Expansions and Dimensions

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Chapter 1

Introduction

1.1 Introduction to the main results of this thesis

Let $\beta \in (1,2)$ and $x \in \mathcal{A}_\beta = [0,(\beta-1)^{-1}]$, we call a sequence $(a_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ a β-expansion or a coding of x provided

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Sidorov [82] proved that Lebesgue almost every point has uncountably many expansions. However, there still exist some points which have unique expansions, see [38]. If the coding of x is unique, then we call x a univoque point. We use $\widetilde{U}$ and U to denote the set of unique β-expansions and the corresponding univoque set, i.e.,

$$U = \{x \in [0,(\beta-1)^{-1}] : x \text{ has a unique coding}\},$$

and

$$\widetilde{U} = \left\{(a_n) \in \{0,1\}^{\mathbb{N}} : x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n},\ x \text{ is a univoque point}\right\}.$$

The dynamical approach is an excellent tool which can generate β-expansions effectively. Define $T_0(x) = \beta x, T_1(x) = \beta x - 1$, see Figure 1.1.

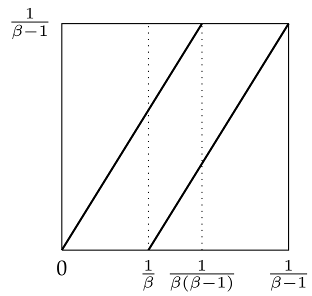


FIGURE 1.1: The dynamical system for $\{T_0, T_1\}$.

Given any $x \in [0,(\beta-1)^{-1}]$, we can consider the orbit of x, i.e $T^n(x)$. Clearly, for any $n \geq 1$, we have

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n} + \frac{T^n(x)}{\beta^n}.$$

Here the digits (a_n) are chosen in the following way: if $T^{n-1}(x) \in [0,\beta^{-1})$, then $a_n = 0$, if $T^{n-1}(x) \in ((\beta-1)^{-1}\beta^{-1},(\beta-1)^{-1}]$, then $a_n = 1$. However, if $T^{n-1}(x) \in [\beta^{-1},(\beta-1)^{-1}\beta^{-1}]$, then we may choose a_n to be 0 or 1. Motivated by this observation, we call

$[\beta^{-1}, (\beta-1)^{-1}\beta^{-1}]$ the switch region, see [15]. Since $T^n(x) \in [0, (\beta-1)^{-1}]$ for any $n \geq 1$, we can find the codings of x, i.e. (a_n), if we iterate the orbit of x for infinitely many times.

From the fractal perspective, we may use an iterated function systems (IFS) to generate β-expansions. Let $\{f_i\}_{i=1}^m$ be an iterated function system of contractive similitudes on $\mathbb{R}$ defined as

$$f_i(x) = r_i x + a_i, \quad i = 1, \ldots, m, \tag{1.1}$$

where $0 < |r_i| < 1$ is the contractive ratio and $b_i \in \mathbb{R}$. Hutchinson proved that there exists a unique non-empty compact set K satisfying

$$K = \cup_{i=1}^m f_i(K).$$

We call K the self-similar set or attractor for the IFS $\{f_j\}_{j=1}^m$, see [46] for further details. An IFS is called homogeneous if all the similarity ratios r_j are equal. For any $x \in K$, there exists a sequence $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ such that

$$x = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(K).$$

We call such a sequence a coding of x. The attractor K defined by (1.1) may equivalently be defined to be the set of points in $\mathbb{R}$ which admit a coding, i.e., we can define a surjective projection map between the symbolic space $\{1, \ldots, m\}^{\mathbb{N}}$ and the self-similar set K by

$$\pi((i_n)_{n=1}^\infty) := \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

In the case of β-expansions, the corresponding IFS is $\left\{f_0(x) = \dfrac{x}{\beta}, f_1(x) = \dfrac{x+1}{\beta}\right\}$. Then $[0, (\beta-1)^{-1}]$ is the unique attractor of $\{f_0, f_1\}$, i.e.

$$[0, (\beta-1)^{-1}] = f_0([0, (\beta-1)^{-1}]) \cup f_1([0, (\beta-1)^{-1}]).$$

For any $a_1 a_2 \cdots a_n \in \{0,1\}^n, n \geq 1$, we call $f_{a_1 a_2 \cdots a_n}([0, (\beta-1)^{-1}])$ a fundamental interval. Now we illustrate how can we generate the codings of x in terms of the IFS. The idea is similar to the dynamical approach. Given any $x \in [0, (\beta-1)^{-1}]$, if $x \in f_0([0, (\beta-1)^{-1}]) \setminus f_1([0, (\beta-1)^{-1}])$, then we choose $a_1 = 0$, if $x \in f_1([0, (\beta-1)^{-1}]) \setminus f_0([0, (\beta-1)^{-1}])$, then we choose $a_1 = 1$, if $x \in f_0([0, (\beta-1)^{-1}]) \cap f_1([0, (\beta-1)^{-1}])$, then we can choose $a_1 = 0$ or $a_1 = 1$. Similarly, for any $n \geq 2$ if $x \in f_{a_1 a_2 \cdots a_n}([0, (\beta-1)^{-1}]) \cap f_{b_1 b_2 \cdots b_n}([0, (\beta-1)^{-1}])$, then the first n digits of the codings of x can be $a_1 a_2 \cdots a_n$ or $b_1 b_2 \cdots b_n$. If there is only one finite sequence $a_1 a_2 \cdots a_n$ such that $x \in f_{a_1 a_2 \cdots a_n}([0, (\beta-1)^{-1}])$, then the first n digits of the codings of x should be $a_1 a_2 \cdots a_n$. In other words, if x lies in the intersection of some fundamental intervals, then we can choose the indices of these fundamental intervals as the first digits of x. Implementing this idea, we can find the codings of x.

The dynamical idea can be utilized to study self-similar sets. Define $T_j(x) := f_j^{-1}(x) = (x - a_j) r_j^{-1}$ for each $1 \leq j \leq m$. We denote the concatenation $T_{i_n} \circ \ldots \circ T_{i_1}(x)$ by $T_{i_1 \ldots i_n}(x)$. The following lemma provides an alternative formulation of codings of elements of K in terms of the maps T_j. It is fundamental when we analyze self-similar sets.

Lemma 1.1.1. *Let $x \in K$. Then $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ is a coding for x if and only if $T_{i_1 \ldots i_n}(x) \in K$ for all $n \in \mathbb{N}$.*

The proof of this lemma can be found in Chapter 3.

In this thesis, we shall use the dynamical and fractal ideas to study β-expansions and self-similar sets. Each of them has its own advantages. Nevertheless, if we combine these two ideas together, then we can obtain many interesting results.

Expansions and dimensions are the central topics of this dissertation. Throughout this thesis, we mainly study the Hausdorff dimension. We have introduced the definitions of expansions and codings. Now we give the detailed definition of the Hausdorff dimension.

Let E be a non-empty subset of $\mathbb{R}^n$ and $s \geq 0$. For all $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) := \inf\left\{\sum_{i=1}^{\infty}|U_i|^s : |U_i| \leq \delta \text{ for all } i,\ E \subset \cup_{i=1}^{\infty} U_i\right\},$$

where $|\cdot|$ stands for the diameter. The s-dimensional Hausdorff measure of the set E is defined as

$$\mathcal{H}^s(E) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(E).$$

The Hausdorff dimension of E, denoted by $\dim_H(E)$, is defined in the following way:

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

The graph-directed self-similar set is a very important tool of this thesis as we mainly utilize this tool to calculate the Hausdorff dimension of self-similar sets. This notation has been introduced by Mauldin and Williams [62]. Here we give the detailed introduction of this tool.

A graph-directed construction in $\mathbb{R}$ consists of the following:

1. A finite union of bounded closed intervals $\cup_{u=1}^n J_u$ such that the interiors of J_u are pairwise disjoint.

2. A directed graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set E. Moreover, we assume that for any $u \in V$ there is some $v \in V$ such that $(u, v) \in E$.

3. For each edge $(u, v) \in E$ there exists a similitude $f_{u,v}(x) = r_{uv}x + a_{uv}$, where $r_{uv} \in (0, 1)$ and $a_{uv} \in \mathbb{R}$. Moreover, for each $u \in V$ the set $\{f_{u,v}(J_v^\circ) : (u, v) \in E\}$ satisfies the open set condition, i.e., there exist n open sets $\{J_u^\circ : u \in V\}$ such that

$$\bigcup_{(u,v)\in E} f_{u,v}(J_v^\circ) \subseteq J_u^\circ,$$

   and the elements of $\{f_{u,v}(J_v^\circ) : (u, v) \in E\}$ are pairwise disjoint, where J° is the interior of J.

As is the case for self-similar sets, we have the following result.

Theorem 1.1.2. *For each graph-directed construction, there exists a unique vector of non-empty compact sets $(C_1, \ldots, C_n)$ such that, for each $u \in V$, $C_u = \bigcup_{(u,v)\in E} f_{u,v}(C_v)$.*

We let $K^* := \cup_{u=1}^n C_u$ and call it the graph-directed self-similar set of this construction. To each graph-directed construction we can associate a weighted adjacency matrix A. This matrix is defined by $A = (r_{u,v})_{(u,v)\in V\times V}$, for simplicity, we assume that $r_{u,v} = 0$ if $(u, v) \notin E$. For each $t \geq 0$ we define another weighted adjacency matrix $A^t = (a_{t,u,v})_{(u,v)\in V\times V}$, where $a_{t,u,v} = r_{u,v}^t$. Let $\Phi(t)$ denote the largest nonnegative eigenvalue of A^t. A graph is strongly connected if for any two vertices $u, v \in V$, there exists a directed path from u to v. A strongly connected component of G is a subgraph C of G such that C is strongly connected, let $SC(G)$ be the set of all the strongly connected components of G. Now we state the main result of [62].

Theorem 1.1.3. *For every graph-directed construction such that G is strongly connected, the Hausdorff dimension of K^* is t_0, where t_0 is uniquely defined by $\Phi(t_0) = 1$.*

If the graph-directed construction G is not strongly connected, we still have a similar result. As is well known, a directed graph G must have a strongly connected component, see [58, section 4.4.]. In which case the following result makes sense.

Theorem 1.1.4. *If the G in our graph-directed construction is not strongly connected, let $t_1 = \max\{t_C : \Phi(t_C) = 1, C \in SC(G)\}$, where $\Phi(t_C)$ is the largest eigenvalue of the adjacency matrix of the strongly connected subgraph C. Then $\dim_H(K^*) = t_1$.*

Finally we introduce the Lipschitz map. Let $(X_i, d_i), i = 1, 2$ be metric spaces. For nonempty sets $A_i \subseteq X_i$ we say they are Lipschitz equivalent, denoted by $A_1 \simeq A_2$, if there exists a bijection $\phi : A_1 \to A_2$ and a constant $c > 0$ such that

$$c^{-1}d_1(x, y) \leq d_2(\phi(x), \phi(y)) \leq cd_1(x, y) \text{ for any } x, y \in A_1.$$

The bijection ϕ is called the Lipschitz map. Lipschitz equivalence can be used to classify fractal sets. It is well-known that if two sets are Lipschitz equivalent, then they have the same Hausdorff dimension, see [30]. We shall use this property in Chapter 6.

The Hausdorff dimension has some good properties, see [34]. For some basic definitions, e.g. the packing dimension and box dimension, we do not give the detailed introduction as we rarely use them. The readers can find them in [34]. Other notation and definitions will be introduced in each chapter.

Now we are ready to introduce the problems and the main results of this thesis. Let K be the attractor of some IFS. Define

$$U_k = \{x \in K : x \text{ has exactly } k \text{ different codings}\},$$

where $k = 1, 2, 3, \cdots, \aleph_0, 2^{\aleph_0}$. One of the aims of this thesis is to investigate this set. Considering this set can assist us in a better understanding of the symbolic space of K. When $k = 1$, in the case of β-expansions, there are many results, see for example [28, 27, 52] and references therein. However, for $k \geq 2$, very few results exist. We consider general self-similar sets as well as β-expansions. For the self-similar sets, we want to find some sufficient conditions under which $\dim_H(U_k) = \dim_H(U_1)$ for any $k \geq 2$, and to find the Hausdorff dimension of the univoque set. These are the main results of Chapters 4 and 6. In the case of β-expansions, this problem is difficult, and up to now there are very few results concerning this problem. In Chapter 5, we study β-expansions in base β over the digit set $\{0, 1, \beta\}$, and give a complete description of this example.

We consider not only problems related to U_k but also study other problems. In Chapter 3, we consider the dimension of U_1 when the self-similar set is an interval. In Chapter 2, we study the interior of a self-affine set defined by two homogeneous affine maps. In Chapter 7, we define a new type of random β-transformation, and investigate the intrinsically ergodic measure for the induced map. Finally, in Chapter 8, we consider the sum of self-similar sets. We now give a simple introduction to our main results.

In Chapter 2, we consider the homogeneous iterated function system generating codings representing simultaneously two different points in two different bases. We show that for β's sufficiently close to 1, the attractor contains a neighbourhood of the origin.

In Chapter 3, given any IFS, we investigate the points of the attractor with unique codings. Such set is called the univoque set. When the attractor is an interval, we prove under mild conditions that the univoque set can be identified with a subshift of finite type. This result enables us to find the Hausdorff dimension of the univoque set. A similar idea can be implemented in higher dimensions. Our main result generalizes some results of by de Vries and Komornik.

In Chapter 4, we consider the case when the attractor is not an interval. We use another approach to characterizing when the attractor can be identified with a subshift of finite type. With this identification, we can find the dimension of the univoque set as well. Moreover, we give some applications of our results. Firstly, we calculate the Hausdorff dimension of the set of points of K with multiple codings. Secondly, in the setting of β-expansions, when the set of all the unique codings is not a subshift of finite type, we can calculate in some cases the Hausdorff dimension of the univoque set. This application generalizes a result of de Vries and Komornik [84]. Thirdly, for the doubling map with asymmetrical holes, we give a sufficient condition such that the attractor can be identified with a subshift of finite type. The third application partially answers a problem posed by Barrera [8]. Fourthly, we can construct a Lipschitz map between two overlapping self-similar sets.

In Chapter 5, for $\beta > 1$ we consider expansions in base β over the alphabet $\{0, 1, \beta\}$. Let U_β be the set of x which have a unique β-expansions. For $k = 2, 3, \cdots, \aleph_0$ let $\mathbb{B}_k$ be the set of bases β for which there exists x having k different β-expansions, and for $\beta \in \mathbb{B}_k$ let $U_\beta^{(k)}$ be the set of all such x's which have k different β-expansions. In this chapter we show that

Theorem 1.1.5.
$$\mathbb{B}_{\aleph_0} = [2, \infty), \quad \mathbb{B}_k = (q_c, \infty) \quad \textit{for any} \quad k \geq 2,$$
where $q_c \approx 2.32472$ is the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Moreover, we show that for any positive integer $k \geq 2$ and any $\beta \in \mathbb{B}_k$ the Hausdorff dimensions of $U_\beta^{(k)}$ and U_β are the same, i.e.,
$$\dim_H U_\beta^{(k)} = \dim_H U_\beta \quad \textit{for any} \quad k \geq 2.$$
Finally, we conclude that the set of x having a continuum of β-expansions has full Hausdorff dimension.

In Chapter 6, we consider a class $\mathcal{E}$ of self-similar sets with overlaps satisfying the following conditions. Let $\{f_i(x)\}_{i=1}^m$ be the IFS of E. Denote by $I = [a, b]$ the convex hull of the self-similar set E. Then

(A) $a = f_1(a) < f_2(a) < \cdots < f_m(a) < f_m(b) = b$.

(B) $f_i(I) \cap f_{i+2}(I) = \emptyset$ for any $1 \leq i \leq m - 2$.

(C) There exist $i, j \in \{1, \cdots, m-1\}$ such that
$$f_i(I) \cap f_{i+1}(I) = \emptyset \quad \text{and} \quad f_j(I) \cap f_{j+1}(I) \neq \emptyset.$$

(D) If $f_i(I) \cap f_{i+1}(I) \neq \emptyset$, then there exist $u, v \geq 1$ such that
$$f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I),$$

where $f_{i_1\cdots i_k}(\cdot) := f_{i_1} \circ \cdots \circ f_{i_k}(\cdot)$.

The intervals $f_i(I), i = 1, \cdots, m$ are called the fundamental intervals of $(E, \{f_i\}_{i=1}^m)$. In this chapter, we have the following results.

Theorem 1.1.6. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E. The following statements are equivalent.*

(i) $f_1(I) \cap f_2(I) \neq \emptyset$ *or* $f_{m-1}(I) \cap f_m(I) \neq \emptyset$.

(ii) $\dim_H U_k = \dim_H U_1$ *for all* $k \geq 1$.

(iii) $f_1(b) \in U_{\aleph_0}$ *or* $f_m(a) \in U_{\aleph_0}$.

(iv) $|U_{\aleph_0}| = \aleph_0$.

Here $|A|$ denotes the cardinality of a set A.

Theorem 1.1.7. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E. The following statements are equivalent.*

(i) $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$.

(ii) $\dim_H U_k = \dim_H U_1$ *if* $k = 2^s$ *for some* $s \geq 1$, *and* $U_k = \emptyset$ *otherwise.*

(iii) $f_1(b) \notin U_{\aleph_0}$ *and* $f_m(a) \notin U_{\aleph_0}$.

(iv) $U_{\aleph_0} = \emptyset$.

These two results imply the following interesting corollaries.

Corollary 1.1.8. *For any $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, we have following dichotomy: either*

$$\dim_H U_k = \dim_H U_1$$

for all $k \geq 1$, or

$$\dim_H U_k = \dim_H U_1$$

if k is 2-power, and $U_k = \emptyset$ otherwise.

Corollary 1.1.9. *For any $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, either $|U_{\aleph_0}| = \aleph_0$ or $U_{\aleph_0} = \emptyset$.*

The above two corollaries yield the following results.

Corollary 1.1.10. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. The following conditions are equivalent.*

- $\dim_H U_k = \dim_H U_1$ *for any* $k \geq 1$.
- $\dim_H(U_2) = \dim_H(U_3)$.
- $\dim_H(U_{2016}) = \dim_H(U_8) = \dim_H(U_{29})$.
- $\dim_H(U_1) = \dim_H(U_{2012}) = \dim_H(U_9) = \dim_H(U_6)$.

Similarly, we have

Corollary 1.1.11. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. The following conditions are equivalent.*

- $\dim_H U_k = \dim_H U_1$ *if k is a 2-power, and* $\dim_H U_k = \dim_H U_3$ *otherwise.*

- $U_{2016} = \emptyset$.
- $U_{2015} = \emptyset$.
- $U_{2014} = \emptyset$.

Finally we prove that

Theorem 1.1.12. $\dim_H(U_{2^{\aleph_0}}) = \dim_H(E)$. *Moreover,* $\dim_H(E)$ *and* $\dim_H(U_1)$ *are calculable. For any* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$,

$$0 < \mathcal{H}^{\dim_H(U_1)}(U_1) < \infty.$$

If $U_k \neq \emptyset$, $k \geq 2$

$$0 < \mathcal{H}^{\dim_H(U_k)}(U_k).$$

For some examples, we can show that the Hausdorff measure of U_k could be infinity. The idea of this chapter can be implemented in higher dimensions and even for self-affine sets. Moreover, even without the Condition (D), the following equation may still hold:

$$\dim_H U_k = \dim_H U_1$$

for any $k \geq 1$. We shall give some examples to demonstrate this point.

In Chapter 7, we define a new type of random β-transformation. For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

In this chapter we define the shrinking random β-transformation K and investigate natural invariant measures for K, and the induced tranformation of K on a special subset of the domain. We prove that both transformations have a unique measure of maximal entropy. However, the measure induced from the intrinsically ergodic measure for K is not the intrinsically ergodic measure for the induced system.

In Chapter 8, we consider the sum of self-similar sets. Let $\beta > 1$. We define a class of similitudes

$$S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.$$

Taking any finite collection of similitudes $\{f_i(x)\}_{i=1}^m$ from S, it is well known that there is a unique self-similar set K_1 satisfying $K_1 = \cup_{i=1}^m f_i(K_1)$. Similarly, another self-similar set K_2 can be generated via the finite contractive maps of S. We call $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$ the arithmetic sum of two self-similar sets. In this chapter, we prove that

Theorem 1.1.13. $K_1 + K_2$ *is either a self-similar set or a unique attractor of some infinite iterated function system.*

As a corollary of this result, we prove that $\dim_P(K_1 + K_2) = \overline{\dim_B}(K_1 + K_2)$, where $\dim_P$ and $\overline{\dim_B}$ denote the packing and upper box dimensions. Using the main result we can calculate the exact Hausdorff dimension of $K_1 + K_2$ under some conditions, which partially provides the dimensional result of $K_1 + K_2$ if the IFS's of K_1 and K_2 fail the irrationality assumption, see Peres and Shmerkin [72].

For the sake of convenience, each chapter is made of self-contained. We will introduce some definitions and results repeatedly.

Chapter 2

Self-affine sets with positive Lebesgue measure

Abstract
Using techniques introduced by C. Güntürk, we prove that the attractors of a family of overlapping self-affine iterated function systems contain a neighbourhood of zero for all parameters in a certain range. This corresponds to giving conditions under which a single sequence may serve as a 'simultaneous β-expansion' of different numbers in different bases.

2.1 Introduction

Given real numbers $1 < \beta_1 < \beta_2$, we define contractions $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T_i(x, y) = \left(\frac{x+i}{\beta_1}, \frac{y+i}{\beta_2} \right).$$

A classical result of Hutchinson [46] asserts that there exists a unique non-empty compact set A_{β_1,β_2} satisfying

$$A_{\beta_1,\beta_2} = T_{-1}(A_{\beta_1,\beta_2}) \cup T_1(A_{\beta_1,\beta_2}).$$

If $\beta_1 \neq \beta_2$ then the contractions T_i are affine contractions and A_{β_1,β_2} is termed a self-affine set. If $\beta_1, \beta_2 < 2$, the two contracted copies $T_{-1}(A_{\beta_1,\beta_2})$ and $T_1(A_{\beta_1,\beta_2})$ overlap. There are many fundamental open questions about the structure of overlapping self-affine sets, see for example [47, 73, 81].

The family A_{β_1,β_2} of sets was studied in [81], where Shmerkin proved that there exists an open set $K \subset (1, 2)^2$ such that for almost every pair $(\beta_1, \beta_2) \in K$ the corresponding set A_{β_1,β_2} has positive Lebesgue measure. This was done by studying the absolute continuity of a certain measure defined on A_{β_1,β_2}. In this article we prove that A_{β_1,β_2} contains a neighbourhood of $(0, 0)$ for all $(\beta_1, \beta_2) \in (1, 1 + C)^2$ for some positive constant C which is explicitly defined later.

In fact this problem is closely related to the problem of 'simultaneous β-expansions' studied by Güntürk in [39]. Given $\beta \in (1, 2)$ and $x \in [\frac{-1}{\beta-1}, \frac{1}{\beta-1}]$, a β-expansion of x is a sequence $\underline{a} \in \{-1, 1\}^\infty$ for which

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = x.$$

This definition can be extended to $\beta > 2$ by letting the digits a_n come from a larger digit set.

For typical x the β-expansion of x is not unique, indeed almost every $x \in [\frac{-1}{\beta-1}, \frac{1}{\beta-1}]$ has uncountably many β-expansions, see [82]. This allows one, given x, to search for β-expansions of x with interesting properties, such as a given digit frequency or that the sequence is a β-expansion of x for more than one β.

In [39], Güntürk proved that given $\beta_1, \beta_2 > 1$ and $(x_1, x_2) \in \mathbb{R}^2$ there exists a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ satisfying

$$\sum_{n=1}^{\infty} a_n \beta_k^{-n} = x_k$$

for each $k \in \{1, 2\}$ whenever a certain algorithm can be implemented, see Proposition 2.2.1. It was claimed without proof[1] that there exist constants $C, \delta > 0$ such that the algorithm can be implemented whenever $\beta_1, \beta_2 \in (1, 1 + C)$ and $(x_1, x_2) \in (-\delta, \delta)^2$. We prove this fact and provide suitable constants C and δ explicitly. We also prove a number of related results including results on finding β-expansions with given digit frequency and finding sequences which serve as multiple expansions for a range of β_1, β_2.

The following is our main theorem.

Theorem 2.1.1. *There exists a constant $C \approx 0.05$ such that for any $1 < \beta_1 < \beta_2 < 1 + C$, there exists $\delta = \delta(\beta_1, \beta_2)$ such that for any pair $(x_1, x_2) \in [-\delta, \delta]^2$, there exists a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$(\sum_{n=1}^{\infty} a_n \beta_1^{-n}, \sum_{n=1}^{\infty} a_n \beta_2^{-n}) = (x_1, x_2). \tag{2.1}$$

In the self-affine setting, this theorem corresponds to saying that the sequence (a_n) is a coding of the pair (x_1, x_2) in A_{β_1,β_2}, and in particular that $(x_1, x_2) \in A_{\beta_1,\beta_2}$. This leads immediately to the following corollary.

Corollary 2.1.2. *For all any $1 < \beta_1 < \beta_2 < 1 + C$ we have that the self-affine fractal A_{β_1,β_2} contains a neighbourhood of $(0, 0)$.*

The constant δ is explicitly computable. If β_1 tends to β_2 the constant δ tends to zero.

Remark 2.1.3. *An important special case of Corollary 2.1.2 is the case $x_1 = x_2$. This was the main motivation of Güntürk for his original article because of its relevance to analogue digital conversion, see [39]. While in general the constant δ depends on β_1, β_2, in the case that $x_1 = x_2$ we can choose $\delta = 0.16$ independently of β_1, β_2 to give that for all $1 < \beta_1 < \beta_2 < 1 + C$ and $x \in [-0.16, 0.16]$ there exists a sequence $(a_i) \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta_1^n} = \sum_{n=1}^{\infty} \frac{a_n}{\beta_2^n}.$$

Using the same techniques, we can also find β-expansions of real numbers which have certain given digit frequencies. It was stated in [39] that the following theorem

[1] Güntürk stated in [39] that details would be provided in a later publication, but has confirmed to us that, due to other commitments, no such publication will be forthcoming. Since the techniques of [39] are rather different from the standard techniques for analysing self-affine sets, and the results are interesting, we take the liberty of providing a proof of the stated results of Güntürk in this article.

should follow by suitably adapting the proof of Theorem 2.1.1, we provide the appropriate adaptation and prove the result giving explicit constants.

Theorem 2.1.4. *Let $C_1 > 0$ satisfy $(1 + C_1) + 2(1 + C_1)^3 = 6$. Then for all $1 < \beta < 1 + C_1$, there exists $\delta = \delta(\beta)$ such that for any $x \in [-\delta, \delta]$ there exists a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ satisfying*

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = x \tag{2.2}$$

and

$$x = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}. \tag{2.3}$$

One can read off limiting digit frequencies of the sequence (a_n) from equation 2.3 by noting that

$$\frac{1 - \frac{a_1+a_2+\cdots+a_n}{n}}{2} = \frac{|\{k \in \{1, \cdots n\} : a_k = -1\}|}{n}.$$

Proofs of Theorems 2.1.1 and 2.1.4 are given in the next two sections. In the final section we state some further corollaries and remarks.

2.2 Proof of Theorem 2.1.1

As stated in the introduction, we are using many of the ideas of [39]. For clarity, we have amalgamated these ideas to form the following proposition, which was proved in [39]. The remainder of our proof of Theorem 3.4.4, which gives conditions under which the algorithm in Proposition 2.2.1 can be implemented, is new.

Proposition 2.2.1. *Given $1 < \beta_1 < \beta_2 < 2$ and $(x_1, x_2) \in \mathbb{R}^2$ suppose that one can implement the following algorithm.*

1. *For $L > 2$ pick real numbers $h_1, \cdots, h_L$ with $h_L \neq 0$ and*

   $$h_{L-1} = h_{L-2} = 0,$$

   such that β_1, β_2 are roots of the polynomial $P(z) = z^L - \sum_{k=1}^{L} h_k z^{L-k}$.

2. *Pick real numbers u_{-L+1}, u_{-L+2} which satisfy the equation*

   $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h_L \begin{pmatrix} \beta_1^{-1} & \beta_1^{-2} \\ \beta_2^{-1} & \beta_2^{-2} \end{pmatrix} \begin{pmatrix} u_{-L+1} \\ u_{-L+2} \end{pmatrix}$$

   Set $u_{-L+3} = \cdots = u_0 = 0$.

3. *Find a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that*

   $$u_n := \sum_{k=1}^{L} h_k u_{n-k} - a_n$$

   satisfies $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$.

Then the sequence $(a_n)_{n=1}^{\infty}$ will satisfy equation (2.1).

In this chapter we give rigorous conditions under which the algorithm of Güntürk can be implemented leading to a proof of Theorem 2.1.1. For completeness we also give the proof of Proposition 2.2.1. We begin by introducing the polynomial P, P was chosen because it has relatively low degree and satisfies the conditions of Proposition 2.2.1, but it is likely that better bounds on C and δ can be obtained by choosing a better polynomial P.

Definition 2.2.2. *Given $\beta_1, \beta_2 > 1$, we define the polynomial P by*

$$P(x) = x^4 - h_1 x^3 - h_2 x^2 - h_3 x - h_4$$

where

$$\begin{aligned} h_1 &= \frac{(\beta_1+\beta_2)(\beta_1^2+\beta_2^2)}{\beta_1^2+\beta_1\beta_2+\beta_2^2} \\ h_2 &= 0 \\ h_3 &= 0 \\ h_4 &= \frac{-(\beta_1\beta_2)^3}{\beta_1^2+\beta_1\beta_2+\beta_2^2}. \end{aligned}$$

We further define the constant C by

$$C := \sqrt[3]{\sqrt{10}-2} \approx 0.05.$$

Lemma 2.2.3. *The polynomial P satisfies $P(\beta_1) = P(\beta_2) = 0$.*

Proof. Defining,

$$b = \frac{\beta_1\beta_2(\beta_1+\beta_2)}{\beta_1^2+\beta_1\beta_2+\beta_2^2},$$

and

$$c = \frac{(\beta_1\beta_2)^2}{\beta_1^2+\beta_1\beta_2+\beta_2^2}$$

gives us that

$$(x-\beta_1)(x-\beta_2)(x^2+bx+c) = x^4 - h_1x^3 - h_2x^2 - h_3x - h_4 = P(x).$$

Then β_1 and β_2 are roots of P. □

Lemma 2.2.4. *For $\beta_1, \beta_2 \in (1, 1+C)$ we have that*

$$\sum_{n=1}^{4} |h_k| = |h_1| + |h_4| \leq 2.$$

Proof. Expanding out, we see that

$$\begin{aligned}\sum_{n=1}^{4}|h_k| &= |h_1|+|h_4| \\ &= \frac{(\beta_1+\beta_2)(\beta_1^2+\beta_2^2)+\beta_1^3\beta_2^3}{\beta_1^2+\beta_1\beta_2+\beta_2^2} \\ &\le \frac{(2+2C)2(1+C)^2+(1+C)^6}{3} \\ &\le 2\end{aligned}$$

whenever $\beta_1, \beta_2 \in (1, 1+C)$, as required. Indeed, C was chosen to be the largest constant such that the above inequalities hold. □

We now prove Theorem 2.1.1 using Proposition 2.2.1.

Proof. We set

$$\begin{aligned}u_{-3} &= \frac{\beta_1^2+\beta_1\beta_2+\beta_2^2}{(\beta_2-\beta_1)\beta_1\beta_2}\left(\frac{x_1}{\beta_2^2}-\frac{x_2}{\beta_1^2}\right), \\ u_{-2} &= \frac{\beta_1^2+\beta_1\beta_2+\beta_2^2}{(\beta_2-\beta_1)\beta_1\beta_2}\left(\frac{x_2}{\beta_1}-\frac{x_1}{\beta_2}\right), \\ u_{-1} &= u_0 = 0.\end{aligned}$$

These choices of u_i ensure that condition (2) of Proposition 2.2.1 is satisfied, i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = h_4 \begin{pmatrix} \beta_1^{-1} & \beta_1^{-2} \\ \beta_2^{-1} & \beta_2^{-2} \end{pmatrix} \begin{pmatrix} u_{-3} \\ u_{-2} \end{pmatrix}.$$

Condition (1) has already been shown to hold for our choice of P by Lemma 2.2.3. It remains to show that condition (3) holds, i.e. that one can choose some sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that defining u_n for $n \in \mathbb{N}$ by

$$u_n := \sum_{k=1}^{L} h_k u_{n-k} - a_n \tag{2.4}$$

gives $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$. Since $h_2 = h_3 = 0$ the above equation for u_n becomes

$$u_n = h_1 u_{n-1} + h_4 u_{n-4} - a_n.$$

We set

$$a_n = \begin{cases} -1 & h_1 u_{n-1} + h_4 u_{n-4} < 0 \\ +1 & h_1 u_{n-1} + h_4 u_{n-4} \ge 0 \end{cases}. \tag{2.5}$$

Now we observe that, if for some $k \in \mathbb{N}$ one has that $u_{k-1}, u_{k-4} \in [-1, 1]$, then it follows from Lemma 2.2.4 that

$$h_1 u_{k-1} + h_4 u_{k-4} \in [-2, 2].$$

Hence it follows that

$$u_k := h_1 u_{k-1} + h_4 u_{k-4} - a_k \in [-1, 1],$$

and hence by induction that $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$.

Now we define

$$\delta = \frac{\beta_1^2\beta_2^2(\beta_2 - \beta_1)}{(\beta_1^2 + \beta_1\beta_2 + \beta_2^2)(\beta_1 + \beta_2)}.$$

We see that $\delta > 0$ whenever $\beta_2 > \beta_1$, but that $\delta \to 0$ as $\beta_2 - \beta_1 \to 0$. From the definition of u_{-3}, u_{-2} we see that for $x_1, x_2 \in [-\delta, \delta]^2$ and $\beta_1, \beta_2 \in (1, 1 + C)$ we have that $u_{-3}, u_{-2} \in [-1, 1]$. Since $u_{-1} = u_0 = 0$ it follows by induction that $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$. Hence conditions $(1), (2)$ and (3) of Proposition 2.2.1 are satisfied, and so the sequence (a_n) satisfies equation 2.1 and Theorem 2.1.1 is proved. □

It remains only to give a formal proof of Proposition 2.2.1.

Proof. We give a proof for the case $L = 4$, which is the case that we have used. From condition (2), we have that

$$x_i = h_4(u_{-3}\beta_i^{-1} + u_{-2}\beta_i^{-2})$$

for $i = 1, 2$. Rewriting condition (3) gives us that $a_n = \sum_{k=0}^{4} h_k u_{n-k}$. Then summing gives us that

$$\sum_{n=1}^{\infty} a_n\beta_i^{-n} = \sum_{n=1}^{\infty}\sum_{k=0}^{4} h_k u_{n-k}\beta_i^{-n} \tag{2.6}$$

where $h_0 = -1$. Since the sequence (u_n) is bounded, we have by Fubini's theorem that

$$\sum_{n=1}^{\infty} a_n\beta_i^{-n} = \sum_{k=0}^{4}\sum_{n=1}^{\infty} h_k u_{n-k}\beta_i^{-n} = \sum_{k=0}^{4} h_k\beta_i^{-k} \sum_{n=-k+1}^{\infty} u_n\beta_i^{-n}$$

Here the first equality involved using equation (2.6) and swapping the order of summation by Fubini. The second equality is just a change of variables. Now, by separating the terms for positive and negative n in the right hand side of the above equation, we have that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n\beta_i^{-n} &= \left(\sum_{k=0}^{4} h_k\beta_i^{-k}\right)\left(\sum_{n=1}^{\infty}\frac{u_n}{\beta_i^n}\right) + h_1\beta_i^{-1}u_0 + h_2\beta_i^{-2}(u_0 + u_{-1}\beta_i) \\ &+ h_3\beta_i^{-3}(u_0 + u_{-1}\beta_i + u_{-2}\beta_i^2) + h_4\beta_i^{-4}(u_0 + u_{-1}\beta_i + u_{-2}\beta_i^2 + u_{-3}\beta_i^3).\end{aligned}$$

Since β_i is the root of $P(x)$ we have $\sum_{k=0}^{4} h_k\beta_i^{-k} = 0$ and so the first term vanishes. From conditions (1) and (2), we have $u_{-1} = u_0 = h_2 = h_3 = 0$. Then, removing the zero terms, the right hand side of the above equation becomes $h_4(u_{-3}\beta_i^{-1} + u_{-2}\beta_i^{-2})$, which by condition (2) is equal to x_i. We conclude that

$$\sum_{n=1}^{\infty} a_n\beta_i^{-n} = h_4(u_{-3}\beta_i^{-1} + u_{-2}\beta_i^{-2}) = x_i$$

as required. This completes the proof of Proposition 2.2.1.

□

Finally we comment that in the case that $x_1 = x_2$ we can give values of δ which are independent of $\beta_1, \beta_2 \in (1, 1+C)$. Our bound on δ was to ensure that $u_{-2}, u_{-3} \in [-1, 1]$.

If $x_1 = x_2$ then

$$\begin{aligned} |u_{-2}| \leq |u_{-3}| &= \left| x_1 \left(\frac{\beta_1^2 + \beta_1\beta_2 + \beta_2^2}{(\beta_1^3\beta_2^3)} \right) (\beta_1 + \beta_2) \right| \\ &\leq |x_1| 6(1+C)^3 \leq 1 \end{aligned}$$

whenever $|x_1| \leq \delta = \frac{1}{6(1+C)^3} \approx 0.16$.

2.3 β-expansions with a given digit frequency.

With some modifications, the algorithm used in the proof of Proposition 2.2.1 can also be utilized to prove Theorem 2.1.4. The following is analagous to Proposition 2.2.1.

Proposition 2.3.1. *Given $1 < \beta < 2$ and $x \in [-\delta, \delta]$ for some δ which will be set in the process of proof, suppose that one can implement the following algorithm.*

1. *For $L > 2$ pick real numbers $h_1, \cdots, h_L$ with $h_L \neq 0$ and*

$$h_{L-1} = h_{L-2} = 0,$$

such that $1, \beta$ are roots of the polynomial $P(z) = z^L - \sum_{k=1}^{L} h_k z^{L-k}$.

2. *Pick a real number u_{-L+1}, which satisfies the equation*

$$x = \frac{h_L(\beta - 1)u_{-L+1}}{\beta(\beta - 2)}.$$

Set $u_{-L+2} = \cdots = u_0 = 0$.

3. *Find a sequence $(a_n) \in \{-1, 1\}^{\mathbb{N}}$ such that*

$$u_n := \left(\sum_{k=1}^{L} h_k u_{n-k} \right) + x - a_n$$

satisfies $u_n \in [-1, 1]$ for each $n \in \mathbb{N}$.

Then the sequence $(a_n)_{n=1}^{\infty}$ satisfies equations (2.2) and (2.3).

Such sequences are known as 'hybrid encoders'. We begin by proving Proposition 2.3.1, this is similar to the proof of Proposition 2.2.1.

Proof. We begin by rearranging condition (3) of Proposition 2.3.1 to give

$$a_n = \left(\sum_{k=1}^{4} h_k u_{n-k} \right) + x - u_n = \left(\sum_{k=0}^{4} h_k u_{n-k} \right) + x.$$

Then we have that

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = \sum_{n=1}^{\infty} \sum_{k=0}^{4} h_k u_{n-k} \beta^{-n} + x \sum_{n=1}^{\infty} \frac{1}{\beta^n} \tag{2.7}$$

where $h_0 = -1$. We now follow the reasoning of the proof of Proposition 2.2.1 exactly, to yield that

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = h_4(u_{-3}\beta^{-1} + u_{-2}\beta^{-2}) + \frac{x}{\beta - 1}.$$

Unlike in Proposition 2.2.1, we also have that $u_{-2} = 0$, so we conclude that

$$\sum_{n=1}^{\infty} a_n \beta^{-n} = h_4(u_{-3}\beta^{-1}) + \frac{x}{\beta - 1},$$

and picking $u_{-3} = \frac{\beta(\beta-2)x}{h_4(\beta-1)}$ yields that

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}.$$

It remains to prove part two of the theorem, that

$$x = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now we have from the condition (3) of Proposition 2.3.1 that

$$\left(\sum_{k=1}^{L} h_k u_{n-k}\right) + x - a_n - u_n = \left(\sum_{k=0}^{L} h_k u_{n-k}\right) + x - a_n = 0.$$

Then

$$\left(\frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k}\right) + x - \frac{1}{N}\sum_{n=1}^{N} a_n = 0.$$

We shall prove that $\sum\limits_{n=1}^{N}\sum\limits_{k=0}^{L} h_k u_{n-k}$ is bounded by some constant independent of N, which will give

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k} = 0$$

and hence that

$$x = \lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

Now we have that

$$\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k} = (u_1 + u_2 + \cdots + u_{N-L})(h_0 + h_1 + \cdots + h_L) + \text{ extra terms},$$

where there are $N(L+1) - ((N-L)(L+1)) = L(L+1)$ extra terms, each of which are bounded in absolute value by

$$(\max_{k\in 0,\cdots,L} |h_k|)(\sup_{n\in\mathbb{N}} u_n) \le \max_{k\in 0,\cdots,L} |h_k| \le M$$

for some constant M. But $h_0 + h_1 + \cdots + h_L = 0$, and so we see that

$$\left|\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k}\right| \leq 0 + L(L+1)M$$

which is independent of N, and so

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{L} h_k u_{n-k} = 0$$

as required. □

Now we prove Theorem 2.1.4.

Proof. The proof of $\sum_{n=1}^{\infty} a_n\beta^{-n} = x$ is almost the same as the proof of Theorem 2.1.1. We follow the construction of Lemma 2.2.3 replacing β_1 by 1 and β_2 by β. This gives

$$P(z) = z^L - \sum_{k=1}^{L} h_k z^{L-k} = (z-1)(z-\beta)(z^2 + az + b),$$

where $b = \dfrac{\beta(1+\beta)}{1+\beta+\beta^2}$ and $c = \dfrac{\beta^2}{1+\beta+\beta^2}$.

Then we have $h_1 = \frac{(1+\beta)(1+\beta^2)}{1+\beta+\beta^2}$ and $h_4 = \frac{-\beta^3}{1+\beta+\beta^2}$. We choose C_1 such that

$$\begin{aligned}\sum_{k=1}^{4}|h_k| &= |h_1| + |h_4| \\ &= \frac{(1+\beta)(1+\beta^2)+\beta^3}{1+\beta+\beta^2} \\ &< \frac{(1+C_1)((1+C_1)^2+1)+(1+C_1)^3}{3} \\ &= 2\end{aligned}$$

where C_1 is the real root of $\frac{(1+x)((1+x)^2+1)+(1+x)^3}{3} = 2$.

