) *for all $m, m', m'' \in M'$, if $m' \not\leq m''$, $m'' \not\leq m'$ and m is the greatest common lower $<$-bound of m' and m'' in M, then $M_m \subseteq M'$.*

Given a BT structure $\mathcal{M}$, let $\mathsf{prun}(\mathcal{M})$ be the set of all prunings of $\mathcal{M}$.

Condition (i) of Def. 3.3, as stated, is a second-order condition as it involves implicit quantification over histories, i.e. over maximal linear sets of moments. The condition is triggered by the need to provide correspondents for maximal consistent transition sets. It ensures that every history in the BT structure

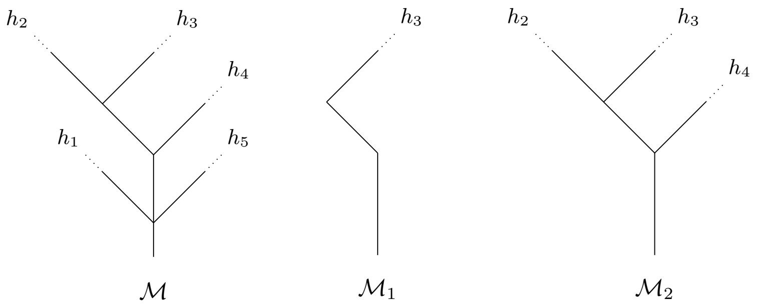


Figure 3.1: Prunings of a given BT structure $\mathcal{M}$.

$\mathcal{M}$, or more precisely, every substructure $\mathcal{M}'$ of $\mathcal{M}$ that is spanned by a single history in $\mathcal{M}$, qualifies as a pruning of $\mathcal{M}$ while none of its proper substructures does.[72] There is no need, however, to formulate the condition as a second-order condition; it is just convenience. Condition (i) of Def. 3.3 can be replaced by the following first-order condition without loss of generality, which is to say that prunings are first-order definable.

(i') M' is a $<$-linear subset of M that is downward closed in M, and there is no $m \in M$ such that for all $m' \in M'$, $m' < m$.

In case we are dealing with a substructure $\mathcal{M}'$ of $\mathcal{M}$ that comprises more than one history, viz. that contains a branching point, the first condition of Def. 3.3 is implied by the second one. Condition (ii) of Def. 3.3 requires that the set $M' \subseteq M$ be both upward and downward closed in M with respect to branching points in the temporal ordering on M'. The condition thereby guarantees that whenever a pruning $\mathcal{M}'$ comprises at least two histories, all of its histories are histories in the BT structure $\mathcal{M}$ as well. Conditions (i) and (ii) of Def. 3.3 then jointly imply that every pruning of the BT structure $\mathcal{M}$ is spanned by a non-empty set of histories in $\mathcal{M}$. No matter whether the substructure $\mathcal{M}'$ of $\mathcal{M}$ comprises but a single or several histories, if $\mathcal{M}'$ is a pruning of $\mathcal{M}$, the

[72]The notion of a pruning bears some resemblance with the definition of a transition put forth in Xu (1997, p.147). There, given a BT structure $\mathcal{M} = \langle M, < \rangle$, a transition is defined as a pair $\langle p, F \rangle$, where F is a non-empty subset of M that is connected and upward closed in M, and p is a non-empty and non-maximal linear set of moments in M which strictly precede any moment in F. It should be noted that in case a BT structure contains a maximal chain of branching points that is not upper bounded in M, Xu's definition does not suffice to single out all histories as transitions in the sense of that definition. For concreteness, consider a BT structure $\mathcal{M} = \langle M, < \rangle$ that contains a history h isomorphic to the natural numbers $\mathbb{N}$ such that for every $m_i \in h$ with $i \in \mathbb{N}$, there is a history $h_i \in \mathsf{hist}(\mathcal{M})$ such that $h \perp_{m_i} h_i$. Here, the history $h \in \mathsf{hist}(\mathcal{M})$ does not qualify as a transition according to Xu's definition.

set $\mathsf{hist}(\mathcal{M}')$ is a subset of $\mathsf{hist}(\mathcal{M})$. Since histories are downward closed, it immediately follows that the set $M' \subseteq M$ is downward closed in M.

Lemma 3.4. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and let $\mathcal{M}' = \langle M', <|_{M'} \rangle$ be a pruning of $\mathcal{M}$. Then the following holds:*

(i) $\mathsf{hist}(\mathcal{M}') \subseteq \mathsf{hist}(\mathcal{M})$*;*

(ii) *for all $m, m' \in M$, if $m' \in M'$ and $m < m'$, then $m \in M'$.*

Proof.

(i) If $\mathcal{M}'$ comprises but a single history, by Def. 3.3 (i) it immediately follows that $\mathsf{hist}(\mathcal{M}') \subseteq \mathsf{hist}(\mathcal{M})$. Now assume that $\mathcal{M}'$ comprises at least two histories, i.e. $\mathsf{hist}(\mathcal{M}') \supseteq \{h', h''\}$, where $h' \neq h''$. We show that $h' \in \mathsf{hist}(\mathcal{M})$. Consider some $m' \in h' \setminus h''$ and some $m'' \in h'' \setminus h'$. Let $m \in M$ be the greatest common lower $<$-bound of m' and m'' in M. By Def. 3.3 (ii) it follows that $M_m \subseteq M'$. Now consider some history $h \in \mathsf{hist}(\mathcal{M})$ s.t. $h' \subseteq h$. Since histories are downward closed, we have $m \in h' \subseteq h$ and hence $h \subseteq M_m$. This implies that $h = h'$ because $M_m \subseteq M' \subseteq M$. Consequently, $h' \in \mathsf{hist}(\mathcal{M})$.

(ii) Let $m \in M$ and $m' \in M'$ s.t. $m < m'$. Consider some history $h \in \mathsf{hist}(\mathcal{M}')$ s.t. $m' \in h$. By (i) it follows that $h \in \mathsf{hist}(\mathcal{M})$. Since histories are downward closed, this implies that $m \in h$ and hence $m \in M'$. □

So, every pruning of a BT structure $\mathcal{M}$ is spanned by a non-empty subset of $\mathsf{hist}(\mathcal{M})$, as captured by Lem. 3.4. Consequently then, every pruning of a BT structure is itself a BT structure. All relevant properties of BT structures are preserved: if $\mathcal{M}' = \langle M', <|_{M'} \rangle$ is a pruning of $\mathcal{M} = \langle M, < \rangle$, the relation $<|_{M'}$ is just the restriction of the left-linear, strict partial order $<$ on M to the non-empty subset M' of M; and by Lem. 3.4, the relation $<|_{M'}$ is serial and jointed, just as the relation $<$ on M is.

Lemma 3.5. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure and let $\mathcal{M}'$ be a pruning of $\mathcal{M}$. Then $\mathcal{M}'$ is again a BT structure.*

Proof. Follows from Lem. 3.4. □

3.2.1.3 The form of a pruning

The name ‘pruning’ of course originates from the fact that prunings arise from a given BT structure by cutting out, viz. pruning, certain branches of that structure. While every history of a pruning is a history in the original BT structure, not all histories of the latter need still be histories in the pruning. Some branches may have dropped off. In order for the structure that results from pruning branches in a given BT structure to qualify as a pruning according to our definition, branches cannot be cut out arbitrarily. In this section, we will take a closer look at the precise form of a pruning.

By Lem. 3.4 it is already evident that in order to obtain a pruning, we cannot simply trim the top ends of the histories of a BT structure. Rather, we can only cut out entire branches, i.e. we have to prune branches directly at the branching points where they emerge as a set of undivided histories. And, by the first condition of the definition of a pruning (Def. 3.3 (i)), we have to spare at least one history.

The second condition of the definition of a pruning (Def. 3.3 (ii)) places even further restrictions on the pruning of branches. First, it requires that if we cut out a branch at the branching point where it emerges, we have to make sure that we also cut out all branches that branch off from the least history that we are going to retain at some earlier moment. Second, condition (ii) of Def. 3.3 demands that if we crop a branch at a branching point at which the relation of undividedness does not only trigger a bifurcation but induces a partition into more than two cells, all but one of the emerging branches at that point need to be cropped. If one of those two pruning requisites is violated, the resulting structure contains a branching point but is not upward closed above that branching in the original BT structure and hence does not qualify as a pruning by Def. 3.3. The two substructures, $\mathcal{M}_3$ and $\mathcal{M}_4$, pictured in Fig. 3.2 fail to be prunings of the given BT structure for precisely those reasons. Each of them violates one of the two pruning requisites that are consequences of Def. 3.3 (ii).

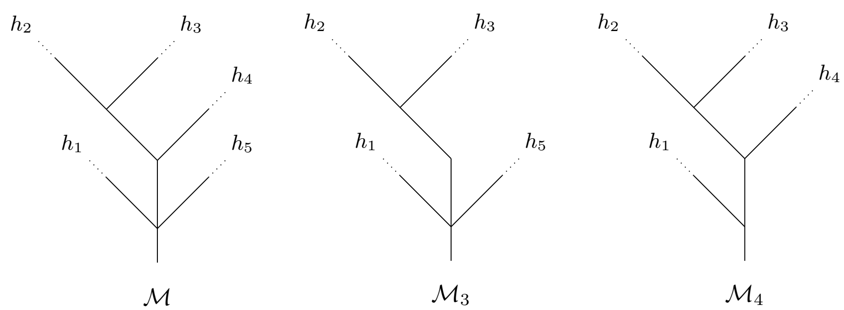


Figure 3.2: Violations of the definition of a pruning.

In order for a structure $\mathcal{M}'$ to be a proper pruning of a given BT structure $\mathcal{M}$, some branch of $\mathcal{M}$ must have dropped off. Yet, given the above pruning requisites, no branch must have dropped off at a moment in the future of a branching point in $\mathcal{M}'$. Nor must a branch have dropped off at a moment at which two histories in $\mathcal{M}'$ branch. Every branching point in $\mathcal{M}'$—if any—has to be strictly later than any moment at which a branch of $\mathcal{M}$ has dropped off. This is to say that all histories in $\mathcal{M}'$ have to be undivided at every moment at which a branch has been cropped. If the pruning comprises but a single history, the requirement is of course trivially fulfilled.

Lemma 3.6. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let $\mathcal{M}' = \langle M', <|_{M'} \rangle$ be a pruning of $\mathcal{M}$ such that $\mathsf{hist}(\mathcal{M}') \subsetneq \mathsf{hist}(\mathcal{M})$. Let $h \in \mathsf{hist}(\mathcal{M}) \setminus \mathsf{hist}(\mathcal{M}')$, $h' \in \mathsf{hist}(\mathcal{M}')$ and $m \in M$ s.t. $h \perp_m h'$. Then $\mathsf{hist}(\mathcal{M}') \subseteq [h']_m$.*

Proof. Let $h \in \mathsf{hist}(\mathcal{M}) \setminus \mathsf{hist}(\mathcal{M}')$, $h' \in \mathsf{hist}(\mathcal{M}')$ and $m \in M$ s.t. $h \perp_m h'$. We can then consider some moment $m' \in h \setminus M'$ s.t. $m < m'$. Assume for reductio that there is some $h'' \in \mathsf{hist}(\mathcal{M}')$ s.t. $h'' \notin [h']_m$. This implies that there is some $m'' \in M$ s.t. $m'' \leq m$ and $h' \perp_{m''} h''$. By Def. 3.3 (ii) we then have $M_{m''} \subseteq M'$. Since $m'' \leq m < m'$, it follows that $m' \in M'$, which contradicts our assumption. □

Even though it is not the case in general that the intersection of all histories in a BT structure is non-empty,[73] from Lem. 3.6 it follows that for every BT structure $\mathcal{M}'$ that is a proper pruning of another BT structure $\mathcal{M}$, it actually holds that $\bigcap \mathsf{hist}(\mathcal{M}') \neq \emptyset$. Given that $\mathcal{M}'$ is a proper pruning of $\mathcal{M}$, at least one branch of $\mathcal{M}$ must have dropped off, and every moment at which a branch has dropped off is contained in all histories in $\mathcal{M}'$ as those histories are undivided at that moment.

Corollary 3.7. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let $\mathcal{M}' = \langle M', <|_{M'} \rangle$ be a pruning of $\mathcal{M}$ such that $\mathcal{M}' \neq \mathcal{M}$. Then $\bigcap \mathsf{hist}(\mathcal{M}') \neq \emptyset$.*

Proof. Given that $\mathcal{M}'$ is a pruning of $\mathcal{M}$ with $\mathcal{M}' \neq \mathcal{M}$, by Lem. 3.4 (i) we have $\mathsf{hist}(\mathcal{M}') \subsetneq \mathsf{hist}(\mathcal{M})$. Let $h \in \mathsf{hist}(\mathcal{M}) \setminus \mathsf{hist}(\mathcal{M}')$ and let $h' \in \mathsf{hist}(\mathcal{M}')$. Then there is some $m \in M$ s.t. $h \perp_m h'$. By Lem. 3.6 it follows that $\mathsf{hist}(\mathcal{M}') \subseteq [h']_m$, which implies that $m \in \bigcap \mathsf{hist}(\mathcal{M}')$ and hence $\bigcap \mathsf{hist}(\mathcal{M}') \neq \emptyset$. □

In case $\mathcal{M}' = \langle M', <|_{M'} \rangle$ is a proper pruning of a given BT structure $\mathcal{M} = \langle M, < \rangle$, the intersection $\bigcap \mathsf{hist}(\mathcal{M}')$ of all histories in $\mathcal{M}'$ is thus a non-empty,

[73]Consider, for example, a BT structure $\mathcal{M}$ that contains a history h isomorphic to the integers $\mathbb{Z}$ such that at every moment $m_i \in h$ with $i \in \mathbb{Z}$, there is a history $h_i \in \mathsf{hist}(\mathcal{M})$ such that $h \perp_{m_i} h_i$. In this case, we have $\bigcap \mathsf{hist}(\mathcal{M}) = \emptyset$.

$<$-linear subset of M, and, obviously, it is downward closed in M as well. We call the non-empty, $<$-linear set $\bigcap \mathsf{hist}(\mathcal{M}')$ the *trunk* of the proper pruning $\mathcal{M}'$. All branches of $\mathcal{M}$ that have dropped off, must have dropped off from the trunk $\bigcap \mathsf{hist}(\mathcal{M}')$ at some non-final point. If the trunk $\bigcap \mathsf{hist}(\mathcal{M}')$ contains a $<$-maximal element, that greatest element is a branching point in $\mathcal{M}'$, and hence, by Lem. 3.6, no branch can have dropped off at that moment.[74]

3.2.1.4 The pruning relation

The notion of a pruning is not an absolute notion but is essentially relational in nature. A pruning is always a pruning of some given BT structure. In this section, we investigate the properties of the relation of being a pruning. We show that the pruning relation provides a partial ordering among BT structures, which when restricted to the set of prunings of a given BT structure, collapses into the substructure relation.

As noted in Lem. 3.5, every pruning of a BT structure is itself a BT structure. We show that the notion of a pruning induces an order relation among BT structures. By the definition of a pruning (Def. 3.3) it is immediately evident that the pruning relation is reflexive. Obviously, every BT structure is a pruning of itself. It is also straightforward by definition that the pruning relation is antisymmetric: since being a pruning involves being a substructure, no BT structure can be a pruning of one of its proper prunings. We prove that the pruning relation is transitive as well.

Lemma 3.8. *Let $\mathcal{M} = \langle M, < \rangle$, $\mathcal{M}' = \langle M', <' \rangle$ and $\mathcal{M}'' = \langle M'', <'' \rangle$ be BT structures. If $\mathcal{M}''$ is a pruning of $\mathcal{M}'$ and $\mathcal{M}'$ is a pruning of $\mathcal{M}$, then $\mathcal{M}''$ is a pruning of $\mathcal{M}$.*

Proof. Given that $\mathcal{M}''$ is a pruning of $\mathcal{M}'$, by Def. 3.3 (i), there is some history $h \in \mathsf{hist}(\mathcal{M}') \cap \mathsf{hist}(\mathcal{M}'')$. Under the assumption that $\mathcal{M}'$ is again a pruning of $\mathcal{M}$, it follows by Lem. 3.4 (i) that $h \in \mathsf{hist}(\mathcal{M})$ and hence we have that $\mathsf{hist}(\mathcal{M}) \cap \mathsf{hist}(\mathcal{M}'') \neq \emptyset$. This shows that $\mathcal{M}$ and $\mathcal{M}''$ satisfy condition (i) of Def. 3.3.

[74] There are proper prunings $\mathcal{M}'$ whose trunk does not contain a $<$-maximum, so that consequently all histories in $\mathcal{M}'$ are undivided at every moment in $\bigcap \mathsf{hist}(\mathcal{M}')$. For concreteness, consider a BT structure $\mathcal{M} = \langle M, < \rangle$ that contains a history h isomorphic to the union of two disjoint but adjacent copies $\mathbb{Z}_1$ and $\mathbb{Z}_2$ of the integers such that for all $j \in \mathbb{Z}_1$ and $k \in \mathbb{Z}_2$, $m_j < m_k$. Assume that for every $m_i \in h$ with $i \in \mathbb{Z}_1 \cup \mathbb{Z}_2$, there is some history $h_i \in \mathsf{hist}(\mathcal{M})$ such that $h \perp_{m_i} h_i$ and $\mathsf{hist}(\mathcal{M}) = \{h\} \cup \{h_i \mid i \in \mathbb{Z}_1 \cup \mathbb{Z}_2\}$. Let $X = \{h\} \cup \{h_i \mid i \in \mathbb{Z}_2\}$. The structure $\mathcal{M}' = \langle \bigcup X, <|_{\bigcup X} \rangle$ is a proper pruning of $\mathcal{M}$ with trunk $\bigcap \mathsf{hist}(\mathcal{M}') = \bigcap X = \mathbb{Z}_1$.

Now assume that there are $m', m'' \in M'' \subseteq M' \subseteq M$ s.t. $m' \not\leq m''$ and $m'' \not\leq m'$ and let m be the greatest common lower $<$-bound of m' and m'' in M. Since $\mathcal{M}'$ is a pruning of $\mathcal{M}$, by Def. 3.3 (ii) it follows that $M_m \subseteq M'$, which in turn implies that $M_m \subseteq M'_m$. Moreover, since $\mathcal{M}''$ is again a pruning of $\mathcal{M}'$, Def. 3.3 (ii) yields $M'_m \subseteq M''$. Hence, we have $M_m \subseteq M''$, which shows that condition (ii) of Def. 3.3 is fulfilled by $\mathcal{M}$ and $\mathcal{M}''$ as well. □

Being reflexive, antisymmetric and transitive, the pruning relation establishes a partial order among BT structures. Obviously, the order induced by the pruning relation among BT structures is a subset of the substructure relation. We show that on the set of prunings of a given BT structure, the pruning relation actually is just the substructure relation. That is, for a pruning $\mathcal{M}''$ of a given BT structure $\mathcal{M}$ to be a pruning of another pruning $\mathcal{M}'$ of that structure $\mathcal{M}$, it is not only necessary but also sufficient that $\mathcal{M}''$ be a substructure of $\mathcal{M}'$.

Lemma 3.9. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let $\mathcal{M}' = \langle M', <|_{M'} \rangle$ and $\mathcal{M}'' = \langle M'', <|_{M''} \rangle$ be prunings of $\mathcal{M}$. If $\mathcal{M}'' \subseteq \mathcal{M}'$, then $\mathcal{M}''$ is a pruning of $\mathcal{M}'$.*

Proof. Assume that $\mathcal{M}', \mathcal{M}'' \in \mathsf{prun}(\mathcal{M})$ with $\mathcal{M}'' \subseteq \mathcal{M}'$. Let $h \in \mathsf{hist}(\mathcal{M}'')$. Then by Lem. 3.4 (i) we have $h \in \mathsf{hist}(\mathcal{M})$. Since $M'' \subseteq M' \subseteq M$, it follows that $h \in \mathsf{hist}(\mathcal{M}')$. Hence $\mathsf{hist}(\mathcal{M}') \cap \mathsf{hist}(\mathcal{M}'') \neq \emptyset$, which shows that $\mathcal{M}'$ and $\mathcal{M}''$ fulfill condition (i) of Def. 3.3.
Now assume that there are $m', m'' \in M''$ s.t. $m' \not\leq m''$ and $m'' \not\leq m'$. Let m be the greatest common lower $<$-bound of m' and m'' in M. By Def. 3.3 (ii) it follows that $M_m \subseteq M''$. Since $M' \subseteq M$, we then also have $M'_m \subseteq M''$, which shows that $\mathcal{M}'$ and $\mathcal{M}''$ satisfy condition (ii) of Def. 3.3. □

Lem. 3.9 implies that if two prunings of a given BT structure are themselves not pruning related either way, neither of them can be a substructure of the other. From this it follows that the restriction of the pruning relation to the set of prunings of a given BT structure $\mathcal{M}$ is right-linear: two prunings of $\mathcal{M}$ cannot have a pruning in common unless they are themselves pruning related one way or another. For, two prunings of $\mathcal{M}$ neither of which is embeddable in the other cannot even share a single history.

Lemma 3.10. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure, and let $\mathcal{M}' = \langle M', <|_{M'} \rangle$, $\mathcal{M}'' = \langle M'', <|_{M''} \rangle$ and $\mathcal{M}''' = \langle M''', <|_{M'''} \rangle$ be prunings of $\mathcal{M}$. If $\mathcal{M}'''$ is a pruning of both $\mathcal{M}'$ and $\mathcal{M}''$, then either $\mathcal{M}'$ is a pruning of $\mathcal{M}''$ or $\mathcal{M}''$ is a pruning of $\mathcal{M}'$.*

Proof. Let $\mathcal{M}', \mathcal{M}'', \mathcal{M}''' \in \mathsf{prun}(\mathcal{M})$ s.t. $\mathcal{M}''' \in \mathsf{prun}(\mathcal{M}') \cap \mathsf{prun}(\mathcal{M}'')$. Assume for reductio that neither $\mathcal{M}'$ is a pruning of $\mathcal{M}''$ nor $\mathcal{M}''$ is a pruning of $\mathcal{M}'$. By Lem. 3.9 this implies that $\mathcal{M}' \not\subseteq \mathcal{M}''$ and $\mathcal{M}'' \not\subseteq \mathcal{M}'$. It follows that there is some history $h' \in \mathsf{hist}(\mathcal{M}') \setminus \mathsf{hist}(\mathcal{M}'')$ and some history $h'' \in \mathsf{hist}(\mathcal{M}'') \setminus \mathsf{hist}(\mathcal{M}')$. We can then consider some moment $m' \in h' \setminus M''$ and some moment $m'' \in h'' \setminus M'$. Let $h''' \in \mathsf{hist}(\mathcal{M}''')$. Then there is some $m_1 \in M$ s.t. $m_1 < m'$ and $h' \perp_{m_1} h'''$ and there is some $m_2 \in M$ s.t. $m_2 < m''$ and $h'' \perp_{m_2} h'''$. It follows that either $m_1 \leq m_2$ or $m_2 \leq m_1$. If $m_1 \leq m_2$, then $m'' \in M'$ because by Def. 3.3 it holds that $M_{m_1} \subseteq M'$ and we have $m_1 \leq m_2 < m''$. This contradicts our assumption. Likewise, if $m_2 \leq m_1$, then, contrary to our assumption, $m' \in M''$ because by Def. 3.3 it holds that $M_{m_2} \subseteq M''$ and we have $m_2 \leq m_1 < m'$. □

3.2.2 The correspondence between transition sets and prunings

In the previous section, we have introduced the notion of a pruning. A pruning $\mathcal{M}'$ of a given BT structure $\mathcal{M}$ is a substructure of $\mathcal{M}$ that is spanned by a non-empty set of histories in $\mathcal{M}$ and is upward closed above branching points in $\mathcal{M}'$. In this section, we show that, given a BT structure $\mathcal{M}$, there is a natural one-to-one correspondence between the various transition sets in $\mathsf{dcts}(\mathcal{M})$ and the various prunings in $\mathsf{prun}(\mathcal{M})$. In section 3.2.2.1, we show that to every transition set $T \in \mathsf{dcts}(\mathcal{M})$, there uniquely corresponds a pruning in $\mathsf{prun}(\mathcal{M})$, and we establish the converse correspondence in section 3.2.2.2. In section 3.2.2.3, we finally prove that the respective mappings are in fact inverses of each other and thus induce a bijection between transition sets and prunings.

3.2.2.1 From transition sets to prunings

In this section, we make a transition from transition sets to prunings. More precisely, we show that, given a BT structure $\mathcal{M}$, every substructure $\mathcal{M}'$ of $\mathcal{M}$ that is spanned by the set $\mathsf{H}(T)$ of histories admitted by some transition set $T \in \mathsf{dcts}(\mathcal{M})$ qualifies as a pruning of $\mathcal{M}$ according to our definition. We moreover illustrate that the correspondence between transition sets and prunings is injective and reverses the inclusion relation among transition sets.

Given a BT structure $\mathcal{M} = \langle M, < \rangle$, every transition set $T \in \mathsf{dcts}(\mathcal{M})$ determines a unique set of histories, viz. the set $\mathsf{H}(T)$ of histories admitted by the transition set T, and each set of histories in $\mathcal{M}$ gives rise to a substructure of $\mathcal{M}$. Starting out with a given transition set $T \in \mathsf{dcts}(\mathcal{M})$, we show that the structure $\mathcal{M}' = \langle \bigcup \mathsf{H}(T), < |_{\bigcup \mathsf{H}(T)} \rangle$ that is made up from all and only those moments that are contained in at least one history in the corresponding set $\mathsf{H}(T)$ is a pruning of the BT structure $\mathcal{M}$. To this end, we first of all show that by taking the union of the set $\mathsf{H}(T)$ of histories admitted by the given transition set T, no new histories emerge, so that $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(T)$.

Proposition 3.11. *Let* $\mathcal{M} = \langle M, < \rangle$ *be a BT structure, and let* $T \in \mathsf{dcts}(\mathcal{M})$ *be a set of transitions. Then the substructure* $\mathcal{M}' = \langle \bigcup \mathsf{H}(T), < |_{\bigcup \mathsf{H}(T)} \rangle$ *of* $\mathcal{M}$ *is a pruning of* $\mathcal{M}$ *and* $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(T)$.

Proof. Let $\mathcal{M}' = \langle \bigcup \mathsf{H}(T), < |_{\bigcup \mathsf{H}(T)} \rangle$. We first show that $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(T)$. Obviously, $\mathsf{H}(T) \subseteq \mathsf{hist}(\mathcal{M}')$. We prove that $\mathsf{hist}(\mathcal{M}') \subseteq \mathsf{H}(T)$ holds as well. Assume for reductio that there is some history $h \in \mathsf{hist}(\mathcal{M}') \setminus \mathsf{H}(T)$. This implies that there is some transition $\langle m \rightarrowtail H \rangle \in T$ such that $h \notin H$. It follows that there is some moment $m' \in h$ s.t. $m' \notin \bigcup H$. Since $\mathsf{H}(T) \subseteq H$, we then have that $m' \notin \bigcup \mathsf{H}(T)$, which contradicts our assumption that $h \in \mathsf{hist}(\mathcal{M}')$. We now show that the substructure $\mathcal{M}'$ is a pruning of $\mathcal{M}$. Since $T \in \mathsf{dcts}(\mathcal{M})$ is consistent, i.e. $\mathsf{H}(T) \neq \emptyset$, and $\mathsf{H}(T) \subseteq \mathsf{hist}(\mathcal{M}')$, it immediately follows that condition (i) of Def. 3.3 is fulfilled. We prove that condition (ii) of Def. 3.3 is fulfilled as well. Assume that there are $m', m'' \in \bigcup \mathsf{H}(T)$ s.t. $m' \not\leq m''$ and $m'' \not\leq m'$. Consider two histories $h', h'' \in \mathsf{hist}(\mathcal{M}')$ s.t. $m' \in h' \setminus h''$ and $m'' \in h'' \setminus h'$. Then there is some moment $m \in M$ s.t. $h' \perp_m h''$. Since $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(T) \supseteq \{h', h''\}$, for every transition $\langle m^* \rightarrowtail H^* \rangle \in T$ it must hold that $m^* < m$. This implies that $\mathsf{H}_m \subseteq \mathsf{H}(T)$. Consequently, $M_m \subseteq \bigcup \mathsf{H}(T)$, which shows that condition (ii) of Def. 3.3 is satisfied. □

Prop. 3.11 establishes a correspondence between transition sets, on the one hand, and prunings, on the other, in a given BT structure $\mathcal{M}$: every transition set $T \in \mathsf{dcts}(\mathcal{M})$ is associated with the structure that is spanned by the set $\mathsf{H}(T)$ of admitted histories. We can capture the correspondence given in Prop. 3.11 by a function $\chi : \mathsf{dcts}(\mathcal{M}) \to \mathsf{prun}(\mathcal{M})$ that maps every transition set $T \in \mathsf{dcts}(\mathcal{M})$ onto the structure $\langle \bigcup \mathsf{H}(T), < |_{\bigcup \mathsf{H}(T)} \rangle \in \mathsf{prun}(\mathcal{M})$. An example of a transition set and the corresponding pruning of the respective BT structure is provided in Fig. 3.3.

Since our considerations are restricted exclusively to consistent, downward closed transition sets, the correspondence between transition sets and prunings induced by the function χ is injective and order-reversing. This is due to the fact that, by Lem. 2.23, for any two transition sets $T', T'' \in \mathsf{dcts}(\mathcal{M})$ such that $T' \neq T''$, we have $\mathsf{H}(T') \neq \mathsf{H}(T'')$; and, in particular, it holds that $T' \subsetneq T''$ if and only if $\mathsf{H}(T'') \subseteq \mathsf{H}(T')$. Consequently, the function χ associates different transition sets with different prunings, and whenever T'' is a proper

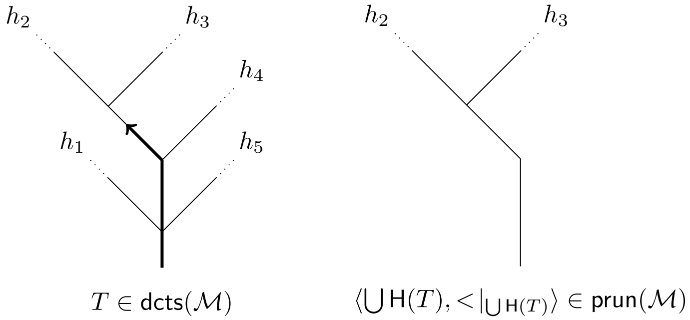


Figure 3.3: Correspondence between transition sets and prunings.

extension of T', the structure $\langle \bigcup \mathsf{H}(T''), <|_{\bigcup \mathsf{H}(T'')} \rangle$ is a proper substructure of $\langle \bigcup \mathsf{H}(T'), <|_{\bigcup \mathsf{H}(T')} \rangle$, and *vice versa*.

Lemma 3.12. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure. The function*

$$\begin{array}{rccl} \chi : & \mathsf{dcts}(\mathcal{M}) & \to & \mathsf{prun}(\mathcal{M}) \\ & T & \mapsto & \langle \bigcup \mathsf{H}(T), <|_{\bigcup \mathsf{H}(T)} \rangle \end{array}$$

is injective. In particular, for all $T', T'' \in \mathsf{dcts}(\mathcal{M})$, we have $T' \subsetneq T''$ iff $\chi(T'') \subsetneq \chi(T')$.

Proof. We show that χ is injective. Let $T', T'' \in \mathsf{dcts}(\mathcal{M})$ s.t. $T' \neq T''$. By Lem. 2.23 it follows that $\mathsf{H}(T') \neq \mathsf{H}(T'')$, which implies that $\bigcup \mathsf{H}(T') \neq \bigcup \mathsf{H}(T'')$ and hence $\chi(T') \neq \chi(T'')$.
The claim that $T' \subsetneq T''$ iff $\chi(T'') \subsetneq \chi(T')$ can be proven along the same lines by making use of the equivalence $T' \subsetneq T''$ iff $\mathsf{H}(T'') \subsetneq \mathsf{H}(T')$ established in Lem. 2.23. □

Recall that by Lem. 3.9 the restriction of the pruning relation to the set $\mathsf{prun}(\mathcal{M})$ coincides with the substructure relation on that set. From Lem. 3.12 it then follows that the inclusion relation between the various transition sets in $\mathsf{dcts}(\mathcal{M})$ is just the converse of the pruning relation between the corresponding χ-images in $\mathsf{prun}(\mathcal{M})$: for all $T', T'' \in \mathsf{dcts}(\mathcal{M})$, we have $T' \subsetneq T''$ if and only if $\langle \bigcup \mathsf{H}(T''), <|_{\bigcup \mathsf{H}(T'')} \rangle$ is proper pruning of $\langle \bigcup \mathsf{H}(T'), <|_{\bigcup \mathsf{H}(T')} \rangle$. Obviously, in case $T' = T''$, the structure $\langle \bigcup \mathsf{H}(T'), <|_{\bigcup \mathsf{H}(T')} \rangle$ is a pruning of itself.

The correspondence χ between transition sets and prunings in a given BT structure $\mathcal{M}$ embraces two limiting cases. First, the pruning corresponding to the empty transition set $\emptyset_{\mathsf{Tr}} \in \mathsf{dcts}(\mathcal{M})$ is the BT structure $\mathcal{M} \in \mathsf{prun}(\mathcal{M})$

itself. While the empty transition set $\emptyset_{\mathsf{Tr}}$ is the unique $\subseteq$-minimal element in $\mathsf{dcts}(\mathcal{M})$, the BT structure $\mathcal{M}$ is the unique $\subseteq$-maximal element in $\mathsf{prun}(\mathcal{M})$. Second, to every maximal consistent transition set in $\mathsf{dcts}(\mathcal{M})$, there corresponds a pruning that is spanned by a single history, viz. the history admitted by the maximal consistent transition set. Since the correspondence χ is order-reversing, maximal consistent transition sets, which constitute $\subseteq$-maximal elements in $\mathsf{dcts}(\mathcal{M})$, are mapped onto $\subseteq$-minimal elements in $\mathsf{prun}(\mathcal{M})$.

3.2.2.2 From prunings to transition sets

Having shown that to every transition set in a given BT structure $\mathcal{M}$, there corresponds a pruning of that structure, we now deal with the converse transition, viz. the transition from prunings to transition sets. We show that every pruning of a BT structure $\mathcal{M}$ can be viewed as being spanned by the set $\mathsf{H}(T)$ of histories admitted by some transition set $T \in \mathsf{dcts}(\mathcal{M})$. The correspondence between prunings and transition sets that we establish here will again be shown to be injective and order-reversing.

We have seen that every pruning $\mathcal{M}'$ of a BT structure $\mathcal{M}$ is spanned by a set of histories in $\mathcal{M}$, all of which are undivided at every moment at which a branch of $\mathcal{M}$ has dropped off. And at each such moment at which a branch of $\mathcal{M}$ has dropped off, there is an indeterministic transition $\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M})$ whose initial m is an element of the trunk $\bigcap \mathsf{hist}(\mathcal{M}')$ of $\mathcal{M}'$ and whose outcome H includes the set $\mathsf{hist}(\mathcal{M}')$ of all histories in $\mathcal{M}'$. We show that the set $T = \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\}$ of all those transitions is consistent and downward closed and is thus contained in $\mathsf{dcts}(\mathcal{M})$. What is more, we show that the transition set T admits precisely those histories of $\mathcal{M}$ that are at the same time histories in $\mathcal{M}'$, i.e. $\mathsf{H}(T) = \mathsf{hist}(\mathcal{M}')$. Note that for all $\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M})$, the condition $\mathsf{hist}(\mathcal{M}') \subseteq H$ obviously implies $m \in \bigcap \mathsf{hist}(\mathcal{M}')$, whereas the converse implication does not hold.

Proposition 3.13. *Let $\mathcal{M} = \langle M, <\rangle$ be a BT structure and $\mathcal{M}' = \langle M', <|_{M'}\rangle$ a pruning of $\mathcal{M}$. Then $T = \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\}$ is a set of transitions in $\mathsf{dcts}(\mathcal{M})$ and $\mathsf{H}(T) = \mathsf{hist}(\mathcal{M}')$.*

Proof. Let $T = \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\}$. Obviously, we have $\mathsf{hist}(\mathcal{M}') \subseteq \mathsf{H}(T)$, from which it follows that T is consistent. It is also straightforward that T is downward closed in $\mathsf{trans}(\mathcal{M})$ via the transition ordering $\prec$. For, an indeterministic transition $\langle m' \rightarrowtail H'\rangle$ precedes another $\langle m'' \rightarrowtail H''\rangle$ in the $\prec$-ordering on $\mathsf{trans}(\mathcal{M})$ if and only if $H'' \subsetneq H'$. Hence, we have $T \in \mathsf{dcts}(\mathcal{M})$.

We prove that $\mathsf{H}(T) \subseteq \mathsf{hist}(\mathcal{M}')$. Two cases can be considered. Case (i): If $\mathcal{M}' = \mathcal{M}$, then $\mathsf{hist}(\mathcal{M}') = \mathsf{hist}(\mathcal{M})$. Since $\mathsf{trans}(\mathcal{M})$ comprises indeterministic transitions only, there is no transition $\langle m' \rightarrowtail H' \rangle \in \mathsf{trans}(\mathcal{M})$ s.t. $\mathsf{hist}(\mathcal{M}) \subseteq H'$ and hence $T = \emptyset$. Then $\mathsf{H}(T) = \mathsf{hist}(\mathcal{M}')$ because $\mathsf{H}(\emptyset_{\mathsf{Tr}}) = \mathsf{hist}(\mathcal{M})$. Case (ii): Let $\mathcal{M}' \neq \mathcal{M}$. Assume for reductio that there is some history $h \in \mathsf{H}(T) \setminus \mathsf{hist}(\mathcal{M}')$. Let $h' \in \mathsf{hist}(\mathcal{M}')$. Then there is some moment $m \in M$ s.t. $h \perp_m h'$. By Lem. 3.6 it follows that $\mathsf{hist}(\mathcal{M}') \subseteq [h']_m$. This implies that $\langle m \rightarrowtail [h']_m \rangle \in T$. Since $h \notin [h']_m$, we have $h \notin \mathsf{H}(T)$, which contradicts our assumption. □

Prop. 3.13 provides a correspondence between prunings and transition sets in a given BT structure $\mathcal{M}$: Every pruning $\mathcal{M}'$ of $\mathcal{M}$ is associated with the transition set $T \in \mathsf{dcts}(\mathcal{M})$ that contains all and only those transitions whose outcome includes the set $\mathsf{hist}(\mathcal{M}')$. The correspondence established in Prop. 3.13 can be captured by a function $\xi : \mathsf{prun}(\mathcal{M}) \to \mathsf{dcts}(\mathcal{M})$ that maps every pruning $\mathcal{M}'$ of $\mathcal{M}$ onto the transition set $\{\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\}$. In Fig. 3.4, an example of a pruning of a BT structure and the transition set associated with that pruning is given.

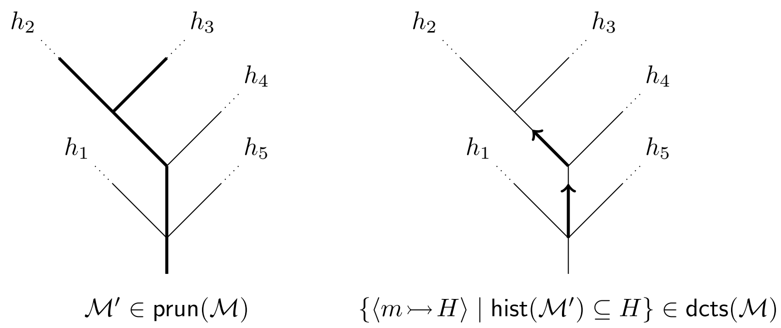


Figure 3.4: Correspondence between prunings and transition sets.

Due to the fact that by Prop. 3.13, for all prunings $\mathcal{M}'$ of a given BT structure $\mathcal{M}$, we have $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(\xi(\mathcal{M}'))$, the correspondence ξ is injective and order-reversing. Different prunings are associated with different transition sets, and whenever a pruning $\mathcal{M}'$ of the BT structure $\mathcal{M}$ is a proper substructure of another pruning $\mathcal{M}''$ of that structure, then the transition set $\xi(\mathcal{M}'')$ is a proper subset of the transition set $\xi(\mathcal{M}')$. As in the case of the correspondence χ (cf. Lem. 3.12), the proof again crucially rests on Lem. 2.23 according to which for any two transition sets $T', T'' \in \mathsf{dcts}(\mathcal{M})$ it holds that $T' = T''$ if and only if $\mathsf{H}(T') = \mathsf{H}(T'')$; and, in particular, we have that $T' \subsetneq T''$ if and only if $\mathsf{H}(T'') \subseteq \mathsf{H}(T')$.

Lemma 3.14. *Let $\mathcal{M} = \langle M, < \rangle$ be a BT structure. The function*

$$\begin{array}{rrcl} \xi: & \mathsf{prun}(\mathcal{M}) & \to & \mathsf{dcts}(\mathcal{M}) \\ & \mathcal{M}' & \mapsto & \{\langle m \rightarrowtail H \rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\} \end{array}$$

is injective. In particular, for all $\mathcal{M}', \mathcal{M}'' \in \mathsf{prun}(\mathcal{M})$, we have $\mathcal{M}' \subsetneq \mathcal{M}''$ iff $\xi(\mathcal{M}'') \subsetneq \xi(\mathcal{M}')$.

Proof. We show that ξ is injective. Let $\mathcal{M}', \mathcal{M}'' \in \mathsf{prun}(\mathcal{M})$ s.t. $\mathcal{M}' \neq \mathcal{M}''$. Then $M' \neq M''$ and hence $\mathsf{hist}(\mathcal{M}') \neq \mathsf{hist}(\mathcal{M}'')$. Since by Prop. 3.13 we have $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(\xi(\mathcal{M}'))$ and $\mathsf{hist}(\mathcal{M}'') = \mathsf{H}(\xi(\mathcal{M}''))$, this implies that $\mathsf{H}(\xi(\mathcal{M}')) \neq \mathsf{H}(\xi(\mathcal{M}''))$, from which it follows that $\xi(\mathcal{M}') \neq \xi(\mathcal{M}'')$.
The claim that $\mathcal{M}' \subsetneq \mathcal{M}''$ iff $\xi(\mathcal{M}'') \subsetneq \xi(\mathcal{M}')$ can be proven along the same lines by making use of the equivalences $\mathcal{M}' \subsetneq \mathcal{M}''$ iff $\mathsf{hist}(\mathcal{M}') \subsetneq \mathsf{hist}(\mathcal{M}'')$ and $\xi(\mathcal{M}') \subsetneq \xi(\mathcal{M}'')$ iff $\mathsf{H}(\xi(\mathcal{M}'')) \subsetneq \mathsf{H}(\xi(\mathcal{M}'))$. □

Since on the set $\mathsf{prun}(\mathcal{M})$ of all prunings of a given BT structure, the pruning relation just is the substructure relation, from Lem. 3.14 it follows that the inclusion relation on the ξ-image of $\mathsf{prun}(\mathcal{M})$ is the converse of the pruning relation on the set $\mathsf{prun}(\mathcal{M})$. For all $\mathcal{M}', \mathcal{M}'' \in \mathsf{prun}(\mathcal{M})$, we have that $\mathcal{M}'$ is a proper pruning of $\mathcal{M}''$ if and only if the transition set $\xi(\mathcal{M}'')$ is properly included in the transition set $\xi(\mathcal{M}')$.

The correspondence ξ between $\mathsf{dcts}(\mathcal{M})$ and $\mathsf{prun}(\mathcal{M})$ embraces again two limiting cases that parallel the ones that we have discussed in the case of the correspondence χ. First, since all transition sets in the $\mathsf{dcts}(\mathcal{M})$ are exclusively built up from indeterministic transitions, the transition set corresponding to the BT structure $\mathcal{M}$ itself is the empty transition set $\emptyset_{\mathsf{Tr}} \in \mathsf{dcts}(\mathcal{M})$. That is, the unique $\subseteq$-maximal in $\mathcal{M} \in \mathsf{prun}(\mathcal{M})$ is mapped upon the unique $\subseteq$-minimal element $\emptyset_{\mathsf{Tr}} \in \mathsf{dcts}(\mathcal{M})$. Second, every pruning $\mathcal{M}'$ of $\mathcal{M}$ that is spanned by a single history h in $\mathcal{M}$ is associated with a maximal consistent transition set in $\mathsf{dcts}(\mathcal{M})$, viz. the set $\mathsf{Tr}(h)$ of transitions characterizing the history h. To every $\subseteq$-minimal element in $\mathsf{prun}(\mathcal{M})$, there corresponds thus a $\subseteq$-maximal element in $\mathsf{dcts}(\mathcal{M})$.

3.2.2.3 A one-to-one correspondence

In section 3.2.2.1, we have established a correspondence $\chi : \mathsf{dcts}(\mathcal{M}) \to \mathsf{prun}(\mathcal{M})$ with $\chi(T) = \langle \bigcup \mathsf{H}(T), < |_{\bigcup \mathsf{H}(T)} \rangle$ and $\mathsf{H}(T) = \chi(T)$ between the various transition sets and prunings in a given BT structure $\mathcal{M}$. And in section 3.2.2.2 we have provided a correspondence $\xi : \mathsf{prun}(\mathcal{M}) \to \mathsf{dcts}(\mathcal{M})$ with $\xi(\mathcal{M}') = \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\}$ and $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(\xi(\mathcal{M}'))$. Both correspondences, χ and ξ, have been show to be injective and order-reversing. We now finally prove that the functions χ and ξ are inverses of each other, which concludes our proof that there is a one-to-one correspondence between transition sets and prunings.[75]

Proposition 3.15. *Let $\mathcal{M} = \langle M, <\rangle$ be a BT structure. The functions*

$$\begin{array}{rrcl} \chi : & \mathsf{dcts}(\mathcal{M}) & \to & \mathsf{prun}(\mathcal{M}) \\ & T & \mapsto & \langle \bigcup \mathsf{H}(T), <\!|_{\bigcup \mathsf{H}(T)}\rangle \end{array}$$

and

$$\begin{array}{rrcl} \xi : & \mathsf{prun}(\mathcal{M}) & \to & \mathsf{dcts}(\mathcal{M}) \\ & \mathcal{M}' & \mapsto & \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{hist}(\mathcal{M}') \subseteq H\} \end{array}$$

are bijections with χ and ξ being inverses of each other.

Proof. By Lem. 3.12 and Lem. 3.14 the functions χ and ξ are injective. We first prove that $\xi(\chi(T)) = T$ for all $T \in \mathsf{dcts}(\mathcal{M})$. Let $T \in \mathsf{dcts}(\mathcal{M})$ and assume that $\chi(T) = \mathcal{M}'$. Then by Prop. 3.11 we have that $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(T)$ and hence $\xi(\chi(T)) = \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{H}(T) \subseteq H\}$. We thus have to show that $\{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{H}(T) \subseteq H\} = T$.
Obviously, for every transition $\langle m' \rightarrowtail H'\rangle \in T$ it holds that $\mathsf{H}(T) \subseteq H'$ and hence we have that $T \subseteq \{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{H}(T) \subseteq H\}$. We prove that $\{\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M}) \mid \mathsf{H}(T) \subseteq H\} \subseteq T$ holds as well. Let

[75] There is a close link to Galois connections. For $\mathcal{M} = \langle M, <\rangle$ a BT structure, let $\mathsf{ts}(\mathcal{M})$ be the set of all consistent sets of indeterministic transitions in $\mathcal{M}$, i.e. $\mathsf{ts}(\mathcal{M}) = \{T \subseteq \mathsf{trans}(\mathcal{M}) \mid \mathsf{H}(T) \neq \emptyset\}$, and let $\mathsf{hs}(\mathcal{M})$ be the set of all non-empty sets of histories in $\mathcal{M}$, i.e. $\mathsf{hs}(\mathcal{M}) = \{H \subseteq \mathsf{hist}(\mathcal{M}) \mid H \neq \emptyset\}$. Both $\mathsf{ts}(\mathcal{M})$ and $\mathsf{hs}(\mathcal{M})$ are partially ordered sets; and the order-reversing functions

$$\begin{array}{rrcl} \mathsf{H} : & \mathsf{ts}(\mathcal{M}) & \to & \mathsf{hs}(\mathcal{M}) \\ & T & \mapsto & \mathsf{H}(T) \end{array}$$

and

$$\begin{array}{rrcl} \mathsf{Tr} : & \mathsf{hs}(\mathcal{M}) & \to & \mathsf{ts}(\mathcal{M}) \\ & H & \mapsto & \{\langle m \rightarrowtail H'\rangle \in \mathsf{trans}(\mathcal{M}) \mid H \subseteq H'\} \end{array}$$

form an antitone Galois connection between those partial orders: $H \subseteq \mathsf{H}(T)$ if and only if $T \subseteq \mathsf{Tr}(H)$. The set of downward closed transition sets $\mathsf{dcts}(\mathcal{M}) \subseteq \mathsf{ts}(\mathcal{M})$ equals the set of closed elements under the associated closure operator $\mathsf{Tr} \circ \mathsf{H}$, i.e. $\mathsf{dcts}(\mathcal{M}) = \{T \in \mathsf{ts}(\mathcal{M}) \mid \mathsf{Tr}(\mathsf{H}(T)) = T\}$. Let $\mathsf{cl}(\mathsf{hs}(\mathcal{M}))$ be the set of closed elements under the associated closure operator $\mathsf{H} \circ \mathsf{Tr}$, i.e. $\mathsf{cl}(\mathsf{hs}(\mathcal{M})) = \{H \in \mathsf{hs}(\mathcal{M}) \mid \mathsf{H}(\mathsf{Tr}(H)) = H\}$. Both the restriction of H to $\mathsf{dcts}(\mathcal{M})$ and the restriction of Tr to $\mathsf{cl}(\mathsf{hs}(\mathcal{M}))$ are bijections, and they are inverses of each other. The notion of a pruning provides a general characterization of the elements of the set $\mathsf{cl}(\mathsf{hs}(\mathcal{M}))$ by spelling out the properties of the substructures of $\mathcal{M}$ that are spanned by those elements. We are thankful to an anonymous reviewer of Rumberg (2016) for pointing out the link to Galois connections in this context.

$\langle m \rightarrowtail H\rangle \in \mathsf{trans}(\mathcal{M})$ s.t. $\mathsf{H}(T) \subseteq H$. Then there is some $h \in \mathsf{H}_m \setminus H$, and it holds that $h \notin \mathsf{H}(T)$. Let $h' \in \mathsf{H}(T) \subseteq H$. Then $h' \in \mathsf{H}_m$. It follows that $h \perp_m h'$. Since $h' \in \mathsf{H}(T)$ but $h \notin \mathsf{H}(T)$, there must be a transition $\langle m' \rightarrowtail H'\rangle \in T$ s.t. $h' \in H'$ and $h \notin H'$. Because h and h' branch at m, the latter requires that $\langle m \rightarrowtail H\rangle \preceq \langle m' \rightarrowtail H'\rangle$. Since T is downward closed in $\mathsf{trans}(\mathcal{M})$ via $\prec$, this implies that $\langle m \rightarrowtail H\rangle \in T$.
We now prove that $\chi(\xi(\mathcal{M}')) = \mathcal{M}'$ for all $\mathcal{M}' \in \mathsf{prun}(\mathcal{M})$. Let $\mathcal{M}' \in \mathsf{prun}(\mathcal{M})$ and assume that $\xi(\mathcal{M}') = T$. Then by Prop. 3.13 we have that $\mathsf{hist}(\mathcal{M}') = \mathsf{H}(T)$ and hence $\chi(\xi(\mathcal{M}')) = \langle \bigcup \mathsf{hist}(\mathcal{M}'), < |_{\bigcup \mathsf{hist}(\mathcal{M}')}\rangle$. We thus have to show that $\langle \bigcup \mathsf{hist}(\mathcal{M}'), < |_{\bigcup \mathsf{hist}(\mathcal{M}')}\rangle = \mathcal{M}'$, which is trivially the case. □

3.3 Transition Structures and Index Structures

Neither Ockhamism nor the transition semantics presented in chapter 2 are genuine Kripke-style semantics. In both frameworks, the semantic evaluation is not only relativized to moments, which form the basic constituents of a branching time structure. Rather, the indices of evaluation correspond to pairs consisting of a moment and some additional parameter of truth that, as a defined notion, specifies a possible course of events. What is more, both Ockhamism and the transition semantics rely on semantic clauses that involve quantification over that second parameter of truth. The crucial difference between Ockhamism and the transition semantics consists in the fact that in Ockhamism, the second parameter of truth is given by a history whereas in the transition semantics it is provided by a consistent, downward closed set of indeterministic transitions, which enables the language to be enriched by an additional kind of intensional operator—over and above temporal and modal ones.

The aim of this section is to define a class of Kripke structures whose basic elements correspond to the indices of evaluation employed in the transition semantics and which comprise primitive accessibility relations for the three different kinds of intensional operators of the transition semantics. We will refer to the relevant class of structures as *index structures*. In an index structure, quantification over transition sets dissolves into restricted quantification over the basic elements of the structure. As said, the notion of an index structure builds the basis for a completeness proof for the transition framework since the models that result from the chronicle construction are models on index structures. We will show here, as a first step toward completeness, that index structures validate exactly those sentences that are validities in the transition semantics. That is, we prove that a sentence of the transition language $\mathcal{L}_{\mathsf{t}}$ is satisfiable in the transition semantics if and only if it is true at some index in a model on an index structure.

The challenge we are facing of course consists in providing an appropriate characterization of the notion of an index structure. In the Ockhamist case, the relevant index structures are Kripke structures $\langle W, \lhd, \sim \rangle$ with two primitive accessibility relations $\lhd$ and $\sim$, one for the temporal operators and one for the modal operators. Various characterizations of those structures are available in the literature, and there is a close correspondence with the notion of a Kamp frame that we have discussed in chapter 1 below.[76] Basically, the set W is supposed to be the disjoint union of $\lhd$-linear orders that are transitive and asymmetric, and the relation $\sim$ is required to be an equivalence relation that is preserved toward the past (i.e. for all $w, w', v, \in W$, if $w' \lhd w \sim v$, then there exists some $v' \in W$ such that $w' \sim v' \lhd v$). Moreover, we demand that the relations $\lhd$ and $\sim$ be mutually exclusive (i.e. $\sim \cap \lhd = \emptyset$) and that for any two $\lhd$-linear orders $l, l' \subseteq W$, we have $\sim \cap (l \times l') \neq \emptyset$.[77]

It is fairly straightforward to see that there is a one-to-one correspondence, up to isomorphism, between those Ockhamist index structures and bundled trees. The relevant correspondences are provided in Fig. 3.5. The first picture illustrates that we can define an Ockhamist index structure on the set of Ockhamist indices of evaluation in a given BT structure by exploiting the relations obtaining among moments and histories in that structure. The second picture shows that from any order isomorphic counterpart of the resulting index structure, we can again reconstruct an isomorphic BT structure by establishing an ordering among the respective $\sim$-equivalence classes. Yet, when moving from an arbitrary Ockhamist index structure to a BT structure, we are facing the problem of emergent histories. A one-to-one correspondence up to isomorphism can therefore in general only be established between Ockhamist index structures and bundled trees instead of BT structures, i.e. complete bundled trees. Validity with respect to the class of Ockhamist index structures accordingly coincides with bundled tree validity rather than with Ockhamist validity. In order to obtain full Ockhamist validity, further modifications are needed (cf. Reynolds 2003).

[76] See the definition of a Kamp frame in Reynolds (2002, 2003) or the definition of an Ockhamist frame in Zanardo (1985, 1991, 1996).

[77] The last condition ensures that the BT structure corresponding to an Ockhamist index structure is connected. In order to guarantee that it is even jointed, we have to require in addition that the intersection $\sim \cap (l \times l')$ always contains a $\lhd$-maximal element.

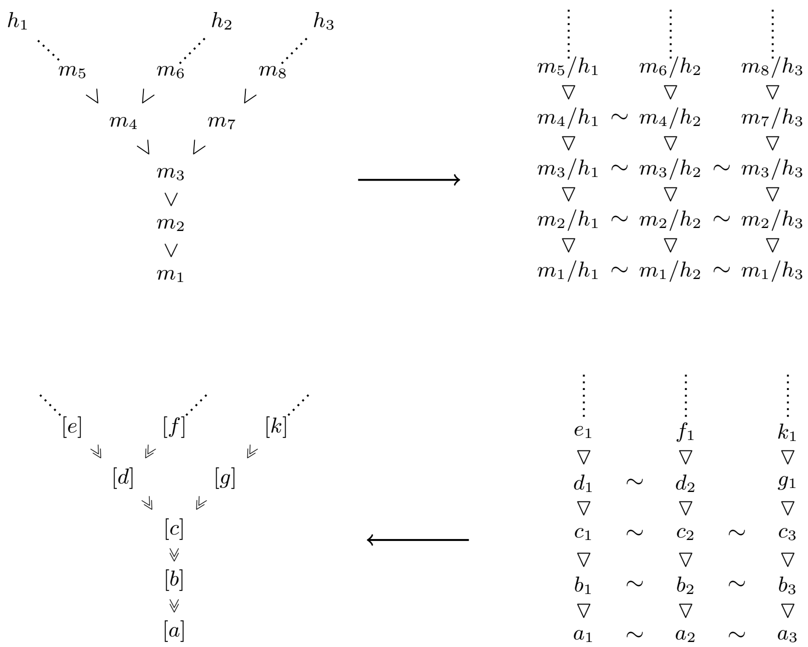


Figure 3.5: BT structures and Ockhamist index structures.

In the case of the transition framework, the definition of an index structure will have to be more complex of course. This is due to the fact that the transition language extends the language of the Ockhamist semantics by an additional kind of intensional operator, viz. the stability operator. An index structure will then be a Kripke structure $\langle W, \lhd, \sim, \sqsubseteq \rangle$ with three primitive accessibility relations $\lhd$, $\sim$ and $\sqsubseteq$, one for the temporal operators, one for the modal operators and another one for the stability operator and its dual. In order to avoid the problem of emerging histories from the start and in order to remain general enough so as to be able to incorporate the Peircean and the Ockhamist case as well, we aim at a notion of an index structure that gives rise to a one-to-one correspondence with the overall class of transition structures rather than with the class of BT structures, i.e. full transition structures only. The notion of a transition structure is our equivalent of an Ockhamist bundled tree. It is a BT structure $\mathcal{M} = \langle M, < \rangle$ together with a set $ts \subseteq \mathsf{dcts}(\mathcal{M})$ of transition sets that is sufficient to cover the entire structure.[78]

[78]Note that if we set up the notion of an index structure in such a way that is corresponds one-to-one to the notion of a transition structure, the underlying BT structure may always,

In order to arrive at an appropriate characterization of the notion of an index structure, we start out with the notion of a transition structure and develop the notion of an index structure from there. Our approach runs parallel to the two pictures provided in Fig. 3.5 above for the Ockhamist case. In section 3.3.1, we make a transition from transition structures to index structures. We first consider the set of all possible indices of evaluation in a given transition structure. We define different relations on that set in terms of the relations holding among moments and transition sets in the underlying BT structure and study the properties of the resulting structure. In section 3.3.2, we then define an index structure as a Kripke structure that has precisely those properties that we have discovered by investigating the structure on the set of indices of evaluation in a transition structure. In section 3.3.3, we make a transition back from index structures to transition structures. That is, we show that given an index structure, we can again reconstruct a transition structure from there. In section 3.3.4, we prove that our back and forth transitions induce a one-to-one correspondence up to isomorphism between transition structures and index structure. Finally, in section 3.3.5, validity with respect to the class of index structures will be shown to coincide with validity with respect to the class of transition structures.

3.3.1 From Transition Structures to Index Structures

In chapter 2, we have provided a general definition of the notion of a transition structure (cf. Def. 2.33). A transition structure has been defined as a triple $\mathcal{M}^{ts} = \langle M, <, ts \rangle$, where $\mathcal{M} = \langle M, < \rangle$ is a BT structure and $ts \subseteq \mathsf{dcts}(\mathcal{M})$ is a non-empty set of consistent, downward closed transition sets in $\mathsf{trans}(\mathcal{M})$. The set ts of transition sets is thereby required to be such that it covers the entire set M of moments: every moment $m \in M$ must be compatible with at least one transition set $T \in ts$. Recall that a moment $m \in M$ is said to be compatible with a transition set $T \in ts$ whenever the transition set admits at least one history that contains m, i.e. if $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$.

The semantic evaluation on a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ is relativized to pairs m/T consisting of a moment $m \in M$ and a transition set $T \in \mathsf{dcts}(\mathcal{M})$ compatible with that moment. The transition language contains three different kinds of intensional operators: temporal, modal and stability operators. While the temporal operators shift the moment parameter, both

i.e. also in the finite case, comprise additional transition sets, whereas in the Ockhamist case, new histories can only emerge in the limit step of the completeness construction.

the modal operators and the stability operators quantify over the transition parameter. The domain of quantification of the stability operators is thereby but a subset of the domain of quantification of the modal operators.

In this section, we start out with some arbitrary transition structure and consider the set of indices of evaluation in that structure. On that set, we define four relations in terms of the relations that hold among moments and transition sets, respectively, in virtue of the properties of the underlying BT structure. We will thoroughly investigate the properties of those different relations and their interrelation step by step. As it turns out that one of the relations is reducible to the remainder, we will finally end up with a Kripke structure on the set of indices of evaluation of the transition structure we have started out with that comprises three accessibility relations, one for each kind of intensional operator.

3.3.1.1 The set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$

Given a transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$, we denote the *set of indices of evaluation in* $\mathcal{M}^{ts}$ by $\mathsf{Ind}(\mathcal{M}^{ts})$. The set $\mathsf{Ind}(\mathcal{M}^{ts})$ then comprises all m/T pairs with $m \in M$ and $T \in ts$ such that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. On the basis of the relations obtaining among moments and transition sets in the modal-temporal order of the branching time structure $\mathcal{M}$, we can define four different relations among the m/T-pairs contained in $\mathsf{Ind}(\mathcal{M}^{ts})$: $\lhd$, $\sim$, $\sqsubseteq$ and $\approx$. The relation $\lhd$ is the relation underlying the temporal operators. It reflects the earlier-later relation among moments. The relation $\sim$, on the other hand, expresses sameness of moment and hence corresponds to the modal operators, while the relation $\sqsubseteq$ represents the inclusion relation among transition sets that plays a role in the interpretation of the stability operators. Finally, the relation $\approx$ captures sameness of the transition parameter. Unlike the other three relations, the relation $\approx$ does not connect to some kind of intensional operator, and we will later see that it can be analyzed in terms of the other ones.

Definition 3.16 (The set of indices of evaluation in $\mathcal{M}^{ts}$). *Given a transition structure* $\mathcal{M}^{ts} = \langle M, <, ts\rangle$, *let*

$$\mathsf{Ind}(\mathcal{M}^{ts}) := \{m/T \mid m \in M,\ T \in ts \text{ and } \mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset\}$$

be the set of indices of evaluation in $\mathcal{M}^{ts}$.

We define the following relations on the set $\mathsf{Ind}(\mathcal{M}^{ts})$. *For all indices of evaluation* m/T, $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$, *we set:*

($\lhd$) $m/T \lhd m'/T'$ *iff* $T = T'$ *and* $m < m'$;

($\sim$) $m/T \sim m'/T'$ *iff* $m = m'$;

($\sqsubseteq$) $m/T \sqsubseteq m'/T'$ *iff* $m = m'$ *and* $T \subseteq T'$;

($\approx$) $m/T \approx m'/T'$ *iff* $T = T'$.

Since for a BT structure together with a set of transitions sets to constitute a transition structure, it is required that for every moment, there be some compatible transition set, the definition of a transition structure guarantees that the set of indices of evaluation is never empty. That is, for any arbitrary transition structure $\mathcal{M}^{ts}$, we have $\mathsf{Ind}(\mathcal{M}^{ts}) \neq \emptyset$. Furthermore, given the definition of the two relations $\sim$ and $\approx$, it is straightforward that both are equivalence relations on the set $\mathsf{Ind}(\mathcal{M}^{ts})$. The relation $\lhd$ is obviously a subset of the equivalence relation $\approx$, and just as the earlier-later relation $<$ on M, it is a strict partial order that is left-linear and serial. The relation $\sqsubseteq$, on the other hand, is a subset of the equivalence relation $\sim$. Reflecting the inclusion relation on the set $ts \subseteq \mathsf{dcts}(\mathcal{M})$, whose elements are linear chains of transitions that are closed toward the past, it is a left-linear partial order. We summarize those observations in Prop. 3.17 below. We use $m/T \unlhd m'/T'$ to stand for ($m/T \lhd m'/T'$ or $m/T = m'/T'$); and we take $m/T \sqsubset m'/T'$ to mean that ($m/T \sqsubseteq m'/T'$ and $m/T \neq m'/T'$).

Proposition 3.17. *Let* $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ *be a transition structure, and let* $\mathsf{Ind}(\mathcal{M}^{ts})$ *be the set of indices of evaluation in* $\mathcal{M}^{ts}$. *The following holds:*

(i) $\mathsf{Ind}(\mathcal{M}^{ts}) \neq \emptyset$;

(ii) *the relation* $\lhd$ *is a strict partial order on* $\mathsf{Ind}(\mathcal{M}^{ts})$ (*i.e. an irreflexive, asymmetric and transitive relation*) *that is left-linear and serial;*

(iii) *the relation* $\sim$ *is an equivalence relation on* $\mathsf{Ind}(\mathcal{M}^{ts})$ (*i.e. a relation that is reflexive, symmetric and transitive*)*;*

(iv) *the relation* $\sqsubseteq$ *is a partial order on* $\mathsf{Ind}(\mathcal{M}^{ts})$ (*i.e. a reflexive, antisymmetric and transitive relation*) *that is left-linear;*

(v) *the relation* $\approx$ *is an equivalence relation on* $\mathsf{Ind}(\mathcal{M}^{ts})$ (*i.e. a relation that is reflexive, symmetric and transitive*)*;*

(vi) $\lhd \; \subseteq \approx$;

(vii) $\sqsubseteq \; \subseteq \sim$.

Proof.

(i) By Def. 1.1 we have $M \neq \emptyset$. Let $m \in M$. By Def. 2.33 it holds that there is some $T \in ts$ s.t. $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$, which implies that $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ and hence $\mathsf{Ind}(\mathcal{M}^{ts}) \neq \emptyset$.

(ii) Being defined in terms of the relation $<$ on M, which by Def. 1.1 is a strict partial order that is left-linear, the relation $\lhd$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ is a left-linear, strict partial order as well. The seriality of $\lhd$ follows from the seriality of $<$ along the following lines: $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ implies that there is some $h \in \mathsf{H}(T) \cap \mathsf{H}_m$, from which it follows by the seriality of $<$ and Def.1.2 that there is some $m' \in h$ s.t. $m' > m$. Then $m'/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ and $m/T \lhd m'/T$.

(iii) Being defined in terms of the identity $=$ between elements of M, which is an equivalence relation, the relation $\sim$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ is an equivalence relation as well.

(iv) Being defined in terms of the subset relation $\subseteq$ on ts, which is a partial order, the relation $\sqsubseteq$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ is a partial order as well. The left-linearity of $\sqsubseteq$ follows from the fact that every transition set $T \in ts$ is linearly ordered and downward closed in $\mathsf{trans}(\mathcal{M})$ via the transition ordering $\prec$.

(v) Being defined in terms of the identity $=$ between elements of ts, which is an equivalence relation, the relation $\approx$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ is an equivalence relation as well.

(vi) Straightforward by definition.

(vii) Straightforward by definition. □

We then have two different equivalence relations, $\approx$ and $\sim$, on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a given transition structure $\mathcal{M}^{ts}$. In the subsequent sections, we will discuss the properties of the structures arising on the respective equivalence classes in virtue of the order relations $\lhd \ \subseteq \approx$ and $\sqsubseteq \subseteq \sim$ and study the interrelation between those structures. But before we move there, some final remark on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ of a transition structure $\mathcal{M}^{ts}$ is in order here.

As we know (cf. Lem. 1.3), the properties of a branching time structure $\mathcal{M} = \langle M, < \rangle$ guarantee that histories are downward closed: hence, for all moments $m, m' \in M$ such that $m' < m$, we have $\mathsf{H}_m \subseteq \mathsf{H}_{m'}$. From this it follows that if a moment $m \in M$ is compatible with a given transition set

$T \in \mathsf{dcts}(\mathcal{M})$, all moments $m' < m$ in its past are so as well. As a consequence then, the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ of a transition structure $\mathcal{M}^{ts}$ contains for every pair $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, all pairs m'/T with $m' < m$ as well. Likewise, since for all transition sets $T, T' \in \mathsf{dcts}(\mathcal{M})$ such that $T' \subseteq T$, we have $\mathsf{H}(T) \subseteq \mathsf{H}(T')$, whenever $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, then for all $T' \in ts$ such that $T' \subseteq T$, we also have $m/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$.

LEMMA 3.18. *Let $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ be a transition structure, and let $\mathsf{Ind}(\mathcal{M}^{ts})$ be the set of indices of evaluation in $\mathcal{M}^{ts}$. The following holds:*

(i) *If $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, then for all $m' \in M$ such that $m' < m$, we have $m'/T \in \mathsf{Ind}(\mathcal{M}^{ts})$;*

(ii) *If $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, then for all $T' \in \mathsf{dcts}(\mathcal{M})$ such that $T' \subseteq T$, we have $m/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$.*

Proof.

(i) Let $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$. Then $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. Assume that there is some $m' \in M$ s.t. $m' < m$. Then $\mathsf{H}_m \subseteq \mathsf{H}_{m'}$ and hence $\mathsf{H}(T) \cap \mathsf{H}_{m'} \neq \emptyset$. From this it follows that $m'/T \in \mathsf{Ind}(\mathcal{M}^{ts})$.

(ii) Let $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$. Then $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. Assume that there is some $T' \in \mathsf{dcts}(\mathcal{M})$ s.t. $T' \subseteq T$. Then $\mathsf{H}(T) \subseteq \mathsf{H}(T')$ and hence $\mathsf{H}(T') \cap \mathsf{H}_m \neq \emptyset$. From this it follows that $m/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$. □

Note that Lem. 3.18 implies the following closure properties. By condition (i), for all $m/T, m'/T, m/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m'/T \lhd m/T \sim m/T'$, there exists some $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m'/T \sim m'/T' \lhd m/T'$. Since $\sqsubseteq \subseteq \sim$, the claim obviously also remains true if we substitute $\sim$ by $\sqsubseteq$ or $\sqsupseteq$. And as a consequence of condition (ii), we have in addition that for all indices of evaluation $m/T, m'/T, m/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m'/T \rhd m/T \sqsupseteq m/T'$, there exists some $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m'/T \sqsupseteq m'/T' \rhd m/T'$. This kind of closure properties become important in the chronicle construction of a completeness proof.

3.3.1.2 The structure $\langle [\mathbf{T}]_{\approx}, \lhd|_{[\mathbf{T}]_{\approx}} \rangle$

In this section, we will consider the structures arising on the $\approx$-equivalence classes of the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$. As an equivalence relation, the relation $\approx$ induces a partition $\mathsf{Ind}(\mathcal{M}^{ts})/\approx$ of the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$, and the relation

$\lhd \ \subseteq \approx$ provides an ordering on each $\approx$-equivalence class in $\mathsf{Ind}(\mathcal{M}^{ts})/\approx$. In what follows, we will investigate the properties of the resulting structures.

In accordance with the definition of the relation $\approx$ provided above, the $\approx$-equivalence class of an element $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ groups together all pairs $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ whose second entry equals T. Since the $\approx$-equivalence class of an element $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ thus depends solely on the respective transition set, we use $[T]_{\approx}$ to stand for the $\approx$-equivalence class of the pair m/T.

DEFINITION 3.19 (The equivalence class $[T]_{\approx}$). *For $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ a transition structure and $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, let*

$$[T]_{\approx} := \{m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts}) \mid m/T \approx m'/T'\}.$$

Due to the fact that each $\approx$-equivalence class $[T]_{\approx} \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$ is a subset of $\mathsf{Ind}(\mathcal{M}^{ts})$, for every pair $m/T \in [T]_{\approx}$, it obviously holds that $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. Each $\approx$-equivalence class $[T]_{\approx} \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$ then contains exactly for every moment $m \in M$ that is compatible with the given transition set $T \in ts$, the pair m/T. To put it differently, the equivalence class $[T]_{\approx}$ is the subset of $\mathsf{Ind}(\mathcal{M}^{ts})$ that contains all m/T-pairs whose moment m is part of at least one history admitted by the transition set T, i.e. for all $m \in M$, $m/T \in [T]_{\approx}$ iff $m \in \bigcup \mathsf{H}(T)$. Since the set $ts \subseteq \mathsf{dcts}(\mathcal{M})$ comprises consistent transition sets only and since different transition sets naturally give rise to different $\approx$-equivalence classes, to every transition set $T \in ts$, there corresponds one-to-one an $\approx$-equivalence class in $[T]_{\approx} \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$.

The relation $\lhd \ \subseteq \approx$ induces a strict partial order on each equivalence class $[T]_{\approx} \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$. The order established by $\lhd$ is above all left-linear and serial. We show that the resulting structure $\langle [T]_{\approx}, \lhd|_{[T]_{\approx}}\rangle$ is in fact a BT structure. To this end, we have to prove that the relation $\lhd|_{[T]_{\approx}}$ on an $\approx$-equivalence class $[T]_{\approx}$ is jointed.

The jointedness of the relation $\lhd|_{[T]_{\approx}}$ on an $\approx$-equivalence class $[T]_{\approx}$ is a consequence of the jointedness of the relation $<$ on M since by Lem. 3.18 (i) the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ is closed downward with respect to the earlier-later relation among moments. Whenever $m/T, m'/T \in [T]_{\approx}$ and there is some moment $m'' \in M$ such that $m \geq m'' \leq m'$, then $m''/T \in [T]_{\approx}$ and $m/T \unrhd m''/T \unlhd m'/T$. Since the relation $<$ on M is jointed, it is then straightforward that also any two pairs $m/T, m'/T \in [T]_{\approx}$ have a greatest common lower $\lhd$-bound in $[T]_{\approx}$.

Lemma 3.20. *For $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ a transition structure and $T \in ts$ a set of transitions, the relation $\lhd|_{[T]_\approx}$ on $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$ is jointed (i.e. for all $m/T, m'/T \in [T]_\approx$, there is some $m''/T \in [T]_\approx$ s.t. $m/T \unrhd m''/T \unlhd m'/T$ and for all $m'''/T \in [T]_\approx$ s.t. $m/T \unrhd m'''/T \unlhd m'/T$, we have $m'''/T \unlhd m''/T$).*

Proof. Follows from the jointedness of the relation $<$ on M (Def. 1.1 (ii)) and Lem. 3.18 (i). □

For every $\approx$-equivalence class $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$, the structure $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ then satisfies all the conditions that we have imposed on the notion of a BT structure: the relation $\lhd|_{[T]_\approx}$ on $[T]_\approx$ is a strict partial order that is left-linear, jointed and serial, and obviously $[T]_\approx \neq \emptyset$.

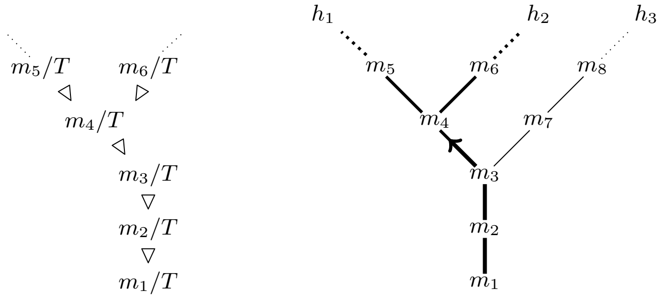
Corollary 3.21. *For $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ a transition structure and $T \in ts$, the structure $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ is a BT structure.*

Proof. Follows from Prop. 3.17 (ii) and (v) in conjunction with Lem. 3.20. □

What is more, for every $\approx$-equivalence class $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$, the BT structure $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ is order isomorphic to a pruning of the original BT structure $\mathcal{M}$. In order to see this, consider the function $\theta : \mathsf{Ind}(\mathcal{M}^{ts}) \to M$ that maps every pair $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ to the corresponding moment $m \in M$. The restriction $\theta|_{[T]_\approx}$ of the function θ to an $\approx$-equivalence class $[T]_\approx \subseteq \mathsf{Ind}(\mathcal{M}^{ts})$ is obviously injective; and its image equals the set $\bigcup \mathsf{H}(T)$, for $[T]_\approx$ is just the set of m/T pairs such that $m \in \bigcup \mathsf{H}(T)$. Since, furthermore, the relation $\lhd$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ accurately reflects the temporal ordering $<$ on M, the function $\theta|_{[T]_\approx}$ is order preserving. Hence, $\theta|_{[T]_\approx}$ is an order isomorphism from $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ onto $\langle \bigcup \mathsf{H}(T), <|_{\bigcup \mathsf{H}(T)}\rangle$, and in section 3.2.2, the latter structure has been shown to be a pruning of $\mathcal{M}$. The order isomorphism between the BT structure $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ and the corresponding pruning $\langle \bigcup \mathsf{H}(T), <|_{\bigcup \mathsf{H}(T)}\rangle$ of $\mathcal{M}$ induced by the function θ is illustrated in Fig. 3.6.

Lemma 3.22. *Let $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ be a transition structure, and let $\theta : \mathsf{Ind}(\mathcal{M}^{ts}) \to M$ be the function with $\theta(m/T) = m$. For every $T \in ts$ the restriction $\theta|_{[T]_\approx}$ is an order isomorphism from $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ onto a pruning of $\mathcal{M} = \langle M, <\rangle$.*

Proof. Obviously, the function $\theta|_{[T]_\approx}$ is injective, and it is also order preserving: for all $m'/T, m''/T \in [T]_\approx$, we have $m'/T \lhd m''/T$ iff $\theta(m'/T) < \theta(m''/T)$. Moreover, $\theta([T]_\approx) = \bigcup \mathsf{H}(T)$, and by Prop. 3.11 it follows that the structure $\langle \bigcup \mathsf{H}(T), <|_{\bigcup \mathsf{H}(T)}\rangle$ is a pruning of $\mathcal{M}$. □



$\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ $\quad$ $\langle \bigcup \mathsf{H}(T), <|_{\bigcup \mathsf{H}(T)} \rangle \in \mathsf{prun}(\mathcal{M})$

Figure 3.6: Order isomorphism from $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ onto $\langle \bigcup \mathsf{H}(T), <|_{\bigcup \mathsf{H}(T)} \rangle$.

In Lem. 3.20 we have shown that the the relation $\lhd|_{[T]_\approx}$ on $[T]_\approx$ is jointed; and being jointed, it is obviously also connected. Together with the fact that $\lhd$ is a subset of $\approx$, this opens up the possibility to define the equivalence relation $\approx$ in terms of the order relation $\lhd$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ altogether: two elements in $\mathsf{Ind}(\mathcal{M}^{ts})$ are related via $\approx$ if and only if they are linked by a $\rhd\lhd$-path. In other words, an equivalence class $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$ can be defined as the set that groups together all and only those pairs that have a common lower $\lhd$-bound.

Proposition 3.23. *For $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ a transition structure, the relation $\approx$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ can be defined in terms of the relation $\lhd$ on $\mathsf{Ind}(\mathcal{M}^{ts})$ along the following lines: for all $m/T, m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$, we have $m/T \approx m'/T'$ iff there is some $m''/T'' \in \mathsf{Ind}(\mathcal{M}^{ts})$ s.t. $m/T \unrhd m''/T'' \unlhd m'/T'$.*

Proof. Let $m/T, m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$. If $m/T \approx m'/T'$, then $T = T'$. By Lem. 3.20 it follows that there is some index of evaluation $m''/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ s.t. $m/T \unrhd m''/T \unlhd m'/T' = m'/T$. If, on the other hand, there is some index $m''/T'' \in \mathsf{Ind}(\mathcal{M}^{ts})$ s.t. $m/T \unrhd m''/T'' \unlhd m'/T'$, then, by Prop. 3.17 (v) and (vi), we have $m/T \approx m''/T'' \approx m'/T'$ and hence $m/T \approx m'/T'$. □

The upshot of this section then is as follows: given a transition structure $\mathcal{M}^{ts} = \langle M, <\ ts \rangle$, the equivalence relation $\approx$ defined on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in Def. 3.16 above is reducible to the relation $\lhd$ on $\mathsf{Ind}(\mathcal{M}^{ts})$. What is more, for every transition set $T \in ts$, the structure $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ established by the relation $\lhd\ \subseteq\ \approx$ is a BT structure and a pruning of $\mathcal{M}$. BT structures based on different elements of $\mathsf{Ind}(\mathcal{M}^{ts})/\approx$ are obviously pairwise disjoint, and their respective orderings reflect the temporal ordering of moments in the BT structure $\mathcal{M}$.

3.3.1.3 The structure $\langle [\mathbf{m}]_\sim, \sqsubseteq|_{[\mathbf{m}]_\sim} \rangle$

Let us now take a look at the structures that arise on the $\sim$-equivalence classes of the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$. The equivalence relation $\sim$ provides a partition $\mathsf{Ind}(\mathcal{M}^{ts})/\sim$ of the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$, just as the relation $\approx$ does. And while the $\approx$-equivalence classes are internally ordered by the relation $\lhd \ \subseteq \approx$, the $\sim$-equivalence classes receive an ordering by the relation $\sqsubseteq \subseteq \sim$.

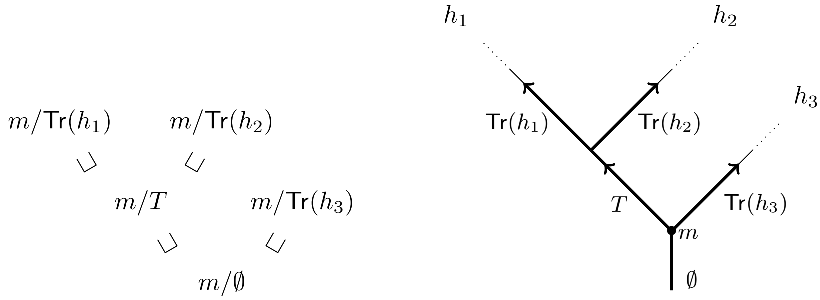
Whereas an $\approx$-equivalence class groups together all index pairs in $\mathsf{Ind}(\mathcal{M}^{ts})$ that share the same transition set, the $\sim$-equivalence class of an element $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ groups together all pairs in $\mathsf{Ind}(\mathcal{M}^{ts})$ that share the moment m. Being only moment-dependent, the $\sim$-equivalence class of a pair $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ is accordingly denoted by $[m]_\sim$.

Definition 3.24 (The equivalence class $[m]_\sim$). *For $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ a transition structure and $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, let*

$$[m]_\sim := \{m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts}) \mid m/T \sim m'/T'\}.$$

The $\sim$-equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ is just the subset of $\mathsf{Ind}(\mathcal{M}^{ts})$ that contains for every transition set $T \in ts$ that is compatible with the given moment m, the pair m/T, i.e. for all $T \in ts$, $m/T \in [m]_\sim$ iff $\mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset$. Since the definition of a transition structure guarantees that every moment $m \in M$ is compatible with at least one transition $T \in ts$ and since different moments lead to different $\sim$-equivalence classes, there is a natural one-to-one correspondence between the moments $m \in M$ and the $\sim$-equivalence classes $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$.

The relation $\sqsubseteq \subseteq \sim$ on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ provides an internal ordering on each equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$. The respective ordering is a left-linear, partial order that reflects the inclusion relation between all transition sets in ts compatible with the moment m. The correspondence between the structure $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim} \rangle$ that arises on a given $\sim$-equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ and the inclusion ordering on the set $\{T \in ts \mid \mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset\}$ of all transition sets that are compatible with the moment $m \in M$ is illustrated in Fig. 3.7.



$\langle [m]_\sim, \sqsubset\!|_{[m]_\sim} \rangle$ $\{T \in ts \mid \mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset\}$

Figure 3.7: Correspondence between $\langle [m]_\sim, \sqsubset\!|_{[m]_\sim} \rangle$ and the order induced by set inclusion on the set $\{T \in ts \mid \mathsf{H}(T) \cap \mathsf{H}_m \neq \emptyset\}$.

Unlike the temporal order $\lhd$ on the $\approx$-equivalence classes in $\mathsf{Ind}(\mathcal{M}^{ts})/\!\approx$, the internal ordering of an $\sim$-equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\!\sim$ by the relation $\sqsubseteq$ does in general not have to be connected, let alone jointed. This rules out the possibility to reduce the equivalence relation $\sim$ to the order relation $\sqsubseteq \subseteq \sim$. If, and only if, the set $ts \subseteq \mathsf{dcts}(\mathcal{M})$ provided by the given transition structure $\mathcal{M}^{ts}$ contains for any two transition sets $T, T' \in ts$, a transition set T'' that is a subset of both, does it hold that any two any pairs $m/T, m/T' \in [m]_\sim$ have a common lower $\sqsubseteq$-bound in $[m]_\sim$. Recall that by Lem. 3.18 (ii), the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ is closed downward with respect to the inclusion ordering among transition sets in ts. Hence, if $m/T, m/T' \in [m]_\sim$, then for all $T'' \in ts$ such that $T \supseteq T'' \subseteq T'$, it holds that $m/T'' \in [m]_\sim$ as well with $m/T \sqsupseteq m/T'' \sqsubseteq m/T'$.

Lemma 3.25. *Let $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ be a transition structure such that for all $T, T' \in ts$, there is some $T'' \in ts$ such that $T'' \subseteq (T \cap T')$. Then for all $m \in M$, the relation $\sqsubseteq\!|_{[m]_\sim}$ on $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\!\sim$ is connected (i.e. for all $m/T, m/T' \in [m]_\sim$, there is some $m/T'' \in [m]_\sim$ s.t. $m/T \sqsupseteq m/T'' \sqsubseteq m/T'$).*

Proof. Follows from Lem. 3.18 (ii). □

Note that as a consequence of Lem. 3.25, the ordering induced by the relation $\sqsubseteq$ on each of the $\sim$-equivalence classes $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\!\sim$ is always connected whenever the set ts contains the empty transition set $\emptyset_{\mathsf{Tr}}$. Ockhamist transition structures, on the other hand, where $ts = \mathsf{mcts}(\mathcal{M})$, provide a paradigm example of a case in which the internal ordering of the various $\sim$-equivalence classes via the relation $\sqsubseteq$ is not connected: in an Ockhamist transition structure, any two pairs m/T and m/T' that involve different transition sets lack a common lower $\sqsubseteq$-bound.

In case the internal order relation $\sqsubseteq$ on each of the $\sim$-equivalence classes $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ is connected, every structure $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ forms a tree with a single trunk. Furthermore, the following is true: if the set $ts \subseteq \mathsf{dcts}(\mathcal{M})$ is closed under intersection, i.e. for all $T, T' \in ts$, we have $T \cap T' \in ts$, then the restriction of the relation $\sqsubseteq$ to each of the $\sim$-equivalence classes $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ is even jointed. Obviously, in case $T \cap T' \in ts$, the pair $m/(T \cap T')$ constitutes the greatest common lower $\sqsubseteq$-bound of m/T and m/T' in $[m]_\sim$. For every full transition structure, i.e. each transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ for which it holds that $ts = \mathsf{dcts}(\mathcal{M})$, the internal ordering induced by the relation $\sqsubseteq$ on the various $\sim$-equivalence classes $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ is thus jointed in each case. Still, the resulting structures $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ are not BT structures according to Def. 1.1. First of all, the relation $\sqsubseteq$ is reflexive rather irreflexive, and, what is more important, the ordering induced by the relation $\sqsubseteq$ on each of the $\sim$-equivalence classes $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ does not only contain a minimum, viz. $m/\emptyset_{\mathsf{Tr}}$, but also $\sqsubseteq$-maximal elements that are provided by the maximal consistent transition sets compatible with the moment m. The best we can say is that in case $ts = \mathsf{dcts}(\mathcal{M})$, the structures $\langle [m]_\sim, \sqsubset|_{[m]_\sim}\rangle$ yielded by the strict ordering $\sqsubset$ are finite BT structures.

In the general case the following holds: given an arbitrary transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$, for every $m \in M$, the structure $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ established by the relation $\sqsubseteq \subseteq \sim$ on the equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ is a left-linear, partial order. Structures based on different $\sim$-equivalence classes in $\mathsf{Ind}(\mathcal{M}^{ts})/\sim$ are pairwise disjoint, and their internal ordering reflects the inclusion relation among transition sets compatible with the respective moment.

3.3.1.4 The structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq\rangle$

In section 3.3.1.1, we have defined four relations on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$: $\lhd$, $\sim$, $\sqsubseteq$ and $\approx$. In section 3.3.1.2 we have shown that the equivalence relation $\approx$ is reducible to the order relation $\lhd\ \subseteq \approx$. Each of the three remaining relations corresponds to a certain kind of intensional operator of the transition language $\mathcal{L}_\mathsf{t}$, and together they form a structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq\rangle$ on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$. While we have so far discussed separately the structures arising on the cells of the partitions $\mathsf{Ind}(\mathcal{M}^{ts})/\!\approx$ and $\mathsf{Ind}(\mathcal{M}^{ts})/\!\sim$ in virtue of the order relations $\lhd\ \subseteq\ \approx$ and $\sqsubseteq\ \subseteq\ \sim$ respectively, we will now take a closer look at their interrelation.

Given a transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$, for every $T \in ts$, there is a BT structure $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ which figures as a pruning of $\mathcal{M}$, and for every $m \in M$, the corresponding structure $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ is a left-linear, partial order. Every m/T-pair contained in the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ is thus an element of two different structures: one based on the $\approx$-equivalence class $[T]_\approx$ and another based on the $\sim$-equivalence class $[m]_\sim$. Hence, for every pair $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, the corresponding structures $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ and $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ are closely entangled. If, on the other hand, a moment $m \in M$ is incompatible with a given transition set $T \in ts$ so that the corresponding pair does not constitute an admissible index of evaluation, the structures $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ and $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ do not have a single element in common. For every moment $m \in M$ and every transition set $T \in ts$, the intersection $[m]_\sim \cap [T]_\approx$ is either empty or contains but a single element, viz. the pair m/T, depending on whether the moment m is compatible with the transition set T or not.

Lemma 3.26. *For $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ a transition structure, $m \in M$ and $T \in ts$, we have either $[m]_\sim \cap [T]_\approx = \emptyset$ or $[m]_\sim \cap [T]_\approx = \{m/T\}$.*

Proof. Let $m \in M$ and $T \in ts$. Assume that $[m]_\sim \cap [T]_\approx \neq \emptyset$. We can then consider some $m'/T' \in [m]_\sim \cap [T]_\approx$. By Def. 3.16 it follows that $m' = m$ and $T' = T$. Hence, $m'/T' = m/T$. □

Each of the structures $\langle [m]_\sim, \sqsubseteq|_{[m]_\sim}\rangle$ based on an $\sim$-equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\!\sim$ naturally induces a left-linear, partial order among the BT structures $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ on the $\approx$-equivalence classes $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\!\approx$ for which it holds that $[m]_\sim \cap [T]_\approx \neq \emptyset$, as illustrated in Fig. 3.8.

Actually, for any two $\approx$-equivalence classes $[T], [T'] \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$, there is in fact an $\sim$-equivalence class $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ with which they both intersect non-emptily. That is, for all $[T], [T'] \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$, the restriction $\sim \cap ([T]_\approx \times [T']_\approx) \neq \emptyset$. The overall structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq\rangle$ is even jointed in the following sense: for any two pairs $m/T \in [T]_\approx$ and $m'/T' \in [T']_\approx$, there exists a $\lhd$-maximal pair of elements $m_0/T \in [T]_\approx$ and $m_0'/T' \in [T']_\approx$ such that $m/T \trianglerighteq m_0/T \sim m_0'/T' \trianglelefteq m'/T'$. The existence of such a $\lhd$-maximal

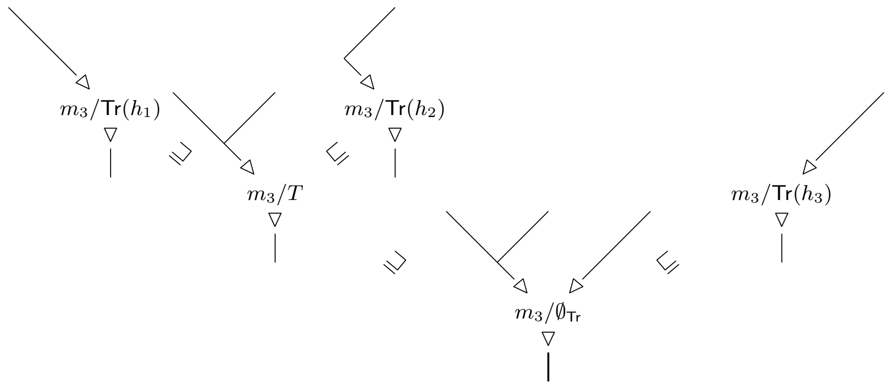


Figure 3.8: The interrelation of the relations $\sim$ and $\approx$.

pair of $\sim$-related elements is a straightforward consequence of the jointedness of the relation $<$ on M, given that the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ is downward closed with respect to the earlier-later relation among moments (cf. Lem. 3.18 (i)).

Proposition 3.27. *Let $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ be a transition structure. For all $m/T, m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$, there is some $m_0/T \in [T]_\approx$ and some $m'_0/T' \in [T']_\approx$ such that (i) $m_0/T \trianglelefteq m/T$ and $m'_0/T' \trianglelefteq m'/T'$, (ii) $m_0/T \sim m'_0/T'$ and (iii) for all $m_1/T, m'_1/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m_1/T \trianglelefteq m/T$, $m'_1/T' \trianglelefteq m'/T'$ and $m_1/T \sim m'_1/T'$, we have $m_1/T \trianglelefteq m_0/T$ and $m'_1/T' \trianglelefteq m'_0/T'$.*

Proof. Let $m/T, m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$. Then $m, m' \in M$. Let m_0 be the greatest common lower $<$-bound of m and m' in M. By Lem. 3.18 (i) it follows that $m_0/T, m_0/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$. Obviously, $m_0/T \in [T]_\approx$ and $m_0/T' \in [T']_\approx$, and we have $m_0/T \trianglelefteq m/T$, $m_0/T' \trianglelefteq m'/T'$ and $m_0/T \sim m_0/T'$. Moreover, since m_0 is the greatest common lower $<$-bound of m and m' in M, by Lem. 3.18 (i) it immediately follows that for all $m_1/T, m'_1/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ s.t. $m_1/T \trianglelefteq m/T$, $m'_1/T' \trianglelefteq m'/T'$ and $m_1/T \sim m'_1/T'$, we have $m_1/T \trianglelefteq m_0/T$ and $m'_1/T' \trianglelefteq m_0/T'$. □

Note that Lem. 3.20, which gives expression to the idea that the restriction of the relation $\lhd$ to an $\approx$-equivalence class $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$ is always jointed, is just a special case of Prop. 3.27, viz. the case in which $T = T'$.

Prop. 3.27 guarantees that for any arbitrary pair of $\approx$-equivalence classes $[T]_\approx, [T']_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$, the intersection $\sim \cap ([T]_\approx \times [T']_\approx)$ is non-empty. In what follows, we will consider how the relation that obtains between the transition sets T and T' in ts bears on the relation $\sim \cap ([T]_\approx \times [T']_\approx)$. Two different cases can be considered: either one of the transition sets T and T' is a subset of the other or not.

Let $T, T' \in ts$ and assume that $T \subseteq T'$. In this case, it does not only hold that the intersection $\sim \cap ([T]_\approx \times [T']_\approx)$ is non-empty, as predicted by Prop. 3.27, but the intersection $\sqsubseteq \cap ([T]_\approx \times [T']_\approx)$ is non-empty as well. We show that the relation $\sqsubseteq \cap ([T]_\approx \times [T']_\approx)$ is even right-total so that actually every element in $[T']_\approx$ is $\sim$-related to some element in $[T]_\approx$. More precisely, we prove that the converse relation $\sqsupseteq \cap ([T']_\approx \times [T]_\approx)$ is the graph of an order isomorphism from $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ onto a pruning of $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$. To this end, consider the function $\pi : [T']_\approx \to [T]_\approx$ that maps every pair $m/T' \in [T']_\approx$ onto the corresponding pair $m/T \in [T]_\approx$, holding the respective moment fixed. Recall that $T \subseteq T'$ implies $\mathsf{H}(T') \subseteq \mathsf{H}(T)$. It is easy to see that the function π is in fact an order isomorphism from $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ onto its image. Moreover, both $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ are order isomorphic to prunings of the original BT structure $\mathcal{M}$; and obviously, the pruning corresponding to the latter structure is a substructure of the pruning corresponding to the former one. Since on the set $\mathsf{prun}(\mathcal{M})$, the pruning relation is just the substructure relation, it then follows that the BT structure $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ is order isomorphic to a pruning of the structure $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$.

Proposition 3.28. *Let $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ be a transition structure and m/T, $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m'/T' \sqsupseteq m/T$. The relation $\sqsupseteq \cap ([T']_\approx \times [T]_\approx)$ is the graph of an order isomorphism π from $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ onto a pruning of $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$.*

Proof. Let $m/T, m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ s.t. $m'/T' \sqsupseteq m/T$. Then $m = m'$ and $T' \supseteq T$. Consequently, $\mathsf{H}(T') \subseteq \mathsf{H}(T)$ and hence $\bigcup \mathsf{H}(T') \subseteq \bigcup \mathsf{H}(T)$. Consider the function $\pi : [T']_\approx \to [T]_\approx$ with $\pi(m^*/T') = (m^*/T)$. Obviously, the function π is injective and also $\lhd$-preserving, and we have that $\pi([T']_\approx) = \{m^*/T \in [T]_\approx \mid m^* \in \bigcup \mathsf{H}(T')\}$.
We show that $\langle \pi([T']_\approx), \lhd|_{\pi([T']_\approx)} \rangle$ is a pruning of $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$. By Lem. 3.22, the mappings $\theta|_{[T]_\approx} : m^*/T \mapsto m^*$ and $\theta|_{[T']_\approx} : m^*/T' \mapsto m^*$ are order isomorphisms from $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$, respectively, onto prunings of $\mathcal{M} = \langle M, < \rangle$. Since $\mathsf{H}(T') \subseteq \mathsf{H}(T)$, the structure $\langle \theta([T']_\approx), \lhd|_{[T']_\approx} \rangle$ is a substructure of $\langle \theta([T]_\approx), \lhd|_{[T]_\approx} \rangle$. From this it follows by Lem. 3.9 that $\langle \theta([T']_\approx), \lhd|_{[T']_\approx} \rangle$ is a pruning of $\langle \theta([T]_\approx), \lhd|_{[T]_\approx} \rangle$. As $\theta(m^*/T) = \theta(m^*/T')$ for all $m^* \in \bigcup \mathsf{H}(T)$, the function π is just the composition $\theta|^{-1}_{[T]_\approx} \circ \theta|_{[T']_\approx}$, which implies that $\langle \pi([T']_\approx), \lhd|_{\pi([T']_\approx)} \rangle$ is a pruning of $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$. □

Note that $T \subsetneq T'$ implies $\mathsf{H}(T') \subsetneq \mathsf{H}(T)$. Consequently, whenever a transition set $T \in ts$ is a proper subset of a transition set $T' \in ts$, the BT structure $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ is order isomorphic to a proper pruning of the structure $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$. If, on the other hand, we have $T = T'$, then, in accordance with the reflexivity of the pruning relation, the BT structure $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ is simply a pruning of itself. An illustration of the result established in Prop. 3.28 is provided in Fig. 3.9, where the underlying BT structure is exactly the same one as in the previous figures considered in this section. The idea expressed in Prop. 3.28 also is apparent in Fig. 3.8.

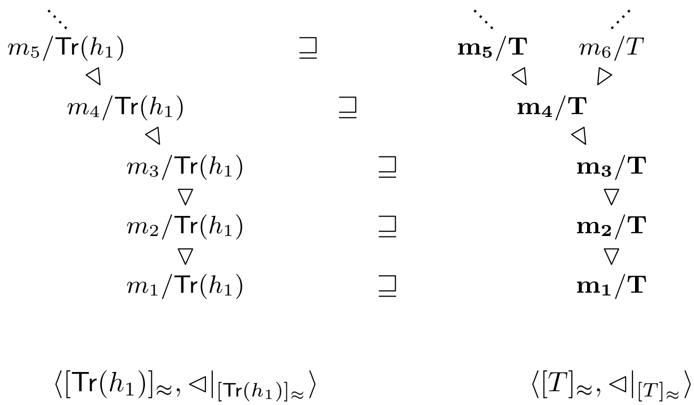


Figure 3.9: Order isomorphism from $\langle [\mathsf{Tr}(h_1)]_\approx, \lhd|_{[\mathsf{Tr}(h_1)]_\approx} \rangle$ onto a pruning of $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$.

Let us now consider two transition sets $T, T' \in ts$ such that $T \not\subseteq T'$ and $T \not\subseteq T'$. By Prop. 3.27 we again have $\sim \cap ([T]_\approx \times [T']_\approx) \neq \emptyset$. But now both the intersection $\sqsubseteq \cap ([T]_\approx \times [T']_\approx)$ as well as the intersection $\sqsupseteq \cap ([T]_\approx \times [T']_\approx)$ equal the empty set. This is to say that there is some $m/T \in [T]_\approx$ and some $m'/T' \in [T']_\approx$ such that $m/T \sim m'/T'$ while $m/T \not\sqsubseteq m'/T'$ and $m/T \not\sqsupseteq m'/T'$. We show that in this case, the relation $\sim \cap ([T]_\approx \times [T']_\approx)$ is the graph of an order isomorphism between proper trunks of the BT structures $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$, as indicated in Fig. 3.10.

First of all, given that $T \not\subseteq T'$ and $T' \not\subseteq T$, we have that $\mathsf{H}(T') \not\subseteq \mathsf{H}(T)$ and $\mathsf{H}(T) \not\subseteq \mathsf{H}(T')$. As a consequence, neither of the structures $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ can be embedded in the other, in such a way that corresponding elements are $\sim$-related. There is an element $m_0/T \in [T]_\approx$ that lacks a

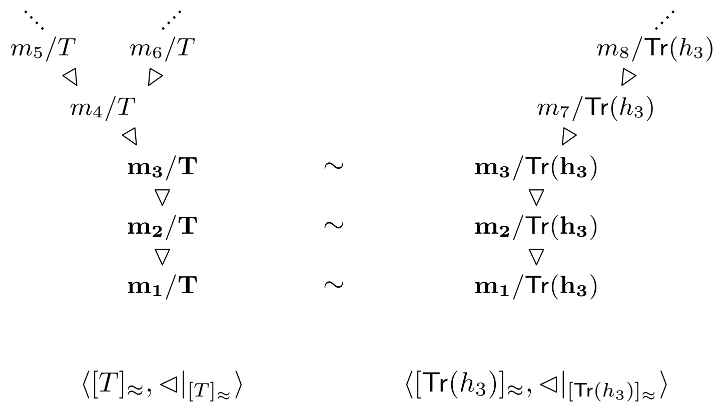


Figure 3.10: Order isomorphism between proper trunks of $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [\mathsf{Tr}(h_3)]_\approx, \lhd|_{[\mathsf{Tr}(h_3)]_\approx} \rangle$.

$\sim$-correspondent in $[T']_\approx$ so that $[m_0]_\sim \cap [T']_\approx = \emptyset$; and, conversely, there is an element $m_0' \in [T'] \approx$ that lacks a $\sim$-correspondent in $[T]_\approx$, i.e. $[m_0']_\sim \cap [T]_\approx \neq \emptyset$.

However, whenever two elements $m_1/T \in [T]_\approx$ and $m_1'/T' \in [T']_\approx$ are $\sim$-related, the relation $\sim$ induces a $\lhd$-preserving order isomorphism between their respective pasts: for every element $m_2/T \in [T]_\approx$ such that $m_2/T \unlhd m_1/T$, there is an element $m_2'/T' \in [T']_\approx$ such that $m_2/T \sim m_2'/T'$ and $m_2'/T \unlhd m_1'/T'$, and *vice versa*.

The order isomorphic pasts of a pair of $\sim$-related elements $m_1/T \in [T]_\approx$ and $m_1'/T' \in [T']_\approx$ thereby never contain an element that constitutes a branching point in the $\lhd$-orders on the respective $\approx$-equivalence classes. Rather, those pasts form part of the trunks of the corresponding structures $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$, and they are even proper subsets of those trunks. If one of the two $\approx$-equivalence classes, say $[T]_\approx$, contains two elements $m/T, m'/T \in [T]_\approx$ that are incomparable by $\lhd$, their greatest common lower $\lhd$-bound m''/T in $[T]_\approx$ cannot be $\sim$-related to some element in the other $\approx$-equivalence class. This requirement derives from the fact that by Lem. 3.22 both $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ are isomorphic to prunings of the original BT structure $\mathcal{M} = \langle M, < \rangle$; and so, every branch that has dropped off, must have dropped off from the trunk at some non-final point.

The upshot then is this: while in case $T \not\subseteq T'$ and $T' \not\subseteq T$, the relation $\sim$ does not induce an embedding between the structures $\langle [T]_\approx, \lhd|_{[T]_\approx} \rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx} \rangle$ either way, it defines an order isomorphism between proper subsets of their respective trunks. We summarize the relevant properties in Prop. 3.29 below. Note that Prop. 3.27 guarantees that the intersection $\sim \cap ([T]_\approx \times [T']_\approx)$ on the proper trunks of the structures $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx}\rangle$ always contains a greatest element.

Proposition 3.29. *For $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ a transition structure and m/T', $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ such that $m/T \sim m'/T'$, $m/T \not\trianglelefteq m'/T'$ and $m/T \not\trianglerighteq m'/T'$, we have the following:*

(a) *there is some $m_0/T \in [T]_\approx$ such that $[m_0]_\sim \cap [T']_\approx = \emptyset$, and there is some $m_0'/T' \in [T']_\approx$ such that $[m_0']_\sim \cap [T]_\approx = \emptyset$;*

(b) *for all $m_1/T \in [T]_\approx$ and for all $m_1'/T' \in [T']_\approx$ such that $m_1/T \sim m_1'/T'$, the relation $\sim \cap (\{m_2/T \in [T]_\approx \mid m_2/T \trianglelefteq m_1/T\} \times \{m_2'/T' \in [T']_\approx \mid m_2'/T' \trianglelefteq m_1'/T'\})$ is the graph of a $\lhd$-preserving order isomorphism from $\{m_2/T \in [T]_\approx \mid m_2/T \trianglelefteq m_1/T\}$ onto $\{m_2'/T' \in [T']_\approx \mid m_2'/T' \trianglelefteq m_1'/T'\}$;*

(c) *for all $m_3/T, m_4/T, m_5/T \in [T]_\approx$ such that $m_3/T \ntrianglelefteq m_4/T$, $m_4/T \ntrianglelefteq m_3/T$ and m_5/T is the greatest common lower $\lhd$-bound of m_3/T and m_4/T in $[T]_\approx$, we have $[m_5]_\sim \cap [T']_\approx = \emptyset$; and for all $m_3'/T', m_4'/T', m_5'/T' \in [T']_\approx$ such that $m_3'/T' \ntrianglelefteq m_4'/T'$, $m_4'/T' \ntrianglelefteq m_3'/T'$ and m_5'/T' is the greatest common lower $\lhd$-bound of m_3'/T' and m_4'/T' in $[T']_\approx$, we have $[m_5']_\sim \cap [T]_\approx = \emptyset$.*

Proof. Let $m/T, m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$ s.t. $m/T \sim m'/T'$, $m/T \not\trianglelefteq m'/T'$ and $m/T \not\trianglerighteq m'/T'$. Then $m = m', T \not\subseteq T'$ and $T' \not\subseteq T$.

(a) Since $T \not\subseteq T'$ and $T' \not\subseteq T$, we have $\mathsf{H}(T) \not\subseteq \mathsf{H}(T')$ and $\mathsf{H}(T') \not\subseteq \mathsf{H}(T)$. We can then consider some $m_0 \in \bigcup \mathsf{H}(T) \setminus \bigcup \mathsf{H}(T')$ and some $m_0' \in \bigcup \mathsf{H}(T') \setminus \bigcup \mathsf{H}(T)$. It follows that $m_0/T \in [T]_\approx$ and $m_0'/T' \in [T']_\approx$, and we have $[m_0]_\sim \cap [T']_\approx = \emptyset$ and $[m_0']_\sim \cap [T]_\approx = \emptyset$.

(b) Let $m_1/T \in [T]_\approx$ and $m_1'/T' \in [T']_\approx$ s.t. $m_1/T \sim m_1'/T'$. By Lem. 3.22 the mappings $\theta|_{[T]_\approx} : m^*/T \mapsto m^*$ and $\theta|_{[T']_\approx} : m^*/T' \mapsto m^*$ are order isomorphisms from $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx}\rangle$, respectively, onto prunings of $\mathcal{M} = \langle M, <\rangle$. We have $\theta(m_1/T) = m_1 = m_1' = \theta(m_1'/T')$ because $m_1/T \sim m_1'/T'$. Since by Lem. 3.4 (ii) prunings are downward closed, this implies that the restriction of the composition $\theta|_{[T']_\approx}^{-1} \circ \theta|_{[T]_\approx}$ to $\{m_2/T \in [T]_\approx \mid m_2/T \trianglelefteq m_1/T\}$ is a $\lhd$-preserving order isomorphism onto $\{m_2'/T' \in [T']_\approx \mid m_2'/T' \trianglelefteq m_1'/T'\}$ with graph $\sim \cap (\{m_2/T \in [T]_\approx \mid m_2/T \trianglelefteq m_1/T\} \times \{m_2'/T' \in [T']_\approx \mid m_2'/T' \trianglelefteq m_1'/T'\})$.

(c) Assume that there are $m_3/T, m_4/T \in [T]_\approx$ s.t. $m_3/T \ntrianglelefteq m_4/T$ and $m_4/T \ntrianglelefteq m_3/T$, and let m_5/T be the greatest common lower $\lhd$-bound

of m_3/T and m_4/T in $[T]_\approx$, which exists by Lem. 3.20. By (a) there is some $m_0/T \in [T]_\approx$ s.t. $[m_0]_\sim \cap [T']_\approx = \emptyset$ and there is some $m'_0/T' \in [T']_\approx$ s.t. $[m'_0]_\sim \cap [T]_\approx = \emptyset$. And, as said under (b), by Lem. 3.22 the mappings $\theta|_{[T]_\approx} : m^*/T \mapsto m^*$ and $\theta|_{[T']_\approx} : m^*/T' \mapsto m^*$ are order isomorphisms from $\langle [T]_\approx, \lhd|_{[T]_\approx}\rangle$ and $\langle [T']_\approx, \lhd|_{[T']_\approx}\rangle$, respectively, onto prunings of $\langle M, <\rangle$.
Assume for reductio that $[m_5]_\sim \cap [T']_\approx \neq \emptyset$. By Lem. 3.26 it follows that $[m_5]_\sim \cap [T']_\approx = \{m_5/T'\}$. We show that $m'_0/T' \ntrianglelefteq m_5/T'$ and $m'_0/T' \not\rhd m_5/T'$. If $m'_0/T' \trianglelefteq m_5/T'$, then by (b) it follows, contrary to our assumption, that $[m'_0]_\sim \cap [T]_\approx \neq \emptyset$ because $m_5/T \sim m_5/T'$. If, on the other hand, $m'_0/T' \rhd m_5/T'$, we also get $[m'_0]_\sim \cap [T]_\approx \neq \emptyset$ since by Def. 3.3 (ii) it holds that $M_{m_5} \subseteq \theta([T])_\approx = \bigcup \mathsf{H}(T)$ and we have $m_5 < m'_0$. Consequently, $m'_0/T' \ntrianglelefteq m_5/T'$ and $m'_0/T' \not\rhd m_5/T'$.
Let m_6/T' be the greatest common lower $\lhd$-bound of m'_0/T' and m_5/T' in $[T']_\approx$, which exists by Lem. 3.20. By (b) it follows that $m_6/T \in [T]_\approx$ because $m_5/T \sim m_5/T'$ and $m_6/T' \trianglelefteq m_5/T'$. We show that $m_0/T \ntrianglelefteq m_6/T$ and $m_0/T \not\rhd m_6/T$. If $m_0/T \trianglelefteq m_6/T$, then by (b) it follows, contrary to our pre-condition, that $[m_0]_\sim \cap [T']_\approx \neq \emptyset$ because $m_6/T \sim m_6/T'$. If, on the other hand, $m_0/T \rhd m_6/T$, we also get $[m_0]_\sim \cap [T']_\approx \neq \emptyset$ since by Def. 3.3 (ii) it holds that $M_{m_6} \subseteq \theta'([T'])_\approx = \bigcup \mathsf{H}(T')$ and we have $m_6 < m_0$. Consequently, $m_0/T \ntrianglelefteq m_6/T$ and $m_0/T \not\rhd m_6/T$.
Let m_7/T be the greatest common lower $\lhd$-bound of m_0/T and m_6/T in $[T]_\approx$, which exists by Lem. 3.20. Since by Def. 3.3 (ii) it holds that $M_{m_7} \subseteq \theta([T]_\approx) = \bigcup \mathsf{H}(T)$ and $m_7 \leq m_6 \leq m'_0$, it follows that $[m'_0]_\sim \cap [T]_\approx \neq \emptyset$, which contradicts our pre-condition.
The claim for $m'_3/T', m'_4/T', m'_5/T' \in [T']_\approx$ follows in the same way. □

3.3.2 Index Structures

In the previous section, our starting point was some arbitrary transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$. We have considered the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in that structure and investigated the properties of the structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd\ \sim, \sqsubseteq\rangle$ that can be defined on the set $\mathsf{Ind}(\mathcal{M}^{ts})$ of those indices. In this section, we put forth the definition of an *index structure*, which generalizes our findings in the previous section. Any arbitrary Kripke structure on a non-empty set W endowed with three accessibility relations, $\lhd$, $\sim$ and $\sqsubseteq$, that satisfies the relevant properties that we have discovered by investigating the structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd\ \sim, \sqsubseteq\rangle$, will be called an index structure. Whereas in section 3.3.1, we have derived the relations $\lhd$, $\sim$ and $\sqsubseteq$ on the set $\mathsf{Ind}(\mathcal{M}^{ts})$ and their properties from the relations that obtain among moments and transition sets in the underlying BT structure $\mathcal{M}$, we now impose those characteristics as conditions on the notion of an index structure. Naturally then, for every transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$, the corresponding structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd \sim, \sqsubseteq \rangle$ is a paradigm example of an index structure. It what follows, we will spell out the general definition of the notion of an index structure and discuss its implications. We conclude the section with a brief remark on a special kind of index structures, which allow for a simpler characterization.

3.3.2.1 The definition of an index structure

An *index structure* is defined as a Kripke structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ that is required to fulfill ten conditions, all of which find their analogues in the properties provided in Prop. 3.17 through Prop. 3.29 in section 3.3.1. We state the definition of the notion of an index structure outright and discuss the various conditions subsequently.

DEFINITION 3.30 (Index structure). *An* index structure *is an ordered quadruple $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ such that*

(i) $W \neq \emptyset$;

(ii) *the relation $\lhd$ is a strict partial order on W (i.e. an irreflexive, asymmetric and transitive relation) that is left-linear;*

(iii) *the relation $\lhd$ on W is serial;*

(iv) *the relation $\sim$ is an equivalence relation on W (i.e. a relation that is reflexive, symmetric and transitive).*
For all $w \in W$, let $[w]_\sim := \{w' \in W \mid w \sim w'\}$;

(v) *the relation $\sqsubseteq$ is a partial order on W (i.e. a reflexive, antisymmetric and transitive relation) that is left-linear;*

(vi) $\sqsubseteq \subseteq \sim$;

(vii) *a relation $\approx$ on W is defined along the following lines: for all $w, w' \in W$, we have $w \approx w'$ iff there is some $z \in W$ such that $w \unrhd z \unlhd w'$.*
For all $w \in W$, let $[w]_\approx := \{w' \in W \mid w \approx w'\}$;

(viii) *for all $w, w' \in W$, there is some $x \in [w]_\approx$ and there is some $x' \in [w']_\approx$ such that (i) $x \unlhd w$ and $x' \unlhd w'$, (ii) $x \sim x'$ and (iii) for all $y, y' \in W$ such that $y \unlhd w$, $y' \unlhd w'$ and $y \sim y'$, we have $y \unlhd x$ and $y' \unlhd x'$;*

(ix) *for all* $w, w' \in W$ *such that* $w \sqsupset w'$, *the relation* $\sqsupset \cap ([w]_{\approx} \times [w']_{\approx})$ *is the graph of an order isomorphism* τ *from* $\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle$ *onto a proper pruning of* $\langle [w']_{\approx}, \lhd|_{[w']_{\approx}} \rangle$;

(x) *for all* $w, w' \in W$ *such that there is some* $x \in [w]_{\approx}$ *and there is some* $x' \in [w']_{\approx}$ *such that* $x \sim x'$, $x \not\sqsubseteq x'$ *and* $x \not\sqsupseteq x'$, *we have the following:*

(a) *there is some* $x_0 \in [w]_{\approx}$ *such that* $[x_0]_{\sim} \cap [w']_{\approx} = \emptyset$, *and there is some* $x'_0 \in [w']_{\approx}$ *such that* $[x'_0]_{\sim} \cap [w]_{\approx} = \emptyset$;

(b) *for all* $x_1 \in [w]_{\approx}$ *and for all* $x'_1 \in [w']_{\approx}$ *such that* $x_1 \sim x'_1$, *the relation* $\sim \cap \{x_2 \in [w]_{\approx} \mid x_2 \unlhd x_1\} \times \{x'_2 \in [w']_{\approx} \mid x'_2 \unlhd x'_1\}$ *is the graph of a* $\lhd$*-preserving order isomorphism from* $\{x_2 \in [w]_{\approx} \mid x_2 \unlhd x_1\}$ *onto* $\{x'_2 \in [w']_{\approx} \mid x'_2 \unlhd x'_1\}$;

(c) *for all* $x_3, x_4, x_5 \in [w]_{\approx}$ *such that* $x_3 \not\unlhd x_4$, $x_4 \not\unlhd x_3$ *and* x_5 *is the greatest common lower* $\lhd$*-bound of* x_3 *and* x_4 *in* $[w]_{\approx}$, *we have* $[x_5]_{\sim} \cap [w']_{\approx} = \emptyset$; *and for all* $x'_3, x'_4, x'_5 \in [w']_{\approx}$ *such that* $x'_3 \not\unlhd x'_4$, $x'_4 \not\unlhd x'_3$ *and* x'_5 *is the greatest common lower* $\lhd$*-bound of* x'_3 *and* x'_4 *in* $[w']_{\approx}$, *we have* $[x'_5]_{\sim} \cap [w]_{\approx} = \emptyset$.

An index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ is a quadruple consisting of a set W and three primitive relations $\lhd$, $\sim$ and $\sqsubseteq$. Let us go over the ten conditions that we impose on that structure one by one.

To begin with, condition (i) places restrictions on the set W. It demands that the base set W be non-empty. And we have seen that the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts}$ also always contains at least on element (cf. Prop 3.17 (i)).

Conditions (ii) through (vi) concern the basic properties of the relations $\lhd$, $\sim$ and $\sqsubseteq$, and in (vii), an additional relation $\approx$ is introduced via definition. The conditions parallel our results pertaining to the corresponding relations defined on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts}$ that we have established in Prop. 3.17 (ii)-(vi) and Prop. 3.23. Condition (ii) requires the relation $\lhd$ to be a left-linear, strict partial order on W, and by condition (iii), the relation $\lhd$ must in addition be serial as well.[79] We use $w' \unlhd w$ in order to indicate that ($w' \lhd w$ or $w' = w$). Condition (iv) ensures that the relation $\sim$ is an equivalence relation on the set W. As usual, the equivalence class of an element $w \in W$ with respect to the relation $\sim$ is denoted by $[w]_{\sim}$.

[79]The reason for listing the seriality of the relation $\lhd$ as a separate condition stems from the fact that in the chronicle construction of a completeness proof finite index structures, which violate condition (iii), play an important role.

Conditions (v) and (vi) lay down that the relation $\sqsubseteq$ is a left-linear, partial order and forms a subset of the equivalence relation $\sim$. The elements of each $\sim$-equivalence class are thus internally ordered via the relation $\sqsubseteq$. We use $w' \sqsubset w$ as an abbreviation for ($w' \sqsubseteq w$ and $w' \neq w$). In (vii), a relation $\approx$ on the set W is defined in terms of the order relation $\lhd$ on W: two elements $w, w' \in W$ are said to be $\approx$-related if and only if they have a common lower $\lhd$-bound in W, which is to say that they need to be connected via some possible $\rhd\lhd$-path. We will see that the relation $\approx$ constitutes an equivalence relation on the set W (cf. Lem. 3.32), and we accordingly use $[w]_\approx$ to stand for the equivalence class of an element $w \in W$ with respect to the relation $\approx$. By the definition of the relation $\approx$, it is straightforward that the order relation $\lhd$ is a subset of the equivalence relation $\approx$, as noted before in Prop. 3.17 (vii). It will be shown to be provable from the the definition of an index structure that the order established by the relation $\lhd$ on an $\approx$-equivalence class $[w]_\approx \subseteq W$ yields a BT structure $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ on that set (cf. Cor. 3.35).

Conditions (viii) through (x) deal with the interrelation of the miscellaneous equivalence classes on the set W and their internal order relations. Condition (viii) is the counterpart of Prop. 3.27. It ensures that for all $w, w' \in W$, there is a $\lhd$-maximal pair of elements $x \in [w]_\approx$ and $x' \in [w']_\approx$ such that $w \unrhd x \sim x' \unlhd w'$. The condition thereby guarantees that for any two elements $w, w' \in W$, the intersection $\sim \cap ([w]_\approx \times [w']_\approx)$ is non-empty. Condition (ix) then concerns the case in which one of the intersections $\sqsupset \cap ([w]_\approx \times [w']_\approx)$ and $\sqsupset \cap ([w']_\approx \times [w]_\approx)$ is non-empty as well, while condition (x) covers the the case in which both those intersections equal the empty set. Condition (ix) gives expression to the idea put forth in Prop. 3.28. Whenever for any two elements $w, w' \in W$ it holds that $w \sqsupset w'$, then the relation $\sqsupset \cap ([w]_\approx \times [w']_\approx)$ is left-total and constitutes the graph of an order isomorphism from $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ onto a proper pruning of $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$. Obviously, in case $w = w'$, the structure $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ is a pruning of itself. Hence, for all $w, w' \in W$ with $w \sqsupseteq w'$, it holds that the structure $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ is order isomorphic to a pruning of $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$. Condition (x) finally constitutes the analogue of Prop. 3.29. It ensures that whenever both $\sqsupset \cap ([w]_\approx \times [w']_\approx) \neq \emptyset$ and $\sqsupset \cap ([w']_\approx \times [w]_\approx) \neq \emptyset$, the relation $\sim \cap ([w]_\approx \times [w']_\approx)$ induces an order isomorphism between proper subsets of the trunks of the structures $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ and $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$.