Since $\sum_{k=1}^{4}|h_k| = |h_1| + |h_4| < 2$, we can choose $\delta_0 > 0$ such that $\sum_{k=1}^{4}|h_k| = |h_1| + |h_4| \leq 2 - \delta_0 < 2$, thus for any x satisfying $|x| \in [0, \delta_0]$ we have

$$\sum_{n=1}^{4}|h_k| = |h_1| + |h_4| \leq 2 - \delta_0 \leq 2 - |x| < 2$$

The next step is to prove the boundness of u_n. Choosing $\delta_1 = \frac{h_4(\beta-1)}{\beta(\beta-2)}$ we have that

$$|u_{-3}| = \left|\frac{\beta(\beta-2)x}{h_4(\beta-1)}\right| \leq 1.$$

Finally, if we take $\delta = \min\{\delta_0, \delta_1\}$, then this choice can ensure that

$$\sum_{n=1}^{4} |h_k| = |h_1| + |h_4| \leq 2 - x$$

and $|u_{-3}| \leq 1$ hold simultaneously. We also have that $u_{-2} = u_{-1} = u_0 = 0$. We let the sequence (a_n) be chosen as follows:

$$a_n = \begin{cases} -1 & \sum_{k=1}^{L} h_k u_{n-k} + x < 0 \\ +1 & \sum_{k=1}^{L} h_k u_{n-k} + x \geq 0 \end{cases} . \tag{2.8}$$

Then by induction we have that $u_n \in [-1, 1]$ for all $n \in \mathbb{N}$, and hence the conditions of Proposition 2.3.1 are fulfilled and Theorem 2.1.4 is proved. □

2.4 Further Remarks

We have the following further remarks.

(i) We have proved that if β_1 and β_2 are very close to 1 then A_{β_1,β_2} has an interior, but it is unlikely that our bounds are optimal, see for example the diagrams in [39]. Our proof was based on choosing an expansion $(a_n)_{n=1}^{\infty}$ of pairs (x_1, x_2) using equation (3). Perhaps by using a more sophisticated algorithm one may hope to gain a truer picture of the conditions under which our technique can be made to work.

(ii) The IFS which we study is a little different to that studied by Shmerkin in [81], since we use digit set $\{-1, 1\}$ rather than $(-\frac{1}{\gamma}, -\frac{1}{\lambda})$ and $(\frac{1}{\gamma}, \frac{1}{\lambda})$. However such changes of digit set do not affect whether the attractor of the corresponding IFS has an interior.

(iii) We note that if $\beta_1\beta_2 > 2$ then A_{β_1,β_2} cannot have an interior. Güntürk gave a volume covering argument to prove this. In [31] Falconer proved that

$$\overline{\dim_B}(A_{\beta_1,\beta_2}) \leq 1 + \frac{\log \frac{2}{\beta_1}}{\log \beta_2} < 2$$

whenever $\beta_1\beta_2 > 2$ and $1 < \beta_1 < \beta_2 < 2$.

(iv) Our approach to generating sequences $\underline{a}$ which satisfy the conditions of Theorem 2.1.1 is in some sense dynamical, we have an algorithm which chooses a value of (a_n) based on the vector $(u_n, u_{n-1}, u_{n-2}, u_{n-3})$, and then maps this vector to the vector $(u_{n+1}, u_n, u_{n-1}, u_{n-2})$ and repeats the operation. This system is reminiscent of shift radix systems, see [2], except that we have a displacement by a_n.

Our algorithm is far less simple than corresponding algorithms for generating expansions in the one dimensional case, such as the random β-transformation of [15]. It would be nice to have an analogue of the random β-transformation for the higher dimensional case which produces expansions of pairs (x_1, x_2) in a more direct and understandable way.

(v) One can use Remark 2.1.3 to consider when, for a specific sequence (a_n) and real number x, there exist β_1, β_2 such that

$$x = \sum_{i=1}^{\infty} a_i \beta_1^{-i} = \sum_{i=1}^{\infty} a_i \beta_2^{-i}.$$

Given $\underline{a} \in \{-1, 1\}^{\mathbb{N}}$ we define the function $f_{\underline{a}} : (1, 2] \to \mathbb{R}$ by

$$f_{\underline{a}}(\beta) = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

The function $f_{\underline{a}}$ is continuous and differentiable. We call a sequence $\underline{a} = (a_n)$ a *simultaneous encoder* of x if there exists there exist $1 < \beta_1 < \beta_2 < 2$ such that $x = f_{\underline{a}}(\beta_1) = f_{\underline{a}}(\beta_2)$. By Remark 2.1.3, for any $1 < \beta_1 < \beta_2 < 1 + C$ and any $x \in [-0.16, 0.16]$ one can find a simultaneous encoder $\underline{a}$ of x satisfying $x = f_{\underline{a}}(\beta_1) = f_{\underline{a}}(\beta_2)$. By the extreme value theorem, the function $f_{\underline{a}}$ has global extrema in $[\beta_1, \beta_2]$. We let $\beta, \beta_0 \in [\beta_1, \beta_2]$ be the values where the global minimum and global maximum take place. Let $y_1 = f_{\underline{a}}(\beta)$ and $y_2 = f_{\underline{a}}(\beta_0)$. Then by the intermediate value theorem, the sequence $\underline{a}$ is a simultaneous encoder for all $z \in (y_1, y_2)$. Thus, if we define the set

$$E_{\underline{a}} := \{x \in \mathbb{R} : \underline{a} \text{ is a simultaneous encoder of } x\}.$$

then the above argument shows that either $E_{\underline{a}}$ is empty, or is a single point or contains an interval.

(vi) In [42, 43, 41], Hare and Sidorov investigated similar problems. They were able to improve the bound of C, i.e. in Corollary 2.1.2 our C is about 0.05 while their bound is around 0.2. Moreover, they considered the higher dimensional case and posed many interesting problems in their papers.

Chapter 3

Univoque points for self-similar sets

Abstract

Let $K \subseteq \mathbb{R}$ be the unique attractor of an iterated function system. We consider the case where K is an interval and study those elements of K with a unique coding. We prove under mild conditions that the set of points with a unique coding can be identified with a subshift of finite type. As a consequence of this, we can show that the set of points with a unique coding is a graph-directed self-similar set in the sense of Mauldin and Williams [62]. The theory of Mauldin and Williams then provides a method by which we can explicitly calculate the Hausdorff dimension of this set. Our algorithm can be applied generically, and our result generalizes the work of [19], [49], [48], and [84].

3.1 Introduction

Let $\{f_j\}_{j=1}^m$ be an iterated function system (IFS) of similitudes which are defined on $\mathbb{R}$ by

$$f_j(x) = r_j x + a_j,$$

where the similarity ratios satisfy $0 < r_j < 1$ and the translation parameter $a_j \in \mathbb{R}$. It is well known that there exists a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^{m} f_j(K). \tag{3.1}$$

We call K the self-similar set or attractor for the IFS $\{f_j\}_{j=1}^m$, see [46] for further details. We refer to the elements of $\{f_j(K)\}_{j=1}^m$ as first-level intervals when K is an interval. An IFS is called homogeneous if all the similarity ratios r_j are equal. For any $x \in K$, there exists a sequence $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ such that

$$x = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(K).$$

We call such a sequence a coding of x. The attractor K defined by (3.1) may equivalently be defined to be the set of points in $\mathbb{R}$ which admit a coding, i.e., we can define a surjective projection map between the symbolic space $\{1, \ldots, m\}^{\mathbb{N}}$ and the self-similar set K by

$$\pi((i_n)_{n=1}^\infty) := \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

An $x \in K$ may have many different codings, if $(i_n)_{n=1}^\infty$ is unique then we call x a univoque point. The set of univoque points is called the univoque set and we denote it by $U_{\{f_j\}_{j=1}^m}$, i.e.,

$$U_{\{f_j\}_{j=1}^m} := \Big\{x \in K : \text{there exists a unique } (i_n)_{n=1}^\infty \in \{1,\ldots,m\}^{\mathbb{N}} \text{ satisfying } x = \lim_{n\to\infty} f_{i_1}\circ\cdots\circ f_{i_n}(0)\Big\}.$$

Let $\widetilde{U}_{\{f_j\}_{j=1}^m} := \pi^{-1}(U_{\{f_j\}_{j=1}^m})$. If there is no risk of confusion, we denote $U_{\{f_j\}_{j=1}^m}$ and $\widetilde{U}_{\{f_j\}_{j=1}^m}$ by U and $\widetilde{U}$ respectively. With a little effort, it may be shown that π is a homeomorphism between the set of unique codings $\widetilde{U}$ and the univoque set U. In this paper we present a general algorithm for determining the Hausdorff dimension of U when K is an interval. Unless stated otherwise, in what follows we will always assume that our IFS is such that K is an interval.

Part of our motivation comes from the study of β-expansions. Given $\beta > 1$ and $x \in \big[0, (\lceil\beta\rceil - 1)(\beta-1)^{-1}\big]$ there exists a sequence $(a_n)_{n=1}^\infty \in \{0,\ldots,\lceil\beta\rceil-1\}^{\mathbb{N}}$ such that

$$x = \sum_{n=1}^{\infty} a_n \beta^{-n}.$$

We call such a sequence a β-expansion of x. Expansions in non-integer bases were pioneered in the papers of Renyi [78] and Parry [70]. For more information, see [26], [15], [84] and the references therein.

We can study β-expansions via the IFS

$$g_j(x) = \frac{x+j}{\beta},\ j \in \{0,\ldots,\lceil\beta\rceil-1\}.$$

The self-similar set for this IFS is the interval $\mathcal{A}_\beta := \big[0, (\lceil\beta\rceil-1)(\beta-1)^{-1}\big]$. For β-expansions, it is clear that any first-level interval $g_j(\mathcal{A}_\beta)$ intersects at most two other first-level intervals simultaneously. For any $M \in \mathbb{N}$, it is straightforward to show that

$$g_{i_1}\circ\cdots\circ g_{i_M}(0) = \sum_{n=1}^{M} i_n\beta^{-n}.$$

Therefore, $\lim_{n\to\infty} g_{i_1}\circ g_{i_2}\circ\ldots\circ g_{i_n}(0) = x$ if and only if $(i_n)_{n=1}^\infty$ is a β-expansion of x. Much work has been done on the set of points with a unique β-expansion. Glendinning and Sidorov classified in [38] those $\beta \in (1,2)$ for which the Hausdorff dimension of the univoque set is positive. However, their approach did not allow them to calculate the Hausdorff dimension. This result was later generalized to arbitrary $\beta > 1$ in [55]. Daróczy and Kátai [19] offered an approach to the problem of calculating the dimension when $\beta \in (1,2)$, but they could only calculate the dimension when β is a special purely Parry number (β is a Parry number if the β-expansions of 1 in base β is eventually periodic). Making use of similar ideas, Kallós [49], [48] showed that for $\beta > 2$:

(1) If $\beta \in [\lceil\beta\rceil - 1, (\lceil\beta\rceil - 1 + \sqrt{(\lceil\beta\rceil)^2 - 2\lceil\beta\rceil + 5})]$, then the Hausdorff dimension of the univoque set is equal to $(\log(\lceil\beta\rceil - 2))(\log\beta)^{-1}$.

(2) If $\beta \in [(\lceil\beta\rceil - 1 + \sqrt{(\lceil\beta\rceil)^2 - 2\lceil\beta\rceil + 5}), \lceil\beta\rceil)$, and β is a purely Parry number, Kallós can still find the dimensional result.

Zou, Lu and Li [90] considered the univoque set for a class of homogeneous self-similar sets with overlaps. Their motivation was to generalize Glendinning and Sidorov's result [38]. In some cases, they provide an explicit formula for the dimension of the univoque set. What made the work of Zou, Lu and Li different from the work of Glendinning and Sidorov, was that the self-similar sets they considered were of Lebesgue measure zero. Their approach was similar to Glendinning and Sidorov's, the crucial technique is finding a new characterisation of the univoque set. Recently, in the setting of β-expansions, Kong and Li [54] generalised Kallós' results, their approach made use of different techniques which were based on the admissible blocks introduced by Komornik and de Vries [84]. They were able to calculate the dimension of the univoque set for β within certain intervals. These intervals cover almost all β, even some bases for which $\widetilde{U}$ is not a subshift of finite type.

In the papers mentioned above, the approaches given always have two points in common. The first is that their method depends on finding a symbolic characterisation of the univoque set via the greedy algorithm. For general self-similar sets such a characterization is not possible. The second point is that in their setup every first-level interval has at most two adjacent first-level intervals intersecting it. For general self-similar sets, some first-level intervals may intersect many first-level intervals simultaneously. As such their methods do not simply translate over and we have to find a new approach.

The goal of this chapter is to give a general algorithm for calculating the Hausdorff dimension of the univoque set when the self-similar set is an interval. When this algorithm can be implemented it identifies the univoque set with a subshift of finite type. With this new symbolic representation, we can use a directed graph to represent the set $\widetilde{U}$, see for example [58]. We then show that U is a graph-directed self-similar set in the sense of Mauldin and Williams [62]. Using the results of [62] we can then calculate $\dim_H(U)$ explicitly. This algorithm can be implemented in a generic sense that we will properly formalise later.

The structure of the chapter is as follows. In section 3.2 we describe the self-similar set via a dynamical system and state Theorem 3.2.4 which is our main result. In section 3.3 we prove Theorem 3.2.4 and demonstrate that for most cases, the hypothesis of Theorem 3.2.4 is satisfied (Corollary 3.3.1). In section 3.4 we restrict to β-expansions and provide an alternative methodology for determining the subshift of finite type representation of $\widetilde{U}$. In section 3.5 we introduce the definition of a graph-directed self-similar set and illustrate how to calculate the dimension of the univoque set using this tool. In section 3.6 we give a worked example.

After completion of this work we were made aware of the work of Bundfuss, Krüger and Troubetzkoy [10]. They were concerned with iterating maps on a manifold M and the set of $x \in M$ that were never mapped into some hole. Theorem 3.2.4 is essentially a consequence of Proposition 4.1 [10]. However, all of our results regarding calculating $\dim_H(U)$ and the identification of the univoque set with a graph-directed self-similar set are completely new.

3.2 Preliminaries and Main Results

In this section we describe the elements of our attractor in terms of a dynamical system. Recall that $K = [a, b] \subseteq \mathbb{R}$ is the attractor of our IFS $\{f_j\}_{j=1}^m$, i.e.,

$$K = \bigcup_{j=1}^{m} f_j(K).$$

Define $T_j(x) := f_j^{-1}(x) = (x - a_j)r_j^{-1}$ for each $1 \leq j \leq m$. We denote the concatenation $T_{i_n} \circ \ldots \circ T_{i_1}(x)$ by $T_{i_1 \ldots i_n}(x)$. The following lemma provides an alternative formulation of codings of elements of K in terms of the maps T_j.

Lemma 3.2.1. *Let $x \in K$. Then $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ is a coding for x if and only if $T_{i_1 \ldots i_n}(x) \in K$ for all $n \in \mathbb{N}$.*

Proof. Assume $x \in K$ has a coding $(i_n)_{n=1}^\infty$. By the continuity of the maps f_j the following equation holds for all $n \in \mathbb{N}$:

$$T_{i_1 \ldots i_n}(x) = \lim_{M \to \infty} f_{i_{n+1}} \circ \cdots \circ f_{i_M}(0).$$

Obviously the right hand side of the above equation is an element of K. As such we have deduced the rightwards implication.

Now let us assume that $(i_n)_{n=1}^\infty$ is such that $T_{i_1 \ldots i_n}(x) \in K$ for all $n \in \mathbb{N}$. Let $x_n = T_{i_1 \ldots i_n}(x)$. We observe the following:

$$|f_{i_1} \circ \cdots \circ f_{i_n}(0) - x| = |f_{i_1} \circ \cdots \circ f_{i_n}(0) - f_{i_1} \circ \cdots \circ f_{i_n}(x_n)| \leq r^n |x_n|.$$

Where $r = \max_{1 \leq j \leq m} r_j$. By our assumption $x_n \in K$, in which case $|x_n|$ can be bounded above by a constant independent of x and n. It follows that $\lim_{n \to \infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = x$ and $(i_n)_{n=1}^\infty$ is a coding for x. □

The dynamical interpretation provided by Lemma 3.2.1 will make our proofs and exposition far more succinct. The following proposition is a straightforward consequence of Lemma 3.2.1.

Proposition 3.2.2. *Let $x \in K$. There exists $(i_n)_{n=1}^N \in \{1, \ldots, m\}^N$ and distinct $k, l \in \{1, \ldots, m\}$ satisfying $T_{i_1 \cdots i_N k}(x) \in K$ and $T_{i_1 \cdots i_N l}(x) \in K$ if and only if $x \notin U$.*

Let $I_j = f_j(K)$, I_j is precisely the set of points that are mapped back into K by T_j. The following reformulation of U is a consequence of Proposition 3.2.2:

$$U = \Big\{ x \in K : \nexists 1 \leq k < l \leq m \text{ and } (i_n)_{n=1}^N \text{ such that } T_{i_1 \cdots i_N}(x) \in I_k \cap I_l \Big\}. \tag{3.2}$$

By Lemma 3.2.1 we know that every $x \in K$ has an infinite sequence of maps which under finite iteration always map x back into K. What (3.2) states is that if $x \in U$, then each of these finite iterations always avoid the intersections of the $I_j' s$.

In what follows we always assume that there are s pairs $(i_k, j_k) \in \{1, \ldots, m\}^2$ such that $H_k := I_{i_k} \cap I_{j_k} \neq \emptyset$ and $i_k \neq j_k$. In fact we will always assume that we are in the case where each $H_k := [a_k, b_k]$ is a nontrivial interval and is contained in the interior of K. There is no loss of generality in making this assumption. If for some $[a_k, b_k]$ it is true that $a_k = a$ or $b_k = b$, then the conclusion of Theorem 3.2.4 is still true under an appropriately modified hypothesis. The argument required is the same as that given below except for an additional notational consideration. We may also assume that the elements of $\{H_k\}$ are pairwise disjoint and that they are located from left to right in K. In the dynamical literature these regions H_k are commonly referred to as switch regions, see for example [15]. We give a simple example to illustrate the above.

Example 3.2.3. *Let $[0, 1/(\beta-1)]$ be the attractor of $\{f_0(x) = \beta^{-1}x,\ f_1(x) = \beta^{-1}(x+1)\}$, where $1 < \beta < 2$. Then we define $T_0(x) = \beta x,\ T_1(x) = \beta x - 1$, see Figure 3.1.*

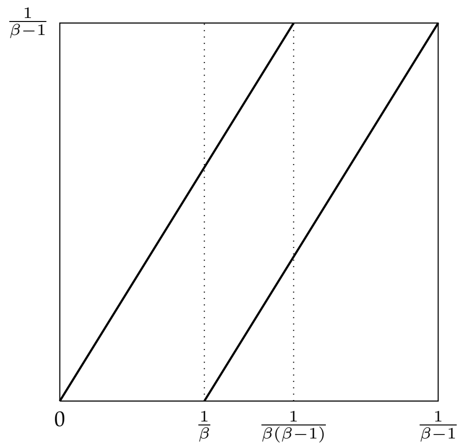


FIGURE 3.1: The dynamical system for $\{T_0,\ T_1\}$

From this figure, we know that $f_0([0, 1/(\beta-1)]) \cap f_1([0, 1/(\beta-1)]) = [1/\beta,\ 1/\beta(\beta-1)]$. For any $x \in [1/\beta,\ 1/\beta(\beta-1)]$ both T_0 and T_1 map x into $[0, 1/(\beta-1)]$.

Now we can state our first result. Recall that $\widetilde{U}$ is defined to be the set of symbolic codings of points in U.

Theorem 3.2.4. *For each a_k and b_k, suppose there exist two finite sequences $(\eta_1 \dots \eta_P) \in \{1, \dots, m\}^P$, $(\omega_1 \dots \omega_Q) \in \{1, \dots, m\}^Q$ such that*

$$T_{\eta_1 \dots \eta_P}(a_k) \in \bigcup_{i=1}^{s} (a_i,\ b_i) \tag{3.3}$$

and

$$T_{\omega_1 \dots \omega_Q}(b_k) \in \bigcup_{i=1}^{s} (a_i,\ b_i), \tag{3.4}$$

Then $\widetilde{U}$ is a subshift of finite type.

3.3 Proof of Theorem 3.2.4

We give a constructive proof of Theorem 3.2.4.

Proof. By our assumptions and the continuity of the T_j's, we can find $\delta_{a_k} > 0$ and $\delta_{b_k} > 0$ such that

$$T_{\eta_1 \dots \eta_P}(a_k - \delta_{a_k}, a_k) \subset \bigcup_{i=1}^{s} (a_i,\ b_i)$$

and

$$T_{\omega_1\ldots\omega_Q}(b_k, b_k+\delta_{b_k}) \subset \bigcup_{i=1}^{s}(a_i,\, b_i).$$

Moreover, we may assume that $[a_k-\delta_{a_k}, b_k+\delta_{b_k}] \cap [a_j-\delta_{a_j}, b_j+\delta_{b_j}] = \emptyset$ for each $1 \le k < j \le s$. Let $\delta = \min_{1\le k\le s}\{\delta_{a_k},\, \delta_{b_k}\}$ and $H = \cup_{i=1}^{s}[a_i-\delta, b_i+\delta]$. By the monotonicity of the T_j's and Proposition 4.2.2 it is clear that any element of H is mapped into the switch region, therefore H is in the complement of the univoque set. We partition K via the iterated function system. For any L we have

$$K = \bigcup_{(i_1,\ldots,i_L)\in\{1,\ldots,m\}^L} f_{i_1}\circ\cdots\circ f_{i_L}(K).$$

We also assume L is sufficiently large such that $|f_{i_1}\circ\cdots\circ f_{i_L}(K)| < \delta$ for all $(i_1,\ldots,i_L) \in \{1,\ldots,m\}^L$. We have a corresponding partition of the symbolic space $\{1,\ldots,m\}^{\mathbb{N}}$ provided by the cylinders of length L. For each $(i_1,\ldots,i_L) \in \{1,\ldots,m\}^L$ let

$$C_{i_1\ldots i_L} = \Big\{(x_n) \in \{1,\ldots,m\}^{\mathbb{N}} : x_n = i_n \text{ for } 1 \le n \le L\Big\}.$$

The set $\{C_{i_1\ldots i_L}\}_{(i_1,\ldots,i_L)\in\{1,\ldots,m\}^L}$ is a partition of $\{1,\ldots,m\}^{\mathbb{N}}$, and $f_{i_1}\circ\cdots\circ f_{i_L}(K) = \pi(C_{i_1\ldots i_L})$. Let

$$\mathbb{F} = \Big\{(i_1,\ldots,i_L) \in \{1,\ldots,m\}^L : f_{i_1}\circ\cdots\circ f_{i_L}(K) \cap \bigcup_{k=1}^{s} H_k \neq \emptyset\Big\}$$

and

$$\mathbb{F}' = \bigcup_{(i_1,\ldots,i_L)\in\mathbb{F}} \pi(C_{i_1\ldots i_L}).$$

By our assumptions on the size of our cylinders the following inclusions hold

$$\bigcup_{k=1}^{s} H_k \subset \mathbb{F}' \subset H.$$

Using these inclusions it is a straightforward observation that $x \notin U$ if and only if there exists $(\theta_1,\ldots,\theta_{n_1}) \in \{1,\ldots,m\}^{n_1}$ such that $T_{\theta_1\ldots\theta_{n_1}}(x) \in \mathbb{F}'$. Showing there exists $(\theta_1,\ldots,\theta_{n_1}) \in \{1,\ldots,m\}^{n_1}$ such that $T_{\theta_1\ldots\theta_{n_1}}(x) \in \mathbb{F}'$ if and only if x has a coding containing a block from $\mathbb{F}$ is straightforward. If $x \notin U$ then by the above observation, there exists $(\theta_1,\ldots,\theta_{n_1}) \in \{1,\ldots,m\}^{n_1}$ such that $T_{\theta_1\ldots\theta_{n_1}}(x) \in \mathbb{F}'$. Therefore, x has a coding containing a block from $\mathbb{F}$. Going in the opposite direction, suppose that x has a coding $(x_n)_{n=1}^{\infty}$ such that $x_{M+1}\ldots x_{M+L} \in \mathbb{F}$ for some $M \in \mathbb{N}$, then $T_{x_1\ldots x_M}(x) \in \mathbb{F}'$. However, $\mathbb{F}' \subset H$, and as previously remarked $H \subset U^c$, therefore $T_{x_1\ldots x_M}(x) \notin U$ and $x \notin U$. Taking $\mathbb{F}$ to be the set of forbidden words defining a subshift of finite type we see that $\widetilde{U}$ is a subshift of finite type. □

The conditions in Theorem 3.2.4 are met for a large class of self-similar sets, provided that the attractor is an interval. We recall the definition of a universal coding. A coding of x, $(d_n)_{n=1}^{\infty} \in \{1,\ldots,m\}^{\mathbb{N}}$ is called a universal coding for x if given any finite block $(\delta_1,\ldots,\delta_k) \in \{1,\ldots,m\}^k$, there exists j such that $d_{j+i} = \delta_i$ for $1 \le i \le k$. Theorem 1.4 from [4] implies that Lebesgue almost every $x \in K$ has a universal coding. This result implies the following corollary.

Corollary 3.3.1. *For Lebesgue almost every $x \in K$, there exists a sequence $(i_n)_{n=1}^N$ and H_k such that $T_{i_1 \dots i_N}(x)$ is in the interior of H_k.*

Let $\Lambda \subset K$ be the set of full measure described by Corollary 3.3.1. It follows that the hypothesis of Theorem 3.2.4 fails only when an endpoint of an H_k is contained in $K \setminus \Lambda$. There are no obvious obstacles to the endpoints of H_k being members of Λ. As such we expect the conditions of Theorem 3.2.4 to be satisfied most of the time. As we will see in section 4, a stronger statement holds when we restrict to β-expansions.

Remark 3.3.2. *In [19],[49],[48] and [54], they all consider homogeneous IFS's. We however allow the similarity ratios to be different. Another advantage of our method is that we can find the forbidden blocks quickly and uniformly.*

Remark 3.3.3. *The method used in Theorem 3.2.4 cannot easily be implemented when K is not an interval. The key difficulty is that when we construct the neighborhoods of a_k and b_k, the images of these neighborhoods may not be mapped into $\cup_{k=1}^s H_k$ by the same maps that worked for a_k and b_k.*

In higher dimensions we can prove an analogous result. The proof requires a minor modification. We give the detailed proof. Let H be the union of the switch regions, and ∂H be the boundary of H.

Theorem 3.3.4. *Let $\{f_i(x)\}_{i=1}^m$ be the self-similar set of $K \subset \mathbb{R}^d, d \geq 2$. Assume K is a cube or a rectangle. For any $x \in \partial H$, if there exists $(i_1 i_2 \cdots, i_n)$ such that $T_{i_1 i_2 \cdots, i_n}(x) \in H^\circ$, then $\tilde{U}$ is a subshift of finite type.*

The main difference between one dimension and higher dimensions is that for the later case the boundary of the switch regions has infinitely many points.

Proof. We assume without loss of generality that there is only one switch region, i.e. H has only one component. It is easy to see that ∂H is a compact set. By the assumption of the theorem and the continuity of each $T_i, 1 \leq i \leq m$, it follows that for any $x \in \partial H$, we can find some $(i_1 i_2 \cdots i_n)$ and $\delta_x > 0$ such that

$$T_{i_1 i_2 \cdots i_n}(B(x, \delta_x)) \subset H^\circ.$$

On the other hand,

$$\cup_{x \in \partial H} B(x, \delta_x)$$

is a cover of ∂H. By the compactness of ∂H, we can find a finite cover $\cup_{i=1}^N B(x_i, \delta_{x_i})$ such that $\partial H \subset \cup_{i=1}^N B(x_i, \delta_{x_i})$ and that for any $x \in \partial H$, there exists i_0, $x \in B(x_{i_0}, \delta_{i_0})$. Now the remaining proof is almost the same as the proof of Theorem 3.2.4. □

For self-affine sets which are simple sets, for instance, rectangles, cubes (see the definition of self-affine sets in [34]), our theorem still holds. However, in this case we do not know whether an analogue of Corollary 3.3.1 is true.

Using a similar idea to the proof of Theorem 3.2.4, we can prove following theorem:

Theorem 3.3.5. *If our attractor K is an interval, then U is closed if and only if $\widetilde{U}$ is a subshift of finite type.*

We give a simple proof of this interesting result.

Proof. If $\widetilde{U}$ is a subshift of finite type, then $\widetilde{U}$ is closed as the forbidden blocks cannot appear in the limit of sequences of $\widetilde{U}$. Hence U is also closed due to the fact that U is homeomorphic to $\widetilde{U}$.

Conversely, suppose U is closed, or equivalently U^c is open. For each interval H_k the endpoints a_k and b_k are in U^c. It follows that there exists $\delta_{a_k} > 0$ and $\delta_{b_k} > 0$ such that $(a_k - \delta_{a_k}, a_k) \subset U^c$ and $(b_k, b_k + \delta_{b_k}) \subset U^c$. The remaining proof, i.e., finding the forbidden blocks, is the same as the proof of Theorem 3.2.4. □

This theorem generalises Komornik and de Vries' statement, see the corresponding equivalent statements in [84, Theorem 1.8]. Moreover, in higher dimensions similar results still holds.

Corollary 3.3.6. *If K is a rectangle or a cube, then U is closed if and only if $\widetilde{U}$ is a subshift of finite type.*

3.4 β-expansions case

In this section we restrict to β-expansions and give an alternative method for determining the subshift of finite type representation of $\widetilde{U}$. Firstly, we recall the relevant IFS for studying β-expansions. Given $\beta > 1$ define the IFS:

$$g_j(x) = \frac{x+j}{\beta},\ j \in \{0, \dots, \lceil\beta\rceil - 1\}.$$

The self-similar set for this IFS is the interval $\mathcal{A}_\beta = [0, (\lceil\beta\rceil - 1)(\beta - 1)^{-1}]$.

We now define greedy and lazy expansions.

Definition 3.4.1. *The greedy map $G : \mathcal{A}_\beta \to \mathcal{A}_\beta$, is defined by*

$$G(x) = \begin{cases} \beta x \mod 1, & x \in [0,1) \\ \beta x - [\beta], & x \in \left[1, \frac{\lceil\beta\rceil - 1}{\beta - 1}\right] \end{cases}$$

For any $n \geq 1$ and $x \in \mathcal{A}_\beta$, we define $a_n(x) = [\beta G^{n-1}(x)]$, where $[y]$ denotes the integer part of $y \in \mathbb{R}$. We then have

$$\begin{aligned} x &= \frac{a_1(x)}{\beta} + \frac{G(x)}{\beta} \\ &= \frac{a_1(x)}{\beta} + \frac{a_2(x)}{\beta^2} + \frac{G^2(x)}{\beta^2} \\ &\vdots \\ &= \sum_{n=1}^{\infty} \frac{a_n(x)}{\beta^n}. \end{aligned}$$

The sequence $(a_n)_{n=1}^\infty \in \{0, \dots, \lceil\beta\rceil - 1\}^{\mathbb{N}}$ generated by G is called the greedy expansion or greedy coding. The orbit $\{G^n(x)\}_{n=1}^\infty$ is called the greedy orbit of x.

Similarly, we define the lazy map and the corresponding lazy expansion as follows.

Definition 3.4.2. *The lazy map $L : \mathcal{A}_\beta \to \mathcal{A}_\beta$, is defined by*

$$L(x) = \begin{cases} \beta x & x \in \left[0, \frac{(\lceil\beta\rceil-1)}{(\beta(\beta-1))}\right] \\ \beta x - b_j & x \in \left(\frac{\lceil\beta\rceil-1}{\beta(\beta-1)} + \frac{b_j-1}{\beta}, \frac{\lceil\beta\rceil-1}{\beta(\beta-1)} + \frac{b_j}{\beta}\right] \textit{for } b_j \in \{1,\ldots,[\beta]\} \end{cases}$$

Here b_j is an element of our set of digits. By Lemma 4.2.1, for each $x \in \mathcal{A}_\beta$ we can generate a β-expansion for x by iterating L. The β-expansion generated by L is called the lazy expansion of x. The orbit $\{L^n(x)\}_{n=1}^\infty$ is called the lazy orbit of x.

Given $i \in \{0,\ldots,\lceil\beta\rceil-1\}$ it is a simple calculation to show that $g_i(\mathcal{A}_\beta) \cap g_j(\mathcal{A}_\beta) \neq \emptyset$ if and only if $j = i-1, i, i+1$. In which case the nontrivial switch regions are of the form:

$$S_l = \Big[\frac{l}{\beta}, \frac{\lceil\beta\rceil-1}{\beta(\beta-1)} + \frac{l-1}{\beta}\Big],$$

for some $1 \leq l \leq \lceil\beta\rceil - 1$. We remark that the greedy and lazy maps only differ on the intervals S_l. Clearly an $x \in \mathcal{A}_\beta$ is a univoque point if and only if it is never mapped into an interval S_l. This implies the following important technical result.

Proposition 3.4.3. *Given $x \in K$, we have that $x \in U$ if and only if its greedy and lazy expansions coincide.*

This simple observation will be a powerful tool, it allows us to give a lexicographic characterisation of $\widetilde{U}$ which will help us determine our subshift of finite type representation.

Each element of $U \setminus \{0, (\lceil\beta\rceil-1)(\beta-1)^{-1}\}$ is eventually mapped into $[(\lceil\beta\rceil-\beta)(\beta-1)^{-1}, 1]$ by G and L (as by definition the orbits of G and L coincide for univoque points). Moreover, once inside this interval they are not mapped out, see [38, page 2]. Therefore, due to the countable stability of Hausdorff dimension ([34, page 32]), to determine the Hausdorff dimension of U, we only need to find the Hausdorff dimension of $U \cap [(\lceil\beta\rceil-\beta)(\beta-1)^{-1}, 1]$. We denote $U \cap [(\lceil\beta\rceil-\beta)(\beta-1)^{-1}, 1]$ and $\pi^{-1}(U \cap [(\lceil\beta\rceil-\beta)(\beta-1)^{-1}, 1])$ by U_β and $\widetilde{U}_\beta$ respectively.

Let $(\alpha_n)_{n=1}^\infty$ be the greedy expansion of 1 and $(\varepsilon_n)_{n=1}^\infty = (\overline{\alpha_n})_{n=1}^\infty = (\lceil\beta\rceil-1-\alpha_n)_{n=1}^\infty$. We are interested in giving conditions when $\widetilde{U}_\beta$ is a subshift of finite type. In this paper, we consider only the collection of β's such that the greedy expansion of 1 is infinite. If the greedy expansion of 1 is finite, then $\widetilde{U}_\beta$ may not be a subshift of finite type, the good examples are Tribonacci numbers, see Theorems 1.2 and 1.5 from [84]. Let σ denote the usual shift map. We now introduce the lexicographic ordering on infinite sequences, given $(a_n)_{n=1}^\infty, (b_n)_{n=1}^\infty \in \{0,\ldots,\lceil\beta\rceil-1\}^{\mathbb{N}}$ we say that $(a_n)_{n=1}^\infty < (b_n)_{n=1}^\infty$ if there exists $M \in \mathbb{N}$ such that $(a_1,\ldots,a_M) = (b_1,\ldots,b_M)$ and $a_{M+1} < b_{M+1}$. There also exists a lexicographic ordering on finite sequences, this is defined in the obvious way.

Theorem 3.4.4. *If there exists $M \in \mathbb{N}$ such that $(\varepsilon_{M+n})_{n=1}^\infty > (\alpha_n)_{n=1}^\infty$ then $\widetilde{U}_\beta$ is a subshift of finite type. More specifically, there exists $p > M$ such that*

$$\widetilde{U}_\beta = \Big\{(d_n)_{n=1}^\infty : (\varepsilon_1,\ldots,\varepsilon_p,(\lceil\beta\rceil-1)^\infty) < \sigma^k((d_n)_{n=1}^\infty) < (\alpha_1,\ldots,\alpha_p,(0)^\infty) \textit{for any } k \geq 0\Big\}.$$

The hypothesis of Theorem 3.4.4 is in fact equivalent to that of Theorem 3.2.4. We omit the details of this equivalence as it hinders our exposition. The spirit of this proof is similar to the proof of Theorem 3.2.4. Heuristically speaking, we are giving an equivalent argument but expressed in the language of sequences. When expressed in this language the proof becomes more concise and provides a more efficient method for determining the set of forbidden words.

The following criterion of the unique codings is pivotal. In fact, in [19], [49], [48] and [54], their approaches strongly depend on this criterion.

Theorem 3.4.5. *Let $(a_n)_{n=1}^\infty$ be a coding of $x \in [0, (\lceil\beta\rceil)(\beta-1)^{-1}]$. Then $(a_n)_{n=1}^\infty \in \widetilde{U}_\beta$ if and only if*

$$(a_{k+1}a_{k+2}\cdots) < (\alpha_n)_{n=1}^\infty$$

wherever $a_k < \alpha_1$,

$$\overline{(a_{k+1}a_{k+2}\cdots)} < (\alpha_n)_{n=1}^\infty$$

wherever $a_k > 0$.

This theorem is a corollary of Theorem 1.1 [84].

Proof of Theorem 3.4.4. From Theorem 3.4.5 we know that

$$\widetilde{U}_\beta = \{(a_n)_{n=1}^\infty : (\varepsilon_n)_{n=1}^\infty < \sigma^k((a_n)_{n=1}^\infty) < (\alpha_n)_{n=1}^\infty \text{ for any } k \geq 0\}.$$

Let M be as in the statement of Theorem 3.4.4, there exists $p > M$ such that

$$(\varepsilon_{M+1}, \ldots, \varepsilon_p) > (\alpha_1, \ldots, \alpha_{p-M}).$$

Recall $(\varepsilon_n) = (\overline{\alpha_n})$, thus we equivalently have

$$(\varepsilon_1, \ldots, \varepsilon_{p-M}) > (\alpha_{M+1}, \ldots, \alpha_p).$$

We shall prove that $\widetilde{U}_\beta = U'_\beta$ where

$$U'_\beta := \left\{(a_n)_{n=1}^\infty : (\varepsilon_1, \ldots, \varepsilon_p, (\lceil\beta\rceil - 1)^\infty) < \sigma^k((a_n)_{n=1}^\infty) < (\alpha_1, \ldots, \alpha_p, (0)^\infty) \text{ for any } k \geq 0\right\}.$$

By Theorem 3.4.5 we have $U'_\beta \subseteq \widetilde{U}_\beta$, therefore it suffices to prove the opposite inclusion.

Let $(a_n)_{n=1}^\infty \in \widetilde{U}_\beta$ and assume that $(a_n)_{n=1}^\infty \notin U'_\beta$. Therefore, we have $\sigma^{k_0}((a_n)_{n=1}^\infty) \geq (\alpha_1, \ldots, \alpha_p, (0)^\infty)$ or $(\varepsilon_1, \ldots, \varepsilon_p, (\lceil\beta\rceil - 1)^\infty) \geq \sigma^{k_0}((a_n)_{n=1}^\infty)$ for some $k_0 \geq 0$. But this is not possible. For instance, if $(\varepsilon_1, \ldots, \varepsilon_p, (\lceil\beta\rceil - 1)^\infty) \geq \sigma^{k_0}((a_n)_{n=1}^\infty)$ then $(a_{k_0+1}, \ldots, a_{k_0+p}) = (\varepsilon_1, \ldots, \varepsilon_p)$ since $(a_n)_{n=1}^\infty \in \widetilde{U}_\beta$. Hence,

$$(a_{k_0+M+1}, \ldots, a_{k_0+p}) = (\varepsilon_{M+1}, \ldots, \varepsilon_p) > (\alpha_1, \ldots, \alpha_{p-M})$$

but this contradicts the fact that $(a_n)_{n=1}^\infty \in \widetilde{U}_\beta$. The other case is proved similarly. As such we may conclude that $\widetilde{U}_\beta \subseteq U'_\beta$. □

Remark 3.4.6. *Theorem 3.4.4 implies that when the greedy orbit of 1 falls into the switch region, then $\widetilde{U}_\beta$ is a subshift of finite type. This theorem is a little weaker than Komornik and de Vries' statement, see [84, Theorem 1.8]. However, we can find the forbidden blocks more quickly. It is not necessary to use Theorem 3.4.5 to find the subshift of finite type, while Komornik and de Vries' method depends on it. We have proved in Theorem 3.2.4 that for self-similar sets a similar idea still works. Moreover, we mentioned in Theorem 3.3.5 that U is closed if and only if $\widetilde{U}$ is a subshift of finte type. As such Theorem 3.2.4 can be interpreted as a generalization of Komornik and de Vries' result to the setting of self-similar sets.*

Remark 3.4.7. *In [48], Kallós used similar ideas to prove a similar theorem. However, the argument in the proof of Theorem 3.4.4 may not be applied in other complicated settings as generally we cannot find criteria for unique codings in terms of a symbolic representation.*

In the setting of β-expansions, let

$$A = \{\beta \in (1, \infty) : \text{the expansion of 1 in base } \beta \text{ is unique}\}.$$

When $1 < \beta < 2$, Erdös and Joó showed that the Lebesgue measure of A is zero. Later Schmeling [80] proved the Lebesgue measure of A is zero. In fact Schmeling proved a much stronger result. This statement implies the following corollary.

Corollary 3.4.8. *For almost every $\beta \in (1, \infty)$ the hypothesis of Theorem 3.4.4 are satisfied.*

This should be compared with Corollary 3.3.1. Unlike Corollary 3.3.1, Corollary 3.4.8 allows us to conclude that we can apply Theorem 3.4.4 for a Lebesgue generic parameter in an appropriate parameter space.

3.5 Hausdorff dimension of univoque set

3.5.1 Graph-directed self-similar sets

Before demonstrating how to calculate the dimension of a univoque set, we introduce the notion of a graph-directed self-similar set. The terminology we use is taken from [62].

A graph-directed construction in $\mathbb{R}$ consists of the following.

1. A finite union of bounded closed intervals $\cup_{u=1}^{n} J_u$ such that the J_u are pairwise disjoint.

2. A directed graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and edge set E. Moreover, we assume that for any $u \in V$ there is some $v \in V$ such that $(u, v) \in E$.

3. For each edge $(u, v) \in E$ there exists a similitude $f_{u,v}(x) = r_{uv}x + a_{uv}$, where $r_{uv} \in (0, 1)$ and $a_{uv} \in \mathbb{R}$. Moreover, for each $u \in V$ the set $\{f_{u,v}(J_v) : (u, v) \in E\}$ satisfies the strong separation condition, i.e.,

$$\bigcup_{(u,v)\in E} f_{u,v}(J_v) \subseteq J_u,$$

   and the elements of $\{f_{u,v}(J_v) : (u, v) \in E\}$ are pairwise disjoint.

As is the case for self-similar sets, we have the following result.

Theorem 3.5.1. *For each graph-directed construction, there exists a unique vector of non-empty compact sets $(C_1, \ldots, C_n)$ such that, for each $u \in V$, $C_u = \bigcup_{(u,v)\in E} f_{u,v}(C_v)$.*

We let $K^* := \cup_{u=1}^{n} C_u$ and call it the graph-directed self-similar set of this construction. To each graph-directed construction we can associate a weighted incidence matrix A. This matrix is defined by $A = (r_{u,v})_{(u,v)\in V\times V}$, for simplicity, we assume that $r_{u,v} = 0$ if $(u, v) \notin E$. For each $t \geq 0$ we define another adjacency matrix $A^t = (a_{t,u,v})_{(u,v)\in V\times V}$, where $a_{t,u,v} = r_{u,v}^t$. Let $\Phi(t)$ denote the largest nonnegative eigenvalue of A^t. A graph is strongly connected if for any two vertices $u, v \in V$, there exists a directed path from u to v. A strongly connected component of G is a subgraph C of G such that C is strongly connected, let $SC(G)$ be the set of all the strongly connected components of G. Now we state the main result of [62].

Theorem 3.5.2. *For every graph-directed construction such that G is strongly connected, the Hausdorff dimension of K^* is t_0, where t_0 is uniquely defined by $\Phi(t_0) = 1$.*

If the graph-directed construction G is not strongly connected, we still have a similar result. As is well known, a directed graph G must have a strongly connected component, see [58, section 4.4.]. In which case the following theorem makes sense.

Theorem 3.5.3. *If the G in our graph-directed construction is not strongly connected, let $t_1 = \max\{t_C : \Phi(t_C) = 1, C \in SC(G)\}$, where $\Phi(t_C)$ is the largest eigenvalue of the adjacency matrix of the strongly connected subgraph C. Then $\dim_H(K^*) = t_1$.*

Proof. We can decompose G into several subgraphs which are each strongly connected, then this theorem holds due to Theorem 3.5.2 and the countable stability of the Hausdorff dimension. □

3.5.2 Calculating the dimension of the univoque set

Now we show how to construct a graph-directed self-similar set using the subshift of finite type representation of $\widetilde{U}$ obtained in Theorem 3.2.4. As we will see, in this case, the graph-directed self-similar set K^* mentioned above will in fact equal U.

Recall the projection map $\pi : \{1, \ldots, m\}^{\mathbb{N}} \to K$ is defined by

$$\pi((i_n)_{n=1}^{\infty}) = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

We use the same notation as in the proof of Theorem 3.2.4. Let $\mathbb{F}$ be the set of finite forbidden blocks and $W = \{1, \ldots, m\}^L \setminus \mathbb{F}$. The set of vertices in our directed graph will be:

$$V = \Big\{(a_1, \ldots, a_{L-1}) \in \{1, \ldots, m\}^{L-1} : \text{there exists } a_L \in \{1, \ldots, m\} \text{ such that } (a_1, \ldots, a_{L-1}, a_L) \in W\Big\}.$$

We now define our edges. For any two vertices $u, v \in V$, $u = (u_1, \ldots, u_{L-1})$, $v = (v_1, \ldots v_{L-1})$, we draw an edge from u to v and label this edge (u, v), if $(u_2, \ldots, u_{L-1}) = (v_1, \ldots v_{L-2})$ and $(u_1, \ldots, u_{L-1}, v_{L-1}) \in W$. Here we should note that the vertices u, v which are from V are blocks, while in the definition of a graph-directed construction u and v refer to integers.

Now we have defined our edges and hence we have constructed a directed graph $G = (V, E)$. If there exists a vertex $u \in V$ for which there is no $v \in V$ satisfying $(u, v) \in E$, then we remove u from our vertex set. Removing this u does not change any of the latter results, so without loss of generality we may assume that for every $u \in V$ there exists $v \in V$ for which (u, v) is an allowable edge. In which case we satisfy 2 in the above definition of a graph-directed construction.

Before showing that we satisfy 1 and 3 in the definition of a graph-directed construction we recall an important result from [58]. We define an infinite path in our graph G to be a sequence $((u^n, v^n))_{n=1}^{\infty} \in E^{\mathbb{N}}$ such that $v^n = u^{n+1}$ for all $n \in \mathbb{N}$, where $u^n = u_1^n u_2^n \cdots u_{L-1}^n$ Define X_G to be

$$X_G := \Big\{(y_n)_{n=1}^{\infty} \in \{1, \ldots, m\}^{\mathbb{N}} : \text{there exists an infinite path } ((u^n, v^n))_{n=1}^{\infty} \in E^{\mathbb{N}} \text{ such that } y_n = u_1^n \text{ for all } n \in \mathbb{N}\Big\}.$$

Theorem 2.3.2 of [58] states the following.

Theorem 3.5.4. *Let G be the directed graph as constructed above. Then $\widetilde{U} = X_G$.*

We define

$$K_u := \left\{ x = \pi((d_n)_{n=1}^{\infty}) : d_i = u_i \text{ for } 1 \leq i \leq L-1 \text{ and } (d_n)_{n=1}^{\infty} \in \widetilde{U} \right\}$$

and

$$J_u := \text{conv}(K_u).$$

Here $u = (u_1, \ldots, u_{L-1}) \in V$ and $\text{conv}(\cdot)$ denotes the convex hull.

Lemma 3.5.5. *Let $u, v \in V$ and $u \neq v$. Then $J_u \cap J_v = \emptyset$.*

Proof. J_u and J_v are the convex hulls of K_u and K_v respectively, as such they are both intervals. We assume that $J_u = [c, d]$ and $J_v = [e, f]$. As K_u is compact, the endpoints of J_u are elements of K_u. Similarly, $e, f \in K_v$. Now we prove that $[c, d] \cap [e, f] = \emptyset$.

If $[c, d]$ and $[e, f]$ intersect in a point then this point must be an endpoint. Without loss of generality assume $d = e$, then $d \in K_u \cap K_v$. However, $K_u \subset U$ and we have a contradiction as $u \neq v$.

Now let us assume J_u and J_v intersect in an interval. Without loss of generality, we assume that $c < e < d$. Since e is a univoque point in K_v, we know by Proposition 3.2.2 that there exists a unique sequence of T_j's of length $L-1$ that map e into K. As $e \in K_v$ this sequence of transformations must be $T_{v_1 \cdots v_{L-1}}$. By our assumption $c < e < d$, therefore by the monotonicity of the maps T_j we have that $T_{u_1 \cdots u_{L-1}}(c) < T_{u_1 \cdots u_{L-1}}(e) < T_{u_1 \cdots u_{L-1}}(d)$. Both $T_{u_1 \cdots u_{L-1}}(c), T_{u_1 \cdots u_{L-1}}(d) \in K$, but as K is an interval this implies $T_{u_1 \cdots u_{L-1}}(e) \in K$, a contradiction. □

By Lemma 3.5.5 we can take $\{J_u\}_{u \in V}$ to be the set of bounded closed intervals required in 1 of the definition of a graph-directed construction.

It remains to show that we satisfy 3 of the definition of a graph-directed construction. First of all we define our similitudes, given an edge $(u, v) \in E$ we define $f_{uv}(x) = r_{u_1} x + a_{u_1}$. The following lemma proves that we satisfy 3 of the graph-directed construction.

Lemma 3.5.6. *Fix $u \in V$. Then $\bigcup_{(u,v) \in E} f_{uv}(J_v) \subseteq J_u$ and $f_{uv}(J_v) \cap f_{uv'}(J_{v'}) = \emptyset$, for all distinct pairs of edges.*

Proof. For the first statement, it is sufficient to prove $\bigcup_{(u,v) \in E} f_{uv}(K_v) \subseteq K_u$. Suppose $(u, v) \in E$ and $x = f_{uv}(y)$ where $y \in K_v$. Let $(y_n)_{n=1}^{\infty} \in \widetilde{U}$ be the unique coding of y. By Theorem 3.5.4 we know that $(y_n)_{n=1}^{\infty} \in X_G$. Let $(x_n)_{n=1}^{\infty}$ be such that $x_1 = u_1$ and $x_i = y_{i-1}$ for $i \geq 2$, then $(x_n)_{n=1}^{\infty}$ is a coding of x. Since $(u, v) \in E$ we have that $(u_2, \ldots, u_{L-1}) = (v_1, \ldots, v_{L-2})$. Moreover, as $(u, v) \in E$ and $(y_n)_{n=1}^{\infty} \in X_G$ then $(x_n)_{n=1}^{\infty} \in X_G$. Using Theorem 3.5.4 again we know that $(x_n)_{n=1}^{\infty} \in \widetilde{U}$, which combined with the observation $(x_1, \ldots, x_{L-1}) = (u_1, \ldots, u_{L-1})$ implies $x \in K_u$.

The second statement is an immediate consequence of Lemma 3.5.5 and the fact that our similitudes are bijections from $\mathbb{R}$ to $\mathbb{R}$ that do not depend on v. □

We have satisfied all of the criteria for a graph-directed construction and may therefore conclude that Theorem 3.5.1 holds. We now show that for our graph construction $K^* = U$. We begin by showing that the K_u's are precisely the C_u's in Theorem 3.5.1.

Lemma 3.5.7. *For each $u \in V$ we have $K_u = \bigcup_{(u,v)\in E} f_{uv}(K_v)$.*

Proof. Let $x \in K_u$ and $(x_n)_{n=1}^{\infty}$ be the unique coding for x. Then $x_n = u_n$ for $1 \leq n \leq L-1$. Let

$$v = (v_1, \ldots, v_{L-1}) = (x_2, \ldots, x_L) = (u_2, \ldots, u_{L-1}, x_L).$$

By Theorem 3.5.4 we have $(x_n)_{n=1}^{\infty} \in X_G$. Therefore $v \in V$ and $(u,v) \in E$. Let $y \in K$ have coding $(x_{n+1})_{n=1}^{\infty}$, $(x_{n+1})_{n=1}^{\infty} \in X_G$ and by Theorem 3.5.4 we know that $(x_{n+1})_{n=1}^{\infty} \in \widetilde{U}$. As $(x_2, \ldots, x_L) = (v_1, \ldots, v_{L-1})$ we can deduce that $y \in K_v$. As $f_{uv}(y) = x$ we have shown that $K_u \subseteq \bigcup_{(u,v)\in E} f_{uv}(K_v)$. The inverse inclusion is proved in Lemma 3.5.6. □

By the uniqueness part of Theorem 3.5.1 we may conclude from Lemma 3.5.7 that the set $\cup_{u=1}^{n} C_u$ in the statement equals $\cup_{u\in V} K_u$. The fact that $U = \cup_{u\in V} K_u$ is immediate from the definition of K_u. As such $U = K^*$ and is the graph directed self-similar set for our construction. Therefore, Theorem 3.5.1 and Theorem 3.5.3 apply and we use them to calculate the Hausdorff dimension of U. We include an explicit calculation in section 6.

Now we give a final remark to finish this section. In [54, 52] Komornik, Kong and Li proved the following interesting result.

Theorem 3.5.8. *There exist intervals for which the function mapping β to the Hausdorff dimension of the univoque set is strictly decreasing.*

This result is somewhat counterintuitive. As β gets the larger, the corresponding switch regions shrink. As such, one might expect that the set of points whose orbits avoid the switch regions, i.e. the univoque set, would be larger. However, Theorem 3.5.8 shows that in terms of Hausdorff dimension this is not always the case.

A similar idea to the proof of Theorem 3.2.4 allows us to recover Theorem 3.5.8 quickly. We only give an outline of this argument. A straightforward manipulation of the formulas given in Theorem 3.5.2 and Theorem 3.5.3, yields that $\dim_H(U_\beta) = \frac{\log \lambda}{\log \beta}$, where λ is the largest eigenvalue of the transition matrix defining our subshift of finite type. Using similar ideas to those given in the proof of Theorem 3.2.4, we can show that if β satisfies the hypothesis of Theorem 3.2.4, then the hypothesis is also satisfied for β' sufficiently close to β. Moreover, a more delicate argument implies that for β' sufficiently close to β, the set of forbidden words for β' equals the set of forbidden words for β. In other words, the subshift of finite type defining the univoque set for β' equals the subshift of finite type defining the univoque set for β. Observing that $\dim_H(U_\beta)$ is decreasing on some sufficiently small interval containing β now follows from the formula stated above.

3.6 Example

In this section, we give an example to show how to calculate the dimension of a univoque set.

Example 3.6.1. *Let $[0, \frac{1}{\beta-1}]$ be the self-similar set with IFS: $\{f_0(x),\ f_1(x)\}$ where*

$$f_0(x) = \frac{x}{\beta},\ f_1(x) = \frac{x+1}{\beta}$$

Let β^* be the unique $\beta \in (1,2)$ satisfing the equation $(111(00001)^\infty)_\beta = 1$. In this case $\beta^* \approx 1.84$. We now calculate $\dim_H(U_{\beta^*})$.

The greedy expansion of 1 in this base is $(\alpha_n)_{n=1}^\infty = (111(00001)^\infty)$, we observe that $(\varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7) > (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. In which case, by Theorem 3.4.4 we have that $\widetilde{U}$ is given by a subshift of finite type. Moreover, in the statement of Theorem 3.4.4 we can take $p = 7$. We now construct the relevant directed graph. In this case our set W is:

$$W = \{(a_1, \ldots, a_7) : (0001111) < (a_1, \ldots, a_7) < (1110000)\},$$

moreover the set of vertices equals

$$V = \{(a_1, \ldots, a_6) : (000111) < (a_1, \ldots, a_6) < (111000)\}.$$

We now construct the edge set in accordance with the construction given in section 3.5.2. In total there are 26 vertices:

$$v_1 = (001001),\ v_2 = (001010),\ v_3 = (001011),\ v_4 = (001100),\ v_5 = (001101),$$

$$v_6 = (010010),\ v_7 = (010011),\ v_8 = (010100),\ v_9 = (010101),\ v_{10} = (010110),$$

$$v_{11} = (011001),\ v_{12} = (011010),\ v_{13} = (011011),\ v_{14} = (100100),\ v_{15} = (100101),$$

$$v_{16} = (100110),\ v_{17} = (101001),\ v_{18} = (101010),\ v_{19} = (101011),\ v_{20} = (101100),$$

$$v_{21} = (101101),\ v_{22} = (110010),\ v_{23} = (110011),\ v_{24} = (110100),\ v_{25} = (110101),$$

$$v_{26} = (110110).$$

We now follow the Mauldin and William approach and construct a 26×26 matrix $(A_{i,j})$, where $A_{i,j} = \dfrac{1}{(\beta^*)^t}$ if there is an edge from vertex v_i to v_j, otherwise, $A_{i,j} = 0$. A computer calculation then yields $\dim_H(U_{\beta^*}) \approx 0.79$.

Chapter 4

Subshift of finite type and self-similar sets

Abstract

Let $K \subset \mathbb{R}$ be a self-similar set generated by some iterated function system. In this chapter we prove, under some assumptions, that K can be identified with a subshift of finite type. With this identification, we can calculate the Hausdorff dimension of K as well as the set of elements in K with unique codings using the machinery of Mauldin and Williams [62]. We give several different applications of our main result. Firstly, we calculate the Hausdorff dimension of the set of points of K with multiple codings. Secondly, in the setting of β-expansions, when the set of all the unique codings is not a subshift of finite type, we can calculate in some cases the Hausdorff dimension of the univoque set. This application generalizes a result of de Vries and Komornik [84]. Thirdly, for the doubling map with asymmetrical holes, we give a sufficient condition such that the attractor can be identified with a subshift of finite type. The third application partially answers a problem posed by Barrera [8]. Fourthly, we can construct the Lipschitz map between two overlapping self-similar sets.

4.1 Introduction

Let $\{f_i\}_{i=1}^m$ be an iterated function system of contractive similitudes on $\mathbb{R}$ defined as

$$f_i(x) = r_i x + a_i, \quad i = 1, \ldots, m, \tag{4.1}$$

where $0 < |r_i| < 1$ is the contractive ratio and $b_i \in \mathbb{R}$. We call K the self-similar set or attractor for the IFS $\{f_j\}_{j=1}^m$, see [46] for further details. An IFS is called homogeneous if all the similarity ratios r_j are equal. For any $x \in K$, there exists a sequence $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ such that

$$x = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0) = \bigcap_{n=1}^{\infty} f_{i_1} \circ \cdots \circ f_{i_n}(K).$$

We call such a sequence a coding of x. The attractor K defined by (4.1) may equivalently be defined to be the set of points in $\mathbb{R}$ which admit a coding, i.e., we can define a surjective projection map between the symbolic space $\{1, \ldots, m\}^{\mathbb{N}}$ and the self-similar set K by

$$\pi((i_n)_{n=1}^\infty) := \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

A point $x \in K$ may have many different codings. If x has a unique coding, then we call x a univoque point. The set of univoque points is called the univoque set, and we denote it by $U_{\{f_j\}_{j=1}^m}$, i.e.,

$$U_{\{f_j\}_{j=1}^m} := \Big\{x \in K : \text{there exists a unique } (i_n)_{n=1}^\infty \in \{1,\dots,m\}^{\mathbb{N}} \text{ satisfying} \\ x = \lim_{n\to\infty} f_{i_1}\circ\cdots\circ f_{i_n}(0)\Big\}.$$

Let $\widetilde{U}_{\{f_j\}_{j=1}^m} := \pi^{-1}(U_{\{f_j\}_{j=1}^m})$. If there is no risk of confusion, we denote $U_{\{f_j\}_{j=1}^m}$ and $\widetilde{U}_{\{f_j\}_{j=1}^m}$ by U and $\widetilde{U}$ respectively.

Calculating the Hausdorff dimension of a self-similar set is a crucial problem in fractal geometry. Usually it is difficult to find the dimension of a self-similar set, especially when serious overlaps occur, see [56, 68, 7, 45] and references therein. In this paper we offer an effective methodology which enables us to find the Hausdorff dimension of many self-similar sets. We give a brief introduction of our idea here. Firstly, we assume that for any $f_i(K)\cap f_j(K)\neq\emptyset$, $i\neq j$, $f_i(K)\cap f_j(K)$ is the union of some exact overlaps (see Definition 4.2.6) and that the endpoints of each $f_i(K), 1\leq i\leq m$, have periodic orbits (see Definitions 4.2.4, 4.2.5). The periodicity of the orbits of the endpoints allows us to give a Markov partition of the self-similar set K. This step is essential as dynamically we transform the original full shift into a subshift of finite type, which allows us to view our attractor as a graph-directed self-similar set [62]. The graph-directed self-similar set satisfies the open set condition due to the Markov property of the partition. As such we can calculate explicitly the Hausdorff dimension of K. A similar idea enables us to find the Hausdorff dimension of the univoque set as well. Analogous results still hold in higher dimensions.

A first application of our main idea is the investigation of the following set,

$$U_k := \{x \in K : x \text{ has exactly } k \text{ codings}\}, k = 1, 2, \cdots, \aleph_0.$$

Considering this set can assist us in a better understanding of the coding space of K. In this chapter we investigate only one example, see Theorem 4.2.21. For more examples, see the next chapter. We give a simple introduction to our result. Suppose that K is the attractor of the following IFS,

$$\left\{f_1(x)=\lambda x, f_2(x)=\lambda x+2\lambda, f_3(x)=\lambda x+3\lambda-\lambda^2, f_4(x)=\lambda x+1-\lambda\right\}.$$

where $0<\lambda<\dfrac{5-\sqrt{21}}{2}$.

Our main result states that for any $k\geq 1$, $\dim_H(U_{2^k}) = \dim_H(U)$. Moreover, $U_i = \emptyset, i\neq 2^s, s\geq 1$, and $U_{\aleph_0}=\emptyset$. As a corollary, we have that any $x\in K$ has either exactly 2^k codings, $k\geq 0$, or has uncountably many codings.

The second application is in the setting of β-expansions. We are able to calculate $\dim_H(U)$ for some cases for which $\tilde{U}$ is not a subshift finite type. We first introduce some notation. Let $\beta\in(1,2)$ and $x\in\mathcal{A}_\beta=[0,(\beta-1)^{-1}]$, we call a sequence $(a_n)_{n=1}^\infty\in\{0,1\}^{\mathbb{N}}$ a β-expansion of x provided

$$x=\sum_{n=1}^\infty \frac{a_n}{\beta^n}.$$

Sidorov [82] proved that Lebesgue almost every point has uncountably many expansions. We still use $\tilde{U}$ and U to denote the set of unique β-expansions and the corresponding univoque set. The dynamical approach is an excellent tool which can generate β-expansions effectively. Define $T_0(x) = \beta x, T_1(x) = \beta x - 1$, see Figure 1. All

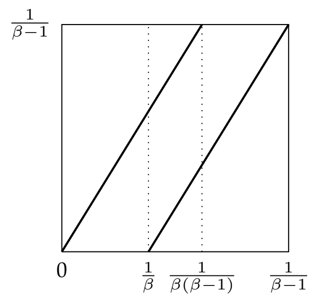


FIGURE 4.1: The dynamical system for $\{T_0, T_1\}$

possible β-expansions can be generated via these two maps, see [15, 17]. We call the common domain of T_0 and T_1, i.e. $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, the switch region. Much work has been done regarding U, see for example [38, 84]. One of the motivations of this paper is to continue the investigation of the Hausdorff dimension of U from the dynamical point of view. In [6], we proved that $\tilde{U}$ is a subshift of finite type if the orbits of the boundary points of the switch region eventually fall into its interior. Schmeling [80] proved that for almost every $\beta \in (1,2)$, the orbit of 1 is dense, which implies that for almost every β, $\tilde{U}$ is a subshift of finite type. On the other hand, if β is a Pisot number, then the associated orbit of 1 may not hit the interior of the switch region. In this case, the idea in [6] cannot be implemented, and we have to find a new approach.

De Vries and Komornik proved in [84] that $\tilde{U}$ is a subshift of finite type if and only if $\beta \in (1,2) \setminus \overline{\mathcal{U}}$, where $\mathcal{U}$ is the set of β for which the β-expansion of 1 is unique. This result allows them to calculate the Hausdorff dimension of U if $\tilde{U}$ is a subshift of finite type. In this chapter, we give a necessary and sufficient condition which can characterize when $\tilde{U}$ is a subshift of finite type in terms of the quasi-greedy orbit of 1, see Theorem 4.3.8. Using this result together with [6, Theorem 2.4], we can give an algorithm to find $\dim_H(U)$ in this case. In some cases when the greedy orbit of 1 is periodic, we are able to calculate $\dim_H(U)$ even when U is not a subshift of finite type. As such we generalize de Vries and Komornik's result concerning the calculation of $\dim_H(U)$.

Some dimensional problems in the open dynamical systems are involved in the third application of this chapter. We only study the doubling map with holes. Let $D(x) = 2x \mod 1$ be the doubling map defined on $[0,1)$. Given any $(a,b) \subset [0,1)$, let

$$J(a,b) = \{x \in [0,1) : D^n(x) \notin (a,b), \forall\, n \geq 0\}.$$

Glendinning and Sidorov [37] gave a complete description of $J(a,b)$. More precisely, given any hole $(a,b) \subset (0,1)$ they can characterize when $J(a,b)$ is of zero or positive Hausdorff dimension. Clark [12] made use of the same techniques and obtained similar results when T is the greedy map. However, both of these papers do not offer an approach to find the exact Hausdorff dimension of $J(a,b)$. The main idea of this paper allows us to give a partial solution to the dimension of the attractor. Moreover, we partially answer Barrera's question [8], i.e. we prove that if the orbits of a and b are eventually periodic, no matter where the hole is located, $J(a,b)$ is isomorphic to a subshift of finite type. Finally, using the main idea of the chapter, we can construct a Lipschitz map between two self-similar sets with overlaps.

This chapter is organized as follows. In section 4.2, we state our main results and give the proofs. We also investigate an example concerning the Hausdorff dimension of U_k. In section 4.3 and section 4.4, we study the case of β-expansions. Firstly we give, from dynamical perspective, an equivalent statement of a result of de Vries and Komornik. Then we implement similar idea which is utilized in section 4.2, and calculate $\dim_H(U)$ when the greedy orbit of 1 is eventually periodic. In section 4.5, we partially answer one problem posed by Barrera. In section 4.6, we discuss the application of constructing a Lipschitz map between two self-similar sets in terms of our main idea. Finally, we give some examples which show the effectiveness of our idea.

4.2 Hausdorff dimension of K and U

4.2.1 Dimension of K

In this section we shall state the main results of our paper. To begin with, we introduce some basic notation. Define $T_j(x) := f_j^{-1}(x) = (x-a_j)r_j^{-1}$ for $x \in f_j(K)$ and $1 \leq j \leq m$. We denote the concatenation $T_{i_n} \circ \ldots \circ T_{i_1}(x)$ by $T_{i_1 \ldots i_n}(x)$. The following lemma was proved in Chapter 3, see also in [6]. The main hypothesis in Chapter 3 is that K is an interval. We emphasize that all the self-similar sets in this paper are not necessarily intervals. The following lemma is still true if K is a general self-similar set.

Lemma 4.2.1. *Let $x \in K$. Then $(i_n)_{n=1}^{\infty} \in \{1, \ldots, m\}^{\mathbb{N}}$ is a coding for x if and only if $T_{i_1 \ldots i_n}(x) \in K$ for all $n \in \mathbb{N}$.*

The dynamical interpretation provided by Lemma 4.2.1 will make our proofs and expositions far more succinct. The following proposition is a straightforward consequence of Lemma 4.2.1.

Proposition 4.2.2. *Let $x \in K$. There exist $(i_n)_{n=1}^{N} \in \{1, \ldots, m\}^N$ and distinct $k, l \in \{1, \ldots, m\}$ satisfying $T_{i_1 \cdots i_N k}(x) \in K$ and $T_{i_1 \cdots i_N l}(x) \in K$ if and only if $x \notin U$.*

Let $I_j = f_j(K)$, I_j is precisely the set of points that are mapped back into K by T_j. The following reformulation of U is a consequence of Proposition 4.2.2:

$$U = \Big\{ x \in K : \nexists 1 \leq k < l \leq m \text{ and } (i_n)_{n=1}^{N} \text{ such that } T_{i_1 \cdots i_N}(x) \in I_k \cap I_l \Big\}. \tag{4.2}$$

By Lemma 4.2.1 we know that every $x \in K$ has an infinite sequence of maps which under finite iterations always map x back into K. What (4.2) states is that if $x \in U$, then each of these finite iterations always avoid the intersections of the $I_j's$.

Motivated by Lemma 4.2.1, we may define the orbits of the points of K.

Definition 4.2.3. *Let $x \in K$ with a coding $(i_n)_{n=1}^{\infty}$, we call the set*

$$\{T_{i_1 \ldots i_n}(x) : n \geq 0\}$$

an orbit set of x.

It is easy to see that for different codings, the orbits of x may be distinct. Now we give some important definitions.

Definition 4.2.4. *Let $A \subset \mathbb{R}$ be a bounded set. We call the minimal and maximal elements of $conv(A)$ the endpoints of A, where $conv(A)$ denotes the convex hull of A.*

Definition 4.2.5. *Let the endpoints of* $\{f_j(K)\}_{j=1}^m$ *be* $a_1 < a_2 < \cdots < a_{n+1}$. *We say* a_i *is a periodic point if the following condition is satisfied: there exists a uniform constant* $M > 0$ *such that for any orbit set of* a_i, *say* F, *the cardinality of* F *is less than* M.

Definition 4.2.6. *We call* $f_{i_1\cdots i_n}(K)$ *an exact overlap if there exists a different*

$$(j_1 \cdots j_t) \in \{1, 2, 3, \cdots, m\}^t$$

for some $t \geq 1$ *such that* $f_{i_1\cdots i_n}(K) = f_{j_1\cdots j_t}(K)$.

Definition 4.2.7. *We say an IFS is an exact overlapping IFS , if for any* $f_i(K) \cap f_j(K) \neq \emptyset$, $i \neq j$, $f_i(K) \cap f_j(K)$ *is the union of some exact overlaps. More precisely, there exist* $\mathbf{t}_q \in \{1, 2 \cdots, m\}^{k_q}, 1 \leq q \leq p$ *such that* $f_i(K) \cap f_j(K) = \cup_{q=1}^p f_{\mathbf{t}_q}(K)$, *where each* $f_{\mathbf{t}_q}(K) = f_{i\, i_2 i_3 \cdots i_{k_q}}(K) = f_{j\, j_2 j_3 \cdots j_r}(K)$ *for some* r.

In section 6, we give some examples of exact overlapping IFS. In this section, we always assume that our IFS is an exact overlapping IFS and that the endpoints of $f_i(K), 1 \leq i \leq m$ are periodic.

The following definition is motivated by the lazy β-expansions [28].

Definition 4.2.8. *Suppose that our IFS is an exact overlapping IFS, and the endpoints of* $f_i(K), 1 \leq i \leq m$ *are periodic. Let* $x \in K$ *with a coding* $(i_n)_{n=1}^\infty$, *and associated orbit*

$$\{T_{i_1 \ldots i_k}(x) : k \geq 0\}.$$

We say $(i_n)_{n=1}^\infty$ *is a lazy coding of* x *if whenever* $T_{i_1 i_2 \cdots i_k}(x) = T_{i_k} \circ \cdots \circ T_{i_1}(x)$ *falls into* $f_i(K) \cap f_j(K), i < j$, *then we implement* T_i *on* $T_{i_1 i_2 \cdots i_k}(x)$, *so the next iteration is* $T_i(T_{i_1 i_2 \cdots i_k}(x))$. *In other words, in every step we choose the smallest digit and iterate the orbit. The associated orbit of* x *is called the lazy orbit . We may denote the lazy orbit of* x *by* $\{T^n(x)\}_{n=1}^\infty$ *if there is no fear of confusion.*

In order to avoid some ambiguity, we adjust our algorithm for some points, i.e. if $f_i(K) \cap f_{i+1}(K) \neq \emptyset$ for some $1 \leq i \leq m$. Let $E_i = f_i(\max K)$, we implement T_{i+1} on E_i. In the remaining of this section, except for these points, we always choose the lazy orbits of points in K. Suppose the lazy orbits of the endpoints of each $f_i(K), 1 \leq i \leq m$, i.e. $a_1 < a_2 < \cdots < a_{n+1}$, are periodic. Since our IFS is an exact overlapping IFS, we may partition the self-similar set via the lazy periodic orbits of $a_1, a_2, \cdots, a_{n+1}$. Without loss of generality we assume the periodic orbits of $\{a_1, a_2, \cdots, a_{n+1}\}$ are $\{b_1, b_2, \cdots, b_{s+1}\}$, i.e.

$$K = \bigcup_{i=1}^{s} ([b_i, b_{i+1}] \cap K).$$

As the endpoints of each $f_i(K), 1 \leq i \leq m$ are periodic, for any $1 \leq i \leq m$ and $1 \leq j \leq s$, we can find some k such that

$$T_i(A_j) = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k},$$

where $A_j = [b_j, b_{j+1}]$. Hence, the collection $\{B_i = [b_i, b_{i+1}] \cap K : 1 \leq i \leq s\}$ is a Markov partition of K. If for some j, $(b_j, b_{j+1}) \cap K$ is empty, then we delete this set from the partition. We suppose without loss of generality that

$$K = \cup_{j=1}^s B_j,$$

and that each set of the partition is not empty. We call the sets $B_j = A_j \cap K, 1 \le j \le s$ the blocks of the Markov partition . We point out here that for the original definition of $T_i, 1 \le i \le m$, the domains are $f_i(K), 1 \le i \le m$. However, since $T_i, 1 \le i \le m$ are linear maps, we can extend the domains to $\mathbb{R}$.

The following lemma is important as it allows us to find the Markov partition of K.

Lemma 4.2.9. *Let our IFS be an exact overlapping IFS. Suppose that the endpoints of each $f_i(K), 1 \le i \le m$ are periodic. Then, for any $1 \le i \le m$ and $1 \le j \le s$, we can find some k such that*

$$T_i(A_j \cap K) = (A_{i_1} \cap K) \cup (A_{i_2} \cap K) \cup \cdots \cup (A_{i_k} \cap K),$$

where

$$T_i(A_j) = A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}$$

for some $i_1, \cdots, i_k$.

Proof.

$$\begin{aligned} T_i(A_j \cap K) &= T_i(A_j) \cap T_i(K) \\ &= (A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_k}) \cap T_i(K) \\ &= (A_{i_1} \cap T_i(K)) \cup (A_{i_2} \cap T_i(K)) \cup \cdots \cup (A_{i_k} \cap T_i(K)) \\ &= (A_{i_1} \cap K) \cup (A_{i_2} \cap K) \cup \cdots \cup (A_{i_k} \cap K) \end{aligned}$$

The first equality holds as each T_i is a bijection. The last equation is due to the fact that for any $1 \le i \le m$, $A_{i_j} \cap T_i(K) = A_{i_j} \cap K, 1 \le j \le k$. □

We denote by S the adjacency matrix of the Markov partition $\{B_i = [b_i, b_{i+1}] \cap K : 1 \le i \le s\}$.

The following example may assist us in understanding the various definitions above.

Example 4.2.10. *Let $q > \dfrac{3+\sqrt{5}}{2}$ be any real number and $\rho = q^{-1}$. Consider the iterated function system defined by*

$$\left\{ f_0(x) = \frac{x}{q}, f_1(x) = \frac{x+1}{q}, f_q(x) = \frac{x+q}{q} \right\}.$$

The convex hull of K is $E = [0, (1-\rho)^{-1}]$. Note that

$$f_0(E) = \left[0, \frac{\rho}{1-\rho}\right], f_1(E) = \left[\rho, \frac{2\rho - \rho^2}{1-\rho}\right], f_q(E) = \left[1, \frac{1}{1-\rho}\right].$$

It is easy to check that $f_0(E) \cap f_1(E) \neq \emptyset$ and $f_0(E) \cap f_q(E) = \emptyset, f_1(E) \cap f_q(E) = \emptyset$ as $q > \dfrac{3+\sqrt{5}}{2}$, see Figure 4.2. Our IFS is an exact overlapping IFS as we have $f_0(K) \cap f_1(K) = f_0 \circ f_q(K) = f_1 \circ f_0(K)$.

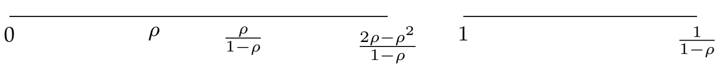


FIGURE 4.2: First iteration

We partition K via $K = A \cup B \cup C \cup D$ where

$$A = [0, \rho] \cap K, B = \left[\rho, \frac{\rho}{1-\rho}\right] \cap K, C = \left[\frac{\rho}{1-\rho}, \frac{2\rho-\rho^2}{1-\rho}\right] \cap K, D = \left[1, \frac{1}{1-\rho}\right] \cap K$$

Simple calculation yields that

$$T_0(A) = A \cup B \cup C, T_0(B) = D, T_1(C) = C \cup D, T_q(D) = A \cup B \cup C \cup D.$$

The adjacency matrix is

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Let Σ be the subshift of finite type generated by S, i.e.

$$\Sigma = \{(i_k)_{k=1}^{\infty} \in \{1, 2, 3, \cdots, s\}^{\mathbb{N}} : S_{i_k, i_{k+1}} = 1 \text{ for any } k\}.$$

We now state one of the main results of this chapter which allows us to calculate the Hausdorff dimension of many fractal sets.

Theorem 4.2.11. *Let K be the self-similar attractor of an exact overlapping IFS. If the endpoints of each $f_j(K)$ are periodic, then K is a graph-directed self-similar set satisfying the open set condition. In particular, the Hausdorff dimension of K can be calculated explicitly in terms of the adjacency matrix S. Moreover, if the associated directed graph of S is strongly connected, then the associated dimensional Hausdorff measure of K is positive and finite.*

Remark 4.2.12. *We may implement an analogous idea in higher dimensions. In [45, Theorem 1.5] Hochman proved the following result: let the IFS on $\mathbb{R}$ be defined by algebraic parameters (the contractive ratios and translations are algebraic numbers), there is a dichotomy: either there are exact overlaps (see the Definition 4.2.6) or the attractor K satisfies $\dim_H(K) = \min\{1, \dim_s(K)\}$, where $\dim_s(K)$ is the similarity dimension which is the unique solution s of $\sum_{i=1}^{m} |r_i^s| = 1$. We will note that our algorithm is effective when the attractor is generated by an exact overlapping IFS, see some examples in section 4.6.*

Before demonstrating how to calculate the dimension of K, we recall the notation of graph-directed self-similar set, see Chapter 3. By virtue of Lemma 4.2.9, we can define an adjacency matrix S which characterizes the relationship between the different $B_i = A_i \cap K$. Equivalently, we may define a directed graph (V, E) such that each B_i is associated with a vertex V_i in this graph. Using the Markov property of the partition, one has an edge from vertex V_i to vertex V_j if there exists some $T_{i,j} \in \{T_1, T_2, \cdots, T_m\}$ such that $B_j \subset T_{i,j}(B_i)$. This allows us to find a similitude, say $g_{i,j}$, associated to the edge $V_i \to V_j$. For instance, if $T_{i,j}(x) = T_k(x)$ for some $1 \leq k \leq m$, then $g_{i,j}(x) = f_k(x)$. Recall that $A_i = [b_i, b_{i+1}], 1 \leq i \leq s$, and note that $\{A_i\}_{i=1}^{s}$ are disjoint except for the endpoints of the intervals, and also forms a Markov partition for the system $\{T_1, T_2, \cdots, T_m\}$. By means of Theorem 3.5.1, for the directed graph (V, E) with associated similitudes between vertices, there exists a unique vector of non-empty compact sets $(C_1, \ldots, C_s)$ such that, for each $u \in V$, $C_u = \bigcup_{(u,v)\in E} g_{u,v}(C_v)$. We denote $K^* = \cup_{i=1}^{s} C_i$ and have following result.

Lemma 4.2.13. $K^* = K$.

Proof. For any $x \in K$, then there exists a lazy coding (i_n) such that

$$x = \lim_{n\to\infty} f_{i_1} \circ \cdots \circ f_{i_n}(0).$$

Dynamically we can find the lazy orbit of x, i.e.

$$\{T_{i_1\ldots i_k}(x) : k \geq 0\},$$

such that for each k, $T_{i_1\ldots i_k}(x)$ falls into some $B_i = A_i \cap K, 1 \leq i \leq s$. As such in the graph (V, E), we can find an infinite path

$$V_{j_1} \xrightarrow{g_{j_1,j_2}} V_{j_2} \xrightarrow{g_{j_2,j_3}} V_{j_3} \xrightarrow{g_{j_3,j_4}} V_{j_4} \cdots V_{j_n} \xrightarrow{g_{j_n,j_{n+1}}} \cdots .$$

such that $x = \lim_{n\to\infty} g_{j_1,j_2} \circ \cdots \circ g_{j_n,j_{n+1}}(0)$. Each $g_{j_n,j_{n+1}} \in \{f_1, \cdots, f_m\}$. Hence $x \in K^*$, which yields $K \subset K^*$. The converse statement $K^* \subset K$ is proved similarly. □

Now we prove that K^* satisfies the open set condition in terms of the Markov property of the partition.

Lemma 4.2.14. *Let $\{O_i = (b_i, b_{i+1})\}_{i=1}^s$. Then for each O_i, we have that $\bigcup_j g_{i,j}(O_j) \subset O_i$, and the union is disjoint, i.e., the graph-directed self-similar set K^* satisfies the open set condition, where $g_{i,j}(x) = f_k(x)$ for some $1 \leq k \leq m$.*

Proof. The union is disjoint as $\{O_j\}$ is. It remains to prove the inclusion. However, this is due to the Markov property of the partition. More precisely, we have $T_{i,j}(O_i) = \bigcup_j(O_j)$, i.e., $\bigcup_j T_{i,j}^{-1}(O_j) \subset O_i$, where $T_{i,j} \in \{T_1, T_2, \cdots, T_m\}$. Hence, we may define some $g_{i,j}(x) = f_k(x), 1 \leq k \leq m$ satisfying $\bigcup_j g_{i,j}(O_j) \subset O_i$. □

Proof of Theorem 4.2.11. The proof follows from Lemma 4.2.13 and Lemma 4.2.14. □

Remark 4.2.15. *We can apply our idea to many different IFS's. In section 6 we give various examples that were investigated earlier by other authors, but with different methods. We however use the same argument which enables us to find the Hausdorff dimension of K quickly, uniformly as well as effectively.*

4.2.2 Dimension of U

We turn to demonstrating how to find the dimension of the univoque set. In Chapter 3, we gave a method which can calculate $\dim_H(U)$. However, in that chapter we assumed K is an interval. In this section, the self-similar sets are not necessarily intervals. We consider the dimension of U under the same assumptions, i.e. our IFS is an exact overlapping IFS, and the endpoints of $f_i(K), 1 \leq i \leq m$ are periodic. However for simplicity we assume that the periodic orbits of the endpoints of $f_i(K), 1 \leq i \leq m$, never fall into $\cup_{i\neq j} f_i(K) \cap f_j(K)$ as under this assumption each non-empty $f_i(K) \cap f_j(K)$ is a set of the Markov partition. If the periodic orbits hit $\cup_{i\neq j} f_i(K) \cap f_j(K)$, then we can discuss the Hausdorff dimension of U in a similar way. We assume that there are t non-empty $f_i(K) \cap f_j(K)$. We call them the switch regions, and denote them by $\widehat{B_1}, \widehat{B_2}, \cdots, \widehat{B_t}$, more precisely, $\widehat{B_j} = B_{i_j}, 1 \leq j \leq t$. These sets are the $i_1, i_2, \cdots i_t$-th sets of the Markov partition.

Recall the adjacency matrix S corresponding to the Markov partition. The corresponding subshift of finite type is

$$\Sigma = \{(i_k)_{k=1}^{\infty} \in \{1, 2, 3, \cdots, s\}^{\mathbb{N}} : S_{i_k,i_{k+1}} = 1 \text{ for any } k\}.$$

We note that the digit $1 \leq i \leq s$ is associated with the block B_i.

Definition 4.2.16. *Suppose K is the attractor of an exact overlapping IFS, and the endpoints of each $f_i(K), 1 \leq i \leq m$ are periodic. Let S be the adjacency matrix of the Markov partition, recall that the indices $i_1, \cdots, i_t$ correspond to the switch regions $B_{i_1} = \widehat{B_1}, B_{i_2} = \widehat{B_2}, \cdots, B_{i_t} = \widehat{B_t}$. In the matrix S, remove the i_j-th row and i_j-th column for $1 \leq j \leq t$, and denote this new matrix by $S^{'}$. The corresponding subshift of finite type is denoted by $\Sigma^{'}$.*

Now we can give the dimensional result for U.

Theorem 4.2.17. *If the IFS of K is an exact overlapping IFS, and the orbits of endpoints of each $f_i(K), 1 \leq i \leq m$ are periodic, then apart from a countable set, U is a graph-directed self-similar set satisfying the open set condition.*

Lemma 4.2.18. $\dim_H(U) \leq \dim_H(\pi(\Sigma^{'}))$.