It is worthwhile to note that the notion of an index structure is first-order definable because the notion of a pruning employed in condition (ix) is so (cf.

section 3.2.1). And obviously, all other conditions straightforwardly translate into first-order conditions as well.

Since we have imposed precisely those conditions on the notion of an index structure that we have discovered by investigating the properties of the structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq \rangle$ defined on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts}$, it is straightforward that for any arbitrary transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$, the corresponding structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq \rangle$ qualifies an index structure according to Def. 3.30. The following theorem captures that result.

Theorem 3.31. *For $\mathcal{M}^{ts} = \langle M, <, ts \rangle$ a transition structure, the structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq \rangle$ is an index structure and for all indices of evaluation m/T, $m'/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$, the following holds:*

($\lhd$) $m/T \lhd m'/T'$ *iff* $T = T'$ *and* $m < m'$;

($\sim$) $m/T \sim m'/T'$ *iff* $m = m'$;

($\sqsubseteq$) $m/T \sqsubseteq m'/T'$ *iff* $m = m'$ *and* $T \subseteq T'$.

Proof. The structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq \rangle$ fulfills the conditions (i)-(x) of Def. 3.30: conditions (i)-(vi) are ensured by Prop. 3.17, condition (vii) is guaranteed by Prop. 3.23, condition (viii) by Prop. 3.27, condition (ix) by Prop. 3.28 and condition (x) by Prop. 3.29. The properties ($\lhd$), ($\sim$) and ($\sqsubseteq$) are consequences of Def. 3.16. □

3.3.2.2 Some implications of the definition

In this section, we will mention some implications that flow from the definition of an index structure. Some of those implications have already been alluded to in section 3.3.2.1 above, and all them are already familiar from our investigations in section 3.3.1. Most importantly, we will show that the relation $\approx$ on the base set W of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ is in fact an equivalence relation that includes the order relation $\lhd$ and that the partitions $W/\sim$ and $W/\approx$ are orthogonal to each other. We can then show that for every equivalence class $[w]_\approx \in W/\approx$, the structure $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ constitutes a BT structure. Moreover, we will discuss some closure properties of index structures.

Let us first focus on the properties of the defined relation $\approx$ on the base set W of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$. By condition (vii) of Def. 3.30, being $\approx$-related is equivalent to having a common lower $\lhd$-bound. While the reflexivity as well as the symmetry of the relation $\approx$ are immediate consequences of the definition itself, the transitivity of the relation $\approx$ follows from the left-linearity of the order relation $\lhd$, in terms of which it is defined. The relation $\approx$ then constitutes an equivalence relation on the set W, and obviously, the order relation $\lhd$ is a subset of the relation $\approx$, just as the order relation $\sqsubseteq$ is a subset of the equivalence relation $\sim$. The properties of the relation $\approx$ in an index structure then perfectly match the properties of that relation that we have encountered by considering the structures on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in a transition structure $\mathcal{M}^{ts}$ in the previous section (cf. Prop. 3.17 (v) and (vi)).

Lemma 3.32. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ be an index structure. The relation $\approx$ on W is an equivalence relation (i.e. a relation that is reflexive, symmetric and transitive), and we have $\lhd \ \subseteq \approx$.*

Proof. Obviously, the relation $\approx$ on W defined in Def. 3.30 (vii) is reflexive and symmetric. The transitivity of the relation $\approx$ follows from the left-linearity of the relation $\lhd$ (cf. Def. 3.30 (ii)). It is straightforward by definition that $\lhd \ \subseteq \approx$. □

We then have two different equivalence relations on the base set W of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, viz. $\sim$ and $\approx$. And each of those equivalence relations provides a partition on the set W. We show that the partitions $W/\!\sim$ and $W/\approx$ induced by the equivalence relations $\sim$ and $\approx$ are orthogonal to each other: the intersection of any two cells taken from different partitions is either empty or contains but a single element. Two different elements that are $\approx$-related cannot at the same time be $\sim$-related, and *vice versa*. In particular, for all $w \in W$, we have $[w]_\sim \cap [w]_\approx = \{w\}$. Lem. 3.33 then is the perfect analogue of Lem. 3.26.

Lemma 3.33. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $w, w' \in W$, we have either $[w]_\sim \cap [w']_\approx = \emptyset$ or $[w]_\sim \cap [w']_\approx = \{v\}$ for some $v \in W$.*

Proof. Assume that $[w]_\sim \cap [w']_\approx \neq \emptyset$. Then there exists some $v \in W$ s.t. $v \in [w]_\sim \cap [w']_\approx$. Assume for reductio that $[w]_\sim \cap [w']_\approx \supseteq \{v, u\}$. It follows that $v \sim u$ and hence we have either (i) $v \sqsupseteq u$, (ii) $u \sqsupseteq v$ or (iii) $v \not\sqsupseteq u$ and $u \not\sqsupseteq v$.
If (i) $v \sqsupseteq u$, then by Def. 3.30 (ix) the relation $\sqsupseteq \cap ([v]_\approx \times [u]_\approx)$ is the graph of an order isomorphism τ from $\langle [v]_\approx, \lhd|_{[v]_\approx} \rangle$ onto a pruning of $\langle [u]_\approx, \lhd|_{[u]_\approx} \rangle$. Since $u, v \in [w']_\approx$, we have $[u]_\approx = [v]_\approx$ and hence τ is the identity. Consequently, $v \sqsupseteq u$ implies $v = u$. The assumption (ii) $u \sqsupseteq v$ yields the same result.

Now assume that (iii) $v \not\trianglerighteq u$ and $u \not\trianglerighteq v$. Since $u, v \in [w']_{\approx}$, we have $[u]_{\approx} = [v]_{\approx}$, which immediately contradicts Def. 3.30 (x.a). □

The relation $\lhd \subseteq \approx$ induces an ordering on each of the $\approx$-equivalence classes $[w]_{\approx} \in W/\approx$. On the basis of Lem. 3.33, we can show that for every equivalence class $[w]_{\approx} \in W/\approx$, the resulting structure $\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle$ is a BT structure according to Def. 1.1, which parallels our result in Cor. 3.21. To this end, we only have to prove that in a given index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, the restriction of the relation $\lhd$ to an $\approx$-equivalence class $[w]_{\approx} \in W/\approx$ is jointed in each case. For, by Def. 3.30 (ii) and (iii), it is already guaranteed that the ordering established by the relation $\lhd$ on an $\approx$-equivalence class is a strict partial order that is left-linear and serial, and obviously, a class $[w]_{\approx} \in W/\approx$ is never empty.

The jointedness of the relation $\lhd|_{[w]_{\approx}}$ on an $\approx$-equivalence class $[w]_{\approx} \in W/\approx$ is a consequence of condition (viii) of Def. 3.30. As a special case, the condition guarantees in combination with Lem. 3.33 that any two elements contained within the same $\approx$-equivalence class have a greatest common lower $\lhd$-bound in that class: for every $\approx$-equivalence class $[w]_{\approx} \in W/\approx$ and all $w', w'' \in [w]_{\approx}$, there exists by Def. 3.30 (viii) a maximal pair of elements $x', x'' \in [w]_{\approx}$ such that $w' \trianglerighteq x' \sim x'' \trianglelefteq w''$, and Lem. 3.33 implies that $x' = x''$.

Lemma 3.34. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $w \in W$, the relation $\lhd|_{[w]_{\approx}}$ on $[w]_{\approx} \subseteq W$ is jointed (i.e. for all $x, y \in [w]_{\approx}$, there is some $z \in [w]_{\approx}$ such that $x \trianglerighteq z \trianglelefteq y$ and for all $z' \in [w]_{\approx}$ such that $x \trianglerighteq z' \trianglelefteq y$, we have $z' \trianglelefteq z$).*

Proof. Follows from Def. 3.30 (vii) and Lem. 3.33. □

We then have as a corollary that for every $w \in W$, the structure $\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle$ is a BT structure.

Corollary 3.35. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $w \in W$, the structure $\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle$ is a BT structure.*

Proof. Follows from Def. 3.30 (ii)-(iii) and Lem. 3.34. □

Whereas the restriction of the order relation $\lhd \subseteq \approx$ to an $\approx$-equivalence class $[w]_{\approx} \in W/\approx$ is always jointed and hence also connected, the left-linear, partial order established by the relation $\sqsubseteq$ on each of the $\sim$-equivalence classes $[w]_{\sim} \in W/\sim$ does in general not have to be connected. Think of the index structures corresponding to Ockhamist transition structures. In section 3.3.2.3, we will see that if, in a given index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, the restriction of the relation $\sqsubseteq$ to an $\sim$-equivalence class is connected in each case, the conditions imposed on an index structure are no longer entirely independent. But before we move there, let us have a look at the closure properties of an index structure.

An index structure $\mathcal{W} = \langle W, \lhd, \sim, \approx \rangle$ can be said to be closed in the following respects. First, whenever for any two elements $w, w' \in W$, we have $w' \lhd w$, then for all $x \in [w]_\sim$, there is some $x' \in [w']_\sim$ such that $x' \lhd x$. To put it differently, the existence of a single pair of elements $w, w' \in W$ such that $w' \lhd w$ guarantees that the relation $\lhd \cap ([w']_\sim \times [w]_\sim)$ is right-total. Likewise, the existence of a single pair of elements $w, w' \in W$ such that $w' \sqsubseteq w$ suffices for the relation $\sqsubseteq \cap ([w']_\approx \times [w]_\approx)$ to be right-total. If for any two elements $w, w' \in W$, it holds that $w' \sqsubseteq w$, then for all $x \in [w]_\approx$, there is some $x' \in [w']_\approx$ such that $x' \sqsubseteq x$. Lem. 3.36 is the counterpart of Lem. 3.18 established in the previous section.

Lemma 3.36. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $w, w' \in W$, the following holds:*

(i) *if $w' \lhd w$, then for all $x \in [w]_\sim$, there is some $x' \in [w']_\sim$ such that $x' \lhd x$;*

(ii) *if $w' \sqsubseteq w$, then for all $x \in [w]_\approx$, there is some $x' \in [w']_\approx$ such that $x' \sqsubseteq x$.*

Proof.

(i) Let $w' \lhd w$ and let $x \in [w]_\sim$. Then we have either (1) $w \sqsupseteq x$, (2) $x \sqsupseteq w$ or (3) $w \not\sqsupseteq x$ and $x \not\sqsupseteq w$.
If (1) $w \sqsupseteq x$, then by Def. 3.30 (ix) the relation $\sqsupseteq \cap ([w]_\approx \times [x]_\approx)$ is the graph of an order isomorphism τ from $\langle [w]_\approx, \lhd | [w]_\approx \rangle$ onto a pruning of $\langle [x]_\approx, \lhd|_{[x]_\approx} \rangle$. It follows that there is some $x' \in [x]_\approx$ s.t. $x' = \tau(w')$ and $x' \lhd x$, and by Def. 3.30 (vi) it holds that $x' \in [w']_\sim$.
If (2) $x \sqsupseteq w$, then by Def. 3.30 (ix) the relation $\sqsupseteq \cap ([x]_\approx \times [w]_\approx)$ is the graph of an order isomorphism τ from $\langle [x]_\approx, \lhd|_{[x]_\approx} \rangle$ onto a pruning of $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$. Since by Lem. 3.4 (ii) prunings are downward closed and $w \in \tau([x]_\approx)$, there is some $x' \in [x]_\approx$ s.t. $\tau(x') = w'$ and $x' \lhd x$, and by Def. 3.30 (vi) it holds that $x' \in [w']_\sim$.
If (3) $w \not\sqsupseteq x$ and $x \not\sqsupseteq w$, then by Def. 3.30 (x.b) it follows that the relation $\sim \cap (\{v \in [w]_\approx \mid v \unlhd w\} \times \{v' \in [x]_\approx \mid v' \unlhd x\})$ is the graph of a $\lhd$-preserving order isomorphism from $\{v \in [w]_\approx \mid v \unlhd w\}$ onto $\{v' \in [x]_\approx \mid v' \unlhd x\}$. This implies that there is some $x' \in [x]_\approx$ s.t. $w' \sim x'$ and $x' \lhd x$.

(ii) Let $w' \sqsubseteq w$. Then by Def. 3.30 (ix) the relation $\sqsupseteq \cap ([w]_{\approx} \times [w']_{\approx})$ is the graph of an order isomorphism τ from $\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle$ onto a pruning of $\langle [w']_{\approx}, \lhd|_{[w']_{\approx}} \rangle$. Hence, for all $x \in [w]_{\approx}$, there is some $x' \in [w']_{\approx}$ s.t. $x' = \tau(x)$ and hence $x' \sqsubseteq x$. □

3.3.2.3 A possible simplification of the definition

Some final remark is in order here that concerns the independence of the conditions stated in the definition of an index structure in a special case. As said, given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, it does not necessarily have to be the case that the restriction of the order relation $\sqsubseteq$ to an $\sim$-equivalence class $[w]_{\sim} \in W/\sim$ is always connected. We show that if in a given index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, for every $\sim$-equivalence class $[w]_{\sim} \in W/\sim$, the relation $\sqsubseteq|_{[w]_{\sim}}$ actually is connected, then conditions (x.b) and (x.c) of Def. 3.30 can be derived from the remainder, viz. from conditions (i)-(x.a).

Note that, by Def. 3.30 (ix), the internal order of an $\sim$-equivalence class via the relation $\sqsubseteq$ induces an ordering among the $\lhd$-structures on the orthogonal $\approx$-equivalence classes: whenever any two $\sqsubseteq$-incomparable elements x and x' of an $\sim$-equivalence class $[w]_{\sim} \in W/\sim$ have a common lower $\sqsubseteq$-bound x'' in $[w]_{\sim}$, both $\langle [x]_{\approx}, \lhd|_{[x]_{\approx}} \rangle$ and $\langle [x']_{\approx}, \lhd|_{[x']_{\approx}} \rangle$ are order isomorphic to prunings of the BT structure $\langle [x'']_{\approx}, \lhd|_{[x'']_{\approx}} \rangle$. Under the assumption that neither of the structures $\langle [x]_{\approx}, \lhd|_{[x]_{\approx}} \rangle$ and $\langle [x']_{\approx}, \lhd|_{[x']_{\approx}} \rangle$ can be embedded in the other, for both to be order isomorphic to prunings of the structure $\langle [x'']_{\approx}, \lhd|_{[x'']_{\approx}} \rangle$, the relation $\sim \cap ([x]_{\approx} \times [x']_{\approx})$ must induce an order isomorphism between proper subsets of their trunks, as we have already seen in Prop. 3.29 above.[80]

Lemma 3.37. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ be an index structure such that for all $v \in W$, it holds that the relation $\sqsubseteq|_{[v]_{\sim}}$ on $[v]_{\sim}$ is connected (i.e. for all $x, y \in [v]_{\sim}$, there is some $z \in [v]_{\sim}$ such that $x \sqsupseteq z \sqsubseteq y$). Then conditions (x.b) and (x.c) of Def. 3.30 are consequences of conditions (i)-(x.a) of Def. 3.30.*

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[80] It is worthwhile to point out that condition (x.a) of Def. 3.30 cannot likewise be deduced from the fact that the relation $\sqsubseteq|_{[w]_{\sim}}$ on each $\sim$-equivalence class $[w]_{\sim} \in W/\sim$ is connected. Consider again a pair of elements $x, x' \in [w]_{\sim}$ that are incomparable by $\sqsubseteq$ and let $x'' \in [w]_{\sim}$ be a proper common lower $\sqsubseteq$-bound of x and x'. Assume for reductio that the relation $\sim \cap ([x]_{\approx} \times [x']_{\approx})$ is left-total. It then indeed follows by Def. 3.30 (ix) and Lem. 3.9 that the structure $\langle [x]_{\approx}, \lhd|_{[x]_{\approx}} \rangle$ is order isomorphic to a pruning of $\langle [x']_{\approx}, \lhd|_{[x']_{\approx}} \rangle$. Yet, from there we cannot infer that $x \sqsupseteq x'$. Condition (x.a) is needed in order to rule out such situations in which a structure $\langle [x]_{\approx}, \lhd|_{[x]_{\approx}} \rangle$ is order isomorphic to a pruning of $\langle [x']_{\approx}, \lhd|_{[x']_{\approx}} \rangle$ although $\sqsupseteq \cap ([x]_{\approx} \times [x']_{\approx}) = \emptyset$.

Proof. Assume that the structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ satisfies conditions (i)-(x.a) of Def. 3.30. Assume moreover that for every $v \in W$, it holds that the relation $\sqsubseteq|_{[v]_\sim}$ on $[v]_\sim$ is connected. We show that conditions (b) and (c) of Def. 3.30 (x) are fulfilled as well. Note that since the proofs of Lem. 3.33 and Lem. 3.34 do not rely on the conditions (x.b) and (x.c) of Def. 3.30, we can use those lemmas freely in what follows.

Let $w, w' \in W$ s.t. there is some $x \in [w]_\approx$ and there is some $x' \in [w']_\approx$ s.t. $x \sim x'$, $x \not\sqsubseteq x'$ and $x' \not\sqsubseteq x$. Since by assumption $[x]_\sim$ is connected, there is some $x'' \in [x]_\sim$ s.t. $x \sqsupset x'' \sqsubset x'$. By Def. 3.30 (ix) it follows that the relation $\sqsupset \cap ([w]_\approx \times [x'']_\approx)$ is the graph of an order isomorphism τ from $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ onto a proper pruning of $\langle [x'']_\approx, \lhd|_{[x'']_\approx} \rangle$; and likewise, the relation $\sqsupset \cap ([w']_\approx \times [x'']_\approx)$ is the graph of an order isomorphism τ' from $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$ onto a proper pruning of $\langle [x'']_\approx, \lhd|_{[x'']_\approx} \rangle$.

Condition (x.b): Let $x_1 \in [w]_\approx$ and $x'_1 \in [w']_\approx$ s.t. $x_1 \sim x'_1$. Then $\tau(x_1)$, $\tau'(x'_1) \in [x'']_\approx$ with $x_1 \sqsupset \tau(x_1)$ and $x'_1 \sqsupset \tau'(x'_1)$ and hence by Def. 3.30 (iv) and (vi) we have that $\tau(x_1) \sim \tau'(x'_1)$. By Lem. 3.33 it follows that $\tau(x_1) = \tau'(x'_1)$. Since by Lem. 3.4 (ii) prunings are downward closed, this implies that the restriction of the composition $\tau'^{-1} \circ \tau$ to the set $\{x_2 \in [w]_\approx \mid x_2 \unlhd x_1\}$ is a $\lhd$-preserving order isomorphism onto $\{x'_2 \in [w']_\approx \mid x'_2 \unlhd x'_1\}$ with graph $\sim \cap \{x_2 \in [w]_\approx \mid x_2 \unlhd x_1\} \times \{x'_2 \in [w']_\approx \mid x'_2 \unlhd x'_1\}$.

Condition (x.c): Let $x_3, x_4, x_5 \in [w]_\approx$ s.t. $x_3 \ntrianglelefteq x_4$, $x_4 \ntrianglelefteq x_3$ and let x_5 be the greatest common lower $\lhd$-bound of x_3 and x_4 in $[w]_\approx$, which exists by Lem. 3.34. By Def. 3.30 (x.a) there is some $x_0 \in [w]_\approx$ s.t. $[x_0]_\sim \cap [w']_\approx = \emptyset$, and there is some $x'_0 \in [w']_\approx$ s.t. $[x'_0]_\sim \cap [w]_\approx = \emptyset$.

Assume for reductio that $[x_5]_\sim \cap [w']_\approx \neq \emptyset$. By Lem. 3.33 it follows that $[x_5]_\sim \cap [w']_\approx = \{x'_5\}$ for some $x'_5 \in [w']_\approx$. We show that $x'_0 \ntrianglelefteq x'_5$ and $x'_0 \ntriangleright x'_5$. If $x'_0 \unlhd x'_5$, then by (b) it follows, contrary to our assumption, that $[x'_0]_\sim \cap [w]_\approx \neq \emptyset$ because $x_5 \sim x'_5$. If, on the other hand, $x'_0 \rhd x'_5$, we also get $[x'_0]_\sim \cap [w]_\approx \neq \emptyset$ since by Def. 3.3 (ii) for all $w^* \in [x'']_\approx$ s.t. $w^* \unrhd \tau(x_5)$, it holds that $w^* \in \tau([w]_\approx)$ and we have $\tau'(x'_0) \rhd \tau'(x'_5) = \tau(x_5)$. Consequently, $x'_0 \ntrianglelefteq x'_5$ and $x'_0 \ntriangleright x'_5$.

Let x'_6 be the greatest common lower $\lhd$-bound of x'_0 and x'_5 in $[w']_\approx$, which exists by Lem. 3.34. By (b) it follows that $[x'_6]_\sim \cap [w]_\approx \neq \emptyset$ because $x_5 \sim x'_5$ and $x'_6 \unlhd x'_5$. Then by Lem. 3.33 it holds that $[x'_6]_\sim \cap [w]_\approx = \{x_6\}$ for some $x_6 \in [w]_\approx$. We show that $x_0 \ntrianglelefteq x_6$ and $x_0 \ntriangleright x_6$. If $x_0 \unlhd x_6$, by (b) it follows, contrary to our pre-condition, that $[x_0]_\sim \cap [w']_\approx \neq \emptyset$ because $x_6 \sim x'_6$. If, on the other hand, $x_0 \rhd x_6$, we also get $[x_0]_\sim \cap [w']_\approx \neq \emptyset$ since by Def. 3.3 (ii) for all $w^* \in [x'']_\approx$ s.t. $w^* \unrhd \tau'(x'_6)$, it holds that $w^* \in \tau'([w']_\approx)$ and we have $\tau(x_0) \rhd \tau(x_6) = \tau'(x'_6)$. Consequently, $x_0 \ntrianglelefteq x_6$ and $x_0 \ntriangleright x_6$.

Let x_7 be the greatest common lower $\lhd$-bound of x_0 and x_6 in $[w]_\approx$, which exists by Lem. 3.34. Since by Def. 3.3 for all $w^* \in [x'']_\approx$ s.t. $w^* \unrhd \tau(x_7)$, we have $w^* \in \tau([w]_\approx)$ and $\tau'(x'_0) \unrhd \tau'(x'_6) = \tau(x_6) \unrhd \tau(x_7)$, it follows that $[x'_0]_\sim \cap [w]_\approx \neq \emptyset$, which contradicts our pre-condition. □

3.3.3 From Index Structures to Transition Structures

In section 3.3.1, we have made a transition from transition structures to index structures, which has culminated in the definition of an index structure in section 3.3.2 above. By imposing precisely those conditions on the notion of an index structure that we discovered by investigating the properties of the structures definable on the set of indices of evaluation in a transition structure, we have established a correspondence between transition structures and index structures: for every transition structure $\mathcal{M}^{ts} = \langle M, <, ts \rangle$, the corresponding structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq \rangle$ is an index structure (cf. Theorem 3.31). In this section, we make a transition back from index structures to transition structures. That is, we show that to every index structure, there corresponds again a transition structure.

We first show that, given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, we can reconstruct a set of moments and a temporal ordering among those moments from the set $W/\sim$, such that the resulting structure is a BT structure (section 3.3.3.1). In a second step, we then illustrate that the quotient set $W/\approx$ determines a set of transition sets in that BT structure (section 3.3.3.2). In section 3.3.3.3, we finally prove that the BT structure based on the set $W/\sim$ together with the set of transition sets extracted from the set $W/\approx$ is in fact a transition structure, whose indices of evaluation correspond one-to-one to the elements of W.

3.3.3.1 Lifting the set of moments

In this section, we consider the quotient set $W/\sim$ of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$. We define a relation $\ll$ on $W/\sim$ in terms of the order relation $\lhd$ on W and show that the structure $\mathcal{T} = \langle W/\sim, \ll \rangle$ that is lifted in this way is a BT structure. That is, the set $W/\sim$ provides a set of moments, and the defined relation $\ll$ induces a temporal earlier-later relation among those moments.

Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, the relation $\sim$ on W is an equivalence relation, and as such it induces a partition $W/\sim$ of the set W. We derive a relation $\ll$ on the set $W/\sim$ from the relation $\lhd \ \subseteq \approx$ on W, which provides an internal ordering on each of the orthogonal $\approx$-equivalence classes in W. The relation $\ll$ is defined as follows: for all $w, w' \in W$, we set $[w]_\sim \ll [w']_\sim$ if and only if there is some $x \in [w]_\sim$ and some $x' \in [w']_\approx$ such that $x \lhd x'$.

DEFINITION 3.38. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, we define a relation $\ll$ on $W/\sim$ along the following lines: for all $w, w' \in W$, let $[w]_\sim \ll [w']_\sim$ iff there is some $x \in [w]_\sim$ and some $x' \in [w']_\sim$ such that $x \lhd x'$.*

By Lem. 3.36 (i) it holds that whenever there exist $x \in [w]_\sim$ and $x' \in [w']_\sim$ such that $x \lhd x'$, the relation $\lhd \cap ([w]_\sim \times [w']_\sim)$ is right-total. The purely existential condition provided in Def. 3.38 is thus equivalent to the requirement that, in order for $[w]_\sim \ll [w']_\sim$ to hold, for every $x' \in [w']_\sim$, there be some $x \in [w]_\sim$ such that $x \lhd x'$.

LEMMA 3.39. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $w, w' \in W$, we have $[w]_\sim \ll [w']_\sim$ iff for all $x' \in [w']_\sim$, there is some $x \in [w]_\sim$ such that $x \lhd x'$.*

Proof. Follows from Def. 3.38 and Lem. 3.36 (i). □

Having defined a relation $\ll$ on the set $W/\sim$, we now show step by step that the resulting structure $\mathcal{T} = \langle W/\sim, \ll \rangle$ qualifies as a BT structure according to Def. 1.1. In Prop. 3.40 below, we prove, first of all, that the relation $\ll$ on $W/\sim$ is a left-linear, strict partial order, establishing its jointedness and seriality shortly after. The irreflexivity, transitivity and left-linearity of the relation $\ll$ on $W/\sim$ follow from the corresponding properties of the relation $\lhd$ on W on the basis of Def. 3.38 and Lem. 3.39, respectively. Being irreflexive and transitive, the relation $\ll$ it obviously asymmetric as well and hence a strict partial order. We use $[w]_\sim \lessapprox [w']_\sim$ to stand for ($[w]_\sim \ll [w']_\sim$ or $[w]_\sim = [w']_\sim$).

PROPOSITION 3.40. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, the relation $\ll$ is a strict partial order on $W/\sim$ (i.e. an irreflexive, asymmetric and transitive relation) that is left-linear.*

Proof. We first show that the relation $\ll$ is irreflexive. Assume for reductio that there is some $[w]_\sim \in W/\sim$ s.t. $[w]_\sim \ll [w]_\sim$. By Def. 3.38 it follows that there exist $x, y \in [w]_\sim$ s.t. $x \lhd y$. By Lem. 3.33 this implies that $x = y$, which contradicts the irreflexivity of $\lhd$ (Def. 3.30 (ii)).
We now prove that the relation $\ll$ is transitive. Let $[w]_\sim, [w']_\sim, [w'']_\sim \in W/\sim$ s.t. $[w]_\sim \ll [w']_\sim$ and $[w']_\sim \ll [w'']_\sim$. Consider some arbitrary $x'' \in [w'']_\sim$. By Lem. 3.39 it holds that there is some $x' \in [w']_\sim$ s.t. $x' \lhd x''$ and that there is again some $x \in [w]_\sim$ s.t. $x \lhd x'$. Since $\lhd$ is transitive (Def. 3.30 (ii)), it follows that $x \lhd x''$, which implies by Def. 3.38 that $[w]_\sim \ll [w'']_\sim$.
Being irreflexive and transitive, the relation $\ll$ on $W/\sim$ is asymmetric as well: assume that there are $[w]_\sim, [w']_\sim \in W/\sim$ s.t. $[w]_\sim \ll [w']_\sim$ and $[w']_\sim \ll [w]_\sim$.

By the transitivity of $\ll$ it follows that $[w]_\sim \ll [w]_\sim$, which contradicts the irreflexivity of $\ll$.
We finally show that the relation $\ll$ is left-linear. Let $[w]_\sim, [w']_\sim, [w'']_\sim \in W/\!\sim$ s.t. $[w']_\sim \ll [w]_\sim$ and $[w'']_\sim \ll [w]_\sim$. Consider some arbitrary $x \in [w]_\sim$. By Lem. 3.39 it follows that there is some $x' \in [w']_\sim$ s.t. $x' \lhd x$ and likewise that there is some $x'' \in [w'']_\sim$ s.t. $x'' \lhd x$. This implies by the left-linearity of $\lhd$ (Def. 3.30 (ii)) that $x' \unlhd x''$ or $x'' \unlhd x'$, and hence by Def. 3.38 it follows that we have either $[w']_\sim \leqq [w'']_\sim$ or $[w'']_\sim \leqq [w']_\sim$. □

Moreover, the relation $\ll$ on $W/\!\sim$ is jointed, as required by the definition of a BT structure: any two equivalence classes $[w]_\sim, [w']_\sim \in W/\!\sim$ have a greatest common lower $\ll$-bound in $W/\!\sim$. The jointedness of the relation $\ll$ on $W/\!\sim$ is a consequence of condition (viii) of the definition of an index structure (Def. 3.30). The greatest common lower $\ll$-bound of $[w]_\sim$ and $[w']_\sim$ in $W/\!\sim$ is provided by the $\sim$-equivalence class of either element of the $\lhd$-maximal pair $x, x' \in W$ such that $w \unrhd x \sim x' \unlhd w'$.

Proposition 3.41. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure, the relation $\ll$ on $W/\!\sim$ is jointed (i.e. for all $[w]_\sim, [w']_\sim \in W/\!\sim$, there is some $[v]_\sim$ such that $[w]_\sim \geqq [v]_\sim \leqq [w']_\sim$ and for all $[v']_\sim \in W/\!\sim$ such that $[w]_\sim \geqq [v']_\sim \leqq [w']_\sim$, we have $[v']_\sim \leqq [v]_\sim$.)*

Proof. Let $[w]_\sim, [w']_\sim \in W/\!\sim$. By Def. 3.30 (viii) there is some $x \in [w]_\approx$ and some $x' \in [w]_\approx$ s.t. $x \unlhd w$, $x' \unlhd w'$, $x \sim x'$ and for all $y, y' \in W$ s.t. $y \unlhd w$, $y' \unlhd w'$ and $y \sim y'$, we have $y \unlhd x$ and $y' \unlhd x'$. By Def. 3.38 it follows that $[w]_\sim \geqq [x]_\sim = [x']_\sim \leqq [w']_\sim$. Since for all $y, y' \in W$ s.t. $y \unlhd w$, $y' \unlhd w'$ and $y \sim y'$, it holds that $y \unlhd x$ and $y' \unlhd x'$, by Lem. 3.39 it follows that $[x]_\sim$ is the greatest common lower $\ll$-bound of $[w]_\sim$ and $[w']_\sim$ in $W/\!\sim$. □

Being jointed, the relation $\ll$ on $W/\!\sim$ is obviously also connected. We finally show that it is serial as well, just as the underlying relation $\lhd$ on W is: there are no maximal elements in the temporal ordering $\ll$ on $W/\!\sim$.

Proposition 3.42. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure, the relation $\ll$ on $W/\!\sim$ is serial (i.e. for all $[w]_\sim \in W/\!\sim$, there is some $[w']_\sim \in W/\!\sim$ such that $[w]_\sim \ll [w']_\sim$).*

Proof. Let $[w]_\sim \in W/\!\sim$. From the seriality of $\lhd$ (Def. 3.30 (iii)) it follows that there is some $w' \in W$ s.t. $w \lhd w'$, which implies by Def. 3.38 that $[w]_\sim \ll [w']_\sim$. □

From Prop. 3.40, Prop. 3.41 and Prop. 3.42 it follows immediately that the structure $\mathcal{T} = \langle W/\!\sim, \ll\rangle$ is a BT structure according to Def. 1.1. Obviously, the set $W/\sim$ is non-empty, and the relation $\lll$ on $W/\sim$ defined in Def. 3.38 has been shown to be a left-linear, strict partial order that is jointed and serial.

Theorem 3.43. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, the structure $\mathcal{T} = \langle W/\sim, \lll \rangle$ is a BT structure.*

Proof. By Def. 3.30 (i) it holds that $W \neq \emptyset$, which implies by Def. 3.30 (iv) that $W/\sim \neq \emptyset$. By Prop. 3.40 the relation $\lll$ is a strict partial order on $W/\sim$ (i.e. an irreflexive, asymmetric and transitive relation) that is left-linear, and by Prop. 3.41 it is jointed as well. The seriality of $\lll$ is guaranteed by Prop. 3.42. □

Theorem 3.43 captures the idea that to every index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, there corresponds a BT structure $\mathcal{T} = \langle W/\sim, \lll \rangle$. Every equivalence class $[w]_\sim \in W/\sim$ represents a moment in the final BT structure $\mathcal{T} = \langle W/\sim, \lll \rangle$, and the relation $\lll$ on $W/\sim$, which has been derived from the temporal ordering $\lhd$ on W, reflects the earlier-later relation among those moments.

3.3.3.2 Lifting the set of transition sets

Having reconstructed a BT structure $\mathcal{T} = \langle W/\sim, \lll \rangle$ on the quotient set $W/\sim$ of a given index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, we will now show that the set $W/\approx$ determines a set of transition sets in $\mathcal{T}$. We thereby make use of the correspondence between prunings and transition sets, which we have established in section 3.2 above.

Generating a set of transition sets in $\mathcal{T}$ from the set $W/\approx$ is a step-by-step procedure. We first define a left-linear, partial order relation $\Subset$ on the set $W/\approx$ (section 3.3.3.2.1). We then show that to every equivalence class in $W/\approx$, there naturally corresponds a BT substructure of $\mathcal{T}$ and that, on the level of those substructures, the relation $\Subset$ defined on $W/\approx$ amounts to the converse of the pruning relation (section 3.3.3.2.2). In a further step, we prove that every substructure corresponding to an equivalence in $W/\approx$ is a pruning of the final BT structure $\mathcal{T}$. This allows us to eventually conclude, by Prop. 3.13, that to every equivalence class in $W/\approx$, there actually corresponds a transition set in $\mathsf{dcts}(\mathcal{T})$ (section 3.3.3.2.3).

3.3.3.2.1 An order relation on $\mathbf{W}/\approx$ In this section, we focus on the quotient set $W/\approx$ of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$. We define a relation $\Subset$ on $W/\approx$ in terms of the order relation $\sqsubseteq$ on W and show that the relation $\Subset$ induces a left-linear, partial order on the set $W/\approx$.

Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, the relation $\approx$, which is defined in terms of the relation $\lhd$, is an equivalence relation on W, just as the relation $\sim$ is, and hence provides a partition $W/\approx$ of the set W. Each equivalence class $[w]_\approx \in W/\approx$ is internally ordered by the relation $\lhd \subseteq \approx$, and the corresponding structure $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ is a BT structure (cf. Cor. 3.35). We now derive a relation between the various $\approx$-equivalence classes in $W/\approx$ from the relation $\sqsubseteq \subseteq \sim$, which provides an internal ordering on each of the orthogonal $\sim$-equivalence classes. For all $w, w' \in W$, we set $[w]_\approx \Subset [w']_\approx$ if and only if there is some $x \in [w]_\approx$ and some $x' \in [w']_\approx$ such that $x \sqsubseteq x'$.

Definition 3.44. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, we define a relation $\Subset$ on $W/\approx$ along the following lines: for all $w, w' \in W$, let $[w]_\approx \Subset [w']_\approx$ iff there is some $x \in [w]_\approx$ and some $x' \in [w']_\approx$ such that $x \sqsubseteq x'$.*

By Def. 3.30 (ix) it holds that whenever for any two elements $x \in [w]_\approx$ and $x' \in [w']_\approx$, we have $x \sqsubseteq x'$, then the converse relation $\sqsupseteq \cap ([w']_\approx \times [w]_\approx)$ is the graph of an order isomorphism from $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$ onto a pruning of $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$. This shows, first of all, that $[w]_\sim \Subset [w']_\approx$ implies that the BT structure $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$ is order isomorphic to a pruning of $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$. At the same time, it illustrates that the existence of a single pair of elements $x \in [w]_\approx$ and $x' \in [w']_\approx$ such that $x \sqsubseteq x'$ guarantees that the relation $\sqsubseteq \cap ([w]_\approx \times [w']_\approx)$ is right-total (cf. Lem. 3.36 (ii)). We can then replace the condition provided in Def. 3.44 by the following stronger one: for $[w]_\approx \Subset [w']_\approx$ to obtain, it is not only sufficient but also necessary that for every $x' \in [w']_\approx$, there be some $x \in [w]_\approx$ such that $x \sqsubseteq x'$.

Lemma 3.45. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $w, w' \in W$, we have $[w]_\approx \Subset [w']_\approx$ iff for all $x' \in [w']_\approx$, there is some $x \in [w]_\approx$ such that $x \sqsubseteq x'$.*

Proof. Follows from Def. 3.44 and Lem. 3.36 (ii). □

We now show that the relation $\Subset$ on $W/\approx$, which we have defined above, is a left-linear, partial order, just as the relation $\sqsubseteq$ on W is. The reflexivity, antisymmetry, transitivity and left-linearity of $\Subset$ flow from the corresponding properties of the relation $\sqsubseteq$ on W on the basis of Def. 3.44 and Lem. 3.45, respectively. We take $[w]_\approx \Subset [w']_\approx$ to mean that ($[w]_\approx \Subset [w']_\approx$ and $[w]_\approx \neq [w']_\approx$).

Proposition 3.46. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, the relation $\Subset$ is a partial order on $W/\sim$ (i.e. a reflexive, antisymmetric and transitive relation) that is left-linear.*

Proof. We first show that the relation $\Subset$ is reflexive. Let $[w]_\approx \in W/\approx$ and consider some arbitrary $x \in [w]_\approx$. By the reflexivity of $\sqsubseteq$ (Def. 3.30 (v)) we have $x \sqsubseteq x$, from which it follows by Def. 3.44 that $[w]_\approx \Subset [w]_\approx$.
We prove that the relation $\Subset$ is antisymmetric. Let $[w]_\approx, [w']_\approx \in W/\approx$ s.t. $[w]_\approx \Subset [w']_\approx$. By Def. 3.44 it follows that there is some $x \in [w]_\approx$ and some $x' \in [w']_\approx$ s.t. $x \sqsubset x'$. Assume for reductio that $[w']_\approx \Subset [w]_\approx$ holds as well. By Lem. 3.45 this implies that there is some $x'' \in [w']_\approx$ s.t. $x'' \sqsubset x$. Since $\sqsubseteq \subseteq \sim$, by Lem. 3.33 it follows that $x' = x''$, which contradicts the antisymmetry of $\sqsubseteq$ (Def. 3.30 (v)).
We now show that the relation $\Subset$ is transitive. Let $[w]_\approx, [w']_\approx, [w'']_\approx \in W/\approx$ s.t. $[w]_\approx \sqsubseteq [w']_\approx$ and $[w']_\approx \sqsubseteq [w'']_\approx$. Consider some arbitrary $x'' \in [w'']_\approx$. By Lem. 3.45 it holds that there is some $x' \in [w']_\approx$ s.t. $x' \sqsubseteq x''$ and that there is again some $x \in [w]_\approx$ s.t. $x \sqsubseteq x'$. By the transitivity of $\sqsubseteq$ (Def. 3.30 (v)) it follows that $x \sqsubseteq x''$, which implies by Def. 3.44 that $[w]_\approx \Subset [w'']_\approx$.
We finally show that the relation $\Subset$ is left-linear. Let $[w]_\approx, [w']_\approx, [w'']_\approx \in W/\approx$ s.t. $[w']_\approx \sqsubseteq [w]_\approx$ and $[w'']_\approx \sqsubseteq [w]_\approx$. Consider some arbitrary $x \in [w]_\approx$. By Lem. 3.45 it follows that there is some $x' \in [w']_\approx$ s.t. $x' \sqsubseteq x$ and that there is some $x'' \in [w'']_\approx$ s.t. $x'' \sqsubseteq x$. This implies by the left-linearity of $\sqsubseteq$ (Def. 3.30 (v)) that $x' \sqsubseteq x''$ or $x'' \sqsubseteq x'$, and hence by Def. 3.44 it follows that we have either $[w']_\approx \Subset [w'']_\approx$ or $[w'']_\approx \Subset [w']_\approx$. □

What will become important in section 3.3.3.2.3 below is the fact that the relation $\Subset$ on $W/\approx$ is not necessarily connected: there are index structures for which it does not hold that any two $\approx$-equivalence classes have a common lower $\Subset$-bound in $W/\approx$. Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, the relation $\Subset$ on $W/\approx$ is connected if, and only if, the restriction of $\sqsubseteq$ to an $\sim$-equivalence class is connected in each case, which needs not be the case in general.

Lemma 3.47. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ be an index structure. The relation $\Subset$ on $W/\approx$ is connected (i.e. for all $[w]_\approx, [w']_\approx \in W/\approx$, there is some $[w'']_\approx \in W/\approx$ such that $[w]_\approx \Supset [w'']_\approx \Subset [w']_\approx$) iff for all $v \in W$, the relation $\sqsubseteq|_{[v]_\sim}$ on $[v]_\sim$ is connected.*

Proof. "⇒": Assume that the relation $\Subset$ on $W/\approx$ is connected. Consider some arbitrary $[x]_\sim \in W/\sim$. If $[x]_\sim = \{x\}$, then the relation $\sqsubseteq|_{[x]_\sim}$ on $[x]_\sim$ is trivially connected. Now suppose that $[x]_\sim \supseteq \{x, x'\}$ with $x \neq x'$. Then $[x]_\approx, [x']_\approx \in W/\approx$ and $[x]_\approx \neq [x']_\approx$. Since the relation $\Subset$ on $W/\approx$ is connected, it follows that there is some $[x'']_\approx \in W/\approx$ s.t. $[x]_\approx \Supset [x'']_\approx \Subset [x']_\approx$. This implies by Lem. 3.45 and Lem. 3.33 that there is some $x''' \in [x'']_\approx$ s.t. $x \sqsupseteq x''' \sqsubseteq x'$.
"⇐": Assume that for all $v \in W$, the relation $\sqsubseteq|_{[v]_\sim}$ on $[v]_\sim$ is connected. Let $[w]_\approx, [w']_\approx \in W/\approx$. By Def. 3.30 (viii) there is some $x \in [w]_\approx$ and some

$x' \in [w']_\approx$ s.t. $x \sim x'$. Since, by assumption, the relation $\sqsubseteq|_{[x]_\sim}$ on $[x]_\sim$ is connected, it follows that there is some $y \in [x]_\sim$ s.t. $x \sqsupseteq y \sqsubseteq x'$. This implies by Def. 3.44 that $[w]_\approx \Supset [y]_\approx \Subset [w']_\approx$. □

3.3.3.2.2 The set $\mathsf{tree}(\mathbf{W}/\approx)$ of substructures of $\mathcal{T}$ We have so far established a left-linear, partial order $\Subset$ on the quotient set $W/\approx$ of a given index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$. We will now in a further step associate each equivalence class in $W/\approx$ with a BT substructure of the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$, which we have carved out in section 3.3.3.1 above. It will be shown that the ordering that the relation $\Subset$ on $W/\approx$ induces on set of those substructures is the converse of the pruning relation.

Consider the function $\sigma : W \to W/\sim$ that maps every element $x \in W$ onto its $\sim$-equivalence class $[x]_\sim \in W/\sim$. If we restrict the domain of σ to an $\approx$-equivalence class $[w]_\approx \subseteq W$, the mapping is obviously injective. For, by Lem. 3.33, it holds that for every $[x]_\sim \in \sigma([w]_\approx)$, the intersection $[x]_\sim \cap [w]_\approx$ contains exactly one element.

Moreover, for every $\approx$-equivalence class $[w]_\approx \in W/\approx$, the corresponding structure $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ is a BT structure (cf. Cor. 3.35), and the σ-image of an $\approx$-equivalence class $[w]_\approx \subseteq W$ is order isomorphic to that structure. This is due to the fact that by Def. 3.38 it holds that for all $w, w' \in [w]_\approx$, we have $w \lhd w'$ if and only if $[w]_\sim \ll [w']_\approx$. Thus, for every $[w]_\approx \in W/\approx$, the restriction $\sigma|_{[w]_\approx}$ is an order isomorphism from $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ onto the structure $\langle \sigma([w]_\approx), \ll|_{\sigma([w]_\approx)}\rangle$. And obviously, the structure $\langle \sigma([w]_\approx), \ll|_{\sigma([w]_\approx)}\rangle$ is a substructure of the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$.

Lemma 3.48. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ be an index structure and $\sigma : W \to W/\sim$ the function with $\sigma(x) = [x]_\sim$. For every $[w]_\approx \in W/\approx$, the restriction $\sigma|_{[w]_\approx}$ is an order isomorphism from the structure $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ onto the substructure $\langle \sigma([w]_\approx), \ll|_{[\sigma([w]_\approx)]}\rangle \subseteq \langle W/\sim, \ll\rangle$.*

Proof. By Lem. 3.33 the mapping $\sigma|_{[w]_\approx} : x \mapsto [x]_\sim$ is injective, and by Def. 3.38 it is also order preserving: for all $x, y \in [w]_\approx$, we have $x \lhd y$ iff $\sigma(x) \ll \sigma(y)$. □

The function σ assigns to each structure $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ based on an equivalence class $[w]_\approx \in W/\approx$, a substructure of the final BT structure of $\mathcal{T} = \langle W/\sim, \ll\rangle$ that is order isomorphic to its pre-image. For every $[w]_\approx \in W/\approx$, we denote the substructure $\langle \sigma([w]_\approx), \ll|_{\sigma([w]_\approx)}\rangle \subseteq \mathcal{T}$ by $\mathcal{T}_{[w]_\approx}$. We use $\mathsf{tree}(W/\approx)$ to stand for the set of all substructures of $\mathcal{T}$ that correspond via σ to an equivalence class in $W/\approx$; so $\mathsf{tree}(W/\approx) := \{\mathcal{T}_{[w]_\approx} \mid [w]_\approx \in W/\approx\}$.

Definition 3.49 (The set $\mathsf{tree}(W/\approx)$ of substructures $\mathcal{T}_{[w]_\approx}$ of $\mathcal{T}$). *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure and $[w]_\approx \in W/\approx$, let*

$$\mathcal{T}_{[w]_\approx} := \langle \sigma([w]_\approx), \ll|_{\sigma([w]_\approx)}\rangle$$

be the substructure of $\mathcal{T} = \langle W/\sim, \ll\rangle$ that is determined by the function $\sigma : W \to W/\sim$ with $\sigma(x) = [x]_\sim$. We let $\mathsf{tree}(W/\approx) := \{\mathcal{T}_{[w]_\approx} \mid [w]_\approx \in W/\approx\}$.

As said, for every $\approx$-equivalence class $[w]_\approx \in W/\approx$, the corresponding structure $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ is a BT structure. Since the restriction of the function σ to an equivalence class $[w]_\approx \in W/\approx$ is an order isomorphism onto its image, for every $[w]_\approx \in W/\approx$, the corresponding substructure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a BT structure as well.

Lemma 3.50. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ be an index structure, and let $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$. Then $\mathcal{T}_{[w]_\approx}$ is a BT structure.*

Proof. Follows from Cor. 3.35 and Lem. 3.48. □

The correspondence between the various equivalence classes $[w]_\approx \in W/\approx$ and the BT structures $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ induced by the function σ is injective: different $\approx$-equivalence classes are associated with different substructures of the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$.

Lemma 3.51. *Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, for all $[w]_\approx, [w']_\approx \in W/\approx$ such that $[w]_\approx \neq [w']_\approx$, we have $\mathcal{T}_{[w]_\approx} \neq \mathcal{T}_{[w']_\approx}$.*

Proof. Follows from Def. 3.30 (xiii), (ix) and (x) on the basis of Lem. 3.45. □

Consider a pair of $\approx$-equivalence classes $[w]_\approx$ and $[w']_\approx$ in $W/\approx$ such that $[w]_\approx \neq [w']_\approx$, and let us investigate how the corresponding substructures $\mathcal{T}_{[w]_\approx}$ and $\mathcal{T}_{[w']_\approx}$ can possibly be related. Two cases can be considered, depending on whether $[w]_\approx$ and $[w']_\approx$ are comparable by $\Subset$ or not.

Let us first examine the case in which $[w]_\approx$ and $[w']_\approx$ are incomparable by $\Subset$. By Def. 3.30 (viii) it follows that there is some $x \in [w]_\approx$ and some $x' \in [w']_\approx$ such that $x \sim x'$, and, by Lem. 3.45, x and x' must be incomparable by $\sqsubseteq$. This implies by Def. 3.30 (x) that the relation $\sim \cap([w]_\approx \times [w']_\approx)$ is an $\lhd$-preserving order isomorphism between proper trunks of the corresponding BT structures $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ and $\langle [w']_\approx, \lhd|_{[w']_\approx}\rangle$. Neither of those BT structures is embeddable in the other: each contains at least one element that lacks a $\sim$-correspondent in the other structure. Consequently, even though the two structures $\mathcal{T}_{[w]_\approx}$ and $\mathcal{T}_{[w']_\approx}$ have a non-empty intersection, none of them can be a substructure of the other.

Let us now focus on the case in which $[w]_\approx$ and $[w']_\approx$ are comparable by $\Subset$. Assume that $[w]_\approx \Subset [w']_\approx$. As already mentioned above, on the basis of Def. 3.30 (ix), this implies that the relation $\sqsupset \cap ([w']_\approx \times [w]_\approx)$ is the graph of an order isomorphism from $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$ onto a proper pruning of $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$. Since $\sqsubseteq\ \subseteq\ \sim$, it follows that $\mathcal{T}_{[w']_\approx}$ is a proper substructure of $\mathcal{T}_{[w]_\approx}$. And what is more, it even holds that $\mathcal{T}_{[w']_\approx}$ is a proper pruning of $\mathcal{T}_{[w]_\approx}$. Note that if $[w]_\approx = [w']_\approx$, the BT structure $\mathcal{T}_{[w]_\approx}$ is obviously a pruning of itself. The converse of the relation $\Subset$ that we have defined on the quotient set $W/\approx$ in terms of the relation $\sqsubseteq$ on W then perfectly mirrors the pruning relation among the corresponding substructures of $\mathcal{T}$ that are elements of $\mathsf{tree}(W/\approx)$: whenever $[w]_\approx \Subset [w']_\approx$, then $\mathcal{T}_{[w']_\approx}$ is a pruning of $\mathcal{T}_{[w]_\approx}$.

Lemma 3.52. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ be an index structure, and let $[w]_\approx, [w']_\approx \in W/\approx$ such that $[w]_\approx \Subset [w']_\approx$. Then $\mathcal{T}_{[w']_\approx}$ is a pruning of $\mathcal{T}_{[w]_\approx}$.*

Proof. Let $[w]_\approx, [w']_\approx \in W/\approx$ s.t. $[w]_\approx \Subset [w']_\approx$ By Def. 3.44 there is some $x \in [w]_\approx$ and some $x' \in [w']_\approx$ s.t. $x \sqsubseteq x'$. This implies by Def. 3.30 (ix) that the relation $\sqsupseteq \cap ([w']_\approx \times [w]_\approx)$ is the graph of an order isomorphism τ from $\langle [w']_\approx, \lhd|_{[w]_\approx} \rangle$ onto a pruning of $\langle [w]_\approx, \lhd|_{[w']_\approx} \rangle$. Moreover, by Lem. 3.48 it holds that the restriction $\sigma|_{[w]_\approx}$ is an order isomorphism from $\langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ onto $\mathcal{T}_{[w]_\approx}$, and likewise that the restriction $\sigma|_{[w']_\approx}$ is an order isomorphism from $\langle [w']_\approx, \lhd|_{[w']_\approx} \rangle$ onto $\mathcal{T}_{[w']_\approx}$. Consequently, the restriction of the composition $\sigma \circ \tau \circ \sigma^{-1}$ to $\mathcal{T}_{[w']_\approx}$ is a $\ll$-preserving order isomorphism onto a pruning of $\mathcal{T}_{[w]_\approx}$. Since by Def. 3.30 (vi) for all $y \in [w]_\approx$ and $y' \in [w']_\approx$ s.t. $\tau(y') = y$, we have $y \sim y'$, the restriction of $\sigma \circ \tau \circ \sigma^{-1}$ to $\mathcal{T}_{[w']_\approx}$ is the identity. □

3.3.3.2.3 The set $\mathsf{ts}(\mathbf{W}/\approx)$ of transition sets in $\mathcal{T}$ Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, we have shown that to every equivalence class $[w]_\approx \in W/\approx$, there corresponds a BT substructure $\mathcal{T}_{[w]_\approx}$ of the final BT structure $\mathcal{T} = \langle W/\sim, \ll \rangle$. Moreover, we have illustrated that on the set $\mathsf{tree}(W/\approx) = \{\mathcal{T}_{[w]_\approx} \mid [w]_\approx \in W/\approx\}$ of all those substructures, the left-linear, partial order relation $\Subset$ that we have defined on $W/\approx$ amounts to the converse of the pruning relation. What we eventually need to show in order to be able to identify a set of transition sets in the final BT structure $\mathcal{T}$, is that for every $[w]_\approx \in W/\approx$, the structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a pruning of $\mathcal{T}$. By the correspondence between prunings and transition sets that we have established in section 3.2, this then allows us to conclude that every equivalence class $[w]_\approx \in W/\approx$ determines a set of transitions in the final BT structure $\mathcal{T}$.

Obviously, since the relation $\approx$ induces a partition of the set W, the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ is the just the union of the various substructures $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$, i.e. $\mathcal{T} = \bigcup \mathsf{tree}(W/\approx)$. The union $\bigcup\{\langle X_i, R_i\rangle \mid i \in I\}$ of a set of structures $\{\langle X_i, R_i\rangle \mid i \in I\}$ (with I an index set) is thereby defined as the pair $\langle \bigcup_{i\in I} X_i, \bigcup_{i\in I} R_i\rangle$. In case the ordering $\Subset$ on the set $W/\approx$ contains a minimum $[w_0]_\approx$ (i.e. a $\Subset$-minimal element such that for all $[w]_\approx \in W/\approx$, we have $[w_0]_\approx \Subset [w]_\approx$), by Lem. 3.52, every BT structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a pruning, and hence a substructure, of the BT structure $\mathcal{T}_{[w_0]_\approx}$ corresponding to that $\sqsubseteq$-minimal element. Consequently, the union $\bigcup \mathsf{tree}(W/\approx)$ is identical to the structure $\mathcal{T}_{[w_0]_\approx} \in \mathsf{tree}(W/\approx)$, and so every BT structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ actually is a pruning of the final BT structure $\mathcal{T}$.

However, it need not be the case in general that the ordering $\Subset$ on the quotient set $W/\approx$ has a minimum. There are index structures that are such that the set $W/\approx$ contains infinitely descending $\Subset$-chains, and the relation $\Subset$ on $W/\approx$ does not necessarily have to be connected. In order to be able to cater for the general case, we proceed in several steps. To begin with, we identify maximal $\Subset$-chains in the ordering on the set $W/\approx$. To each such maximal $\Subset$-chain in $W/\approx$, there corresponds a reverse maximal chain of prunings in $\mathsf{tree}(W/\approx)$ (cf. section 3.3.3.2.3.1). In a first step, we then show that every structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a pruning of the union of at least one such pruning chain in $\mathsf{tree}(W/\approx)$, independently of whether the corresponding $\Subset$-chains contain a first element or descend ad infinitum (cf. section 3.3.3.2.3.2). In a second step, we prove that the union of each pruning chain in $\mathsf{tree}(W/\approx)$ is a pruning of the final tree $\mathcal{T}$, no matter whether the relation $\Subset$ on $W/\approx$ is connected or not (cf. section 3.3.3.2.3.3). From those two steps, we can finally infer that every BT structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a pruning of the final BT structure $\mathcal{T}$ and hence corresponds to a set of transitions in that structure (cf. section 3.3.3.2.3.4).

3.3.3.2.3.1 So let us single out maximal $\Subset$-chains in the ordering on the quotient set $W/\approx$ of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ and take a look at the corresponding chains of prunings in $\mathsf{tree}(W/\approx)$.

As we have seen, the relation $\Subset$ on $W/\approx$, which we have defined in terms of the relation $\sqsubseteq$ on W in section 3.3.3.2.1 above, is a left-linear, partial order on the set $W/\approx$. Within that ordering, we can identify maximal $\Subset$-chains of

$\approx$-equivalence classes, i.e. maximal $\Subset$-linear subset of $W/\approx$. On the basis of the results established in section 3.3.3.2.2, to every link $[w]_\approx$ of such a maximal $\Subset$-chain in $W/\approx$, there corresponds a substructure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ of the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$. The correspondence is induced by the function $\sigma : W \to W/\sim$, which assigns to every BT structure $\langle [w]_\approx, \lhd|_{[w]_\approx}\rangle$ on an $\approx$-equivalence class $[w]_\approx \in W/\approx$ an order isomorphic image $\mathcal{T}_{[w]_\approx} = \langle \sigma([w]_\approx), \ll|_{\sigma([w]_\approx)}\rangle$.

Moreover, the ordering $\Subset$ on the set $W/\approx$ has been shown to coincide with the converse of the pruning relation on the set $\mathsf{tree}(W/\approx)$. For every maximal $\Subset$-linear subset C of $W/\approx$, the corresponding set $C_\mathcal{T} := \{\mathcal{T}_{[x]_\approx} \mid [x]_\approx \in C\}$ is thus a maximal linear subset of $\mathsf{tree}(W/\approx)$ ordered by the pruning relation. For any two BT structures contained in $C_\mathcal{T}$, it holds that one is a pruning of the other, while no proper superset of $C_\mathcal{T}$ has that feature. Since every pruning of a BT structure is by definition also a substructure of the latter, the linear order on the set $C_\mathcal{T}$ induced by the pruning relation is an order by inclusion. Given a maximal $\Subset$-chain C in $W/\approx$, any two links $[w]_\approx$ and $[w']_\approx$ of that chain are comparable by $\Subset$, and whenever $[w]_\approx \Subset [w']_\approx$, then the structure $\mathcal{T}_{[w']_\approx}$ is a pruning of $\mathcal{T}_{[w]_\approx}$, and hence $\mathcal{T}_{[w']_\approx} \subseteq \mathcal{T}_{[w]_\approx}$. Note that the transition from C to $C_\mathcal{T}$ is order-reversing: for every descending maximal $\Subset$-chain C in $W/\approx$, the corresponding maximal chain $C_\mathcal{T}$ is a reverse chain of prunings in $\mathsf{tree}(W/\approx)$ and thus an ascending chain of substructures in the final BT structure $\mathcal{T}$.

Given a maximal $\Subset$-chain C in the quotient set $W/\approx$ of an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, we call the subset $C_\mathcal{T} := \{\mathcal{T}_{[x]_\approx} \mid [x]_\approx \in C\} \subseteq \mathsf{tree}(W/\approx)$ a *pruning chain*, and we denote the set of all pruning chains in $\mathsf{tree}(W/\approx)$ by $\mathsf{Ch}(W/\approx)$.

Definition 3.53 (Pruning chain). *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure and C a maximal $\Subset$-linear subset of $W/\approx$ (i.e. a subset of $W/\approx$ that is linearly ordered via $\Subset$ and such that no proper superset $C' \supsetneq C$ in $W/\approx$ is linearly ordered via $\Subset$ as well), we call the set $C_\mathcal{T} := \{\mathcal{T}_{[x]_\approx} \mid [x]_\approx \in C\} \subseteq \mathsf{tree}(W/\approx)$ a* pruning chain. *Let $\mathsf{Ch}(W/\approx)$ be the set of all pruning chains in* $\mathsf{tree}(W/\approx)$.

3.3.3.2.3.2 Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, a pruning chain in $\mathsf{tree}(W/\approx)$ then is a chain of substructures in the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ that is linked by the pruning relation. Having analyzed the final BT structure into its constituent pruning chains, we now show that every link of such a pruning chain constitutes a pruning of the union of that chain.

Take some arbitrary maximal $\Subset$-linear subset $C \subseteq W/\approx$ and consider the corresponding pruning chain $C_{\mathcal{T}} \subseteq \mathsf{tree}(W/\approx)$. In case the $\Subset$-chain C contains a minimum $[w_0]_\approx$, the corresponding pruning chain $C_{\mathcal{T}}$ contains a greatest BT structure $\mathcal{T}_{[w_0]_\approx}$ that includes every link $\mathcal{T}_{[w]_\approx}$ of the chain $C_{\mathcal{T}}$ as a pruning. The claim that we aim to prove then follows immediately: every link $\mathcal{T}_{[w]_\approx}$ of the pruning chain $C_{\mathcal{T}}$ is a pruning of the union $\bigcup C_{\mathcal{T}}$ of that chain since, obviously, the union $\bigcup C_{\mathcal{T}}$ just is the greatest BT structure $\mathcal{T}_{[w_0]_\approx} \in C_{\mathcal{T}}$.

However, if the $\Subset$-chain C does not contain a minimum but descends ad infinitum, the corresponding pruning chain $C_{\mathcal{T}}$ does not contain a greatest BT structure. Rather, it is an infinitely ascending chain of substructures of the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ connected by the pruning relation. We show that for any arbitrary pruning chain $C_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)$, be it upper bounded or not, it holds that every BT structure $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$ is a pruning of $\bigcup C_{\mathcal{T}}$.

PROPOSITION 3.54. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure, let C be a maximal $\Subset$-linear subset of $W/\approx$ and let $C_{\mathcal{T}} = \{\mathcal{T}_{[x]_\approx} \mid [x]_\approx \in C\} \subseteq \mathsf{tree}(W/\approx)$ be the corresponding pruning chain. Then every $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$ is a pruning of $\bigcup C_{\mathcal{T}}$.*

Proof. Let C be a maximal $\Subset$-linear subset of $W/\approx$. Assume that C contains a $\Subset$-minimal element $[w_0]_\approx$, i.e. for all $[w]_\approx \in C$, it holds that $[w_0]_\approx \Subset [w]_\approx$. By Lem. 3.52 this implies that every $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$ is a pruning and hence a substructure of $\mathcal{T}_{[w_0]_\approx}$. Consequently, $\bigcup C_{\mathcal{T}} = \mathcal{T}_{[w_0]_\approx}$, which concludes the proof that every $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$ is a pruning of $\bigcup C_{\mathcal{T}}$.
Now assume that C does not contain a $\Subset$-minimal element and consider some arbitrary $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$. Obviously, $\mathcal{T}_{[w]_\approx}$ is a substructure of $\bigcup C_{\mathcal{T}}$. We prove that $\mathcal{T}_{[w]_\approx}$ is also a pruning of $\bigcup C_{\mathcal{T}}$ according to Def. 3.3.
Let $h \in \mathsf{hist}(\mathcal{T}_{[w]_\approx})$ and let $h' \in \mathsf{hist}(\bigcup C_{\mathcal{T}})$ s.t. $h \subseteq h'$. We show that $h = h'$. Assume for reductio that there is some $[v]_\sim \in h' \setminus h$. Since $h \in \mathsf{hist}(\mathcal{T}_{[w]_\approx})$, this implies that $[v]_\sim \notin \sigma([w]_\approx)$. Consequently, there must be some $\mathcal{T}_{[w']_\approx} \in C_{\mathcal{T}}$ s.t. $[v]_\sim \in \sigma([w']_\approx)$. From $[v]_\sim \in \sigma([w']_\approx)\setminus\sigma([w]_\approx)$ it follows that $[w]_\approx \Supset [w']_\approx$ and hence by Lem. 3.52 we have that $\mathcal{T}_{[w]_\approx}$ is a pruning of $\mathcal{T}_{[w']_\approx}$. By Lem. 3.4 (i) this implies that $h \in \mathsf{hist}(\mathcal{T}_{[w']_\approx})$, which contradicts our assumption that $[v]_\sim \notin h$. This shows that condition (i) of Def. 3.3 is fulfilled.
Assume that there are $[z']_\sim, [z'']_\sim \in \sigma([w]_\approx)$ s.t. $[z']_\sim \not\ll [z'']_\sim$ and $[z'']_\sim \not\ll [z']_\sim$, and let $[z]_\sim$ be the greatest common lower $\ll$-bound of $[z']_\sim$ and $[z'']_\sim$ in $\bigcup_{\mathcal{T}_{[x]_\approx} \in C_{\mathcal{T}}} \sigma([x]_\approx)$, which exists by Prop. 3.41 and Lem. 3.39. Assume for reductio that there is some $[y]_\sim \in \bigcup_{\mathcal{T}_{[x]_\approx} \in C_{\mathcal{T}}} \sigma([x]_\approx)$ s.t. $[y]_\sim \gg [z]_\sim$ and $[y]_\sim \notin \sigma([w]_\approx)$. Then there must be some $\mathcal{T}_{[w']_\approx} \in C_{\mathcal{T}}$ s.t. $[y]_\sim \in \sigma([w']_\approx)$. From $[y]_\sim \in \sigma([w']_\approx) \setminus \sigma([w]_\approx)$ it follows that $[w]_\approx \Supset [w']_\approx$ and hence by Lem. 3.52 we have that $\mathcal{T}_{[w]_\approx}$ is a pruning of $\mathcal{T}_{[w']_\approx}$. This implies that

$[y]_\sim \in \sigma([w]_\approx)$, which contradicts our assumption. This shows that condition (ii) of Def. 3.3 is satisfied. □

3.3.3.2.3.3 For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, the following holds: every substructure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ of the final BT structure $\mathcal{T} = \langle W/\sim, \ll \rangle$ is part of at least one pruning chain in $\mathsf{Ch}(W/\approx)$, and the final BT structure $\mathcal{T}$ equals the union of the unions of all pruning chains in $\mathsf{Ch}(W/\approx)$; i.e. $\mathcal{T} = \bigcup_{C_\mathcal{T} \in \mathsf{Ch}(W/\approx)} (\bigcup C_\mathcal{T})$. Having established that every link of a pruning chain $C_\mathcal{T} \in \mathsf{Ch}(W/\approx)$ is a pruning of the union of that chain, we now show in a further step that the union of each pruning chain is a pruning of the final tree $\mathcal{T}$.

In case the relation $\Subset$ on $W/\approx$ is connected, the intersection of any two pruning chains in $\mathsf{Ch}(W/\approx)$ is non-empty. For, $[w']_\approx \sqsupseteq [w]_\approx \sqsubseteq [w'']_\approx$ implies $\mathcal{T}_{[w']_\approx} \subseteq \mathcal{T}_{[w]_\approx} \supseteq \mathcal{T}_{[w'']_\approx}$ for all $[w]_\approx, [w']_\approx, [w'']_\approx \in W/\approx$. By the left-linearity of the relation $\Subset$ on $W/\approx$ it then follows that if the relation $\Subset$ on $W/\approx$ is connected, the unions of all pruning chains $\mathsf{Ch}(W/\approx)$ are the same. Consequently, whenever any two $\approx$-equivalence classes in $W/\approx$ have a common lower $\Subset$-bound in $W/\approx$, the final BT structure $\mathcal{T}$ is identical to the union of either pruning chain in $\mathsf{Ch}(W/\approx)$. It is then straightforward that the union of each pruning chain in $\mathsf{Ch}(W/\approx)$ is a pruning of the final BT structure $\mathcal{T}$ because every BT structure is a pruning of itself.

Yet, it does not necessarily have to be the case that the relation $\Subset$ on $W/\approx$ is connected, and hence it need not be the case in general that any two pruning chains in $\mathsf{Ch}(W/\approx)$ converge. In section 3.3.3.2.1, we have shown that if and only if the restriction of the relation $\sqsubseteq$ to an $\sim$-equivalence class in $W/\sim$ is connected in each case, do any two $\approx$-equivalence classes in $W/\approx$ have a common lower $\Subset$-bound in $W/\approx$. In case $\mathsf{Ch}(W/\approx)$ contains a pair $C_\mathcal{T}$ and $C'_\mathcal{T}$ of disjoint pruning chains, any two BT structures $\mathcal{T}_{[w]_\approx} \in C_\mathcal{T}$ and $\mathcal{T}_{[w']_\approx} \in C'_\mathcal{T}$ have some proper trunk in common, but neither of them is a substructure of the other; for, the fact that $C_\mathcal{T}$ and $C'_\mathcal{T}$ are disjoint presupposes that $[w]_\approx$ and $[w']_\approx$ are incomparable by $\Subset$ (cf. section 3.3.3.2.2).

We prove that for every arbitrary index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, the union of each pruning chain in $\mathsf{Ch}(W/\approx)$ is a pruning of the final BT structure $\mathcal{T} = \langle W/\sim \rangle, \ll \rangle$, viz. of the union of the unions of all those chains, no matter whether any two pruning chains $\mathsf{Ch}(W/\approx)$ intersect with each other or not.

Proposition 3.55. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure, let C be a maximal $\Subset$-linear subset of $W/\approx$ and let $C_{\mathcal{T}} = \{\mathcal{T}_{[x]_\approx} \mid [x]_\approx \in C\} \subseteq \mathsf{tree}(W/\approx)$ be the corresponding pruning chain. Then $\bigcup C_{\mathcal{T}}$ is a pruning of $\mathcal{T} = \langle W/\sim, \ll\rangle$.*

Proof. Obviously, $\mathcal{T} = \bigcup_{C_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)}(\bigcup C_{\mathcal{T}})$. Assume that the relation $\Subset$ on $W/\approx$ is connected. Then for every pair of pruning chains $C'_{\mathcal{T}}, C''_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)$, we have $C'_{\mathcal{T}} \cap C''_{\mathcal{T}} \neq \emptyset$, which implies by the left-linearity of $\Subset$ that $\bigcup C'_{\mathcal{T}} = \bigcup C''_{\mathcal{T}}$. Let $C_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)$ be some arbitrary pruning chain. Then $\bigcup C_{\mathcal{T}} = \mathcal{T}$. Since by Def. 3.3 every BT structure is a pruning of itself, it follows that $\bigcup C_{\mathcal{T}}$ is a pruning of $\mathcal{T}$.

Now assume that the relation $\Subset$ on $W/\approx$ is not connected. Consider again some arbitrary pruning chain $C_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)$. Let $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$ and let $h \in \mathsf{hist}(\mathcal{T}_{[w]_\approx})$. Since by Prop. 3.54 the structure $\mathcal{T}_{[w]_\approx}$ is a pruning of $\bigcup C_{\mathcal{T}}$, it follows by Lem. 3.4 (i) that $h \in \mathsf{hist}(\bigcup C_{\mathcal{T}})$. Let $h' \in \mathsf{hist}(\mathcal{T})$ s.t. $h \subseteq h'$. We show that $h = h'$. Assume for reductio that there is some $[v]_\sim \in h' \setminus h$. Since $h \in \mathsf{hist}(\bigcup C_{\mathcal{T}})$, this implies that $[v]_\sim \notin \bigcup_{\mathcal{T}_{[x]_\approx} \in C_{\mathcal{T}}} \sigma([x]_\approx)$. Then there is some $C'_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)$ s.t. $\bigcup C_{\mathcal{T}} \neq \bigcup C'_{\mathcal{T}}$ and $[v]_\sim \in \bigcup_{\mathcal{T}_{[x']_\approx} \in C'_{\mathcal{T}}} \sigma([x']_\approx)$. This implies that there is some $\mathcal{T}_{[w']_\approx} \in C'_{\mathcal{T}}$ s.t. $[v]_\sim \in \sigma([w']_\approx)$. From $\bigcup C_{\mathcal{T}} \neq \bigcup C'_{\mathcal{T}}$ it follows that $[w]_\approx \not\Subset [w']_\approx$ and $[w']_\approx \not\Subset [w]_\approx$. Consequently, there is some $y \in [w]_\approx$ and some $y' \in [w']_\approx$ s.t. $y \sim y'$ but $y \not\sqsubseteq y'$ and $y' \not\sqsubseteq y$. Assume that $\mathsf{hist}(\mathcal{T}_{[w]_\approx}) = \{h\}$. By Def. 3.30 (x.a) it follows that there is some $y_0 \in [w]_\approx$ s.t. $[y_0]_\sim \cap [w']_\approx = \emptyset$. Then $[y_0]_\sim \in h$ and since histories are downward closed, it follows that $[y_0]_\sim \ll [v]_\sim$, which by Lem. 3.39 contradicts $[y_0]_\sim \cap [w']_\approx = \emptyset$. If, on the other hand, $\mathsf{hist}(\mathcal{T}_{[w]_\approx}) \supset \{h, h''\}$, then there is some $[y_1]_\sim \in \sigma([w]_\approx)$ s.t. $h \perp_{[y_1]_\sim} h''$. By Def. 3.30 (x.c) we have that $[y_1]_\sim \cap [w']_\approx = \emptyset$. Since $[y_1]_\sim \in h$ and histories are downward closed, it follows that $[y_1]_\sim \ll [v]_\sim$, which by Lem. 3.39 contradicts $[y_1]_\sim \cap [w']_\approx = \emptyset$. Consequently, $h = h'$, which shows that condition (i) of Def. 3.3 is fulfilled.

Assume that there are $[z']_\sim, [z'']_\sim \in \bigcup_{\mathcal{T}_{[x]_\approx} \in C_{\mathcal{T}}} \sigma([x]_\approx)$ s.t. $[z']_\sim \not\leqq [z'']_\sim$ and $[z'']_\sim \not\leqq [z']_\sim$. Then there is some $\mathcal{T}_{[w]_\approx} \in C_{\mathcal{T}}$ s.t. $\sigma([w]_\approx) \supseteq \{[z']_\sim, [z'']_\sim\}$. Let $[z]_\sim$ be the greatest common lower $\ll$-bound of $[z']_\sim$ and $[z'']_\sim$ in $\sigma([w]_\approx)$, which exists by Prop. 3.41 and Lem. 3.39. Assume for reductio that there is some $[y]_\sim \in W/\sim$ s.t. $[y]_\sim \gg [z]_\sim$ and $[y]_\sim \notin \bigcup_{\mathcal{T}_{[x]_\approx} \in C_{\mathcal{T}}} \sigma([x]_\approx)$. Then there is some $C'_{\mathcal{T}} \in \mathsf{Ch}(W/\approx)$ s.t. $\bigcup C_{\mathcal{T}} \neq \bigcup C'_{\mathcal{T}}$ and $[y]_\sim \in \bigcup_{\mathcal{T}_{[x']_\approx} \in C'_{\mathcal{T}}} \sigma([x']_\approx)$. This implies that there is some $\mathcal{T}_{[w']_\approx} \in C'_{\mathcal{T}}$ s.t. $[y]_\sim \in \sigma([w']_\approx)$. From $\bigcup C_{\mathcal{T}} \neq \bigcup C'_{\mathcal{T}}$ it follows that $[w]_\approx \not\Subset [w']_\approx$ and $[w']_\approx \not\Subset [w]_\approx$. Consequently, there is some $u \in [w]_\approx$ and some $u' \in [w']_\approx$ s.t. $u \sim u'$ but $u \not\sqsubseteq u'$ and $u' \not\sqsubseteq u$. By Def. 3.30 (x.c) it then follows that $[z]_\sim \cap [w']_\approx = \emptyset$, which by Lem. 3.39 contradicts $[y]_\sim \gg [z]_\sim$. This shows that condition (ii) of Def. 3.3 is satisfied. □

3.3.3.2.3.4 In Prop. 3.54 above, we have shown that, given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, every link $\mathcal{T}_{[w]_\approx}$ of a pruning chain $C_\mathcal{T} \in \mathsf{Ch}(W/\approx)$ is a pruning of the union $\bigcup C_\mathcal{T}$ of that chain; and in Prop. 3.55, we have established, in a further step, that the union $\bigcup C_\mathcal{T}$ of each pruning chain $C_\mathcal{T} \in \mathsf{Ch}(W/\approx)$ is a pruning of the final tree $\mathcal{T} = \langle W/\sim, \ll \rangle$. Since every BT structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a link of at least one pruning chain in $\mathsf{Ch}(W/\approx)$ and since, by Lem. 3.8, the pruning relation is transitive, Prop. 3.54 and Prop. 3.55 jointly imply that every BT substructure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ of the final BT structure $\mathcal{T}$ is in fact a pruning of $\mathcal{T}$.

Proposition 3.56. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ be an index structure, and let $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$. Then $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a pruning of $\mathcal{T} = \langle W/\sim, \ll \rangle$.*

Proof. Let $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$. Then there is some pruning chain $C_\mathcal{T} \in \mathsf{Ch}(W/\approx)$ s.t. $\mathcal{T}_{[w]_\approx} \in C_\mathcal{T}$. By Prop. 3.54 it holds that $\mathcal{T}_{[w]_\approx}$ is a pruning of $\bigcup C_\mathcal{T}$, and by Prop. 3.55 we have that $\bigcup C_\mathcal{T}$ is a pruning of $\mathcal{T}$. Since by Lem. 3.8 the pruning relation is transitive, it follows that $\mathcal{T}_{[w]_\approx}$ is a pruning of $\mathcal{T}$. □

Prop. 3.56 guarantees that for every $\approx$-equivalence class $[w]_\approx \in W/\approx$, the corresponding BT structure $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ is a pruning of the final BT structure $\mathcal{T} = \langle W/\sim, \ll \rangle$. On the basis of the correspondence between prunings and transition sets established in section 3.2.2.2, it then follows immediately that every $\approx$-equivalence class $[w]_\approx \in W/\approx$ determines a set of transitions in $\mathcal{T}$. The correspondence between prunings and transition sets is induced by the function $\xi : \mathsf{prun}(\mathcal{T}) \to \mathsf{dcts}(\mathcal{T})$ that assigns to each pruning $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx) \subseteq \mathsf{prun}(\mathcal{T})$ of the final BT structure $\mathcal{T}$, the set of all transitions in $\mathcal{T}$ whose outcomes include the set $\mathsf{hist}(\mathcal{T}_{[w]_\approx})$. Recall that by Prop. 3.13, it thereby holds that $\mathsf{hist}(\mathcal{T}_{[w]_\approx}) = \mathsf{H}(\xi(\mathcal{T}_{[w]_\approx}))$. For every $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$, we denote the transition set $\xi(\mathcal{T}_{[w]_\approx}) \in \mathsf{dcts}(\mathcal{T})$ corresponding to the pruning $\mathcal{T}_{[w]_\approx} \in \mathsf{prun}(\mathcal{T})$ by $T_{[w]_\approx}$. We use $\mathsf{ts}(W/\approx)$ to stand for the set that contains the ξ-images of all prunings in $\mathsf{tree}(W/\approx)$; so $\mathsf{ts}(W/\approx) := \{T_{[w]_\approx} \mid [w]_\approx \in W/\approx\}$.

Definition 3.57. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure and $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$, let*

$$T_{[w]_\approx} := \xi(\mathcal{T}_{[w]_\approx}).$$

We define $\mathsf{ts}(W/\approx) := \{T_{[w]_\approx} \mid [w]_\approx \in W/\approx\}$.

Theorem 3.58. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ an index structure, the set $\mathsf{ts}(W/\approx)$ is a subset of $\mathsf{dcts}(\mathcal{T})$.*

Proof. Let $[w]_\approx \in W/\approx$. By Prop. 3.56 it holds that the structure $\mathcal{T}_{[w]_\approx}$ is a pruning of $\mathcal{T} = \langle W/\sim, \ll\rangle$. From this it follows by Prop. 3.13 that $\xi(\mathcal{T}_{[w]_\approx})$ is a set of transitions in $\mathsf{dcts}(\mathcal{T})$. □

To every $\approx$-equivalence class $[w]_\approx \in W/\approx$, there thus corresponds a set $T_{[w]_\approx}$ of transition sets in $\mathsf{dcts}(W/\approx)$. The correspondence between the various $\approx$-equivalence classes $[w]_\approx \in W/\approx$ and the transition sets $T_{[w]_\approx} \in \mathsf{ts}(W/\approx) \subseteq \mathsf{dcts}(W/\approx)$ is injective: different $\approx$-equivalence classes determine different transition sets. This is due to the fact that both the correspondence between the $\approx$-equivalence classes $[w]_\approx \in W/\approx$ and the BT structures $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$ as well as the function ξ are injective.

Lemma 3.59. *Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, for all $[w]_\approx, [w']_\approx \in W/\approx$ such that $[w]_\approx \neq [w']_\approx$, we have $T_{[w]_\approx} \neq T_{[w']_\approx}$.*

Proof. Follows from Lem. 3.14 and Lem. 3.51. □

It is worthwhile to point out that the set $\mathsf{ts}(W/\approx)$ does not necessarily have to exhaust the entire range of possible transition sets in $\mathsf{dcts}(\mathcal{T})$. If, and only if, the set $\mathsf{tree}(W/\approx)$ equals the set $\mathsf{prun}(\mathcal{T})$ of all possible prunings of the final BT structure, does it hold that $\mathsf{ts}(W/\approx) = \mathsf{dcts}(\mathcal{T})$. In the next section, we will show that while it is not excluded that the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ comprises additional transition sets over and above those contained in $\mathsf{ts}(W/\approx)$, the transition sets contained in $\mathsf{ts}(W/\approx)$ suffice in order to cover the entire BT structure.

3.3.3.3 Lifting the transition structure

Starting out with an arbitrary transition structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, we have first specified a BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ on the quotient set $W/\sim$ (section 3.3.3.1), and we have subsequently shown that the set $W/\approx$ determines a set $\mathsf{ts}(W/\approx) \subseteq \mathsf{dcts}(\mathcal{T})$ of consistent, downward closed transition sets in that structure (section 3.3.3.2). We now prove that the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ together with the set $\mathsf{ts}(W/\approx) \subseteq \mathsf{dcts}(\mathcal{T})$ of transition sets in fact constitutes a transition structure. Furthermore, we illustrate that there is a one-to-one correspondence between the elements of W and the indices of evaluation in the lifted transition structure and that, moreover, our primitive relations $\lhd, \sim$ and $\sqsubseteq$ match the relations that figure in the interpretation of the intensional operators of the transition language $\mathcal{L}_\mathsf{t}$ on that structure.

In order to prove that the final BT structure $\mathcal{T} = \langle W/\sim, \ll\rangle$ together with the set $\mathsf{ts}(W/\approx) \subseteq \mathsf{dcts}(\mathcal{T})$ of transition sets qualifies as a transition structure according to Def. 2.33, we have to check whether every moment, viz. every $\sim$-equivalence class $[w]_\sim \in W/\sim$, is compatible with at least one transition set $T_{[w']_\approx} \in \mathsf{ts}(W/\approx)$. Since every moment $[w]_\sim \in W/\sim$ is contained in at least one BT structure in $\mathsf{tree}(W/\approx)$, namely, in the BT structure $\mathcal{T}_{[w]_\approx}$, and since by Prop. 3.13, it holds that $\mathsf{hist}(\mathcal{T}_{[w]_\approx}) = \mathsf{H}(T_{[w]_\approx})$, it is straightforward that the condition is actually fulfilled: for every moment $[w]_\sim \in W/\sim$, there is at least one transition set $T_{[w']_\approx} \in \mathsf{ts}(W/\approx)$ such that $\mathsf{H}(T_{[w']_\approx}) \cap \mathsf{H}_{[w]_\sim} \neq \emptyset$. And hence, the structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \lll, \mathsf{ts}(W/\approx)\rangle$ lifted from the index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ is a transition structure.

Theorem 3.60. *For $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ an index structure, the structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \lll, \mathsf{ts}(W/\approx)\rangle$ is a transition structure.*

Proof. By Theorem 3.43, the structure $\mathcal{T} = \langle W/\sim, \lll\rangle$ is a BT structure, and by Theorem 3.58, the set $\mathsf{ts}(W/\approx)$ is a set of transition sets that is included in $\mathsf{dcts}(\mathcal{T})$. We have $W/\sim = \bigcup \mathsf{tree}(W/\approx)$, and by Prop. 3.13, for every $\mathcal{T}_{[w]_\approx} \in \mathsf{tree}(W/\approx)$, it holds that $\mathsf{hist}(\mathcal{T}_{[w]_\approx}) = \mathsf{H}(T_{[w]_\approx})$. From this it follows immediately that for every $[w]_\sim \in W/\sim$, there is some $T_{[w]_\approx} \in \mathsf{ts}(W/\approx)$ s.t. $\mathsf{H}(T_{[w']_\approx}) \cap \mathsf{H}_{[w']_\sim} \neq \emptyset$. □

Theorem 3.60 establishes a correspondence between index structures and transition structures along the following lines: for every index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, the structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \lll, \mathsf{ts}(W/\approx)\rangle$ is a transition structure. We will show that the correspondence between index structures and transition structures is such that it induces a bijection between the set W and the set of indices of evaluation $\mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$ in the corresponding transition structure. Furthermore, we will illustrate that the relations $\lhd$, $\sim$ and $\sqsubseteq$ accurately reflect the relations between moments and transition sets that underly the intentional operators of the transition language $\mathcal{L}_{\mathsf{t}}$.

As a preliminary result, we first show that, given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$, a pair consisting of an $\sim$-equivalence class $[w']_\sim \in W/\sim$ and a transition set $T_{[w'']_\approx} \in \mathsf{ts}(W/\approx)$ constitutes an index of evaluation in the transition structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \lll, \mathsf{ts}(W/\approx)\rangle$ if and only if the intersection $[w]_\sim \cap [w'']_\approx$ of the two different equivalence classes is non-empty.

Lemma 3.61. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ be an index structure, and let $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \lll, \mathsf{ts}(W/\approx)\rangle$ be the corresponding transition structure. Then for all $[w']_\sim \in W/\sim$ and $T_{[w'']_\approx} \in \mathsf{ts}(W/\approx)$, we have $\mathsf{H}(T_{[w'']_\approx}) \cap \mathsf{H}_{[w']_\sim} \neq \emptyset$ iff $[w']_\sim \cap [w'']_\approx \neq \emptyset$.*

Proof. Let $[w']_{\sim} \in W/\sim$, and let $T_{[w'']_{\approx}} \in \mathsf{ts}(W/\approx)$. Assume that we have $\mathsf{H}(T_{[w'']_{\approx}}) \cap \mathsf{H}_{[w']_{\sim}} \neq \emptyset$. Since, by Prop. 3.13, $\mathsf{H}(T_{[w'']_{\approx}}) = \mathsf{hist}(\mathcal{T}_{[w'']_{\approx}})$, this implies that there is some history $h \in \mathsf{hist}(\mathcal{T}_{[w'']_{\approx}})$ s.t. $[w']_{\sim} \in h$. Consequently, $[w']_{\sim} \in \sigma([w'']_{\approx})$ and hence $[w']_{\sim} \cap [w'']_{\approx} \neq \emptyset$.
Now assume that $[w']_{\sim} \cap [w'']_{\approx} \neq \emptyset$. Then $[w']_{\sim} \in \sigma([w'']_{\approx})$, which implies that $\mathsf{hist}(\mathcal{T}_{[w'']_{\approx}}) \cap \mathsf{H}_{[w']_{\sim}} \neq \emptyset$. Since $\mathsf{hist}(\mathcal{T}_{[w'']_{\approx}}) = \mathsf{H}(T_{[w'']_{\approx}})$, it follows that $\mathsf{H}(T_{[w'']_{\approx}}) \cap \mathsf{H}_{[w']_{\sim}} \neq \emptyset$. □

By Lem. 3.61, for all $[w']_{\sim} \in W/\sim$ and for all $[w'']_{\approx} \in W/\approx$, it holds that $[w']_{\sim}/T_{[w'']_{\approx}} \in \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$ iff $[w']_{\sim} \cap [w'']_{\approx} \neq \emptyset$. Since by Lem. 3.33, the intersection $[w']_{\sim} \cap [w'']_{\approx} \neq \emptyset$ of any two different equivalence classes is either empty or contains but a single element, it follows that every index of evaluation $[w']_{\sim}/T_{[w'']_{\approx}} \in \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$ is identical to a pair $[w]_{\sim}/T_{[w]_{\approx}}$ for some $w \in W$. We can then consider a function $\nu : W \to \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$ that maps every element $w \in W$ onto the corresponding index of evaluation $[w]_{\sim}/T_{[w]_{\approx}} \in \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$. By Lem. 3.61, it is straightforward that the function ν is surjective, and it is injective as well: by Lem. 3.59, $[w']_{\sim}/T_{[w']_{\approx}} = [w'']_{\sim}/T_{[w'']_{\approx}}$ implies $[w']_{\sim} = [w'']_{\sim}$ and $[w'']_{\approx} = [w']_{\approx}$, from which it follows that $w' = w''$.

The correspondence between an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ and its corresponding transition structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \ll, \mathsf{ts}(W/\approx) \rangle$ is moreover such that the primitive relations $\lhd$, $\sim$ and $\sqsubseteq$ of the index structure $\mathcal{W}$ mirror the accessibility relations of the various kinds of intensional operators of the transition language $\mathcal{L}_{\mathsf{t}}$ in the transition structure $\mathcal{T}^{\mathsf{ts}(W/\approx)}$. The relation $\lhd$ reflects the temporal earlier-later relation among the moments in $W/\sim$ that underlies the temporal operators. The relation $\sim$ expresses sameness of moment and hence corresponds to the relation underlying the modal operators. And the relation $\sqsubseteq$, finally, captures the inclusion relation among the transition sets in $\mathsf{ts}(W/\approx)$, which figures in the interpretation of the stability operators.

Lemma 3.62. *Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ be an index structure and let $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \ll, \mathsf{ts}(W/\approx) \rangle$ be the corresponding transition structure. The function $\nu : W \to \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$ with $\nu(w) = [w]_{\sim}/T_{[w]_{\approx}}$ is a bijection and for all $w, w' \in W$ the following holds:*

- ($\lhd$) *$w \lhd w'$ iff $[w]_{\sim} \ll [w']_{\sim}$ and $T_{[w]_{\approx}} = T_{[w']_{\approx}}$;*
- ($\sim$) *$w \sim w'$ iff $[w]_{\sim} = [w']_{\sim}$;*
- ($\sqsubseteq$) *$w \sqsubseteq w'$ iff $[w]_{\sim} = [w']_{\sim}$ and $T_{[w]_{\approx}} \subseteq T_{[w']_{\approx}}$.*

Proof. That the function ν is injective is a straightforward consequence of Lem. 3.33 and Lem. 3.59. We prove that ν is surjective. Consider some arbitrary $[w]_\sim/T_{[w']_\approx} \in \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$. By Lem. 3.61 it follows that we have $[w]_\sim \cap [w']_\approx \neq \emptyset$. This implies by Lem. 3.33 that $[w]_\sim \cap [w']_\approx = \{z\}$ for some $z \in W$. Then $[w]_\sim = [z]_\sim$ and $[w']_\approx = [z]_\approx$, and hence $[w]_\sim/T_{[w']_\approx} = [z]_\sim/T_{[z]_\approx}$. Consequently, $[w]_\sim/T_{[w']_\approx} = \nu(z)$.

($\lhd$) Let $w \lhd w'$. By Def. 3.38 it follows that $[w]_\sim \ll [w']_\sim$. Moreover, by Lem. 3.32, $w \lhd w'$ implies $w \approx w'$, and hence we have $T_{[w]_\approx} = T_{[w']_\approx}$.
Now assume that $[w]_\sim \ll [w']_\sim$ and $T_{[w]_\approx} = T_{[w']_\approx}$. By Lem. 3.59 it follows that $[w]_\approx = [w']_\approx$, and by Def. 3.38 and Lem. 3.33, $[w]_\approx = [w']_\approx$ and $[w]_\sim \ll [w']_\sim$ entails that $w \lhd w'$.

($\sim$) Straightforward.

($\sqsubseteq$) Let $w \sqsubseteq w'$. By Def. 3.30 (vi), $w \sqsubseteq w'$ implies $w \sim w'$, and hence we have $[w]_\sim = [w']_\sim$. Moreover, by Def. 3.44 it follows that $[w]_\approx \Subset [w']_\approx$, which implies by Lem. 3.52 that $\mathcal{T}_{[w']_\approx}$ is a pruning of $\mathcal{T}_{[w]_\approx}$. By Lem. 3.14 we then have that $T_{[w]_\approx} = \xi(\mathcal{T}_{[w]_\approx}) \subseteq \xi(\mathcal{T}_{[w']_\approx}) = T_{[w']_\approx}$.
Now assume that $[w]_\sim = [w']_\sim$ and $T_{[w]_\approx} \subseteq T_{[w']_\approx}$. Then $w \sim w'$. Moreover, since $T_{[w]_\approx} = \xi(\mathcal{T}_{[w]_\approx})$ and $T_{[w']_\approx} = \xi(\mathcal{T}_{[w']_\approx})$, by Lem. 3.14 it follows that $\mathcal{T}_{[w']_\approx} \subseteq \mathcal{T}_{[w]_\approx}$. This implies that $\sigma([w']_\approx) \subseteq \sigma([w]_\approx)$. Consequently, $w \not\sqsupset w'$, for otherwise $\sigma([w]_\approx) \subsetneq \sigma([w']_\approx)$. In conjunction with $w \sim w'$, it then follows by Def. 3.30 (x.a) that $w \sqsubseteq w'$. □

3.3.4 A one-to-one correspondence up to isomorphism

By Theorem 3.31, to every transition structure there corresponds an index structure, and in Theorem 3.60, we have shown that, conversely, to every index structure, there corresponds a transition structure. While those correspondences do not directly induce a bijection between transition structures and index structures, they do give rise to a one-to-one correspondence between the two classes of structures up to isomorphism.

Let $\mathscr{C}$ be the class of all transition structures, and let $\mathscr{I}$ be the class of all index structures. The correspondence established in Theorem 3.31 can be captured by a function $\lambda : \mathscr{C} \to \mathscr{I}$ that maps every transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ onto its corresponding index structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq\rangle$. Likewise, the correspondence between index structures and transition structures resulting from Theorem 3.60 can be described by a function $\zeta : \mathscr{I} \to \mathscr{C}$ that maps every index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ onto the corresponding transition structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \ll, \mathsf{ts}(W/\approx)\rangle$. Obviously, both the functions λ and ζ are injective. We show that they provide a one-to-one correspondence up to isomorphism between the classes $\mathscr{C}$ and $\mathscr{I}$: every transition structure in $\mathscr{C}$ is shown to be order isomorphic to the ζ-image of some index structure in $\mathscr{I}$, and every index structure in $\mathscr{I}$ is shown to be order isomorphic to the λ-image of some transition structure in $\mathscr{C}$. More precisely, we prove that every transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ is order isomorphic to the transition structure $\langle \mathsf{Ind}(\mathcal{M}^{ts})/\sim, \ll, \mathsf{ts}(\mathsf{Ind}(\mathcal{M}^{ts})/\approx)\rangle$ that corresponds via ζ to the index structure $\mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq\rangle$ definable on the set of indices of evaluation $\mathsf{Ind}(\mathcal{M}^{ts})$ in $\mathcal{M}^{ts}$. Also, we prove that every index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ is order isomorphic to the index structure $\langle \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)}), \lhd, \sim, \sqsubseteq\rangle$ that corresponds via λ to the transition structure $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \ll, \mathsf{ts}(W/\approx)\rangle$ lifted from $\mathcal{W}$.

Theorem 3.63. *Consider the functions*

$$\begin{array}{lccc} \lambda: & \mathscr{C} & \to & \mathscr{I} \\ & \mathcal{M}^{ts} = \langle M, <, ts\rangle & \mapsto & \mathcal{X} = \langle \mathsf{Ind}(\mathcal{M}^{ts}), \lhd, \sim, \sqsubseteq\rangle \end{array}$$

and

$$\begin{array}{lccc} \zeta: & \mathscr{I} & \to & \mathscr{C} \\ & \mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle & \mapsto & \mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\sim, \ll, \mathsf{ts}(W/\approx)\rangle. \end{array}$$

The following holds:

(i) *for all $\mathcal{M}^{ts} \in \mathscr{C}$, the structure $\zeta(\lambda(\mathcal{M}^{ts}))$ is order isomorphic to $\mathcal{M}^{ts}$;*

(ii) *for all $\mathcal{W} \in \mathscr{I}$, the structure $\lambda(\zeta(\mathcal{W}))$ is order isomorphic to $\mathcal{W}$.*

Proof.

(i) Let $\mathcal{M}^{ts} = \langle M, <, ts\rangle$ be a transition structure. Then $\zeta(\lambda(\mathcal{M}^{ts})) = \langle \mathsf{Ind}(\mathcal{M}^{ts})/\sim, \ll, \mathsf{ts}(\mathsf{Ind}(\mathcal{M}^{ts})/\approx)\rangle$. Since for every $m \in M$, we have $[m]_\sim \in \mathsf{Ind}(\mathcal{M}^{ts})/\sim$, we can define a function $\rho : M \to \mathsf{Ind}(\mathcal{M}^{ts})/\sim$ with $\rho(m) = [m]_\sim$ for all $m \in M$. By Def. 3.16 it is straightforward that ρ is a bijection. We show that ρ is order preserving. Let $m, m' \in M$ s.t. $m < m'$. By Def. 2.33 there is some $T \in ts$ s.t. $m'/T \in \mathsf{Ind}(\mathcal{M}^{ts})$, and by Lem. 3.18 (i) it follows that $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ as well. Moreover, by Def. 3.16 we then have $m/T \lhd m'/T$, which implies by Def. 3.38 that $[m]_\sim \ll [m']_\sim$.
Since for every $T \in ts$, we have $[T]_\approx \in \mathsf{Ind}(\mathcal{M}^{ts})/\approx$, we can define a function $\rho' : ts \to \mathsf{ts}(\mathsf{Ind}(\mathcal{M}^{ts})/\approx)$ with $\rho'(T) = \xi(\langle \sigma([T]_\approx), \ll|_{\sigma([T]_\approx)}\rangle)$ for all $T \in ts$. By Def. 3.16 and Lem. 3.59 it is straightforward that ρ' is a bijection. We show that ρ' is order preserving as well. Let

$T, T' \in ts$ s.t. $T \subseteq T'$. Since T' is consistent, there is some $m \in M$ s.t. $m/T' \in \mathsf{Ind}(\mathcal{M}^{ts})$, and by Lem. 3.18 (ii) it follows that $m/T \in \mathsf{Ind}(\mathcal{M}^{ts})$ as well. Moreover, by Def. 3.16 we have $m/T \sqsubseteq m/T'$, which implies by Def. 3.44 that $[T]_{\approx} \Subset [T']_{\approx}$. Then by Lem. 3.52, $\langle \sigma([T']_{\approx}), \ll|_{\sigma([T']_{\approx})}\rangle$ is a pruning of $\langle \sigma([T]_{\approx}), \ll|_{\sigma([T]_{\approx})}\rangle$, from which it follows by Lem. 3.14 that $\xi(\langle \sigma([T]_{\approx}), \ll|_{\sigma([T]_{\approx})}\rangle) \subseteq \xi(\langle \sigma([T']_{\approx}), \ll|_{\sigma([T']_{\approx})}\rangle)$.

(ii) Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq\rangle$ be an index structure. Then $\lambda(\zeta(\mathcal{W})) = \langle \mathsf{Ind}(\mathcal{T}^{/,\mathsf{ts}(W/\approx)}), \lhd\ \ \sim, \sqsubseteq\rangle$. By Lem. 3.62 it holds that the function $\nu : W \to \mathsf{Ind}(\mathcal{T}^{\,\mathsf{ts}(W/\approx)})$ with $\nu(w) = [w]_{\sim}/T_{[w]_{\approx}}$ is a bijection. That the function ν preserves the relations $\lhd$, $\sim$ and $\sqsubseteq$ follows by successive application of the conditions for those relations provided in Lem. 3.62 and Def. 3.16, respectively. For illustration, we spell out the two steps for the case of the relation $\lhd$: let $w, w' \in W$ and assume that $w' \lhd w$. By Lem. 3.62 it follows that $[w']_{\sim} \ll [w]_{\sim}$ and $T_{[w']_{\approx}} = T_{[w]_{\approx}}$, which implies by Def. 3.16 that $[w']_{\sim}/T_{[w']_{\approx}} \lhd [w]_{\sim}/T_{[w]_{\approx}}$. □

3.3.5 Validity with respect to $\mathscr{C}$ and $\mathscr{I}$

In section 3.3.2, we have provided a general definition of the notion of an index structure, and in the previous section, we have shown that there is a one-to-one correspondence up to isomorphism between the class of transition structures $\mathscr{C}$ and the class of index structures $\mathscr{I}$. While we have so far focused exclusively on the properties of the respective structures, in this section, we finally turn to the semantic notions of truth and validity. What we eventually aim to show is that validity with respect to the class of transition structures $\mathscr{C}$ is equivalent to validity with respect to the class of index structures $\mathscr{I}$.

The need to come up with the characterization of the notion of an index structure has been triggered by the fact that the transition semantics is not a genuine Kripke-style semantics. In the transition semantics, the semantic evaluation does not solely depend on a moment parameter but is in addition relativized to a transition set, which figures as a defined notion in a transition structure. And what is more, the transition language $\mathcal{L}_{\mathsf{t}}$ comprises intensional operators that involve quantification over the transition parameter. With the notion of an index structure, we have provided a class of Kripke structures in which sentences are to be evaluated at the basic points of the underlying structure and in which quantification over transition sets accordingly dissolves into restricted quantification over those points. Those are the structures that the models that are built up in the completeness construction rest upon.

In this section, we first of all provide the definition of a model of the transition language $\mathcal{L}_{\mathsf{t}}$ on an index structure (section 3.3.5.1). We then show that the transition from index structures to transition structures that we have made in section 3.3.3 preserves satisfiability (section 3.3.5.2). On the basis of the one-to-one correspondence between $\mathscr{C}$ and $\mathscr{I}$ up to isomorphism that we have established in section 3.3.4, this allows us to conclude that a sentence of the transition language $\mathcal{L}_{\mathsf{t}}$ is satisfiable in a transition structure if and only if it is satisfiable in an index structure.

3.3.5.1 Index models

Unlike in a transition structure, in an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, the semantic evaluation is relativized to the elements of the base set of that structure. In this section, we introduce the notion of an index model and spell out the truth conditions for sentences of the transition language.

An *index model* $\mathfrak{W} = \langle W, \lhd \sim, \sqsubseteq, v_{\mathsf{i}} \rangle$ is an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ together with a valuation function v_{i} that assigns truth values to the propositional variables $p \in \mathsf{At}$ relative to an index $w \in W$. As usual, the valuation assigned to the propositional variables $p \in \mathsf{At}$ is then extended to arbitrary sentences of the transition language $\mathcal{L}_{\mathsf{t}}$ by recursive semantic clauses. The three primitive relations $\lhd$, $\sim$ and $\sqsubseteq$ of the index structure $\mathcal{W}$ thereby lay down the accessibility relations of the three different kinds of intensional operators of the transition language. The temporal operators shift the index of evaluation forward or backward, respectively, along the relation $\lhd$, the modal operators shift the index of evaluation along the relation $\sim$, and the stability operators shift the index of evaluation forward along the relation $\sqsubseteq$. Note that while quantification over transition sets is replaced by quantification over the basic elements of the index structure at hand, the semantic clause for the strong future operator F still involves quantification over histories. We use $\mathfrak{W}, w \vDash_{\mathsf{i}} \phi$ in order to indicate that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true in an index model $\mathfrak{W} = \langle W, \lhd \sim, \sqsubseteq, v_{\mathsf{i}} \rangle$ at an index $w \in W$.

DEFINITION 3.64 (Index model). *An* index model *is an ordered quintuple* $\mathfrak{W} = \langle W, \lhd, \sim, \sqsubseteq, v_{\mathsf{i}} \rangle$ *where* $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ *is an index structure and* $v_{\mathsf{i}} : \mathsf{At} \times W \to \{0, 1\}$ *a valuation function.*

(At) $\mathfrak{W}, w \vDash_{\mathsf{i}} p$ iff $v_{\mathsf{i}}(p, w) = 1$;

(¬) $\mathfrak{W}, w \vDash_{\mathsf{i}} \neg\phi$ iff $\mathfrak{W}, w \nvDash_{\mathsf{i}} \phi$;

(∧) $\mathfrak{W}, w \vDash_{\mathsf{i}} \phi \wedge \psi$ iff $\mathfrak{W}, w \vDash_{\mathsf{i}} \phi$ and $\mathfrak{W}, w \vDash_{\mathsf{i}} \psi$;

(P) $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{P}\phi$ iff there is some $w' \in W$ s.t. $w' \lhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \phi$;

(f) $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{f}\phi$ iff there is some $w' \in W$ s.t. $w' \rhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \phi$;

(F) $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{F}\phi$ iff for all $h \in \mathsf{hist}(\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle)$ s.t. $w \in h$, there is some $w' \in h$ s.t. $w' \rhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \phi$;

(□) $\mathfrak{W}, w \vDash_{\mathsf{i}} \Box\phi$ iff for all $w' \sim w$, $\mathfrak{W}, w' \vDash_{\mathsf{i}} \phi$;

(S) $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{S}\phi$ iff for all $w' \sqsupseteq w$, $\mathfrak{W}, w' \vDash_{\mathsf{i}} \phi$.

3.3.5.2 Preservation of satisfiability

With the notion of an index model in place, we can now show that the class of transition structures $\mathscr{C}$ and the class of index structures $\mathscr{I}$ validate exactly the same $\mathcal{L}_{\mathsf{t}}$-sentences. To this end, we prove that the correspondence between index structure and transition structures that we have established step by step throughout section 3.3.3 is such that it preserves satisfiability. The claim then follows immediately because, by Theorem 3.63, every transition structure is order isomorphic to a transition structure that has been lifted from some index structure.

Consider the correspondence $\zeta : \mathscr{I} \to \mathscr{C}$ between index structures and transition structures as stated in Theorem 3.63. Let $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ be an arbitrary index structure in $\mathscr{I}$, and let $\mathcal{T}^{\mathsf{ts}(W/\approx)} = \langle W/\approx, \ll, \mathsf{ts}(W/\approx) \rangle$ be the corresponding transition structure in $\mathscr{C}$. In Lem. 3.62, we have shown that the correspondence ζ induces a bijection $\nu : W \to \mathsf{Ind}(\mathcal{T}^{\mathsf{ts}(W/\approx)})$ with $\nu(w) = [w]_{\sim}/T_{[w]_{\approx}}$ between the base set of the index structure $\mathcal{W}$ and set of indices of evaluation in the corresponding transition structure $\zeta(\mathcal{W}) = \mathcal{T}^{\mathsf{ts}(W/\approx)}$. Since, in the index structure $\mathcal{W}$, the semantic evaluation is relativized to the elements of W, it follows that the function $\mu : \langle \mathcal{W}, v_{\mathsf{i}} \rangle \mapsto \langle \zeta(\mathcal{W}), v_{\mathsf{t}} \rangle$ with $v_{\mathsf{t}}(p, [w]_{\sim}/T_{[w]_{\approx}}) = v_{\mathsf{i}}(p, w)$ for all $p \in \mathsf{At}$ and $w \in W$ is a bijection between index models on $\mathcal{W}$ and transition models on $\zeta(\mathcal{W})$. We show that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is true at a point $w \in W$ in an index model $\mathfrak{W} = \langle W, \lhd, \sim, \sqsubseteq, v_{\mathsf{i}} \rangle$ on $\mathcal{W}$ if and only if ϕ is true at the corresponding pair $\nu(w) = [w]_{\sim}/T_{[w]_{\approx}}$ in the transition model $\mu(\mathfrak{W})$ on $\zeta(\mathcal{W})$.

Proposition 3.65. *Let* $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ *be an index structure. The mapping* $\mu : \langle \mathcal{W}, v_{\mathsf{i}} \rangle \mapsto \langle \zeta(\mathcal{W}), v_{\mathsf{t}} \rangle$ *with* $v_{\mathsf{t}}(p, [w]_\sim / T_{[w]_\approx}) = v_{\mathsf{i}}(p, w)$ *for all* $p \in \mathsf{At}$ *and* $w \in W$ *is a bijection between index models on* $\mathcal{W}$ *and transition models on* $\zeta(\mathcal{W})$*. The following holds: given an index model* $\mathfrak{W} = \langle W, \lhd, \sim, \sqsubseteq, v_{\mathsf{i}} \rangle$*, for every* $\phi \in \mathcal{L}_{\mathsf{t}}$ *and every* $w \in W$*:*

$$\mathfrak{W}, w \vDash_{\mathsf{i}} \phi \textit{ iff } \mu(\mathfrak{W}), [w]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \phi.$$

Proof. The proof runs by induction on the structure of a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ and rests on the induction hypothesis that the claim holds for every proper subformula ψ of ϕ. Given the correspondence μ, the base clause is straightforward. The cases for the truth-functional connectives $\neg$ and $\wedge$ are trivial, and we restrict ourselves here to the cases for the temporal and modal operators and the stability operator.

(P) Assume that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{P}\psi$. Then by Def. 3.64 there is some $w' \in W$ s.t. $w' \lhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. On the basis of Lem. 3.62 and the induction hypothesis it follows that $[w']_\sim \ll [w]_\sim$, $T_{[w']_\approx} = T_{[w]_\approx}$ and $\mu(\mathfrak{W}), [w']_\sim / T_{[w']_\approx} \vDash_{\mathsf{t}} \psi$, which by the standard semantic clause for P implies that $\mu(\mathfrak{W}), [w]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \mathsf{P}\psi$.
Now assume that $\mu(\mathfrak{W}), [w]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \mathsf{P}\psi$. Then by the standard semantic clause for P there is some $[v]_\sim \in W/\sim$ s.t. $[v]_\sim \ll [w]_\sim$ and $\mu(\mathfrak{W}), [v]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \psi$. By Lem. 3.39, there is some $w' \in [v]_\sim$ s.t. $w' \lhd w$. Then $[v]_\sim = [w']_\sim$ and $[w]_\approx = [w']_\approx$. On the basis of the induction hypothesis it follows that $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$, which implies by Def. 3.64 that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{P}\psi$.

(f) Assume that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{f}\psi$. Then by Def. 3.64 there is some $w' \in W$ s.t. $w' \rhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. On the basis of Lem. 3.62 and the induction hypothesis it follows that $[w']_\sim \gg [w]_\sim$, $T_{[w']_\approx} = T_{[w]_\approx}$ and $\mu(\mathfrak{W}), [w']_\sim / T_{[w']_\approx} \vDash_{\mathsf{t}} \psi$, which by the standard semantic clause for f implies that $\mu(\mathfrak{W}), [w]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \mathsf{f}\psi$. Note that by Lem. 3.62 it is guaranteed that $\mathsf{H}(T_{[w]_\approx}) \cap \mathsf{H}_{[w']_\sim} \neq \emptyset$.
Now assume that $\mu(\mathfrak{W}), [w]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \mathsf{f}\psi$. Then by the standard semantic clause for f it follows that there is some $[v]_\sim \in W/\sim$ s.t. $[v]_\sim \gg [w]_\sim$, $\mathsf{H}(T_{[w]_\approx}) \cap \mathsf{H}_{[v]_\sim} \neq \emptyset$ and $\mu(\mathfrak{W}), [v]_\sim / T_{[w]_\approx} \vDash_{\mathsf{t}} \psi$. Let $w' \in W$ be the unique element in $[v]_\sim \cap [w]_\approx$, which exists by Lem. 3.33. Then $[v]_\sim = [w']_\sim$ and $[w]_\approx = [w']_\approx$. On the basis of Lem. 3.62 and the induction hypothesis it follows that $w' \rhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$, which implies by Def. 3.64 that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{f}\psi$.

(F) Assume that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{F}\psi$. Then by Def. 3.64 for all $h \in \mathsf{hist}(\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle)$ s.t. $w \in h$, there is some $w' \in h$ s.t. $w' \rhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. By Lem. 3.14 and Prop. 3.13, for any such history $h \in \mathsf{hist}(\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle)$ and for any such future witness $w' \in h$, we have $\sigma(h) \in \mathsf{H}(T_{[w]_{\approx}}) \cap \mathsf{H}_{[w]_{\sim}}$ and $[w']_{\sim} \in \sigma(h)$. On the basis of Lem. 3.62 and the induction hypothesis it moreover follows that $[w']_{\sim} \gg [w]_{\sim}$, $T_{[w']_{\approx}} = T_{[w]_{\approx}}$ and $\mu(\mathfrak{W}), [w']_{\sim}/T_{[w']_{\approx}} \vDash_{\mathsf{t}} \psi$. Since by Lem. 3.14 every history in $\mathsf{H}(T_{[w]_{\approx}})$ that contains $[w]_{\sim}$ is identical to the σ-image of some history $h \in \mathsf{hist}(\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle)$ with $w \in h$, this implies by the standard semantic clause for F that $\mu(\mathfrak{W}), [w]_{\sim}/T_{[w]_{\approx}} \vDash_{\mathsf{t}} \mathsf{F}\psi$.
Now assume that $\mu(\mathfrak{W}), [w]_{\sim}/T_{[w]_{\approx}} \vDash_{\mathsf{t}} \mathsf{F}\psi$. Then by the standard semantic clause for F for all $h \in \mathsf{H}(T_{[w]_{\approx}})$ s.t. $[w]_{\sim} \in h$, there is some $[v]_{\sim} \in h$ s.t. $[v]_{\sim} \gg [w]_{\sim}$ and $\mu(\mathfrak{W}), [v]_{\sim}/T_{[w]_{\approx}} \vDash_{\mathsf{t}} \psi$. Consider some such history $h \in \mathsf{H}(T_{[w]_{\approx}})$ and with some such future witness $[v]_{\sim} \in h$. Let $w' \in W$ be the unique element in $[v]_{\sim} \cap [w]_{\approx}$, which exists by Lem. 3.33. Then $[v]_{\sim} = [w']_{\sim}$ and $[w]_{\approx} = [w']_{\approx}$. On the basis of Lem. 3.62 and the induction hypothesis it follows that $w' \rhd w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. Since by Lem. 3.14 and Prop. 3.13 the σ-image of any history in $\mathsf{hist}(\langle [w]_{\approx}, \lhd|_{[w]_{\approx}} \rangle)$ that contains w, is a history in $\mathsf{H}(T_{[w]_{\approx}})$ that contains $[w]_{\sim}$, this implies by Def. 3.64 that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{F}\psi$.

($\Box$) Assume that $\mathfrak{W}, w \vDash_{\mathsf{i}} \Box\psi$. Then by Def. 3.64 for all $w' \in W$ s.t. $w \sim w'$, we have $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. By Lem. 3.62 and the induction hypothesis, for any such $w' \in W$ it holds that $[w']_{\sim} = [w]_{\sim}$ and $\mu(\mathfrak{W}), [w']_{\sim}/T_{[w']_{\approx}} \vDash_{\mathsf{t}} \psi$. Since by Lem. 3.62 it is guaranteed that for all $T_{[v]_{\approx}} \in \mathsf{ts}(W/\approx)$ s.t. $\mathsf{H}(T_{[v]_{\approx}}) \cap \mathsf{H}_{[w]_{\sim}} \neq \emptyset$, there is some $w' \in W$ s.t. $w \sim w'$, this implies by the standard semantic clause for $\Box$ that $\mu(\mathfrak{W}), [w]_{\sim}/T_{[w]_{\approx}} \vDash_{\mathsf{t}} \Box\psi$.
Now assume that $\mu(\mathfrak{W}), [w]_{\sim}/T_{[w]_{\approx}} \vDash_{\mathsf{t}} \Box\psi$. Then by the standard semantic clause for $\Box$ for all $T_{[v]_{\approx}} \in W/\approx$ s.t. $\mathsf{H}(T_{[v]_{\approx}}) \cap \mathsf{H}_{[w]_{\sim}} \neq \emptyset$, we have $\mu(\mathfrak{W}), [w]_{\sim}/T_{[v]_{\approx}} \vDash_{\mathsf{t}} \psi$. Consider some such $T_{[v]_{\approx}} \in \mathsf{ts}(W/\approx)$. Let $w' \in W$ be the unique element in $[w]_{\sim} \cap [v]_{\approx}$, which exists by Lem. 3.33. Then $[w]_{\sim} = [w']_{\sim}$ and $[v]_{\approx} = [w']_{\approx}$, and by the induction hypothesis it follows that $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. Since by Lem. 3.62 for all $w' \in W$ s.t. $w \sim w'$, there is some $T_{[w']_{\approx}} \in \mathsf{ts}(W/\approx)$ s.t. $\mathsf{H}(T_{[w']_{\approx}}) \cap \mathsf{H}_{[w]_{\sim}} \neq \emptyset$, this implies by Def. 3.64 that $\mathfrak{W}, w \vDash_{\mathsf{i}} \Box\psi$.

(S) Assume that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{S}\psi$. Then by Def. 3.64 for all $w' \in W$ s.t. $w' \sqsupseteq w$ we have $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. On the basis of Lem. 3.62 and the induction hypothesis, for any such $w' \in W$ it holds that $[w']_{\sim} = [w]_{\sim}$, $T_{[w']_{\approx}} \supseteq T_{[w]_{\approx}}$ and $\mu(\mathfrak{W}), [w']_{\sim}/T_{[w']_{\approx}} \vDash_{\mathsf{t}} \psi$. Since by Lem. 3.62 it is guaranteed that for all $T_{[v]_{\approx}} \in \mathsf{ts}(W/\approx)$ s.t. $T_{[v]_{\approx}} \supseteq T_{[w]_{\approx}}$ and $\mathsf{H}(T_{[v]_{\approx}}) \cap \mathsf{H}_{[w]_{\sim}} \neq \emptyset$, there is some $w' \in W$ s.t. $w' \sqsupseteq w$, this implies by the standard semantic clause for S that $\mu(\mathfrak{W}), [w]_{\sim}/T_{[w]_{\approx}} \vDash_{\mathsf{t}} \mathsf{S}\psi$.

Now assume that $\mu(\mathfrak{W}), [w]_\sim/T_{[w]_\approx} \vDash_{\mathsf{t}} \mathsf{S}\psi$. Then by the standard semantic clause for S for all $T_{[v]_\approx} \in \mathsf{ts}(W/\approx)$ s.t. $T_{[v]_\approx} \supseteq T_{[w]_\approx}$ and $\mathsf{H}(T_{[v]_\approx}) \cap \mathsf{H}_{[w]_\sim} \neq \emptyset$, we have $\mu(\mathfrak{W}), [w]_\sim/T_{[v]_\approx} \vDash_{\mathsf{t}} \psi$. Consider some such $T_{[v]_\approx} \in \mathsf{ts}(W/\approx)$. Let $w' \in W$ be the unique element in $[w]_\sim \cap [v]_\approx$, which exists by Lem. 3.33. Then $[w]_\sim = [w']_\sim$ and $[v]_\approx = [w']_\approx$. On the basis of Lem. 3.62 and the induction hypothesis it follows that $w' \sqsupseteq w$ and $\mathfrak{W}, w' \vDash_{\mathsf{i}} \psi$. Since by Lem. 3.62 for all $w' \in W$ s.t. $w' \sqsupseteq w$, there is some $T_{[w']_\approx} \in \mathsf{ts}(W/\approx)$ s.t. $T_{[w']_\approx} \supseteq T_{[w]_\approx}$ and $\mathsf{H}(T_{[w']_\approx}) \cap \mathsf{H}_{[w]_\sim} \neq \emptyset$, this implies by Def. 3.64 that $\mathfrak{W}, w \vDash_{\mathsf{i}} \mathsf{S}\psi$. □

From Prop. 3.65 it follows that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is satisfiable in an index structure $\mathcal{W} \in \mathscr{I}$ if and only if it is satisfiable in the corresponding transition structure $\zeta(\mathcal{W}) \in \mathscr{C}$. Or, to put it differently, for every sentence $\phi \in \mathcal{L}_{\mathsf{t}}$, it holds that ϕ is valid in the index structure $\mathcal{W}$ if and only if it is valid in the transition structure $\zeta(\mathcal{W})$. Since every transition structure in $\mathscr{C}$ is order isomorphic to the ζ-image of some index structure in $\mathscr{I}$, it then follows that a sentence $\phi \in \mathcal{L}_{\mathsf{t}}$ is valid with respect to the class of index structures $\mathscr{I}$ if and only if it is valid with respect to the class of transition structures $\mathscr{C}$.

Theorem 3.66. *For every $\phi \in \mathcal{L}_{\mathsf{t}}$,*

$$\vDash_{\mathsf{i}} \phi \textit{ iff } \vDash_{\mathsf{t}} \phi.$$

Proof. "⇒": Assume that $\nvDash_{\mathsf{t}} \phi$. Then there is some transition structure $\mathcal{M}^{ts} \in \mathscr{C}$ s.t. $\mathcal{M}^{ts} \nvDash_{\mathsf{t}} \phi$. By Theorem 3.63 it follows that there is an index structure $\mathcal{W} \in \mathscr{I}$ s.t. $\mathcal{M}^{ts} = \zeta(\mathcal{W})$, and by Prop. 3.65 we have that $\mathcal{W} \nvDash_{\mathsf{i}} \phi$. This implies that $\nvDash_{\mathsf{i}} \phi$.
"⇐": Assume that $\nvDash_{\mathsf{i}} \phi$. Then there is some index structure $\mathcal{W} \in \mathscr{I}$ s.t. $\mathcal{W} \nvDash_{\mathsf{i}} \phi$, which implies by Prop. 3.65 that $\zeta(\mathcal{W}) \nvDash_{\mathsf{t}} \phi$. Consequently, it follows that $\nvDash_{\mathsf{t}} \phi$. □

3.4 Concluding remarks

In this chapter, we have provided a general definition of the notion of an index structure, and we have shown that there is a one-to-one correspondence up to isomorphism between the class of index structures $\mathscr{I}$ and the class of transition structures $\mathscr{C}$ that preserves satisfiability. That is, a sentence of the transition language $\mathcal{L}_{\mathsf{t}}$ is valid with respect to the class of transition structures $\mathscr{C}$ if and only if it is valid with respect to the class of index structures $\mathscr{I}$. While in a transition structure $\mathcal{M}^{ts} = \langle M, <, ts\rangle$, sentences are evaluated at pairs consisting of a moment $m \in M$ and a transition set $T \in ts$, which figures as a defined notion, the semantic evaluation on an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ is relativized to the points $w \in W$ of the structure, and quantification over transition sets dissolves into restricted quantification over those points. We said that the notion of an index structure builds the basis for a completeness proof for the transition framework, and we will conclude this chapter with a sketch of the role of index structures in the completeness construction.