Proof. For any $x \in U$, there exists a unique coding of x, say $(i_n) \in \{1, \cdots, m\}^{\mathbb{N}}$, such that $T_{i_1 i_2 \cdots i_n}(x) \notin \widehat{B_1} \cup \cdots \cup \widehat{B_t}$ for any $n \geq 1$. By the definition of $\Sigma^{'}$, we have $\dim_H(U) \leq \dim_H(\pi(\Sigma^{'}))$. □

Lemma 4.2.19. *There exists a countable set C such that*

$$\pi(\Sigma^{'}) \subset U \cup C.$$

Proof. Let $x \in \pi(\Sigma^{'})$. By the definition of $\Sigma^{'}$, there exists $(i_n) \in \{1, \cdots, m\}^{\mathbb{N}}$, such that $T_{i_1 i_2 \cdots i_n}(x) \notin \widehat{B_1} \cup \cdots \cup \widehat{B_t}$ for any $n \geq 1$. If the orbit of x never hits the endpoints of each $B_j, 1 \leq j \leq s$, then x is a univoque point, i.e. $x \in U$. If there exists some n_0 such that $T_{i_1 i_2 \cdots i_{n_0}}(x)$ hits some endpoint of $B_j, 1 \leq j \leq s$, then we have

$$x \in \cup_{n=1}^{\infty} \cup_{(i_1 \cdots i_n) \in \{1,2,\cdots,m\}^n} f_{i_1 \cdots i_n}(F),$$

where F is the set of all the endpoints of $\cup_{j=1}^{s} B_j$. □

Now the remaining proof of Theorem 4.2.17 is analogous to Theorem 4.2.11.

Remark 4.2.20. *In the proof of Theorems 4.2.17 and 4.2.11, we choose the lazy orbits. This is not necessary, for instance, we may choose the greedy algorithm, see the definition in the next section.*

4.2.3 Points in K with multiple codings

In this subsection we give an application of Theorem 4.2.17. Let

$$U_k := \{x \in K : x \text{ has exactly } k \text{ codings}\}, k = 1, 2, \cdots, \aleph_0.$$

Simple analysis enables us to find the Hausdorff dimension of U_k for some cases. For simplicity, we consider an example. This example contains some key ideas which are useful to analyze U_k. Suppose that K is the attractor of the following IFS,

$$\left\{ f_1(x) = \lambda x, f_2(x) = \lambda x + 2\lambda, f_3(x) = \lambda x + 3\lambda - \lambda^2, f_4(x) = \lambda x + 1 - \lambda \right\}.$$

where $0 < \lambda < \dfrac{5 - \sqrt{21}}{2}$.

The convex hull of K is $E = [0, 1]$.

$$f_1(E) = [0, \lambda], f_2(E) = [2\lambda, 3\lambda],$$

$$f_3(E) = [3\lambda - \lambda^2, 4\lambda - \lambda^2], f_4(E) = [1 - \lambda, 1].$$

The first iteration of this IFS is

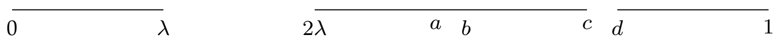


FIGURE 4.3: First iteration of K

where $a = 3\lambda - \lambda^2, b = 3\lambda, c = 4\lambda - \lambda^2, d = 1 - \lambda$.

Note that $f_2 \circ f_4 = f_3 \circ f_1$. Hence, we can partition K via

$$K = A \cup B \cup C \cup D \cup E,$$

where $A = [0, \lambda] \cap K, B = [2\lambda, 3\lambda - \lambda^2] \cap K, C = [3\lambda - \lambda^2, 3\lambda] \cap K, D = [3\lambda, 4\lambda - \lambda^2] \cap K, E = [1 - \lambda, 1] \cap K$. These blocks have following relations: $T_1(A) = A \cup B \cup C \cup D \cup E$, $T_2(B) = A \cup B \cup C \cup D, T_2(C) = E, T_3(D) = B \cup C \cup D \cup E, T_4(E) = A \cup B \cup C \cup D \cup E$. The adjacency matrix is

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Since $C = [3\lambda - \lambda^2, 3\lambda] \cap K = f_2 \circ f_4(K) = f_3 \circ f_1(K) = f_2(K) \cap f_3(K)$, we can define

$$S' = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We have the following result.

Theorem 4.2.21. *For any $k \geq 1$, $\dim_H(U_{2^k}) = \dim_H(U) = \dfrac{\log(2 + \sqrt{2})}{-\log \lambda}$, where $2 + \sqrt{2}$ is the spectral radius of S'. Moreover, $U_i = \emptyset, i \neq 2^k, k \geq 0$, and $U_{\aleph_0} = \emptyset$.*

Corollary 4.2.22. *For any $x \in K$, x has $2^k, k \geq 0$, codings or has uncountable codings.*

To prove the above results, we need several lemmas.

Lemma 4.2.23. *Suppose $x \in [0, \lambda] \cap K$ has exactly k codings, $k \geq 1$. Then $x + 1 - \lambda \in [1 - \lambda, 1] \cap K$ also has exactly k codings. Similarly, we can prove that if a point $y \in [1 - \lambda, 1] \cap K$ has exactly k different codings, then $y - (1 - \lambda)$ also has exactly k different codings.*

Proof. Suppose that $x \in [0, \lambda] \cap K$ has k codings. Every coding begins with digit 1, i.e. $(1 i_2 i_3 i_4 \cdots)$. We define a corresponding coding $(4 i_2 i_3 i_4 \cdots)$. Note that

$$x = f_1(\lim_{n \to \infty} f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_n}(0)),$$

hence, we have

$$x + 1 - \lambda \quad = \quad f_4(\lim_{n\to\infty} f_{i_2} \circ f_{i_3} \circ \cdots \circ f_{i_n}(0))$$

Therefore, $(4x_2x_3x_4\cdots)$ is a coding of $x+1-\lambda$. By the definition of $T_i, 1 \leq i \leq 4$, it follows that $T_1(x) = T_4(x+1-\lambda)$. In other words, after the first iteration of the expanding maps, the orbits of x and $x+1-\lambda$ coincide. Thus, if $x \in [0,\lambda] \cap K$ has exactly k codings, then $x+1-\lambda \in [1-\lambda, 1] \cap K$ also has precisely k codings. □

Lemma 4.2.24. $\dim_H(U_2) = \dim_H(U \cap (U+1-\lambda))$.

Proof. It is sufficient to prove that

$$U_2 = \cup_{n=1}^{\infty} \cup_{(i_1 i_2 \cdots i_n)\in D} f_{i_1 i_2 \cdots i_n} \circ f_2(U \cap (U+1-\lambda)),$$

where D is the collection of all possible finite blocks appearing in unique codings of K.

For any $x \in U_2$, there exists a finite block $(i_1 i_2 \cdots i_n) \in D$ such that $T_{i_1 i_2 \cdots i_n}(x)$ falls into $f_2(K) \cap f_3(K)$ for the first time. Since x has exact two codings, it follows that $T_2(T_{i_1 i_2\cdots i_n}(x)), T_3(T_{i_1 i_2\cdots i_n}(x)) \in U$. By the definition of $T_i, 1 \leq i \leq 4$, we have $T_2(x) = T_3(x) + 1 - \lambda$. Hence, $T_3(T_{i_1 i_2 \cdots i_n}(x)) + 1 - \lambda = T_2(T_{i_1 i_2\cdots i_n}(x))$, which yields $T_2(T_{i_1 i_2 \cdots i_n}(x)) \in U \cap (U+1-\lambda)$, i.e. $x \in f_{i_1 i_2 \cdots i_n} \circ f_2(U \cap (U+1-\lambda))$. The converse inclusion can be proved similarly. □

Lemma 4.2.23 together with Lemma 4.2.24 imply the following lemma.

Lemma 4.2.25. *$U \cap (U+1-\lambda)$ is exactly all the univoque points in $f_4(K)$, i.e.*

$$U \cap (U+1-\lambda) = \{x \in K : x \text{ has a unique coding which has form } (4x_2x_3\cdots)\},$$

where $(x_2x_3\cdots)$ is a unique coding. Moreover,

$$\dim_H(U_2) = \dim_H(U \cap (U+1-\lambda)) = \dim_H(U).$$

Lemma 4.2.26. *For any $k \geq 1$, $\dim_H(U_{2^k}) \leq \dim_H(U)$.*

Proof. The lemma immediately follows from the following inclusion:

$$U_{2^k} \subset \cup_{n=1}^{\infty} \cup_{(i_1 i_2\cdots i_n)\in\{1,2,3,4\}^n} f_{i_1 i_2 \cdots i_n}(U).$$

□

We now prove by induction that for any $k \geq 2$, $\dim_H(U_{2^k}) = \dim_H(U)$.

Lemma 4.2.27. *For any $k \geq 1$, we have $f_2 \circ f_4(U_{2^k}) \subset U_{2^{k+1}}$.*

Proof. Given any $x \in f_2 \circ f_4(U_{2^k})$, since $f_2 \circ f_4 = f_3 \circ f_1$, and $f_2 \circ f_4(K)$ does not intersect with $f_i \circ f_j(K) \in \cup_{(p,q)\notin\{(2,4),(3,1)\}}\{f_p \circ f_q(K)\}$, it follows that the first two digits of x should be 24 or 31. Therefore,

$$x = f_2 \circ f_4(y) = f_3 \circ f_1(y),$$

where $y \in U_{2^k}$. It is easy to see that $T_2(x) = f_4(y)$ and $T_3(x) = f_1(y)$ have exactly 2^k expansions, respectively. Hence, x has precisely 2^{k+1} expansions. □

The first statement of Theorem 4.2.21 now is a straightforward result of Lemma 4.2.27, 4.2.26 and 4.2.25. It remains to prove that $U_i = \emptyset, i \neq 2^k, k \geq 1$. First we show that

Lemma 4.2.28. $U_{2i+1} = \emptyset, i \in \mathbb{N}$.

Proof. We start with proving $U_3 = \emptyset$. If $x \in U_3 \neq \emptyset$, then we can find some finite block $(i_1 i_2 \cdots i_n)$ such that $T_{i_1 i_2 \cdots i_n}(x)$ falls into $f_2(K) \cap f_3(K)$ for the first time. Since $x \in U_3$, it follows that either $T_3(T_{i_1 i_2 \cdots i_n}(x))$ has a unique coding and $T_2(T_{i_1 i_2 \cdots i_n}(x))$ has exactly two codings, or $T_2(T_{i_1 i_2 \cdots i_n}(x))$ has a unique coding and $T_3(T_{i_1 i_2 \cdots i_n}(x))$ has exactly two codings. We prove that these two cases are impossible. By Lemma 4.2.23, it is enough to consider the case $T_3(T_{i_1 i_2 \cdots i_n}(x))$ has a unique coding, and $T_2(T_{i_1 i_2 \cdots i_n}(x))$ has two codings. Note that

$$T_3((T_{i_1 i_2 \cdots i_n}(x))) + 1 - \lambda = T_2(T_{i_1 i_2 \cdots i_n}(x)).$$

As $T_{i_1 i_2 \cdots i_n}(x) \in f_2(K) \cap f_3(K)$, we have $T_3((T_{i_1 i_2 \cdots i_n}(x))) \in [0, \lambda] \cap K$. By Lemma 4.2.23, we know that

$$T_3((T_{i_1 i_2 \cdots i_n}(x))) + 1 - \lambda = T_2(T_{i_1 i_2 \cdots i_n}(x))$$

has a unique coding, which leads to a contradiction. Thus, $U_3 = \emptyset$. For a general odd number $2k+1$, the proof is similar. For if $U_{2k+1} \neq \emptyset$, then we can find a point $x \in U_{2k+1}$ and a finite sequence $(i_1 i_2 \cdots i_n)$ such that $T_{i_1 i_2 \cdots i_n}(x)$ falls into $f_2(K) \cap f_3(K)$ for the first time. Then, $T_2(T_{i_1 i_2 \cdots i_n}(x))$ has exactly a_0 expansions while $T_3(T_{i_1 i_2 \cdots i_n}(x))$ has $2k + 1 - a_0$ expansions, where $1 \leq a_0 \leq 2k$. However, by the same argument as above $T_2(T_{i_1 i_2 \cdots i_n}(x))$ and $T_3(T_{i_1 i_2 \cdots i_n}(x))$ have exactly the same number of different expansions, leading to $a_0 = 2k + 1 - a_0$ which is impossible. Hence, $U_{2k+1} = \emptyset$. □

Lemma 4.2.29. $U_{2i} = \emptyset, i \geq 3$*, where* $2i \neq 2^s$ *for any* $s \geq 1$.

Proof. By assumption $2i = 2^\ell(2m + 1)$ for some $\ell \geq 1$ and $m \geq 1$. For each fixed m, the proof is done by induction on ℓ. For $\ell = 1$ we have $2i = 2(2m+1)$. If $x \in U_{2(2m+1)}$, then there exists a finite sequence $(j_1 j_2 \cdots j_n)$ such that $T_{j_1 j_2 \cdots j_n}(x)$ falls into $f_2(K) \cap f_3(K)$ for the first time. By a similar argument as in Lemma 4.2.28, the points $T_2(T_{j_1 j_2 \cdots j_n}(x))$ and $T_3(T_{j_1 j_2 \cdots j_n}(x))$ have exactly $i = 2m + 1$ expansions, which is impossible since $U_{2m+1} = \emptyset$. Suppose it is true for $2i = 2^\ell(2m + 1)$, i.e. $U_{2^\ell(2m+1)} = \emptyset$, and assume $x \in U_{2^{\ell+1}(2m+1)}$. Then there exists a finite sequence $(j_1 j_2 \cdots j_n)$ such that $T_{j_1 j_2 \cdots j_n}(x)$ falls into $f_2(K) \cap f_3(K)$ for the first time, and as above $T_2(T_{j_1 j_2 \cdots j_n}(x))$ and $T_3(T_{j_1 j_2 \cdots j_n}(x))$ have exactly $i = 2^\ell(2m+1)$ expansions, which is impossible since $U_{2^\ell(2m+1)} = \emptyset$. Thus, $U_{2^{\ell+1}(2m+1)} = \emptyset$. □

Finally, we end the proof of Theorem 4.2.21 with the following lemma.

Lemma 4.2.30. $U_{\aleph_0} = \emptyset$.

Proof. Note that the switch region is $f_2 \circ f_4(K) = f_3 \circ f_1(K) = f_2(K) \cap f_3(K)$. We decompose the switch region by

$$f_2(K) \cap f_3(K) = \Big[\{3\lambda - \lambda^2\} \cup \{3\lambda\}\Big] \cup \Big[(f_2(K) \cap f_3(K)) \setminus (\{3\lambda - \lambda^2\} \cup \{3\lambda\})\Big].$$

Note that $3\lambda - \lambda^2$ has exactly two codings (241^∞) and (31^∞), and 3λ also has exactly two codings (24^∞) and (314^∞). Assume $x \in K$ has infinitely many codings, we will show x has uncountably many codings. By hypothesis, we can find an orbit of x, for simplicity we denote it by $\{T^n(x)\}_{n=0}^\infty$, that hits the switch region infinitely many times. The orbit $\{T^n(x)\}_{n=0}^\infty$ cannot hit $3\lambda - \lambda^2$ or 3λ as $3\lambda - \lambda^2$ and 3λ have two exactly codings, respectively. Hence, $\{T^n(x)\}_{n=0}^\infty$ hits $\{(f_2(K) \cap f_3(K)) \setminus (\{3\lambda - \lambda^2\} \cup \{3\lambda\})\}$ for infinitely many times. Since $f_2 \circ f_4(K) = f_3 \circ f_1(K) = f_2(K) \cap f_3(K)$, we can replace the finite block 24 by 31 for infinitely many times in the coding of x, implying that x has uncountably many codings. So either x has a finite number of expansions or x has uncountably many, hence $U_{\aleph_0} = \emptyset$. □

Remark 4.2.31. *Although for different self-similar sets, analyzing the multiple codings of K may vary from each other, the main ideas, i.e. Lemmas 4.2.24, 4.2.23, 4.2.26 are very useful to study the multiple codings of K.*

4.3 Dynamical discription of $\tilde{U}$

In this section we concentrate on β-expansions, and give a new dynamical criterion under which $\tilde{U}$ is a subshift of finite type. The motivation of this section is to generalize a result of de Vries and Komornik regarding the calculation of $\dim_H(U)$. We begin with the definition of the greedy expansions and quasi-greedy expansions.

Definition 4.3.1. *Let $1 < \beta < 2$, the greedy map $G : \mathcal{A}_\beta = [0, (\beta-1)^{-1}] \to \mathcal{A}_\beta$, is defined by*

$$G(x) = \begin{cases} \beta x \mod 1 & x \in [0,1) \\ \beta x - 1 & x \in \left[1, \frac{1}{\beta-1}\right] \end{cases}$$

For any $n \geq 1$ and $x \in \mathcal{A}_\beta$, we define $a_n(x) = \lfloor \beta G^{n-1}(x) \rfloor$, where $\lfloor y \rfloor$ denotes the integer part of $y \in \mathbb{R}$. We then have

$$\begin{aligned} x &= \frac{a_1(x)}{\beta} + \frac{G(x)}{\beta} = \frac{a_1(x)}{\beta} + \frac{a_2(x)}{\beta^2} + \frac{G^2(x)}{\beta^2} \\ &\vdots \\ &= \sum_{n=1}^{\infty} \frac{a_n(x)}{\beta^n} \end{aligned}$$

The sequence $(a_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$ generated by G is called the greedy expansion or greedy coding. The orbit $\{G^n(x)\}_{n=1}^\infty$ is called the greedy orbit of x.

Remark 4.3.2. *In this section, we consider only $\frac{1+\sqrt{5}}{2} < \beta < 2$ as $\tilde{U} = \{0, (\beta-1)^{-1}\}$ if $1 < \beta \leq \frac{1+\sqrt{5}}{2}$, see [38].*

Similarly, we can define the quasi-greedy orbit of 1.

Definition 4.3.3. *If the greedy orbit of 1 is infinite, we identify the quasi-greedy orbit of 1 with the greedy one, i.e., $\{Q^n(1)\}_{n=1}^\infty = \{G^n(1)\}_{n=1}^\infty$. Otherwise, let $(a_1 a_2 \cdots a_n 0^\infty)$ be the greedy expansion of 1, we define the quasi-greedy coding of 1 by $(a_1 a_2 \cdots a_n^-)^\infty$, where $a_n^- = a_n - 1$. In this case, the quasi-greedy orbit of 1 is defined by*

$$Q^i(1) = (\sigma^i(a_1 a_2 \cdots a_n^-)^\infty))_\beta, 1 \leq i \leq n$$

and $Q^{kn+i}(1) = Q^i(1)$ *for any* $k \geq 0$*, where* $(b_n)_\beta := \sum_{n=1}^{\infty} b_n \beta^{-n}$*. For simplicity, throughout this section we always assume the quasi-greedy expansion of* 1 *is* $(\eta_i)_{i=1}^{\infty}$*.*

Definition 4.3.4. *We say the quasi-greedy orbit of* 1 *hits* $\beta^{-1}(\beta-1)^{-1}$ (β^{-1}) *for the first time if there exists a minimal number* $k \geq 1$ *such that* $Q^k(1) = (\beta^{-1}(\beta-1)^{-1})$ $(Q^k(1) = \beta^{-1})$ *and*

$$Q^i(1) \in [0, \beta^{-1}) \cup (\beta^{-1}(\beta-1)^{-1}, (\beta-1)^{-1}],\ 1 \leq i \leq k-1.$$

Remark 4.3.5. *Similarly, we can define the quasi-greedy orbit of* 1 *hits* $(\beta^{-1}, \beta^{-1}(\beta-1)^{-1})$ *for the first time.*

Lemma 4.3.6. *If the quasi-greedy orbit of* 1 *hits* $\beta^{-1}(\beta-1)^{-1}$ *for the first time, then the greedy orbit of* 1 *is finite. Moreover there exists some* n_0 *such that the quasi-greedy expansion of* 1 *is*

$$(\eta_i) = (a_1 a_2 \cdots a_{n_0} \overline{a_1 a_2 \cdots a_{n_0}})^{\infty}$$

for some $(a_1 a_2 \cdots a_{n_0})$*, where* $a_{n_0} = 1$*.*

Proof. Note that if $G^k(1) \in [0, \beta^{-1}] \cup [\beta^{-1}(\beta-1)^{-1}, 1]$ for all $1 \leq k \leq n$, then $G^k(\bar{1}) + G^k(1) = (\beta-1)^{-1}$, where $\bar{1} = (\beta-1)^{-1} - 1$. If the quasi-greedy orbit of 1 hits $\beta^{-1}(\beta-1)^{-1}$ for the first time, then there exists $t_0 \geq 1$ such that $G^{t_0}(1) = \beta^{-1}(\beta-1)^{-1}$. Hence, we have that $G^{t_0+1}(1) = \bar{1}$. By symmetry, after t_0 step the greedy orbit of $\bar{1}$ will hit β^{-1} as $\beta^{-1} + \beta^{-1}(\beta-1)^{-1} = (\beta-1)^{-1}$. Hence the greedy orbit of 1 is finite. The second statement follows immediately from the first statement if we take $n_0 = t_0 + 1$. □

In [84], de Vries and Komornik proved that

Theorem 4.3.7. $\widetilde{U}$ *is a subshift of finite type if and only if* $\beta \in (1,2) \setminus \overline{\mathcal{U}}$*, where* $\mathcal{U}$ *is the set of* β *for which the* β*-expansion of* 1 *is unique.*

Equivalentlly we shall prove

Theorem 4.3.8. $\widetilde{U}$ *is a subshift of finite type if and only if the quasi-greedy orbit of 1 falls into the interval* $\left(\dfrac{1}{\beta}, \dfrac{1}{\beta(\beta-1)}\right]$*.*

We partition the proof of this theorem into several lemmas.

Lemma 4.3.9. *If the quasi-greedy orbit of* 1 *hits* $\beta^{-1}(\beta-1)^{-1}$ *for the first time, then* $\tilde{U}$ *is a SFT.*

Before we prove this lemma, we review one important result, see [53].

Theorem 4.3.10. *Let* (η_n) *be the quasi-greedy expansion of* 1*, then* $\beta \in \overline{\mathcal{U}}$ *if and only if*

$$\overline{\eta_k \eta_{k+1} \eta_{k+2} \cdots} < \eta_1 \eta_2 \eta_3 \cdots$$

for any $k \geq 1$*.*

By Theorems 4.3.7 and 4.3.10, to prove Lemma 4.3.9, it is enough to prove that there exists k_0 such that $\overline{\eta_{k_0} \eta_{k_0+1} \eta_{k_0+2} \cdots} = (\eta_n)$.

Proof of Lemma 4.3.9. By Lemma 4.3.6, we may assume the quasi-greedy expansion of 1 is

$$(\eta_i) = (a_1 a_2 \cdots a_{n_0} \overline{a_1 a_2 \cdots a_{n_0}})^{\infty}$$

for some $(a_1 a_2 \cdots a_{n_0})$. From this it follows that $\sigma^{n_0}\overline{(\eta_i)} = (\eta_i)$, and Lemma 4.3.9 is proved. □

Lemma 4.3.11. *If there exists $k_0 \geq 1$ such that*

$$\overline{\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots} > \eta_1\eta_2\eta_3\cdots,$$

then the quasi-greedy orbit of 1 *hits the interior of the switch region.*

Proof. Suppose

$$\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots < \overline{\eta_1\eta_2\eta_3\cdots}$$

for some $k_0 \geq 1$, and let (α_i) be the quasi-greedy expansion of $\bar{1}$. Since the quasi-greedy expansion is the largest infinite sequence in the sense of lexicographical ordering, we have

$$\overline{\eta_1\eta_2\eta_3\cdots} \leq (\alpha_i).$$

Thus,

$$\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots < (\alpha_i).$$

By monotonicity of the quasi-greedy expansion [84], it follows that

$$(\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots)_\beta := \sum_{j=1}^{\infty} \eta_{j+k_0}\beta^{-j} < \bar{1}.$$

Hence, the quasi-greedy orbit of 1 must fall into $(\beta^{-1}, \beta^{-1}(\beta-1)^{-1})$. □

Lemma 4.3.12. *If the quasi-greedy orbit of* 1 *hits* β^{-1} *for the first time, then we have that*

$$\overline{\eta_{k+1}\eta_{k+2}\eta_{k+3}\cdots} < \eta_1\eta_2\eta_3\cdots,$$

for any $k \geq 1$. Consequently, $\tilde{U}$ is not a subshift of finite type.

Proof. Suppose the quasi-greedy orbit of 1 hits β^{-1} for the first time, but

$$\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots \leq \overline{\eta_1\eta_2\eta_3\cdots}$$

for some $k_0 \geq 1$. By Lemma 4.3.11, we have

$$\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots \geq \overline{\eta_1\eta_2\eta_3\cdots}$$

implying

$$\eta_{k_0+1}\eta_{k_0+2}\eta_{k_0+3}\cdots = \overline{\eta_1\eta_2\eta_3\cdots}.$$

Therefore there exists $k_0 \geq 1$ such that $\sigma^{k_0}(\overline{\eta_i}) = (\eta_i)$. By the assumption of the lemma, we know that the greedy orbit of 1 is finite. Hence, the quasi-greedy expansion of 1 has the form $(\eta_i) = (a_1a_2\cdots a_n^-)^\infty$ for some $(a_1a_2\cdots a_n^-)$, where $a_n = 1$. Since $(\eta_i) = (a_1a_2\cdots a_n^-)^\infty$, it follows that the quasi-greedy orbit of 1 never hits the point $(\beta)^{-1}(\beta-1)^{-1}$, and subsequently never hits the point $\bar{1} = (2-\beta)(\beta-1)^{-1}$. However, we have $\sigma^{k_0}(\eta_i) = (\overline{(\eta_i)})$, which yields that the quasi-greedy orbit of 1 hits $\bar{1}$, leading a contradiction. □

Proof of Theorem 4.3.8. If the quasi-greedy orbit of 1 hits the interior of switch region or $(\beta)^{-1}(\beta-1)^{-1}$ for the first time, then $\tilde{U}$ is a subshift of finite type in terms of Theorem 3.2.4 and Lemma 4.3.9. Conversely, if $\tilde{U}$ is a subshift of finite type, then we have $\beta \notin \overline{\mathcal{U}}$, which implies that the quasi-greedy expansion of 1 hits the switch region eventually. By Lemma 4.3.12, it follows that the quasi-greedy orbit of 1 hits the interior of switch region or $(\beta)^{-1}(\beta-1)^{-1}$ for the first time. □

4.4 Univoque set for β-expansions

In this section, we give another application of our result to β-expansions. Our main goal is to calculate $\dim_H(U)$ when $\tilde{U}$ is not a subshift of finite type, in general this is a hard task. However, in this scenario the technique of periodic orbits allows us to find $\dim_H(U)$ in many cases. By Theorem 3.2.4 we have that $\tilde{U}$ is a subshift of finite type if the orbit of 1 falls into $(\beta^{-1}, \beta^{-1}(\beta-1)^{-1})$ and that $\dim_H(U)$ can be calculated in terms of Mauldin and Williams' work [62]. Hence we suppose in this section that the greedy orbit of 1 is eventually periodic and that it never hits $(\beta^{-1}, \beta^{-1}(\beta-1)^{-1})$.

One can express the univoque set U as

$$U = \left\{x : G^n(x) \notin [\beta^{-1}, \beta^{-1}(\beta-1)^{-1}] \text{ for any } \; n \geq 0\right\}.$$

It is sufficient to consider the univoque points in $[0,1)$ since the greedy orbits of the points in $[1, (\beta-1)^{-1}]$ eventually fall $[0,1)$ and remain there. In this section, we denote as before the set of all the univoque points in $[0,1)$ by U.

Now we partition $[0,1)$ via the eventual periodic greedy orbits of 1 and $(2-\beta)(\beta-1)^{-1}$. Let $\bar{1} = (2-\beta)(\beta-1)^{-1}$ be the reflection of 1. Since the greedy orbit of 1 takes only finitely many values, by symmetry it follows that the greedy orbit of $\bar{1}$ also takes only finitely many values, see the first statement in the proof of Lemma 4.3.6. Without loss of generality, we may assume that the values of the greedy orbits of 1 and $\bar{1}$ are $c_1 < c_2 < \cdots < c_p$. If β^{-1}, $\beta^{-1}(\beta-1)^{-1}$, 0 and 1 are not in $\{c_1, c_2, \cdots, c_p\}$, then we add these four points to this set. For simplicity, we still assume that our partition points of $[0,1)$ are $\{c_1, c_2, \cdots, c_p\}$. Now we have that

$$[0,1) = \bigcup_{i=1}^{p-1} [c_i, c_{i+1}).$$

We emphasize that 1 is only the endpoint, it is not in the partition. This adjustment will not affect our main result. It is easy to see that the image of each interval of the partition is the union of some elements of the Markov partition, see the following example.

Example 4.4.1. *Let β be the largest positive root of $x^3 = x^2 + x + 1$. The greedy orbit of 1 takes three vaules, i.e. $G(1) = \beta - 1, G^2(1) = \beta^{-1}, G^n(1) = 0, n \geq 3$. Similarly, $G(\bar{1}) = (\beta-1)^{-1}\beta^{-2}$, $G^2(\bar{1}) = (\beta-1)^{-1}\beta^{-1}$, $G^3(\bar{1}) = \bar{1}$.*

Let $A = \left[0, \dfrac{2-\beta}{\beta-1}\right)$, $B = \left[\dfrac{2-\beta}{\beta-1}, \dfrac{1}{(\beta-1)\beta^2}\right)$, $C = \left[\dfrac{1}{(\beta-1)\beta^2}, \dfrac{1}{\beta}\right)$, $D = \left[\beta^{-1}, \dfrac{1}{(\beta-1)\beta}\right)$, $E = \left[\dfrac{1}{(\beta-1)\beta}, \beta-1\right)$, $F = [\beta-1, 1)$. Hence we give a Markov partition of $[0,1)$ and they have following relations:

$T_0(A) = A \cup B$, $T_0(B) = C \cup D$, $T_0(C) = E \cup F$, $T_1(D) = A$, $T_1(E) = B \cup C$, $T_1(F) = D \cup E$. The corresponding adjacency matrix is given by

$$S = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Recall the Markov partition of $[0,1) = \bigcup_{i=1}^{p-1}[c_i, c_{i+1})$, we identify each intervals $[c_i, c_{i+1})$ with a block A_i. In this section, we use the set of blocks $\{A_i\}_{i=1}^{p-1}$ representing the intervals $\{[c_i, c_{i+1})\}_{i=1}^{p-1}$. Let S be the corresponding adjacency matrix of the Markov partition, and Σ the associated subshift of finite type, i.e.

$$\Sigma = \{(i_n) \in \{1, 2, 3 \cdots, p-1\}^{\mathbb{N}} : S_{i_n, i_{n+1}} = 1\}.$$

Recall that we are assuming that the greedy orbit of 1 never hits $(\beta^{-1}, \beta^{-1}(\beta-1)^{-1})$. Hence, by the definition of Markov partition, the switch region is one of the sets of this partition. We denote its associated block by A_i.

Definition 4.4.2. *Let S be the underlying adjacency matrix, and the associated block of the switch region be A_i. We remove the i-th row and i-th column of S, and denote this new matrix by $S^{'}$.*

For Example 4.4.1 we mentioned above,

$$S^{'} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Now we state the main result of this section.

Theorem 4.4.3. *Let $1 < \beta < 2$, if the greedy expansion of 1 in base β is eventually periodic, then we have $\dim_H(U) = \dfrac{\log r}{\log \beta}$, where r is the spectral radius of matrix $S^{'}$.*

The proof of this theorem is analogous to Theorem 4.2.17, we omit the details.

Remark 4.4.4. *We explain here why we generalize de Vries and Komornik's result regarding the calculation of $\dim_H(U)$. In the last section, we proved a necessary and sufficient condition which can determine when $\tilde{U}$ is a subshift of finite type. If the quasi-greedy orbit of 1 hits $(\beta^{-1}, \beta^{-1}(\beta-1)^{-1})$, then $\tilde{U}$ is a subshift of finite type. For this case we can implement the idea of Theorem 3.2.4, and find $\dim_H(U)$ in terms of Mauldin and Williams' work [62], see the details in the last chapter. If the quasi-greedy orbit of 1 hits $\beta^{-1}(\beta-1)^{-1}$ for the first time, then $\tilde{U}$ is also a subshift of finite type. For this case, the greedy expansion of 1 is finite (Lemma 4.3.6), and we can make use of Theorem 4.4.3 calculating $\dim_H(U)$. However, Theorem 4.4.3 enables us to calculate some cases such that $\tilde{U}$ is not a subshift of finite type, see some examples in section 6. As such we generalize the result of de Vries and Komornik. Recently, Komornik, Kong and Li proved that $\dim_H(U) = h(\tilde{U})(\log \beta)^{-1}$, where $h(\tilde{U})$ is the entropy of $\tilde{U}$, essentially their result allows them to calculate almost every β. However, our idea here is still effective if the attractor is an interval, i.e. the self-similar set of the overlapping IFS is an interval. Roughly speaking, the univoque set can be identified with an open dynamical system. Finding the dimension of the univoque set is to find the dimension of the survivor set of some open dynamical system. Our method is still working for other IFS's, not just for the case of β-expansions.*

4.5 Doubling map with holes

In this section, we consider the doubling map with holes . Our main motivation is to answer partially one problem posed in the PhD thesis of Barrera [8]. We start with some definitions. Let $D(x) = 2x \mod 1$ be the doubling map defined on $[0, 1)$. Given any $(a, b) \subset [0, 1)$, define

$$J(a, b) := \{x \in [0, 1) : D^n(x) \notin (a, b), \forall\, n \geq 0\}.$$

Barrera posed the following question in his PhD thesis.

Question 4.5.1. *For which holes $(a, b) \subset (0, \frac{1}{2})$ with the property that the orbits of a and b do not hit (a, b), is $J(a, b)$ is isomorphic to a subshift of finite type ?*

Our partial answer to this question is the following theorem.

Theorem 4.5.2. *For any hole $(a, b) \subset (0, \frac{1}{2})$, if the orbits of a and b are eventually periodic, then $J(a, b)$ is isomorphic to a subshift of finite type.*

Remark 4.5.3. *Analogous result is still correct for several holes. It is easy to see that any point $x \in [0, 1)$ can be approximated by a sequence $(x_n)_{n=1}^{\infty} \in [0, 1)$ such that each x_n has an eventually periodic coding. With this observation, the event that $J(a, b)$ is isomorphic to a subshift of finite type happens for many cases. Calculating the Hausdorff dimension of $J(a, b)$ is indeed analogous to Theorem 4.4.3. We do not give the detailed proof for this problem.*

The proof of Theorem 4.5.2 follows the same idea which is utilized in section 2. Since the orbits of a and b are eventually periodic, we can partition $[0, 1)$ by the orbits of a and b. Denote the values of the orbits of a and b by $D = \{d_1, d_2, \cdots, d_p\}$. If $0, 2^{-1}, a, b$ and 1 are not in this set, we put these points in D. Without loss of generality, we still use the set $D = \{0 = d_1, d_2, \cdots, d_p = 1\}$. These points give a partition of $[0, 1)$, i.e.,

$$[0, 1) = \cup_{i=1}^{p-1} [d_i, d_{i+1}).$$

Note that 1 is not a point of the partition. We still use the notation above as it will not effect our main result. It is not difficult to find that each $[d_i, d_{i+1})$ is mapped into an interval which is the union of some sets of the partition, see the following example.

Example 4.5.4. *Let $a = (01010)_2^{\infty}$ and $b = (10010)_2^{\infty}$, then the orbit of a is*

$$\left\{a = \frac{10}{31}, D(a) = \frac{20}{31}, D^2(a) = \frac{9}{31}, D^3(a) = \frac{18}{31}, D^4(a) = \frac{5}{31}\right\}$$

the orbit of b is

$$\left\{b = \frac{18}{31}, D(b) = \frac{5}{31}, D^2(b) = \frac{10}{31}, D^3(b) = \frac{20}{31}, D^4(b) = \frac{9}{31}\right\},$$

We partition $[0, 1)$ as follows,

$$\begin{aligned} [0, 1) &= \left[0, \frac{5}{31}\right) \cup \left[\frac{5}{31}, \frac{9}{31}\right) \cup \left[\frac{9}{31}, \frac{10}{31}\right) \cup \left[\frac{10}{31}, \frac{1}{2}\right) \cup \left[\frac{1}{2}, \frac{18}{31}\right) \cup \left[\frac{18}{31}, \frac{20}{31}\right) \cup \left[\frac{20}{31}, 1\right) \\ &= A \cup B \cup C \cup D \cup E \cup F \cup G \end{aligned}$$

This is a Markov partition of $[0, 1)$ as we have

$D(A) = A \cup B \cup C, D(B) = D \cup E, D(C) = E, D(D) = G, D(E) = A, D(F) = B$ and $D(G) = C \cup D \cup E \cup F \cup G$. Subsequently we can define an adjacency matrix

$$S = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Recall the Markov partition of $[0,1) = \bigcup_{i=1}^{p-1}[d_i, d_{i+1})$, we identify each intervals $[d_i, d_{i+1})$ with a block B_i. In what follows, we use the set of blocks $\{B_i\}_{i=1}^{p-1}$ to represent the intervals $\{[d_i, d_{i+1})\}_{i=1}^{p-1}$. The corresponding adjacency matrix S generates a subshift of finite type, which we denote it by Σ, i.e.

$$\Sigma = \{(i_n) \in \{1, 2, \cdots, p-1\}^{\mathbb{N}} : S_{i_n, i_{n+1}} = 1\}.$$

Let

$$P = \{(i_n) \in \Sigma : \exists\, k \geq 0 \,\text{with}\, \sigma^k(i_n) = (p-1)^\infty\}.$$

In this section, we always assume that the binary expansion of $x \in [0,1)$ is generated by the doubling map, i.e. any binary expansion $(a_n) \in \{0,1\}^{\mathbb{N}}$ cannot end with 1^∞. Hence we should remove P from Σ.

Note that for each $1 \leq i \leq p-1$, it has an associated B_i which is one of the sets of the Markov partition. For any $x \in [0,1)$, we can find its associated coding in Σ in terms of the orbit of x, i.e. we can find some sequence $(j_k) \in \Sigma \setminus P$ such that $x \in B_{j_1}, D(x) \in B_{j_2}, \cdots, D^k(x) \in B_{j_{k+1}}, \cdots$. Conversely, for any $(j_k) \in \Sigma \setminus P$, we may also find some point $x \in [0,1)$ such that $D^k(x) \in B_{j_{k+1}}, k \geq 0$.

The following lemma is clear.

Lemma 4.5.5. $\dim_H(J(a,b)) = \dim_H(J[a,b))$, *where*

$$J[a,b) = \{x \in [0,1) : D^n(x) \notin [a,b), \forall\, n \geq 0\},$$

moreover, $J(a,b) = J[a,b)$ except for a countable set.

Proof. The lemma follows the following simple inclusions.

$$J[a,b) \subset J(a,b) \subset \cup_{n=1}^{\infty}\{x \in [0,1) : D^n(x) = a\} \cup J[a,b).$$

□

Since the Hausdorff dimension of a countable set is zero, in the remaining of this section, we only consider the set $J[a,b)$ instead of $J(a,b)$.

By virtue of the definition of Markov partition, the hole $[a,b)$ is the union of some sets of this partition. We denote its associated blocks by $B_{i_1}, B_{i_2}, \cdots, B_{i_s}$. For simplicity we denote these blocks by $\widehat{B_1} = B_{i_1}, \cdots, \widehat{B_s} = B_{i_s}$.

Definition 4.5.6. *Let S be the adjacency matrix, and the associated blocks of the hole are $B_{i_1}, B_{i_2}, \cdots, B_{i_s}$. We remove the i_j-th row and i_j-th column of S, $1 \leq j \leq s$, and denote this new matrix by S'. Similarly, the subshift of finite type generated by S' is denoted by Σ'.*

For Example 4.5.4 we mentioned above,

$$S' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Lemma 4.5.7. *There is a bijection between the set of all binary expansions of points in* $[0,1)$ *and* $\Sigma \setminus P$.

Proof. Given $(i_n) \in \Sigma \setminus P$, for i_1 we can find a corresponding interval B_{i_1}. B_{i_1} is the subdomain of $T_0 = 2x$ or $T_1 = 2x - 1$. If B_{i_1} is the subdomain of T_0, then we let $a_1 = 0$, otherwise, we set $a_1 = 1$. By the definition of Markov partition, $B_{i_{n+1}} \subset D(B_{i_n})$ for any $n \geq 1$. Then similarly we identify (i_n) with a sequence $(a_n) \in \{0,1\}^{\mathbb{N}}$ in the following way

$$a_n = \begin{cases} 0 & \text{if } B_{i_n} \text{ is the subdomain of } T_0 = 2x \\ 1 & \text{if } B_{i_n} \text{ is the subdomain of } T_1 = 2x - 1 \end{cases}$$

Define $x = \sum_{n=1}^{\infty} a_n 2^{-n}$, we know that (a_n) is a binary expansion of x, which cannot end with 1^{∞} as $(i_n) \notin P$. Conversely, given any binary coding (a_n) which does not end with 1^{∞}, we can define a point $x = \sum_{n=1}^{\infty} a_n 2^{-n}$. For any k, $D^k(x)$ falls into some $B_j, 1 \leq j \leq p-1$. Therefore we may find a unique associated sequence $(i_n) \in \Sigma \setminus P$. Now we can define a bijection

$$\phi : \{\text{all the binary expansions under the doubling map}\} \to \Sigma \setminus P$$

by

$$\phi(a_n) = (i_n),$$

if $D^n(x) \in B_{i_{n+1}}, n \geq 0$, where $x = \sum_{n=1}^{\infty} a_n 2^{-n}$. □

In order to construct an isomorphism between $J[a,b)$ and Σ', we have to remove some points from these two sets respectively. Let $E = \{d_1 = 0, d_2, \cdots, d_{p-1}\}$, define

$$J[a,b) \setminus (\cup_{n=0}^{\infty} D^{-n}(E)).$$

By Lemma 7.2.2, every point in $x \in [0,1)$ has a unique coding in $\Sigma \setminus P$. Since $\cup_{n=0}^{\infty} D^{-n}(E)$ is a countable set, the corresponding codings of this set in $\Sigma \setminus P$ is also a countable set. Denote this set by Q.

For any $x \in J[a,b) \setminus (\cup_{n=0}^{\infty} D^{-n}(E))$, we consider the orbit of x, i.e. we can find an infinite sequence $\{B_{j_k}\}_{k=1}^{\infty}$ such that $x \in B_{j_1}, D(x) \in B_{j_2}, \cdots, D^k(x) \in B_{j_{k+1}}, \cdots$. Hence, we obtain a unique $(j_k) \in \Sigma' \setminus (P \cup Q)$. Subsequently we can define a map

$$\phi : J[a,b) \setminus (\cup_{n=0}^{\infty} D^{-n}(E)) \to \Sigma' \setminus (P \cup Q)$$

by

$$\phi(x) = (j_k).$$

Since Σ' is a subshift of finite type, we can define an invariant ergodic measure on it, see [75]. Denote this measure by μ. Now we can prove following result.

Theorem 4.5.8. $(J[a,b), D, \mu \circ \phi)$ *is isomorphic to* (Σ', σ, μ).

Proof. It is easy to check that $\sigma \circ \phi = \phi \circ T$. Hence, it remains to prove that ϕ is a bijection between $J[a,b) \setminus (\cup_{n=0}^{\infty} D^{-n}(E))$ and $\Sigma' \setminus (P \cup Q)$. Firstly ϕ is one-to-one. For any $x, y \in J[a,b) \setminus (\cup_{n=0}^{\infty} D^{-n}(E))$, if $\phi(x) = \phi(y)$, then we can implement the idea, which is used in the proof of Lemma 7.2.2, to find the binary codings of x and y. Since $\phi(x) = \phi(y)$, it follows that their associated binary codings also coincide. Hence, $x = y$. On the other hand, for any $(j_k) \in \Sigma' \setminus (P \cup Q)$, we can find a point x such that $D^k(x) \in B_{j_{k+1}}, k \geq 0$ (also see the proof of Lemma 7.2.2). By the definition of $\Sigma' \setminus (P \cup Q)$, $D^k(x)$ cannot hit $\cup_{n=0}^{\infty} D^{-n}(E)$, and $\{B_{j_{k+1}}, k \geq 0\}$ does not consist of any $\widehat{B_1}, \cdots, \widehat{B_s}$. Therefore, $x \in J[a,b) \setminus (\cup_{n=0}^{\infty} D^{-n}(E))$. □

Now Theorem 4.5.2 follows from Theorem 4.5.8 and Lemma 4.5.5.

4.6 Lipschitz equivalence of self-similar sets with overlaps

Let $A, B \subset \mathbb{R}^d$ be two bounded sets. We say A and B are Lipschitz equivalent if there exist a bijection f from A to B and a constant $c > 0$ such that

$$c^{-1}|x-y| \leq |f(x) - f(y)| \leq c|x-y|.$$

We denote $A \simeq B$ if A and B are Lipschitz equivalent, and call f a Lipschitz map.

Lipschitz equivalence of self-similar sets is an important subject in fractal geometry. To construct a Lipschitz map of two self-similar sets is a difficult problem, especially if the serious overlaps occur. In [77], Rao, Ruan and Xi constructed a Lipschitz map between two self-similar sets with the open set condition. From then on, many papers were dedicated to this respect, see [87, 79, 32, 33, 57, 60] and references therein. However, a major assumption of these papers is that K satisfies the open set condition. The main reason is that usually it is difficult to construct a Lipschitz map of self-similar sets. To the best of our knowledge, there are very few papers considering the Lipschitz map of overlapping self-similar sets. Hence, the motivation of this section is to investigate this problem. In [40], Guo et al. constructed a Lipschitz map between two self-similar sets having exact overlaps. Their crucial idea is still from [77]. We however use different idea from [77] and [59]. The idea of Theorem 4.2.11 gives a sufficient condition for a self-similar set to be seen as a graph-directed self-similar set satisfying the open set condition or even the strong separation condition.

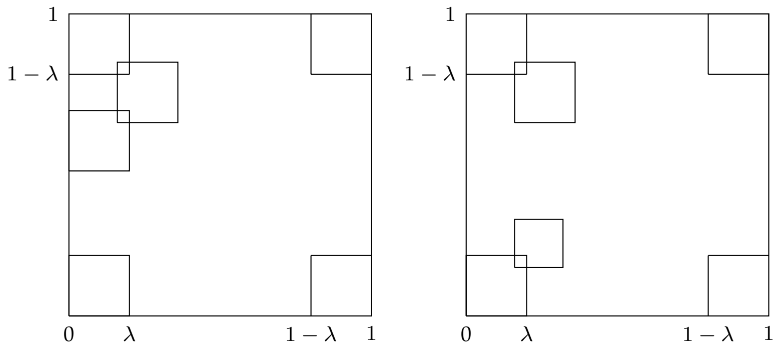
In this section, we consider one example which is from [11]. For further result, we refer to [11].

Example 4.6.1. *Let $0 < \lambda < (2-\sqrt{2})/2$. Consider two IFSs $\{f_i : 1 \leq i \leq 6\}$ and $\{g_i : 1 \leq i \leq 6\}$ where*

$$\begin{aligned} &f_1(x,y) = \lambda(x,y), && f_2(x,y) = \lambda(x,y) + (1-\lambda, 0), \\ &f_3(x,y) = \lambda(x,y) + (1-\lambda, 1-\lambda), && f_4(x,y) = \lambda(x,y) + (0, 1-\lambda), \\ &f_5(x,y) = \lambda(x,y) + (\lambda(1-\lambda), (1-\lambda)^2), && f_6(x,y) = \lambda(x,y) + (0, (1-\lambda)(1-2\lambda)), \end{aligned}$$

and $g_6(x,y) = \lambda(x,y) + (\lambda(1-\lambda), \lambda(1-\lambda))$ with $g_i(x,y) = f_i(x,y)$ for $1 \leq i \leq 5$. Let F and G be the self-similar sets generated by IFSs $\{f_i : 1 \leq i \leq 6\}$ and $\{g_i : 1 \leq i \leq 6\}$, respectively. Then $F \simeq G$.

The following figures show locations of squares $f_i([0,1]^2), 1 \leq i \leq 6$ and squares $g_i([0,1]^2), 1 \leq i \leq 6$.



We now recall the definition of the graph-directed self-similar set, see the introduction in section 2.

Before proving the Lipschitz equivalence of F and G, we have following lemma. The proof of this result is similar with [77, Theorem 2.1].

Lemma 4.6.2. *Let $\{E_i\}_{i=1}^N$ and $\{F_i\}_{i=1}^N$ be the graph-directed sets with the same directed graph G. That is, $E_i = \bigcup_{j=1}^N \bigcup_{e\in V_{i,j}} f_e(E_j)$, $F_i = \bigcup_{j=1}^N \bigcup_{e\in V_{i,j}} g_e(F_j)$. If for each e of G, the similitudes f_e and g_e have the same ratio ρ_e, and $\{E_i\}_{i=1}^N$ and $\{F_i\}_{i=1}^N$ are dust-like (Here dust-like means each E_i (F_i) is piecewise disjoint). Then $E^* = \bigcup_{i=1}^N E_i$ and $F^* = \bigcup_{i=1}^N F_i$ are Lipschitz equivalent.*

Proof. Since $\{E_i\}_{i=1}^N$ are dust-like, it follows that for any $x \in E_i$ there exists a unique infinite path $e_1e_2\cdots$ starting from vertex i such that

$$x = \bigcap_{k=1}^{\infty} f_{e_1\cdots e_k}(E_{i_k})$$

We say $e_1e_2\cdots$ is a coding of x. Define $\phi : E_i \to F_i$ by

$$\phi(x) = \bigcap_{k=1}^{\infty} g_{e_1\cdots e_k}(E_{i_k})$$

If $x, y \in E_i$, then we can construct a bi-Lipschitz map ϕ such that

$$c_2|x-y| \le |\phi(x)-\phi(y)| \le c_1|x-y|.$$

This is the main result of [77, Theorem 2.1]. Suppose $x \in E_i$ and $y \in E_j$, $i \neq j$. Since E_i and E_j are disjoint, therefore we can easily find two constants c_1, c_2 such that

$$c_2|x-y| \le |\phi(x)-\phi(y)| \le c_1|x-y|.$$

□

Now we can construct a Lipschitz map between F and G. Let $F_i = f_i(F)$ for $i = 1, 2, 3, 5$, $F_4 = f_4(F) \setminus f_4 \circ f_2(F)$ and $F_6 = f_6(F) \setminus f_6 \circ f_3(F)$. Then $F_i, 1 \le i \le 6$, are pairwise disjoint nonempty compact sets such that $F = \bigcup_{1\le i\le 6} F_i$ since $f_4 \circ f_2 = f_5 \circ f_4$ and $f_6 \circ f_3 = f_5 \circ f_1$.

Thus we have

$$\begin{cases} F_i = f_i(F_1) \cup f_i(F_2) \cup f_i(F_3) \cup f_i(F_4) \cup f_i(F_5) \cup f_i(F_6) & \text{for } i = 1, 2, 3, 5 \\ F_4 = f_4(F_1) \cup f_4(F_3) \cup f_4(F_4) \cup f_4(F_5) \cup f_4(F_6) \\ F_6 = f_6(F_1) \cup f_6(F_2) \cup f_6(F_4) \cup f_6(F_5) \cup f_6(F_6) \end{cases}$$

By the same way as above let $G_1 = g_5(G)$, $G_2 = g_2(G)$, $G_3 = g_3(G)$, $G_4 = g_4(G) \setminus g_4 \circ g_2(G)$, $G_5 = g_6(G)$ and $G_6 = g_1(G) \setminus g_1 \circ g_3(G)$. We have $G_i, 1 \le i \le 6$, are pairwise disjoint nonempty compact sets with $G = \bigcup_{1 \le i \le 6} G_i$ and satisfy

$$\begin{cases} G_1 = g_5(G_1) \cup g_5(G_2) \cup g_5(G_3) \cup g_5(G_4) \cup g_5(G_5) \cup g_5(G_6) \\ G_2 = g_2(G_1) \cup g_2(G_2) \cup g_2(G_3) \cup g_2(G_4) \cup g_2(G_5) \cup g_2(G_6) \\ G_3 = g_3(G_1) \cup g_3(G_2) \cup g_3(G_3) \cup g_3(G_4) \cup g_3(G_5) \cup g_3(G_6) \\ G_5 = g_6(G_1) \cup g_6(G_2) \cup g_6(G_3) \cup g_6(G_4) \cup g_6(G_5) \cup g_6(G_6) \\ G_4 = g_4(G_1) \cup g_4(G_3) \cup g_4(G_4) \cup g_4(G_5) \cup g_4(G_6) \\ G_6 = g_1(G_1) \cup g_1(G_2) \cup g_1(G_4) \cup g_1(G_5) \cup g_1(G_6). \end{cases}$$

Thus $F \simeq G$ by Lemma 4.6.2.

4.7 Examples

In this section we give various examples which demonstrate the effectiveness of our simple algorithm. We start with one example considered by other authors but with different methods. What makes our result different is that we can calculate the Hausdorff dimension of K and U simultaneously, uniformly and quickly.

The following example was investigated by [68]. We give a very short calculation.

Example 4.7.1. *Let K be the self-similar set of the following IFS:*

$$\left\{ f_1(x) = \frac{x}{3},\ f_2(x) = \frac{x}{9} + \frac{8}{27},\ f_3(x) = \frac{x+2}{3} \right\}.$$

Let $J = [0,1]$, $f_1(J) = \left[0, \frac{1}{3}\right]$, $f_2(J) = \left[\frac{8}{27}, \frac{11}{27}\right]$, $f_3(J) = \left[\frac{2}{3}, 1\right]$. After some calculation, we find that $f_1(K) \cap f_2(K) = f_{1331}(K) \cup f_{1332}(K) \cup f_{1333}(K)$, where $f_{1331}(K) = f_{211}(K), f_{1332}(K) = f_{212}(K), f_{1333}(K) = f_{213}(K)$, i.e. $f_1(K) \cap f_2(K)$ is the union of some exact overlaps. Hence the IFS is an exact overlapping IFS, and the endpoints of $f_i(K), 1 \le i \le 3$, are periodic. We partition $K = A \cup B \cup C \cup D \cup E$, where

$$A = \left[0, \frac{8}{27}\right] \cap K, B = \left[\frac{8}{27}, \frac{1}{3}\right] \cap K, C = \left[\frac{1}{3}, \frac{11}{27}\right] \cap K, D = \left[\frac{2}{3}, \frac{8}{9}\right] \cap K, E = \left[\frac{8}{9}, 1\right] \cap K.$$

Then we have

$$T_1(A) = A \cup B \cup C \cup D, T_1(B) = E,$$

and

$$T_2(C) = C \cup D \cup E,\ T_3(D) = A \cup B \cup C,\ T_3(E) = D \cup E.$$

The weighted adjacency matrix for K is

$$A^t = \begin{pmatrix} 3^{-t} & 3^{-t} & 3^{-t} & 3^{-t} & 0 \\ 0 & 0 & 0 & 0 & 3^{-t} \\ 0 & 0 & 9^{-t} & 9^{-t} & 9^{-t} \\ 3^{-t} & 3^{-t} & 3^{-t} & 0 & 0 \\ 0 & 0 & 0 & 3^{-t} & 3^{-t} \end{pmatrix}$$

Since the block $B = f_1(K) \cap f_2(K)$ is the union of some exact overlaps, then the weighted adjacency matrix for the univoque set is

$$A^t = \begin{pmatrix} 3^{-t} & 3^{-t} & 3^{-t} & 0 \\ 0 & 9^{-t} & 9^{-t} & 9^{-t} \\ 3^{-t} & 3^{-t} & 0 & 0 \\ 0 & 0 & 3^{-t} & 3^{-t} \end{pmatrix}$$

Therefore the Hausdorff dimension of K is $\frac{\log \lambda}{\log 9} = \alpha$, where λ is the largest solution of $x^3 - 6x^2 + 5x^2 - 1 = 0$, and the dimension of U is $\frac{\log r}{\log 9} = \gamma$, where r is the largest positive root of

$$x^5 - 6x^4 + 9x^3 - 8x^2 + 4x - 1 = 0$$

Moreover, we have

$$0 < \mathcal{H}^\alpha(K) < \infty.$$

and

$$0 < \mathcal{H}^\gamma(U) < \infty.$$

The following example is considered in the paper [56]. Again we can find the dimension of U easily.

Example 4.7.2.

$$S_1(x) = \rho x, S_2(x) = rx + \rho(1-r), S_3(x) = rx + 1 - r,$$

where $0 < \rho < 1, 0 < r < 1, \rho + 2r - \rho r \leq 1$, *then* $\dim_H(K) = \alpha$, *where* α *is the solution of* $1 - 2b + ab - a = 0$, $a = \rho^\alpha$ *and* $b = r^\alpha$, *and* $\dim_H(U) = \gamma$, *where* γ *is the solution of* $(1-a)(1-2b) - ab^2 = 0$, *where* $a = \rho^\gamma, b = r^\gamma$. *Moreover, we have*

$$0 < \mathcal{H}^\alpha(K) < \infty,$$

and

$$0 < \mathcal{H}^\gamma(U) < \infty.$$

Let $E = [0,1]$, $S_1(E) = [0, \rho]$, $S_2(E) = [\rho(1-r), r + \rho(1-r)]$, $S_3(E) = [1-r, 1]$. Note that $S_1 \circ S_3 = S_2 \circ S_1$. Again we partition the attractor K in the following way:

$$A = [0, \rho(1-r)] \cap K, B = [\rho(1-r), \rho] \cap K$$

and

$$C = [\rho, r + \rho(1-r)] \cap K, D = [1-r, 1] \cap K.$$

These blocks have following relations:

$$T_1(A) = A \cup B \cup C$$

$$T_1(B) = D$$

$$T_2(C) = C \cup D$$

$$T_3(D) = A \cup B \cup C \cup D.$$

The weighted adjacency matrix for K is

$$A^t = \begin{pmatrix} \rho^t & \rho^t & \rho^t & 0 \\ 0 & 0 & 0 & \rho^t \\ 0 & 0 & r^t & r^t \\ r^t & r^t & r^t & r^t \end{pmatrix}$$

As $B = [\rho(1-r), \rho] \cap K = S_1 \circ S_3$ is an exact overlap. Therefore, the weighted adjacency matrix for U is

$$A^t = \begin{pmatrix} \rho^t & \rho^t & 0 \\ 0 & r^t & r^t \\ r^t & r^t & r^t \end{pmatrix}$$

By Theorem 4.2.11 and Theorem 4.2.17, we can find $\dim_H(K)$ and $\dim_H(U)$ simultaneously.

When $\rho = r = 1/3$ this example is related to the well-known $\{0, 1, 3\}$ problem. Let K be the attractor of following IFS.

$$\phi_1(x) = \lambda x, \phi_2(x) = \lambda(x+1), \phi_3(x) = \lambda(x+3)$$

[76] proved that for almost every $\lambda \in [4^{-1}, 3^{-1}]$, $\dim_H(K) = \dfrac{\log 3}{-\log \lambda}$. Our idea allows us to calculate many exceptional cases. The following example is another exceptional case for Pollicott and Simon's result.

Example 4.7.3. *Let the IFS be*

$$\phi_1(x) = \lambda x, \phi_2(x) = \lambda(x+1), \phi_3(x) = \lambda(x+3)$$

with contractive ratio $\lambda = \dfrac{\sqrt{13}-3}{2} \in [4^{-1}, 3^{-1}]$. *Then* $\dim_H(K) = \dfrac{\log(1+\sqrt{3})}{-\log \lambda} = \alpha$. *Moreover, we have*

$$0 < \mathcal{H}^\alpha(K) < \infty.$$

Again we partition $K = A \cup B \cup C \cup D \cup E$, where $A = [0, \lambda] \cap K$, $B = \left[\lambda, \dfrac{3\lambda^2}{1-\lambda}\right] \cap K$, $C = \left[\dfrac{3\lambda^2}{1-\lambda}, \lambda + \dfrac{3\lambda^2}{1-\lambda}\right] \cap K$, $D = [3\lambda, 1] \cap K$ and $E = \left[1, \dfrac{3\lambda}{1-\lambda}\right] \cap K$. This Markov partition has following relations:

$$T_1(A) = A \cup B \cup C \cup D$$

$$T_1(B) = E$$

$$T_2(C) = B \cup C \cup D \cup E$$

$$T_3(D) = A$$

$$T_3(E) = B \cup C \cup D \cup E$$

Hence the weighted adjacency matrix is

$$A^t = \begin{pmatrix} \lambda^t & \lambda^t & \lambda^t & \lambda^t & 0 \\ 0 & 0 & 0 & 0 & \lambda^t \\ 0 & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & 0 & 0 & 0 & 0 \\ 0 & \lambda^t & \lambda^t & \lambda^t & \lambda^t \end{pmatrix}$$

and $\dim_H(K) = \dfrac{\log(1+\sqrt{3})}{-\log\lambda}$.

The following example is analyzed in [25].

Example 4.7.4. $S_1(x) = r_1x + 1$, $S_2(x) = r_2x + 1 + 3r_1$, $S_3(x) = r_3x + 3$, $S_4(x) = r_4x$. *where* $0 < r_2, r_4 \le r_1 \le 4^{-1}$, $r_1r_3 = r_2r_4$.

Let $J = [0, 3(1-r_3)^{-1}]$. Then we have $S_4(J) = [0, 3r_4(1-r_3)^{-1}], S_1(J) = [1, 1 + 3r_1(1-r_3)^{-1}], S_2(J) = [1+3r_1, 1+3r_1+3r_2(1-r_3)^{-1}]$, $S_3(J) = [3, 3(1-r_3)^{-1}]$. Since $S_1 \circ S_3 = S_2 \circ S_4$, we partition the attractor via

$$K = A \cup B \cup C \cup D \cup E,$$

where

$$A = [0, 3r_4(1-r_3)^{-1}] \cap K, B = [1, 1+3r_1] \cap K, C = [1+3r_1, 1+3r_1(1-r_3)^{-1}] \cap K,$$

$$D = [1+3r_1(1-r_3)^{-1}, 1+3r_1+3r_2(1-r_3)], E = [3, 3(1-r_3)^{-1}]$$

This Markov partition has following relations:

$$T_4(A) = A \cup B \cup C \cup D \cup E$$

$$T_1(B) = A \cup B \cup C \cup D$$

$$T_1(C) = E$$

$$T_2(D) = B \cup C \cup D \cup E$$

$$T_3(E) = A \cup B \cup C \cup D \cup E$$

The weighted adjacency matrix of K is

$$A^t = \begin{pmatrix} r_4^t & r_4^t & r_4^t & r_4^t & r_4^t \\ r_1^t & r_1^t & r_1^t & r_1^t & 0 \\ 0 & 0 & 0 & 0 & r_1^t \\ 0 & r_2^t & r_2^t & r_2^t & r_2^t \\ r_3^t & r_3^t & r_3^t & r_3^t & r_3^t \end{pmatrix}$$

Therefore, $\dim_H(K) = \alpha$, where α is the unique solution of $r_1^\alpha + r_2^\alpha + r_3^\alpha + r_4^\alpha - (r_1r_3)^\alpha = 1$. The weighted adjacency matrix of U is

$$A^t = \begin{pmatrix} r_4^t & r_4^t & r_4^t & r_4^t \\ r_1^t & r_1^t & r_1^t & 0 \\ 0 & r_2^t & r_2^t & r_2^t \\ r_3^t & r_3^t & r_3^t & r_3^t \end{pmatrix}$$

as the block $C = S_1 \circ S_3(K)$. Thus $\dim_H(U) = \gamma$, where γ is the unique solution of $1 - a - b - c - d + 2ac = 0$, where $a = r_1^\gamma, b = r_2^\gamma, c = r_3^\gamma, d = r_4^\gamma$. Moreover, we have

$$0 < \mathcal{H}^\alpha(K) < \infty,$$

and

$$0 < \mathcal{H}^\gamma(U) < \infty.$$

The following example is the two-dimensional case. We give the partition and the rest proof of our statement follows Theorem 4.2.11.

Example 4.7.5.

$$f_1(x, y) = (\lambda x, \lambda y),\ f_2(x, y) = (\lambda x + 1 - \lambda, \lambda y),\ f_3(x, y) = (\lambda x + 1 - \lambda, \lambda y + 1 - \lambda),$$

and

$$f_4(x, y) = (\lambda x, \lambda y + 1 - \lambda),\ f_5(x, y) = (\lambda x + \lambda(1 - \lambda), \lambda y + (1 - \lambda)^2)$$

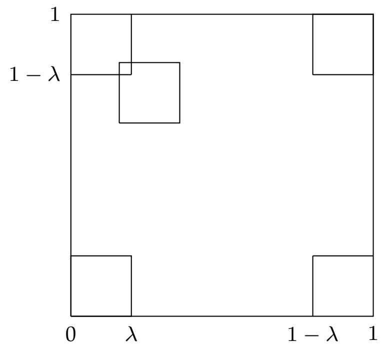
where $0 < \lambda < 1$ *and* $\lambda^2 - 3\lambda + 1 > 0$. *Then the Hausdorff dimension of* K *and* U *are* $\dim_H(K) = \dfrac{\log r}{-\log \lambda} = \alpha$, $\dim_H(U) = \dfrac{\log s}{-\log \lambda} = \gamma$, *where* r *is the largest root of* $x^2 - 5x + 1 = 0$, *and* s *is the largest positive root of* $x^3 - 5x^2 + 2x - 1 = 0$. *Moreover, we have*

$$0 < \mathcal{H}^\alpha(K) < \infty,$$

and

$$0 < \mathcal{H}^\gamma(U) < \infty.$$

The first iteration of $\{f_i([0, 1]^2)\}_{i=1}^5$ *can be found in the next page.*



Let

$$A_1 = ([0, \lambda] \times [0, \lambda]) \cap K, A_2 = ([1 - \lambda, 1] \times [0, \lambda]) \cap K, A_3 = ([1 - \lambda, 1] \times [1 - \lambda, 1]) \cap K,$$

$$A_4 = ([0, \lambda] \times [1 - \lambda, 1] \setminus (\lambda(1 - \lambda), \lambda] \times [1 - \lambda, \lambda^2 - \lambda + 1)) \cap K$$

$$A_5 = ([\lambda(1 - \lambda), \lambda] \times [1 - \lambda, \lambda^2 - \lambda + 1]) \cap K$$

and

$$A_6 = ([\lambda(1-\lambda), \lambda+\lambda(1-\lambda)] \times [(1-\lambda)^2, \lambda^2-\lambda+1] \setminus [\lambda(1-\lambda), \lambda) \times (1-\lambda, \lambda^2-\lambda+1]) \cap K.$$

These blocks have following relations:

$$T_1(A_1) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$

$$T_2(A_2) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$
$$T_3(A_3) = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$
$$T_4(A_4) = A_1 \cup A_3 \cup A_4 \cup A_5 \cup A_6$$
$$T_4(A_5) = A_2$$
$$T_5(A_6) = A_1 \cup A_2 \cup A_3 \cup A_6$$

Therefore the weighted adjacency matrix of K is

$$A^t = \begin{pmatrix} \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & 0 & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ 0 & \lambda^t & 0 & 0 & 0 & 0 \\ \lambda^t & \lambda^t & \lambda^t & 0 & 0 & \lambda^t \end{pmatrix}.$$

and the weighted adjacency matrix for U is

$$A^t = \begin{pmatrix} \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & \lambda^t & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & 0 & \lambda^t & \lambda^t & \lambda^t \\ \lambda^t & \lambda^t & \lambda^t & 0 & \lambda^t \end{pmatrix}.$$

Hence we can calculate the dimension of K and U respectively.

The following two-dimensional example was investigated in [56]. Again our algorithm allows us to find the dimension of K.

Example 4.7.6. *Let $S_1(x, y) = (\rho x, \rho y)$, $S_2(x, y) = (rx + \rho - \rho r, ry)$, $S_3(x, y) = (rx + 1 - r, ry)$, $S_4(x, y) = (rx, ry + 1 - r)$. where $0 < \rho < r < 1$ and $\rho + 2r - \rho r \le 1$.*

Let $A = ([0, \rho - \rho r] \times [0, \rho]) \cap K$, $B = (S_2(E)) \cap K$, $C = (S_3(E)) \cap K$, $D = (S_4(E)) \cap K$. After simple calculation, we have

$$T_1(A) = A \cup B \cup D$$
$$T_2(A) = A \cup B \cup C \cup D$$
$$T_3(A) = A \cup B \cup C \cup D$$
$$T_4(A) = A \cup B \cup C \cup D$$

Hence, the weighted adjacency matrix is

$$A^t = \begin{pmatrix} \rho^t & \rho^t & 0 & \rho^t \\ r^t & r^t & r^t & r^t \\ r^t & r^t & r^t & r^t \\ r^t & r^t & r^t & r^t \end{pmatrix}$$

and we have $\dim_H(U) = \alpha$, where α is the solution of $1 - 3r^\alpha - \rho^\alpha + (\rho r)^\alpha = 0$. Moreover, we have

$$0 < \mathcal{H}^\alpha(K) < \infty.$$

Now we give some examples in the setting of β-expansions.

Example 4.7.7. *Let $\beta \approx 1.8393$ be the appropriate root of $x^3 = x^2 + x + 1$. Then*

$$\dim_H(U) = \frac{\log G}{\log \beta},$$

where G is the golden mean.

In view of Theorem 4.4.3, it remains to calculate the spectral radius of S'.

From Example 4.4.1, we know that

$$S' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

therefore the spectral radius of this matrix is G.

Remark 4.7.8. *In [38], Glendenning and Sidorov stated (without proof) that for any multinacci number c_n, i.e. the largest positive root of $x^n = \sum_{i=1}^{n-1} x^i, n \geq 3$, $\dim_H(U) = \frac{\log c_{n-1}}{\log c_n}$. We may calculate all these cases in terms of Theorem 4.4.3. For this class, the quasi-greedy orbit of 1 hits β^{-1} for the first time, which means that $\tilde{U}$ is not a subshift of finite type.*

Example 4.7.9. *Let $\beta \approx 1.8668$ be the Pisot number satisfying $x^4 - 2x^3 + x - 1 = 0$, then $\dim_H(U) = \frac{\log G}{\log \beta}$, where G is the golden mean.*

$$S' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

For this example the quasi-greedy orbit of 1 hits $\beta^{-1}(\beta - 1)^{-1}$ for the first time, and $\tilde{U}$ is a subshift of finite type.

Example 4.7.10. *Let β be the largest positive root of $x^4 - x^3 - 2x^2 + 1 = 0$, then $\dim_H(U) = \frac{\log r}{\log \beta} \approx \frac{\log 1.7693}{\log \beta}$, where r is the real root of $x^3 - 2x - 2 = 0$.*

$$S' = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The spectral radius of this matrix is the real root of $x^3 - 2x - 2 = 0$. The solution is about 1.7693. For this example, 1 has a unique coding in base β, and this β is a Pisot number.

Remark 4.7.11. *Recently, Komornik, Kong and Li [52] gave a uniform formula of $\dim_H(U)$, i.e., $\dim_H(U) = \frac{h(\tilde{U})}{\log \beta}$, where $h(\tilde{U})$ is the entropy of $\tilde{U}$. In fact, Glendinning and Sidorov [38]* *mentioned this result in their paper (without proof). It is very difficult to calculate the entropy of $\tilde{U}$ for every $\beta > 1$. However, our main results, namely Theorem 3.2.4, Theorem 4.4.3 and Theorem 4.3.8 allow us to calculate $\dim_H(U)$ for almost $\beta > 1$. Similar result is also obtained in [54] by a different method.*

The final example is the attractor generated by the doubling map with hole.

Example 4.7.12. *Let $a = \dfrac{1}{31}$ and $b = \dfrac{2}{31}$, the hole $(a, b) \subset (0, 2^{-1})$. Then*

$$S' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

the spectral radius of this matrix is the largest solution of $x^4 = x^3 + x^2 + x + 1$, we denote it by α. Hence we have that $\dim_H(J(a, b)) = \dfrac{\log \alpha}{\log 2}$.

Chapter 5

Multiple expansions over digit set $\{0, 1, \beta\}$

Abstract

For $\beta > 1$ we consider expansions in base β over the alphabet $\{0, 1, \beta\}$. Let U_β be the set of x which have a unique β-expansions. For $k = 2, 3, \cdots, \aleph_0$ let $\mathbb{B}_k$ be the set of bases β for which there exists x having k different β-expansions, and for $\beta \in \mathbb{B}_k$ let $U_\beta^{(k)}$ be the set of all such x's which have k different β-expansions. In this paper we show that

$$\mathbb{B}_{\aleph_0} = [2, \infty), \quad \mathbb{B}_k = (q_c, \infty) \quad \text{for any} \quad k \geq 2,$$

where $q_c \approx 2.32472$ is the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Moreover, we show that for any positive integer $k \geq 2$ and any $\beta \in \mathbb{B}_k$ the Hausdorff dimensions of $U_\beta^{(k)}$ and U_β are the same, i.e.,

$$\dim_H U_\beta^{(k)} = \dim_H U_\beta \quad \text{for any} \quad k \geq 2.$$

Finally, we conclude that the set of x having a continuum of β-expansions has full Hausdorff dimension.

5.1 Introduction

Expansions in non-integer bases were pioneered by Rényi [78] and Parry [70]. It is well-known that typically a real number has a continuum of expansions (cf. [82, 16]). However, there still exist reals having a unique expansion (cf. [28, 38, 55]). Recently, de Vries and Komornik [84] investigated the topological properties of unique expansions. Komornik et al. [52] considered the Hausdorff dimension of unique expansions, and conclude that the dimension function behaves like a devil's staircase. Interestingly, for $k = 2, 3, \cdots$ or $\aleph_0$ it was first discovered by Erdős et al. [26, 27] that there exists x having k different expansions. For more information on expansions in non-integer bases we refer to [83, 3, 89], and survey [51].

In this chapter we consider expansions with digits set $\{0, 1, \beta\}$. Given $\beta > 1$, an infinite sequence (d_i) is called a *β-expansion* of x if

$$x = ((d_i))_\beta := \sum_{i=1}^{\infty} \frac{d_i}{\beta^i}, \quad d_i \in \{0, 1, \beta\}, \quad i \geq 1.$$

Here we point out that the expansion is over the *alphabet* $\{0, 1, \beta\}$ which depends on the base β.

For $\beta > 1$ let E_β be the set of x which have a β-expansion. Then E_β is the attractor of the *iterated function system* (cf. [46])

$$\phi_d(x) = \frac{x+d}{\beta}, \quad d \in \{0, 1, \beta\},$$

i.e., E_β is the non-empty compact set satisfying $E_\beta = \bigcup_{d\in\{0,1,\beta\}} \phi_d(E_\beta)$.

The set E_β is a *self-similar set* with overlaps, and it attracts great attention since the work of Ngai and Wang [68] for giving an explicit formulae for the Hausdorff dimension of E_β:

$$\dim_H E_\beta = \frac{\log q^*}{\log \beta} \quad \text{for any} \quad \beta > q^*, \tag{5.1}$$

where $q^* = (3 + \sqrt{5})/2$. Moreover, Yao and Li [88] considered all of its generating iterated function systems of the set E_β. Recently, Zou et al. [90] considered the set of points in E_β which have a unique β-expansion. Then it is natural to ask what can we say for points in E_β having multiple β-expansions?

For $k = 1, 2, \cdots, \aleph_0$ or $2^{\aleph_0}$, let $\mathbb{B}_k$ be the set of bases $\beta > 1$ such that there exists $x \in E_\beta$ having k different q-expansions. Accordingly, for $\beta \in \mathbb{B}_k$ let $U_\beta^{(k)}$ be the set of $x \in E_\beta$ having k different β-expansions. Then $\mathbb{B}_k = \{\beta > 1 : U_\beta^{(k)} \neq \emptyset\}$, and for $\beta \in \mathbb{B}_k$

$$U_\beta^{(k)} = \{x \in E_\beta : x \text{ has } k \text{ different } \beta\text{-expansions}\}.$$

For simplicity, we write $U_\beta := U_\beta^{(1)}$ for the set of $x \in E_\beta$ having a unique β-expansion, and denote by U'_β the set of corresponding expansions.

First we consider the set $\mathbb{B}_k$ for $k = 1, 2, \cdots, \aleph_0$ or $2^{\aleph_0}$. Clearly, when $k = 1$ we have $\mathbb{B}_1 = (1, \infty)$, since 0 always has a unique β-expansion for any $\beta > 1$.

When $k = 2, 3, \cdots, \aleph_0$ or $2^{\aleph_0}$ we have the following theorem.

Theorem 5.1.1. *Let $q_c \approx 2.32472$ be the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$. Then*

$$\mathbb{B}_{2^{\aleph_0}} = (1, \infty), \quad \mathbb{B}_{\aleph_0} = [2, \infty), \quad \mathbb{B}_k = (q_c, \infty) \quad \textit{for any} \quad k \geq 2.$$

In terms of Theorem 5.1.1 it follows that for $\beta \in [2, q_c]$ any $x \in E_\beta$ can only have a unique β-expansion, countably infinitely many β-expansions, or a continuum of β-expansions.

For $k \geq 1$ and $q \in \mathbb{B}_k$ we consider the set $U_\beta^{(k)}$. When $k = 1$, the following theorem for the *univoque set* $U_\beta = U_\beta^{(1)}$ was shown in [90].

Theorem 5.1.2 ([90]).
- *If $\beta \in (1, q_c]$, then $U_\beta = \{0, \beta/(\beta - 1)\}$;*
- *If $\beta \in (q_c, q^*)$, then U_β contains a continuum of points;*
- *If $\beta \in [q^*, \infty)$, then $\dim_H U_\beta = \log q_c / \log \beta$.*

In the following theorem we show that the Hausdorff dimensions of $U_\beta^{(k)}$ are the same for any integer $k \geq 1$.

Theorem 5.1.3. *For any integer $k \geq 2$ and any $\beta \in \mathbb{B}_k$ we have*

$$\dim_H U_\beta^{(k)} = \dim_H U_\beta.$$

Moreover, $\dim_H U_\beta > 0$ if and only if $\beta > q_c$.

In terms of Theorem 5.1.3 it follows that q_c is indeed the *critical base* in the sense that $U_\beta^{(k)}$ has positive Hausdorff dimension if $\beta > q_c$, while $U_\beta^{(k)}$ has zero Hausdorff dimension if $\beta \le q_c$. In fact, by Theorems 5.1.1 and 5.1.2 it follows that for $\beta \le q_c$ the set $U_\beta = \{0, \beta/(\beta-1)\}$ and $U_\beta^{(k)} = \emptyset$ for any integer $k \ge 2$.

In the following theorem we consider $U_\beta^{(\aleph_0)}$ and $U_\beta^{(2^{\aleph_0})}$.

Theorem 5.1.4.

- *Let* $q \in \mathbb{B}_{\aleph_0} \setminus (q_c, q^*)$. *Then* $U_\beta^{(\aleph_0)}$ *contains countably infinitely many points;*
- *Let* $\beta > 1$. *Then* $U_\beta^{(2^{\aleph_0})}$ *has full Hausdorff dimension, i.e.,*

$$\dim_H U_\beta^{(2^{\aleph_0})} = \dim_H E_\beta.$$

Remark 5.1.5. *In fact, we show in Lemma 5.5.5 that the Hausdorff measures of* $U_\beta^{(2^{\aleph_0})}$ *and* E_β *are the same for any* $\beta > 1$, *i.e.,*

$$\mathcal{H}^s(U_\beta^{(2^{\aleph_0})}) = \mathcal{H}^s(E_\beta) \in (0, \infty),$$

where $s = \dim_H E_\beta$.

The rest of the paper is arranged in the following way. In Section 5.2 we recall some properties of unique β-expansions. The proof of Theorem 5.1.1 for the sets $\mathbb{B}_k$ will be presented in Section 5.3, and the proofs of Theorems 5.1.3 and 5.1.4 for the sets $U_\beta^{(k)}$ will be given in Sections 5.4 and 5.5, respectively. Finally, in Section 5.6 we give some examples and end the paper with some questions.

5.2 Unique expansions

In this section we recall some properties of the univoque set U_β. Recall that

$$q_c \approx 2.32472, \quad q^* = \frac{3+\sqrt{5}}{2}.$$

Here q_c is the appropriate root of the equation $x^3 - 3x^2 + 2x - 1 = 0$. Then for $\beta \in (1, q^*]$ the attractor $E_\beta = [0, \beta/(\beta-1)]$ is an interval. However, for $\beta > q^*$ the attractor E_β is a Cantor set which contains neither interior nor isolated points.

The following characterization of the univoque set U_β for $\beta > q^*$ was established in [90, Lemma 3.1].

Lemma 5.2.1. *Let* $\beta > q^*$. *Then* $(d_i) \in U'_\beta$ *if and only if*

$$\begin{cases} (d_{n+i}) < \beta 0^\infty & \text{if } \ d_n = 0, \\ (d_{n+i}) > 1^\infty & \text{if } \ d_n = 1. \end{cases}$$

In the following we consider unique q-expansions with $\beta \le q^*$. For $\beta \in (1, q^*]$ we denote by

$$\alpha(\beta) = (\alpha_i(\beta))$$

the *quasi-greedy* β-expansion of $\beta - 1$, i.e., the lexicographical largest infinite β-expansion of $\beta - 1$. Here an expansion (d_i) is called *infinite* if $d_i \ne 0$ for infinitely many indices $i \ge 1$.

In terms of Theorem 5.1.2 it is interesting to consider the set U'_β of unique expansions for $\beta \in (q_c, q^*]$. The following lemma was obtained in [90, Lemmas 3.1 and 3.2].

Lemma 5.2.2. *Let $\beta \in (q_c, q^*]$. Then*

$$A_\beta \subseteq U_\beta{}' \subseteq B_\beta,$$

where A_β is the set of sequences $(d_i) \in \{0,1,\beta\}^\infty$ satisfying

$$\begin{cases} (d_{n+i}) < 1\alpha(\beta) & \text{if } \; d_n = 0, \\ 1^\infty < (d_{n+i}) < \alpha(\beta) & \text{if } \; d_n = 1, \\ (d_{n+i}) > 0\beta^\infty & \text{if } \; d_n = \beta, \end{cases} \tag{5.2}$$

and B_β is the set of sequences $(d_i) \in \{0,1,\beta\}^\infty$ satisfying the first two inequalities in (5.2).

For $\beta > 1$ let $\Phi : \{0,1,\beta\}^\infty \to \{0,1,2\}^\infty$ be defined by

$$\Phi((d_i)) = (d_i'),$$

where $d_i' = d_i$ if $d_i \in \{0,1\}$, and $d_i' = 2$ if $d_i = \beta$. Clearly, the map Φ is continuous and bijective.

The following monotonicity of $\Phi(\alpha(\beta))$ was given in [90, Lemma 3.2].

Lemma 5.2.3. *The map $\beta \to \Phi(\alpha(\beta))$ is strictly increasing in $(1, q^*]$.*

5.3 The range of $\mathbb{B}_k$

In this section we will investigate the set $\mathbb{B}_k$ of bases $\beta > 1$ in which there exists $x \in E_\beta$ having k different β-expansions. When $k = 1$ it is obviously that $\mathbb{B}_1 = (1, \infty)$ because $0 \in E_\beta$ always has a unique q-expansion 0^∞ for any $\beta > 1$. In the following we consider $\mathbb{B}_k$ for $k = 2, 3, \cdots, \aleph_0$ or $2^{\aleph_0}$.

The following lemma was established in [90, Theorem 4.1] and [36, Theorem 1.1].

Lemma 5.3.1. *Let $\beta \in (1,2)$. Then any $x \in E_\beta$ has either a unique β-expansion, or a continuum of β-expansions. Moreover, for $\beta = 2$ any $x \in E_\beta$ can only have a unique β-expansion, countably infinitely many β-expansions, or a continuum of β-expansions.*

For $\beta > 1$ we recall that $\phi_d(x) = (x+d)/\beta, d = 0, 1, \beta$. Let

$$S_\beta := (\phi_0(E_\beta) \cap \phi_1(E_\beta)) \cup (\phi_1(E_\beta) \cap \phi_\beta(E_\beta)) \,. \tag{5.3}$$

Here S_β is called the *switch region*, since any $x \in S_\beta$ has at least two β-expansions. Clearly, any $x \in \phi_0(E_\beta) \cap \phi_1(E_\beta)$ has at least two β-expansions: one beginning with the word 0 and one beginning with the word 1. Accordingly, any $x \in \phi_1(E_\beta) \cap \phi_\beta(E_\beta)$ also has at least two β-expansions: one starting at the word 1 and one starting at the word β. We point out that the union in (5.3) is disjoint if $\beta > 2$. In fact, for $\beta > q^*$ the intersection $\phi_1(E_\beta) \cap \phi_\beta(E_\beta) = \emptyset$.

For $x \in E_\beta$ let $\Sigma(x)$ be the set of all β-expansions of x, i.e.,

$$\Sigma(x) := \{(d_i) \in \{0,1,\beta\}^\infty : ((d_i))_\beta = x\},$$

and denote by $|\Sigma(x)|$ its cardinality.

We recall from [3] that a point $x \in S_\beta$ is called a *β-null infinite point* if x has an expansion (d_i) such that whenever

$$x_n := (d_{n+1}d_{n+2}\cdots)_\beta \in S_\beta,$$

one of the following quantities is infinity, and another two are finite:

$$\left|\Sigma(\phi_0^{-1}(x_n))\right|, \quad \left|\Sigma(\phi_1^{-1}(x_n))\right| \quad \text{and} \quad \left|\Sigma(\phi_\beta^{-1}(x_n))\right|.$$

Clearly, any β-null infinite point has countably infinitely many β-expansions. In order to investigate $\mathbb{B}_{\aleph_0}$ we need the following relationship between $\mathbb{B}_{\aleph_0}$ and β-null infinite points which was established in [3] (see also, [89]).