Assume that we are given a list of axioms and sound inference rules for the transition framework and that we want to prove the completeness of that system.[81] The proof consists in the step-by-step construction of a model on an index structure, starting out with some arbitrary maximal consistent set of $\mathcal{L}_\mathsf{t}$-sentences that is supposed to be shown to be satisfiable. More precisely, what is constructed is an infinite sequence of index structures $\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2 \ldots$ each of which is endowed with a function that assigns to every point of the respective structure a maximal consistent set of sentences of the transition language $\mathcal{L}_\mathsf{t}$. The construction is initiated by the maximal consistent set that is to be proven to be satisfiable, and the model we are striving for is the limit of the construction, i.e. union of all intermediate steps. It is an index structure together with a valuation function that is determined by the assignment of maximal consistent sets of sentences.

At each single step of the completeness construction, we are provided with a finite index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ whose points are associated with maximal consistent sets of sentences. That an index structure is finite thereby just means that the relation $\lhd$ on W is not serial: there are always maximal elements in the temporal ordering. The assignment of maximal consistent sets of $\mathcal{L}_\mathsf{t}$-sentences to the various points of an index structure can of course not be arbitrary but has to respect the relations holding among those points. For each of the primitive relations $\lhd$, $\sim$ and $\sqsubseteq$ of an index structure, we can define a corresponding relation on the set of maximal consistent sets of $\mathcal{L}_\mathsf{t}$-sentences, which in the following, we denote by mcs. We lift the relation $\lhd$ on the set mcs from the temporal operators, the relation $\sim$ from the modal operators and the relation $\sqsubseteq$ from the stability operators.

[81] In fact, a useful way to go forward actually follows the opposite direction. For, we are not in possession of a list of axioms yet. Rather, the axioms are developed along the way in the completeness construction. Whenever, during the construction, we encounter a counterexample that we are not able to eliminate by the means that we have at our disposal so far, that counterexample is a pointer to a missing axiom.

Definition 3.67 (The set mcs of maximal consistent sets). *Let* mcs *be the* set of maximal consistent sets *of sentences in* $\mathcal{L}_{\mathsf{t}}$.

We define the following relations $\lhd$, $\sim$ *and* $\sqsubseteq$ *on the set* mcs *of maximal consistent sets. For all* $\Gamma, \Sigma \in \mathsf{mcs}$, *we set:*

($\lhd$) $\Gamma \lhd \Sigma$ *iff* $\Sigma \subseteq \{\phi \mid \mathsf{f}\phi \in \Gamma\}$ *iff* $\Gamma \subseteq \{\phi \mid \mathsf{P}\phi \in \Sigma\}$;

($\sim$) $\Gamma \sim \Sigma$ *iff* $\{\phi \mid \Box\phi \in \Gamma\} \subseteq \Sigma$;

($\sqsubseteq$) $\Gamma \sqsubseteq \Sigma$ *iff* $\{\phi \mid \mathsf{S}\phi \in \Gamma\} \subseteq \Sigma$.

Given an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, an assignment $C : W \to \mathsf{mcs}$ of maximal consistent sets of $\mathcal{L}_{\mathsf{t}}$-sentences to the points of the structure that preserves the relations $\lhd$, $\sim$ and $\sqsubseteq$ is called a *chronicle*.

Definition 3.68 (Chronicle). *Given an index structure* $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$, *a* chronicle *is a function* $C : W \to \mathsf{mcs}$ *such that for all* $w, w' \in W$, *the following holds:*

($\lhd$) *if* $w \lhd w'$, *then* $C(w) \lhd C(w')$;

($\sim$) *if* $w \sim w'$, *then* $C(w) \sim C(w')$;

($\sqsubseteq$) *if* $w \sqsubseteq w'$, *then* $C(w) \sqsubseteq C(w')$.

Note that the chronicle function C is not required to be injective. Different points of an index structure can be associated with the very same maximal consistent set. Moreover, the definition of a chronicle does not rule out a case in which the C-images of elements in W are linked via either of the relations $\lhd$, $\sim$ or $\sqsubseteq$ even though their respective pre-images are not. There can, for instance, be $w, w' \in W$ with $C(w) \lhd C(w')$ while $w \not\lhd w'$; and similarly for the relations $\sim$ and $\sqsubseteq$.

What we are given at the outset is a maximal consistent set of sentences, say Λ, which we want to show to be satisfiable. In the base step of the completeness construction, we then start out with an index structure $\mathcal{W}_0 = \langle \{w_0\}, \emptyset, \{\langle w_0, w_0 \rangle\}, \{\langle w_0, w_0 \rangle\} \rangle$ that is made up from just a single element, and we define a chronicle C_0 on $\mathcal{W}_0$ that assigns to that single element the maximal consistent set Λ, i.e. we let $C_0 : \{w_0\} \to \mathsf{mcs}$ be the function with $C_0(w_0) = \Lambda$.

In the step-by-step construction, the transition from an index structure $\mathcal{W}_n$ with chronicle C_n to the next index structure $\mathcal{W}_{n+1}$ with chronicle C_{n+1} is based on the technique of elimination of counterexamples. To be concrete, we consider an infinite list of all sentences of the transition language $\mathcal{L}_\mathsf{t}$ that have existential import. That is, we consider an infinite sequence $\alpha_0, \alpha_1, \alpha_2, \ldots$ of $\mathcal{L}_\mathsf{t}$-sentences of the form $\mathsf{P}\phi$, $\mathsf{f}\phi$, $\mathsf{F}\phi$, $\Diamond\phi$ and $\mathsf{2}\phi$ in which every sentence occurs infinitely often. The index structure $\mathcal{W}_{n+1}$ with chronicle C_{n+1} is then obtained from the index structure $\mathcal{W}_n$ with chronicle C_n by eliminating all possible counterexamples pertaining to the sentence $\alpha_n \in \mathcal{L}_\mathsf{t}$ in the structure $\mathcal{W}_n$, which requires adding new points to the structure $\mathcal{W}_n$ and adjusting both the relations and the chronicle accordingly. A counterexample in an index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ with chronicle C is thereby defined as a pair (w, α) consisting of a point $w \in W$ and a sentence $\alpha \in \mathcal{L}_\mathsf{t}$ that is of one of the mentioned forms and that is contained in $C(w)$ but lacks an appropriate witness in the index structure $\mathcal{W}$ with chronicle C.

DEFINITION 3.69 (Counterexample). *Let* $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$ *be an index structure with chronicle* C *and let* $w \in W$. *We say that* (w, α) *is a* counterexample *in* $\mathcal{W}$ *iff* $\alpha \in \mathcal{L}_\mathsf{t}$ *is of one of the following forms and the corresponding condition is fulfilled:*

($\mathsf{P}\phi$) $\mathsf{P}\phi \in C(w)$ *and there is no* $w' \in W$ *such that* $w' \lhd w$ *and* $\phi \in C(w')$;

($\mathsf{f}\phi$) $\mathsf{f}\phi \in C(w)$ *and there is no* $w' \in W$ *such that* $w' \rhd w$ *and* $\phi \in C(w')$;

($\mathsf{F}\phi$) $\mathsf{F}\phi \in C(w)$ *and there is some history* $h \in \langle [w]_\approx, \lhd|_{[w]_\approx} \rangle$ *such that there is no* $w' \in h$ *such that* $w' \rhd w$ *and* $\phi \in C(w')$;

($\Diamond\phi$) $\Diamond\phi \in C(w)$ *and there is no* $w' \in W$ *such that* $w \sim w'$ *and* $\phi \in C(w')$;

($\mathsf{2}\phi$) $\mathsf{2}\phi \in C(w)$ *and there is no* $w' \in W$ *such that* $w \sqsubseteq w'$ *and* $\phi \in C(w')$.

The possibility of the step-by-step construction rests on the following conjecture, which states that all possible counterexamples are in fact eliminable step by step. It ensures that for every index structure $\mathcal{W}_n$ with chronicle C_n there is always a finite index structure $\mathcal{W}_{n+1} \supseteq \mathcal{W}_n$ with chronicle C_{n+1} in which any alleged counterexample (w, α_n) in $\mathcal{W}_n$ does not constitute a counterexample anymore.

CONJECTURE. *For every finite index structure* $\mathcal{W}$ *with chronicle* C *and every counterexample* (w, α) *in* $\mathcal{W}$, *there exists a finite index structure* $\mathcal{W}' \supseteq \mathcal{W}$ *with chronicle* C' *in which* (w, α) *is no longer a counterexample.*

The proof of that theorem requires, for each possible counterexample (w, α) in $\mathcal{W}$, the specification of a general construction method that captures the transition from the index structure $\mathcal{W}$ with chronicle C to the respective index structure $\mathcal{W}' \supseteq \mathcal{W}$ with chronicle C'. In order to illustrate what the specification of such a construction method amounts to, let us consider the case of a counterexample of the form $(w, \mathsf{P}\phi)$ in the index structure $\mathcal{W} = \langle W, \lhd, \sim, \sqsubseteq \rangle$. So, assume that $\mathsf{P}\phi \in C(w)$ and that there is no $w' \in W$ such that $w' \lhd w$ and $\phi \in C(w')$. Without loss of generality, we can assume that w is the $\lhd$-smallest element in W such that $\mathsf{P}\phi \in C(w)$. Now two possible cases can be considered. Either w does not have a $\lhd$-predecessor at all, or there exists an immediate predecessor $w'' \lhd w$ and $\neg\phi \wedge \mathsf{H}\neg\phi \in C(w'')$. In the first case, we just add a new point x below w, i.e., we let $x \lhd^* w$, while in the second case, we have to insert x in between w'' and w, i.e., we set $w'' \lhd^* x \lhd^* w$; and in each case, we need to associate our new point x with a maximal consistent set that contains ϕ and that satisfies $C^*(x) \lhd C(w)$ or $C(w'') \lhd C^*(x) \lhd C(w)$, respectively. As the resulting structure is required to be an index structure again, adding the new point x forces us to add even further points. To wit, we have to add corresponding past nodes for every point that is $\sim$-related to w in such a way that all relevant structural properties are preserved; and of course, we have to associate each of the new points with a maximal consistent set in such a way that the resulting assignment forms a chronicle. The existence of the required maximal consistent sets is to be guaranteed by the axioms, and by means of the Gabbay Irreflexivity Rule it can be ensured that the respective pasts of all $\sim$-related points are order isomorphic in each case. The index structure $\mathcal{W}'$ with chronicle C' that results from the construction is thus obtained from the index structure $\mathcal{W}$ with chronicle C via the addition of new points and corresponding extensions of the relations and the chronicle function, i.e., $W' = W \cup \{x, \dots\}$, the relation $\lhd'$ is the smallest strict partial order on W' containing $\lhd \cup \lhd^*$, the relation $\sim'$ is the smallest equivalence relation containing $\sim \cup \sim^*$, the relation $\sqsubseteq'$ is the smallest partial order containing $\sqsubseteq \cup \sqsubseteq^*$, and finally, for the chronicle, we have $C' = C \cup C^*$.

Whether we can construct an infinite sequence of index structures $\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2, \dots$ with chronicles $C_0 \subseteq C_1 \subseteq C_2, \dots$ by the technique of elimination of counterexamples, starting out with an arbitrary maximal consistent set of $\mathcal{L}_\mathrm{t}$-sentences, crucially hinges on the provability of the above conjecture. Obstacles in the proof of that conjecture are pointers to missing axioms. If the chronicle construction initiated by some arbitrary maximal consistent set Λ succeeds, its limit, which is just the union of all intermediate steps, is an index structure $\mathcal{W}_\infty = \langle W_\infty, \lhd_\infty, \sim_\infty, \sqsubseteq_\infty \rangle$ with chronicle C_∞ that is entirely free of counterexamples. Every intermediate index structure is a substructure of $\mathcal{W}_\infty$, and the chronicle of each such intermediate index structure is included in C_∞. The chronicle C_∞ naturally induces a valuation v_i on the index structure $\mathcal{W}_\infty$. For all $p \in \mathsf{At}$ and $w \in W_\infty$, we set $v_\mathsf{i}(w,p) = 1$ if and only if $p \in C_\infty(w)$. The resulting model $\mathfrak{W}_\infty = \langle \mathcal{W}_\infty, v_\mathsf{i} \rangle$ is the model we were striving for. It can easily be shown that for every arbitrary sentence $\phi \in \mathcal{L}_\mathsf{t}$ and every $w \in W_\infty$ it holds that ϕ is true at w in the model $\mathfrak{W}_\infty$ if and only if ϕ is an element of $C_\infty(w)$.

LEMMA 3.70. *Let the index structure $\mathcal{W}_\infty = \langle W_\infty, \lhd_\infty, \sim_\infty, \sqsubseteq_\infty \rangle$ with chronicle C_∞ be the limit of the infinite sequence of index structures $\mathcal{W}_0 \subseteq \mathcal{W}_1 \subseteq \mathcal{W}_2, \ldots$ with chronicles $C_0 \subseteq C_1 \subseteq C_2, \ldots$ generated by the technique of elimination of counterexamples, and let $\mathfrak{W}_\infty = \langle \mathcal{W}_\infty, v_\mathsf{i} \rangle$ be a model on $\mathcal{W}_\infty$ with $v_\mathsf{i}(w,p) = 1$ iff $p \in C_\infty(w)$. Then for every $\phi \in \mathcal{L}_\mathsf{t}$ and for every $w \in W_\infty$, the following holds:*

$$\mathfrak{W}_\infty, w \vDash_\mathsf{i} \phi \quad \text{iff} \quad \phi \in C_\infty(w).$$

Since, as we have shown, there is a one-to-one correspondence up to isomorphism between the class of index structures $\mathscr{I}$ and the class of transition structures $\mathscr{C}$, from Lem. 3.70 it follows immediately—under the above conjecture—that the maximal consistent set of $\mathcal{L}_\mathsf{t}$-sentences Λ that we have started out with and whose satisfiability was to be proven is in fact satisfiable in the transition semantics. Once a proof of our conjecture and, with it, all required axioms and inferences rules are in place, this concludes the completeness proof.

* * *

Chapter 4

Branching Time Models for Real Possibility

4.1 Introduction

The overall aim of this thesis is to develop a semantics for real possibility. In chapter 1, we have introduced the notion of real possibility as a special kind of possibility, contrasting it with epistemic possibility and the prevalent group of alethic possibilities: logical, metaphysical and physical possibility. Real possibilities are alternative possibilities for the future in an indeterministic world. Just as logical, metaphysical and physical possibilities, real possibilities are alethic possibilities. They are possibilities in an ontological sense. Yet, in contradistinction to logical, metaphysical and physical possibilities, real possibilities are concretely anchored in time, and they are dynamic. They represent temporal alternatives for a dynamic actuality to evolve rather than mere modal alternatives to a given actuality. What is more, the notion of reality that is at the heart of real possibility does not exhaust itself in the logical form, the nature of things or the laws of nature. Real possibilities are stronger tied up with the world than logical, metaphysical and physical possibilities are. Real possibilities in addition depend on the concrete momentary circumstances. What is really possible in a situation is what can temporally evolve from that concrete situation against the background of what the world is like—logically, metaphysically and physically.

The Prior-Thomason theory of branching time provides an adequate formal framework for modeling real possibilities as dynamic future possibilities. In a branching time structure, the modal-temporal structure of the world is represented as a tree of moments that branches toward an open future. At any given moment, the past is fixed, while there may be alternative possibilities for the future to evolve. In chapter 2, we have presented a novel branching time semantics for a propositional language that makes use of parameters of truth that are suited for capturing the temporal dynamics of real possibilities, and we have investigated the properties of the resulting structures on the indices of evaluation in chapter 3. Our semantics builds on the notion of a transition, which, as a structural element, captures the local change at a moment. Building on sets of transitions, the transition semantics exploits the structural resources a branching time structure has to offer and enables a fine-grained picture of the interrelation of modality and time. It adequately reflects the temporal aspect of real possibilities.

The definition of a branching time structure allows for uncountably many structures, and each branching time structure allows for several models. Not every branching time model that is possible from a logical point of view is, however, a model for real possibility. Real possibilities are alternative future possibilities in an indeterministic world, and as such they presuppose models that bear an intimate relation to reality. A branching time model for real possibility needs to incorporate a close link between the language, the branching time structure and the world. Each history depicted in a branching time model for real possibility must represent a course of events that is possible against the background of the prevailing laws of nature. In particular, the model has to fit the underlying branching time structure: there must be a match between the arity of the local branching structure and the momentary circumstances at hand.

In this chapter, our aim is to systematically single out those branching time models that are in fact models for real possibility, i.e. that adequately accommodate the worldly aspect of real possibilities. While the search for appropriate parameters of truth, that we have focused on so far, constitutes a logical challenge, the search for appropriate models poses a metaphysical one. Narrowing down the space of possibilities to real possibilities requires a metaphysical explanation. In section 4.2, we will perform a transition from logical to metaphysical considerations and make a first step in the direction of branching time models for real possibility by tying the language to the world. The challenge that remains and that constitutes the main concern of this chapter is how to guarantee the fit between the model and the underlying branching time structure.

In section 4.3, we will argue that in order to provide a systematic account of branching time models for real possibility, we cannot start out with a given branching time structure and restrict the possible models on that structure to models for real possibility by means of certain additional metaphysical constraints. Rather, the model and its underlying structure need to be developed hand in hand. In particular, we will show that the model together with its underlying structure must be lifted in dynamic fashion, in accordance the with the temporal dynamics of real possibilities. To this end, some modal element is needed.

In section 4.4, we will present a dynamic, modal explanation of branching time models for real possibility in terms of potentialities. The core idea is this: by manifesting their potentialities, objects become causally efficacious and jointly give direction to the possible future courses of events. The manifestations of potentialities will be construed as transitions that point toward the future. Unlike the notion of a transition employed in the transition semantics, the manifestations of potentialities are, first of all, not part of a branching time structure but are abstract transitions from a state of affairs to another state of affairs or into a process. We will show how those transitions allow us to successively lift a branching time model for real possibility together with its underlying structure from a single momentary circumstance. By grounding real possibilities in potentialities, we provide a systematic account of branching time models for real possibility that elucidates why all possible courses of events are in accordance with the prevailing laws of nature.

4.2 From Logical to metaphysical considerations

Our endeavor is the search for a semantics for the notion of real possibility. As a basis of our investigations, we make use of the framework of branching time, which allows for a perspicuous representation of real possibilities as alternative future possibilities.

In chapter 2, we have devised a modal-temporal language that is suited for reasoning about real possibilities and provided an appropriate interpretation of that language on a branching time structure. The framework of branching time has been developed for precisely those semantic purposes. It finds its application in the semantics of languages containing temporal and modal operators. The languages of study are usually formal languages, in which complex sentences are recursively built up from a given stock of propositional variables and a fixed set of primitive logical connectives and intensional operators. The interpretation of a language on a branching time structure involves the notion of a model, which ties the language to the structure by fixing the truth values of the propositional variables on the structure. The interpretation can then be extended to complex sentences by means of recursive semantic clauses.

The task of providing a semantics for the notion of real possibility is twofold: there is a logical and a metaphysical challenge. From a logical point of view, the challenge consists in the search for appropriate parameters of truth. The choice of parameters of truth bears on the interaction of temporal and modal operators and thus has implications for the representation of the interrelation of modality and time, which is at the heart of the notion of real possibility. The notion of real possibility is itself, however, not a logical notion but rather a metaphysical one. Real possibilities are future possibilities in an indeterministic world—this is essentially where the motivation of the branching time framework comes from.[82] The models of our language only gain significance for the notion of real possibility when they properly relate to reality. There needs to be a pertinent link between the interpretation of the language, the branching time structure and the world. From a metaphysical point of view, the challenge is the search for appropriate models.

Up to now, the logical challenge has been in the focus of our investigations, and we will briefly reflect on the precise character of those investigations in section 4.2.1 below. In section 4.2.2, we then turn toward the metaphysical challenge, which forms the topic of the present chapter. By tying the language to the world, we pave the way for the search for branching time models for real possibility and reveal what the key challenge consists in.

[82] For the original motivation of the framework of branching time, see Ploug and Øhrstrøm (2012).

4.2.1 A logical challenge: The search for appropriate parameters of truth

The result of our search for appropriate parameters of truth is the transition semantics presented in chapter 2. The parameters of truth employed in that framework are tailored for modeling real possibilities as alternative future possibilities. In particular, they allow us to adequately capture the temporal dynamics of real possibilities.

We have seen that branching time structures allow for a variety of different semantic approaches that differ with respect to which structural elements—over and above a moment—are employed as parameters of truth in the semantic evaluation. While on the Peircean account, truth is relativized to a moment parameter only, Ockhamism and the transition semantics make use of a second parameter of truth that specifies some possible course of events. On the Ockhamist account, the second parameter of truth is provided by a history, which singles out a complete possible course of events, whereas in the transition semantics, the second parameter of truth is a consistent, downward closed set of indeterministic transitions, which can represent an incomplete possible course of events as well. The choice of parameters of truth has an impact on the interpretability of temporal and modal operators and hence affects the expressivity of the respective language. While on the Peircean account, only temporal operators are interpretable, on the Ockhamist account, modal operators can be interpreted as well, and on the transition account, in addition to temporal and modal operators, a stability operator enters the picture.

How the temporal and modal operators are to be interpreted in each case is laid down by recursive semantic clauses. Together with the semantic clauses for the primitive logical connectives, those semantic clauses extend the interpretation of the propositional variables in a model to any arbitrary sentence of the language. They specify in completely general terms under which conditions a sentence is true in a model at an index of evaluation. In contrast to the semantic clauses for the truth-functional connectives, the recursive semantic clauses for the temporal and modal operators involve a shift of the parameters of truth. They operate on the set of all possible indices of evaluation, and in doing so, they make use of the structural relations holding between those indices in a branching time structure.

Different parameters of truth allow for different witnesses for future possibilities. In section 2.2, we have illustrated how the choice of parameters of truth affects the interpretation of future possibilities and motivated the prospects of our approach. While the Peircean account allows for alternative future possibilities, it does not allow for a notion of plain future truth and hence is unable to make explicit what needs to happen for a future possibility to come about. The Ockhamist accounts does better in this respect. By making use of a history as a second parameter of truth next to the moment parameter, Ockhamism establishes a link between actuality and possibility. The transition semantics further enhances the Ockhamist idea: in contrast to Ockhamism, it allows for local witnesses for future possibilities. The stability operator allows us to specify exactly how and how far the future has to unfold for it to be settled that a future possibility will be realized, and thereby enables a perspicuous treatment of real possibilities. The dynamics of the transition semantics adequately reflects the dynamics of real possibilities: their transition from contingency to stability.

What is key to the interpretation of future possibilities are the structural relations holding between the possible indices of evaluation. In the structure on those indices the interrelation of modality and time comes to the fore. And the logical challenge consists in the choice of parameters of truth that give rise to a structure that adequately represent that interrelation. In the transition semantics, the logical challenge is resolved by replacing the Ockhamist history parameter by a dynamic transition parameter. The transition semantics exploits the full resources a branching time structure has to offer and allows for a fine-grained, dynamic picture of the interrelation of possibility and time. The resulting structure on the indices of evaluation is a tree of substructures of branching time structures that accommodates the idea of the passage of time, rather than a single branching time structure or a pluriverse of parallel linear possible worlds. The temporal dynamics of real possibilities can be captured on purely structural grounds, abstracting away from particular models. Accommodating the peculiar temporal aspect of real possibilities is, first and foremost, a matter of choice of which structural elements to employ as parameters of truth in the semantic evaluation, and as such it is primarily a logical matter.

4.2.2 A metaphysical challenge: The search for appropriate branching time models

There is one aspect of the notion of real possibility that we have not taken into account so far, and that is its intimate relation to the world. Real possibilities are worldly possibilities. What is really possible at a moment is what can temporally evolve from the concrete situation obtaining at that moment against the background of the prevailing laws of nature. This aspect of real possibilities cannot be warranted on purely structural grounds but places restrictions on the admissible models. There must be a close and systematic connection between the interpretation of the language, the branching time structure and the world. For a branching time model to constitute a model for real possibility, each history must depict a possible course of events that conforms to the prevailing laws of nature, and all courses of events that are possible under the prevailing laws must be represented. In particular, the model must fit the underlying branching time structure. For example, if the situation obtaining at a moment is such that the laws of nature allow for three possible future continuations, the local branching structure must be a tripartition. Singling out those branching time models that are in fact models for real possibility is not a logical enterprise but requires a transition from logical to metaphysical considerations.

In order to be able to systematically narrow down the space of possibilities to real possibilities, we first of all have to establish some link between the atomic sentences of the language and the world. Only once that link is in place, can we, in a second step, address the question how to guarantee the fit between the model and its underlying structure. The first step involves, first, that we relativize the interpretation of the propositional variables of the language on a branching time structure to a moment parameter only, second, that we break the atomic sentences down into their constituent parts and, third, that we conceive of their constituent parts as being properly tied up with some fundamental ontology. In doing so, we restrict the overall range of branching time models that are in principle possible from a logical point of view to those in which the totality of atomic sentences that are true at a moment represents a momentary, metaphysical possibility in each case. The challenge that remains and that constitutes the main concern of this chapter is how to further delimit the range of branching time models obtained that way to branching time models for real possibility; that is, how to eventually single out those branching time models in which each succession of momentary, metaphysical possibilities is in accordance with the laws of nature. In the remainder of this section, we will devote ourselves for a start to the first step, discussing one by one the three amendments it involves.

4.2.2.1 Moment-relative truth

When concerned with the question which structural elements to employ as parameters of truth in the semantic evaluation, our focus was primarily on complex sentences, and we have not paid any special attention to the atomic ones. We have treated propositional variables as arbitrary sentences, and we have relativized their truth values to the same parameters of truth as those of complex sentences. In other words, we have assumed that the sequence of parameters of truth on which the valuation of the propositional variables in a model depends perfectly matches the sequence of parameters of truth to which the recursive semantic machinery is relativized. Accordingly, we have distinguished between Peircean, Ockhamist and transition models. If one is mainly interested in the validities that capture the interrelation of the various kinds of intensional operators, treating propositional variables as arbitrary sentences is advantageous as it preserves the substitution property of the resulting logic.

From a metaphysical point of view, on the other hand, it suggests itself to have the truth values of the atomic sentences in a model on a branching time structure depend solely on a moment parameter. Moments are the basic building blocks of branching time structures, and atomic sentences can be conceived of as providing the fundamental—non-temporal and non-modal—facts that can possibly obtain at those basic building blocks.[83] A moment-relative branching time model then fixes what ultimately is the case at the various building blocks

[83]Sometimes, as, for example, in Reynolds (2002) and Thomason (1984), the metaphysical assumption that atomic sentences represent fundamental facts and do not contain traces of futurity is built directly into the logical system, and the truth values of propositional variables are relativized to a moment parameter only, irrespective of whether the recursive semantic machinery makes use of a second parameter of truth. The assumption that atomic sentences never contain traces of futurity is, however, not entirely uncontroversial. Some authors claim that at least for some atomic sentences, this is not the case. They therefore require the truth values of atomic sentences at a moment to depend on a second parameter of truth. Along those lines, Ming Xu, for instance, proposes that in Ockhamism, the atomic sentences should be assigned truth values relative to moment-history pairs with the additional constraint that their truth values at a moment cannot vary with respect to histories that are undivided at that moment (cf. Zanardo 1996, pp. 2f.). Prior contemplates the idea to have a two sorted language that contains two different kinds of atomic sentences that presuppose a different treatment, depending on whether they contain traces of futurity or not (see Prior 1967, pp. 124ff.).

of the underlying structure. In what follows, we presuppose that the atomic sentences of our language do not contain traces of futurity, and we use AT to refer to the respective set of atomic sentences—regardless of whether we are dealing with a propositional or a first-order language.

Definition 4.1 (A moment-relative branching time model). *For* AT *a set of atomic sentences, a* moment-relative branching time model *is an ordered triple* $\mathfrak{M} = \langle M, <, v\rangle$, *where* $\mathcal{M} = \langle M, <\rangle$ *is a branching time structure and* $v : \mathsf{AT} \times M \to \{0, 1\}$ *a valuation function.*

Branching time models in which the truth values of the atomic sentences depend only on a moment parameter can serve as a basis for any of the different semantic approaches that we have discussed.[84] In case the recursive semantic machinery is in addition relativized to a second parameter of truth next to the moment parameter, as in Ockhamism and likewise in the transition semantics, the moment-dependent interpretation of the atomic sentences in the model can be straightforwardly extended to that second parameter. An atomic sentence can be defined true at an index pair that comprises at least a moment of evaluation if and only if it is true at the given moment with respect to any value of the second parameter of truth, which given the semantic clauses for the respective modal operators, amounts to its truth at that moment being settled.

4.2.2.2 Atomic sentences as structured entities

So far, we have been exclusively concerned with propositional languages, in which complex sentences are recursively built up from a given stock of propositional variables and a fixed set of primitive logical connectives and temporal and/or modal operators. Atomic sentences are the ultimate constituents of a propositional language. They are represented by propositional variables and are not further decomposable.

The transition to metaphysical considerations, however, creates the need to be able to reason about objects, which have certain properties and stand in certain relations. This requires that we make the internal structure of atomic sentences explicit and break them down into their constituent parts. In other words, we will conceive of atomic sentences as structured entities that are built up from a given stock of individual constants, predicates and n-ary relation

[84] Since on the Peircean account, the semantic evaluation depends only on a moment parameter, the notion of a moment-relative branching time model coincides with the notion of a Peircean branching time model (Def. 1.9).

symbols. Each atomic sentence is composed of either a predicate or an n-ary relation symbol and a suitable number of individual constants. Formally, this means that we move from a propositional language to a quantifier-free first-order language.

Definition 4.2 (Structured atomic sentences). *Assume that, instead of a set of propositional variables, the alphabet of our language contains a denumerable stock of the following symbols:*

- *individual constants:* $c_1, c_2, c_3, \ldots$;
- *predicates:* $P_1, P_2, P_3, \ldots$;
- *n-ary relation symbols (for every $n > 1$):* $R^n_1, R^n_2, R^n_3, \ldots$.[85]

Structured atomic sentences are formed along the following lines:

- *if φ is a predicate and α is an individual constant, then $\varphi(\alpha)$ is a structured atomic sentence;*
- *if φ^n is an n-ary relation symbol and $\alpha_1, \ldots, \alpha_n$ are individual constants, then $\varphi^n(\alpha_1, \ldots, \alpha_n)$ is a structured atomic sentence.*

Whereas in the propositional case, the valuation function of a branching time model immediately assigns truth values to the atomic sentences relative to a moment, in the first-order case, a model fixes the truth values of the atomic sentences at a moment by assigning appropriate extensions to the individual constants, predicates and n-ary relation symbols relative to that moment on the basis of some overall domain of objects. More precisely, at each moment, the constituents of the atomic sentences receive a set-theoretic interpretation against the background of a given domain of objects, and recursive semantic clauses are provided that specify under which conditions an atomic sentence is true at a moment, by reference to the relations obtaining between the set-theoretic entities associated with its constituent parts at that moment.

Definition 4.3 (A first-order branching time model). *For* AT *a set of structured atomic sentences, a* first-order branching time model *is an ordered quintuple* $\mathfrak{M} = \langle M, <, D, d, I \rangle$*, where*

- $\mathcal{M} = \langle M, < \rangle$ *is a branching time structure;*
- D *is a non-empty domain of objects;*

[85] From a logical point of view, predicates are just unary relation symbols. That is, we could simply subsume predicates under relations, treating them as relations of arity one. However, from a metaphysical point of view, a separate treatment of predicates and relations seems favorable.

- $d : M \to \mathsf{Pow}(D)$;
- *I is an interpretation function such that*
    - $I(\alpha, m) \in D$ *for every individual constant* α*;*
    - $I(\varphi, m) \subseteq d(m)$ *for every predicate* φ*;*
    - $I(\varphi^n, m) \subseteq d(m)^n$ *for every n-ary relation symbol* φ^n*.*

The following semantic clauses specify under which conditions a structured atomic sentence is true in a first-order branching time model $\mathfrak{M}$ *at a moment* $m \in M$*:*

- $\mathfrak{M}, m \vDash \varphi(\alpha)$ *iff* $I(\alpha, m) \in I(\varphi, m)$*;*
- $\mathfrak{M}, m \vDash \varphi^n(\alpha_1, \ldots, \alpha_n)$ *iff* $\langle I(\alpha_1, m), \ldots, I(\alpha_n, m)\rangle \in I(\varphi^n, m)$*.*

Objects persist through time, and that means that we have to be able to identify the same object across different moments. In the following, we will assume that individual constants amount to proper names and treat them as rigid designators that refer to the same object at every moment.[86] That is, we focus on first-order branching time models whose interpretation function satisfies the following constraint: for every individual constant α and for all $m, m' \in M$, we have $I(\alpha, m) = I(\alpha, m')$.[87] As a consequence, every object is considered to be wholly present at each moment at which it exists. Or, to put it differently, we assume that objects endure through time.[88]

4.2.2.3 Tying the language to the world

The modal-temporal languages that we have considered up to now are purely formal languages. They are mere syntactic constructs. The sentences of our languages are recursively built up from a given alphabet according to fixed rules.

[86] The term ‘rigid designator’ was coined by Kripke. In its original meaning, a rigid designator is a term that refers to the same entity in all possible worlds in which the entity exists. See Kripke (1980). Note that we assume here that an individual constant refers to the same object at every moment, irrespective of whether the object exists at all those moments.

[87] First-order systems based on the theory of branching time are still rare. Wölfl (1999) presents a predicate logic for branching time. In that framework, individual constants are treated as rigid designators along the same lines as we do here. Rather than conceiving of individual constants as rigid designators, an intensional account of objects can be given. This requires that we equip our first-order models with an additional set of intensional objects and assign to each individual constant at every moment the same intensional object, viz. a function $int : M \to D$. In such a framework, predication can be either extensional or intensional. For an account that builds on intensional objects and makes use of intensional predication, see Belnap and Müller (2014b,a).

[88] According to Lewis, an object can persist through time by either *enduring* or *perduring*. While endurantism is the thesis that an object is wholly present at each time at which it exists, perdurantism is the idea that objects persists by having different temporal parts at different times. See D. Lewis (1986a, ch. 4.2).

They can be interpreted in a model but are otherwise devoid of any content. The interpretation of complex sentences thereby builds on the interpretation of the atomic sentences of the language, and the latter only receive meaning by being assigned certain truth values relative to a moment in a branching time model—or their constituents being assigned certain extensions at each moment against the background of some overall domain of objects. In the first-order case, the objects contained within the domain are considered bare particulars, and properties and relations are nothing but arbitrary subsets of the domain or its Cartesian product, respectively. In short, on such a formal approach, the atomic sentences and their interpretation do not bear any relation to the world.

Real possibilities, however, are possibilities in an indeterministic world. They are intimately connected with reality. In order for us to be able to reason about real possibilities, the atomic sentences and their interpretation at a moment in a model must properly relate to reality. If the interpretation of the atomic sentences at the various moments in a branching time model is completely detached from the world, we cannot even reasonably ask whether the branching time model is a model for real possibility to begin with, viz. whether each possible course of events conforms to the prevailing laws of nature. Taking into account the worldly aspect of real possibilities requires a transition from a purely formal to a formalized language.[89] The atomic sentences can no longer be conceived of as mere syntactic strings of symbols, but their constituents must be firmly tied up with entities in the world.

4.2.2.3.1 The world What then is the world with which our language is to be tied up? It is crucial to our approach that we remain neutral on that question. Our aim is not to eventually single out a unique branching time model that is supposed to represent this world, the world we actually live in. Nor will we restrict our investigations to this-worldly entities. Our aim is rather to provide a systematic account of branching time models for real possibility based on some fundamental ontology.

Our starting point is some stock of fundamental objects, properties and relations. This basic stock provides the furniture of our world. It lays down our fundamental ontology. The entities from which our fundamental ontology is made up determine by their very nature which maximal configurations of

[89]The distinction between formal and formalized languages has been famously invoked by Tarski. See Tarski (1936, sec. 2).

objects, properties and relations are simultaneously possible in space, and we will refer to those possible configurations as *circumstances*. A circumstance specifies a way the world might possibly be at a certain point in time. It is a cluster of objects that have certain properties and stand in certain relations. Once the fundamental ontology is in place, the range of possible circumstances is fixed.

Circumstances represent momentary possibilities. They are inclusive in space but not in time. Each circumstance depicts some momentary configuration of objects, properties and relations that spans all of space simultaneously. While being spatially extended, circumstances are temporally flat. A circumstance provides a snapshot of what can possibly be the case at some moment in a branching time model.

Circumstances represent metaphysical possibilities. Each configuration of objects, properties and relations depicted by a circumstance is not merely logically but also metaphysically possible. Which configurations are possible does not solely depend on the logical form of the entities of our fundamental ontology, i.e. on whether a given fundamental entity is an object, a property or a relation, but rather on their metaphysical makeup. It is in the very nature of the fundamental entities that are part of the furniture of our world that certain configurations are possible, while others are not. The range of possible circumstances mirrors the space of momentary possibilities that are in accordance with what our fundamental entities are, viz. with their respective identity or essence.[90]

It is not the case that every object has to exist in every possible circumstance. There might be circumstances in which a given object lacks existence. Moreover, objects can have different properties and stand in different relations in different circumstances. Representing what is possible in virtue of the very nature of our fundamental entities, the range of possible circumstances does in general, however, not exhaust the entire range of logical possibilities. It lies in the essence of our fundamental entities that an object may have some of its properties contingently, others necessarily and that certain properties exclude each other; the same for relations. There cannot be a single circumstance, for

[90] Here, essence is not accounted for in terms of metaphysical possibility. Rather, the former is considered to be prior to the latter: the essence of the entities of our fundamental ontology determines the range metaphysical possibilities. For an account of metaphysical possibility along those lines, see Fine (1994, 2002). For a defense of the idea of essentialism, see, for example, Mulder (2013).

instance, in which the same thing is both red all over and green all over, and there cannot be a single circumstance in which something is both a frog and a cat.[91] If something is a frog in one possible circumstance, it is even excluded that it is a cat in some other possible circumstance. For, being a frog—and likewise, being a cat—is an essential property of an object.

4.2.2.3.2 The language Provided with some basic stock of fundamental objects, properties and relations, we can set up a language that perfectly matches our fundamental ontology. In the alphabet of our language, we include an individual constant for every fundamental object, a predicate for every fundamental property and a suitable n-ary relation symbol for every fundamental n-ary relation. Atomic sentences are structured entities that are formed in the usual way by combining a predicate or an n-ary relation symbol with a suitable number of individual constants. Given the alphabet of the language and the formation rules, the set of atomic sentences of the language is fixed. The range of complex sentences depends on which logical connectives and temporal and/or modal operators we adopt into the alphabet of our language. In the following, we will assume that the alphabet contains at least a sign for negation.

The crucial difference with the approach that we have pursued in chapter 2 consists in the fact that we do not conceive of our language as a purely formal one but rather as a formalized one. In other words, the atomic sentences are no longer considered mere syntactic strings of symbols, but their constituents are viewed as being firmly tied up with the fundamental entities of our ontology for which the respective symbols have been introduced. When interpreting the atomic sentences at a moment in a model, we keep that link between the constituents of the atomic sentences and the fundamental entities of our ontology intact. The overall domain of objects against which the constituents of the atomic sentences are interpreted at the various moments in a model is now provided by the set of fundamental objects that are part of the furniture of our world, and the individual constants pick out the very same fundamental object in each case. The interpretation of the individual constants in a model is fixed from the outset, so to say. The predicates and relation symbols still receive an interpretation at each moment in a model by being assigned certain arbitrary

[91] In his *Tractatus*, Wittgenstein pursued the project of determining which configurations of fundamental entities are possible in virtue of the logical form of those entities; see Wittgenstein (1922). The approach taken by Wittgenstein was criticized in Ramsey (1923) for only taking into account logical possibility, and the 'color exclusion problem' alluded to above is discussed there. See also our discussion of logical possibility in section 1.2.1.2.1 above.

set-theoretic extensions on the basis of the overall domain. Yet, they do not only thereby receive meaning. As they are firmly tied up with our fundamental ontology, they bear meaning independently of being assigned certain extensions. By falling under the extension of a certain predicate or relation symbol at a moment in a model, an object is at the same time ascribed some fundamental property or appointed some fundamental relation in the world. In short, the atomic sentences of our formalized language and their constituent parts receive an interpretation relative to a moment in a model by being assigned certain truth values or extensions, respectively, but they already have meaning by being tied up with the world.[92] The individual constants, predicates and relation symbols refer to the fundamental entities of our ontology.

4.2.2.3.3 The models We can conceive of the moment-relative interpretation of the structured atomic sentences of our formalized language on a branching time structure as a function that assigns to each moment of the structure a maximal consistent set of literals, i.e. a set that contains for every atomic sentence of the language either the sentence itself or its negation but not both. The set of literals associated with a moment is to be understood as containing all and only those atomic sentences that are true at the given moment as well as the negations of all and only those atomic sentences that are considered false at that moment. Let us denote the set of all possible maximal consistent sets of literals by VAL.[93]

By defining a branching time model as a branching time structure $\mathcal{M} = \langle M, < \rangle$ together with a function $V : M \to \mathsf{VAL}$ from the set of moments to the set of all maximal consistent sets of literals, we generalize both the notion of a moment-relative branching time model provided in Def. 4.1 and the notion a first-order order branching time model provided in Def. 4.3. It is straightforward that every valuation function $v : \mathsf{AT} \times M \to \{0, 1\}$ can be translated into a function $V : M \to \mathsf{VAL}$, and *vice versa*. For every atomic sentence $\phi \in \mathsf{AT}$, we set $\phi \in V(m)$ if and only if $v(\phi, m) = 1$, and $\neg\phi \in V(m)$ if and only if $v(\phi, m) = 0$. Moreover, to every first-order branching time model,

[92]In a certain sense, it is misleading to say that the atomic sentences and their constituent parts receive an interpretation relative to a moment in a model given that they already have meaning. They are rather evaluated at a moment in a model, keeping their meaning, i.e. their relation to the world, constant.

[93]Note that provided with the set AT, a specification of which atomic sentences are true at a moment at the same time lays down which atomic sentences are false at that moment. Why we define VAL as a set of literals rather than as a set of atomic sentences, viz. the atomic sentences that are considered true, will become clear in section 4.4.2.2.1.1 below.

there naturally corresponds a function V; and since the individual constants are taken to rigidly refer to the objects of our fundamental ontology, every function $V : M \rightarrow \mathsf{VAL}$ in turn determines an interpretation function I of a first-order branching time model whose domain equals the set of fundamental objects of our ontology by means of the semantic clauses provided in Def. 4.3.[94]

From a logical point of view, all that matters for assigning an interpretation to the atomic sentences at a moment of a branching time structure is the logical form of the fundamental entities of our ontology, which, given the intimate link between the language and the world, is reflected by the range of atomic sentences of the language itself. By merely taking into account the logical form, the objects of our fundamental ontology can, at the very same moment, fall under the extension of any predicate or relation symbol whatsoever. The atomic sentences of the language can be arbitrarily assigned truth values at any given moment. At each moment, some of the atomic sentences can be rendered true, others false. All combinations are logically possible. The truth or falsity of a given atomic sentence has no bearing on the truth or falsity of other atomic sentences. The atomic sentences are logically completely independent of each other. In case the number of atomic sentences of the language equals n, there are 2^n possible combinations concerning the truth and falsity of the atomic sentences. The range of logical possibilities exhausts the entire range of possible maximal consistent sets of literals provided by VAL. From a logical perspective, any combination of a branching time structure and a function $V : M \rightarrow \mathsf{VAL}$ constitutes a possible branching time model.

From a metaphysical point of view, solely taking into account the logical form of the entities of our fundamental ontology is not enough, but we also have to take into account their metaphysical makeup. It is in the very nature of the fundamental entities that are part of the furniture of our world that some configurations of objects, properties and relations are simultaneously possible and hence constitute a metaphysically possible circumstance, while others are not. Since the constituents of the atomic sentences of our formalized language are firmly tied up with the entities of our fundamental ontology, some maximal consistent sets of literals are then ruled out as an interpretation of the atomic sentences on metaphysical grounds, by the essence or identity of our fundamen-

[94] It is plausible to assume that every object that exists at a moment has at least one property or stands in at least some relation. The maximal consistent set of literals associated with a moment then also determines the domain of objects assigned to that moment in the corresponding first-order branching time model.

tal entities. While the atomic sentences are logically independent from each other, metaphysically they are not. Owing to the metaphysical makeup of the fundamental entities, only maximal consistent sets of literals that correspond to a metaphysically possible circumstance are suited for providing an interpretation of the atomic sentences at a moment, and we will denote the set of those metaphysically possible maximal consistent set of literals by $\mathsf{VAL}|_{\mathsf{meta}}$. A branching time structure together with a function $V : M \to \mathsf{VAL}|_{\mathsf{meta}}$ yields a branching time model in which the interpretation of the atomic sentences at each single moment is in accordance with the very nature of the fundamental entities of our ontology with which our language is tied up. We will call such a model in which each moment is associated with some metaphysically possible circumstance—or rather, a maximal consistent set of literals that describes some metaphysically possible circumstance—a *locally metaphysically possible branching time model.*

Definition 4.4 (A locally metaphysically possible branching time model). *For* AT *a set of structured atomic sentences of a formalized language that is tied up with some given fundamental ontology, a* locally metaphysically possible branching time model *is a triple* $\mathfrak{M} = \langle M, <, V \rangle$ *consisting of a branching time structure* $\mathcal{M} = \langle M, < \rangle$ *and a function* $V : M \to \mathsf{VAL}|_{\mathsf{meta}}$.

4.2.2.4 The missing link

In the previous sections, we have made a transition from logical to metaphysical considerations, and we have thereby taken a first step into the direction of branching time models for real possibility. We have abandoned a purely formal approach on which atomic sentences are conceived of as mere syntactic strings of symbols that can be interpreted in a model on a branching time structure but are otherwise devoid of any content. Instead, we have taken a metaphysical stance and tied the language to the world, switching from a purely formal to a formalized language.

We have relativized the interpretation of the atomic sentences in a model to a moment parameter only (section 4.2.2.1), decomposed the atomic sentences into their constituent parts (section 4.2.2.2) and linked those constituent parts to the fundamental entities of some given ontology (section 4.2.2.3). In doing so, we have established an intimate connection between the interpretation of the atomic sentences and the fundamental entities that are part of the furniture of our world at each building block of a branching time structure. In light of the very nature of the fundamental entities of our ontology, we have restricted the range of possible branching time models of our formalized language to locally metaphysically possible ones, in which each moment is assigned a maximal consistent set of literals from $\mathsf{VAL}|_{\mathsf{meta}}$ and hence is associated with some metaphysically possible circumstance.

By tying the language to the world, we have already considerably narrowed down the range of possible branching time models. From all branching time models that are in principle possible from a logical point of view, we have singled out those in which the interpretation of the atomic sentences at each single moment represents a metaphysically possible circumstance and thus bears an intimate relation to the world. Still, we have not yet arrived at branching time models for real possibility. In order for a branching time model to constitute a model for real possibility it is not sufficient that the interpretation of the atomic sentences at each single moment corresponds to a metaphysically possible circumstance, but it is furthermore required that the interpretation of the atomic sentences across the various moments of the underlying structure is such that each history represents a possible course of events that conforms to the prevailing laws of nature. While we have so far established metaphysical dependencies between the interpretation of the atomic sentences at each individual moment, we have not yet established nomological dependencies between the interpretation of the atomic sentences at subsequent moments. In the models that we have singled out so far, the circumstance associated with one moment has no bearing on the circumstances associated with other moments. Any circumstance can follow any circumstance. All combinations are possible.

The challenge that remains and that constitutes the key concern of this chapter is how to establish the missing link between the interpretation of the atomic sentences of our formalized language across different moments. How can we eventually single out from all locally metaphysically possible branching time models those in which each represented succession of metaphysically possible circumstances constitutes a nomologically possible course of events? In particular, how can we guarantee the fit between the model and its underlying structure? How can we ensure that the arity of the local branching time structure fits the momentary circumstance at hand?

Narrowing down the space of locally metaphysically possible branching time models to branching time models for real possibility is again a metaphysical rather than a logical enterprise. The missing link between the interpretation of the atomic sentences across different moments can only be established on metaphysical grounds. From a logical point of view, the interpretation of the atomic sentences at one moment of a branching time structure does not place any restrictions on the interpretation of the atomic sentences at any other moment of the structure. While the truth of sentences containing temporal operators certainly depends on the truth values of other sentences at different moments, the truth values of the atomic sentences can logically vary completely independently across moments. In order to establish a connection between the circumstances associated with different moments some metaphysical element is needed.

One last remark is in order here. It is worthwhile to point out that claiming that just any locally metaphysically possible branching time model constitutes a model for real possibility and that the laws of nature are simply nothing else than what can be read off that model will not do. In that case, we would be missing any systematic connection between the model and its underlying structure. Different moments that are associated with the very same circumstance—and may even be associated with the very same circumstances in their entire past—could each allow for a different range of possible future continuations. Both the number of possible future continuations and the circumstances associated with the respective future moments could vary. This is absurd, given that a branching time model for real possibility is supposed to depict all the possible courses of events that are in accordance with the prevailing laws of nature. For the laws to be laws at all, they must not discriminate two indistinguishable moments with respect to what is really possible. In order to provide a sophisticated account of branching time models for real possibility, we have to establish a systematic link between the interpretation of atomic sentences, the branching time structure and the world.

4.3 Models for real possibility

Our aim is to single out branching time models for real possibility. In the previous section, we have tied our language to some fundamental ontology and thereby restricted the overall range of possible branching time models to those in which each moment is associated with some metaphysically possible circumstance. What we are still missing is a nomological link between the circumstances obtaining at subsequent moments. We are in need of some metaphysical explanation that allows us to further delimit the range of locally metaphysically possible branching time models to models for real possibility. The explanation we are after has to be able to account for the fact that in a branching time model for real possibility, each succession of metaphysically possible circumstances is in accordance with the prevailing laws of nature. In particular, the explanation has to guarantee the fit between the model and its underlying structure.

In this section, we will discuss and discard several possibilities of how to construct branching time models for real possibility. In section 4.3.1, we will argue that starting out with a given branching time structure and restricting the possible models on that structure by means of certain additional metaphysical constraints is not a viable option. If we want to establish a systematic link between the moment-relative interpretation of the atomic sentences, the branching time structure and the world, the model and its underlying structure need to be developed hand in hand. In section 4.3.2, we contemplate the possibility of starting out with a history and developing the model from there. The approach is inspired by divergence models based on the notion of physical possibility. They are the counterparts of branching time models for real possibility in a neo-Humean possible worlds setting, and provide a paradigm example of how a model and its structure can be generated in parallel. Indeterministic divergence models are carved out from an underlying set of metaphysically possible worlds—or histories, in our terminology—by establishing suitable accessibility relations between those possible worlds. We will illustrate that transferring the approach to a branching time setting fails since the kind of indeterminism depicted in each case is a different one. In order to preserve the local kind of indeterminism that is at the heart of branching time models, we have to start out with a single metaphysically possible circumstance obtaining at a moment rather than with an entire history. In section 4.3.3, we show that this in turn implies that branching time models for real possibility and their underlying structures need to be developed hand in hand by establishing temporal relation between momentary circumstances in a dynamic fashion.

4.3.1 Starting from the structure

Inspired by what is familiar to us from logical practice, we may want to start out with a given branching time structure and subsequently try to restrict the range of possible models on that structure to models for real possibility by placing suitable metaphysical constraints on the admissible models. Since we need to establish some nomological link between successive circumstances, the approach requires that, unless we are able to derive the laws of nature from our basic stock of objects, properties and relations, we have to include them as an extra ontological ingredient in our fundamental ontology. The idea then would be this: while the objects, properties and relations ensure that each single moment is associated with some metaphysically possible circumstance, the laws of nature guarantee that the circumstances associated with successive moments constitute a nomologically possible course of events. They provide the missing link.

Yet, such an approach is a hopeless enterprise. There is no systematic way of how we could generate a branching time model for real possibility on a given structure. If we succeed in restricting the interpretation of the atomic sentences at the various moments of the structure in such a way that the resulting model is in fact a branching time model for real possibility, it is a matter of sheer luck. At every single moment, there needs to be a close link between the circumstance associated with that moment and the local branching structure. The circumstance associated with a moment must always be such that the prevailing laws of nature allow for precisely as many possible future continuations as the given branching time structure provides.

Note that it is not even the case that every combination of a branching time structure and a fundamental ontology allows for a branching time model for real possibility at all. Imagine a universe whose ontology is exhausted by a single particle of spin $\frac{1}{2}$ that makes its way through series of magnetic fields. According to the Stern-Gerlach experiment, the laws of nature are such that whenever a spin-$\frac{1}{2}$ particle passes an inhomogenous magnetic field, the number of possible deflections of its trajectory equals 2. Assume that we are given a branching time structure that contains a moment with three possible future continuations. It is simply impossible to provide a model on that structure that depicts the possible trajectories of our spin-$\frac{1}{2}$ particle when passing the magnetic fields.

The above considerations illustrate that starting out with a given branching time structure and placing metaphysical restrictions on the admissible models is not a feasible method for the purpose of systematically singling out branching time models for real possibility. If we manage to come up with a model that in fact fits the given branching time structure, there is no systematic reason for it, and the resulting model is entirely *ad hoc*. What is really possible at a moment depends on what is the case at that moment, and this presupposes a systematic link between the circumstance associated with a moment and the local branching structure. The local branching structure at a moment must depend on the circumstance associated with that moment. In order to establish a pertinent link between the model and its underlying structure, we have to develop the model and its underlying structure hand in hand.

4.3.2 Starting from a history

Divergence models based on the notion of physical possibility in a neo-Humean possible worlds setting are indeterministic models in which all possible courses of events conform to the prevailing laws of nature and share some common past, and as such they are perfect counterparts of branching time models for real possibility. Just as branching time models for real possibility, the corresponding divergence models presuppose a close and systematic link between the model and its underlying structure. Divergence models for real possibility thereby provide a clear case in which a model and its underlying structure are developed hand in hand. In the following, we will first discuss how divergence models are constructed in a neo-Humean possible worlds framework. We will then argue that the approach cannot be adapted to the framework of branching time. The reason why transferring the approach fails brings out the differences between branching and divergence and the kind of indeterminism that is depicted in each case.

4.3.2.1 Divergence models based on physical possibility

So let us make a side trip into a neo-Humean possible worlds pluriverse in order to illustrate how a close and systematic link between the model and its underlying structure can be guaranteed by generating the model and its underlying structure in parallel.

A possible worlds pluriverse is a set of possible worlds, and it is not to be confused with a Kripke frame, even though our discussion here of course relates back to the discussion of the structural properties of the possible worlds framework in chapter 1. Possible worlds fall into the same category as our circumstances.[95] They are populated by objects that have certain properties and stand in certain relations. Each possible world represents a metaphysical

[95] The standard possible worlds framework is surely the one provided in D. Lewis (1986a). It is worthwhile to point out that for present purposes nothing hinges on whether we take possible worlds to be real entities, as adherents of Lewis' modal realism would have it, or whether we view possible worlds as mere Ersatzist constructions.

possibility, just as a circumstance does. And all metaphysical possibilities are represented: there is a principle of plenitude. What makes the framework neo-Humean is the idea that the possible worlds are taken to be modally flat. They are considered a means to reduce modality.

A crucial difference between possible worlds and circumstances consists in the fact that possible worlds are inclusive in both space and time, while circumstances represent momentary possibilities. Each possible world comes with a linear temporal structure. It is a linear sequence of circumstances, if you want. Possible worlds do not branch, and there are no temporal relations between different possible worlds. The notion of time is rather built into the worlds themselves. Each possible world is associated with a linear series of times. Presupposing a suitable connection between the constituents of the atomic sentences of our language and the fundamental entities scattered over all the possible worlds, we can conceive of the possible worlds pluriverse as providing a model for that language on a Kripke frame. And due the principle of plenitude, the model is unique up to isomorphism.[96]

In a possible worlds framework, the notion of physical possibility, which builds the basis of the divergence models we are going to consider, is modeled as restricted quantification over the set of all metaphysically possible worlds.[97] From all metaphysically possible worlds, those that are physically possible from the standpoint of a given world are singled out by an accessibility relation. Something is said to be physically possible in a world if and only if it is the case in some physically accessible world. Unlike the notion of metaphysical possibility, physical possibility is not an absolute notion. Which worlds are physically accessible and hence constitute physical possibilities may vary from world to world. Physical possibility is world-dependent.

Given a possible world, the range of physically accessible worlds is determined by the prevailing laws of nature. A possible world is physically accessible from a given one if and only if what is the case in the former is compatible with the laws of nature of the latter. Nothing in the physically accessible worlds must

[96] Two different conceptions seem to be possible depending on whether one assumes that the time specification is built directly into the atomic sentences or not. In the former case, the respective model is a Kripke frame in which truth is relativized to a world parameter only. In the latter case, one obtains a model in which truth in a world is in addition relativized to a time.

[97] Whether the set of all physically possible worlds is a proper subset of the set of metaphysically possible worlds depends on whether one takes the laws of nature to be metaphysically contingent or necessary. On the view that the laws of nature are metaphysically necessary, physical and metaphysical possibility coincide.

contradict those laws.[98] On a neo-Humean account, where possible worlds are considered modally flat and basically anything can follow anything, the laws of nature holding in a world are taken to supervene on what is the case throughout that world. They are read off from the complete temporal development of that world. The laws of nature are considered the axioms of the best system of all the regularities found in the world, viz. the axioms of the deductive system that strikes the best balance between simplicity and strength.[99] Under the neo-Humean conception of laws of nature, what is physically possible in a world ultimately depends on what is the case throughout that world. What is the case in a world determines the laws of nature holding in that world, which again determine which metaphysically possible worlds are physically accessible. Contrary to what is common in logical practice, there is no physical accessibility relation independently of what is the case throughout the relevant worlds.

Next to the notion of physical possibility, which guarantees conformity with the prevailing laws of nature, the notion of divergence, which is supposed to capture the idea of sameness of the past, is crucial for providing indeterministic models in a possible worlds framework. Two worlds are said to diverge at a certain time if they share the same initial segment up to that time and differ thereafter. That two worlds share some initial segment means that their initial segments are perfect duplicates of each other. Unlike in the case of branching, in the case of divergence the idea of sameness of the past is spelled out in terms of duplication rather than in terms of overlap and hence identity.[100]

By combining physical possibility and divergence, we can discern a divergence structure in the overall model for metaphysical possibility that accommodates the idea of indeterminism. We obtain indeterministic models in which all possible courses of events are compatible with the prevailing laws of nature and share some common past. The divergence model and its underlying struc-

[98]Note that it is in general not required that the laws of nature holding in a world are *laws* in all physically accessible worlds. Since physical possibility is spelled out in terms of compatibility with the laws of nature rather than in terms of identity of laws, the physical accessibility relation is not necessarily an equivalence relation.

[99]According to Lewis, Humean supervenience "is the doctrine that all there is to the world is a vast mosaic of local matters of particular fact, just one little thing and then another." (D. Lewis 1986b, p. xi) For his views on the best system account of laws of nature, see D. Lewis (1973a, 1983, 1994). The view is sometimes referred to as the Mill-Ramsey-Lewis account.

[100]For Lewis's account of divergence, see D. Lewis (1986a, sec. 1.6). In section 4.2 of that work, Lewis presents an argument in favor of divergence and against branching. Lewis's objections against branching are discussed and countered in Placek (2001).

ture are thereby developed hand in hand: the relevant accessibility relation involved is lifted from what is the case throughout the metaphysically possible worlds. A possible world is accessible from another at a certain time if and only if the former is physically accessible from the latter and diverges from the latter only at some later time. Note that, consequently, whenever a world is accessible from another at a certain time, it is also accessible from the latter at any prior time, which is precisely the condition to be found in the definition of a $T \times W$ frame, or more generally, in that of a Kamp frame (cf. Def. 1.12 and Def. 1.13, respectively).[101] Starting out with a given world, the accessibility relation group together, at every time, all those worlds that are physically accessible and have the same initial segment up to and including that time. In order for us to actually obtain an indeterministic model and not end up with just a single world, it is important that the laws of nature holding in the world we start out with are indeterministic. If the laws are deterministic, there cannot be a physically possible world that diverges from the given one at some point in time.[102] To wit, that the laws of nature are deterministic is usually taken to mean nothing else than that any two worlds that obey those laws and share some initial segment coincide in their entire temporal development, i.e., they never diverge.[103]

When generating divergence models based on the notion of physical possibility in a neo-Humean setting, we do not start out with a given divergence structure and subsequently try to restrict the admissible models on that structure in such a way that each possible course of events conforms to the prevailing laws of nature and fits the underlying structure. Rather, the model and its structure are developed hand in hand. Starting out with a single metaphys-

[101] In a possible worlds framework, the laws of nature are a feature of temporally extended worlds as a whole, and the physical accessibility relation is accordingly, first and foremost, a relation between worlds. The notions of divergence and (in)determinism, on the other hand, involve a reference to times. While the mismatch is completely unproblematic from a metaphysical point of view, the issue becomes intricate if we want to provide a semantic account based on that metaphysics. As a relation between worlds, the physical accessibility relation seems to presuppose that truth is relativized to a world parameter only, whereas a semantic account of divergence and likewise (in)determinism presupposes that truth in a world is in addition relativized to a time parameter. Relativizing the physical accessibility relation to time-world pairs as well, however, seems only to be possible once the relation between complete possible worlds is already in place.

[102] D. Lewis (1979) holds on to the idea of determinism, and divergence models are accounted for in terms of miracles rather than in terms of indeterministic laws.

[103] That kind of determinism is usually referred to as ‘Laplacian determinism’, on the basis of Laplace (1820). For a definition of determinism along those lines, see, for example, Earman (1986).

ically possible course of events, the model is carved out from the overall set of metaphysically possible worlds by extracting an accessibility relation from what is the case throughout those worlds. In this way, systematically, a close link between the model and its underlying structure is established. The model perfectly fits the structure.