Lemma 5.3.2. *$\beta \in \mathbb{B}_{\aleph_0}$ if and only if S_β contains a β-null infinite point.*

First we consider the set $\mathbb{B}_{\aleph_0}$.

Lemma 5.3.3. $\mathbb{B}_{\aleph_0} = [2, \infty)$.

Proof. By Lemma 5.3.1 we have $\mathbb{B}_{\aleph_0} \subseteq [2, \infty)$ and $2 \in \mathbb{B}_{\aleph_0}$. So, it suffices to prove $(2, \infty) \subseteq \mathbb{B}_{\aleph_0}$.

Take $\beta \in (2, \infty)$. Note that $0 = (0^\infty)_\beta$ and $\beta/(\beta - 1) \in (\beta^\infty)_\beta$ belong to U_β. We claim that

$$x = (0\beta^\infty)_\beta$$

is a β-null infinite point.

By the words substitution $10 \sim 0\beta$ it follows that all expansions $1^k 0\beta^\infty, k \geq 0$, are β-expansions of x, i.e.,

$$\bigcup_{k=0}^{\infty} \{1^k 0\beta^\infty\} \subseteq \Sigma(x).$$

This implies that $|\Sigma(x)| = \infty$.

Furthermore, since $\beta > 2$, the union in (5.3) is disjoint. This implies

$$x = (0\beta^\infty)_\beta = (10\beta^\infty)_\beta \in \phi_0(E_\beta) \cap \phi_1(E_\beta) \setminus \phi_\beta(E_\beta).$$

Then $\phi_0^{-1}(x) = (\beta^\infty)_\beta \in U_\beta$, $\phi_1^{-1}(x) = x$ and $\phi_\beta^{-1}(x) \notin E_\beta$, i.e.,

$$|\Sigma(\phi_0^{-1}(x))| = 1, \quad |\Sigma(\phi_1^{-1}(x))| = \infty, \quad |\Sigma(\phi_q^{-1}(x))| = 0.$$

By iteration it follows that x is a β-null infinite point. Hence, by Lemma 5.3.2 it yields that $\beta \in \mathbb{B}_{\aleph_0}$, and therefore $(2, \infty) \subseteq \mathbb{B}_{\aleph_0}$. □

In the following we will consider $\mathbb{B}_k$. By Lemma 5.3.1 it follows that $\mathbb{B}_k \subseteq (2, \infty)$ for any $k \geq 2$. First we consider $k = 2$. In the following lemma we give a characterization of the set $\mathbb{B}_2$.

Lemma 5.3.4. *Let $\beta > 2$. Then $\beta \in \mathbb{B}_2$ if and only if there exist $(a_i), (b_i) \in U_\beta'$ such that*

$$(1(a_i))_\beta = (0(b_i))_\beta,$$

or there exist $(c_i), (d_i) \in U_\beta'$ such that

$$(1(c_i))_\beta = (\beta(d_i))_\beta.$$

Proof. First we prove the necessity. Take $\beta \in \mathbb{B}_2$. Suppose $x \in E_\beta$ has two different β-expansions, say

$$((a_i))_\beta = x = ((b_i))_\beta.$$

Then there exists a least integer $k \geq 1$ such that $a_k \neq b_k$. Then

$$(a_k a_{k+1} \cdots)_\beta = (b_k b_{k+1} \cdots)_\beta \in S_\beta, \quad \text{and} \quad (a_{k+i})_\beta, (b_{k+i})_\beta \in U_\beta. \tag{5.4}$$

Since $\beta > 2$, it gives that the union in (5.3) is disjoint. Then the necessity follows by (5.4).

Now we turn to prove the sufficiency. Without loss of generality we assume $(1(a_i))_\beta = (0(b_i))_\beta$ with $(a_i), (b_i) \in U'_\beta$. Note by $\beta > 2$ that the union in (5.3) is disjoint. Then

$$x = (1(a_i))_\beta = (0(b_i)) \in \phi_0(E_\beta) \cap \phi_1(E_\beta) \setminus \phi_\beta(E_\beta).$$

This implies that x has two different β-expansions, i.e., $\beta \in \mathbb{B}_2$. □

Recall that $q_c \approx 2.32472$ is the appropriate root of $x^3 - 3x^2 + 2x - 1 = 0$, and $q^* = (3 + \sqrt{5})/2$. By a direct computation one can verify that

$$\alpha(q_c) = q_c 1^\infty, \quad \alpha(q^*) = (q^*)^\infty. \tag{5.5}$$

In the following lemma we consider the set $\mathbb{B}_2$.

Lemma 5.3.5. $\mathbb{B}_2 = (q_c, \infty)$.

Proof. First we show that $\mathbb{B}_2 \subseteq (q_c, \infty)$. By Lemma 5.3.1 it suffices to prove that $(2, q_c]$ is not contained in $\mathbb{B}_2$. Take $\beta \in (2, q_c]$. Then by Theorem 5.1.2 it gives that $U_\beta = \{(0^\infty)_\beta, (\beta^\infty)_\beta\}$. In terms of Lemma 5.3.4 it follows that if $\beta \in \mathbb{B}_2 \cap (2, q_c]$ then β must satisfies one of the following equations

$$(10^\infty)_\beta = (0\beta^\infty)_\beta \quad \text{or} \quad (1\beta^\infty)_\beta = (\beta 0^\infty)_\beta.$$

This is impossible since neither equation has a solution in $(2, q_c]$. Hence, $\mathbb{B}_2 \subseteq (q_c, \infty)$.

Now we turn to prove $(q_c, \infty) \subseteq \mathbb{B}_2$. In terms of Lemmas 5.2.1 and 5.3.4 one can verify that for any $\beta > q^*$ the number

$$x = (0\beta 0^\infty)_\beta = (10^\infty)_\beta$$

has two different q-expansions. This implies that $(q^*, \infty) \subseteq \mathbb{B}_2$.

In the following it suffices to prove $(q_c, q^*] \subset \mathbb{B}_2$. Take $\beta \in (q_c, q^*]$. Note by (5.5) that $\alpha(q_c) = q_c 1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. Then by Lemma 5.2.3 there exists a large integer m such that

$$\alpha(\beta) > \beta 1^m \beta 0^\infty.$$

Hence, by Lemmas 5.2.2 and 5.3.4 one can verify that

$$y = (0\beta(1^{m+1}\beta)^\infty)_\beta = (10(1^{m+1}\beta)^\infty)_\beta$$

has two different β-expansions. This implies that $(q_c, q^*] \subseteq \mathbb{B}_2$, and completes the proof. □

Lemma 5.3.6. $\mathbb{B}_k = (q_c, \infty)$ *for any* $k \geq 3$.

Proof. First we prove $\mathbb{B}_k \subseteq \mathbb{B}_2$ for any $k \geq 3$. By Lemma 5.3.1 it follows that $\mathbb{B}_k \subseteq (2, \infty)$. Take $\beta \in \mathbb{B}_k$ with $k \geq 3$. Suppose $x \in E_\beta$ has k different β-expansions. Since $\beta > 2$, the union in (5.3) is disjoint. This implies that there exists a word $d_1 \cdots d_n$ such that

$$\phi_{d_1}^{-1} \circ \cdots \circ \phi_{d_n}^{-1}(x)$$

has two different q-expansions, i.e., $\beta \in \mathbb{B}_2$. Hence, $\mathbb{B}_k \subseteq \mathbb{B}_2$ for any $k \geq 3$.

Now we turn to prove $\mathbb{B}_2 \subseteq \mathbb{B}_k$ for any $k \geq 3$. In terms of Lemma 5.3.5 it suffices to prove

$$(q_c, \infty) \subseteq \mathbb{B}_k.$$

First we prove $(q^*, \infty) \subseteq \mathbb{B}_k$. Take $\beta \in (q^*, \infty)$. We only need to show that for any $k \geq 1$,

$$x_k = (0\beta^{k-1}(1\beta)^\infty)_\beta$$

has k different β-expansions. We will prove this by induction on k.

For $k = 1$ one can easily check by using Lemma 5.2.1 that $x_1 = (0(1\beta)^\infty)_\beta \in U_\beta$. Suppose x_k has exactly k-different β-expansions. Now we consider x_{k+1}, which can be written as

$$x_{k+1} = (0\beta^k(1\beta)^\infty)_\beta = (10\beta^{k-1}(1\beta)^\infty)_\beta.$$

By Lemma 5.2.1 we have $\beta^k(1\beta)^\infty \in U'_\beta$. Moreover, by the induction hypothesis

$$(0\beta^{k-1}(1\beta)^\infty)_\beta = x_k$$

has exactly k different β-expansions. Then x_{k+1} has at least $k+1$ different β-expansions. On the other hand, note by $\beta > q^* > 2$ that the union in (5.3) is disjoint. Then

$$x_{k+1} \in \phi_0(E_\beta) \cap \phi_1(E_\beta) \setminus \phi_\beta(E_\beta).$$

This implies that x_{k+1} indeed has $k + 1$ different q-expansions.

In the following we prove $(q_c, q^*] \subseteq \mathbb{B}_k$. Take $\beta \in (q_c, q^*]$. Then by (5.5) and Lemma 5.2.3 there exists a sufficiently large integer $m \geq 1$ such that

$$\alpha(\beta) > \beta 1^m \beta 0^\infty. \tag{5.6}$$

We will finish the proof by inductively showing that

$$y_k = (0\beta^{k-1}(1^{m+1}\beta)^\infty)_\beta$$

has k different β-expansions.

If $k = 1$, then by using (5.6) in Lemma 5.2.2 it gives that $y_1 = (0(1^{m+1}\beta)^\infty)_\beta$ has a unique β-expansion. Suppose y_k has exactly k-different β-expansions. Now we consider y_{k+1}. Clearly,

$$y_{k+1} = (10\beta^{k-1}(1^{m+1}\beta)^\infty)_\beta = (0\beta^k(1^{m+1}\beta)^\infty)_\beta.$$

By (5.6) and Lemma 5.2.2 it yields that $\beta^k(1^{m+1}\beta)^\infty \in U'_\beta$. By induction we know that $(0\beta^{k-1}(1^{m+1}\beta)^\infty)_\beta = y_k$ has exactly k different β-expansions. This implies that y_{k+1} has at least $k + 1$ different β-expansions. On the other hand, note that $\beta > q_c > 2$, and therefore the union in (5.3) is disjoint. So,

$$y_{k+1} \in \phi_0(E_\beta) \cap \phi_1(E_\beta) \setminus \phi_\beta(E_\beta),$$

which implies that y_{k+1} indeed has $k+1$ different β-expansions. □

Proof of Theorem 5.1.1. In terms of Lemmas 5.3.3, 5.3.5 and 5.3.6 it suffices to prove $\mathbb{B}_{2^{\aleph_0}} = (1, \infty)$. This can be verified by observing that

$$x = ((100)^\infty)_\beta \in U_\beta^{(2^{\aleph_0})}$$

for any $\beta > 1$. Because by the word substitution $10 \sim 0\beta$ one can show that x indeed has a continuum of different β-expansions.

□

5.4 Finite expansions

In this section we are going to investigate the Hausdorff dimension of $U_\beta^{(k)}$. First we show that $q_c \approx 2.32472$ is the critical base for U_β.

Lemma 5.4.1. *Let $\beta > 1$. Then $\dim_H U_\beta > 0$ if and only if $\beta > q_c$.*

Proof. The necessity follows by Theorem 5.1.2. Now we consider the sufficiency. Take $\beta \in (q_c, \infty)$. If $\beta > q^*$, then by Theorem 5.1.2 we have

$$\dim_H U_\beta = \frac{\log q^*}{\log \beta} > 0.$$

Then it suffices to prove $\dim_H U_\beta > 0$ for any $\beta \in (q_c, q^*]$.

Take $\beta \in (q_c, q^*]$. Recall from (5.5) that $\alpha(q_c) = q_c 1^\infty$ and $\alpha(q^*) = (q^*)^\infty$. Then by Lemma 6.2.4 it follows that there exists a sufficiently large integer $m \geq 1$ such that

$$\alpha(\beta) > \beta 1^m \beta 0^\infty.$$

Whence, by Lemma 6.2.3 one can verify that all sequences in

$$\Delta_m' := \prod_{i=1}^\infty \{\beta 1^{m+1}, 1^{m+2}\}$$

excluding those ending with 1^∞ belong to U_β'. This implies that

$$\dim_H U_\beta \geq \dim_H \Delta_m(\beta), \tag{5.7}$$

where $\Delta_m(\beta) = \{((d_i))_\beta : (d_i) \in \Delta_m'\}$.

Note that $\Delta_m(\beta)$ is a self-similar set generated by the IFS

$$f_1(x) = \frac{x}{\beta^{m+2}} + (\beta 1^{m+1} 0^\infty)_\beta, \quad f_2(x) = \frac{x}{\beta^{m+2}} + (1^{m+2} 0^\infty)_\beta,$$

which satisfies the open set condition (cf. [30]). Therefore, by (5.7) we conclude that

$$\dim_H U_\beta \geq \dim_H \Delta_m(\beta) = \frac{\log 2}{(m+2)\log \beta} > 0.$$

□

In the following we will consider the Hausdorff dimension of $U_\beta^{(k)}$ for any $k \geq 2$, and prove $\dim_H U_\beta^{(k)} = \dim_H U_\beta$. First we consider the upper bound of $\dim_H U_\beta^{(k)}$.

Lemma 5.4.2. *Let* $\beta > 1$. *Then* $\dim_H U_\beta^{(k)} \le \dim_H U_\beta$ *for any* $k \ge 2$.

Proof. Recall that $\phi_d(x) = (x+d)/\beta$ for $d \in \{0,1,\beta\}$. Then the lemma follows by observing that for any $k \ge 2$ we have

$$U_\beta^{(k)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1\cdots d_n \in \{0,1,\beta\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n}(U_\beta).$$

□

Now we consider the lower bound of $\dim_H U_\beta^{(k)}$. By Lemmas 5.4.1 and 5.4.2 it follows that

$$\dim_H U_\beta^{(k)} = 0 = \dim_H U_\beta$$

for any $\beta \le q_c$. So, in the following it suffices to consider $\beta > q_c$.

For $\beta > q_c$ let

$$F_\beta^{'}(1) := \{(d_i) \in U_\beta^{'} : d_1 = 1\}$$

be the *follower set* in $U_\beta^{'}$ generated by the word 1, and let $F_\beta(1)$ be the set of $x \in E_q$ which have a β-expansion in $F_\beta^{'}(1)$, i.e.,

$$F_\beta(1) = \{((d_i))_\beta : (d_i) \in F_\beta^{'}(1)\}.$$

Lemma 5.4.3. *Let* $\beta > q_c$. *Then* $\dim_H U_\beta^{(k)} \ge \dim_H F_\beta(1)$ *for any* $k \ge 1$.

Proof. For $k \ge 1$ and $\beta > q_c$ let

$$\Lambda_\beta^k := \{((d_i))_\beta : d_1 \cdots d_k = 0\beta^{k-1}, (d_{k+i}) \in F_\beta'(1)\}.$$

Then $\Lambda_\beta^k = \phi_0 \circ \phi_\beta^{k-1}(F_\beta(1))$, and therefore

$$\dim_H \Lambda_\beta^k = \dim_H F_\beta(1).$$

Hence, it suffices to prove $\Lambda_\beta^k \subseteq U_\beta^{(k)}$.

Take

$$x_k = (0\beta^{k-1}(c_i))_\beta \in \Lambda_\beta^k \quad \text{with} \quad (c_i) \in F_\beta'(1).$$

We will prove by induction on k that x_k has k different β-expansions.

For $k = 1$, by Lemmas 5.2.1 and 5.2.2 it follows that $x_1 = (0(c_i))_\beta \in U_\beta$. Suppose $x_k = (0\beta^{k-1}(c_i))_\beta$ has k different β-expansions. Now we consider x_{k+1}, which can be expanded as

$$x_{k+1} = (0\beta^k(c_i))_\beta = (10\beta^{k-1}(c_i))_\beta.$$

By Lemmas 5.2.1 and 5.2.2 we have $q^k(c_i) \in U_\beta'$, and by the induction hypothesis it yields that $(0\beta^{k-1}(c_i))_\beta = x_k$ has k different β-expansions. This implies that x_{k+1} has at least $k+1$ different β-expansions. On the other hand, since $\beta > q_c > 2$, it gives that the union in (5.3) is disjoint. Then

$$x_{k+1} \in \phi_0(E_\beta) \cap \phi_1(E_\beta) \setminus \phi_\beta(E_\beta).$$

This implies that x_{k+1} indeed has $k+1$ different β-expansions, and we conclude that $\Lambda_\beta^k \subseteq U_\beta^{(k)}$. □

Lemma 5.4.4. *Let $\beta > q_c$. Then* $\dim_H F_\beta(1) \geq \dim_H U_\beta$.

Proof. First we consider $\beta > q^*$. By Lemma 5.2.1 one can show that U'_β is contained in an irreducible sub-shift of finite type X'_A over the states $\{0,1,\beta\}$ with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{5.8}$$

Moreover, the complement set $X'_A \setminus U'_\beta$ contains all sequences ending with 1^∞. This implies that

$$\dim_H U_\beta = \dim_H X_A(\beta), \tag{5.9}$$

where $X_A(\beta) := \{((d_i))_\beta : (d_i) \in X'_A\}$. Note that $X_A(\beta)$ is a graph-directed set satisfying the open set condition (cf. [90, Theorem 3.4], see also in Chapter 3), and the sub-shift of finite type X'_A is irreducible. Then by (5.9) it follows that

$$\dim_H U_\beta = \dim_H X_A(\beta) = \dim_H F_\beta(1).$$

Now we consider $\beta \in (q_c, q^*]$. Using Lemma 5.2.2, we have

$$U'_\beta \subseteq \{0^\infty\} \cup \bigcup_{k=0}^{\infty} \{\beta^k 0^\infty\} \cup \bigcup_{k=0}^{\infty} \bigcup_{m=0}^{\infty} \{\beta^k 0^m F'_\beta(1)\},$$

where

$$\beta^k 0^m F'_\beta(1) := \{(d_i) : d_1 \cdots d_{k+m} = \beta^k 0^m, (d_{k+m+i}) \in F'_\beta(1)\}.$$

This implies that $\dim_H U_\beta \leq \dim_H F_\beta(1)$. □

Proof of Theorem 5.1.3. The theorem follows directly by Lemmas 5.4.1–5.4.4. □

5.5 Countable expansions and uncountable expansions

In the following we will consider the set $U_\beta^{(\aleph_0)}$ which contains all $x \in E_\beta$ having countably infinitely many β-expansions.

Lemma 5.5.1. *For any $\beta \in \mathbb{B}_{\aleph_0}$ the set $U_\beta^{(\aleph_0)}$ contains infinitely many points.*

Proof. It suffices to show that all of these points

$$z_k := (0^k \beta^\infty)_\beta, \quad k \geq 1,$$

are β-null infinite points, i.e., $z_k \in U_\beta^{(\aleph_0)}$.

Note by Theorem 5.1.1 that $\beta \in \mathbb{B}_{\aleph_0} = [2, \infty)$. If $\beta > 2$, then by the proof of Lemma 6.3.3 it yields that $z_1 = (0\beta^\infty)_\beta$ is a β-null infinite point. Moreover, note that $z_k = \phi_0^{k-1}(z_1) \notin S_\beta$ for any $k \geq 2$. This implies that all of these points $z_k, k \geq 1$, are q-null infinite points. So, $\{z_k : k \geq 1\} \subseteq U_\beta^{(\aleph_0)}$.

If $\beta = 2$, then by using the substitutions

$$0\beta \sim 10, \quad 0\beta^\infty = 1^\infty = \beta 0^\infty,$$

one can also show that z_k is a β-null infinite point. Moreover, all of the β-expansions of $z_k = (0^k \beta^\infty)_\beta$ are of the form

$$0^k \beta^\infty, \quad 0^{k-1} 1^\infty; \quad 0^{k-1} 1^m 0 \beta^\infty, \quad 0^{k-1} 1^{m-1} \beta 0^\infty,$$

where $m \geq 1$. Therefore, $z_k \in U_\beta^{(\aleph_0)}$ for any $k \geq 1$. □

First we consider the set $U_\beta^{(\aleph_0)}$ for $\beta \geq q^*$.

Lemma 5.5.2. *Let* $\beta \geq q^*$. *Then* $U_\beta^{(\aleph_0)}$ *is at most countable.*

Proof. Let $x \in U_\beta^{(\aleph_0)}$. Then x has a β-expansion (d_i) satisfying

$$|\Sigma(x_n)| = \infty$$

for infinitely many integers $n \geq 1$, where $x_n := ((d_{n+i}))_\beta$. This implies that (d_i) can not end in $U_\beta^{'}$.

Note by the proof of Lemma 5.4.4 that $U_\beta^{'} \subseteq X_A'$, where X_A' is a sub-shift of finite type over the state $\{0, 1, \beta\}$ with adjacency matrix A defined in (5.8). Moreover, $X_A' \setminus U_\beta^{'}$ is at most countable (cf. [90, Theorem 3.4]). Therefore, we will finish the proof by showing that the sequence (d_i) must end in X_A'.

Suppose on the contrary that (d_i) does not end in X_A'. Then the word 0β or 10 occurs infinitely many times in (d_i). Using the word substitution $0\beta \sim 10$ this implies that $x = ((d_i))_\beta$ has a continuum of β-expansions, leading to a contradiction with $x \in U_\beta^{(\aleph_0)}$. □

Now we prove that $U_\beta^{(\aleph_0)}$ is also countable for $\beta \in [2, q^*]$.

Lemma 5.5.3. *Let* $\beta \in [2, q_c]$. *Then* $U_\beta^{(\aleph_0)}$ *is at most countable.*

Proof. Take $\beta \in [2, q_c]$. By Theorems 5.1.1 and 5.1.2 it follows that any $x \in E_\beta$ with $|\Sigma(x)| < \infty$ must belong to $U_\beta = \{0, \beta/(\beta - 1)\}$. Suppose $x \in U_\beta^{(\aleph_0)}$. Then there exists a word $d_1 \cdots d_n$ such that

$$\phi_{d_1}^{-1} \circ \cdots \circ \phi_{d_n}^{-1}(x) \in U_\beta.$$

This implies that the set $U_\beta^{(\aleph_0)}$ is at most countable, since

$$U_\beta^{(\aleph_0)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1 \cdots d_n \in \{0,1,\beta\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n}(U_\beta).$$

□

In the following lemma we consider the Hausdorff dimension of the set $U_\beta^{(\aleph_0)}$ for $\beta \in (q_c, q^*)$.

Lemma 5.5.4. *For* $\beta \in (q_c, q^*)$ *we have* $\dim_H U_\beta^{(\aleph_0)} \leq \dim_H U_\beta < 1$.

Proof. Take $\beta \in (q_c, q^*)$. Note that

$$U_\beta^{(\aleph_0)} \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1 \cdots d_n \in \{0,1,\beta\}^n} \phi_{d_1} \circ \cdots \circ \phi_{d_n}(U_\beta).$$

This implies that $\dim_H U_\beta^{(\aleph_0)} \leq \dim_H U_\beta$. In the following it suffices to prove $\dim_H U_\beta < 1$.

Note that $U_\beta' \subseteq X_A'$, where X_A' is the sub-shift of finite type over the state $\{0, 1, \beta\}$ with adjacency matrix A defined in (5.8). Then

$$U_\beta \subseteq X_A(\beta) = \{((d_i))_\beta : (d_i) \in X_A'\}.$$

Note that $X_A(\beta)$ is a graph-directed set (cf. [62]). This implies that

$$\dim_H U_\beta \leq \dim_H X_A(\beta) \leq \frac{\log q_c}{\log \beta} < 1.$$

□

In the following lemma we investigate the set $U_\beta^{(2^{\aleph_0})}$ and show that $U_\beta^{(2^{\aleph_0})}$ has full Hausdorff measure.

Lemma 5.5.5. *Let $\beta > 1$. Then the set $U_\beta^{(2^{\aleph_0})}$ has full Hausdorff measure, i.e.,*

$$\mathcal{H}^s(U_\beta^{(2^{\aleph_0})}) = \mathcal{H}^s(E_\beta) \in (0, \infty),$$

where $s = \dim_H E_\beta$.

Proof. Clearly, for $\beta \in (1, q^*]$ we have $E_\beta = [0, \beta/(\beta - 1)]$, and therefore $s = \dim_H E_\beta = 1$. Moreover, for $\beta > q^*$ we have by (5.1) that $s = \dim_H E_\beta = \log q^* / \log \beta$. Hence, the set E_β has positive and finite Hausdorff measure (cf. [68]), i.e.,

$$0 < \mathcal{H}^s(E_\beta) < \infty \quad \text{for any} \quad \beta > 1. \tag{5.10}$$

Moreover,

$$E_\beta = U_\beta^{(2^{\aleph_0})} \cup U_\beta^{(\aleph_0)} \cup \bigcup_{k=1}^{\infty} U_\beta^{(k)}. \tag{5.11}$$

First we prove the lemma for $\beta \leq q^*$. By Theorems 5.1.1 and 5.1.2 it follows that for any $\beta \in (1, q^*]$

$$\dim_H U_\beta^{(k)} = \dim_H U_\beta < 1 = \dim_H E_\beta \quad \text{for any} \quad k \geq 2.$$

Moreover, by Lemmas 5.5.2–5.5.4 we have

$$\dim_H U_\beta^{(\aleph_0)} < 1.$$

Therefore, by (5.10) and (5.11) we have $\mathcal{H}^s(U_\beta^{(2^{\aleph_0})}) = \mathcal{H}^s(E_\beta) \in (0, \infty)$.

Now we consider $\beta > q^*$. By Theorems 5.1.2, 5.1.3 and (5.1) it follows that

$$\dim_H U_\beta^{(k)} = \frac{\log q_c}{\log \beta} < \frac{\log q^*}{\log \beta} = \dim_H E_\beta$$

for any $k = 1, 2, \cdots$. Moreover, by Lemma 5.5.2 we have $\dim_H U_\beta^{(\aleph_0)} = 0$. Therefore, the lemma follows by (5.10) and (5.11). □

Proof of Theorem 5.1.4. The theorem follows by Lemmas 5.5.1–5.5.3 and 5.5.5. □

5.6 Examples and final remarks

In the section we consider some examples. The first example is an application of Theorems 5.1.1–5.1.4 to expansions with deleted digits set.

Example 5.6.1. *Let $\beta = 3$. We consider β-expansions with digits set $\{0, 1, 3\}$. This is a special case of expansions with deleted digits (cf. [76]). Then*

$$E_3 = \left\{ \sum_{i=1}^{\infty} \frac{d_i}{3^i} : d_i \in \{0, 1, 3\} \right\}.$$

By Theorems 5.1.2 and 5.1.3 we have

$$\dim_H U_3^{(k)} = \dim_H U_3 = \frac{\log q_c}{\log 3} \approx 0.767877$$

for any $k \geq 2$. This means that the set $U_3^{(k)}$ of $x \in E_3$ has k different expansions has the same Hausdorff dimension $\log q_c / \log 3$ for any integer $k \geq 1$.

Moreover, by Theorem 5.1.4 it yields that $U_3^{(\aleph_0)}$ contains countably infinitely many points, and

$$\dim_H U_3^{(2^{\aleph_0})} = \dim_H E_3 = \frac{\log q^*}{\log 3} \approx 0.876036.$$

In terms of Theorem 5.1.2 we have a uniform formula of the Hausdorff dimension of U_β for $\beta \in [q^*, \infty)$. Excluding the trivial case for $\beta \in (1, q_c]$ that $U_\beta = \{0, \beta/(\beta - 1)\}$, it would be interesting to ask whether the Hausdorff dimension of U_β can be explicitly calculated for any $\beta \in (q_c, q^*)$.

In the following we give an example for which the Hausdorff dimension of U_q can be explicitly calculated.

Example 5.6.2. *Let $\beta = 1 + \sqrt{2} \in (q_c, q^*)$. Then*

$$(\beta 0^\infty)_\beta = (1\beta\beta 0^\infty)_\beta \quad \textit{and} \quad \alpha(\beta) = (\beta 1)^\infty.$$

Moreover, the quasi-greedy β-expansion of $\beta - 1$ with alphabet $\{0, \beta - 1, \beta\}$ is $\beta(\beta - 1)^\infty$. Therefore, by Lemmas 3.1 and 3.2 of [90] it follows that U'_β is the set of sequences $(d_i) \in \{0, 1, \beta\}^\infty$ satisfying

$$\begin{cases} d_{n+1}d_{n+2}\cdots < (1\beta)^\infty & \textit{if} \quad d_n = 0, \\ 1^\infty < d_{n+1}d_{n+2}\cdots < (\beta 1)^\infty & \textit{if} \quad d_n = 1, \\ d_{n+1}d_{n+2}\cdots > 01^\infty & \textit{if} \quad d_n = \beta. \end{cases}$$

Let X'_A be the sub-shift of finite type over the states

$$\{00, 01, 11, 1\beta, \beta 0, \beta 1, \beta\beta\}$$

with adjacency matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then one can verify that $U'_\beta \subseteq X'_A$, and $X'_A \setminus U'_\beta$ contains all sequences ending with 1^∞ or $(1\beta)^\infty$. This implies that

$$\dim_H U_\beta = \dim_H X_A(\beta),$$

where $X_A(\beta) = \{((d_i))_\beta : (d_i) \in X'_A\}$.

Note that $X_A(\beta)$ is a graph-directed set satisfying the open set condition (cf. [62]). Then by Theorem 5.1.3 we have

$$\dim_H U_\beta^{(k)} = \dim_H U_\beta = \frac{h(X'_A)}{\log \beta} \approx 0.691404.$$

Furthermore, by the word substitution $\beta 00 \sim 1\beta\beta$ and in a similar way as in the proof of Lemma 5.5.2 one can show that $U_\beta^{(\aleph_0)}$ contains countably infinitely many points.

By Theorem 5.1.4 we have $\dim_H U_\beta^{(2^{\aleph_0})} = \dim_H E_\beta = 1$. Here we should emphasize that the Hausdorff dimension of U_β can be calculated via the method of Chapter 4. We can give a Markov partition of $[0, \beta(\beta-1)^{-1}]$, then the switch regions are two of the elements of the Markov partition. The calculation is straightforward, and we omit the details.

Finally we give some open problems for this chapter.

Question 1. Can we give a uniform formula for the Hausdorff dimension of U_β for $\beta \in (q_c, q^*)$?

In β-expansions we know that the dimension function of the univoque set has a devil's staircase behavior (cf. [52]).

Question 2. Does the dimension function $D(\beta) := \dim_H U_\beta$ have a devil's staircase behavior in the interval (q_c, q^*)?

By Theorem 5.1.4 it gives that $U_\beta^{(\aleph_0)}$ is countable for any $\beta \in \mathbb{B}_2 \setminus (q_c, q^*)$. Moreover, in Lemma 5.5.4 we show that $\dim_H U_\beta^{(\aleph_0)} \le \dim_H U_\beta < 1$ for any $\beta \in (q_c, q^*)$. In terms of Example 5.6.2 we made the following conjecture.

Conjecture. The set $U_\beta^{(\aleph_0)}$ contains countably infinitely many points for any $\beta \in (q_c, q^*)$.

In this chapter we investigated β-expansions over digit $\{0,1,\beta\}$. It is natural to ask similar questions in the case for the classical β-expansions (with digit set $\{0,1\}$). Define

$$U_k = \{x \in [0, (\beta-1)^{-1}] : x \text{ has exactly } k \text{ different expansions}\}, k = 1, 2, \cdots, \aleph_0, 2^{\aleph_0}.$$

We list some of interesting questions of U_k.

Question 1. Give the necessary and sufficient condition for the equality:

$$\dim_H(U_k) = \dim_H(U_1)$$

for any $k \ge 2$.

Question 2. Give the necessary and sufficient condition for the equality:

$$\dim_H(U_2) = \dim_H(U_1).$$

Question 3. How can we calculate the Haudorff dimension of U_2.

Question 4. The topological structure of U_k, especially the fractal structure of this set.

Question 5. For any $1 < \beta < 2$, is $U_{\aleph_0}$ either an empty set or a countable set?

Question 6. What is the relation between the following two cases

1) $U_{\aleph_0}$ is countable set.

2) $\dim_H(U_k) = \dim_H(U_1)$ for any $k \ge 2$.

Do they hold simultaneously all the time when $C < \beta < 2$, here C is the Komornik-Loreti constant?

Chapter 6

Multiple codings for self-similar sets with overlaps

Abstract

In this chapter we consider a class $\mathcal{E}$ of self-similar sets with overlaps. In particular, for a self-similar set $E \in \mathcal{E}$ and $k = 1, 2, \cdots, \aleph_0$ or $2^{\aleph_0}$ we investigate the Hausdorff dimension of the subset $U_k(E)$ which contains all points $x \in E$ having exactly k different codings. This generalizes many results obtained in Chapters 4 and 5.

6.1 Introduction

Non-integer base expansions were pioneered by Rényi [78] and Parry [70]. It is generally believed that a real number x has a continuum of expansions [82]. However, Erdős et al. [27] discovered that there still exist a large number of reals having exactly k different expansions, where $k = 1, 2, \cdots$ or $\aleph_0$. Denote by U_k the set of all such reals. In particular, for $k = 1$ there are many works devoted to U_1 (cf. [84, 54, 52]). However, when $k \geq 2$, very few is known for U_k (see [38, 3, 89]).

In this paper we consider similar questions for self-similar sets with overlaps. For $1 \leq i \leq m$ let $f_i(\cdot)$ be a *similitude* on $\mathbb{R}$ defined by

$$f_i(x) = r_i x + b_i,$$

where $r_i \in (0, 1)$ and $b_i \in \mathbb{R}$. Then there exists a unique non-empty compact set E satisfying (cf. [46])

$$E = \bigcup_{i=1}^{m} f_i(E).$$

In this case, we call the couple $(E, \{f_i\}_{i=1}^m)$ a self-similar iterated function system (SIFS). Accordingly, the compact set E is called a self-similar set generated by $\{f_i\}_{i=1}^m$.

In this chapter we consider a class $\mathcal{E}$ of SIFS $(E, \{f_i\}_{i=1}^m)$ satisfying the following conditions. Denote by $I = [a, b]$ the convex hull of the self-similar set E. Then

(A) $a = f_1(a) < f_2(a) < \cdots < f_m(a) < f_m(b) = b$.

(B) $f_i(I) \cap f_{i+2}(I) = \emptyset$ for any $1 \leq i \leq m - 2$.

(C) There exist $i, j \in \{1, \cdots, m-1\}$ such that

$$f_i(I) \cap f_{i+1}(I) = \emptyset \quad \text{and} \quad f_j(I) \cap f_{j+1}(I) \neq \emptyset.$$

(D) If $f_i(I) \cap f_{i+1}(I) \neq \emptyset$, then there exist $u, v \geq 1$ such that

$$f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I),$$

where $f_{i_1 \cdots i_k}(\cdot) := f_{i_1} \circ \cdots \circ f_{i_k}(\cdot)$.

The intervals $f_i(I), i = 1, \cdots, m$ are called the fundamental intervals of $(E, \{f_i\}_{i=1}^m)$.

Then by Conditions (A)–(D) it follows that the fundamental intervals are located from left to right in the following way: the most left one is $f_1(I)$, and then the second one is $f_2(I)$, and the most right one is $f_m(I)$. Furthermore, there exist two neighbour fundamental intervals having a non-empty intersection, and also exist two neighbour fundamental intervals have an empty intersection. But any three fundamental intervals must have a null intersection. By Condition (D) it follows that a fundamental interval can not be contained in another fundamental interval, and the intersection of fundamental intervals can not be a singleton.

Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Then for any $x \in E$ there exists a sequence $(d_i) = d_1 d_2 \cdots \in \{1, 2, \cdots, m\}^\infty$ such that (cf. [30])

$$x = \lim_{n \to \infty} f_{d_1 \cdots d_n}(0) =: \pi((d_i)). \tag{6.1}$$

The sequence (d_i) is called a coding of x with respect to $\{f_i\}_{i=1}^m$. We point out that $x \in E$ may have multiple codings.

For $k = 1, 2, \cdots, \aleph_0, 2^{\aleph_0}$ and $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$ we set

$$U_k(E) := \{x \in E : x \text{ has exactly } k \text{ different codings w.r.t. } \{f_i\}_{i=1}^m\}.$$

In Chapter 4, we considered the calculation of the Hausdorff dimension of $U_1(E)$. In Chapter 5, we considered a special candidate $(E, \{f_i\}_{i=1}^3) \in \mathcal{E}$, where

$$f_1(x) = \frac{x}{q}, \quad f_2(x) = \frac{x}{q} + 1, \quad f_3(x) = \frac{x+q}{q}$$

with $q > (3 + \sqrt{5})/2$. In particular, it was shown that the Hausdorff dimensions of $U_k(E)$ are the same for all $k \geq 1$.

In this chapter we generalize some results of Chapter 4 and obtain the following results.

Theorem 6.1.1. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E. The following statements are equivalent.*

(i) $f_1(I) \cap f_2(I) \neq \emptyset$ *or* $f_{m-1}(I) \cap f_m(I) \neq \emptyset$.

(ii) $\dim_H U_k(E) = \dim_H U_1(E)$ *for all* $k \geq 1$.

(iii) $f_1(b) \in U_{\aleph_0}(E)$ *or* $f_m(a) \in U_{\aleph_0}$.

(iv) $|U_{\aleph_0}(E)| = \aleph_0$.

Here $|A|$ denotes the cardinality of a set A.

Theorem 6.1.2. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Denote by $I = [a, b]$ the convex hull of E. The following statements are equivalent.*

(i) $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$.

(ii) $\dim_H U_k(E) = \dim_H U_1(E)$ *if* $k = 2^s$ *for some* $s \geq 1$, *and* $U_k(E) = \emptyset$ *otherwise.*

(iii) $f_1(b) \notin U_{\aleph_0}$ *and* $f_m(a) \notin U_{\aleph_0}$.

(iv) $U_{\aleph_0}(E) = \emptyset$.

These two results imply following interesting corollaries.

Corollary 6.1.3. *For any* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, *we have following dichotomy: either*

$$\dim_H U_k(E) = \dim_H U_1(E)$$

for all $k \geq 1$, *or*

$$\dim_H U_k(E) = \dim_H U_1(E)$$

if $k = 2^s$ *for some* $s \geq 1$, *and* $U_k(E) = \emptyset$ *otherwise.*

Corollary 6.1.4. *For any* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, *either* $|U_{\aleph_0}(E)| = \aleph_0$ *or* $U_{\aleph_0}(E) = \emptyset$.

Corollary 6.1.3 gives many sufficient and necessary conditions for the equation

$$\dim_H U_k(E) = \dim_H U_1(E)$$

for all $k \geq 1$.

Corollary 6.1.5. *For any* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, *The following conditions are equivalent.*

- $\dim_H U_k(E) = \dim_H U_1(E)$ *for any* $k \geq 1$.
- $\dim_H(U_2(E)) = \dim_H(U_3(E))$.
- $\dim_H(U_{2016}(E)) = \dim_H(U_8(E)) = \dim_H(U_{29}(E))$.
- $\dim_H(U_1(E)) = \dim_H(U_{2012}(E)) = \dim_H(U_9(E)) = \dim_H(U_6(E))$.

Similarly, we have

Corollary 6.1.6. *For any* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$, *The following conditions are equivalent.*

- $\dim_H U_k(E) = \dim_H U_1(E)$ *if* k *is a 2-power, and* $\dim_H U_k(E) = \dim_H U_3(E)$ *otherwise.*
- $U_{2016}(E) = \emptyset$.
- $U_{2015}(E) = \emptyset$.
- $U_{2014}(E) = \emptyset$..

Finally, we prove that

Theorem 6.1.7. $\dim_H(U_{2^{\aleph_0}}(E)) = \dim_H(E)$. *Moreover,* $\dim_H(E)$ *and* $\dim_H(U_1(E))$ *are calculable. For any* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$,

$$0 < \mathcal{H}^{\dim_H(U_1)}(U_1) < \infty.$$

If $U_k \neq \emptyset$, $k \geq 2$

$$0 < \mathcal{H}^{\dim_H(U_k)}(U_k)$$

To obtain the equation $\dim_H U_k(E) = \dim_H U_1(E)$ for any $k \geq 1$, Condition (D) is not necessary. We shall give some examples for this aspect. In Theorem 6.1.7, we show that if $U_k \neq \emptyset$, $k \geq 2$, then

$$0 < \mathcal{H}^{\dim_H(U_k)}(U_k).$$

In fact, the Hausdorff measure of U_k can be infinity. Generally, we conjecture that for any $U_k \neq \emptyset, k \geq 2$, $\mathcal{H}^{\dim_H(U_k)}(U_k) = \infty$. However, we cannot prove this result. We shall give some examples to demonstrate how to prove this result.

The rest of the chapter is arranged in the following way. In Section 6.2 we consider the set $U_k(E)$ of points having exactly k different codings, and prove the equivalences $(i) \Leftrightarrow (ii)$ in Theorems 6.1.1 and 6.1.2, respectively. In Section 6.3 we investigate the set $U_{\aleph_0}(E)$ which contains $x \in E$ having countably infinitely many codings, and finish the proofs of Theorems 6.1.1 and 6.1.2. The proof of Theorem 6.1.7 will be presented in Section 4. Finally, in Section 6.5 we give some examples and remarks.

6.2 Finite codings

Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. In this section we will consider the set $U_k(E)$ which contains all $x \in E$ having exactly k different codings with respect to $\{f_i\}_{i=1}^m$, and prove the equivalences $(i) \Leftrightarrow (ii)$ in Theorems 6.1.1 and 6.1.2, respectively.

First we give some properties of the SIFS $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$.

Lemma 6.2.1. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_{im^u}(I) = f_{(i+1)1^v}(I)$ for some $1 \leq i \leq m-1$ and $u, v \geq 1$, then $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$.*

Proof. Note that for any $x \in \mathbb{R}$ we can write

$$f_{im^u}(x) = rx + t, \quad f_{(i+1)1^v}(x) = r'x + t', \tag{6.2}$$

for some $r, r' \in (0,1)$ and $t, t' \in \mathbb{R}$. Suppose that $I = [a, b]$. Then by using $f_{im^u}(I) = f_{(i+1)1^v}(I)$ it follows that

$$\begin{aligned} ra + t = f_{im^u}(a) = f_{(i+1)1^v}(a) = r'a + t', \\ rb + t = f_{im^u}(b) = f_{(i+1)1^v}(b) = r'b + t'. \end{aligned}$$

This implies $r = r'$ and $t = t'$. By (6.2) we have $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. □

Lemma 6.2.2. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Suppose*

$$x \in f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I)$$

for some $i \in \{1, \cdots, m-1\}$ and $u, v \geq 1$. Then the codings of x either start with im^{u-1} or $(i+1)1^{v-1}$.

Proof. Let (d_i) be a coding of x w.r.t. $\{f_i\}_{i=1}^m$. Note that $x = \pi((d_i)) \in f_i(I) \cap f_{i+1}(I)$ and that any three fundamental intervals have empty intersection. Then

$$d_1 = i \quad \text{or} \quad i+1.$$

If $u = v = 1$, then we are done. In the following we assume $u, v > 1$, and split the proof into the following two cases.

Case (I). $d_1 = i$. Note that $x = \pi(id_2d_3\cdots) \in f_{im^u}(I)$. Then

$$\pi(d_2d_3\cdots) \in f_{m^u}(I). \tag{6.3}$$

Suppose on the contrary that $d_2 \neq m$. Then by the location of these fundamental intervals we have $d_2 = m-1$. So, by (6.3) and Condition (D) it follows that

$$\pi(d_2d_3\cdots) \in f_{m^u}(I) \cap \big(f_{m-1}(I) \cap f_m(I)\big) \subseteq f_{m^u}(I) \cap f_{m1}(I),$$

leading to a contradiction since $f_1(I) \cap f_m(I) = \emptyset$.

Therefore, $d_2 = m$. By iteration it follows that $d_2 \cdots d_m = m^{u-1}$.

Case (II). $d_1 = i+1$. Note that $x = \pi((i+1)d_2d_3\cdots) \in f_{(i+1)1^v}(I)$. Then

$$\pi(d_2d_3\cdots) \in f_{1^v}(I). \tag{6.4}$$

Suppose on the contrary that $d_2 \neq 1$. Then $d_2 = 2$, and therefore by (6.4) and Condition (D) it follows that

$$\pi(d_2d_3\cdots) \in f_{1^v}(I) \cap \big(f_1(I) \cap f_2(I)\big) \subseteq f_{1^v}(I) \cap f_{1m}(I),$$

leading to a contradiction with $f_1(I) \cap f_m(I) = \emptyset$.

Therefore, $d_2 = 1$. By iteration we conclude that $d_2 \cdots d_v = 1^{v-1}$. □

The upper bound of $\dim_H U_k(E)$ can be deduced directly.

Lemma 6.2.3. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. Then for any $k \geq 1$ we have*

$$\dim_H U_k(E) \leq \dim_H U_1(E).$$

Proof. Take $x \in U_k(E)$. Then all of its codings eventually belong to $U_1'(E) := \{(c_i) \in \{1, \cdots, m\}^\infty : \pi((c_i)) \in U_1(E)\}$. Therefore, the lemma follows by observing that

$$U_k(E) \subseteq \bigcup_{n=1}^{\infty} \bigcup_{d_1\cdots d_n \in \{1,2,\cdots,m\}^n} f_{d_1\cdots d_n}(U_1(E)).$$

□

For the lower bound of $\dim_H U_k(E)$ we split the proof into the following four subsections.

- $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$;
- $f_1(I) \cap f_2(I) = \emptyset$ but $f_{m-1}(I) \cap f_m(I) \neq \emptyset$;
- $f_1(I) \cap f_2(I) \neq \emptyset$ and $f_{m-1}(I) \cap f_m(I) \neq \emptyset$;
- $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$.

6.2.1 $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$

In the following lemma we will show that the set of $x \in U_1(E)$ with its coding starting at $m-1$ has the same Hausdorff dimension of $U_1(E)$.

Lemma 6.2.4. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_{m-1}(I) \cap f_m(I) = \emptyset$, then*

$$\dim_H f_{m-1}(E) \cap U_1(E) = \dim_H U_1(E),$$

Proof. Let

$$\begin{aligned} \phi: \quad U_1(E) &\longrightarrow f_{(m-1)m}(E) \cap U_1(E) \\ x \quad &\mapsto \quad f_{(m-1)m}(x). \end{aligned}$$

First we prove that ϕ is well-defined. Take $x \in U_1(E)$. It suffices to prove $f_{(m-1)m}(x) \in U_1(E)$.

Note that $f_{m-1}(I) \cap f_m(I) = \emptyset$. Then by the locations of the fundamental intervals it yields that

$$f_i(I) \cap f_m(I) = \emptyset \quad \text{for any} \quad i \neq m. \tag{6.5}$$

So, $f_m(x) \in U_1(E)$. Suppose on the contrary that $f_{(m-1)m}(x) \notin U_1(E)$. Then by (6.5) and Condition (D) it follows that

$$f_{(m-1)m}(x) \in f_{m-2}(I) \cap f_{m-1}(I) \subseteq f_{(m-1)1}(I).$$

This implies $f_m(x) \in f_1(I)$, leading to a contradiction with (6.5).

Therefore, ϕ is well-defined. Note that ϕ is a similitude. Then one can easily check that ϕ is bijective. In particular, ϕ is a bi-Lipschitz map between $U_1(E)$ and $f_{(m-1)m}(E) \cap U_1(E)$. Hence,

$$\begin{aligned} \dim_H U_1(E) = \dim_H f_{(m-1)m}(E) \cap U_1(E) &\leq \dim_H f_{m-1}(E) \cap U_1(E) \\ &\leq \dim_H U_1(E). \end{aligned}$$

□

Now we consider the lower bound of $\dim_H U_k(E)$.

Lemma 6.2.5. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$, then for any $k \geq 1$ we have*

$$\dim_H U_k(E) \geq \dim_H U_1(E).$$

Proof. Note that $f_1(I) \cap f_2(I) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I).$$

By Lemma 6.2.1 we have $f_{1m^u}(\cdot) = f_{21^v}(\cdot)$.

Take $= \pi((c_i)) \in f_{m-1}(E) \cap U_1(E)$. Then $c_1 = m - 1$. Now we claim that

$$x_s := \pi(1m^{us}(c_i))$$

has exactly $s + 1$ different codings. We will prove this by induction on s.

Suppose $s = 0$. Then $x_0 = \pi(1(c_i))$. Denote by $I = [a, b]$. Then by (6.5) and Condition (A) it follows that

$$x_0 = \pi(1(m-1)c_2c_3\cdots) \leq f_{1(m-1)}(b) < f_{1m^u}(a) = f_{21^v}(a) = f_2(a).$$

This together with $\pi((c_i)) = \pi((m-1)c_2c_3\cdots) \in U_1(E)$ implies that x_0 has a unique coding.

Now suppose that x_s has $s + 1$ different codings for some $s \geq 0$. We will prove that x_{s+1} has exactly $s + 2$ codings. Note that

$$x_{s+1} = f_{1m^u}(\pi(m^{us}(c_i))) = f_{21^v}(\pi(m^{us}(c_i))) = f_{21^{v-1}}(x_s). \tag{6.6}$$

By the induction hypothesis this implies that x_{s+1} has at least $s+2$ different codings: one is $1m^{u(s+1)}(c_i)$, and the others start at 21^{v-1}.

In the following we will prove that x_{s+1} has exactly $s+2$ codings. Suppose that (d_i) is a coding of x_{s+1}. By (6.6) and Lemma 6.2.2 it follows that

$$d_1 \cdots d_u = 1m^{u-1} \quad \text{or} \quad d_1 \cdots d_v = 21^{v-1}.$$

So, it suffices to prove that $d_1 \cdots d_u = 1m^{u-1}$ implies $(d_i) = 1m^{u(s+1)}(c_i)$.

Suppose $d_1 \cdots d_u = 1m^{u-1}$. Then by (6.6) it gives

$$\pi(d_{u+1}d_{u+2}\cdots) = f_{m^{us+1}}(\pi((c_i))).$$

By (6.5) it follows that

$$d_{u+1} \cdots d_{u(s+1)+1} = m^{us+1}, \quad \pi(d_{u(s+1)+2}d_{u(s+1)+3}\cdots) = \pi((c_i)).$$

Observe that $\pi((c_i)) \in U_1(E)$. Then $(d_i) = 1m^{u(s+1)}(c_i)$.

Hence, we conclude by induction that x_s has exactly $s+1$ codings for any $s \geq 0$. Note that $\pi((c_i))$ is taken from $f_{m-1}(E) \cap U_1(E)$ arbitrarily. Then

$$\{x_s = \pi(1m^{us}(c_i)) : \pi((c_i)) \in f_{m-1}(E) \cap U_1(E)\} \subseteq U_{s+1}(E).$$

By Lemma 6.2.4 it follows that for any $s \geq 0$ we have

$$\dim_H U_{s+1}(E) \geq \dim_H f_{m-1}(E) \cap U_1(E) = \dim_H U_1(E).$$

□

6.2.2 $f_1(I) \cap f_2(I) = \emptyset$ but $f_{m-1}(I) \cap f_m(I) \neq \emptyset$

First we show that the set of $x \in U_1(E)$ with its coding beginning with 2 has the same Hausdorff dimension as $U_1(E)$.

Lemma 6.2.6. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) = \emptyset$, then*

$$\dim_H f_2(E) \cap U_1(E) = \dim_H U_1(E),$$

Proof. In a similar way as in Lemma 6.2.4 one can show that the following map

$$\begin{aligned} \psi : \quad U_1(E) &\longrightarrow f_{21}(E) \cap U_1(E) \\ x \quad &\mapsto \quad f_{21}(x) \end{aligned}$$

is bi-Lipschtiz.

Then

$$\begin{aligned} \dim_H U_1(E) = \dim_H f_{21}(E) \cap U_1(E) &\leq \dim_H f_2(E) \cap U_1(E) \\ &\leq \dim_H U_1(E). \end{aligned}$$

□

Lemma 6.2.7. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If $f_1(I) \cap f_2(I) = \emptyset$ but $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then for any $k \geq 1$ we have*

$$\dim_H U_k(E) \geq \dim_H U_1(E).$$

Proof. Note that $f_{m-1}(I)\cap f_m(I)\neq\emptyset$. Then by Condition (D) there exist $u,v\geq 1$ such that

$$f_{m-1}(I)\cap f_m(I)=f_{(m-1)m^u}(I)=f_{m1^v}(I).$$

By Lemma 6.2.1 we have $f_{(m-1)m^u}(\cdot)=f_{m1^v}(\cdot)$.

Take $\pi(c_1c_2\cdots)\in f_2(E)\cap U_1(E)$. Then $c_1=2$. For $s\geq 0$ we define

$$y_s:=\pi(m1^{vs}(c_i)).$$

In a similar way as in the proof of Lemma 6.2.5 one can prove that y_s has exactly $s+1$ different codings.

Note that $\pi((c_i))$ is taken from $f_2(E)\cap U_1(E)$ arbitrarily. Then by Lemma 6.2.6 it follows that for any $s\geq 0$ we have

$$\begin{aligned}\dim_H U_{s+1}(E) &\geq \dim_H\{y_s=\pi(m1^{vs}(c_i)):\pi((c_i))\in f_2(E)\cap U_1(E)\}\\ &=\dim_H f_2(E)\cap U_1(E)=\dim_H U_1(E).\end{aligned}$$

□

6.2.3 $f_1(I)\cap f_2(I)\neq\emptyset$ and $f_{m-1}(I)\cap f_m(I)\neq\emptyset$

By Condition (C) we may assume that $f_i(I)\cap f_{i+1}(I)=\emptyset$ for some $i\in\{2,\cdots,m-2\}$. In the following lemma we will show that the Hausdorff dimension of $U_1(E)$ is dominated by the subset which contains all $x\in U_1(E)$ with its coding starting at i or $i+1$.

Lemma 6.2.8. *Let $(E,\{f_i\}_{i=1}^m)\in\mathcal{E}$. If $f_i(I)\cap f_{i+1}(I)=\emptyset$ for some $i\in\{2,\cdots,m-2\}$, then*

$$\dim_H U_1(E)=\max\{\dim_H f_i(E)\cap U_1(E),\quad \dim_H f_{i+1}(E)\cap U_1(E)\}.$$

Proof. Note that $U_1(E)=\bigcup_{j=1}^m f_j(E)\cap U_1(E)$. It suffices to prove

$$\dim_H f_i(E)\cap U_1(E)\geq\dim_H\bigcup_{j=i+1}^m f_j(E)\cap U_1(E) \tag{6.7}$$

and

$$\dim_H f_{i+1}(E)\cap U_1(E)\geq\dim_H\bigcup_{j=1}^i f_j(E)\cap U_1(E)$$

Without loss of generality we only prove (6.7). Let

$$\begin{aligned}\varphi:\quad \bigcup_{j=i+1}^m f_j(E)\cap U_1(E) &\longrightarrow f_i(E)\cap U_1(E)\\ x\quad &\mapsto\quad f_i(x).\end{aligned}$$

First we prove that φ is well-defined. Take $x\in\bigcup_{j=i+1}^m f_j(E)\cap U_1(E)$. It suffices to prove that $f_i(x)\in U_1(E)$. Suppose on the contrary that $f_i(x)\notin U_1(E)$. Note that $f_i(I)\cap f_{i+1}(I)=\emptyset$. Then by the locations of the fundamental intervals it follows that

$$f_i(x)\in f_{i-1}(I)\cap f_i(I)\subseteq f_{i1}(I).$$

This implies that $x\in f_1(I)$, leading to contradiction since $f_1(I)\cap\bigcup_{j=i+1}^m f_j(I)=\emptyset$.

Therefore, φ is well-defined. Note that φ is a similitude. Hence,

$$\begin{aligned}\dim_H f_i(E)\cap U_1(E) &\ge \dim_H \varphi\left(\bigcup_{j=i+1}^{m} f_j(E)\cap U_1(E)\right)\\ &= \dim_H \bigcup_{j=i+1}^{m} f_j(E)\cap U_1(E).\end{aligned}$$

This establishes (6.7). □

Lemma 6.2.9. *Let* $(E,\{f_i\}_{i=1}^m)\in\mathcal{E}$. *If* $f_1(I)\cap f_2(I)\ne\emptyset$, $f_{m-1}(I)\cap f_m(I)\ne\emptyset$, *then for any* $k\ge 1$ *we have*

$$\dim_H U_k(E)\ge \dim_H U_1(E).$$

Proof. Suppose $f_i(I)\cap f_{i+1}(I)=\emptyset$ for some $i\in\{2,\cdots,m-2\}$. Then by Lemma 6.2.8 it follows that

$$\dim_H U_1(E)=\max\{\dim_H f_i(E)\cap U_1(E),\ \ \dim_H f_{i+1}(E)\cap U_1(E)\}.$$

Without loss of generality we assume that

$$\dim_H U_1(E)=\dim_H f_i(E)\cap U_1(E). \tag{6.8}$$

Note that $f_1(I)\cap f_2(I)\ne\emptyset$. Then by Condition (D) there exist $u,v\ge 1$ such that

$$f_1(I)\cap f_2(I)=f_{1m^u}(I)=f_{21^v}(I).$$

By Lemma 6.2.1 we have $f_{1m^u}(\cdot)=f_{21^v}(\cdot)$.

Take $\pi((c_i))\in f_i(E)\cap U_1(E)$. Then $c_1=i$. Now we claim that

$$z_s:=\pi(1m^{us}(c_i))$$

has exactly $s+1$ different codings. We will prove this by induction on s.

Suppose $s=0$. Then $z_0=\pi(1(c_i))$. Note that $f_i(I)\cap f_{i+1}(I)=\emptyset$ for some $i\in\{2,\cdots,m-2\}$. Denote by $I=[a,b]$. Then by Condition (A) it follows that

$$z_0=\pi(1ic_2c_3\cdots)\le f_{1i}(b)<f_{1m^u}(a)=f_{21^v}(a)=f_2(a).$$

This together with $\pi((c_i))=\pi(ic_2c_3\cdots)\in U_1(E)$ implies that z_0 has a unique coding.

Now suppose that z_s has $s+1$ different codings for some $s\ge 0$. We will prove that z_{s+1} has exactly $s+2$ codings. Note that

$$z_{s+1}=f_{1m^u}(\pi(m^{us}(c_i)))=f_{21^v}(\pi(m^{us}(c_i)))=f_{21^{v-1}}(z_s). \tag{6.9}$$

By the induction hypothesis this implies that z_{s+1} has at least $s+2$ different codings: one is $1m^{u(s+1)}(c_i)$, and the others start at 21^{v-1}.

In the following we will prove that z_{s+1} has exactly $s+2$ codings. Suppose that (d_i) is a coding of z_{s+1}. Then by (6.9) and Lemma 6.2.2 it follows that

$$d_1\cdots d_u=1m^{u-1}\quad\text{or}\quad d_1\cdots d_v=21^{v-1}.$$

So, it suffices to prove that $d_1\cdots d_u=1m^{u-1}$ implies $(d_i)=1m^{u(s+1)}(c_i)$.

Suppose $d_1 \cdots d_u = 1m^{u-1}$. Then by (6.9) we have

$$\pi(d_{u+1}d_{u+2}\cdots) = f_{m^{us+1}}(\pi((c_i))). \tag{6.10}$$

Suppose $d_{u+1} \neq m$. Then by the locations of the fundamental intervals we have $d_{u+1} = m-1$. Therefore, by (6.10) and Condition (D) it follows that

$$f_{m^{us+1}}(\pi((c_i))) \in f_{m-1}(I) \cap f_m(I) \subseteq f_{m1}(I).$$

This implies that $f_{m^{us}}(\pi((c_i))) \in f_1(I)$, leading to a contradiction since $f_i(I) \cap f_{i+1}(I) = \emptyset$. By iteration and note that $\pi((c_i)) \in U_1(E)$ we conclude by (6.10) that $(d_i) = 1m^{u(s+1)}(c_i)$.

Hence, we conclude by induction that z_s has exactly $s+1$ codings for any $s \geq 0$. Note that $\pi((c_i))$ is taken from $f_i(E) \cap U_1(E)$ arbitrarily. Then

$$\{z_s = \pi(1m^{us}(c_i)) : \pi((c_i)) \in f_i(E) \cap U_1(E)\} \subseteq U_{s+1}(E).$$

Hence, by (6.8) we have for any $s \geq 0$ that

$$\dim_H U_{s+1}(E) \geq \dim_H f_i(E) \cap U_1(E) = \dim_H U_1(E).$$

□

6.2.4 $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$.

Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. First we show that $U_k(E)$ is empty if $k \neq 2^s$ for any $s \geq 1$.

Lemma 6.2.10. *Let $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. If*

$$f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset,$$

then $U_k(E) = \emptyset$ for any $k \neq 2^s$ with $s \geq 1$.

Proof. For $x \in E$ we denote by $N(x)$ the number of different codings of x with respect to $\{f_i\}_{i=1}^m$. Let $k \geq 2$ and take $x \in U_k(E)$. Then $N(x) = k$. So, there exist $i \in \{2, \cdots, m-2\}$ and

$$x_1 \in U_k(E) \cap f_i(I) \cap f_{i+1}(I) \tag{6.11}$$

such that $N(x) = N(x_1)$.

Note that $f_i([0,1]) \cap f_{i+1}([0,1]) \neq \emptyset$. Then by Condition (D) there exist $u, v \geq 1$ such that

$$f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I). \tag{6.12}$$

Observe that $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$. Then by the locations of the fundamental intervals we obtain

$$f_1(I) \cap f_i(I) = \emptyset, \quad f_j(I) \cap f_m(I) = \emptyset \tag{6.13}$$

for any $i \neq 1$ and any $j \neq m$.

Therefore, by (6.11)–(6.13) it follows that the codings of x_1 either start at im^u or begin with $(i+1)1^v$. Note by using (6.12) in Lemma 6.2.1 that $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. Then there exists $y \in E$ such that $x_1 = f_{im^u}(y) = f_{(i+1)1^v}(y)$. Furthermore,

$$N(x_1) = N(f_{m^u}(y)) + N(f_{1^v}(y)) = 2N(y),$$

where the last equality holds by (6.13) that

$$N(f_{m^u}(y)) = N(y) = N(f_{1^v}(y)).$$

Hence, we conclude that $N(x) = N(x_1) = 2N(y)$.

By iteration it follows that $N(x)$ must be of the form 2^s for some $s \geq 1$. This completes the proof. □

Lemma 6.2.11. *Let* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. *If*

$$f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset,$$

then $\dim_H U_{2^s}(E) \geq \dim_H U_1(E)$ *for any* $s \geq 0$.

Proof. We will prove this lemma by induction on s. Clearly, the lemma follows if $s = 0$.

Now we assume that $\dim_H U_{2^s}(E) \geq \dim_H U_1(E)$ for some $s \geq 0$. In the following it suffices to prove that

$$\dim_H U_{2^{s+1}}(E) \geq \dim_H U_{2^s}(E).$$

By Condition (C) there exists $i \in \{2, \cdots, m-2\}$ for which $f_i(I) \cap f_{i+1}(I) \neq \emptyset$. Then by Condition (D) we can find $u, v \geq 1$ such that

$$f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I).$$

By Lemma 6.2.1 this yields that $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. Note that any three fundamental intervals has empty intersection. So, by (6.13) it follows that

$$\{f_{im^u}(x) = f_{(i+1)1^v}(x) : x \in U_{2^s}(E)\} \subseteq U_{2^{s+1}}(E).$$

Hence, $\dim_H U_{2^{s+1}}(E) \geq \dim_H U_{2^s}(E)$.

By induction we conclude that $\dim_H U_{2^s}(E) \geq \dim_H U_1(E)$ for any $s \geq 0$. □

Now we give the proof of the equivalence of (i) and (ii) in Theorems 6.1.1 and 6.1.2.

Proof of Theorems 6.1.1 and 6.1.2 $(i) \Leftrightarrow (ii)$. By Lemmas 6.2.3, 6.2.5, 6.2.7 and 6.2.9 it follows that if $f_1(I) \cap f_2(I) \neq \emptyset$ or $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then

$$\dim_H U_k(E) = \dim_H U_1(E) \quad \text{for all} \quad k \geq 1.$$

On the other hand, if $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$, then by Lemmas 6.2.3, 6.2.10 and 6.2.11 it follows that

$$\begin{cases} \dim_H U_k(E) = \dim_H U_1(E) & \text{if} \quad k = 2^s, \\ U_k(E) = \emptyset & \text{otherwise.} \end{cases}$$

This completes the proof. □

6.3 Countable codings

In this section we will consider the set $U_{\aleph_0}(E)$, and prove the equivalences $(i) \Leftrightarrow (iii) \Leftrightarrow (iv)$ in Theorems 6.1.1 and 6.1.2, respectively. First we prove the equivalence $(i) \Leftrightarrow (iii)$.

Lemma 6.3.1. *Let* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. *Denote by* $I = [a, b]$ *the convex hull of* E.

(A) $f_1(I) \cap f_2(I) \neq \emptyset$ *if and only if* $f_1(b) \in U_{\aleph_0}(E)$.

(B) *$f_{m-1}(I)\cap f_m(I)\neq\emptyset$ if and only if $f_m(a)\in U_{\aleph_0}(E)$.*

Proof. Since the proof of (B) is similar, we only prove (A).

First we consider the sufficiency. Suppose on the contrary that $f_1(I)\cap f_2(I)=\emptyset$. Then by the locations of the fundamental intervals we have

$$f_1(I)\cap f_i(I)=\emptyset \quad \text{for any} \quad i\neq 1.$$

Note that $b=\pi(m^\infty)\in U_1(E)$. Then $f_1(b)\in U_1(E)$.

Now we turn to proving the necessity. Suppose $f_1(I)\cap f_2(I)\neq\emptyset$. Then by Condition (D) there exist $u,v\geq 1$ such that

$$f_1(I)\cap f_2(I)=f_{1m^u}(I)=f_{21^v}(I).$$

By Lemma 6.2.1 it gives that $f_{1m^u}(\cdot)=f_{21^v}(\cdot)$. Then

$$f_1(b)=\pi(1m^\infty)=\pi(21^{v-1}1m^\infty)=\cdots=\pi((21^{v-1})^s1m^\infty)=\cdots. \tag{6.14}$$

This implies that $f_1(b)$ has at least countably infinitely many codings.

In the following we show that $f_1(b)$ indeed has countably infinitely many codings. Suppose that (d_i) is a coding of $f_1(b)$. By (6.14) and Lemma 6.2.2 it follows that $d_1\cdots d_u=1m^{u-1}$ or $d_1\cdots d_v=21^{v-1}$.

- If $d_1\cdots d_u=1m^{u-1}$, then by (6.14) we have

  $$\pi(d_{u+1}d_{u+2}\cdots)=\pi(m^\infty)\in U_1(E).$$

  This implies that $(d_i)=1m^\infty$.

- If $d_1\cdots d_v=21^{v-1}$, then by (6.14) it yields that

  $$\pi(d_{v+1}d_{v+2}\cdots)=\pi(1m^\infty)=f_1(b).$$

By iteration of the above arguments it follows that all the codings of $f_1(b)$ are of the form

$$(21^{v-1})^s1m^\infty, \quad s\geq 0.$$

Hence, $f_1(b)\in U_{\aleph_0}(E)$. This establishes the lemma. □

Proof of Theorems 6.1.1 and 6.1.2 $(i)\Leftrightarrow(iii)$. The equivalences of (i) and (iii) in Theorems 6.1.1 and 6.1.2 follows directly by Lemma 6.3.1. □

Lemma 6.3.2. *Let $(E,\{f_i\}_{i=1}^m)\in\mathcal{E}$. If $f_1(I)\cap f_2(I)=f_{m-1}(I)\cap f_m(I)=\emptyset$, then $U_{\aleph_0}(E)=\emptyset$.*

Proof. Suppose on the contrary that $U_{\aleph_0}(E)\neq\emptyset$. Take $x\in U_{\aleph_0}(E)$. Then x must have a coding (d_i) satisfying

$$x_n:=\pi(d_{n+1}d_{n+2}\cdots)\in E\cap\bigcup_{i=2}^{m-2}f_i(I)\cap f_{i+1}(I) \tag{6.15}$$

for infinitely many $n\geq 1$.

Take n satisfying (6.15) and assume that

$$x_n \in E \cap f_i(I) \cap f_{i+1}(I)$$

for some $2 \leq i \leq m-2$. By Condition (D) there exist $u, v \geq 1$ such that

$$x_n \in f_i(I) \cap f_{i+1}(I) = f_{im^u}(I) = f_{(i+1)1^v}(I). \tag{6.16}$$

Note that $f_1(I) \cap f_2(I) = f_{m-1}(I) \cap f_m(I) = \emptyset$. Then by the locations of the fundamental intervals it follows that

$$f_1(I) \cap f_j(I) = f_\ell(I) \cap f_m(I) = \emptyset \tag{6.17}$$

for any $j \neq 1$ and any $\ell \neq m$. Therefore, by (6.16) and (6.17) it follows that

$$d_{n+1} \cdots d_{n+u+1} = im^u \quad \text{or} \quad d_{n+1} \cdots d_{n+v+1} = (i+1)1^v.$$

Note by using (6.16) in Lemma 6.2.1 we have $f_{im^u}(\cdot) = f_{(i+1)1^v}(\cdot)$. Therefore, we have a substitution in $d_{n+1}d_{n+2}\cdots$ by considering $im^u \sim (i+1)1^v$.

In terms of (6.15) and by iteration it follows that there exist infinitely many independent substitutions in (d_i). This implies that x has a continuum of codings, leading to a contradiction with $x \in U_{\aleph_0}(E)$. □

Lemma 6.3.3. *Let* $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$. *If* $f_1(I) \cap f_2(I) \neq \emptyset$ *or* $f_{m-1}(I) \cap f_m(I) \neq \emptyset$*, then* $|U_{\aleph_0}(E)| = \aleph_0$.

Proof. Without loss of generality we may assume $f_1(I) \cap f_2(I) \neq \emptyset$ but $f_{m-1}(I) \cap f_m(I) = \emptyset$. By Condition (D) there exist $u, v \geq 1$ such that

$$f_1(I) \cap f_2(I) = f_{1m^u}(I) = f_{21^v}(I). \tag{6.18}$$

Then by Lemma 6.2.1 it yields that $f_{1m^u}(\cdot) = f_{21^v}(\cdot)$.

Denote by $I = [a, b]$. First we claim that $f_{1^n}(b) \in U_{\aleph_0}(E)$ for any $n \geq 1$. We will prove this by induction on n. Clearly, for $n = 1$ we have by Lemma 6.3.1 that $f_1(b) \in U_{\aleph_0}(E)$.

Suppose that $f_{1^n}(b) \in U_{\aleph_0}(E)$ for some $n \geq 1$. Now we consider $f_{1^{n+1}}(b)$. If $f_{1^{n+1}}(b) \in f_2(I)$, then by Condition (D) it follows that

$$f_{1^{n+1}}(b) \in f_1(I) \cap f_2(I) \subseteq f_{1m}(I).$$

This implies $f_{1^n}(1) \in f_m(I)$, leading to a contradiction since $f_1(I) \cap f_m(I) = \emptyset$.

Therefore, we conclude by induction that $\{f_{1^n}(b) : n \geq 1\} \subseteq U_{\aleph_0}(E)$. In the following it suffices to prove that any $x \in U_{\aleph_0}(E)$ must have a coding ending with $1m^\infty \sim (21^{v-1})^\infty$.

Take $x \in U_{\aleph_0}(E)$. Suppose on the contrary that all the codings of x do not end with $1m^\infty \sim (21^{v-1})^\infty$. Note that x has a coding (d_i) satisfying

$$\pi(d_{n+1}d_{n+2}\cdots) \in E \cap \bigcup_{i=1}^{m-2} f_i(I) \cap f_{i+1}(I) \tag{6.19}$$

for infinitely many $n \geq 1$.

Fix n satisfying (6.19), and assume that $\pi(d_{n+1}d_{n+2}\cdots)\in f_i(I)\cap f_{i+1}(I)$ for some $i\in\{1,\cdots,m-2\}$. Then by Condition (D) there exist $p,q\geq 1$ such that

$$\pi(d_{n+1}d_{n+2}\cdots)\in f_i(I)\cap f_{i+1}(I)=f_{im^p}(I)=f_{(i+1)1^q}(I). \tag{6.20}$$

By Lemmas 6.2.1 and 6.2.2 it follows that $f_{im^p}(\cdot)=f_{(i+1)1^q}(\cdot)$, and

$$d_{n+1}\cdots d_{n+p}=im^{p-1} \quad \text{or} \quad d_{n+1}\cdots d_{n+q}=(i+1)1^{q-1}.$$

Now we split the proof into the following two cases.

Case (I). $d_{n+1}\cdots d_{n+p}=im^{p-1}$. Then by (6.20) and using $f_{m-1}(I)\cap f_m(I)=\emptyset$ it follows that $d_{n+p+1}=m$. Therefore, we have a substitution by replacing $d_{n+1}\cdots d_{n+p+1}=im^p$ by $(i+1)1^q$.

Case (II). $d_{n+1}\cdots d_{n+q}=(i+1)1^{q-1}$. Then by (6.20) it follows that $d_{n+q+1}=1$ or 2.

- If $d_{n+q+1}=1$, then we have a substitution by replacing $d_{n+1}\cdots d_{n+q+1}=(i+1)1^q$ by im^p.
- If $d_{n+q+1}=2$, then by (6.18) and (6.20) it yields that

  $$\pi(d_{n+q+1}d_{n+q+2}\cdots)\in f_1(I)\cap f_2(I)=f_{1m^u}(I)=f_{(i+1)1^v}(I).$$

  So, by Lemma 6.2.2 it follows that $d_{n+q+1}\cdots d_{n+q+v}=21^{v-1}$. Furthermore, $d_{n+q+v+1}=1$ or 2. Note by the assumption that (d_i) does not end with $(21^{v-1})^\infty$. Then by iteration it follows that there exists $N\geq n+v+1$ such that

  $$d_{N+1}\cdots d_{N+v+1}=21^v.$$

  Hence, we also have a substitution by considering $21^v\sim 1m^u$.

By Cases (I)–(II) and (6.19) it follows that there exist infinitely many independent substitutions in (d_i). This implies that x has a continuum of codings, leading to a contradiction with $x\in U_{\aleph_0}(E)$. □

Proof of Theorems 6.1.1 and 6.1.2 $(i)\Leftrightarrow(iv)$. The equivalences of (i) and (iv) follows directly by Lemmas 6.3.2 and 6.3.3. □

6.4 Uncountable codings and Dimension of U_k

In this section we will consider the set $U_{2^{\aleph_0}}(E)$ which contains all points having a continuum of codings, and prove Theorem 6.1.7. First let us recall that the system $(E,\{f_i\}_{i=1}^m)$ coming from the collection $\mathcal{E}$ described in section 1. We say that i is an admissible initial code of $x\in E$ if $x\in f_i(I)$. Then each $x\in E$ has at least one admissible initial code and at most two admissible initial codes.

As we know, when $f_i(I)\cap f_{i+1}(I)\neq\emptyset$ there exists a unique positive integer pair $(u(i),v(i))$ such that $f_{im^{u(i)}}(I)=f_{(i+1)1^{v(i)}}(I)$. Let $u=\max_i u(i)$ and $v=\max_i v(i)$. Let

$$\mathcal{P}=\bigcup_{i=1}^m\{f_i(a),f_i(b)\}\cup\bigcup_{\ell=1}^v\{f_{1^\ell}(b)\}\cup\bigcup_{\ell=1}^u\{f_{m^\ell}(a)\}.$$

The following lemma is crucial to our analysis. With the help of this lemma, we can give a Markov partition of E.

Lemma 6.4.1. *Let $a = c_1 < c_2 < \cdots < c_{2m} = b$ be the endpoints of $\{f_i(I)\}_{i=1}^m$. Then any orbits of $c_i, 1 \le i \le 2m$ hit finite points. Moreover, this finite set is precisely*

$$\mathcal{P} = \bigcup_{i=1}^{m} \{f_i(a), f_i(b)\} \cup \bigcup_{\ell=1}^{v} \{f_{1^\ell}(b)\} \cup \bigcup_{\ell=1}^{u} \{f_{m^\ell}(a)\}.$$

Proof. Without loss of generality, let $f_i([a,b]) \cap f_{i+1}([a,b]) \neq \emptyset$. By Lemma 6.2.1, there exist $k, l \geq 1$ such that $f_{im^k} = f_{(i+1)1^l}$. Suppose $c_j = f_{i+1}(a), c_{j+1} = f_i(b)$. It suffices to show that any orbits of c_j hit finite points. Since $c_j = f_{i+1}(a)$, it follows that $f_{i+1}^{-1}(c_j) = a, f_1^{-n}(a) = a, n \geq 1$. Therefore, if we pick f_{i+1}^{-1} for the first time, then the orbit of c_j remains in the point a forever.

Suppose that we implement f_i^{-1} for the first time. Note that $c_j = f_{i+1}(a)$,

$$\begin{aligned} f_i^{-1}(c_j) &= f_i^{-1}(f_{i+1}(a)) = f_i^{-1}(f_{i+11^l}(a)) \\ &= f_i^{-1}(f_{im^k}(a)) = f_{m^k}(a) \end{aligned}$$

If $k = 1$, then $f_i^{-1}(c_j) = f_m(a)$ is one of the elements of $\mathcal{P}$. If $k \geq 2$ and $f_{m-1}([a,b]) \cap f_m([a,b]) = \emptyset$, then we can only pick f_m^{-1}. The orbit of $f_i^{-1}(c_j)$ will hit a eventually. If $k \geq 2$ and $f_{m-1}([a,b]) \cap f_m([a,b]) \neq \emptyset$. By Lemma 6.2.1 we know that there exist $p, q \geq 1$ such that $f_{m-1m^p} = f_{m1^q}$. We claim that

$$f_{m^k}(a) \in f_m([a,b]) \setminus f_{m-1}([a,b]).$$

If not, we must have $f_{m^k}(a) < f_{m-1}(b) = f_{m-1m^p}(b) = f_{m1^q}(b)$. Since $k \geq 2$, we have

$$f_{m^{k-1}}(a) < f_{1^q}(b).$$

However, this is impossible as $f_1([a,b]) \cap f_m([a,b]) = \emptyset$. Hence the orbit of $f_{m^k}(a)$ eventually hits the point a, which yields that the orbit of c_j hits finite points. The proof of $c_{j+1} = f_i(b)$ hitting finite points is similar, and we omit the details. □

The first set $\bigcup_{i=1}^m \{f_i(a), f_i(b)\}$ of $\mathcal{P}$ consists of the endpoints of the fundamental intervals $f_i(I), 1 \leq i \leq m$. Now we list the elements of $\mathcal{P}$ in the increasing order and write

$$\mathcal{P} = \{s_j : 1 \leq j \leq \gamma\},$$

where $\gamma = \#\mathcal{P} = 2m + u + v - 2$. Note that $f_{1^2}(b) < f_2(a)$ and $f_{m-1}(b) < f_{m^2}(a)$. Then the first v members and the last u members of $\mathcal{P}$ are

$$s_1 = f_1(a) = a < s_2 = f_{1^v}(b) < s_3 = f_{1^{v-1}}(b) < \cdots < s_v = f_{1^2}(b)$$

and

$$s_{\gamma-u+1} = f_{m^2}(a) < s_{\gamma-u+2} = f_{m^3}(a) < \cdots < s_{\gamma-1} = f_{m^u}(a) < s_\gamma = f_m(b) = b.$$

For two consecutive members s_i and s_{i+1} of $\mathcal{P}$, we call them an *admissible pair* if there exists a j such that

$$s_i, s_{i+1} \in f_j(I). \tag{6.21}$$

Let

$$\mathcal{Q} = \{1 \leq i < \gamma : \{s_i, s_{i+1}\} \text{ is an admissible pair}\}. \tag{6.22}$$

For an admissible pair $\{s_i, s_{i+1}\}$, there exist at most two j's satisfying (6.21) and we denote by $\alpha(i)$ the smaller j. One can verify that $f^{-1}_{\alpha(i)}(s_i), f^{-1}_{\alpha(i)}(s_{i+1}) \in \mathcal{P}$. For $s, t \in \mathcal{P}$ with $s < t$ let

$$\mathcal{V}[s,t] = \{\{s_j, s_{j+1}\} : j \in \mathcal{Q} \text{ and } [s_j, s_{j+1}] \subseteq [s,t]\}$$

For an admissible pair $\{s_i, s_{i+1}\}$ let

$$\mathcal{A}\{s_i, s_{i+1}\} = \mathcal{V}[f^{-1}_{\alpha(i)}(s_i), f^{-1}_{\alpha(i)}(s_{i+1})] \text{ and } [s_i, s_{i+1}]_E = [s_i, s_{i+1}] \cap E.$$

The following properties can be verified:

(I)*We have* $E = \bigcup_{i\in\mathcal{Q}}[s_i, s_{i+1}]_E$.

(II) *The compact sets* $[s_i, s_{i+1}]_E, i \in \mathcal{Q}$ *obey a graph-directed structure:*

$$[s_i, s_{i+1}]_E = \bigcup_{\{s_j,s_{j+1}\}\in\mathcal{A}\{s_i,s_{i+1}\}} f_{\alpha(i)}([s_j, s_{j+1}]_E).$$

In addition, it is clear that the above graph-directed structure satisfies the open set condition with respect to the open sets $\{(s_i, s_{i+1}) : i \in \mathcal{Q}\}$.

We remark that the above properties actually give a way to calculate $\dim_H E$. Now we construct a directed graph $\mathcal{G}$ by taking $\mathcal{V} = \{\{s_j, s_{j+1}\} : i \in \mathcal{Q}\}$ as the vertex set. For two vertices $\{s_i, s_{i+1}\}$ and $\{s_j, s_{j+1}\}$ we connect a directed edge from $\{s_i, s_{i+1}\}$ to $\{s_j, s_{j+1}\}$, denoted by $\{s_i, s_{i+1}\} \to \{s_j, s_{j+1}\}$, if $[s_j, s_{j+1}] \in \mathcal{A}\{s_i, s_{i+1}\}$. We say vertex $\{s_i, s_{i+1}\}$ can be connected to vertex $\{s_j, s_{j+1}\}$ by edges, denoted by $\{s_i, s_{i+1}\}\rightrightarrows\{s_j, s_{j+1}\}$, if either $\{s_i, s_{i+1}\} \to \{s_j, s_{j+1}\}$ or there exist vertices $B_1, \cdots, B_n$ such that

$$\{s_i, s_{i+1}\} \to B_1 \to \cdots \to B_n \to \{s_j, s_{j+1}\}.$$

Lemma 6.4.2. *The directed graph* $\mathcal{G}$ *is strongly connected.*

Proof. Arbitrarily fix two vertices $\{s_i, s_{i+1}\}$ and $\{s_j, s_{j+1}\}$. We need to show $\{s_i, s_{i+1}\} \rightrightarrows \{s_j, s_{j+1}\}$. The proof is divided into three cases.

Case I. $\{s_i, s_{i+1}\} \in \{\{s_1, s_2\}, \{s_{\gamma-1}, s_\gamma\}\}$.

Without loss of generality we assume that $\{s_i, s_{i+1}\} = \{s_1, s_2\} = \{f_1(a), f_{1^v}(b)\}$. The case $\{s_i, s_{i+1}\} = \{s_{\gamma-1}, s_\gamma\} = \{f_{m^u}(a), f_m(b)\}$ can be dealt with in the same way. By definition the following connections hold: $\{s_1, s_2\} \to \{s_1, s_2\}$, and

$$\{s_1, s_2\} \to \{s_2, s_3\} \to \cdots \to \{s_{v-1}, s_v\} = \{f_{1^3}(b), f_{1^2}(b)\}.$$

Note that

$$\mathcal{A}\{f_{1^3}(b), f_{1^2}(b)\} = \begin{cases} \{\{f_{1^2}(b), f_1(b)\}\} & \text{if } f_1(I) \cap f_2(I) = \emptyset \\ \{\{f_{1^2}(b), f_2(a)\}, \{f_2(a), f_1(b)\}\} & \text{if } f_1(I) \cap f_2(I) \neq \emptyset. \end{cases}$$

When $f_1(I) \cap f_2(I) = \emptyset$ we have

$$\{f_{1^3}(b), f_{1^2}(b)\} \to \{f_{1^2}(b), f_1(b)\} \to \{s,t\} \text{ for all } \{s,t\} \in \mathcal{V}[f_1(b), b].$$

Thus the result is correct. When $f_1(I) \cap f_2(I) \neq \emptyset$ we have

$$\{f_{1^3}(b), f_{1^2}(b)\} \to \{f_{1^2}(b), f_2(a)\} \to \{s,t\} \text{ for all } \{s,t\} \in \mathcal{V}[f_1(b), f_{m^{u(1)}}(a)].$$

and

$$\{f_{1^3}(b), f_{1^2}(b)\} \to \{f_2(a), f_1(b)\} \to \{s,t\} \text{ for all } \{s,t\} \in \mathcal{V}[f_{m^{u(1)}}(a), b].$$

This implies the result is correct.

Case II. $\{s_i, s_{i+1}\} \in \mathcal{V}[a, f_1(b)] \cup \mathcal{V}[f_m(a), b]$.

Without loss of generality we assume that $\{s_i, s_{i+1}\} \in \mathcal{V}[a, f_1(b)]$. The case $\{s_i, s_{i+1}\} \in \mathcal{V}[f_m(a), b]$ can be dealt with in the same way.