4.3.2.2 Divergence vs. branching

The construction method employed in the case of divergence models based on the notion of physical possibility in a neo-Humean possible worlds setting cannot be adapted to a branching time setting. In the following, we will illustrate what translating the construction method into a branching time framework would amount to and discuss why the approach is not a viable option for generating branching time models for real possibility. The impossibility of transferring the account derives from the differences between branching and divergence with regard to the kind of indeterminism that is depicted in each case. We show that by transferring the account, we lose the local kind of indeterminism that lies at the heart of branching time structures and makes those structures especially appealing for modeling the notion of real possibility in the first place.

4.3.2.2.1 Transferring the construction method A first difficulty for transferring the construction method employed in the case of divergence models based on the notion of physical possibility to the branching time framework already arises from the fact that in a possible worlds framework, we are provided with a set of metaphysically possible worlds that are inclusive in time and span entire possible temporal developments, while the metaphysically possible circumstances that constitute the basic building blocks of our branching time models merely represent momentary possibilities. Starting out with a set of complete metaphysically possible temporal developments as depicted by possible worlds amounts to starting out with entire metaphysically possible histories rather than with momentary metaphysically possible circumstances. In order to be able to transfer the approach at all, we first have to construct histories from circumstances.

As said, logically speaking, our metaphysically possible circumstances are first of all completely independent from each other with respect to their temporal succession. Any circumstance can follow any circumstance. It is precisely our endeavor to establish a metaphysical—or rather, a nomological—link between subsequent circumstances. Without the laws of nature—or something equipollent—at our disposal, we can only randomly combine momentary circumstances to create histories. In other words, all we can do is build arbitrary infinite sequences of metaphysically possible circumstances, each of which is supposed to represent a locally metaphysically possible history. Fortunately, in a neo-Humean setting, those locally metaphysically possible histories fit the bill. Since according to the neo-Humean doctrine, circumstances are modally flat and anything can follow anything, every arbitrary locally metaphysically possible history constitutes a metaphysically possible history from a neo-Humean perspective. And if we project a linear series of times onto the various metaphysically possible histories, we obtain something quite analogous to the set of metaphysically possible worlds, which have the notion of time built into them.

In accordance with the construction method of divergence models, we pick one of our neo-Humean metaphysically possible histories, we read off the laws of nature from that history and subsequently group together, at every time, all those metaphysically possible histories that are compatible with the respective laws of nature and diverge from the given history only at some later time. In this way, we obtain a model in which each history is physically possible from the perspective of the initial history and accessible at any time at which it still agrees with the latter. The models we arrive at are reminiscent of models on a $T \times W$ frame or Kamp frame, respectively. They accommodate the idea of sameness of the past. And what is more, they only contain histories that are compatible with the prevailing laws of nature.

Since our aim is to construct a branching time model rather than a divergence model, we finally have to identify circumstances that are associated with the same time and are part of an initial segment shared by several histories across those histories rather than treating them as mere duplicates. The time-relative accessibility relation between histories that conform to the laws of nature and share the same past merges into identity.[104] On the surface, the resulting model is a branching time model for real possibility: every course of events is possible against the background of the prevailing laws of nature, viz. the laws holding in the history we have started out with, and all possible courses of events that are in accordance with the laws and share some common

[104]By identifying duplicate circumstances, the time-relative accessibility between histories becomes an equivalence relation while this does not necessarily hold for its possible worlds counterpart (cf. footnote 98).

past are represented. The model has been developed in a systematic way, and it perfectly fits its underlying structure.

4.3.2.2.2 Different kinds of indeterminism Although by transferring the construction method employed in the case of divergence models in a neo-Humean possible worlds framework, we have established indeterministic branching time models that perfectly fit their underlying structure, the strategy is nevertheless not a viable option. Besides the fact that our neo-Humean histories, which allow for any arbitrary succession of circumstances, are rather obscure, the idea to start out with a given history and develop the model from there runs counter to the very idea of branching. It presupposes a God's eye view perspective, a standpoint from the end of time from which it is already determined which of the several alternative histories is realized in the end. To put it in a nutshell, adopting the construction method employed in the case of indeterministic divergence models to the framework of branching time would amount to embracing the existence of a Thin Red Line that represents 'the actual history' (cf. section 1.4.3.1.2). That the method employed in the construction of divergence models based on the notion of physical possibility in a neo-Humean possible worlds framework cannot be transferred to the branching time framework is due to the fact that divergence models and branching time models crucially differ with respect to the kind of indeterminism they depict: branching time models incorporate a local kind of indeterminism, while the kind of indeterminism represented in a divergence model as well as in a branching time model with a Thin Red Line is a global one.[105]

4.3.2.2.2.1 Global indeterminism In a possible worlds framework, indeterminism is modeled in purely modal terms, from a perspective outside of time. The basic building blocks of a divergence model are possible worlds that span entire possible temporal developments and stretch linearly all the way into the future. Possible worlds do not reside in time. The notion of time is rather built into the possible worlds themselves, and there are no temporal relations between different worlds. On a possible worlds account, there is no indeterminism on the temporal level. Temporally, each possible world is deterministic. The temporal development of a world is a linear one. At any time

[105] While we use the terms 'global' and 'local' to distinguish between a modal and a temporal kind of indeterminism, in Belnap (2012), the terms are used to distinguish between branching time and branching space-time indeterminism. Unlike in branching time, in branching space-times, indeterminism is also spatially local. What we mean by local indeterminism is the kind of indeterminism alluded to in Belnap, Perloff, and Xu (2001, p. 137).

within a world, it is fixed what the future will bring. The local perspective from a point in time thus shades into the standpoint of an entire world.

By taking a local standpoint in time, one of the possible worlds is marked as the actual world, and indeterminism is, strictly speaking, not a feature of that world.[106] It is rather a feature of the laws of nature holding in that world and, relatedly, a feature of the model that results from grouping together possible worlds that share some initial segment with the actual world on the basis of the prevailing laws of nature. The respective model is indeterministic, viz. contains at least two diverging worlds, if and only if the laws of nature obtaining in the actual world, the world we start out with, are indeterministic.

The kind of indeterminism depicted by divergence models is a global one. Indeterminism figures exclusively on the modal level and does not affect the temporal structure of worlds. It is reflected by suitable relations between different possible worlds, all of which are linearly extended in time. One of them is the actual world, and the others constitute mere modal alternatives to that actuality. They are worlds that are compatible with the laws of nature obtaining in the actual world and share some common past. The local standpoint in time determines which of the physically possible worlds are presently accessible from the actual world given the past course of events. Yet, those accessible worlds constitute modal alternatives to the actual world as a whole. It is the fact that the kind of indeterminism depicted by a divergence model is an entirely modal one that legitimates the construction method. Owing to the fact that indeterminism is modeled in terms of modal alternatives to a given actuality, the model can be generated from a complete possible linear temporal development.

4.3.2.2.2.2 Local indeterminism What makes the branching time framework especially appealing for modeling the notion of real possibility is that it accommodates a local rather than a global kind of indeterminism. Indeterminism is not spelled out in purely modal terms as a relation between different histories, but it is built directly into the temporal structure of a branching time model. The basic building blocks of a branching time model are momentary circumstances rather than temporally extended worlds or histories, and the modal structure between histories is lifted from the temporal order-

[106] Of course, one can define a world to be (in)deterministic if and only if the laws of nature holding in the world are (in)deterministic, and that is what is often done. Yet, the fact that the standard definition of (in)determinism in terms of possible worlds makes reference to at least two worlds shows that (in)determinism is a feature of a world in only a derived, extrinsic, sense.

ing among circumstances by identifying maximal chains within the tree-like temporal structure.

A branching time model reflects the modal-temporal structure of a single world, and being built directly into the temporal order, indeterminism becomes a feature of that world. Each history represents a possible way of how the world might temporally develop. A branching time model, and with it the world, is indeterministic if and only if the temporal ordering of momentary circumstances that arises against the background of the prevailing laws of nature contains a circumstance with more than one possible future continuation, viz. if the temporal ordering contains a branching point.[107] The alternative future continuations are equal temporal alternatives from the perspective of the given circumstance. The kind of indeterminism involved is a local one. It is reflected by suitable temporal relations between momentary circumstances. It arises on the temporal level and from there carries over to the modal one. The temporal ordering of momentary circumstances gives rise to a tree of histories all of which share some common past and branch toward an open future. Yet, the various histories passing through a given circumstance represent temporal alternatives for the future rather than modal alternatives to some given actuality that embraces a unique future, and they are modally on a par. Given a circumstance, none of the histories passing through that circumstance can be singled out as 'the actual history'. Which of the alternative histories will be realized in the end is objectively indeterminate.

Entering on the temporal level, indeterminism is modeled from the local standpoint of a momentary circumstance within the temporal order rather than from a modal perspective outside of time. In a branching time model, a modal perspective outside the temporal ordering from which it is determined which of the alternative histories is realized in the end is not available. The idea of a Thin Red Line that represents 'the actual history' is in stark contrast with the idea of local indeterminism: in the presence of a Thin Red Line, temporal alternatives turn into modal ones (cf. section 1.4.3.1). What we did, however, when transferring the construction method of indeterministic divergence models to the framework of branching time, was postulating the existence of

[107] Being built into the temporal order, a purely structural notion of indeterminism is available as well. A branching time structure can be said to be indeterministic if and only if it contains a branching point. Note that this kind of indeterminism is first of all completely detached from the world and the particular laws of nature. An illuminating discussion of the issues surrounding the definition of (in)determinism can be found in Müller (2015).

a Thin Red Line and constructing a globally indeterministic model from there by establishing modal accessibility relations between histories. By identifying circumstances in shared initial segments across histories, we finally lifted the temporal order among momentary circumstances from the modal relations between histories rather than the other way around. If we adopt the construction method and establish indeterminism on the temporal level on the basis of a global kind of indeterminism that involves a distinguished history, we exactly lose the local kind of indeterminism that is key to the notion of real possibility and provided the motivation for employing the framework of branching time in the first place.

4.3.3 Starting from a moment

In order to retain the local kind of indeterminism that makes the framework of branching time so appealing for modeling the notion of real possibility, we cannot start out with an entire locally metaphysically possible history but have to develop branching time models for real possibility together with their underlying structures from the local standpoint of a momentary circumstance. Taking some momentary metaphysically possible circumstance rather than a chain of metaphysically possible circumstances as a starting point, we might nevertheless try to hold on to the idea that from that local standpoint the model can be carved out from the underlying set of locally metaphysically possible histories by establishing suitable accessibility relations between those histories. If we establish the accessibility relations between histories from the standpoint of a momentary circumstance, lifting the temporal order from the modal one would no longer stand in the way of local indeterminism. Yet, such an approach is of no avail. A momentary circumstance does by no way suffice for establishing the required modal relations between entire locally metaphysically possible histories. Let us see what the options are and why they are not viable.

First, one could stick to the idea that circumstances are modally flat and that basically any circumstance can follow any circumstance and try to run the neo-Humean program on a partial course of events up to a given momentary circumstance only. Instead of starting out with an entire neo-Humean history that stretches all the way into the future and reading off the laws from that history, one could start out with a momentary circumstance contained in some such metaphysically possible neo-Humean history and attempt to read off the laws of nature from the past course of events up to that circumstance only.[108] One could then follow the standard strategy involved in the construction of divergence models based on the notion of physical possibility and group together, at every time, all those histories that are compatible with the given laws of nature and share the same initial segment up to and including that time, identifying duplicate circumstances. The resulting model would constitute a model for real possibility from the standpoint of the initial circumstance. From that circumstance onwards it would be locally indeterministic, and it naturally fits the underlying structure.

Still, such a model does not constitute a branching time model for real possibility in the proper sense. The laws of nature on which the accessibility relation rests depend on the initial circumstance. The model might therefore contain histories that, while being compatible with the laws of nature read off from the past course of events up to the initial circumstance, are not compatible anymore with the laws of nature read off from the past course of events of some later circumstance. Even if the laws of nature turn out to be deterministic at the end of time, the models obtained at earlier circumstances might comprise several diverging—or rather, branching—histories. Yet, histories that drop off as soon as the supervenience base is extended toward the future, do not really constitute real possibilities, and so the account fails.

Second, one could abandon the neo-Humean doctrine and introduce a primitive kind of modality, making circumstances modally robust. Some locally metaphysically possible histories would then cease to be metaphysically possible. The task would then consist in singling out, from the perspective of a given modally robust circumstance, all those locally metaphysically possible histories that contain that circumstance as an initial and in which the succession of circumstances is nomologically possible. One possibility of making circumstances modally robust is by assuming that among the entities of our fundamental ontology, there are modal properties or relations.[109] Yet, if a single momentary circumstance is supposed to suffice to single out from the range of all locally metaphysically possible courses of events those that are in accordance with the prevailing laws, the laws of nature that govern complete possible temporal developments would have to be completely grounded in the modal properties

[108] For a neo-Humean account along those lines, see Backmann (2013). Hüttemann (2014) introduces the label 'open-future-Humeanism' to refer to those accounts and raises criticism against the view.

[109] One could, for example, assume that among the fundamental properties of our ontology, there are powers, dispositions, etc. A suitable modal relation might be Armstrong's necessitation relation between universals (cf. Armstrong 1978).

and relations of the objects existing in that momentary circumstance—rather than in the fundamental ontology as a whole. This in turn presupposes that all modal properties and relations are instantiated in that very same circumstance. Whether there can be such a circumstance at all crucially depends on our fundamental ontology, and it is very unlikely that every metaphysically possible circumstance will be of that sort. The strategy therefore fails the benchmark of a systematic account that allows us to generate branching time models for real possibility for any metaphysically possible circumstance and any arbitrary fundamental ontology.

Rather than providing the fundamental entities of our ontology with a modal flavor and assuming that the laws of nature can be derived from a single metaphysically possible circumstance made up from those entities, we could of course simply include the laws of nature as an extra ontological ingredient into our fundamental ontology.[110] Given some metaphysically possible circumstance, we would then also be provided with the laws of nature. Starting with some arbitrary circumstance, we could then group together, at every time, all those histories that are compatible with the given laws of nature, contain that circumstance as an initial and diverge only later, identifying duplicate circumstances. The resulting model constitutes a model for real possibility that is locally indeterministic and perfectly fits the structure, and, unlike in the case of the relativized neo-Humean approach, the laws of nature do no longer depend on the choice of the initial circumstance. Yet, if we adopt the laws of nature as an extra ingredient into our fundamental ontology, we lack any explanation of what those laws of nature have to do with our fundamental objects, properties and relations. While the account is systematic, it is not explanatory. If we are to obtain a systematic account that is at the same time explanatory, we cannot add the laws of nature as an extra ontological ingredient but build only on the fundamental entities contained within our basic stock of objects, properties and relations.

From the local standpoint of a moment, it is impossible to establish suitable modal relations between entire locally metaphysically possible histories—at least if we want our account to be explanatory and hence refrain from assuming primitive laws of nature—, and starting out from a complete locally metaphysically possible history runs counter to the idea of local indeterminism that lies

[110] An account on which the laws of nature are considered ontologically primitive is given in Maudlin (2007).

at the heart of branching time models. We then have to altogether abandon the idea to construct branching time models for real possibility by establishing modal relations between locally metaphysically possible histories and derive the temporal relations from there by identifying duplicate circumstances. It suggests itself that the construction of branching time models for real possibility should rather proceed by establishing temporal relations between momentary circumstances and lifting the modal relations from the temporal ordering. In order to be able to extract temporal relations between momentary circumstances from those circumstances themselves, some modal element is needed in our fundamental ontology. In the remainder of this chapter, we will suggest a metaphysical explanation of branching time models for real possibility in terms of potentialities. By establishing temporal relations between momentary circumstances on the basis of the potentialities of objects in a dynamic fashion, we provide a systematic account of branching time models for real possibility that elucidates why all possible courses of events are in accordance with the laws of nature.

4.4 Real possibilities and potentialities

In this section, we will provide a dynamic, modal explanation of branching time models for real possibility in terms of potentialities. Branching time models for real possibility will be lifted in a dynamic fashion from a single momentary, metaphysically possible circumstance by establishing temporal relations between momentary circumstances on the basis of the potentialities of objects.

In the previous section, we have argued that in order to obtain a systematic account for branching time models for real possibility, starting out with a given branching time structure and restricting the possible models on that structure to models for real possibility by means of certain additional metaphysical constraints is not a viable option. Since real possibilities presuppose a close and systematic link between the model and its underlying structure, we have to develop the model and the structure hand in hand. The parallel construction thereby has to proceed in such a way that the local kind of indeterminism that lies at the heart of branching time models is preserved.

Our aim is to lift branching time models for real possibility from a set of metaphysically possible circumstances, just as we have carved out divergence models based on the notion of physical possibility from an underlying set of metaphysically possible worlds or histories. The crucial difference with the divergence approach consists in the fact that we will now proceed in a dynamic fashion, establishing temporal relations between momentary circumstances rather than modal relations between entire worlds that span complete possible courses of events. In this way, the local kind of indeterminism is retained. Instead of invoking a neo-Humean conception of laws of nature that flows from entire possible temporal developments, we will base our construction of branching time models for real possibility on potentialities, which provide a local alternative to laws of nature.

We conceive of potentialities as genuine modal properties of objects that are part of the furniture of our world. Potentialities are to be included in the basic stock of fundamental properties of our ontology. Departing from the literature, in section 4.4.1, we will develop an alternative account of potentialities. Potentialities will be construed as dynamic properties of objects whose manifestations point toward the future and jointly give direction to the alternative possible future courses of events. Having explicated our conception of potentialities, we will put the account to work in section 4.4.2. We will show how potentialities and their manifestations allow us to successively lift a branching time model for real possibility together with its underlying structure.

4.4.1 Potentialities

We use the term 'potentiality' in a rather broad sense. Potentialities are properties of objects and comprise dispositions, powers, potencies, capacities, abilities, etc.[111] Paradigm examples of potentialities include the property of being fragile, being soluble, being lethal, etc. Each potentiality is associated with some characteristic manifestation. Fragility, for instance, is associated with breaking, solubility is associated with dissolving, and being lethal is associated with killing. What is crucial to the notion of a potentiality is that the characteristic manifestation of an object's potentiality concerns the possible behavior of the object in virtue of its potentiality rather than its actual behavior.

In this respect, it might prove helpful to contrast potentialities with categorical properties—although the precise distinction is far from clear. Roughly speaking, a categorical property, such as, for example, the property of being triangular, is always manifest when had by an object. A potentiality, on the other hand, can be had by an object without being manifested. A glass can

[111] The term 'potentiality' is also used as an umbrella term in Vetter (2010, 2015) and related papers by the same author. Abilities bear an intimate relation to agency, which makes them quite intricate. For an overview, see Maier (2014).

be fragile without being broken, a sugar cube can be soluble without being dissolved and poison can be lethal without actually killing someone. Unlike with categorical properties, in the case of potentialities, the instantiation of the property itself and its manifestation come apart.

In the literature, the most extensive treatment of potentialities is provided in the context of the debate on dispositions. In the following, we will briefly review the current debate. We will thereby focus on the counterfactual analysis of disposition ascriptions, which takes center stage in the debate, and discuss the most prominent realist accounts of dispositions. Leaving behind the current debate, we then set out to provide an alternative realist account of potentialities.

4.4.1.1 Extant accounts of dispositions

The debate on dispositions is first and foremost a debate about an appropriate semantic analysis of disposition ascriptions. The search for an appropriate semantic analysis of disposition ascriptions ties in with questions concerning the ontological status of dispositions as well as the kind of modality involved in dispositionality.[112] The most prominent semantic account of disposition ascriptions is provided by the counterfactual analysis. We will discuss the analysis, its shortcomings and remedies below. Subsequently, we will turn to the question of the ontological status of dispositions. In this connection, we will address the most prominent realist accounts of dispositions which differ with respect to which kind of modality is considered to be at the heart of dispositionality.

4.4.1.1.1 The counterfactual analysis It is a widespread assumption that there is a close link between disposition ascriptions and conditionals. The assumption rests on the idea that a disposition cannot only be associated with a characteristic manifestation but also with characteristic stimulus conditions.[113] Identifying the stimulus conditions of a disposition amounts to identifying external factors that (ideally) trigger its manifestation. While dispositions are commonly considered intrinsic properties,[114] viz. properties that an object has independently of the relations it bears to other objects, their characteristic manifestations are subject to external conditions.

[112] For an overview of the debate, see Choi and Fara (2014) and Cross (2012).
[113] For an opposing view, see Vetter (2014).
[114] For counterexamples to the claim that all dispositions are intrinsic properties, see McKitrick (2003).

The fact that dispositions concern possible rather than merely actual behavior suggests that the link between the characteristic stimulus conditions of a disposition and its manifestation cannot be cashed out in terms of the material conditional.[115] It may be that an object has a disposition whose characteristic stimulus conditions are actually not fulfilled so that the manifestation fails to come about. Yet, if the characteristic stimulus conditions were fulfilled, the disposition would display its characteristic manifestation, or so the idea goes. This has led people to think that the kind of conditionals that play a role in the analysis of disposition ascriptions are counterfactual conditionals. An object is said to have a certain disposition if and only if the characteristic manifestation would come about if the respective stimulus conditions were to obtain.[116]

Most dispositional predicates, such as "fragile", "soluble", "lethal", etc., lack any explicit reference to their characteristic stimulus conditions and manifestations. In order to analyze those predicates in terms of counterfactual conditionals, we first need to identify their respective stimulus conditions and manifestations. The counterfactual analysis of disposition ascriptions can accordingly be viewed as proceeding in two steps.[117] In a first step, a dispositional predicate which lacks any explicit reference to its characteristic stimulus conditions and manifestation is associated with a dispositional predicate that makes explicit mention of the characteristic stimulus conditions and the characteristic manifestation. In a second step, the latter is analyzed in terms of a counterfactual conditional. If we use D for the dispositional predicate to be analyzed, S for its characteristic stimulus conditions, M for its characteristic manifestation and x for an arbitrary object, we can describe the counterfactual analysis with its two step approach as follows:

[115] Since the material conditional $S(x) \rightarrow M(x)$ is true whenever $S(x)$ is false, disposition ascriptions cannot be analyzed along the following lines: $D(x) \leftrightarrow (S(x) \rightarrow M(x))$. For otherwise, an object would be ascribed any disposition whose stimulus conditions are not fulfilled. In response to this problem, Carnap (1936) suggested the following reduction sentence: $S(x) \rightarrow (D(x) \leftrightarrow M(x))$. The reduction sentence offers a partial definition. It remains silent in cases in which the stimulus conditions are not fulfilled. A reductive analysis of disposition ascriptions in an extensional language seems to be doomed.

[116] For the standard semantic accounts of counterfactual conditionals, see D. Lewis (1973a,b) and Stalnaker (1968). On Lewis's account, the counterfactual "x would M, if x were (in) S" is true in a world w if and only if some world in which both $S(x)$ and $M(x)$ are true is closer to w than any world in which $S(x)$ is true and $M(x)$ is false. On Stalnaker's account, on the other hand, the counterfactual in question is true in a world w if and only if in the closest world in which $S(x)$ is true, $M(x)$ is true as well. In the debate on dispositions, it is commonly assumed that a world w is always closer to itself than any other possible world, which is referred to as 'strong centering'.

[117] Cf. D. Lewis (1997).

(CA) *Step 1:* x has D iff x is disposed to M when S.
Step 2: x is disposed to M when S iff x would M if x were (in) S.
Hence: x has D iff x would M if x were (in) S.

Following this approach, fragility, for example, is first characterized as the disposition to break when struck, and that disposition is then analyzed along the following lines: x is disposed to break when struck if and only if x would break if x were struck. By combining the two steps we get that an object is fragile if and only if it would break if it were struck.

The counterfactual analysis is, however, not without problems. Various counterexamples have been raised, and an enormous amount of literature has been produced in response to those counterexamples. The most prominent counterexamples to the counterfactual analysis are those treated under the labels finks and masks. A fink is an external factor that makes an object gain a disposition that it previously lacks in circumstances in which the stimulus conditions of that particular disposition are fulfilled. The standard example is an electro-fink, a device that makes a dead wire alive when touched by a conductor. In the presence of the electro-fink, the dead wire would conduct electricity if it were touched by a conductor, and hence would not qualify as dead according to the counterfactual analysis.[118]

We will here focus on masks. A mask is an external factor that prevents the manifestation of an object's disposition although the respective stimulus conditions are fulfilled.[119] A fragile glass that is wrapped in packing material will not break when struck and thus constitutes a counterexample to the counterfactual analysis. Poison is lethal and still, if I ingest it, I will survive in case I also take the antidote. The packing material and the antidote are considered masks. They block the characteristic manifestations of the respective dispositions and hence render the corresponding counterfactual conditionals false in

[118] Finks were introduced in Martin (1994). On the assumption that dispositions are intrinsic properties of objects, scenarios involving a fink require an object to undergo intrinsic change and can hence be blocked as counterexamples to the counterfactual analysis by evaluating the counterfactual against the background of the assumption that the relevant intrinsic properties of an object are retained for a sufficient period of time (cf. D. Lewis 1997). If the wire were to retain its intrinsic property of being dead after the fink sets in, it would not conduct if it were touched by a conductor. There are also reverse-finks. Those involve cases in which an object loses its disposition in circumstances in which the stimulus conditions of that disposition are fulfilled.

[119] Masks were introduced by Johnston (1992) and Bird (1998).

cases in which we wish to ascribe the disposition. It is worthwhile to note that masks are not a rare phenomenon. They are rather pervasive.

4.4.1.1.2 The counterfactual analysis revisited In response to the problem of masks, revised versions of the counterfactual analysis have been proposed. There are basically two possible ways to deal with the problem of masks that allow one to maintain that there is a close link between disposition ascriptions and counterfactual conditionals.

First, one could revise the first step of the counterfactual analysis and retain the second step in unmodified form. Proponents of this strategy hold that in identifying the stimulus conditions of the ordinary dispositional predicate we are to analyze we have not been precise enough.[120] We have characterized fragility as the disposition to break when struck, but really it is the disposition to break when struck in the absence of packing material. Likewise, that poison is lethal does not mean that it is disposed to lead to death when ingested; being lethal rather amounts to the disposition to lead to death when ingested without the antidote. But of course, this will not be enough: the process of refinement will have to go on without end. In order to rule out all possible masking cases, we have to make the characteristic stimulus conditions more and more specific. In the end, the ordinary dispositional predicate we started out with is associated with a rather complex dispositional predicate whose stimulus conditions are specific enough to yield the right results when entering the antecedent of the counterfactual conditional in the second step of the analysis.

Second, one could stick to the characterization provided in the first step of the analysis and argue that the counterfactual conditional in the second step is in need of refinement. Proponents of this strategy argue that the characteristic stimulus conditions we have identified in the first step of the analysis are in fact the appropriate ones but take it that those characteristic stimulus conditions contain some implicit specification of certain standard conditions to which the counterfactual conditional in the second step needs to be explicitly restricted.[121] That a fragile glass is disposed to break when struck is taken to mean that in the absence of packing material, it would break if it were struck. Likewise, poison is disposed to lead to death; for, in the absence of the antidote, it would lead to death, if it were ingested. The standard conditions are supposed to rule

[120] For a response along those lines see, for example, D. Lewis (1997) and Choi (2008).

[121] A strategy along those lines is advocated, for example, by Malzkorn (2000), Mumford (1998) and Steinberg (2010).

out all possible masking cases. An object is ascribed the disposition to M when S if and only if in the absence of masks, it would M if it were (in) S. Standard conditions might be identified with certain normal or ideal conditions or just be captured by adding a *ceteris paribus* clause to the counterfactual conditional.

Both strategies involve a refinement of either of the two steps of the counterfactual analysis. The crucial difference between the two strategies consists in where the respective refinements enter the analysis. According to the first response, the absence of masks is considered part of the characteristic stimulus conditions of the ordinary dispositional predicate to be analyzed, whereas according to the second response, the absence of masks is considered part of certain background conditions against which the counterfactual conditional is to be evaluated. The first strategy amounts to claiming that in masking cases, the manifestation fails to come about because the stimulus conditions are not fulfilled; along the lines of the second strategy, on the other hand, the stimulus conditions are in fact fulfilled, but the manifestation is masked.

The question arises whether it is possible to draw a clear cut distinction between stimulus conditions and background conditions on independent grounds and whether it is plausible to treat all masking cases in a uniform way. In some masking cases, the manifestation seems to be blocked from the outset, as, for instance, in the case of the glass wrapped in packing material, while in other cases, it seems that the manifestation fails to come about because of some interfering factor. The antidote in the poisoning constitutes seems to constitute such a case. In order for there to be room for interfering factors at all, the stimulus must temporally precede the manifestation or at least its supposed outcome. If death sets in as soon as the poison is ingested, there is no room for the antidote to counteract in the first place. Is the absence of the antidote part of the stimulus conditions of being lethal or is it part of a relevant background condition that excludes that the manifestation is interfered with in the course of time? Unfortunately, the temporal succession of stimulus conditions and manifestations and role the of time in that context do not receive much attention when it comes to the assessment of masking cases.

Although the two strategies discussed above crucially differ with respect to where the relevant refinements enter that are supposed to rule out all masking cases at once and in a uniform way, the upshot is nevertheless the same. On both accounts, an ordinary dispositional ascription is ultimately analyzed in terms of a highly specific counterfactual conditional whose antecedent is designed in such a way that masks are blocked as counterexamples. Both accounts face the problem of how to spell out the necessary refinements in a suitable way without rendering the analysis vacuous or having it fall prey to further counterexamples. The task is complicated by the fact that masks may even occur in cases that are paradigmatic for the manifestation of the disposition.[122]

There is an alternative account that avoids the problems just mentioned. Rather than analyzing disposition ascriptions in terms of a single counterfactual conditional with a highly specific antecedent that excludes all masking cases, disposition ascriptions are linked to an entire list of counterfactual conditionals. The characteristic stimulus conditions of a disposition identified in the first step of the counterfactual analysis are taken to determine a range of particular cases, and an object is then said to have the disposition in question if and only if the characteristic manifestation would come about in a suitable proportion of those cases. What counts as a suitable proportion is determined by the context.[123]

The account is not threatened by the problem of triviality, and masks—even if they occur in paradigmatic cases—do not constitute counterexamples to the analysis as they will be outweighed by non-masking cases. By invoking an entire list of counterfactuals conditionals rather than a single highly specific counterfactual, unlike the two approaches discussed above, the account can also deal with dispositional predicates which lack any specific stimulus conditions, such as, for example, a radium atom's disposition to decay.[124] In that case, simply all possible cases are considered. What the account cannot deal with, as it still builds on the assumption that each dispositional predicate is to be associated with certain (possibly empty) characteristic stimulus conditions, are ordinary dispositional predicates that in principle allow for various different stimulus conditions and for which there simply is no such thing as 'the characteristic stimulus conditions'.[125] One might hold that a fragile glass is not only disposed to break when struck, but it is also disposed to break when dropped or stressed, etc. In order to make room for dispositional predicates

[122] For examples, see Manley and Wasserman (2008).

[123] This view is introduced and defended in Manley and Wasserman (2008, 2011). The approach has the advantage that it can deal with comparative disposition ascriptions, such as "a is more fragile than b", which are not uncommon. Note that under the assumption of strong centering, the relevant list of counterfactual conditionals boils down to mere material implications. Dispositions are eventually analyzed by variably strict conditionals.

[124] For the view that there are dispositions that lack any specific stimulus conditions, see Manley and Wasserman (2008) and Molnar (2003).

[125] Dispositions that can be associated with several stimulus-manifestation pairs are referred to as multi-track dispositions. For the view that many ordinary dispositions are multi-track, see Vetter (2013b).

that allow for various different stimulus conditions, an even more fine-grained analysis seems to be needed.

4.4.1.1.3 Dispositional realism If successful, the counterfactual analysis of disposition ascriptions can—though it does not have to—be considered a reductive analysis of dispositions. The problems of finks and masks surrounding the counterfactual analysis have, however, motivated several strands of accounts that consider dispositions genuine modal properties that are part of fundamental reality. The accounts differ with respect to which kind of modality they take to be at the heart of dispositionality.[126]

4.4.1.1.3.1 Dispositional essentialism Among the dispositional realists, we find the so-called dispositional essentialists. Dispositional essentialism is typically characterized by the slogan that all, or at least some, (fundamental) properties have dispositional essences.[127] Naturally, the question arises what it is supposed to mean that a property has a dispositional essence. The idea seems to be that a property with a dispositional essence is a property that comes with a certain dispositional profile and bears that dispositional profile essentially. That a property bears its dispositional profile essentially is thereby taken to mean nothing else than that the property's dispositional profile is constant across all metaphysically possible worlds.[128]

According to Bird (2005), the dispositional profile of a property is adequately captured by an instance of the counterfactual analysis endowed with a *ceteris paribus* clause. That is, the dispositional essence of fragility, for example, is represented by the following equivalence: x is fragile iff *ceteris paribus*, x would break, if x were struck. Since a disposition bears its dispositional profile essentially, the equivalence holds necessarily.

Dispositional essentialists, such as, for example, Bird (2005), then go on and ground the laws of nature in the dispositional essences of fundamental properties. The dispositional essences of properties allow to derive universal generalizations of the following form: for all x, *ceteris paribus*, if x is fragile and x is struck, then x breaks. The resulting laws of nature are *ceteris paribus* laws,

[126] Realist accounts of dispositions have been advocated among others by Bird (2007), Ellis (2001), Martin (2008), Molnar (2003), Mumford (1998) and Vetter (2015). Recents contributions to the debate are included in Marmodoro (2010).

[127] The main proponents of this view include Bird (2007, 2005) and Ellis (2001).

[128] It seems highly questionable whether an account according to which dispositions are considered genuine modal properties should appeal to the eternalist possible worlds pluriverse, which provides, first of all, a neo-Humean picture. It seems that the possible worlds pluriverse of the dispositional realist must be a different one, viz. one that is generated by dispositions.

and, owing to the fact that the dispositional profile of a disposition is constant across all metaphysically possible worlds, they are metaphysically necessary.[129]

4.4.1.1.3.2 Dispositionalism about modality Not all realists about dispositions are wedded to the idea that there is a close link between dispositions and counterfactual modality. Some dispositional realists link dispositionality to some other kind of modality. There are several accounts that propose to relate dispositionality to possibility.[130] The link between dispositions and possibility is established on the basis of the observation that dispositions concern possible rather than actual behavior. A disposition can be had by an object without being manifested, and in this case its manifestation constitutes a mere possibility.

The idea then is to ground possibilities in the dispositions that there actually are. The possibility to be grounded is an atemporal one and it is not anchored in what is the case at some particular point in time. It is timelessly grounded in the dispositions that there are, have been and will be.[131] What is possible is determined by the manifestation of dispositions. In Borghini and Williams (2008, p. 26), the following characterization is provided: "If [and only if] the world contains some disposition such that its manifestation is the state of affairs S, then S is possible". The basic idea is that what is possible is what constitutes the characteristic manifestation of some actual (past, present or future) disposition. On the basis of those considerations, an account of possibility in terms of dispositions can be provided.

Note that the relation between the stimulus conditions and the manifestation of a disposition completely loses its significance if possibility is speled out in this way. It is possible that a glass breaks because it is fragile and thus has the potentiality to break. Under which conditions the glass actually breaks does not matter, as long as there is at least one. It is commonly assumed that the kind of possibility that is to be accounted for is the atemporal and absolute notion of metaphysical possibility. Yet, the question arises what the relevant notion of metaphysical possibility is: what is the range of metaphysical pos-

[129] A virtue of dispositional essentialism is that it is able to provide an explanation of the laws of nature by grounding them in the dispositional essences of fundamental properties. In Jaag (2014), the tenability of that idea is questioned. The basic idea of Jaag's criticism is that nothing can ground what is constitutive of its own essence. For further criticism of the view, see Vetter (2012).

[130] Cf. Vetter (2010, 2011, 2013a, 2014, 2015), Borghini and Williams (2008), Jacobs (2007, 2010) and Pruss (2002).

[131] A time-relative account of possibility is alluded to in Pruss (2002) and mentioned but not pursued in Vetter (2015, pp. 199f.).

sibilities that are supposed to be captured and in fact can be captured? Is it possible for the world as a whole to be different, and is it possible for an object to have alien properties?[132] In order to allow for remote or far-fetched possibilities, the analysis is in need of amendments. The challenge then consists in spelling out those necessary amendments, and still, the question remains how far one can get.

4.4.1.1.3.3 Dispositionality as tendency The accounts we have reviewed so far either link dispositionality to counterfactual modality or to metaphysical possibility. There is another position among the dispositional realists that considers dispositionality a *sui generis* kind of modality that cannot be analyzed in terms of other modal notions. Dispositionality is construed as a tendency: it is stronger than metaphysical possibility but weaker than metaphysical necessity.[133] Dispositions are said to tend toward their manifestations. The account is motivated by the ubiquitous possibility of masking. The idea is that a disposition's manifestation can in principle always be prevented. That is, there is always a possible masking scenario in which manifestation is blocked. The fact that manifestations can in principle always be interfered with renders dispositionality weaker than necessity. Possibility, on the other hand, although a necessary requirement for dispositionality, is itself considered too weak. A fragile glass tends to break when struck: it breaks in some metaphysically possible scenarios but not in all. If the glass is safely packed in styrofoam, for example, it does not break. It is neither impossible nor necessary that the glass breaks.

The account of dispositionality as a mere tendency is then employed by Mumford ad Anjum as the basis of an account of causation. Causation is explained in terms of dispositionality. The idea is that causes dispose toward their effects. Just as dispositionality, causation does not involve necessity but tendency. In the case of causation, different dispositions interact and contribute to an overall effect. Dispositions can combine with different dispositions in different circumstances, and they make the same contribution in each case, but this contribution may be counteracted by other dispositions. Something is puzzling

[132] For arguments to the effect that the range of metaphysical possibilities that a dispositionalist account of modality can account for is rather limited, see, for example, Hawthorne (2001) and Cameron (2008). For a response to objections of that sort, see Jacobs (2007, ch. 6.2.5) and Vetter (2015, ch. 7).

[133] An account of dispositionality as tendency is advocated by Mumford and Anjum (2010, 2011b,a).

here: since dispositions tend toward their manifestation, their manifestations must be distinct from their contributions, and since the overall effect is the result of the interaction of different dispositions, they must also be different from the overall effect. So, what are manifestations then? It seems that the manifestation of a disposition is taken to be the overall effect that is brought about when the disposition acts in isolation, that is, when its manifestation is the sole contribution.

On the whole, there seems to be not much common ground in the debate on dispositions. The debate on dispositions centers around the question of an appropriate semantic analysis of disposition ascriptions, and controversies about dispositions arise with respect to the ontological status of dispositions as well as with respect to the kind of modality involved in dispositionality. In the following section, we will provide an alternative realist account of dispositions—or rather, potentialities, as we call them. Our aim is not to provide a semantic analysis of potentiality ascriptions in terms of some other kind of modality. Our endeavor is to provide a metaphysical explanation of branching time models for real possibility. Our focus is on the metaphysics of potentialities, and we do not assume that potentialities can be analyzed in terms of real possibilities. We assume that each potentiality comes with some modal profile and bears that profile essentially, but we take the modal profile of a potentiality to be far more complex than that it could be captured by a single counterfactual conditional or a list of counterfactual conditionals that build on the notion of stimulus conditions. First of all, we discard the idea that every potentiality can be associated with a characteristic stimulus condition. What is more, we render the notion of the manifestation of a potentiality differently and bring time into the picture: the manifestations of potentialities point toward the future and can naturally be interfered with. We will discuss in section 4.4.3.2 how exactly our account relates to the extant realist accounts of dispositions.

4.4.1.2 Potentialities: An alternative account

We are realists about potentialities. Potentialities are genuine modal properties of objects. They are part of the furniture of our world, and they are ontologically irreducible. Potentialities are to be included in the basic stock of fundamental properties of our ontology and from there they enter into metaphysically possible circumstances. In contrast to neo-Humean worlds, metaphysically possible circumstances are modally robust: they comprise potentialities as genuine modal properties.

We follow the prevailing literature in assuming that every potentiality is associated with some kind of manifestation. Fragility is associated with breaking, solubility is associated with dissolving, and being lethal is associated with some kind of processes that culminate in death. However, while the manifestation of a potentiality is commonly equated with what in fact is the case, i.e. with how things in fact turn out, if the potentiality is manifested, we uncouple the manifestation of a potentiality from what can, would or tends to be the case in virtue of the potentiality being manifested. The manifestation of a potentiality is viewed as a mere pointing toward the future. While we concur with the prevailing literature that the manifestation of an object's potentiality concerns possible rather than actual behavior and is licensed in certain circumstances only, we do not invoke stimulus conditions, and we allow for potentialities that in the very same circumstance can be manifested in several different ways. Each potentiality is associated with a rather complex and fine-grained modal profile that captures the possible manifestations and non-manifestations of the potentiality in all possible circumstances in which an object has that potentiality.

4.4.1.2.1 Manifestations We conceive of potentialities as dynamic properties of objects that account for change. By manifesting their potentialities objects become causally efficacious and give direction to the alternative possible future courses of events.[134] The manifestation of an object's potentiality is thereby understood as a transition toward the future. Objects possess and manifest their potentialities in momentary circumstances; yet, the manifestations of their potentialities point beyond those circumstances. They pull toward the future.

Manifestations are local in two respects. First, manifestations are local in space. The manifestation of an object's potentiality in a circumstance does not amount to a transition that affects the entire circumstance as a whole. It rather captures the transformation of some local state of affairs obtaining in the respective circumstance. In particular, it captures the future behavior of the bearer of the potentiality. We model the manifestation of an object's potentiality in a circumstance as a pair consisting of an initial and a subsequent outcome. The initial is some salient subpart of the given circumstance, a local

[134]Building on the assumption that potentialities account for change, our account gains neo-Aristotelian traits. Cf. Aristotle (1996, book III, 1-2) and Aristotle (1998, book IX).

state of affairs that involves at least the bearer of the potentiality. The outcome can either be again a state of affairs or a process that describes the future temporal evolution of the initial.

Second, manifestations are local in time. Whenever an object manifests its potentiality in a circumstance, the manifestation comes about in that very circumstance. This is, however, not to say that the outcome of the manifestation will occur. An object can manifest its potentiality without the respective outcome ever being realized. The manifestation of an object's potentiality in a circumstance is a mere pointing toward the future rather than the unfolding of some future course of events.[135] It contributes to the latter, but it is not identical to it. What in fact is the case in some possible future course of events depends on the manifestations of other potentialities as well. When ingested in a circumstance, poison manifests its potentiality to lead to death. The manifestation is, however, not a sequence of static circumstances, one following the other. It is rather a dynamic pull toward the future that emanates locally from the momentary circumstances and can be interfered with by the manifestations of other potentialities. The manifestations of potentialities in a circumstance are transitions consisting of an initial and an outcome. They are like little arrows that attach to the momentary circumstance and pull toward the future. They are not to be confused with their respective outcomes, and they are local in time in that they are where their initials are, not where their outcomes are. Manifestations have 'no simple location'.[136]

The notion of a transition that is invoked here closely resembles the notion of a transition found in von Wright (1963) but differs from the notion of a transition that builds the basis of the transition semantics presented in chapter 2 in several respects. Transitions are no longer understood as mere elements of a branching time structure that capture one of the possible future continuations of a moment. First of all, as manifestations of potentialities transitions capture the transformation of particular situations; and the situations on which they operate do not span all of space, like a moment does, but correspond to local states of affairs within a circumstance. What is more, the metaphysically possible circumstances and with them the transitions that capture the manifestations of potentialities within such circumstances are not yet concretely

[135] Manifestations are not flip-book activities, to borrow a term from Ruth Groff. See Groff (2013, ch. 2.3) as well as Groff (2015, 2016). For an argument to the effect that our metaphysical thought involves different kinds of predication, see Mulder (2014).

[136] Cf. Belnap (2005, p.232), who quotes Whitehead (1925, ch.3).

anchored in time. Only once the circumstances to which they adhere are integrated in some temporal order, the transitions become concrete. We will see that in the branching time models that we lift by establishing temporal relations between momentary circumstances on the basis of the potentialities of objects and their local manifestations, we can find back our original notion of a transition under certain conditions: in case a moment is associated with a circumstance in which only a single potentiality is manifested, the transitions representing the possible manifestations of that potentiality in the circumstance in question are isomorphic to the transitions that capture the possible immediate future continuations of the given moment in the underlying branching time structure (cf. section 4.4.2.1.1).

4.4.1.2.2 Licensing of the manifestation A potentiality is a property that is had by an object in a circumstance. It is possible that an object possesses a potentiality in one circumstance but lacks it in another. Whether an object possesses a potentiality in a particular circumstance is an intrinsic matter and does not depend on how things stand with other objects in that circumstance. Potentialities are intrinsic properties. The manifestation of an object's potentiality in a particular circumstance, on the other hand, does depend on how things stand with other objects in that circumstance. It is an extrinsic matter. Objects can manifest their potentialities in certain circumstances only, and whether an object can manifest its potentiality in a given circumstance depends on external factors.

We depart from the prevailing literature in discarding the idea that every potentiality is to be associated with certain characteristic stimulus conditions. Potentialities are individuated in terms of their manifestations only—or rather, in terms of their modal profiles.[137] Some potentialities do not seem to require a particular trigger, i.e. stimulus, at all. Others can manifest themselves under various miscellaneous conditions, and exactly which conditions license their manifestations might in addition depend on the particular bearers of the respective potentialities. Consider a china cup and a clay jug. Both are fragile. Yet, the former will already break if only carelessly knocked over, while the latter generally will not. Rather than building on a notion of characteristic

[137] Vetter (2014) also abandons the idea that every potentiality comes with certain specific stimulus conditions and individuates potentialities solely in terms of their manifestations. Cf. also Vetter (2010, 2015).

stimulus conditions, we directly link the licensing of the manifestation of an object's potentiality to concrete circumstances.

Some circumstances license the manifestation of an object's potentiality, while others may not. The precise range of circumstances in which an object can manifest its potentiality varies with both the particular object and the potentiality in question. Of course, in the relevant circumstances, the object has to exist and possess the potentiality in question in the first place. Furthermore, the licensing of the manifestation of an object's potentiality in a circumstance depends on the object's immediate environment. Whether a given object can manifest its potentiality in a particular circumstance depends on both the presence and absence of other objects in its direct vicinity.

First of all, some objects can only manifest their potentialities if they stand in suitable relations to other objects. The manifestations of their potentialities are licensed only in the presence of certain objects that have certain properties—including potentialities.[138] Imagine a glass standing on a table in an otherwise empty room. The glass is fragile. Yet, in the given circumstance, it cannot manifest its potentiality to break. The licensing of the manifestation requires the presence of someone with a potentiality whose manifestation is licensed and in virtue of which he strikes the glass or throws it off the table so that it falls on the stony ground. Similarly, a sugar cube aground on a dune in a parched desert cannot manifest its potentiality to dissolve, while it is surely soluble. It can only manifest its potentiality if submerged in water or some other liquid that has the potentiality to disorganize the crystalline structure of saccharides and can also exercise that potentiality. If the water is oversaturated, the sugar will not dissolve.

Moreover, the manifestation of an object's potentiality might not only require the presence of other objects in its proximity. Its licensing might also presuppose the absence of certain objects. In particular, nothing in the circumstance must block the manifestation. If our glass on the table is safely packed in styrofoam, it cannot manifest its potentiality to break—even if someone is present who strikes it or throws it off the table. The glass can only manifest its potentiality in the absence of the packing material.

[138] The idea that some potentialities require reciprocal or mutual manifestation partners can already be found in Aristotle's *Metaphysics* (cf. Aristotle 1998, book IX, 1). The idea is prominent in various current accounts of dispositions or powers, such as in Martin (1997), among others.

Just as the manifestation of an object's potentiality itself, its licensing in a circumstance is a local matter, both spatially and temporally. Whether an object can manifest its potentiality in a given circumstance depends on its immediate surroundings in that momentary circumstance. How things stand with objects located in the far distance does not matter. Nor does it play a role what happens in the future. This is not to say that the licensing of the manifestation of an object's potentiality in a circumstance may not depend on whether the manifestations of other objects' potentialities are enabled in that circumstance. As said, a sugar cube can only manifests its potentiality to dissolve if in suitable contact with a liquid that can manifest its potentiality to disorganize the crystalline structure of saccharides. While the licensing of the manifestation of an object's potentiality in a circumstance may very well depend on the manifestations of other objects' potentialities in that circumstance, it does in no way depend on their respective outcomes. Even less does it depend on which potentialities are manifested at a later stage, once a temporal order is in place. The manifestation of an object's potentiality can be licensed in a circumstance even though its outcome is incompatible with the outcomes of the manifestations of other objects' potentialities in that—or a later—circumstance. And in that case, the respective outcomes interfere.

4.4.1.2.3 Deterministic and indeterministic potentialities We invoke a distinction between deterministic and indeterministic potentialities, a distinction that is not commonly drawn in the prevailing literature on dispositions. In the debate on dispositions, the focus is mainly on what we call 'deterministic potentialities'. A kind of dispositions that resembles our indeterministic potentialities in certain respects are propensities, which are probabilistic dispositions.[139] Indeterministic potentialities also bear a certain similarity with the notion of a two-way power that figures in debates on agency and free will.[140]

A deterministic potentiality is a potentiality that in every circumstance can be manifested in but a single way. In other words, if an object that possesses a deterministic potentiality manifests that potentiality in a circumstance, a single transition with a unique outcome comes along. The precise character of the manifestation of an object's deterministic potentiality in a given circumstance depends both on the particular bearer and the circumstance. If an unprotected

[139] Propensities are, for example, discussed in Mellor (1971).

[140] See, for example, Steward (2012) and Alvarez (2013). For a discussion of the role of powers in the debate of agency and free will, see also van Miltenburg (2015).

glass is struck with a certain force, it breaks in a certain way. The breaking of the glass is a different one depending on whether the glass is struck with a wooden slat or with a sledgehammer; and a china cup again breaks differently in those circumstances than the glass does. A sugar cube dissolves if submerged into a liquid that has and can manifest the potentiality to disorganize the crystalline structure of saccharides. It shows a unique manifestation whose precise character depends on the sugar cube's particular crystalline structure, the temperature of the liquid, etc.[141]

Next to those potentialities that can always manifest themselves but in a unique way, there must in an indeterministic world be potentialities that in the very same circumstance can be manifested in several different ways. More precisely, an indeterministic potentiality is a potentiality such that if it is manifested, several transitions with the same initial and mutually exclusive outcomes can come along. All possible manifestations are manifestations of the same potentiality. All of them are equally possible, i.e. they are modally on a par, but only one of them can be realized at once. Note that for a potentiality to be indeterministic, it is not required that it allows for multiple possible manifestations in every circumstance, but just that there is at least one such circumstance. The precise range as well as the precise character of the possible manifestations of an indeterministic potentiality in a given circumstance can vary with both the particular bearer and the circumstance. As the Stern-Gerlach experiment shows, the number of possible deflections of the trajectory of a particle passing an inhomogeneous magnetic field differs depending on the particular spin of the particle, and the precise orientations of the respective deflections vary with the strength of the magnetic field.

Among our indeterministic potentialities, there are some that are such that one of the several different ways in which an object can manifest them is by refraining from bringing about their characteristic manifestation. In other words, omission or forbearance constitutes one of the multiple possible ways in which those indeterministic potentialities can be manifested in a circumstance: omission or forbearance is a possible manifestation. Omission or forbearance can, however, never, be the only possible manifestation of an indeterministic potentiality in a circumstance; it can only be one of several alternatives.[142]

[141] There is a unique manifestation at least on the marcolevel, even though the underlying microdynamics may be indeterministic.

[142] Indeterministic potentialities of this kind play a role in debates on agency and free will. In those debates, indeterministic potentialities that make room for omission and forbearance

Treating omissions or forbearances as possible manifestations of indeterministic potentialities rather than as non-manifestations, allows us to distinguish them from non-manifestations that are due to licensing failure. I have the potentiality to walk, and in virtue of that potentiality I can also refrain from walking. A circumstance in which I refrain from walking is, however, quite different from a circumstance in which I am unable to manifest my potentiality to walk as I am tied up to a chair. On a formal level, the difference between omissions or forbearances on the one hand and non-manifestations due to licensing failure on the other finds its expression in the fact that as manifestations of some indeterministic potentialities omissions and forbearances are transitions—albeit trivial ones, whose respective outcomes are qualitatively identical to their initials—while non-manifestations are not. In the case of non-manifestation due to licensing failure, there is no pull toward the future, and no transition arises. If I am tied up to a chair, I am simply stuck. I can only manifest my potentiality again if someone comes along who has a potentiality in virtue of which he can and does untie me. If, on the other hand, I refrain from walking, I preserve the possibility to walk for the future. The distinction becomes relevant in our construction of branching time models for real possibility. In case all objects existing in a circumstance get stuck as none of them can manifest a potentiality anymore, no new future moment can be determines and time ends, as it were, while if the objects merely refrain from manifesting their potentialities in a characteristic way, time continues (cf. section 4.4.2.1.3).

4.4.1.2.4 The modal profile of a potentiality Potentialities are properties of objects, and, just as any other property, they are possessed by objects within possible circumstances. If we think of potentialities in purely model-theoretic terms, potentialities, like other properties, are just sets of objects. We can associate a potentiality in each possible circumstance with the set of objects that have the potentiality in the respective circumstance.[143]

are usually discussed under the heading 'two-way powers'. See, for example, Steward (2012) and Alvarez (2013). As the name suggests, two-way powers always involve two alternatives. It often remains unclear, however, what the alternatives are, viz. whether omission or forbearance is the possibility of non-manifestation or the possibility to bring about a contrary manifestation. We consider the alternatives underlying the concept of a two-way power to be manifestation vs. non-manifestation, and we conceive of the non-manifestation as one of alternative possible manifestations.

[143] As with properties in general, the extension of a predicate that designates a potentiality is just the set of objects having the potentiality in the respective circumstance. It is worthwhile to note that whether an object falls under the extension of a certain potentiality predicate at a given index of evaluation associated with a circumstance does, on our account, not

As sets of objects, potentialities are first of all something entirely static, as is any other property. What distinguishes potentialities from other properties is that, unlike other properties, each potentiality comes with some specific modal profile. A potentiality does not only identify in each possible circumstance the set of objects that have the potentiality in the respective circumstance but at the same time also determines the possible future behavior of an object in virtue of its potentiality, its pointing toward the future, in all possible circumstances in which the object possesses the potentiality.

Potentialities have a rather complex and fine-grained modal profile. To begin with, the modal profile of a potentiality captures for each possible circumstance in which an object possesses the potentiality whether the object can actually manifest its potentiality in that circumstance. As said, objects can manifest their potentialities in certain circumstances only. The modal profile of a potentiality does, however, not only settle in which circumstances an object can manifest its potentiality and in which it cannot. If in a given circumstance an object can in fact manifest its potentiality, the modal profile also already lays down the precise range of possible manifestations of the object's potentiality in that circumstance. Depending on whether the potentiality is deterministic or indeterministic, more than one manifestation might be possible, and the particular character of the possible manifestations hinges on the respective bearer and the circumstance. The modal profile of a potentiality thus specifies for each possible circumstance in which an object possesses the potentiality whether the manifestation of the object's potentiality is licensed in that circumstance and if so, it provides all possible manifestations. The modal profile of a potentiality is constant across all metaphysically possible circumstances. A potentiality cannot have different modal profiles in different circumstances—otherwise, it would not be the same potentiality. At least, this is the picture we will be working with.

We conceive of the possible manifestations of an object's potentiality in a circumstance as transitions toward the future. The modal profile of a potentiality can accordingly be viewed as an assignment of sets of transitions to objects relative to all those circumstances in which the respective objects possess the potentiality. In other words, in virtue of the modal profile of a potentiality, an object is assigned a set of transitions relative to every circumstance in which it

require an intensional analysis in terms of what is the case in virtue of the potentiality and its manifestation in other possible circumstances.

has the potentiality in question. The set of transitions assigned to an object in a circumstance captures the possible manifestations of the object's potentiality in that circumstance. All transitions contained within the respective set share the same initial and have mutually exclusive outcomes. If in a given circumstance an object cannot manifest its potentiality, it is assigned the empty set of transitions.[144]

We will say that the possibly empty set of transitions that is associated with an object's potentiality in a given circumstance specifies the object's possible *progressings* in virtue of its potentiality in that particular circumstance. A progressing can either be a possible manifestation of the object's potentiality as provided by a transition or correspond to the potentiality's non-manifestation, which is, strictly speaking, nothing and represented by the empty transition set. The possible progressings of an object in a circumstance constitute the object's *de re* possibilities in virtue of its potentiality in that respective circumstance. They capture the object's possible future behavior in virtue of its potentiality, its locally pulling toward the future in one way or another or its not pulling toward the future at all. The modal profile of a potentiality assigns a possibly empty set of transitions to an object relative to every circumstance in which the object possesses the potentiality, and it thereby pins down the object's possible progressings in virtue of the potentiality in each case. Equipped with a modal profile, potentialities endow our otherwise static circumstances with a dynamic component: the modal profile captures the dynamics of an object in virtue of its potentiality.

4.4.2 Lifting branching time models for real possibility

In this section, we will put potentialities to work. That is, we will provide a dynamic, modal explanation for branching time models for real possibility in terms of potentialities. It will be shown how the conception of potentialities developed in the previous section allows us to lift a branching time model for

[144] In Cartwright and Pemberton (2013), causal profile accounts of powers are criticized for not being able to account for the fact that the transformation of a concrete configuration of objects, properties and relations is brought about by a combination of different powers. Our account does not fall prey to the objections raised there. We uncouple the manifestation of an object's potentiality from what in fact is the case, i.e. from how things in fact turn out, if the potentiality is manifested, but conceive of manifestations as transitions toward the future. The modal profile accordingly associates an object's potentiality in a concrete circumstance with its contribution to an overall outcome rather than with the overall outcome itself. What in fact is the case, the overall outcome, is the result of the combination and interaction of the manifestations of different potentialities.

real possibility together with its underlying structure from a single momentary, metaphysically possible circumstance in a dynamic fashion.

We have modeled potentialities as dynamic properties of objects. Although objects possess and manifest their potentialities in momentary circumstances, the manifestations of their potentialities point beyond those circumstances. They are transitions toward the future. In each possible circumstance in which an object possesses a potentiality, it is associated with a possibly empty set of transitions, which captures the possible manifestations of the object's potentiality in that circumstance and thereby specifies its possible progressings. In case the potentiality is indeterministic, the corresponding set of transitions has more than one element, and all of them share the same initial and have mutually exclusive outcomes.

Metaphysically possible circumstances are first of all something entirely static, and on the view developed so far, they are not yet temporally ordered. Owing to the object's potentialities, a dynamic component enters the picture. Endowed with sets of transitions that represent the possible manifestations of the objects' potentialities, our otherwise purely static circumstances become dynamic. They start being in motion and undergo change. Albeit being temporally flat, they are modally loaded and pull toward the future. Construed as transitions that point toward the future, the possible manifestations of the objects' potentialities allow us to establish temporal relations between momentary circumstances. In particular, they enable us to lift the immediate possible future continuations of a circumstance from the circumstance itself, and they thereby guarantee conformity with the laws of nature. That a circumstance may allow for more than one nomologically possible future continuation is accounted for in terms of indeterministic potentialities.

Branching time models for real possibility are the result of different potentialities acting in concert. At every moment, several objects can manifest their potentialities, and their respective possible manifestations are local in space and time. Only jointly do they give direction to the alternative possible future courses of events. In order to get a grasp on how the possible future continuations of a circumstance can be lifted from the various sets of transitions that arise from the objects' potentialities in that circumstance, in section 4.4.2.1, we will first have a look at a simple toy example before considering the general case in section 4.4.2.2. Having provided a construction method for branching time models for real possibility, we will conclude the chapter by reflecting on the precise nature of the interrelation of potentialities and real possibilities and address the differences between our account and other realist accounts of dispositions.

4.4.2.1 A toy example

Real possibilities are possibilities in an indeterministic world, and the branching time models we construct are always branching time models for real possibility relative to some given fundamental ontology that provides the furniture of our world. The range of metaphysically possible circumstances is determined by the fundamental entities of our ontology, and the prevailing laws of nature are encoded by the potentialities included in our basic stock of properties. Presupposing a formalized language that perfectly matches our fundamental ontology, the range of metaphysically possible circumstances places restrictions on the interpretation of the atomic sentences of the language at the various moments of a branching time model, and we will now show how the potentialities and their manifestations establish the missing nomological link between the interpretation of the atomic sentences at successive moments.

The construction of the branching time model resembles a chronicle construction as employed in completeness proofs (cf. chapter 3). What we establish is a temporal ordering among moments, where each moment is associated with a maximal consistent set of literals that specifies some metaphysically possible circumstance. The existence of additional future moments and the assignment of maximal consistent sets of literals to those future moments is governed by the set of transitions that represent the possible manifestations of the objects' potentialities. At each moment, the possible manifestations of the potentialities of all powerful objects combine and jointly give direction to the alternative possible future courses of events.

In order to illustrate how the construction of branching time models for real possibility proceeds, we will consider a simple toy example and increase its complexity step by step. In section 4.4.2.1.1, we will start by considering the case in which there is but a single object that has but a single indeterministic potentiality. In this case, the possible future continuations of a circumstance are solely determined by the transitions that represent the possible progressings of that single object. Subsequently, in section 4.4.2.1.2, we will see what happens if a second objects that is likewise equipped with but a single indeterministic potentiality enters the stage, and we will discuss how the possible manifestations of the objects' potentialities combine and interact. Finally, in section 4.4.2.1.3, we will address the difference between omissions and forbearances as possible manifestations of indeterministic potentialities on the one hand and non-manifestations of potentialities due to licensing failure on the other and illustrate how this difference bears on the construction of branching time models for real possibility.

4.4.2.1.1 The tale of Tim the frog So, let us delve into a little toy universe. We first specify the fundamental ontology of our toy universe, which we will enrich in the following sections, and set up a language that perfectly fits our fundamental ontology. Once the range of metaphysically possible circumstances is fixed, the set of maximal consistent sets of literals that are suited for providing an interpretation of the atomic sentences at a moment is given as well, and we can start the construction. We start out with some arbitrary metaphysically possible circumstance and illustrate how we can successively build up a branching time model by successively lifting the immediate possible future continuations of a moment from the transitions that represent the possible manifestations of the object's indeterministic potentiality in the corresponding circumstance. The possible manifestations will thereby be construed as pairs consisting of two consecutive states of affairs so that the resulting model will be discrete. We finally show that the result of our construction is in fact a model for real possibility.

4.4.2.1.1.1 World and language Imagine that all there is to the world is a chessboard and a little frog. Assume moreover that the range of fundamental properties of our toy universe is exhausted by the 64 properties of being located on either square of the chessboard and the potentiality to jump.[145] While each of the properties of being located on one particular square of the chessboard constitutes a contingent property of our little frog, we will assume that he possesses the potentiality to jump essentially. Under the assumption that our frog is always located on some square and cannot be co-located on two different squares at the same time, there are 64 metaphysically possible circumstances. Each metaphysically possible circumstance represents the frog as being located on some particular square on the chessboard, and in all those circumstances our little frog has the potentiality to jump. One of the meta-

[145] For the sake of simplicity, we will neglect that there is also the property of being a frog. One might say that in such a poor world, all there is to the property of being a frog is the potentiality to jump.

physically possible circumstances, namely the circumstance in which our frog is located on the square B2 of the chessboard, is provided in Fig. 4.1.[146]

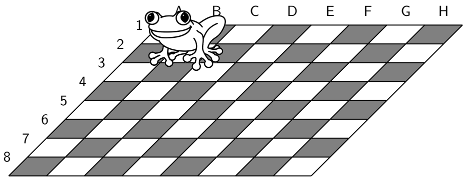

Figure 4.1: Frog on square B2.

We will assume that the frog's potentiality to jump is an indeterministic potentiality that comes with the following modal profile. In a nutshell, the frog can jump diagonally left or diagonally right unless located on an edge of the chessboard.

(i) if the frog is located on a square neither on the leftmost file (A) or rightmost file (H) nor on the eighth rank, he can manifest his potentiality to jump by jumping toward either the square on the left-adjacent file on the next higher rank or to the square on the right-adjacent file on the next higher rank;

(ii) if the frog is located on a square on the leftmost file (A), he can manifest his potentiality to jump by jumping toward the square on the right-adjacent file on the next higher rank;

(iii) if the frog is located on a square on the rightmost file (H), he can manifest his potentiality to jump by jumping toward the square on the left-adjacent file on the next higher rank;

(iv) if the frog is located on a square on the eighth rank, he cannot manifest his potentiality to jump.

The modal profile captures the possible manifestations of the frog's indeterministic potentiality to jump in every possible circumstance, that is, given any of his possible positions on the chessboard, and thereby specifies the frog's

[146] Experts in chess will notice that our chessboard is mirror inverted. Recall that our aim is not to construct models of the real world, and the mirror inverted chess board is an emblem of that.

possible progressings in all those circumstances. A possible manifestation of the frog's potentiality to jump in a given circumstance is a transitions whose initial consists in the frog's position in that circumstance and whose outcome is provided by some possible next position. Each possible manifestation is a transition from one state of affairs to another, where the latter state is considered an immediate successor of the former. The potentiality to jump is an indeterministic potentiality that in certain circumstances can be manifested in several different ways. In any circumstance in which the frog is located on a square neither on one of the outermost files nor on the eighth rank, he can manifest its potentiality to jump by jumping diagonally left or diagonally right. The precise range of possible manifestations in a circumstance depends on the concrete circumstance. If the frog is located on a square on one of the outermost files, there remains, due to external conditions, but one possible manifestation; and if he is located on a square on the eighth rank, he cannot manifest his potentiality at all. In a circumstance in which the frog is located on a square on the eighth rank, the manifestation of his potentiality is not licensed.

We can easily come up with a formalized first-order language that perfectly matches our fundamental ontology. The alphabet of our language contains an individual constant, Tim, that rigidly refers to our little frog, a one-place predicate for each of the 64 properties of being located on either square of the chessboard, $\mathsf{A1}$, $\mathsf{A2}$, …, $\mathsf{B1}$, etc. and another one-place predicate, J, for the potentiality to jump. Atomic sentences are formed in the usual way by combining the individual constant Tim with some arbitrary predicate φ of the alphabet of our language and take the form '$\varphi(\mathsf{Tim})$'.

Assuming that the alphabet of our language is in addition equipped with a sign for negation, the range of metaphysically possible circumstances determines the set $\mathsf{VAL}|_{\mathsf{meta}}$ that comprises all and only those maximal consistent sets of literals that, given the link between the language and the world, are suited for providing an interpretation of the atomic sentences of our formalized language at a moment. While from a logical perspective, there are $2^{65} = 36893488147419103232$ possibilities for the truth or falsity of the 65 atomic sentences of our language, the set $\mathsf{VAL}|_{\mathsf{meta}}$ merely contains 64 maximal consistent sets of literals. Each set contained in $\mathsf{VAL}|_{\mathsf{meta}}$ comprises 65 literals, among which there are always precisely two non-negated atomic sentences: one that ascribes to our little frog Tim the potentiality to jump, which he has essentially, and another that specifies Tim's contingent position on the chessboard.

The latter suffices to distinguish between the 64 elements of $\mathsf{VAL}|_{\mathsf{meta}}$, and an element of $\mathsf{VAL}|_{\mathsf{meta}}$ will in the following be indicated by a set Γ indexed by a specification of Tim's respective position on the chessboard.

Our formalized language does not contain expressions that refer to the possible manifestations of Tim's potentiality to jump in the various metaphysically possible circumstances. It is only suited for describing those circumstances themselves as something entirely static. The possible manifestations of Tim's potentiality to jump, which provide those static circumstances with a dynamic component, can nevertheless be given a linguistic rendering. A transition that captures a possible manifestation of Tim's potentiality to jump in a given circumstance can be modeled as a pair $\langle \varphi(\mathsf{Tim}), \varphi'(\mathsf{Tim}) \rangle$ consisting of two atomic sentences: the initial $\varphi(\mathsf{Tim})$ specifies Tim's position on the chessboard in the given circumstance, and the outcome $\varphi'(\mathsf{Tim})$ specifies some possible next position.

4.4.2.1.1.2 The construction Having set the stage, we will now consider how we can successively lift a branching time model for real possibility from a single circumstance on the basis of Tim's potentiality to jump. Let us start out with the metaphysically possible circumstance depicted in Fig. 4.1, in which Tim the frog is located on the square B2 of our chessboard. We take some arbitrary moment m_0, and we set $M_0 := \{m_0\}$, and, not having established any temporal relations yet, we let the temporal ordering $<_0 \subseteq M_0 \times M_0$ on M_0 be the empty set $\emptyset$. We furthermore define a function $V_0 : M_0 \to \mathsf{VAL}|_{\mathsf{meta}}$ that assigns to m_0 the set of literals Γ_{B2}, which corresponds to our initial circumstance.

According to the modal profile of the potentiality to jump, being located on the square B2, Tim can manifest his potentiality in two different ways. The modal profile provides a set of transitions that contains two transitions with the same initial but mutually exclusive outcomes, viz. the transition set $\{\langle \mathsf{B2}(\mathsf{Tim}), \mathsf{A3}(\mathsf{Tim}) \rangle, \langle \mathsf{B2}(\mathsf{Tim}), \mathsf{C3}(\mathsf{Tim}) \rangle\}$. The transitions emerge in the circumstance obtaining at the moment m_0 and point toward the future. They specify Tim's alternative possible progressings in virtue of his potentiality to jump in the given circumstance, and Tim can exercise only one of them at once. Each of the two transitions specifies one possible future continuation of m_0, and for each transition we add a future moment above m_0. That is, we add two new future moments m_1 and m_2 to M_0, and we put them in suitable temporal relations to m_0, thereby creating a branching point. More formally, we set $M_1 := M_0 \cup \{m_1, m_2\}$ and $R_1 := <_0 \cup \{\langle m_0, m_1\rangle, \langle m_0, m_2\rangle\}$. Since the temporal ordering relation we are to establish is supposed to be transitive, we take $<_1$ to be the transitive closure R_1^* of R_1. While, so far, this does not make any difference, it will become important later on.

We denote the set of transitions that spans the possible future continuation of m_0 corresponding to the new future moment m_1 by $\mathsf{poss}_{\langle m_0, m_1\rangle}$ and the set of transitions that specifies the possible future continuation of m_0 corresponding to m_2 by $\mathsf{poss}_{\langle m_0, m_2\rangle}$. That is:

- $\mathsf{poss}_{\langle m_0, m_1\rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{A3(Tim)}\rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_2\rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{C3(Tim)}\rangle\}$.

Both sets are singletons. That is, every possible future transformation of the circumstance obtaining at m_0 is captured by a single transition only. In each case, the unique transition determines what is the case at the respective future moment by providing Tim's future position on the chessboard, leaving his potentiality to jump untouched. At the moment m_1, Tim is located on the square A3, and at the moment m_2, he is located on the square C3. We define an extension V_1 of the function along the following lines: we let $V_1|_{M_0} := V_0$, $V_1(m_1) := \Gamma_{\mathrm{A3}}$ and $V_1(m_2) := \Gamma_{\mathrm{C3}}$. The tree resulting from this construction step is depicted in Fig. 4.2.

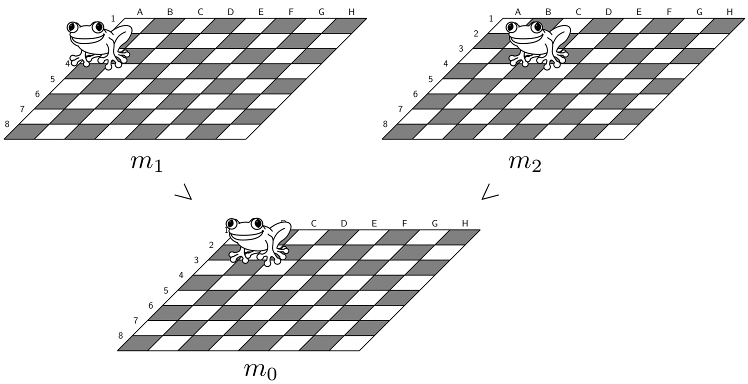

Figure 4.2: The first construction step.

The construction can of course be iterated. In the next step, we consider some $<_1$-maximal moment in M_1. In our case, there are two such moments, and we can proceed with either of them. Our construction method does not reflect the flow of time, and the precise order does not matter. With that said, let us consider the moment m_1, at which Tim is located on the square A3. As this square lies on the leftmost file of the chessboard, according to the modal profile of the potentiality to jump, Tim can manifest his potentiality in the given circumstance in only a single way: he can only jump diagonally right, the corresponding transition set being $\{\langle \mathsf{A3(Tim)}, \mathsf{B4(Tim)}\rangle\}$. In this case, only one new moment, m_3, emerges in the future of m_1, and we have $\mathsf{poss}_{\langle m_1, m_3\rangle} = \{\langle \mathsf{A3(Tim)}, \mathsf{B4(Tim)}\rangle\}$. We set $M_2 := M_1 \cup \{m_3\}$, $R_2 := <_1 \cup \{\langle m_1, m_3\rangle\}$ and $<_2 := R_2^*$. Moreover, we let $V_2|_{M_1} := V_1$ and $V_2(m_3) := \Gamma_{\mathrm{B4}}$.

In the circumstance obtaining at m_2, Tim is located on the square C3 of the chessboard. Given the modal profile of the potentiality to jump, in this circumstance, he can again manifest his potentiality in two different ways, as specified by the following transition set: $\{\langle \mathsf{C3(Tim)}, \mathsf{B4(Tim)}\rangle, \langle \mathsf{C3(Tim)}, \mathsf{D4(Tim)}\rangle\}$. We add two new future moments, m_4 and m_5, above m_2, and we let:

- $\mathsf{poss}_{\langle m_2, m_4\rangle} = \{\langle \mathsf{C3(Tim)}, \mathsf{B4(Tim)}\rangle\}$;
- $\mathsf{poss}_{\langle m_2, m_5\rangle} = \{\langle \mathsf{C3(Tim)}, \mathsf{D4(Tim)}\rangle\}$.

We set $M_3 := M_2 \cup \{m_4, m_5\}$, $R_3 := <_2 \cup \{\langle m_2, m_4\rangle, \langle m_2, m_5\rangle\}$ and $<_3 := R_3^*$. Moreover, we define $V_3|_{M_2} := V_2$ and $V_3(m_4) := \Gamma_{\mathrm{B4}}$ and $V_3(m_5) := \Gamma_{\mathrm{D4}}$. The result of our construction is depicted in Fig. 4.3. Note that both at the moment m_3 and m_4, Tim is located on the square B4. The circumstances obtaining at those different moments are precisely the same, and their respective possible future continuations will likewise be identical. However, the circumstances obtaining at m_3 and m_4 differ with respect to their past. The very same circumstance is part of two different possible courses of events. This shows that, strictly speaking, we are not establishing temporal relations between concrete metaphysically possible circumstances but rather between instances thereof.

It should be obvious by now how the construction proceeds, and we will refrain from executing any further steps. In case there is only a single object with a single potentiality, at each step of the construction there arises but a single set of transitions, and each of the transitions spans one possible future continuation. For each transition, we add a new future moment, and assign to that future moment a maximal consistent set of literals from $\mathsf{VAL}|_{\mathsf{meta}}$, which is determined by the outcome of the respective transition. In our little toy example, the construction can be continued until, at any topmost moment, Tim

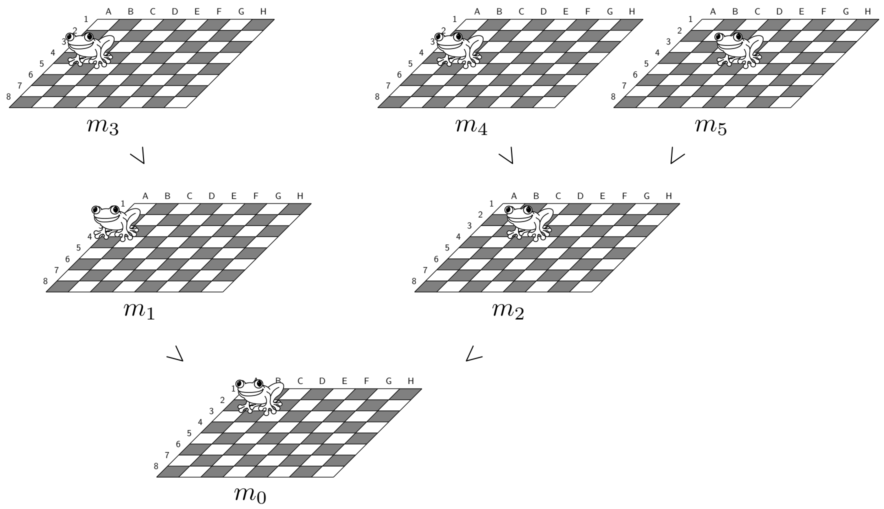

Figure 4.3: A tree for Tim.

has reached a square on the eighth rank. If Tim is located on a square on the eighth rank, he cannot manifest his potentiality to jump anymore. Hence, no new transitions emerge, no future moments arise, and the construction comes to an end.

4.4.2.1.1.3 The result By following the above construction method, we have established a temporal ordering among moments, each of which is associated with some element from $\mathsf{VAL}|_{\mathsf{meta}}$ and hence corresponds to a metaphysically possible circumstance. The temporal ordering has been raised in a dynamic fashion from the transitions that capture the possible manifestations of Tim's potentiality to jump in the respective circumstances. The result is a branching time model for real possibility.

The construction, first of all, yields a set of moments M with a temporal ordering relation $<$. The temporal order on M is by construction transitive, and since the transitions from which it is lifted are future-directed, also irreflexive, asymmetric and left-linear. Due to the fact that the manifestations of potentialities are local in time, the temporal order is jointed as well. The set of moments M together with its temporal ordering relation $<$ thus constitutes a branching time structure $\mathcal{M} = \langle M, < \rangle$, according to Def. 1.1, albeit a finite one: all possible branches ultimately come to an end. No matter where

Tim starts out and which route he takes, he will sooner or later end up on a square on the eighth rank. Since the transitions from which the structure is lifted are pairs whose initials and outcomes correspond to two consecutive states of affairs, the temporal ordering is moreover discrete. Each immediate possible future continuation of a moment is determined by a single transition that captures one of the possible manifestations of Tim's potentiality to jump, and those single transitions can be found back in the emerging branching time structure. On the structural level, every transition that captures a possible manifestation of Tim's potentiality to jump is isomorphic to a pair of two consecutive moments and hence corresponds to a discrete transition in the sense in which the notion is employed in the transition semantics presented in chapter 2. Recall that in the transition semantics, only indeterministic transitions, which capture the local change at a branching point, were taken into account since, from the abstract point of view of logic, structurally trivial transitions do not add to the course of events represented by a transition set. From a metaphysical perspective, however, structurally trivial transitions are interesting in their own right as they are now filled with content. Structurally, the transition from the moment m_1 to the moment m_2 is a trivial one because the moment m_1 is not a branching point. It describes Tim as jumping from A3 to A4 and thus it is not trivial as well qualitatively.

The branching time structure $\mathcal{M} = \langle M, < \rangle$ resulting from the construction is equipped with a function $V : M \to \mathsf{VAL}|_{\mathsf{meta}}$ that assigns to each moment of the temporal order a maximal consistent set of literals that corresponds to a metaphysically possible circumstance and provides an interpretation of the atomic sentences of the language at every single moment. Together with the function V, the branching time structure $\mathcal{M}$ yields a locally metaphysically possible branching time model $\mathfrak{M} = \langle M, <, V \rangle$, according to Def. 4.4. The interpretation of the atomic sentences takes into account the very nature of the entities of our fundamental ontology with which the language is tied up.

The branching time model generated by our construction is developed from the standpoint of the initial circumstance. It is a branching time model relative to the metaphysically possible circumstance we started out with. Had we started out with another metaphysically possible circumstance, we would have obtained a different branching time model. Notwithstanding the particular choice of the initial circumstance, however, each branching time model that results from our construction is always a model for real possibility. Everything that is the case at some moment in some possible future course of events can temporally evolve from the initial circumstance against the background of the laws of nature prevailing in our little toy universe. The laws of nature are encoded by the objects' potentialities—or more precisely, by the potentialities' modal profiles. They enter the construction locally, with the potentialities of the objects existing in a given circumstance instead of flowing from complete possible temporal developments as is the case on a neo-Humean account. As Anscombe said, "the laws are, rather, like the rules of chess" (Anscombe 1981a, p. 143). They capture which moves are possible in each case on the basis of the potentialities of the chess pieces, or in our case, on the basis of Tim's potentiality to jump. By successively lifting the alternative possible courses of events from Tim's initial position on the chessboard on the basis of the possible manifestations of his potentiality to jump, we ensure that each history represents a possible temporal development that conforms to the laws of nature, which are encoded by the modal profile of his potentiality. In particular, the model perfectly fits its underlying structure, and the local kind of indeterminism that lies at the heart of branching time models is preserved.

The construction method allows us to delimit the range of all locally metaphysically possible branching time models to branching time models for real possibility. A branching time model which contains a possible course of events in which Tim is located on the square B2 of the chessboard at one moment and reaches the square F8 only two moments later, for instance, is ruled out, owing to the modal profile of Tim's potentiality to jump. While on the basis of the modal profile of the potentiality to jump, it is really possible that Tim reaches the square F8 at some future moment, if he starts out on the square B2, he can reach that square only six moments later. The branching time model that is lifted from the initial circumstance in which Tim is located on the square B2 actually contains several possible courses of events in which Tim reaches the square F8 six moments later. In fact, in that model exactly six histories emerge in which this is the case.

Branching is accounted for in terms of the indeterministic potentialities of objects. It is the fact that in some possible circumstances Tim can manifest his potentiality to jump in several different ways that explains why there are moments at which histories branch. There is a close and systematic link between the circumstance obtaining at a given moment and the local branching structure. If histories branch at a moment, there is always a possible manifestation that makes the difference. Every possible future continuation of a moment is due to some possible manifestation of Tim's potentiality to jump. That histories branch at a certain moment in a certain way—rather than in another—is accounted for by the modal profile of Tim's potentiality. Since the modal profile of the potentiality to jump does not allow for a circumstance in which Tim can manifest his potentiality in three different ways, a branching time model for real possibility which contains a branching point with three possible future continuations is excluded from the outset—irrespective of which metaphysically possible circumstance we start out with.

Given the modal profile of Tim's potentiality, in every possible circumstance, Tim can jump at the utmost diagonally left or diagonally right. Therefore, he will always remain on light squares, if he starts out on a light square, and likewise, he will always stay on dark squares, if he starts out on a dark square. In Tim's universe, there cannot be a possible branching time model for real possibility that contains a history in which Tim is located on a light square at one moment and on a dark square at some later moment, or the other way around. We will see that things change as soon as another object enters the chessboard, in which case Tim's jumping is no longer always entirely undisturbed.

4.4.2.1.2 Tim's friend Tom We have so far focused on the case in which there is only a single object with a single indeterministic potentiality. The branching time model for real possibility that we have constructed was lifted from an initial circumstance on the basis of that's objects potentiality alone. At every step of the construction, we were provided with a single set of transitions capturing the possible manifestations of the object's potentiality in the corresponding circumstance, and each transition of the set was taken to span one possible immediate future continuation. We will now consider what happens if another powerful object, again equipped with an indeterministic potentiality, enters the scene. That is, we will illustrate how the possible immediate future continuations are now determined by the combination and interaction of the possible manifestations of the objects' respective potentialities in each case.

4.4.2.1.2.1 World and language Assume that our little frog Tim is not alone, but that there is another little frog jumping around on the chessboard. That is, we enrich the fundamental ontology of our toy universe by another object, leaving the range of fundamental properties and potentialities intact. Let us assume that, just as Tim, his friend possesses the potentiality to jump essentially, while each of the properties of being located on some particular square on the chessboard constitutes a contingent property of him. We furthermore assume that the modal profile of the potentiality to jump does not discriminate between the two frogs, i.e. the modal profile that we provided in section 4.4.2.1.1.1 also applies to Tom. On the assumption that each frog is always located on exactly one square of the chessboard and that they can never occupy the same square at once, there are $\frac{64!}{64-2!} = 4032$ metaphysically possible circumstances. Every metaphysically possible circumstance represents the respective positions of the two frogs, and in every metaphysically possible circumstance both frogs have the potentiality to jump.

Introducing a new object into our fundamental ontology, we also need to extend the alphabet of our first-order language in a suitable way. In other words, we need to add another individual constant that rigidly refers to Tim's new friend: we will give him the name Tom. Given that the language is firmly tied up with our fundamental ontology, corresponding to the range of metaphysically possible circumstances, we obtain the set $\mathsf{VAL}|_{\mathsf{meta}}$. We will distinguish the elements Γ from $\mathsf{VAL}|_{\mathsf{meta}}$ by double-indexing them: the first index specifies Tim's contingent position on the chessboard and the second specifies Tom's contingent position on the chessboard.

4.4.2.1.2.2 The construction With these preliminaries in place, we can now again construct a branching time model for real possibility. As a starting point, we take the circumstance in which Tim is located on the square B2, as before, and Tom is located on the square F2. We take some arbitrary moment m_0, we set $M_0 := m_0$, $<_0 := \emptyset$ and $V_0(m_0) := \Gamma_{\mathrm{B2,F2}}$.

In the given circumstance, both frogs can manifest their potentiality to jump in two different ways. We obtain two sets of transitions: one set capturing Tim's possible manifestations and one set capturing Tom's possible manifestations. We denote the set of transition sets arising at the moment m_0 on the basis of the frogs' potentialities in the corresponding circumstance by $\mathsf{PROG}(m_0)$. The set $\mathsf{PROG}(m_0)$ contains the following elements:

- $\{\langle \mathsf{B2(Tim)}, \mathsf{A3(Tim)} \rangle, \langle \mathsf{B2(Tim)}, \mathsf{C3(Tim)} \rangle\}$;
- $\{\langle \mathsf{F2(Tom)}, \mathsf{E3(Tom)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{G3(Tom)} \rangle\}$.

Being alternative possible manifestations of an object's indeterministic potentiality, the transitions contained within the same set always share the same initial but have mutually exclusive outcomes. Both frogs can exercise only one of their respective transitions at once. Yet, no matter which of his transitions Tim exercises, Tom can simultaneously exercise either of his transitions; and *vice versa*. The possible progressings of Tim and Tom are completely independent from each other: every possible progressing of Tim can be combined with any possible progressing of Tom. Every possible combination over the sets of transitions in $\mathsf{PROG}(m_0)$ yields a transition set that spans one possible immediate future continuations of m_0, and we denote the set of those transition sets by $\mathsf{POSS}(m_0)$. Each set in $\mathsf{POSS}(m_0)$ is made up from two transitions, one for Tim and one for Tom, and there are exactly four such sets:

- $\{\langle \mathsf{B2(Tim)}, \mathsf{A3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{E3(Tom)} \rangle\}$;
- $\{\langle \mathsf{B2(Tim)}, \mathsf{A3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{G3(Tom)} \rangle\}$;
- $\{\langle \mathsf{B2(Tim)}, \mathsf{C3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{E3(Tom)} \rangle\}$;
- $\{\langle \mathsf{B2(Tim)}, \mathsf{C3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{G3(Tom)} \rangle\}$.

Hence, four possible immediate future continuations emerge, and we add four new future moments, m_1, m_2, m_3 and m_4, above m_0. We set $M_1 := M_0 \cup \{m_1, m_2, m_3, m_4\}$, $R_1 := <_0 \cup \{\langle m_0, m_1 \rangle, \langle m_0, m_2 \rangle, \langle m_0, m_3 \rangle, \langle m_0, m_4 \rangle\}$ and $<_1 := R_1^*$.

We associate each transition set in $\mathsf{PROG}(m_0)$ with one of our new future moments. We let:

- $\mathsf{poss}_{\langle m_0, m_1 \rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{A3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{E3(Tom)} \rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_2 \rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{A3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{G3(Tom)} \rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_3 \rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{C3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{E3(Tom)} \rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_4 \rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{C3(Tim)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{G3(Tom)} \rangle\}$.

Unlike in the case, in which we were dealing with only one powerful object, the transition sets that span the possible immediate future continuations of the moment m_0 now contain two transitions in each case, and those transitions jointly determine what is the case at the respective future moments. Each transition captures a possible progressing of one of the frogs, and together they determine the transformation of the circumstance as a whole. Since each set of $\mathsf{PROG}(m_0)$ is such that the outcomes of the transitions it contains are perfectly compatible and can be realized at once, what is the case at the corresponding future moment is basically nothing else than the conjunction of those outcomes. We let $V_1|_{M_0} := V_0$ and set $V_1(m_1) := \Gamma_{\mathrm{A3,E3}}$, $V_1(m_2) := \Gamma_{\mathrm{A3,G3}}$, $V_1(m_3) := \Gamma_{\mathrm{C3,E3}}$ and $V_1(m_4) = \Gamma_{\mathrm{C3,G3}}$. The result of this construction step is depicted in Fig. 4.4.

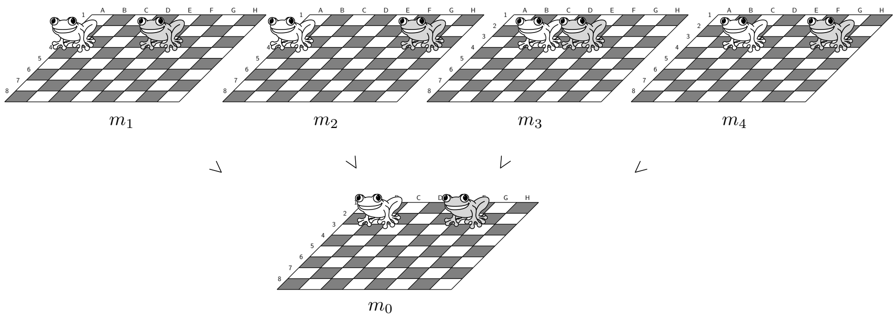

Figure 4.4: The first step of the construction.

Let us now consider the circumstance obtaining at m_3. Recall that the order in which we proceed does not matter. In the circumstance obtaining at m_3, Tim is located on the square C3, and Tom is located on the square E3. The set $\mathsf{PROG}(m_3)$ contains two transition set, each containing two transitions:

- $\{\langle \mathsf{C3(Tim)}, \mathsf{B4(Tim)}\rangle, \langle \mathsf{C3(Tim)}, \mathsf{D4(Tim)}\rangle\}$;
- $\{\langle \mathsf{E3(Tom)}, \mathsf{D4(Tom)}\rangle, \langle \mathsf{E3(Tom)}, \mathsf{F4(Tom)}\rangle\}$.

Due to the independence of the progressings of our two frogs, again all possible combinations over the two sets of transitions in $\mathsf{PROG}(m_3)$ are possible. The set $\mathsf{POSS}(m_3)$ accordingly contains four sets, each of which specifies one possible future continuation of m_3. We add four new future moments, m_5, m_6, m_7 and m_8. We set $M_2 := M_1 \cup \{m_5, m_6, m_7, m_8\}$, $R_2 := {<_1} \cup \{\langle m_3, m_5\rangle, \langle m_3, m_6\rangle, \langle m_3, m_7\rangle, \langle m_3, m_8\rangle\}$ and ${<_2} := R_2^*$, and we let:

- $\mathsf{poss}_{\langle m_3, m_5\rangle} = \{\langle \mathsf{C3(Tim)}, \mathsf{B4(Tim)}\rangle, \langle \mathsf{E3(Tom)}, \mathsf{D4(Tom)}\rangle\}$;
- $\mathsf{poss}_{\langle m_3, m_6\rangle} = \{\langle \mathsf{C3(Tim)}, \mathsf{B4(Tim)}\rangle, \langle \mathsf{E3(Tom)}, \mathsf{F4(Tom)}\rangle\}$;
- $\mathsf{poss}_{\langle m_3, m_7\rangle} = \{\langle \mathsf{C3(Tim)}, \mathsf{D4(Tim)}\rangle, \langle \mathsf{E3(Tom)}, \mathsf{D4(Tom)}\rangle\}$;
- $\mathsf{poss}_{\langle m_3, m_8\rangle} = \{\langle \mathsf{C3(Tim)}, \mathsf{D4(Tim)}\rangle, \langle \mathsf{E3(Tom)}, \mathsf{F4(Tom)}\rangle\}$.

Contrary to what has been the case so far, the $\mathsf{POSS}(m_3)$ comprises a set of transitions whose outcomes are not compatible with each other: the outcomes of the two transitions contained in the set $\mathsf{poss}_{\langle m_3,m_7\rangle}$ cannot both be realized at the moment m_7. Both frogs jump toward the square D4, but they cannot both end up on that square at the moment m_7 as it is, as we said, metaphysically impossible that both frogs occupy the same square. It is important to note that the fact that the outcomes of the transitions in $\mathsf{poss}_{\langle m_3,m_7\rangle}$ are incompatible does not imply that those transitions cannot be exercised simultaneously. Both the manifestations and their licensing are local in time. That the combination of the outcomes of the transitions in $\mathsf{poss}_{\langle m_3,m_7\rangle}$ is metaphysically impossible just means that the transitions interfere with each other and produce a joint outcome that differs from the conjunction of their individual outcomes. Here it becomes crucial that a manifestation is a transition toward the future and not an unfolding of a future course of events.

Now, it happens to be an inherent feature of the potentiality to jump that if two frogs exercise their potentiality to jump by jumping toward the very same square of the chessboard, they bump into each other, recoil and each of them ends up on the respective immediate square on the same file and on the next higher rank from where he started. That is, if in the circumstance obtaining at m_3 both frogs exercise their respective transitions as provided by the set $\mathsf{poss}_{\langle m_3,m_7\rangle}$, at the moment m_7, Tim will be located on the square C4 and Tom on the square E4. We set $V_2(m_7) := \Gamma_{\mathrm{C4,E4}}$. All other cases are defined as usual. We let $V_2|_{m_2} := V_1$, $V_2(m_5) := \Gamma_{\mathrm{B4,D4}}$, $V_2(m_6) := \Gamma_{\mathrm{B4,F4}}$ and $V_2(m_8) := \Gamma_{\mathrm{D4,F4}}$. The result of our construction is provided in Fig. 4.5. The construction can be continued until, at every topmost moment, both frogs have reached a square on the eighth rank.

4.4.2.1.2.3 The result The result of our construction is again a branching time model for real possibility. We obtain an ordering of moments that constitutes a branching time structure and a function that assigns to each moment of the structure a maximal consistent set of literals from $\mathsf{VAL}|_{\mathsf{meta}}$. Since the model and the underlying structure are lifted from the possible manifestations of the objects' potentialities, each possible course of events is compatible with the prevailing laws of nature as encoded by the potentialities, their modal profile and their interaction.

The crucial difference with the case in which there is just a single object with a single potentiality consists in the fact that when a second powerful object enters the scene, at every step of the construction, we are provided with two transition sets, one for each object. On the assumption that the possible

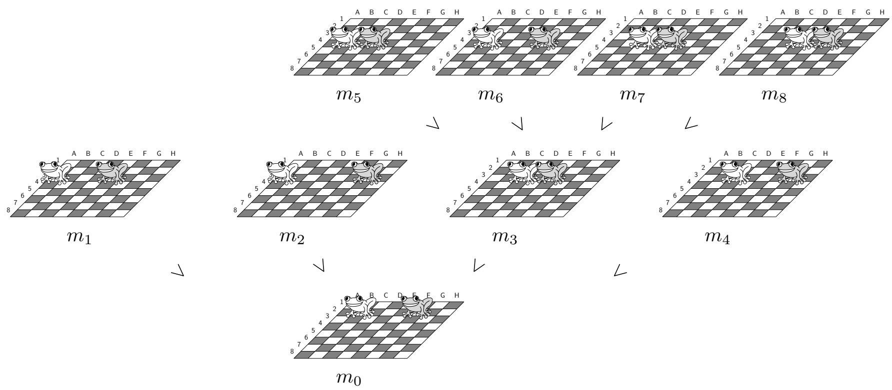

Figure 4.5: A tree for Tim and Tom.

progressings of the two objects in virtue of their potentialities are always completely independent from each other, the range of immediate possible future continuations of a moment is determined by the possible combinations over the sets of transitions associated with the objects potentialities in the corresponding circumstance. An immediate possible future continuation is now spanned by a two-elementary set of transitions rather than by a single transition. What is the case at a future moment is accordingly determined by the combination of two transitions.

The example illustrates that the branching time model that is generated on the basis of the potentialities of two objects is not simply the combination of the branching time models that had been obtained by treating each object and its respective potentiality separately. In other words, what is the case at a future moment is not always the mere conjunction of the outcomes of the transitions making up the corresponding immediate future continuation. While there are cases in which the respective transitions in fact combine in such a way, this does not hold in general. As a transition, a possible manifestation of a potentiality is a mere pointing toward the future and can be naturally interfered with. The possible manifestations of the objects' potentialities that span a possible immediate future continuation of a moment can interact with each other, and the outcome they jointly produce does not have to be identical to the conjunction of their respective individual outcomes. They contribute to what is the case at the corresponding future moment, but they are not identical to it.

From the interaction of the possible manifestation of different potentialities new possibilities may emerge. As said, if Tim is located on the square C3 and Tom on E3 and both manifest their potentiality to jump by jumping toward the square D4, none of them will actually make it to his destination. Tim ends up on the dark square in from of him, and the same holds for Tom. None of them could have reached the respective square in the absence of the other. It is not in their individual power to do so. A possible future course of events in which Tim and Tom each end up on a dark square, although having started out on a light one, can only arise if during the construction there emerges some moment with a circumstance in which Tim and Tom are located in such a way that the possible manifestations of their potentialities can interact. In a branching time model in which Tim starts out on a light square, whereas Tom starts out on a dark square that possibility is blocked from the outset.

4.4.2.1.3 The possibility and impossibility of refraining When laying down the modal profile of our frogs' potentiality to jump, we have assumed that each possible circumstance allows for at most two possible progressings: at the utmost our frogs can jump diagonally left or diagonally right, but they always have to move if they can. What we have neglected so far is the possibility that our frogs may simply refrain from jumping. That is, we have not yet taken into account omission or forbearance as a possible manifestation of the potentiality to jump. In this section, we will deal with omissions and forbearances as possible manifestations of indeterministic potentialities. In particular, we will discuss the difference between the possibility of omission or forbearance, on the one hand, and the impossibility of manifestation due to licensing failure, on the other.

4.4.2.1.3.1 World and language In order to make room for omission or forbearance, we have to revise the modal profile of the potentiality to jump. Strictly speaking, we thereby make it a different potentiality. For the sake of simplicity, we will not include omission or forbearance as a third possible manifestation into the modal profile of the potentiality to jump as this would needlessly increase the space of possibilities. We rather assume that in virtue of their potentiality to jump, Tim and Tom can either jump toward the immediate square on the same file on the next higher rank or refrain from jumping, in every circumstances in which they can manifest their potentiality at all.

To bring out the difference between omission and forbearance, on the one hand, and non-manifestation due to licensing failure, on the other, we also make a slight modification to our chessboard. Imagine that last night a kobold came along, daubed several squares of the chessboard with glue and disappeared again. Let us assume that the squares affected by the kobold's nightly glue attack are all the squares on the third and sixth rank. Now, if a frog reaches a square daubed with glue, he will be stuck to it and can no longer manifest his potentiality to jump. In such a circumstance, he cannot even refrain from jumping, since omission or forbearance can never be the only possible manifestation of an indeterministic potentiality. So here is the modal profile of our new potentiality to jump:

(i') if the frog is located on a square that is neither on the eighth rank nor daubed with glue, he can manifest his potentiality to jump by either jumping toward the immediate square on the same file on the next higher rank or by refraining from jumping;

(ii') if the frog is located on a square that is on the eighth rank or daubed with glue, he cannot manifest his potentiality to jump.

Our formalized language remains unaltered.

4.4.2.1.3.2 The construction We will now again construct a branching time model for real possibility, and we will illustrate how the difference between omissions and forbearances, on the one hand, and non-manifestations, on the other, bears on our construction. Let us start out with the metaphysically possible circumstance in which Tim is located on the square B2 of the chessboard and Tom is located on the square F2. We take some arbitrary moment m_0, we set $M_0 := \{m_0\}$, $<_0 := \emptyset$ and $V_0(m_0) := \Gamma_{\mathrm{B2,F2}}$. In the given circumstance, both frogs can manifest their potentiality to jump in two different ways: they can either jump one square ahead or refrain from jumping. The set $\mathsf{PROG}(m_0)$ accordingly contains two transition sets, each equipped with two transitions: one capturing a possible movement, the other representing omission or forbearance. Omissions or forbearances are possible manifestations of indeterministic potentialities, and like all other possible manifestations, they are conceived of as transitions toward the future. Omissions and forbearances are only special as so far as they correspond to transitions whose respective outcomes are qualitatively identical to their initials. The transition sets contained in $\mathsf{PROG}(m_0)$ are the following:

- $\{\langle \mathsf{B2(Tim)}, \mathsf{B3(Tim)}\rangle, \langle \mathsf{B2(Tim)}, \mathsf{B2(Tim)}\rangle\}$;
- $\{\langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)}\rangle\}$.

Since the possible progressings of Tim and Tom are completely independent from each other, there are four possible combinations over the sets in $\mathsf{PROG}(m_0)$, and those combinations make up the elements of the set $\mathsf{POSS}(m_0)$:

- $\{\langle \mathsf{B2(Tim)}, \mathsf{B3(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)}\rangle\}$;
- $\{\langle \mathsf{B2(Tim)}, \mathsf{B3(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)}\rangle\}$;
- $\{\langle \mathsf{B2(Tim)}, \mathsf{B2(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)}\rangle\}$;
- $\{\langle \mathsf{B2(Tim)}, \mathsf{B2(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)}\rangle\}$.

Each transition set contained in $\mathsf{POSS}(m_0)$ captures one possible immediate future continuation of m_0, and accordingly four new moments m_1, m_2, m_3 and m_4, emerge in the future of m_0. We set $M_1 := M_0 \cup \{m_1, m_2, m_3, m_4\}$, $R_1 := <_0 \cup \{\langle m_0, m_1\rangle, \langle m_0, m_2\rangle, \langle m_0, m_3\rangle, \langle m_0, m_4\rangle\}$ and $<_1 := R_1^*$. Moreover, we associate each element of $\mathsf{POSS}(m_0)$ with one of the new future moments. We set:

- $\mathsf{poss}_{\langle m_0, m_1\rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{B3(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)}\rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_3\rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{B3(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)}\rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_2\rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{B2(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)}\rangle\}$;
- $\mathsf{poss}_{\langle m_0, m_4\rangle} = \{\langle \mathsf{B2(Tim)}, \mathsf{B2(Tim)}\rangle, \langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)}\rangle\}$.

Each set of $\mathsf{POSS}(m_0)$ is such that the outcomes of the transitions it contains are perfectly compatible. We then set $V_1|_{M_0} := V_0$, $V_1(m_1) := \Gamma_{\text{B3,F3}}$, $V_1(m_2) := \Gamma_{\text{B3,F2}}$, $V_1(m_3) := \Gamma_{\text{B2,F3}}$ and $V_1(m_4) := \Gamma_{\text{B2,F2}}$.

Since the entire third rank is daubed with glue, in the circumstance obtaining at m_1, neither Tim not Tom can manifest their potentiality to jump. The manifestation of their potentiality is not licensed. In this case, no transitions arise. Both the set capturing Tim's possible progressing and the set capturing Tom's possible progressings equal the empty set of transitions. Consequently, the set $\mathsf{POSS}(m_1)$, which specifies the range of immediate possible future continuations of m_1, is empty as well. Hence, no new moments emerge in the future of m_1, and the branch comes to an end.

In the circumstance obtaining at m_2, Tim is stuck. He is located on a square daubed with glue and accordingly cannot manifest his potentiality to jump.

Tom, on the other hand, who has refrained from jumping at the moment m_0, is still located on the square F2 and can again manifest his potentiality to jump in two different ways: he can jump or refrain from jumping. The set $\mathsf{PROG}(m_2)$ contains two transition sets. While the set of transitions corresponding to Tim's possible manifestations is empty, Tom's transition set contains two transitions. The elements of the set $\mathsf{PROG}(m_2)$ are the following:

- $\emptyset$;
- $\{\langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)} \rangle, \langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)} \rangle\}$.

There are only two possible combinations over the sets of $\mathsf{PROG}(m_2)$, and the set $\mathsf{POSS}(m_2)$ accordingly contains two elements. Its elements are singletons: each of them contains one of Tom's possible progressings. We add two new future moments, m_5 and m_6, and we set $M_2 := M_1 \cup \{m_5, m_6\}$, $R_2 := <_1 \cup \{\langle m_2, m_5 \rangle, \langle m_2, m_6 \rangle\}$ and $<_2 := R_2^*$. We let:

- $\mathsf{poss}_{\langle m_2, m_5 \rangle} = \{\langle \mathsf{F2(Tom)}, \mathsf{F3(Tom)} \rangle\}$;
- $\mathsf{poss}_{\langle m_2, m_6 \rangle} = \{\langle \mathsf{F2(Tom)}, \mathsf{F2(Tom)} \rangle\}$.

Tim's position remains unaltered, and we get $V_2|_{M_1} := V_1$, $V_2(m_5) = \Gamma_{\mathrm{B3,F3}}$ and $V_2(m_6) = \Gamma_{\mathrm{B3,F2}}$. In the circumstance obtaining at m_5, both frogs are stuck on glue. Consequently, they cannot manifest their potentiality to jump anymore, and the branch comes to an end, as it does at the moment m_1. The circumstance obtaining at m_6, on the other hand, is exactly the same as the one obtaining at m_2, and in the future continuation of that moment, the construction step carried out at the moment m_2 will be repeated over and over again.

The circumstance obtaining at m_3 is similar to the one obtaining at m_2, just that the roles of Tim and Tom are switched. This time, Tom is stuck, while Tim can still manifest his potentiality to jump. We add two future moments, m_7 and m_8. We set $M_3 := M_2 \cup \{m_7, m_8\}$, $R_3 := <_2 \cup \{\langle m_3, m_7 \rangle, \langle m_3, m_8 \rangle\}$ and $<_3 := R_3^*$, and we let $V_3|_{M_2} := V_2$, $V_3(m_7) = \Gamma_{\mathrm{B3,F3}}$ and $V_3(m_8) = \Gamma_{\mathrm{B3,F2}}$.

Finally, at the moment m_4, both Tim and Tom are still in their respective original positions. Neither of them has moved. Both of them have refrained from doing so at the moment m_0. However, unlike in the case in which they are both stuck on glue, by refraining from jumping, Tim and Tom have contributed to the future course of events by preserving the possibility to jump for the future. A new future moment has emerged, viz. m_4, and the circumstance obtaining at that moment is identical to the one obtaining at our initial moment m_0, giving rise to the same possibilities. At the moment m_4, the entire construction starts over again. The result of our construction is depicted in Fig. 4.6. Dotted lines indicate that the branch will be continued, while missing dotted lines indicate that the branch has come to an end.

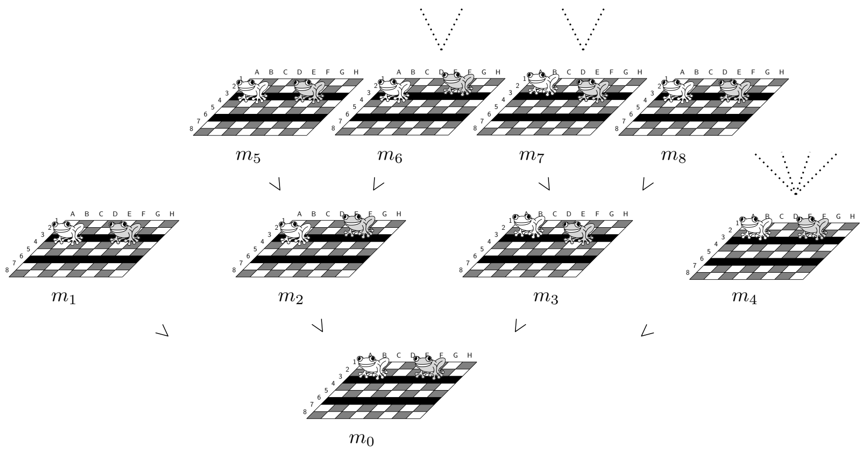

Figure 4.6: Possibility and impossibility of refraining.

4.4.2.1.3.3 The result Owing to the kobold's nightly glue attack, in the branching time model we have constructed, branches come to an end whenever both frogs have reached a square on the third rank. When ending up on a square on the third rank, Tim and Tom are stuck. They can no longer manifest their potentiality to jump. They cannot even refrain from jumping. The transition sets associated with their potentiality in those circumstances are empty, and thus no new future moments arise. On the picture we are working with then, time comes to a standstill. There is no time without change, and there is no change without objects manifesting their potentialities.[147]

[147]Our example is of course a simple toy example. Our frogs have but the potentiality to jump. If our frogs also had the potentiality to breathe, they could and probably would still manifest that potentiality when stuck on glue. In this case, non-empty sets of transitions would arise in those circumstances after all, and the respective branches would accordingly be continued although none of the frogs can manifest its potentiality to jump and is damned to remain where he is.

Although several branches will abruptly come to an end, the construction goes on forever.[148] This is due to the fact that rather than jumping toward a square that is daubed with glue and getting stuck, Tim and Tom can just as well refrain from jumping, and in this case, non-empty sets of transitions arise, and the branch is continued. In contradistinction to non-manifestations, omissions and forbearances are transitions toward the future—albeit trivial ones, whose respective outcomes are identical to their initials. By refraining from jumping, Tim and Tom contribute to the future course of events in as so far as they preserve the possibility to jump for the future—and with it also the possibility to refrain from jumping. If they refrain from jumping, they remain where they are, but they are not stuck. They preserve the status quo. The resulting branching time model contains infinite branches with successions of the very same circumstance that result from carrying out the very same construction step over and over again. In particular, it contains one infinite branch in which each single moment is associated with an instance of the initial circumstance. But such a branch is only possible because at every moment, there are alternative future possibilities.

It should be noted that not every branching time model for real possibility that contains a circumstance in which an object possesses an indeterministic potentiality that it can manifest by refraining from bringing about the characteristic manifestation contains infinite branches. Although by refraining from bringing about the characteristic manifestation, the object preserves the possibility of manifestation for the future by retaining the status quo, this is not to say that the object can actually manifest its potentiality in the future. For, by manifesting their potentialities, other objects may change the prevailing circumstance in such a way that the manifestation of the object's potentiality is no longer licensed, in which case even omission or forbearance are no longer possible. Just as any other possible manifestation of a potentiality, omission and forbearance is a mere pointing toward the future and can be interfered with by the manifestations of other potentialities. Suppose that you are sitting on a chair and refrain from getting up. And now image what happen if someone comes and ties you up. Neither getting up nor refraining from doing so is any longer a possibility for you.

[148] Of course, this is only true under the assumption that Tim and Tom are immortal. On our inductive evidence, they are.

4.4.2.2 The construction method

In the previous section, we have illustrated by means of a simple toy example how a branching time model for real possibility can be lifted from a single circumstance in a dynamic fashion on the basis of the potentialities of objects and their manifestations. At each step of the construction, the immediate possible future continuations of some topmost moment emerge from the possible manifestations of the objects' potentialities in the corresponding circumstance.

The possible manifestations of an object's potentiality in a given circumstance are mutually exclusive and local in space and time. At each moment, the possible manifestations of the potentialities of all powerful objects combine and jointly give direction to the alternative possible future courses of events. In our toy example, we have first considered how the construction of a branching time model for real possibility proceeds if there is just a single object with but a single indeterministic potentiality, Tim, and we have then discussed how the construction is affected if a second powerful object, Tom, equipped with the very same potentiality, enters the stage. We have illustrated how the possible manifestations of their potentialities combine and interact, how they can interfere with each other, producing a joint outcome. In general, it may of course be the case that the possible manifestations of more than two objects' potentialities combine and act in concert. Just imagine what happens if you dump an entire bucket of frogs over a chessboard and allow them to jump criss-cross. To be sure, our world is not a chessboard and its inhabitants are not exhausted by frogs whose sole potentiality is the potentiality to jump. Overall, our fundamental ontology can comprise various different kinds of objects each of which may possess various different potentialities at once.

In our little toy example, we have moreover construed the possible manifestations of Tim and Tom's potentiality to jump as transitions which point from one state of affairs to some immediate future state of affairs. Not all manifestations of potentialities need to correspond, however, to pairs consisting of two consecutive states of affairs. The possible manifestations of some—if not all—potentialities may involve processes of some kind. That is, they may correspond to pairs whose initials are states of affairs while theirs outcomes are processes rather than consecutive future states of affairs.

We will now generalize the construction of branching time models that we have developed in our little toy example. In section 4.4.2.2.1, we will first adapt our construction method to the case of an arbitrarily rich fundamental ontology, sticking to state-based transitions. Subsequently, in section 4.4.2.2.2 we will discuss the implications of extending the construction method to the case in which the possible manifestations of potentialities are conceived of as process-based transitions.

4.4.2.2.1 Branching time models for real possibility with state-based transitions In this section, we will describe the construction method for branching time models for real possibility that we have introduced on the basis of our toy example in more general terms, presupposing some arbitrary fundamental ontology. As in the case of our toy example, we will construe the possible manifestations of the objects' potentialities as state-based transitions.

4.4.2.2.1.1 World and language Our starting point is some basic stock of objects, properties, including potentialities, and relations. This basic stock provides the fundamental ontology of our world and lays down once and for all the set of all momentary, metaphysically possible circumstances.[149] We furthermore presuppose a formalized first-order language that perfectly matches our fundamental ontology. The alphabet of our language contains for each object of our fundamental ontology some rigidly designating individual constant, for each property and potentiality a corresponding predicate and for each n-ary relation an appropriate n-ary relation symbol. We assume that in addition, the alphabet comprises at least a sign for negation. Atomic sentences and their negations are formed in the usual way, and given that the language is firmly tied up with the world, there is a one-to-one correspondence between metaphysically possible circumstances and maximal consistent sets of literals in $\mathsf{VAL}|_{\mathsf{meta}}$.

Every potentiality comes along with a modal profile that captures the possible manifestations of an object in virtue of its potentiality in all possible circumstances in which the object has the potentiality. The possible manifestations of an object's potentiality in a circumstance are thereby conceived of as state-based transitions, i.e. as pairs whose respective initials and outcomes are consecutive states of affairs in each case. Although our language does not contain expressions that refer to the possible manifestations of the potentiali-

[149] In branching time, a moment is usually taken to span all of space simultaneously. We therefore assume that circumstances are inclusive in space. However, nothing in our construction hinges on that assumption. We could just as well start out with a set of circumstances that captures the momentary, metaphysical possibilities of some spatially local system only. The branching time model for real possibility arising from such a local configuration represents what in Cartwright and Pemberton (2013) is called a 'nomological machine'—albeit a possibly indeterministic one.

ties of objects, manifestations can be given a linguistic rendering. Unlike in the case of our little toy example, we will now, however, construe the initials and outcomes as sets of literals rather than as single atomic sentences. By allowing initials and outcomes to comprise negated atomic sentences as well, we make room for manifestations that involve an object acquiring or losing a property or affect its relations to other objects. Conceiving of initials and outcomes as entire sets of literals, moreover, allows us to accommodate cases in which the manifestation of an object's potentiality affects more than one of the object's properties or relations or even affects other objects' properties and relations as well. A transition in a circumstance is then viewed as a pair consisting of two non-empty sets of atomic sentences that represent two consecutive, metaphysically possible states of affairs. While the initial is always a suitable subset of the maximal consistent set of literals corresponding to the circumstance at hand, the outcome can in principle be any subset of a maximal consistent set of literals from $\mathsf{VAL}|_{\mathsf{meta}}$.

4.4.2.2.1.2 Getting the construction started We will now describe the construction method for branching time models for real possibility in general terms. That is, we will show how the objects' potentialities and their manifestations allow us to successively lift a branching time model for real possibility together with its underlying structure from a single metaphysically possible circumstance.

We start out with some arbitrary metaphysically possible circumstance c_0. We take some arbitrary moment m_0. We set $M_0 := \{m_0\}$ and $<_0 := \emptyset$, and we define a function $V_0 : M_0 \to \mathsf{VAL}|_{\mathsf{meta}}$ that assigns to m_0 the maximal consistent set of literals $\Gamma_{c_0} \in \mathsf{VAL}|_{\mathsf{meta}}$ that corresponds to the circumstance c_0. This is the base step of the construction.

The construction procedure is iterative, and in what follows we will specify, for some arbitrary step n, how the construction proceeds.[150] At the n-th step of the construction, we are provided with some set of moments M_n, some temporal ordering $<_n$ on M_n and some function V_n that assigns to each moment $m \in M_n$ some maximal consistent set of literals $\Gamma_c \in \mathsf{VAL}|_{\mathsf{meta}}$ that corresponds to some metaphysically possible circumstance c. We consider some $<_n$-maximal moment $m_i \in M_n$, i.e. some moment m_i for which there is no $m' \in M_n$ such that $m_i <_n m'$. Assume that $V_n(m_i) = \Gamma_{c_i}$.

[150]The models that can be constructed by the method we provide here are exclusively finite models. In order to allow for infinite models as well, transfinite induction is needed.

Given that the circumstance c_i obtaining at m_i is populated by powerful objects, a dynamic component enters the picture. Owing to the potentialities' modal profiles, every object that possesses a potentiality in the circumstance c_i is associated with a set of transitions that captures the possible manifestations of the object's potentiality in that circumstance and thereby specifies the object's possible progressings in virtue of its potentiality at the moment m_i. Formally, a set $\mathsf{PROG}(m_i)$ emerges that contains for each object and every potentiality possessed by that object in the circumstance c_i obtaining at m_i a possibly empty set of transitions.

The sets of transitions contained in $\mathsf{PROG}(m_i)$ can be singletons or consist of several transitions, depending on whether the corresponding potentiality is deterministic or indeterministic. In case a transition set in $\mathsf{PROG}(m_i)$ contains more than one transition, the transitions all share the same initial but have mutually exclusive outcomes. They represent alternative possible manifestations of the indeterministic potentiality of some particular object in a concrete circumstance, and only one of them can be exercised at once.

4.4.2.2.1.3 Extending the structure The range of immediate possible future continuations of the moment m_i is determined by the possible combinations of transitions over the various sets in $\mathsf{PROG}(m_i)$. The totality of all such possible combinations is given by the set $\mathsf{POSS}(m_i)$. The set $\mathsf{POSS}(m_i)$ spans the space of immediate future possibilities: each transition set contained in $\mathsf{POSS}(m_i)$ specifies one possible future continuation of m_i, and for each such set, we add some new future moment to M_n. Assume that the number of transition sets in $\mathsf{POSS}(m_i)$ equals k. We then add k new future moments $m_{j_1}, \ldots, m_{j_k}$ above m_i. We set $M_{n+1} := M_n \cup \{m_{j_1}, \ldots, m_{j_k}\}$, we let $R_{n+1} := <_n \cup \{\langle m_i, m_{j_1}\rangle, \ldots, \langle m_i, m_{j_k}\rangle\}$ and we define $<_{n+1}$ as the transitive closure R^*_{n+1} of R_{n+1}.

Under the assumption that every transition contained within some set of $\mathsf{PROG}(m_i)$ can co-occur with any transition contained within some other set of $\mathsf{PROG}(m_i)$, the set of transitions sets $\mathsf{POSS}(m_i)$ is identical to the full unordered product over the transition sets in $\mathsf{PROG}(m_i)$. If we denote the unordered product $\{\{x_1, \ldots, x_n\} \mid x_1 \in X_1, \ldots, x_n \in X_n\}$ over the sets of some set $X = \{X_1, \ldots, X_n\}$ by $\prod X$, we accordingly have that $\mathsf{POSS}(m_i) = \prod \mathsf{PROG}(m_i)$. In this case, the cardinality of $\mathsf{POSS}(m_i)$—and hence the number of possible future continuations of m_i—equals the product over the cardinalities of the various sets in $\mathsf{PROG}(m_i)$.

That every combination of transitions over the sets in $\mathsf{PROG}(m_i)$ is possible and constitutes an element of $\mathsf{POSS}(m_i)$ means that the possible manifestations of an object's potentiality in the circumstance obtaining at m_i do in no way depend on the possible manifestations of other objects' potentialities or on the possible manifestations of other potentialities of the very same object in the circumstance at hand. In other words, every object can realize any of the possible manifestations of its potentiality, no matter which of the possible manifestations of their respective potentialities other objects realize and no matter in which way it realizes any of its other potentialities.[151]

When discussing the licensing of the manifestation of an object's potentiality in section 4.4.1.2.2 above, we have seen, however, that there are cases in which the possible manifestation of one object's potentiality depends on the manifestations of other objects' potentialities. A fragile glass can only manifest its potentiality to break in the presence of someone with a potentiality that he can exercise and in virtue of which he strikes the glass.[152] Now assume that refraining from striking the glass constitutes another possible manifestation of the striker's potentiality. In this case, whether the glass can manifest its potentiality to break depends on which of the two possible manifestations of his potentiality the striker exercises. Only striking and breaking go together; refraining from striking and breaking do not. Here is another example that involves different potentialities of one and the same object. You both have the indeterministic potentiality to sit and the indeterministic potentiality to walk. Yet, you cannot exercise those potentialities simultaneously by both sitting and walking. You can refrain from one and do the other, or you can refrain from both, but you cannot both sit and walk at the same time. If in a circumstance, the possible manifestation of an object's potentiality depends on the manifestations of other objects' potentialities or other potentialities of the same object, not all combinations of transitions over the sets in $\mathsf{PROG}(m_i)$ are possible anymore. In case there are dependencies between different transitions from different sets in $\mathsf{PROG}(m_i)$, the number of future possibilities encoded by the set $\mathsf{POSS}(m_i)$ diminishes accordingly.

[151] This condition closely corresponds to the principle of *Independence of Agency* as it occurs in *stit* logics for agency. The Principle of independence requires that every choice of one agent can be combined with every choice of any other agent at a given moment. Technically, this amounts to the requirement that the intersection of the various possible choices of different agents at a given moment must contain at least one history. Cf. Belnap, Perloff, and Xu (2001, ch.7C4).

[152] We are presupposing here that the glass manifests its potentiality to break as soon as our striker manifests his respective potentiality.

4.4.2.2.1.4 Extending the model The set $\mathsf{POSS}(m_i)$ spans the entire range of possible immediate future continuations of the moment m_i, and for each set contained within $\mathsf{POSS}(m_i)$, we have added a new future moment above m_i to M_n. What is the case at our new future moments is now determined by the various transitions of the corresponding set and ultimately depends on what is the case at m_i. Let us in the following focus on one transition set in $\mathsf{POSS}(m_i)$ and the future moment it gives rise to and see how the value of the function V_{n+1} at that moment comes about. Assume that the set we are considering is the one corresponding to the new future moment m_{j_l} $(1 \leq l \leq k)$, and we will accordingly denote it by $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle}$. The transitions contained within our set $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle} \in \mathsf{POSS}(m_i)$ each belong to a different set of $\mathsf{PROG}(m_i)$ and hence either each correspond to a possible manifestation of a different potentiality or relate to a different object or both. All of them point locally—both spatially and temporally—toward the future.

First of all, as possible manifestations of objects' potentialities, the transitions in $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle}$ are local in space. Each of them captures the transformation of some local state of affairs obtaining in the circumstance c_i at m_i rather than the transformation of the entire circumstance as a whole. Both the initials and the outcomes of our transitions correspond to local states of affairs: both are mere subsets of sets of literals in $\mathsf{VAL}|_{\mathsf{meta}}$ that fully describe a circumstance. Being local in space, the transitions in $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle}$ need to combine. Only jointly can they determine the future transformation of the entire circumstance c_i, which spans all of space.

Moreover, the transitions in $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle}$ are local in time. Each of them points toward some immediate outcome, some future state of affairs; and its pointing toward a particular outcome is different from the actual unfolding of the corresponding future course of events. A transition's pointing toward some immediate future state of affairs is not to be equated with the obtaining of that future state of affairs at the immediate future moment m_{j_l}. While there are cases in which all outcomes of the transitions that jointly span an immediate future course of events are in fact brought about at the corresponding future moment, this need not generally be the case. The transitions in $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle}$ interact, possibly interfere with each other, and produce a joint outcome that may be different from the mere conjunction of their individual outcomes. Each transition in $\mathsf{poss}_{\langle m_i, m_{j_l} \rangle}$ contributes to what is the case at m_{j_l}, but this does not imply that its outcome will be realized at that later moment. What is in fact the case at the moment m_{j_l} depends on how the various transitions in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ interact. How exactly the possible manifestations of potentialities interact is inherent to the potentialities themselves. Even though the interaction of the possible manifestations of different potentialities is not encoded by the modal profiles of those potentialities themselves, the interaction principles are provided with the potentialities. All potentialities and their possible manifestations form an interrelated network.

Since every transition in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ is local in space and time, it is but their combination and interaction that gives direction to what is the case at the future moment m_{j_l}. By combining and acting in concert, all the transitions in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ together jointly determine the future transformation of the circumstance c_i obtaining at the moment m_i. The value of the function V_{n+1} at the moment m_{j_l} is the result of the combination and interaction of all the transitions in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ applied to the circumstance c_i obtaining at m_i, or the set $V_n(m_i)$, respectively.[153] If we denote the combination and interaction of the various transitions in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ by $\biguplus \mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ and view it as a function from $\{V_n(m_i)\}$ to $\mathsf{VAL}|_{\mathsf{meta}}$,[154] we can define the extension V_{n+1} of the function V_n along the following lines: we set $V_{n+1}|_{M_n} := V_n$ and $V_{n+1}(m_{j_l}) = \biguplus \mathsf{poss}_{\langle m_i, m_{j_l}\rangle}(V_n(m_i))$. This concludes the n-th step of the construction.

One last remark is in order here. In section 4.4.1.2.1, we have commented on the fact that the notion of a transition as a possible manifestation of an object's potentiality differs from the notion of a transition that builds the basis of the transition semantics developed in chapter 2. While the former captures the transformation of local states of affairs that are part of metaphysically possible circumstances, the latter is a purely structural notion that represents the possible future continuations of a moment in a branching time structure. Since the circumstance c_i from which the transitions contained in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ arise is tied up with the moment m_i that is part of a temporal ordering, the local transitions in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ become concrete, and their combination and inter-

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153 The idea that the manifestations of different potentialities combine in order to produce a joint outcome is also to be found in, e.g., Cartwright and Pemberton (2013), Lowe (2010), Molnar (2003) and Mumford and Anjum (2011b).

154 The fact that the codomain of the function $\biguplus \mathsf{poss}_{\langle m_i, m_{j_l}\rangle} : \{V_n(m_i)\} \to \mathsf{VAL}|_{\mathsf{meta}}$ is given by $\mathsf{VAL}|_{\mathsf{meta}}$ rather than by VAL presupposes that the possible manifestations of the objects' potentialities can never shift us outside the realm of what is metaphysically possible. If they could, the fundamental ontology we have started out would have to have been inconsistent from the outset.

action $\biguplus \mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ affects the entire circumstance obtaining at the moment m_i. The sole element of the graph of the function $\biguplus \mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ is a transition $\langle V_n(m_i), V_{n+1}(m_{j_l})\rangle$ from the circumstance obtaining at m_i to the circumstance obtaining at m_{j_l}. On the structural level, the transition $\langle V_n(m_i), V_{n+1}(m_{j_l})\rangle$ corresponds to the transition $\langle m_i, m_{j_l}\rangle$ and thus to the discrete version of the notion of transition that employed in the transition semantics. In particular, in case $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ comprises just a single element, that transition is itself isomorphic to the structural transition $\langle m_i, m_{j_l}\rangle$. Furthermore, if $k = 1$, i.e. if $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ is the only transition set in $\mathsf{POSS}(m_i)$, the structural counterpart $\langle m_i, m_{j_l}\rangle$ of $\langle V_n(m_i), V_{n+1}(m_{j_l})\rangle$ is a trivial transition.

4.4.2.2.1.5 When the construction ends By specifying the base step and the n-th step of the construction, we have provided a complete description of our construction method. Once the base step is set, the construction proceeds by mere iteration of the n-th step. At each step of the construction, we consider some $<_n$-topmost moment m_i in M_n. The circumstance c_i associated with the set $V_n(m_i)$ provides a set $\mathsf{PROG}(m_i)$ of possibly empty transition sets on the basis of the modal profiles of the objects' potentialities in c_i, and by rearranging the various transitions contained within those sets we obtain a possibly empty set $\mathsf{POSS}(m_i)$. For each transition set contained within $\mathsf{POSS}(m_i)$, we add a new future moment above m_i and assign to that future moment the set of literals that is jointly determined by the transitions on the basis of $V_n(m_i)$. The model and its underlying structure are lifted successively by establishing temporal relations between momentary circumstances by means of transitions that represent the possible manifestations of the potentialities of objects.

The construction proceeds as long as there is some topmost moment m_i such that $\mathsf{PROG}(m_i)$ contains at least one non-empty transition set. There are cases in which for a given moment m, the set $\mathsf{PROG}(m)$ fails to contain a non-empty set of transitions. First, if the moment m is associated with a circumstance in which there exists not a single object that has a potentiality, the set $\mathsf{PROG}(m)$ is the empty set. Second, if the moment m is associated with a circumstance that is in fact endowed with powerful objects but does not license the manifestation of their respective potentialities, each transition set contained within $\mathsf{PROG}(m)$ is empty, even though the set $\mathsf{PROG}(m)$ itself it not. In both cases, the set $\mathsf{POSS}(m)$, which is determined on the basis of $\mathsf{PROG}(m)$, equals the empty set. Given that $\mathsf{POSS}(m)$ does not contain a single transition set, no new future moments emerge above m, and the corresponding branch comes to an end. Without objects manifesting their potentialities, there is no change, no pull toward the future; and without change there is no time.[155] A moment at which no potentiality is manifested is a last moment, a final point in the temporal ordering. At such a moment, time ends. Now, if we arrive at a step of the construction at which there is no topmost node m_i for which $\mathsf{PROG}(m_i)$ contains at least one non-empty transition set, the construction stops. It can only be continued as long as at some topmost moment objects can manifest their potentialities. It should be noted that if at a given moment, some particular object cannot manifest its potentiality, this does not imply that it will never ever be able to again manifest its potentiality. As long as other objects can still manifest their respective potentialities, there is the possibility that the circumstances change in such a way that the manifestation of the object's potentiality will again be licensed at a later moment. This possibility disappears, however, in case no single object can manifest its potentiality anymore, and then the branch—and with it time—stops.

4.4.2.2.1.6 The result By carrying out the construction method, we obtain a branching time model for real possibility. The model and its underlying structure are developed hand in hand. At each step of the construction, we are provided with a non-empty set of moments M that is endowed with a temporal order $<$ and a function $V : M \to \mathsf{VAL}|_{\mathsf{meta}}$.

The temporal order $<$ is first of all irreflexive, asymmetric, left-linear and jointed. The relation $<$ has those properties owing to the transitions from which it is lifted, which are future-directed and local in time. By definition, the relation $<$ is also transitive because in every step of the construction, we have taken the transitive closure of the relation that emerges from the transitions. The set of moments M together with its temporal order $<$ thus constitutes a branching time structure $\mathcal{M} = \langle M, < \rangle$—albeit a possibly finite one, as there may be last moments. Since we construed the manifestation of objects' potentialities as state-based transitions, the temporal ordering is moreover discrete.

The function $V : M \to \mathsf{VAL}|_{\mathsf{meta}}$ assigns to each moment $m \in M$ a maximal consistent set of literals $\Gamma \in \mathsf{VAL}|_{\mathsf{meta}}$. It thereby specifies for each atomic sentence of our language whether that atomic sentence is true or false at m. Equipped with the function V, the branching time structure $\mathcal{M} = \langle M, < \rangle$ constitutes a branching time model $\langle M, <, V \rangle$. The branching time model is a

[155] The idea that there is no time without change can already be found in Aristotle's *Physics* (Aristotle 1996, book IV, 11), and it has been made famous by Leibniz (1956).

locally metaphysically possible one, according to Def. 4.4: each moment is associated with some metaphysically possible circumstance. In section 4.2.2.3.3, we have seen that to each such model, there corresponds a moment-relative branching time model with a valuation function $v : M \times \mathsf{At} \rightarrow \{0, 1\}$ (see Def. 4.1) and likewise a first-order branching time model (see Def. 4.3), in which the constituents of the atomic sentences are assigned, at each moment, appropriate extensions in set-theoretic terms on the basis of the domain of objects of our fundamental ontology.

The branching time models that are generated by following the construction method are branching time models for real possibility. Every possible course of events that is depicted in such a model conforms to the prevailing laws of nature in so far as it arises from the potentialities of objects. The laws of nature are coded by the potentialities, their modal profiles and the interaction of their possible manifestations.[156] The construction proceeds in such a way that everything that is the case at some moment within some possible future course of events can in fact temporally evolve from the initial circumstance on the basis of the objects' potentialities and the interaction of their possible manifestations. Branching is accounted for in terms of indeterministic potentialities. They elucidate why histories branch at a certain moment in a certain way. There is a systematic link between what is the case at a moment and the local branching structure, and the fit between the model and its underlying structure is guaranteed. Our construction method provides a metaphysical explanation of branching time models for real possibility that respects the local kind of indeterminism that lies at the heart of branching time structures and yields a criterion that allows us to demarcate branching time models that are models for real possibility from those that are not. On the basis of the construction method, we can delimit the overall range of locally metaphysically possible branching time models to models for real possibility.

In the base step, we can start out with any metaphysically possible circumstance whatsoever, and for every metaphysically possible circumstance we obtain a branching time model for real possibility. The branching time model that is constructed from a given metaphysically possible circumstance by following the construction method is always uniquely determined. Given some initial circumstance, the potentialities of objects fix once and for all the range

[156]The idea that the laws of nature are basically nothing else than the potentialities when abstracted from the concrete circumstances, is also to be found in Anscombe (1981b), see especially Anscombe (1981a) in that collection. The idea is also present in Lowe (2009).

of real possibilities. What is really possible is not random but enabled and at the same time constrained by the objects' potentialities. The result is a limited kind of indeterminism.[157] Each metaphysically possible circumstance gives rise to exactly one branching time model for real possibility. The order in which the construction proceeds, on the other hand, is not fixed. At each step of the construction, any of the topmost moments can equally serve as a basis for the next step. The very same model can be constructed in several different ways.

Our construction method does not reflect the flow of time. The flow of time is rather a building block of the branching time model that is to be constructed. It enters locally and dynamically at a moment via the possible manifestations of the objects' potentialities, which point beyond that moment and pull toward the future, and merges into the eternal, static temporal ordering of moments once the branching time model is in place. To put it in McTaggart's terms of *A*-theory and *B*-theory: we construct a *B*-theoretic model by *A*-theoretic means.[158]

4.4.2.2.2 Branching time models for real possibility with process-based transitions We have so far focused on how a branching time model for real possibility can be lifted from a single circumstance on the basis of the potentialities of objects when the possible manifestations of those potentialities are modeled as state-based transitions. More often than not, however, the outcome of the possible manifestation of an object's potentiality is a process rather than a state of affairs. In this section, we will discuss how a shift from state-based to process-based transitions affects the construction of branching time models. That is, we will consider how the construction proceeds if the possible manifestations of the objects' potentialities are viewed as pairs whose initial is a state of affairs and whose outcome is a process.

4.4.2.2.2.1 World and language The basic setup is the same as before: we start out with some arbitrary fundamental ontology and a formalized first-order language that perfectly matches our ontology. The modal profiles of potentialities again provide for each object and each circumstance in which an object possesses a potentiality, a possibly empty set of transitions. However, while we have so far assumed that the set of transitions that is associated with an object's potentiality in a circumstance is a set of state-based transitions in each case, we will now assume that the respective elements of those sets are

[157] The term 'limited indeterminism' was coined by Prior (1962).

[158] For McTaggart's categories of *A*-theory and *B*-theory, see McTaggart (1908).

process-based transitions. We will focus on transitions whose outcomes are discrete processes, viz. sequences of consecutive states of affairs. Rather than modeling the possible manifestation of an object's potentiality in a circumstance as a pair consisting of two sets of literals that represent two consecutive states of affairs, we represent a possible manifestation as a pair whose initial is a set of literals and whose outcome is a possibly infinite sequence of sets of literals. Each set of literals captures a local state of affairs, and a sequence of such sets describes a local process. As in the case of state-based transitions, the initial of a process-based transition captures a local state of affairs obtaining in the respective circumstance at hand; and the first element of the sequence figuring as outcome is considered an immediate successor of the initial.

4.4.2.2.2.2 Getting the construction started The base step of our construction remains unaltered. We start out with some arbitrary metaphysically possible circumstance c_0 and some arbitrary moment m_0. We set $M_0 := \{m_0\}$ and $<_0 := \emptyset$, and we define a function $V_0 : M_0 \to \mathsf{VAL}|_{\mathsf{meta}}$ that assigns to m_0 the set of literals $\Gamma_{c_0} \in \mathsf{VAL}|_{\mathsf{meta}}$ corresponding to c_0.

In the n-th step of the construction, we are given a set of moments M_n, a temporal order $<_n$ on M_n and a function $V_n : M_n \to \mathsf{VAL}|_{\mathsf{meta}}$. As before, we consider some $<_n$-topmost moment m_i in M_n, and—under the assumption that the circumstance c_i obtaining at m_i comprises powerful objects—we obtain a set $\mathsf{PROG}(m_i)$, which in turn determines a set $\mathsf{POSS}(m_i)$. The elements of $\mathsf{POSS}(m_i)$ are sets of transitions, and each such set specifies one possible future continuation of m_i.

Given that the outcomes of our transitions are now entire sequences of states of affairs, the question arises how the extensions M_{n+1}, $<_{n+1}$ and V_{n+1} of V_n are to be defined. We can either add, for each set in $\mathsf{POSS}(m_i)$, an entire sequence of moments above m_i and define the values of the function V_{n+1} at those moments in terms of the combination and interaction of the respective transitions of that set over time, or we can add just a single moment in each case and define V_{n+1} on those moments only. The fact that transitions are local in time suggests that we proceed stepwise and just add a single future moment for each set in $\mathsf{POSS}(m_i)$. Since the outcomes of our transitions are now processes, the transitions cannot only interfere with the transitions contained within the same set from $\mathsf{POSS}(m_i)$ but also with the transitions arising at a later moment.

4.4.2.2.2.3 Another toy example In order to get a clear grasp on what is at stake here, let us consider a simple example. Assume that at the moment m the following circumstance obtains: there are three balls arranged on a chessboard, all of which are at rest. Ball_1 is located on the square A1 of the chessboard, ball_2 on the square G4 and ball_3 on the square D3. Now, consider the following possible future continuation of the circumstance obtaining at m: imagine that at the moment m, ball_1 is pushed so that it manifests its deterministic potentiality to roll. The ball's rolling is a process, and we can represent the manifestation of the balls potentiality to roll as a process-based transition. Assume that the ball is pushed in such a way that the transition corresponding to the manifestation of its potentiality in the circumstance at hand is given by $\langle \mathsf{A1}(\mathsf{ball}_1), [\mathsf{B2}(\mathsf{ball}_1), \mathsf{C3}(\mathsf{ball}_1), \mathsf{D4}(\mathsf{ball}_1), \mathsf{E5}(\mathsf{ball}_1), \ldots] \rangle$. Assume moreover that as part of the very same future continuation also ball_2 is pushed at m and consequently manifests its potentiality to roll, the corresponding transition being $\langle \mathsf{G4}(\mathsf{ball}_2), [\mathsf{F4}(\mathsf{ball}_2), \mathsf{E4}(\mathsf{ball}_2), \mathsf{D4}(\mathsf{ball}_2), \mathsf{C4}(\mathsf{ball}_2), \ldots] \rangle$.[159] The circumstance obtaining at the moment m is depicted in Fig. 4.7.

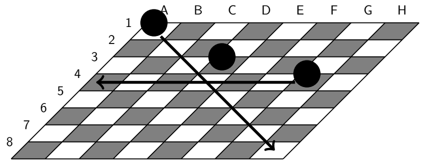

Figure 4.7: Balls at the moment m.

Given the trajectories of our balls, as things stand at m, ball_1 and ball_2 will collide three moments later on the square D4. Their transitions interfere with each other. Here it becomes important again that the manifestations of potentialities as well as their licensing are local in time: both transitions can be exercised at the moment m even though their outcomes are incompatible. The transitions interact and produce a joint outcome that is different from the mere conjunction of their individual outcomes. We could now add an entire sequence of future moments $m' < m'' < m''' < m'''' < \ldots$ above m and specify for each

[159] Note that the transition set that specifies our possible future continuation of m will contain further transitions next to the given ones, which we will neglect here for the sake of simplicity. In addition to the transitions capturing the movements of the balls, the set must also contain transitions that represent the manifestations of the potentiality in virtue of which the balls are pushed.

such moment what is the case at that moment given how the transitions that span our possible future continuation of m interact over time. However, instead of doing so, we just add a single future moment $m' > m$ only. Why we choose that option over the alternative will become clear in a second.

In the circumstance obtaining at our new future moment m', ball_1 is located on the square B4, and ball_2 is located on the square F4, and each of them pulls in a certain direction. Assume that at the moment m', ball_3, which so far has been at rest, is eventually pushed as well, manifests its potentiality to roll and thereby gives rise to the transition $\langle \mathsf{D3}(\mathsf{ball}_3), [\mathsf{C3}(\mathsf{ball}_3), \mathsf{B3}(\mathsf{ball}_3), \ldots] \rangle$. The positions of the balls and their respective trajectories at the moment m' are provided in Fig. 4.8 As things stand at the moment m', ball_1 and ball_3 will collide at the next moment on the square C3. Also here it is important to note that due to the local nature of manifestations, ball_3 can in fact manifest its potentiality to roll at m' although ball_1 is approaching and their trajectories intersect. As a result of ball_3 manifesting its potentiality to roll, ball_1 is deflected on the square C3 and hence will never make it to the square D4, where it would have collided with ball_2, had ball_3 not intervened.

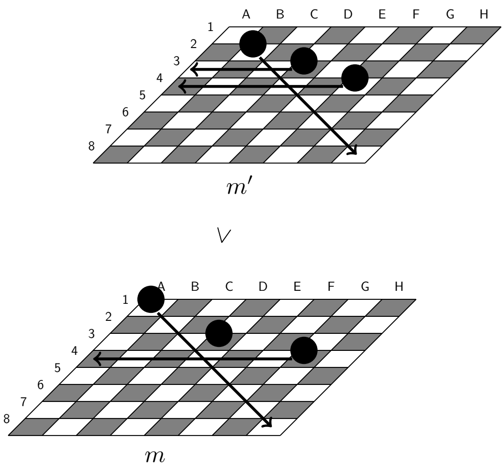

Figure 4.8: Balls at m'.

4.4.2.2.2.4 Extending structure and model The example illustrates that since process-based transitions can be intervened on by transitions arising at later moments, we cannot add entire sequences of future moments at once but have to add new future moments step by step. For, what will in fact be the case at remote future moments might be determined only later on. The circumstances we would have associated with the moments succeeding m' on the basis of the transitions provided at m are not the circumstances that need to be associated with those moments given the transitions emerging at m'. For each set in $\mathsf{POSS}(m_i)$ we thus add but a single future moment, as we did in the case of state-based transitions. Given that the cardinality of $\mathsf{POSS}(m_i)$ equals k, we add k future moments above m_i. We set $M_{n+1} := M_n \cup \{m_{j_1}, \ldots, m_{j_k}\}$ and let $<_{n+1}$ be the transitive closure of $R_{n+1} = <_n \cup \{\langle m_i, m_{j_1}\rangle, \ldots, \langle m_i, m_{j_k}\rangle\}$, i.e. $<_{n+1} := R^*_{n+1}$.

The example also illustrates that what is the case at some new future moment m_{j_l} $(1 \leq l \leq k)$ is not solely determined by the combination and interaction of the transitions contained within the transition set $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle} \in \mathsf{POSS}(m_i)$, which spans the possible future continuation containing that moment. In addition to those transitions, the transitions that have emerged at earlier moments and contribute to the same future continuation have to be taken into account as well. For determining what is the case in the possible future continuation of the moment m' that we were considering in our example, the transition arising at that moment is not enough. That transition only captures the movement of ball_3 that would come about, were ball_1 not approaching. What in fact is the case, viz. ball_1 and ball_3 colliding, depends also on the transitions arising at the earlier moment m and figuring into the same course of events. The value of the function V_{n+1} at the future moment m_{j_l} is then the result of applying the combination of all the transitions in $\mathsf{poss}_{\langle m_i, m_{j_l}\rangle}$ and any set $\mathsf{poss}_{\langle m'_i, m'_{j_l}\rangle}$ for $m'_{j_l} \leq m_i$ to $V_n(m_i)$, i.e. $V_{n+1}(m_{j_l}) := \biguplus(\mathsf{poss}_{\langle m_i, m_{j_l}\rangle} \cup \bigcup_{m'_{j_l} \leq m_i} \mathsf{poss}_{\langle m'_i, m'_{j_l}\rangle})(V_n(m_i))$.

4.4.2.2.2.5 Time and continuity Our investigations in this section were concerned exclusively with process-based transitions whose outcomes constitute discrete processes rather than continuous ones. The temporal ordering of moments underlying the branching time model that is generated by means of those transitions is a discrete order, just as in the case of state-based transitions.

It is worthwhile to note that in our construction, we have refrained from identifying the emerging moments with times. It is plausible to assume that the transitions that represent the possible manifestations of objects' potentialities themselves imply such an identification. Whether it takes a frog one or two seconds to jump one square is an inherent feature of the frog's potentiality to jump. Likewise, whether it takes a ball one or two seconds to cover a distance of one square when pushed in a certain way is an inherent feature of the ball's potentiality to roll.

Under the assumption that the identification of moments with times is provided by the possible manifestations of the objects' potentialities themselves, we can include a function *times* into our construction that assigns to each moment an appropriate clock time (cf. Def. 1.6). Just as the function V, the function *times* would need to be extended in each step of the construction as the set of moments grows. Given that the sequel of times is inherent to the transitions themselves, the processes figuring as outcomes of process-based transitions can then be construed as functions in time that capture the temporal evolution of the initial. If the processes involved are supposed to be continuous, process-based transitions cannot even be modeled otherwise. Technically, the continuous case is further complicated by the fact that the construction cannot proceed step by step: there is no immediate next future moment that could build the basis for the next step. Processes would have to emerge and merge continuously.

4.4.3 Real possibilities, potentialities and their interrelation

In this chapter, we set out to provide a metaphysical explanation of branching time models for real possibility. The explanation we provided is a dynamic, modal one. We showed how branching time models for real possibility can be lifted from a single momentary, metaphysically possible circumstance in a dynamic fashion on the basis of the potentialities of objects and their manifestations. We start out with some initial circumstance, and in each step of the construction, we lift the possible immediate future continuations of some topmost moment from the possible manifestations of the objects' potentialities in the corresponding circumstance. In this section, we will address the question how real possibilities and potentialities relate, both ontologically and with respect to the kind of modality they involve, and in that context, we will also discuss the differences between our account and extant accounts of dispositions.

4.4.3.1 Real possibilities grounded in potentialities

Real possibilities are grounded in potentialities.[160] The alternative future possibilities arising at a moment are immediately—although not completely—grounded in the possible manifestations of the objects' potentialities in the circumstance obtaining at that moment. The complete ground also comprises the concrete circumstance at hand, and in case manifestations are process-based transitions, the possible manifestations of the objects' potentialities at earlier moments become relevant as well. Ultimately, the entire range of real possibilities depicted in a branching time model is grounded in the initial circumstance and the potentialities of the objects existing in that circumstance. What is really possible ultimately depends on the circumstance we start out with and the possible manifestations of the objects' potentialities in that circumstance.

Real possibilities and potentialities are not on a par. Real possibilities ontologically depend on potentialities. They owe their existence to potentialities. What is really possible is so in virtue of the objects's potentialities. The ontological status of real possibilities derives from the ontological status of potentialities. Real possibilities exist because potentialities exist, and unlike potentialities, which have real existence in the initial circumstance from which the model is lifted, real possibilities only exist as possibilities. The branching time models for real possibility we construct are not models of the world: the world does not branch. What our models depict is the modal-temporal structure of the world, the range of real possibilities, from the standpoint of some initial circumstance.

Both potentialities and real possibilities are modal notions. Yet, the kind of modality is a different one in each case. Every potentiality comes along with some unique modal profile that broadly ranges over all metaphysically possible circumstances and captures the possible progressings of an object in virtue of its potentiality in all those possible circumstances. The possible progressings of an object in a metaphysically possible circumstance specifies the object's local *de re* possibilities in virtue of its potentiality in that particular circumstance, its possible manifestations or non-manifestation, its pulling toward the future in one way or another or its not pulling towards the future at all. The range

[160] The term 'grounding' refers to some kind of metaphysical explanation. An illuminating introduction to the notion of metaphysical grounding as well as recent contributions to the debate are included in Correia and Schnieder (2012). Grounding is commonly understood as a dependency relation between facts or propositions. If a proposition or fact grounds another, then the latter is often said to hold in virtue of the former.

of possible progressings of an object in virtue of its potentiality depends on a metaphysically possible circumstance but they are time-independent. The modal profile of a potentiality itself is constant across all metaphysically possible circumstances, and it is an entirely atemporal notion. Real possibilities, on the other hand, are tied to concrete circumstances, and they are concretely anchored in time. They are future possibilities that arise from a concrete circumstance that is anchored in the temporal order on the basis of the combination and interaction of the objects' possible progressings in virtue of their potentialities in that circumstance.

4.4.3.2 Linking back to extant accounts of dispositions

Advancing the view that possibility can be accounted for in terms of potentialities, we take sides with dispositionalist accounts of modality. However, while dispositionalists about modality typically aim at providing an account of the absolute and atemporal notion of metaphysical possibility, the kind of possibility we seek to ground is real possibility. Real possibility and metaphysical possibility crucially differ in one important aspect: real possibility is time-dependent, and what is really possible at a certain point in time furthermore depends on the concrete situation. Since real possibility is a situated kind of possibility, real possibilities cannot simply be lifted from the potentialities that are instantiated, at a particular point in time, without taking into consideration the concrete situation.[161]

In order to make room for real possibilities, we have to both relativize the relevant potentialities to a momentary circumstance and take into account how the respective circumstance bears on the manifestations of those potentialities. That in some metaphysically possible circumstance, objects can bring about a certain outcome in virtue of their potentialities is not sufficient for the outcome to be really possible. For determining what is really possible it does not matter how objects could or would behave in virtue of their potentialities in some other metaphysically possible circumstance, but how they can possibly behave in the concrete circumstance at hand. Consider a fragile glass. It might be said to be metaphysically possible that the glass breaks because it is fragile and thus has the potentiality to break. Yet, the fragility of the glass by itself does

[161] A disposition-based notion of possibility that resembles our notion of real possibility is the tensed notion of 'strong possibility' mentioned in Vetter (2015). Roughly, something is strongly possible at a certain time if and only if at that time all objects together have the potentiality to bring it about.

not entail that the breaking of the glass constitutes a real possibility. If the glass is situated in a circumstance in which it is untouched or safely packed in styrofoam, it cannot manifest its potentiality to break and hence the real possibility cannot arise. The concrete circumstance matters.[162]

Just as the dispositional essentialists, we assume that potentialities have dispositional essences. The dispositional essence of a potentiality consists in its modal profile, which is constant across all metaphysically possible circumstances. We do not assume, however, that the modal profile of a potentiality can be captured by a single counterfactual conditional endowed with a *ceteris paribus clause*; nor can it be fully captured by a list of counterfactual conditionals. Potentialities are equipped with rather complex and fine-grained modal profiles, which is not to say, however, that generalizations are not possible at all. We have abandoned the idea that every potentiality can be associated with certain characteristic stimulus conditions. The modal profiles of potentialities are taken to capture the objects' possible progressings in virtue of their potentialities, their possible manifestations as well as their non-manifestation, in all metaphysically possible circumstances in which the objects have the potentialities. More importantly, we have changed the notion of the manifestation of a potentiality. We have uncoupled the manifestation of a potentiality from what in fact is the case, i.e. from how things in fact turn out, if the potentiality is manifested. The manifestation of a potentiality is conceived of as a local pointing toward a future outcome. Whenever an object can manifest its potentiality, the manifestation comes about, but this is not to say that the outcome of the manifestation occurs. What in fact is the case depends on the manifestations of other potentialities as well. Manifestations can be intervened on.

In section 4.4.1.1, we have seen that the counterfactual analysis is faced by the problem of masks, and we have discussed the two main strategies that have been suggested in order to overcome that difficulty. One strategy consists in refining the stimulus conditions, while the other suggests a refinement of the counterfactual conditional by a *ceteris paribus* clause or something similar. By conceiving of the manifestation of a potentiality as a local pointing toward the future, we gain a more differentiated view on masking cases. We can differentiate between external factors that prevent the manifestation of an object's potentiality from the outset and external factors that prevent the outcome of

[162]The notion of possibility underlying the nomological machines discussed in Cartwright and Pemberton (2013) is a situated kind of possibility, just as real possibility. In the case of nomological machines, the initial configuration matters as well. See also Pemberton (2016).

the manifestation of an object's potentiality as they interfere with the manifestation in the course of time. A fragile glass that is safely packed in styrofoam cannot manifest its potentiality to break, even if it is struck. Poison, on the other hand, can and does manifest its potentiality to lead to death when ingested, even in cases in which an antidote is taken as well. The antidote does not prevent the poison from manifesting its potentiality but rather interferes with its manifestation that points towards death and prevents the supposed outcome. There seems to be a fundamental difference between the packing material and the antidote: the former bears on the licensing of the manifestation, and hence places restrictions on the stimulus conditions if you want, while the latter places restrictions on the background conditions that need to be preserved in the course of time.

Like the tendency account of dispositions, our account gives way to interventions. On our account, however, interventions do not figure on the level of potentialities themselves. They do not concern the possible manifestations that are associated with an object's potentiality in a circumstance due to the potentiality's modal profile. Since manifestations are local in time, there is no room for interventions on that level. Interventions rather affect the outcomes of manifestations. Manifestations may be intervened on by the manifestations of other potentialities. In the course of time, the manifestations of different potentialities interact and produce a joint outcome that may be different from the mere conjunction of the individual outcomes. The manifestation of a potentiality is its pointing towards the future and hence its contribution to the overall outcome, rather than the overall outcome itself. Potentialities do not tend toward their manifestations; rather, the manifestations tend toward their outcomes. This is not to say, however, that the occurrence of the outcome of the manifestation of an object's potentiality is always really possible and never really necessary. What is really possible or necessary in a circumstance depends on the presence and the possible manifestations of other potentialities. While interventions might always constitute a metaphysical possibility, interventions are not always really possible. If in a given circumstance, no antidote is available—either because it is at an end, or because it is too far away to get it on time, or because it is not yet discovered—, the ingested poison will necessarily lead to death. Interventions are the result of the interaction of the manifestations of different potentialities in a concrete circumstance: interventions are real possibilities.

4.5 Concluding remarks

In this chapter, we made a transition from logical to metaphysical considerations and focused on the worldly aspect of the notion of real possibility, aiming at incorporating that aspect into our semantic investigations. Real possibilities are future possibilities in an indeterministic world, and their intimate relation to reality places restrictions on the admissible models of the language. Not every possible branching time model is a model for real possibility. There must be a pertinent link between the interpretation of the language, the branching time structure and the world. Every history must represent a course of events that is possible against the background of the prevailing laws of nature. In particular, the model must fit the underlying branching time structure. Our endeavor was the search for a metaphysical explanation that allows us to systematically single out those branching time models that are in fact models for real possibility.

In a first step, we tied the language to the world and thereby restricted the range of branching time models that are possible in a broadly logical sense to those in which the interpretation of the atomic sentences at each single moment represents a metaphysically possible circumstance and hence bears some relation to the world. The key challenge then consisted in establishing the missing nomological link between the interpretation of the atomic sentences at successive moments.

We argued that starting out with a given branching time structure and restricting the possible models on that structure to models for real possibility by means of certain additional metaphysical constraints is not a viable option. The model and its underlying structure need to be developed hand in hand. Our side-trip into a neo-Humean possible worlds universe revealed the differences between branching and divergence with respect to the kind of indeterminism that is modeled in each case. The side trip made clear that in order to preserve the local kind of indeterminism that lies at the heart of branching time models for real possibility, we need to develop the model and its structure in parallel by establishing temporal relations between momentary circumstances in a dynamic fashion rather than by establishing modal relations between entire static worlds. Real possibilities are temporal alternatives for a dynamic actuality to evolve rather than mere modal alternatives to a static actuality.

We finally set out to develop a dynamic, modal explanation of branching time models for real possibility in terms of potentialities, construed as dynamic properties. Our account of potentialities thereby differs from extant accounts in several respects. We discarded the idea that every potentiality is to be associated with characteristic stimulus conditions and attributed to potentialities a rather complex and fine-grained modal profile that captures the possible manifestations and non-manifestation of an object in virtue of its potentiality in all possible circumstances. The manifestations of potentialities are uncoupled from what is the case if the potentiality is manifested and are instead modeled as transitions that point toward the future. They are local in both space and time and can be interfered with by the manifestations of other potentialities.

We showed how branching time models for real possibility can be successively lifted from a single circumstance on the basis of the potentialities of objects and their manifestations. At every moment, the possible manifestations of all powerful objects combine and jointly give direction to the alternative possible future courses of events. Real possibilities emerge from the interaction of the possible manifestations of different potentialities in a dynamic fashion, and they are ultimately grounded in the potentialities of objects existing in some initial circumstance. The result is a limited kind of indeterminism. The resulting branching time model is indeterministic in that it contains branching points that are accounted for by the possible manifestations of indeterministic potentialities. The potentialities, at the same time, enable and constrain what is really possible. They allow us to delimit the overall range of locally metaphysically possible branching time models to branching time models for real possibility.

Both in the logical context of the search for appropriate parameters of truth and in the metaphysical context of the search for appropriate models, the notion of a transition plays a crucial role. The notion of a transition that is employed as a parameter of truth in the transition semantics differs, however, from the notion of a transition that represents a possible manifestation of an object's potentiality. The former are structural elements that represent a possible future continuation of an entire moment in a given branching time structure, while the latter are local in space and filled with content. Nevertheless, the transition semantics harmonizes with our metaphysical explanation of branching time models as it allows us to capture real possibilities locally at the branching points where they arise from the possible manifestations of the objects' potentialities.

* * *

Conclusion

In this thesis, we set out to develop a semantics for the notion of real possibility. Real possibilities are alternative possibilities for the future in an indeterministic world. They are anchored in concrete situations, and they bear an intimate relation to time and to the world. Our aim was to establish a rigorous formal conceptual framework for reasoning about real possibilities, tackling challenges of both a logical and a metaphysical nature. On the logical side, we aimed at devising a formal semantics that adequately captures the temporal dynamics of real possibilities, the interrelation of modality and time. On the metaphysical side, our endeavor was to provide a systematic explanation of semantic models that appropriately accommodates the worldly aspect of real possibilities.

In chapter 1, we introduced the notion of real possibility by locating real possibilities in the vast landscape of possibilities. In our discussion of the varieties of possibility, we have mainly focused on two aspects, namely on the relation that different kinds of possibility bear to the notion of actuality, on the one hand, and to the notion of reality, on the other. We pointed out that real possibilities are alethic rather than epistemic possibilities. They are possibilities in virtue of some objective reality, some real aspect of the world. What distinguishes real possibilities from the standard notions of alethic possibilities—to wit, logical, metaphysical and physical possibility—is that real possibilities are temporal alternatives for a dynamic actuality to evolve rather than modal alternatives to a given actuality.

We argued that the theory of branching time with its tree-like temporal structures adequately captures the peculiar temporal aspect of real possibilities while the possible worlds framework fails in that respect. Combining possible worlds with times is not enough in order to capture the idea that the future is open in a genuine sense. In the possible worlds framework, future possibilities boil down to mere modal alternatives to an overarching actuality. The possible worlds framework certainly is a powerful tool for modeling various different kinds of possibility but it is of no avail for the specific notion of real possibility that we aim to model.

Having established the adequacy of the branching time framework for the formal representation of real possibilities, we turned to the question as to which parameters of truth are suited in order to adequately capture the interrelation of modality and time that is at the heart of the notion of real possibility. Our search for appropriate parameters of truth culminated in the transition semantics presented in chapter 2. Whereas in the standard Ockhamist semantics, sentences are evaluated at a moment with respect to a history, which spans an entire possible temporal development, in our transition semantics, the semantic evaluation at a moment is relativized to a consistent, downward closed set of indeterministic transitions. The static Ockhamist history parameter is replaced by a dynamic transition parameter. Every transition selects one of the immediate future possibilities open at a branching point, and sets of transitions perfectly mirror the forking paths in a branching time structure. As a possibly non-maximal chain of transitions that is complete with respect to the past, a consistent, downward closed set of transitions can in particular represent a complete or incomplete possible course of events. Next to temporal and modal operators, our transition language is equipped with a stability operator that is interpreted as a universal quantifier over the possible future extensions of a given transition set and brings in a dynamic element. The stability operator enables us to capture the behavior of the truth value of a sentence about the future in the course of time—its changing from contingent to stably-true or stably-false—and provides us with local witnesses for future possibilities.

Building on sets of transitions, the transition semantics exploits the structural resources a branching time structure has to offer and provides a fine-grained, dynamic picture of the interrelation of modality and time. It adequately reflects the dynamics of real possibilities. What is more, the semantics developed along those lines allows for great generality and comes with expressive means that are not available on extant accounts. By invoking the general notion of a transition structure, we provided correspondence results that expose both Peirceanism and Ockhamism as limiting cases—or rather, limitations—of the transition approach. Peirceanism is obtained by restricting the semantic evaluation to the empty transition set, while a restriction to maximal consistent transition sets yields Ockhamism. Both Peirceanism and Ockhamism fall short of the transition semantics in terms of expressive strength. On both accounts, stability collapses into truth.

In chapter 3, we took a more detailed look at the indices of evaluation employed in the transition semantics and made a first step toward completeness by providing a general definition of the notion of an index structure. As a preliminary result, we showed that the transition sets that figure as a second parameter of truth in our semantics are set-theoretically less complex than they seem at first glance. Sets of transitions correspond one-to-to to certain substructures of branching time structures, which are first-order definable, viz. to the so-called prunings. In a nutshell, a pruning is a substructure that is obtained from a given branching time structure by cutting out certain branches, namely all those branches that constitute counterfactual possibilities given the course of events represented by the corresponding transition set.

With the notion of a pruning at hand, we set out to study the interrelations among the indices of evaluation in a given transition structure, and we derived the notion of an index structure from there, by generalizing our findings. Index structures are genuine Kripke structures in which quantification over transition sets dissolves into quantification over the base elements of the structure, and validity with respect to the class of index structures coincides with validity with respect to the class of transition structures. Our definition of an index structure builds on the notion of a pruning, and just as prunings themselves, index structures are first-order definable. On the surface, an index structure is nothing but a tree-like—albeit not necessarily connected—arrangement of prunings. A tree of trees, so to say. The definition of an index structure constitutes a significant step toward a future completeness proof for the transition framework, which will have to proceed by a chronicle construction on index structures. However, index structures are also interesting from a metaphysical point of view. They bring to the fore the fine-grained, dynamic picture of the interrelation of modality and time invoked by the transition semantics. In the notion of an index structure, the idea that time passes shines through.

In chapter 4, we made a transition from logical to metaphysical considerations and took up the challenge to provide a systematic explanation of branching time models that accommodates the worldly aspect of real possibilities. The challenge consisted in narrowing down the space of all possible branching time models to models for real possibility, viz. to branching time models in which each possible course of events is in accordance with the prevailing laws of nature. The crux was to guarantee the fit between the model and its underlying structure: at each single moment, there has to be a match between the arity of the local branching structure and the concrete momentary circumstance at hand. A pertinent link between the interpretation of the language, the branching time structure and the world had to be established.

In a first step, we tied the language to the world, and thereby limited the overall range of possible branching time models to models in which each moment is associated with a metaphysical rather than with a logical circumstance. The challenge that remained was to establish the missing link between successive moments. We argued that, contrary to what is familiar to us from logical practice, starting out with a given branching time structure and restricting the admissible models on that structure by means of certain additional metaphysical constraints is not a viable option. Rather, the model and its underlying structure need to be developed hand in hand. Moreover, we illustrated that in accordance with the dynamic nature of real possibilities, the model together with its underlying structure must be generated in a dynamic fashion.

We provided a dynamic, modal explanation of branching time models for real possibility in terms of potentialities, and we showed how a model for real possibility can be lifted dynamically from a single momentary circumstance on the basis of the potentialities of objects. What is crucial to our approach is that we uncoupled the manifestation of a potentiality from what turns out to be the case if the potentiality is manifested. Instead, we construed the manifestations of potentialities as transitions that point toward the future. We thereby made room for interventions as well as for omission and forbearance, and we arrived at a perspicuous account of how the potentialities of objects jointly give direction to the possible future course of events. Real possibilities are ultimately grounded in the potentialities of objects and the interaction of their manifestations. The result is a limited kind of indeterminism.

The future is open for further research. There is one notion that takes center stage throughout this thesis, and that is the notion of a transition. The notion of a transition constitutes the basis of our transition semantics, and it plays a pivotal role in our metaphysical account of potentialities. In both cases, transitions capture the local change at a moment and point toward the future. However, whereas in the transition semantics, transitions are conceived of as purely structural elements, on our metaphysical account of potentialities, transitions are filled with content. A desideratum for future research is to bring the two notions of transition together by establishing a suitable link via the language. In our construction of branching time models for real possibility, structural transitions emerge from the combination of the transitions that capture the manifestations of the potentialities of objects. In other words, with each structural transition, there come along one or more transitions that carry propositional content.

Based on the results established here, one might try the following idea. The transition parameter employed in the transition semantics is a hypothetical parameter. It provides conditional information. And since every structural transition is tied up with one or more transitions that capture the manifestations of potentialities, we gain access to that conditional information. By extending the language and devising an appropriate theory of historical conditionals, we can use the information that is carried by the transitions that capture the manifestations of potentialities in order to extend or shift the transition parameter of the transition semantics. For concreteness, assume that I have bet on a coin toss and that I put all my money on heads. Even though before the coin is tossed, it is contingent, with respect to the past course of events, whether I will win or lose, it is true that *if the coin lands heads, I will win.* Now suppose that time has passed, and I in fact won. Given that actual course of events, it is nonetheless true that *if the coin had landed tails, I would have lost.* In the case of the indicative conditional, the antecedent can be viewed as extending the given transition set such that it stretches further into the future while in the case of the counterfactual conditional, the antecedent takes us back to a past moment and shifts the transition set to a counterfactual course of events. In either case, the consequent is evaluated with respect to the transition set provided by the antecedent.

In that context, it is worthwhile to recall that we have uncoupled the manifestation of a potentiality from what turns out to be the case if the potentiality is manifested. Manifestations are not identical to their outcomes, let alone to the joint outcome produced by the manifestations of different potentialities acting in concert. This opens up the possibility to conditionalize on manifestations as dynamic elements, real happenings and doings, rather than on the outcomes of those happenings and doings. In prospect of such a theory of historical conditionals, interesting questions as to the presuppositions of conditionals, their logic and their relation to stit-accounts of agency arise that seem worth pursuing.

* * *

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Samenvatting in het Nederlands

Welkom in een wereld vol mogelijkheden! Het onderzoek in dit proefschrift is gebaseerd op het idee dat de wereld tal van mogelijkheden biedt, werkelijke mogelijkheden voor de toekomst. Op dit moment zit ik hier in Konstanz aan mijn bureau deze zinnen te typen en is het mogelijk dat ik nog een paar uur door ga met schrijven. Maar er zijn alternatieve mogelijkheden. Ik kan een pauze nemen en gaan zwemmen in het Meer van Konstanz, en ik kan net zo goed naar de ijswinkel om de hoek gaan voor een lekker ijsje. De toekomst is open. Er zijn alternatieve mogelijkheden voor hoe de toekomst zich kan ontvouwen. Het plaatje van de modale structuur van de wereld dat ten grondslag ligt aan ons onderzoek is dat van een boom die naar de toekomst vertakt. De boomachtige structuur representeert het idee dat er aan elke tijdstip meerdere mogelijkheden voor de toekomst kunnen zijn terwijl het verleden vast ligt. Als wij naar de toekomst kijken, staan wij tegenover een doolhof met zich vertakkende paden en de actualiteit baant zich een weg door dit doolhof. Bij elke vertakking slaat de actualiteit een van de alternatieve mogelijke paden in, en alle paden zijn in gelijke mate begaanbaar. Op elk tijdstip wordt een van de onmiddellijke alternatieve mogelijkheden geactualiseerd en de rest verdwijnt. De alternatieve paden die naar de toekomst leiden representeren daarbij geen puur epistemische mogelijkheden die alleen maar uit ons gebrek aan kennis voortkomen over wat de toekomst gaat brengen. Centraal in dit proefschrift staat het idee van objectief indeterminisme. Elk van de alternatieve paden die naar de toekomst leiden representeert een werkelijke mogelijkheid voor de toekomst ingevolge van de hoedanigheid van de wereld. Werkelijke mogelijkheden zijn alternatieve mogelijkheden voor de toekomst in een indeterministische wereld.

Het doel van dit proefschrift is in de titel “Transities naar een semantiek voor werkelijke mogelijkheid” al aangeduid. Het gaat erom een semantiek voor de notie van werkelijke mogelijkheid te ontwerpen. Met andere woorden, ons doel is om een modaal-temporele taal te ontwikkelen om over werkelijke mogelijkheden te kunnen redeneren en een passende interpretatie aan deze taal te geven. De theorie die ten grondslag ligt aan ons onderzoek is de zogenaamde theorie van *branching time*. Deze theorie vormt een adequate formele basis voor het modeleren van de notie van werkelijke mogelijkheid. Een branching time structuur is niets anders dan een boom van momenten die naar de toekomst vertakt. Deze boomachtige structuren veroorloven een directe representatie van alternatieve toekomstmogelijkheden met een gezamenlijk verleden. De zoektocht naar een passende semantiek voor de notie van werkelijke mogelijkheid in de theorie van branching time gaat gepaard met zowel logische als ook metafysische kwesties. Aan de logische kant rijst de vraag hoe wij de interactie van mogelijkheid en tijd op structureel niveau adequaat kunnen beschrijven. Aan de metafysische kant bestaat de uitdaging erin een systematische verklaring van de desbetreffende modellen te geven die verheldert waarom de gerepresenteerde mogelijkheden in overeenstemming zijn met de hoedanigheid van de wereld. Zowel bij ons antwoord op de logische kwestie als ook bij ons antwoord op de metafysische kwestie staat de notie van een transitie centraal. Een transitie kan worden beschouwd als een pijltje dat een mogelijke richting wijst bij een vertakking en een lokale verandering aangeeft. Aan de logische kant kunnen wij middels transities precies beschrijven hoe en hoe ver de toekomst moet vorderen voordat een bepaalde mogelijkheid gerealiseerd wordt. Aan de metafysische kant geven wij een verklaring van de desbetreffende modellen in termen van de potentialiteiten van dingen, waarvan wij de manifestaties als transities opvatten.

Hoofdstuk 1. In het eerste hoofdstuk leggen wij de basis voor ons onderzoek. We introduceren de notie van werkelijke mogelijkheid en de theorie van branching time en tonen de adequaatheid aan van het branching time framework voor de formele representatie van werkelijke mogelijkheden. Er is een grote diversiteit aan verschillende noties van mogelijkheden. Wij bespreken de bijzonderheden van werkelijke mogelijkheden en plaatsen de notie van werkelijke mogelijkheid in het ruime landschap van soorten mogelijkheden. In onze discussie van de verschillende noties van mogelijkheden concentreren wij ons vooral op twee aspecten, namelijk op de relatie waarin de verschillende noties van mogelijkheden staan met de noties van realiteit en actualiteit. Werkelijke mogelijkheden zijn geen epistemische maar alethische mogelijkheden. Dat wil zeggen: zij zijn mogelijkheden op grond van een objectieve realiteit, een reëel aspect van de wereld. Anders dan de standaard noties van alethische mogelijkheden—te weten, logische, metafysische en fysieke mogelijkheden—zijn werkelijke mogelijkheden echter geen modale alternatieven tot een gegeven actualiteit maar temporele alternatieven voor een dynamische actualiteit. Het is logisch, metafysisch en fysiek mogelijk dat ik nu in Utrecht ben terwijl ik op dit moment daadwerkelijk in Konstanz aan mijn bureau deze zinnen zit te typen. Maar dit is geen werkelijke mogelijkheid. Wat er werkelijk mogelijk is is dat ik in de volgende trein naar Utrecht stap en dan over enkele uur in Utrecht ben en het is ook werkelijk mogelijk dat ik gewoon in Konstanz blijf zitten schrijven.

Vaak worden mogelijkheden gelijkgesteld met het idee van mogelijke werelden. Wij beargumenteren dat de theorie van branching time—in tegenstelling tot het standaard mogelijke werelden framework—geschikt is om het temporele aspect van werkelijke mogelijkheden en in het bijzonder de specifieke relatie tussen actualiteit en mogelijkheid adequaat te kunnen vatten. Wij bediscussiëren de wezenlijke verschillen tussen het branching time framework en het mogelijke werelden framework en laten zien waarom het mogelijke werelden framework geen mogelijk alternatief is voor het branching time framework wat betreft de representatie van werkelijke mogelijkheden: in het mogelijke werelden framework komen toekomstmogelijkheden neer op modale alternatieven tot een gegeven actualiteit die tijdelijk volledig is uitgebreid en al een unieke toekomst omvat.

Hoofdstuk 2. In het tweede hoofdstuk presenteren wij een nieuwe semantiek voor het branching time framework: de zogenaamde transitie semantiek. Ons uitgangspunt is de vraag hoe wij de interactie van mogelijkheid en tijd die voor de notie van werkelijke mogelijkheid essentieel is op structureel niveau adequaat kunnen beschrijven. Iets technischer uitgedrukt: welke structuurelementen zijn geschikt als semantische parameters om de dynamiek van werkelijke mogelijkheden nauwkeurig te kunnen vatten? Op elk tijdstip kunnen er alternatieve mogelijkheden voor de toekomst zijn en welke van deze mogelijkheden uiteindelijk gerealiseerd wordt is afhankelijk van hoe de toekomst zich ontvouwt. Om de dynamiek van werkelijke mogelijkheden te kunnen beschrijven en een nauw verband te kunnen leggen tussen actualiteit en mogelijkheid is het niet voldoende om waarheid en onwaarheid met betrekking tot een moment te definiëren, zoals in de zogenaamde Peirceaanse semantiek. Veeleer moeten waarheid en onwaarheid op een moment bovendien aan een mogelijke toekomstige gang van zaken gerelateerd zijn. In de zogenaamde Ockhamistische semantiek zijn waarheid en onwaarheid op een moment afhankelijk van een hele geschiedenis, een volledige mogelijke gang van zaken. Vaak is een partiële gang van zaken echter toereikend om de contingentie ten opzichte van wat de toekomst gaat brengen op te lossen. Stel dat je een lootje hebt gekocht voor de loterij van morgen. Het is werkelijk mogelijk dat je wint en ook dat je verliest, en of je wint of verliest staat vast zodra de trekking heeft plaatsgevonden. Wat er later op de dag gebeurt, laat staan over tweeduizend jaar, speelt geen rol.

In onze transitie semantiek zijn waarheid en onwaarheid op een moment gerelateerd aan een verzameling van transities, die een volledige ofwel een partiële gang van zaken kan voorstellen die zich uitstrekt van het verleden en wijst naar een open toekomst. Transities zijn gedefinieerd als structuurelementen die een mogelijke richting wijzen bij een vertakking en in de verzamelingen die wij bekijken zijn de transities allemaal in een rijtje geordend. Naast operatoren voor het verleden, de toekomst en mogelijkheid bevat de taal van de transitie semantiek een operator voor stabiliteit, die geïnterpreteerd wordt als een universele kwantor over de mogelijke toekomstige uitbreidingen van de respectieve verzameling transities. Met behulp van deze operator kunnen wij precies beschrijven hoe en hoe ver de toekomst zich moet ontvouwen voordat een bepaalde mogelijkheid gerealiseerd wordt. In het loterijvoorbeeld kunnen wij door middel van de stabiliteitsoperator aantonen dat zodra de trekking heeft plaatsgevonden, vast ligt of je wint of verliest terwijl dit voor de trekking nog contingent was. De transitie semantiek maakt gebruik van de rijke structurele bronnen die een branching time structuur te bieden heeft en geeft een nauwkeurige representatie van de interactie van mogelijkheid en tijd waarin de dynamiek van werkelijke mogelijkheden adequaat wordt gereflecteerd. We laten zien dat zowel de Peirceaanse semantiek als ook de Ockhamistische semantiek als grensgevallen van de transitie semantiek kunnen worden beschouwd welke enkel van beperkte structurele bronnen gebruik maken. De respectieve beperkingen maken de limitaties de van deze semantieken duidelijk: stabiliteit valt samen met waarheid.

Hoofdstuk 3. In het derde hoofdstuk kijken wij in wat meer detail naar de semantische parameters van de transitie semantiek en maken een eerste stap naar een toekomstig volledigheidsresultaat voor het transitie framework. De transitie semantiek is geen Kripke semantiek in beperkte zin. Waarheid en onwaarheid zijn niet alleen maar gerelateerd aan momenten, welke de fundamentele bouwstenen vormen van branching time structuren, maar er wordt bovendien gebruik gemaakt van verzamelingen van transities. Verzamelingen transities zijn gedefinieerde structuurelementen en de taal van de transitie semantiek bevat operatoren die als kwantoren over verzamelingen van transities worden geïnterpreteerd. Verder lijken de verzamelingen transities die als tweede semantische parameter fungeren verzamelingstheoretisch vooralsnog vrij complex te zijn.

Wij laten zien dat verzamelingen van transities verzamelingstheoretisch niet zo complex zijn als zij bij de eerste aanblik lijken. Verzamelingen transities corresponderen met bepaalde deelstructuren van branching time structuren, welke wij *prunings* noemen. Een pruning van een branching time structuur is een deelstructuur waar sommige takken zijn terug geknipt, namelijk precies die takken die geen werkelijke mogelijkheden meer voorstellen volgens de gang van zaken die door de corresponderende verzameling transities wordt gerepresenteerd. Een pruning bestaat uit tenminste één geschiedenis en als er een vertakking in zit is de deelstructuur compleet bovenaan deze vertakking.

Vervolgens bestuderen wij de interrelaties tussen de semantische parameters in het transitie framework en geven op basis van dit onderzoek een algemene karakterisering van een klasse van Kripke structuren waarvan wij laten zien dat geldigheid behouden blijft. De gedefinieerde Kripke structuren hebben de volgende eigenschappen: waarheid en onwaarheid zijn gerelateerd aan de basiselementen van de structuur, de structuren bevatten telkens één primitieve toegangsrelatie voor de temporele, de modale en de stabiliteitsoperatoren en kwantificatie over verzamelingen transities is vervangen door kwantificatie over de structuurelementen in overeenstemming met de respectieve toegangsrelaties. De definitie van de desbetreffende Kripke structuren is gebaseerd op de notie van een pruning: de gedefinieerde Kripke structuren zijn boomachtige ordeningen van prunings, bomen van bomen bij wijze van spreken. In deze structuren wordt de dynamiek van de interactie van mogelijkheid en tijd die essentiel is voor de notie van werkelijke mogelijkheid zichtbaar. Het resultaat van dit hoofdstuk kan worden beschouwd als een eerste en significante stap naar een toekomstig volledigheidsbewijs voor het transitie framework, die nu via een zogenaamde chronicle constructie op de gedefinieerde structuren kan verlopen.

Hoofdstuk 4. In het vierde hoofdstuk richten wij ons op het wereldlijke aspect van werkelijke mogelijkheden. Zoals eerder opgemerkt zijn werkelijke mogelijkheden mogelijkheden voor de toekomst in een indeterministische wereld en zijn derhalve eng verbonden met de realiteit. Wat er in een situatie werkelijk mogelijk is is wat er kan voortkomen uit deze situatie ingevolge de hoedanigheid van de wereld. In een branching time model voor werkelijke mogelijkheid moet elke mogelijke gang van zaken in overeenstemming zijn met de bestaande natuurwetten. In het bijzonder moet het model passen bij de onderliggende structuur. Als bijvoorbeeld de situatie op een moment zodanig is dat de natuurwetten drie mogelijke toekomstige gangen van zaken toelaten, moeten er in de onderliggende structuur drie takken naar de toekomst zijn. Niet alle mogelijke branching time modellen zijn modellen voor werkelijke mogelijkheid en de uitdaging bestaat nu daarin de ruimte van mogelijke branching time modellen systematisch te beperken tot branching time modellen voor werkelijke mogelijkheid. Deze taak verlangt een transitie van logische naar metafysische overwegingen.

Wij beargumenteren dat het niet mogelijk is om met een gegeven structuur te beginnen en vervolgens de modellen op deze structuur door middel van bepaalde metafysische eisen te beperken tot modellen voor werkelijke mogelijkheid. Veeleer moeten het model en de onderliggende structuur samen worden opgebouwd. Ook tonen wij aan dat het dynamische karakter van werkelijke mogelijkheden een dynamische constructiemethode veronderstelt en stellen wij een dynamische, modale verklaring van de desbetreffende modellen in termen van de potentialiteiten van dingen voor. Wij geven een formele karakterisering van potentialiteiten en hun manifestaties en laten zien hoe een branching time model voor werkelijke mogelijkheid stap voor stap kan worden opgebouwd uit een enkel moment op grond van de potentialiteiten van de dingen.

Potentialiteiten zijn modale eigenschappen van dingen. Een ding kan een potentialiteit hebben zonder deze potentialiteit te manifesteren. Dingen manifesteren hun potentialiteiten alleen maar onder bepaalde omstandigheden. Een glas is breekbaar ook zonder gebroken te zijn en het breekt alleen maar als je het bijvoorbeeld laat vallen of stukslaat. Wij modelleren de manifestaties van potentialiteiten als transities die een lokale verandering aangeven en naar de toekomst wijzen en onderscheiden tussen deterministische en indeterministische potentialiteiten. Indeterministische potentialiteiten zijn potentialiteiten die een ding in dezelfde omstandigheden op verschillende manieren kan manifesteren. Zij zijn oorzakelijk voor het vertakken van de structuur. Door de manifestaties van potentialiteiten als transities op te vatten maken we ruimte voor interventies, preventies en omissies. Slechts gezamenlijk geven de potentialiteiten van de dingen vorm aan de mogelijke toekomstige gangen van zaken. Door werkelijke mogelijkheden in de potentialiteiten van de dingen te grondvesten geven wij een systematische verklaring van branching time modellen voor werkelijke mogelijkheid die verheldert waarom elke mogelijke gang van zaken in overeenstemming is met de natuurwetten. Het resultaat is een beperkt indeterminisme. De modellen zijn indeterministisch omdat zij naar de toekomst vertakken en het indeterminisme is beperkt omdat er niet zo maar van alles kan gebeuren. Toch is alles wat uit de hoedanigheid van de dingen voortkomt ook werkelijk mogelijk.

* * *

Curriculum Vitae

Antje Susanne Rumberg was born on March 18th 1982 in Sigmaringen, Germany. She studied Philosophy (major), Mathematics (minor) and English Linguistics (minor) at the University of Tübingen, Germany, where she obtained her *Magistra Artium* degree in 2010 with distinction. Subsequently, she held a PhD position in Thomas Müller's research project "What is Really Possible? Philosophical Explorations in Branching-History-Based Real Modality" launched at Utrecht University, and she followed the research project from Utrecht to Konstanz in 2013. Since 2015, she is a researcher in the research project "Alternatives for the Future", which forms part of the research unit "What if ...?" hosted at the University of Konstanz. She has taught courses on modal and temporal logic and the history of logic. Her main research interests include the logic and metaphysics of historical and natural modalities and Bernard Bolzano.

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