When $f_1(I) \cap f_2(I) = \emptyset$, the fact $\{s_i, s_{i+1}\} \rightrightarrows \{f_{m^u}(a), b\}$ can be derived directly from the discussion in case I. So the result is correct.

For the case that $f_1(I) \cap f_2(I) \neq \emptyset$ it suffices to show $\{f_{1^2}(b), f_2(a)\} \rightrightarrows \{f_{m^u}(a), b\}$. Note that

$$\begin{cases} \{f_1(b), f_2(b)\} \in \mathcal{V}[f_1(b), f_{m^{u(1)}}(a)] & \text{when } f_2(I) \cap f_3(I) = \emptyset \\ \{f_3(a), f_2(b)\} \in \mathcal{V}[f_1(b), f_{m^{u(1)}}(a)] & \text{when } f_2(I) \cap f_3(I) \neq \emptyset. \end{cases}$$

Thus either

$$\{f_{1^2}(b), f_2(a)\} \to \{f_1(b), f_2(b)\} \to \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_{1^{v(1)}}(b), b]$$

or

$$\{f_{1^2}(b), f_2(a)\} \to \{f_3(a), f_2(b)\} \to \{s, t\} \text{ for all } \{s, t\} \in \mathcal{V}[f_{m^{u(2)}}(a), b].$$

Thus we have $\{f_{1^2}(b), f_2(a)\} \rightrightarrows \{f_{m^u}(a), b\}$.

Case III. $\{s_i, s_{i+1}\} \in \mathcal{V}[f_1(b), f_m(a)]$.

For this case the admissible pair $\{s_i, s_{i+1}\}$ may occur as the following five forms:

(i) $\{s_i, s_{i+1}\} = \{f_k(a), f_{k+1}(a)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(a), f_{k+1}(a)\} \to \{s, t\} \in \mathcal{V}[a, f_{m^{u(k)}}(a)].$$

This reduces to Case II.

(ii) $\{s_i, s_{i+1}\} = \{f_{k+1}(a), f_k(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_{k+1}(a), f_k(b)\} \to \{s, t\} \in \mathcal{V}[f_{m^{u(k)}}(a), b].$$

This reduces to Case II.

(iii) $\{s_i, s_{i+1}\} = \{f_k(b), f_{k+1}(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(b), f_{k+1}(b)\} \to \{s, t\} \in \mathcal{V}[f_{1^{v(k)}}(b), b].$$

This reduces to Case II.

(iv) $\{s_i, s_{i+1}\} = \{f_k(a), f_k(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(a), f_k(b)\} \to \{s, t\} \in \mathcal{V}[a, b].$$

This reduces to Case II.

(v) $\{s_i, s_{i+1}\} = \{f_k(b), f_{k+2}(a)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(b), f_{k+2}(a)\} \to \{s, t\} \in \mathcal{V}[f_{1^{v(k)}}(b), f_{m^{u(k+1)}}(a)].$$

Note that $[f_1(b), f_m(a)] \subseteq [f_{1^{v(k)}}(b), f_{m^{u(k+1)}}(a)]$. Then there exists an admissible pair $\{s, t\} \in \mathcal{V}[f_1(b), f_m(a)]$ belonging to one of types (i)–(iv), and $\{s_i, s_{i+1}\} \to \{s, t\}$. Thus the result follows by (i)–(iv).

The result now is proved by the above three cases. □

In the following lemma we show that the Hausdorff dimension of $U_1(E)$ is strictly smaller than $\dim_H E$.

Lemma 6.4.3. *Let* $(E, \{f_i\}_{i=1}^m) \in E$. *Then* $\dim_H U_1(E) < \dim_H E$. *Furthermore,*

$$0 < \mathcal{H}^{\dim_H E}(E) < \infty.$$

Proof. By (II) and Lemma 6.4.2 it follows that E is a strongly connected graph-directed set satisfying the open set condition. Then by [62, Theorem 1.3] we obtain

$$0 < \mathcal{H}^{\dim_H E}(E) < \infty.$$

Let $\mathcal{Q}^*$ be the subset of $\mathcal{Q}$ defined in (6.22) by deleting those j for which $\{s_j, s_{j+1}\} = \{f_{\ell+1}(a), f_\ell(b)\}$ for some ℓ. For this $\mathcal{Q}^*$, one can get a graph-directed set E^* for which $\dim_H E^* < \dim_H E$. Moreover, $U_1(E)$ is a subset of E^*. Hence $\dim_H U_1(E) < \dim_H E$. □

Proof of Theorem 6.1.7. Note that

$$E = U_{2^{\aleph_0}}(E) \cup U_{\aleph_0}(E) \cup \bigcup_{k=1}^{\infty} U_k(E).$$

Furthermore, by Theorems 6.1.1–6.1.2 and Lemma 6.4.3 it follows that

$$\dim_H \left(U_{\aleph_0}(E) \cup \bigcup_{k=1}^{\infty} U_k(E) \right) = \dim_H U_1 < \dim_H E.$$

Therefore, by Lemma 6.4.3 we have

$$\dim_H U_{2^{\aleph_0}}(E) = \dim_H E \quad \text{and} \quad 0 < \mathcal{H}^{\dim_H E}(U_{2^{\aleph_0}}(E)) < \infty.$$

□

In the proof of Lemma 6.4.3, we construct a sub graph-directed self-similar sets of E, i.e. E^*. Obviously $\dim_H(\mathcal{U}_1(E)) \le \dim_H(E^*)$, and the Hausdorff dimension of E^* can be calculated as E satisfies the open set condition. However, we can show that

Lemma 6.4.4. $\dim_H(\mathcal{U}_1(E)) \ge \dim_H(E^*)$. *Moreover,* $\mathcal{U}_1(E) = E^*$ *except for a countable set.*

Before we prove this lemma, we give some definitions. Define

$$\mathcal{V}'[s,t] = \{\{s_j, s_{j+1}\} : j \in \mathcal{Q}^* \text{ and } [s_j, s_{j+1}] \subseteq [s,t]\}$$

and

$$\mathcal{A}'\{s_i, s_{i+1}\} = \mathcal{V}'[f_{\alpha(i)}^{-1}(s_i), f_{\alpha(i)}^{-1}(s_{i+1})].$$

For $\mathcal{Q}^*$, we can construct a directed graph, and denote it by G^*. Evidently $\mathcal{U}_1(E) \subset E^*$. Hence it suffices to prove that there exists a countable set, denoted by C, such that $E^* \subset \mathcal{U}_1(E) \cup C$. In order to prove this inclusion, we view our attractor E as a topological dynamical system in the following way.

Lemma 6.4.5. *Let* $x \in E$. *Then* $(i_n)_{n=1}^\infty \in \{1, \ldots, m\}^{\mathbb{N}}$ *is a coding for* x *if and only if* $f_{i_n}^{-1} \circ f_{i_{n-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) \in E$ *for all* $n \in \mathbb{N}$.

The proof of this lemma can be found in Chapter 3. Let (i_n) be a coding of $x \in E$. We call

$$\{f_{i_n}^{-1} \circ f_{i_{n-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) : n \ge 1\}$$

an orbit of x.

Proof of Lemma 6.4.4. Take $x \in E^*$. Then we can find a coding $(i_n) \in \{1, 2, \cdots, m\}^{\mathbb{N}}$ such that

$$f_{i_n}^{-1} \circ f_{i_{n-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) \in \mathcal{V}^*[a, b]$$

for any $n \geq 1$, where $\mathcal{V}^*[a, b] = \{[s_j, s_{j+1}]_E : \{s_j, s_{j+1}\} \in \mathcal{V}^{**}[a, b]\}$, and $\mathcal{V}^{**}[a, b] = \{\{s_j, s_{j+1}\} : j \in \mathcal{Q}^*\}$. If the orbit of x never hits the endpoints of every $[s_j, s_{j+1}]_E \in \mathcal{V}^*[a, b]$, denoted the set of these endpoints by W, then x has a unique coding. Otherwise, suppose that there exists some n_0 such that

$$f_{i_{n_0}}^{-1} \circ f_{i_{n_0-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) \in W.$$

Clearly $W \subset \mathcal{P}$.
Define

$$C := \Big\{x \in E : \text{there exists a sequence } (i_n)_{n=1}^{\infty} \in \{1, \ldots, m\}^{\mathbb{N}} \text{ and } n_0 \\ \text{such that } f_{i_{n_0}}^{-1} \circ f_{i_{n_0-1}}^{-1} \circ \ldots \circ f_{i_1}^{-1}(x) \in W\Big\}.$$

By Lemma 6.4.1, C is a countable set. More precisely,

$$C \subset \cup_{n=1}^{\infty} \cup_{(i_1 i_2 \cdots i_n) \in \{1,2,\cdots,m\}^n} f_{i_1 i_2 \cdots i_n}(\mathcal{P}).$$

Therefore, we have proved that $E^* \subset \mathcal{U}_1(E) \cup C$. □

For Theorem 6.1.7, it remains to show that

$$0 < \mathcal{H}^{\dim_H \mathcal{U}_1(E)}(\mathcal{U}_1(E)) < \infty,$$

and that if $\mathcal{U}_k(E) \neq \emptyset$, then

$$0 < \mathcal{H}^{\dim_H \mathcal{U}_k(E)}(\mathcal{U}_k(E)).$$

By Lemma 6.4.4, it suffices to show that the associated graph of the graph-directed self-similar set E^*, i.e. G^*, is strongly connected as we can identify $\mathcal{U}_1(E)$ with E^*.

Now we give a proof of this statement, the idea is similar with Lemma 6.4.2.

Lemma 6.4.6. *The graph G^* is strongly connected.*

Proof. Without loss of generality we assume $m \geq 5$. For $m = 3$ or 4 it is easy to check that G^* is strongly connected.

Case I. $\{s_i, s_{i+1}\} \in \{\{s_1, s_2\}, \{s_{\gamma-1}, s_\gamma\}\}$.

Without loss of generality we assume that $\{s_i, s_{i+1}\} = \{s_1, s_2\} = \{f_1(a), f_{1^v}(b)\}$. The case $\{s_i, s_{i+1}\} = \{s_{\gamma-1}, s_\gamma\} = \{f_{m^u}(a), f_m(b)\}$ can be discussed in the same way. By definition we have $\{s_1, s_2\} \rightrightarrows \{s_1, s_2\}$, and

$$\{s_1, s_2\} \to \{s_2, s_3\} \to \cdots \to \{s_{v-1}, s_v\} = \{f_{1^3}(b), f_{1^2}(b)\}.$$

Note that

$$\mathcal{A}'\{f_{1^3}(b), f_{1^2}(b)\} = \begin{cases} \{\{f_{1^2}(b), f_1(b)\}\} & \text{if } f_1(I) \cap f_2(I) = \emptyset \\ \{\{f_{1^2}(b), f_2(a)\}\} & \text{if } f_1(I) \cap f_2(I) \neq \emptyset. \end{cases}$$

When $f_1(I)\cap f_2(I)=\emptyset$ we have

$$\{f_{1^3}(b), f_{1^2}(b)\}\to\{f_{1^2}(b), f_1(b)\}\to\{s,t\} \text{ for all } \{s,t\}\in\mathcal{V}^{'}[f_1(b),b],$$

which implies that the vertex can reach any other vertices in G^*. Suppose that $f_1(I)\cap f_2(I)\neq\emptyset$. Since $m\geq 5$, by Conditions (C) there exists some $2\leq i\leq m-1$ such that either $f_i(I)\cap f_j(I)=\emptyset$ for any $j\neq i$ or $f_i(I)\cap f_{i+1}(I)\neq\emptyset$ and $f_{i+1}(I)\cap f_{i+2}(I)=\emptyset$.

For the first case, since $f_i(I)\cap f_j(I)=\emptyset$ for any $j\neq i$ and the fact that $[f_1(b), f_{m^{u(1)}}(a)]_E\supset[f_i(a), f_i(b)]_E$, it follows that $\{s_1,s_2\}$ can reach any vertices in G^*.

For the second case, since $f_i(I)\cap f_{i+1}(I)\neq\emptyset$ and $f_{i+1}(I)\cap f_{i+2}(I)=\emptyset$, by Condition (D) and Lemma 6.2.1, $f_{im^{u(i)}}=f_{(i+1)1^{v(i)}}$. Therefore, $[f_1(b), f_{m^{u(1)}}(a)]_E\supset[f_i(b), f_{i+1}(b)]_E$, and

$$\{f_i(b), f_{i+1}(b)\}\to\{f_{1^{v(i)}}(b), b\}\to\{f_{m^{u(1)}}(a), b\}.$$

Hence the result is correct.

Case II. $\{s_i,s_{i+1}\}\in\mathcal{V}^{'}[a,f_1(b)]\cup\mathcal{V}^{'}[f_m(a),b]$.

We suppose without loss of generality that $\{s_i,s_{i+1}\}\in\mathcal{V}^{'}[a,f_1(b)]$. If $f_1(I)\cap f_2(I)=\emptyset$, for this case it suffices to prove that for the vertex $\{f_{1^2}(b), f_1(b)\}$, it can reach any other vertices in G^*. If there exists some $3\leq i\leq m$ such that $f_i(I)\cap f_j(I)=\emptyset$ for any $j\neq i$, then the statement above is correct. Otherwise, by Condition (C) we can find some $3\leq i\leq m-1$ such that $f_i(I)\cap f_{i+1}(I)\neq\emptyset$ and $f_{i-1}(I)\cap f_i(I)=\emptyset$. By Condition (D) and Lemma 6.2.1 $f_{im^{u(i)}}=f_{(i+1)1^{v(i)}}$. Hence

$$\{f_1(b), b\}\to\{f_i(a), f_{i+1}(a)\}\to\{a, f_{m^{u(i)}}(a)\}\to\{a, f_1(b)\},$$

and the result is correct. If $f_1(I)\cap f_2(I)\neq\emptyset$, then we have to show that $\{f_{1^2}(b), f_2(a)\}$ can reach any vertices in G^*. If there exists some $3\leq i\leq m$ such that $f_i(I)\cap f_j(I)=\emptyset$ for any $j\neq i$, then the result is correct. Otherwise, by Condition (C), there exists some $2\leq i\leq m-2$ such that $f_{i-1}(I)\cap f_i(I)\neq\emptyset$, $f_i(I)\cap f_{i+1}(I)=\emptyset$, and $f_{i+1}(I)\cap f_{i+2}(I)\neq\emptyset$. By Lemma 6.2.1, it follows that

$$f_{(i-1)m^{u(i-1)}}=f_{i1^{v(i-1)}}, f_{(i+1)m^{u(i+1)}}=f_{(i+2)1^{v(i+1)}}.$$

Hence

$$\{f_{1^2}(b), f_2(a)\}\to\{f_1(b), f_{m^{u(1)}}(a)\}\supset\{f_{i-1}(b), f_i(b)\}\cup\{f_{i+1}(a), f_{i+2}(a)\}.$$

However, we have

$$\{f_{i-1}(b), f_i(b)\}\to\{f_{1^{v(i-1)}}(b), b\}$$

and

$$\{f_{i+1}(a), f_{i+2}(a)\}\to\{a, f_{m^{u(i+1)}}(a)\},$$

therefore the result is correct.

Case III. $\{s_i,s_{i+1}\}\in\mathcal{V}^{'}[f_1(b), f_m(a)]$.

For this case the admissible pair $\{s_i,s_{i+1}\}$ may occur as the following four forms:

(i) $\{s_i,s_{i+1}\}=\{f_k(a), f_{k+1}(a)\}$. Then we have

$$\{s_i,s_{i+1}\}=\{f_k(a), f_{k+1}(a)\}\to\{s,t\}\in\mathcal{V}^{'}[a, f_{m^{u(k)}}(a)].$$

This reduces to Case II.

(ii) $\{s_i, s_{i+1}\} = \{f_{k+1}(a), f_k(b)\}$. By the definition of $\mathcal{Q}^*$, we delete it and therefore we do not have to consider this case.

(iii) $\{s_i, s_{i+1}\} = \{f_k(b), f_{k+1}(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(b), f_{k+1}(b)\} \to \{s,t\} \in \mathcal{V}'[f_{1^{v(k)}}(b), b].$$

This reduces to Case II.

(iv) $\{s_i, s_{i+1}\} = \{f_k(a), f_k(b)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(a), f_k(b)\} \to \{s,t\} \in \mathcal{V}'[a, b].$$

This reduces to Case II.

(v) $\{s_i, s_{i+1}\} = \{f_k(b), f_{k+2}(a)\}$. Then we have

$$\{s_i, s_{i+1}\} = \{f_k(b), f_{k+2}(a)\} \to \{s,t\} \in \mathcal{V}'[f_{1^{v(k)}}(b), f_{m^{u(k+1)}}(a)].$$

Note that $[f_{1^{v(k)}}(b), f_{m^{u(k+1)}}(a)]_E \supset [f_1(b), f_m(a)]_E$. Hence this case can also be reduced to Case II.

With the discussion above, we prove that G^* is strongly connected. □

Finally, we show that

Lemma 6.4.7. *If* $\mathcal{U}_k(E) \neq \emptyset$, *then*

$$0 < \mathcal{H}^{\dim_H \mathcal{U}_k(E)}(\mathcal{U}_k(E)).$$

Proof. If $\mathcal{U}_k(E) \neq \emptyset$, then $\dim_H \mathcal{U}_k(E) = \dim_H \mathcal{U}_1(E) < \dim_H(E)$ in terms of Theorems 6.1.1 , 6.1.2 and Lemma 6.4.3.
If $f_1(I) \cap f_2(I) \neq \emptyset$ or $f_{m-1}(I) \cap f_m(I) \neq \emptyset$, then by Lemmas 6.2.5, 6.2.7 and 6.2.9, we know that

$$\mathcal{H}^{\dim_H \mathcal{U}_k(E)}(\mathcal{U}_k(E)) \geq c\mathcal{H}^{\dim_H \mathcal{U}_1(E)}(f_i(E) \cap \mathcal{U}_1(E))$$

for some positive constant c and some $1 \leq i \leq m$. By Lemma 6.4.6 and the positiveness of the Hausdorff measure of $\mathcal{U}_1(E)$, it follows that

$$\mathcal{H}^{\dim_H \mathcal{U}_1(E)}(f_i(E) \cap \mathcal{U}_1(E)) > 0.$$

If $f_1(I) \cap f_2(I) = \emptyset$ and $f_{m-1}(I) \cap f_m(I) = \emptyset$, by virtue of the proof of Lemma 6.2.11, we have

$$\{f_{im^u}(x) = f_{(i+1)1^v}(x) : x \in U_{2^k}(E)\} \subseteq U_{2^{k+1}}(E).$$

for any $k \geq 0$. Hence, using the fact $\dim_H \mathcal{U}_{2^k}(E) = \dim_H \mathcal{U}_1(E)$ for any $k \geq 2$, it follows that

$$\begin{aligned}
\mathcal{H}^{\dim_H \mathcal{U}_{2^k}(E)}(\mathcal{U}_{2^k}(E)) &\geq c_1\mathcal{H}^{\dim_H \mathcal{U}_{2^k}(E)}(\mathcal{U}_{2^{k-1}}(E)) \\
&\geq \cdots \\
&\geq c_k\mathcal{H}^{\dim_H \mathcal{U}_{2^k}(E)}(\mathcal{U}_1(E)) > 0
\end{aligned}$$

where $\{c_i\}$ are some positive constants, as required. □

6.5 Examples and further remarks

Example 6.5.1. *Let E be the self-similar set generated by*

$$f_1(x) = \lambda x, \qquad f_2(x) = \lambda x + \lambda - \lambda^2,$$
$$f_3(x) = \lambda x + 1 - 2\lambda + \lambda^2, \quad f_4(x) = \lambda x + 1 - \lambda,$$

where $0 < \lambda < \dfrac{1}{4}$.

Then $I = [0, 1]$, and one can check that $(E, \{f_i\}_{i=1}^4) \in \mathcal{E}$. In particular, we have

$$f_1(I) \cap f_2(I) = f_{14}(I) = f_{21}(I),$$
$$f_3(I) \cap f_4(I) = f_{34}(I) = f_{41}(I),$$

and $f_2(I) \cap f_3(I) = \emptyset$. Hence, by Theorem 6.1.1 it follows that $|U_{\aleph_0}(E)| = \aleph_0$, and

$$\dim_H U_k(E) = \dim_H U_1(E) = \frac{\log 3}{-\log \lambda}.$$

The calculation of $\dim_H(U_1(E))$ is due to Theorem 6.1.7 and Theorem 4.2.17. We give a simple introduction. Note that

$$\mathcal{P} = \{0, \lambda - \lambda^2, \lambda, 2\lambda - \lambda^2, 1 - 2\lambda + \lambda^2, 1 - \lambda, 1 - \lambda + \lambda^2, 1\}.$$

Let $A_1 = [0, \lambda - \lambda^2] \cap E$, $A_2 = [\lambda - \lambda^2, \lambda] \cap E$, $A_3 = [\lambda, 2\lambda - \lambda^2] \cap E$, $A_4 = [1 - 2\lambda + \lambda^2, 1 - \lambda] \cap E$, $A_5 = [1 - \lambda, 1 - \lambda + \lambda^2] \cap E$, $A_6 = [1 - \lambda + \lambda^2, 1] \cap E$.

It is easy to check that

$$A_1 = f_1(A_1) \cup f_1(A_2) \cup f_1(A_3) \cup f_1(A_4)$$
$$A_2 = f_1(A_5) \cup f_1(A_6)$$
$$A_3 = f_2(A_3) \cup f_2(A_4) \cup f_2(A_5) \cup f_2(A_6)$$
$$A_4 = f_3(A_1) \cup f_3(A_2) \cup f_3(A_3) \cup f_3(A_4)$$
$$A_5 = f_3(A_5) \cup f_3(A_6)$$
$$A_6 = f_4(A_3) \cup f_4(A_4) \cup f_4(A_5) \cup f_4(A_6)$$

Hence the adjacency matrix (see Chapter 4) is

$$S = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Note that A_2, A_5 are the switch regions. Therefore we can define an adjacency matrix for the univoque set, see Chapter 4, i.e.

$$S' = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Using these two matrices, we can explicitly calculate the Hausdorff dimension of E as well as $U_1(E)$, see the main results in Chapter 4.

Example 6.5.2. *Let E be the self-similar set generated by*

$$f_1(x) = \lambda x, \;\; f_2(x) = \lambda x + 2\lambda, \;\; f_3(x) = \lambda x + 3\lambda - \lambda^2, \;\; f_4(x) = \lambda x + 1 - \lambda,$$

where $0 < \lambda < \dfrac{5-\sqrt{21}}{2}$.

Then $I = [0,1]$, and one can check that $(E, \{f_i\}_{i=1}^4) \in \mathcal{E}$. In particular, we can show that

$$f_1(I) \cap f_2(I) = f_3(I) \cap f_4(I) = \emptyset$$

and

$$f_2(I) \cap f_3(I) = f_{24}(I) = f_{31}(I).$$

Then by Theorem 6.1.2 it follows that $U_{\aleph_0}(E) = \emptyset$ for any $k \neq 2^s, s \geq 1$, and

$$\dim_H U_{2^s}(E) = \dim_H U_1(E) = \frac{\log(2+\sqrt{2})}{-\log \lambda}.$$

The calculation of $\dim_H(U_1(E))$ can be found in Chapter 4.

Example 6.5.3. *Let E be a self-similar set generated by*

$$f_1(x) = \frac{x}{3}, \quad f_2(x) = \frac{x}{9} + \frac{8}{27}, \quad f_3(x) = \frac{x+2}{3}.$$

Then $I = [0,1]$, and one can check that $(E, \{f_i\}_{i=1}^3) \in \mathcal{E}$. In particular, we have

$$f_1(I) \cap f_2(I) = f_{133}(I) = f_{21}(I), \quad \textit{and} \quad f_2(I) \cap f_3(I) = \emptyset.$$

Hence, by Theorem 6.1.1 it follows that $|U_{\aleph_0}(E)| = \aleph_0$, and for any $k \geq 2$ we have

$$\dim_H U_k(E) = \dim_H U_1(E) = \frac{\log r}{\log 9},$$

where r is the largest positive root of

$$x^5 - 6x^4 + 9x^3 - 8x^2 + 4x - 1 = 0.$$

We finish this chapter by giving some remarks.

- For the calculation of $\dim_H(E)$ and $\dim_H(U_1(E))$, we refer to Chapter 4. In fact, for the class $\mathcal{E}$ we defined, we can explicitly calculate these two values. Theorem 6.1.7 indeed provides a proof.
- It is easy to find some examples such that although the IFS does not satisfy the conditions $(A) - (D)$, the equality $\dim_H(U_k(E)) = \dim_H(U_1(E))$ still holds for

any finite $k \geq 2$. For instance, let E be the attractor of following IFS, $\{f_1(x) = \lambda x, f_2(x) = \lambda x + \lambda - \lambda^2, f_3, f_4, \cdots, f_{m-1}, f_m(x) = \lambda x + 1 - \lambda\}$, where $0 < \lambda < 1$ and the convex hull of E is $I = [0, 1]$. For the similitudes $f_i, 3 \leq i \leq m-2$, we only assume that $f_i(I) \subset (2\lambda - \lambda^2, 1 - \lambda)$, i.e. these similitudes can overlap with each other seriously. $f_j(I) \cap f_i(I) = \emptyset, j = m-1, m, i \neq j$. It is easy to see that $f_{1m} = f_{21}$. By Lemmas 6.2.3, 6.2.4, 6.2.5, we can prove $\dim_H(U_k(E)) = \dim_H(U_1(E))$ for any $k \geq 2$.

- Similar idea can also be implemented in higher dimensions.
- For the self-affine setting, our idea is still working. We shall make use of the following example to illustrate this point.

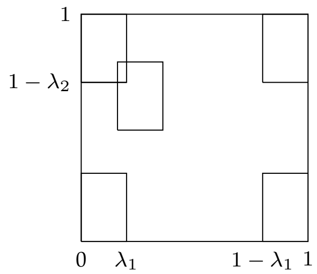
Example 6.5.4. *Let K be the self-affine set generated by the IFS*

$$\{f_1(x, y) = (\lambda_1 x, \lambda_2 y), f_2(x, y) = (\lambda_1 x + 1 - \lambda_1, \lambda_2 y),$$

$$f_3(x, y) = (\lambda_1 x + 1 - \lambda_1, \lambda_2 y + 1 - \lambda_2), f_4(x, y) = (\lambda_1 x, \lambda_2 y + 1 - \lambda_2),$$

$$f_5(x, y) = (\lambda_1 x + \lambda_1(1 - \lambda_1), \lambda_2 y + (1 - \lambda_2)^2)\}$$

where $0 < \lambda_1, \lambda_2 < \dfrac{3 - \sqrt{5}}{2}$. Let $I = [0, 1]^2$, then the first iteration of $\{f_i(I)\}_{i=1}^5$ is the figure in the next page.



For this example, note that $f_{42} = f_{54}$, and $f_i(I) \cap f_j(I) = \emptyset, 1 \leq i \leq 3, j \neq i$. Using similar ideas of Lemmas 6.2.3, 6.2.4, 6.2.5, we can show that

$$\dim_H(U_k(K)) = \dim_H(U_1(K))$$

for any finite $k \geq 2$.

- The Hausdorff measure of U_k could be infinity. Generally for any $(E, \{f_i\}_{i=1}^m) \in \mathcal{E}$ we cannot show this result. Here we give one example to illustrate our main idea.

Example 6.5.5. *Let $q > \dfrac{3 + \sqrt{5}}{2}$ be any real number and $\rho = q^{-1}$. Consider the iterated function system defined by*

$$\left\{ f_0(x) = \frac{x}{q}, f_1(x) = \frac{x + 1}{q}, f_q(x) = \frac{x + q}{q} \right\}.$$

The convex hull of K is $E = [0, (1 - \rho)^{-1}]$. Note that

$$f_0(E) = \left[0, \frac{\rho}{1 - \rho}\right], f_1(E) = \left[\rho, \frac{2\rho - \rho^2}{1 - \rho}\right], f_q(E) = \left[1, \frac{1}{1 - \rho}\right].$$

Clearly, $f_0(E) \cap f_1(E) \neq \emptyset$ *and* $f_0(E) \cap f_q(E) = \emptyset, f_1(E) \cap f_q(E) = \emptyset$ *as* $q > \frac{3+\sqrt{5}}{2}$*, This IFS is an exact overlapping IFS as we have* $f_0(K) \cap f_1(K) = f_0 \circ f_q(K) = f_1 \circ f_0(K)$*.*

Now we shall show that the Hausdorff measure of $U_k, k \geq 2$, is infinity. Firstly, we define the following set

$$\Omega = \{0, 1, q\},$$

$\Omega_1 = \{f_k : k \in \{0, 1, q\}, f_k(x)$ is an exact overlapping similitude$\}, \Omega_1^c = \{0, 1, q\} \setminus \Omega_1,$

$$\Omega_2 = \Big\{f_{\mathbf{k}}(x) : \mathbf{k} \in \Omega_1^c \times \{0, 1, q\},\ f_{\mathbf{k}}(x) \text{ is an exact overlapping similitude,}$$
$$f_{\mathbf{k}}(K) \not\subset f_{\mathbf{j}}(K), \mathbf{j} \in \Omega_1\Big\}.$$

$$\Omega_2^c = \Omega_1^c \times \{0, 1, q\} \setminus \Omega_2.$$

Generally we can define

$$\Omega_{i+1} = \Big\{f_{\mathbf{k}}(x) : \mathbf{k} \in \Omega_i^c \times \{0, 1, q\},\ f_{\mathbf{k}}(x) \text{ is an exact overlapping similitude}$$
$$f_{\mathbf{k}}(K) \not\subset f_{\mathbf{j}}(K), \mathbf{j} \in \Omega_s, 1 \leq s \leq i\Big\}.$$

$$\Omega_{i+1}^c = \Omega_i^c \times \{0, 1, q\} \setminus \Omega_{i+1}.$$

Here the definition of “exact overlapping similitude" can be found in Definition 4.2.6.

It is easy to show that the growth rate of $\mathbf{card}(\Omega_{i+1})$ is determined by the spectral radius of the following matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

After simple calculation, the spectral radius is q_c, where $q^s = q_c$ and $s = \dim_H(U_1)$. Roughly speaking, $\mathbf{card}(\Omega_i) = b_i = C \times q_c^i$ for some positive constant C, when i is very large. By our algorithm, any similitudes from Ω_i are piecewise disjoint. Moreover, take $f_{\mathbf{i}}(x) = f_{t_1 t_2 \cdots t_{i-1} t_i}(x) \in \Omega_i$, note that $t_{i-1} t_i = 0q$. The following lemma can be proved by induction.

Lemma 6.5.6. *Take* $f_{\mathbf{i}}(x) = f_{t_1 t_2 \cdots t_{i-1} t_i}(x) \in \Omega_i$*. Then* $f_{t_1 t_2 \cdots t_{i-2} 0 q^{k-1}}(f_1(E) \cap U_1) \subset U_k$*.*

Using this lemma we have

$$\mathcal{H}^s(U_k) \geq \mathcal{H}^s(\cup_{i=2}^{\infty} \cup_{(f_{\mathbf{k}}) \in \Omega_i} f_{\mathbf{k} 0 q^{k-1}}(f_1(E) \cap U_1)),$$

where the exact overlaps are piecewise disjoint. Hence

$$\begin{aligned}
\mathcal{H}^s(\cup_{i=2}^{\infty} \cup_{(f_{\mathbf{k}})\in\Omega_i} f_{\mathbf{k}0q^{k-1}}(f_1(E)\cap U_1)) &= \sum_{i=2}^{\infty}\sum_{(f_{\mathbf{k}})\in\Omega_i} \mathcal{H}^s(f_{\mathbf{k}0q^{k-1}}(f_1(E)\cap U_1)) \\
&= \sum_{i=2}^{\infty} b_i \frac{1}{q^{s(i+k)}} \mathcal{H}^s(f_1(E)\cap U_1)) \\
&= \sum_{i=2}^{\infty} C q_c^i \frac{1}{q^{s(i+k)}} \mathcal{H}^s(f_1(E)\cap U_1)) \\
&= \sum_{i=2}^{\infty} C q^{-ks} \mathcal{H}^s(f_1(E)\cap U_1)) \\
&= \infty
\end{aligned}$$

Here we note that $q_c = q^s$. The directed graph $\mathcal{G}^*$ is strongly connected, which yields $\mathcal{H}^s(f_1(E)\cap U_1)) > 0$.

- Finally, we give one example, which does not satisfy the Condition (D), to show that our idea can be implemented for many other self-similar sets with overlaps.

Example 6.5.7. *Let K be the attractor of the following IFS,*

$$\{f_1(x) = rx, f_2(x) = rx + r^2(1-r), f_3(x) = rx + 1 - r,\}$$

where $0 < r \leq \frac{1}{3}$. Then $\dim_H(\mathcal{U}_k) = \dim_H(\mathcal{U}_1)$ for any finite $k \geq 2$.

Let the convex hull of K be $I = [0,1]$. After simple calculation, we have

$$f_{111}(I) = \left[0, r^3\right], f_{112}(I) = \left[r^4(1-r), r^3 + r^4 - r^5\right], f_{113}(I) = \left[r^2 - r^3, r^2\right],$$
$$f_{121}(I) = \left[r^3(1-r), 2r^3 - r^4\right], f_{122}(I) = \left[r^3(1-r^2), 2r^3 - r^5\right],$$
$$f_{123}(I) = \left[r^2 - r^4, r^2 + r^3 - r^4\right], f_{212}(I) = \left[r^2 - r^3 + r^4 - r^5, r^2 + r^4 - r^5\right],$$
$$f_{222}(I) = \left[r^2 - r^5, r^2 + r^3 - r^5\right],$$

and

$$f_{113} = f_{211}, f_{123} = f_{221}.$$

For this example, we find that $f_1(I)\cap f_2(I)$ is not an exact overlap. However, the dimension of the set of points with finite multiple codings still coincides with the dimension of the univoque set.

Lemma 6.5.7. *For any $k \geq 2$, $x_k = \pi((113)^k(c_i))$ has precisely $k+1$ codings, where $(c_i) \in \tilde{U}$ and $c_1c_2c_3c_4c_5c_6 = 111111$.*

Proof. Let $k = 1$. Then $x_1 = \pi(113(c_i)) = \pi(211(c_i))$. Hence x_1 has at least two different codings. Suppose that it has an extra coding, denoted by (d_i). Then $d_1 = 1$ or $d_1 = 2$. d_1 cannot be 3 as $f_3(I)\cap f_i(I) = \emptyset, i = 1, 2$. If $d_1 = 1$ then $d_2 = 1$. Otherwise, if $d_2 = 2$ (d_2 cannot be 3 as $f_1(I)\cap f_3(I) = \emptyset$), then $x_1 \in f_{113111111}([0,1]) \cap f_{123}([0,1])$ as $f_{121}(1) < f_{122}(1) < f_{113}(0)$. However, $f_{113111111}(1) < f_{123}(0) = r^2 - r^4$, leading to a contradiction. Hence $d_1d_2 = 11$, which implies $d_3 = 3$ since $f_{111}(1) < f_{112}(1) < f_{113}(0)$. Hence by the uniqueness of (c_i), if $d_1 = 1$, then $(d_i) = 113(c_i)$. Similarly, if $d_1 = 2$, then $d_2 = 1$. However, $d_1 d_2 = 21$ implies that $d_3 = 1$. Otherwise, if $d_3 = 2$, then $x_1 \in f_{113111111}([0,1]) \cap f_{212}([0,1])$. However,

$$f_{113111111}(1) < f_{212}(0) = r^2 - r^3 + r^4 - r^5,$$

leading to contradiction. We have proved that when $k = 1$, the lemma is correct.

Now suppose $x_k = \pi((113)^k(c_i))$ has precisely $k+1$ codings. Then

$$x_{k+1} = \pi((113)^{k+1}(c_i)) = f_{113}(x_k) = f_{211}(x_k).$$

Hence x_{k+1} has at least $k+2$ different codings. Suppose it has an extra coding (d_i).Then $d_1 = 1$ or $d_2 = 2$. If $d_1 = 1$, then $d_2 = 1$. Otherwise, assume $d_2 = 2$ (d_2 cannot be 3 as $f_1(I) \cap f_3(I) = \emptyset$). Then $x_{k+1} \in f_{113113}([0,1]) \cap f_{123}([0,1])$. However, this is impossible as

$$f_{113113}(1) < f_{123}(0).$$

Hence $d_1 d_2 = 11$, which implies $d_3 = 3$ since $f_{111}(1) < f_{112}(1) < f_{113}(0)$. Similarly, if $d_1 = 2$, then $d_1 d_2 d_3 = 211$. Hence we have proved that the extra coding (d_i) is one of the $k+2$ codings for x_{k+1}, as required. □

The following lemma is trivial.

Lemma 6.5.8. $\dim_H(f_{111111}(K) \cap U_1) = \dim_H(U_1)$.

Proof. Since $f_3(I) \cap f_i(I) = \emptyset, i = 1, 2$, it follows that there exist some large $n > 6$ and $(111111a_7a_8 \cdots a_n)$ such that $f_{111111a_7a_8\cdots a_n}(I)$ do not intersect with other fundamental intervals in level n and that $(111111a_7a_8 \cdots a_n)$ does not contain blocks $(113), (211), (123)$ and (221). Hence, we can construct a bi-Lipschitz map between $\mathcal{U}_1$ and

$$f_{111111a_7a_8\cdots a_n}(K) \cap \mathcal{U}_1.$$

Subsequently,

$$\begin{aligned} \dim_H(U_1) &= \dim_H(f_{111111a_7a_8\cdots a_n}(K) \cap \mathcal{U}_1) \\ &\leq \dim_H(f_{111111}(K) \cap \mathcal{U}_1) \\ &\leq \dim_H(U_1), \end{aligned}$$

as required. □

Now the stated result in Example 6.5.7 follows from the lemmas above.

Chapter 7

Shrinking random β-expansions

Abstract

For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

In this chapter we define the shrinking random β-transformation K and investigate natural invariant measures for K, and the induced tranformation of K on a special subset of the domain. We prove that both transformations have a unique measure of maximal entropy. However, the measure induced from the intrinsically ergodic measure for K is not the intrinsically ergodic measure for the induced system.

7.1 Introduction

Let $\beta \in (1,2)$ and $x \in \mathcal{A}_\beta = [0, (\beta-1)^{-1}]$, we call a sequence $(a_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ a β-expansion of x if

$$x = \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Renyi [78] introduced the greedy map, and showed that the greedy expansion $(a_i)_{i=1}^{\infty}$ of $x \in [0,1)$ can be generated by defining $T(x) = \beta x \mod 1$ and letting $a_i = k$ whenever $T^{i-1}(x) \in [k\beta^{-1}, (k+1)\beta^{-1})$. Since then, many papers were dedicated to the dynamical properties of this map, see for example [86, 13, 50, 71, 16, 70] and references therein. However, Renyi's greedy map is not the unique dynamical approach to generate β-expansions. In [15] (see also [17, 16]) a new transformation was introduced, the random β-transformation, that generates all possible β-expansions, see Figure 1. This transformation makes random choices between the maps $T_0(x) = \beta x$ and $T_1(x) = \beta x - 1$ whenever the orbit falls into $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, which we refer to as the switch region.

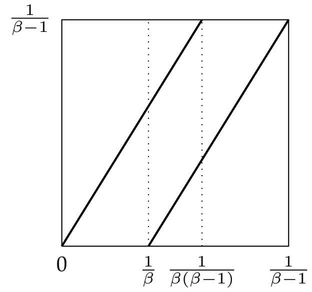


FIGURE 7.1: The dynamical system for $\{T_0(x) = \beta x,\, T_1(x) = \beta x - 1\}$

Although, all possible β-expansions can be generated via the random β-transformation, nevertheless, for some practical problems one would want to make choices only on a subset of the switch region $[\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, for instance, in A/D (analog-to-digit) conversion [20]. This motivates our study of the shrinking random β-transformation described below.

Let $1 < \beta < 2^{-1}(1+\sqrt{5})$, $\Omega = \{0,1\}^{\mathbb{N}}$, and $E = [0, (\beta-1)^{-1}]$. Set $a = (\beta^2-1)^{-1}$, $b = \beta(\beta^2-1)^{-1}$, i.e. $T_0(a) = b, T_1(b) = a$. The shrinking random β-transformation K is defined in the following way.

Definition 7.1.1. *$K : \Omega \times E \to \Omega \times E$ is defined by*

$$K(\omega, x) = \begin{cases} (\omega, \beta x) & x \in [0, a) \\ (\sigma(\omega), \beta x - \omega_1) & x \in [a, b] \\ (\omega, \beta x - 1) & x \in (b, (\beta-1)^{-1}] \end{cases}$$

In Figure 2, the map K is shown. Note that $[a, b] \subset [\beta^{-1}, \beta^{-1}(\beta-1)^{-1}]$, and for any $(\omega, x) \in \Omega \times [0, (\beta-1)^{-1}]$, the orbit of (ω, x) under K eventually hits $\Omega \times [T_1(a), T_0(b)]$ and remains therein. Hence, throughout the chapter we restrict the domain of K to $\Omega \times [T_1(a), T_0(b)]$.

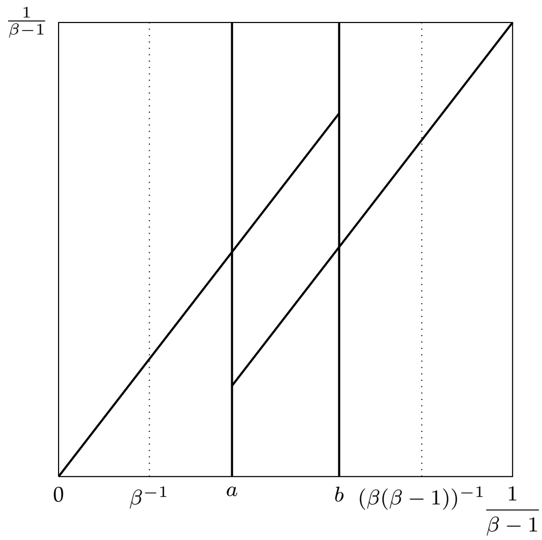


FIGURE 7.2: Shrinking random β-transformation

Given $(\omega, x) \in \Omega \times [a, b]$, the first return time is defined by

$$\tau(\omega, x) = \min\{n \geq 1 : K^i(\omega, x) \notin \Omega \times [a, b], 1 \leq i \leq n-1, K^n(\omega, x) \in \Omega \times [a, b]\},$$

Define $K_{\Omega\times[a,b]}(\omega, x) = K^{\tau(\omega,x)}(\omega, x)$, and denote it for simplicity by I.

We now consider a special family of algebraic bases defined as follows. For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

The following lemma is clear.

Lemma 7.1.2. *For any $n \geq 3$, let $\beta_n > 1$ be the largest positive real root of the equation*

$$x^n = x^{n-2} + x^{n-3} + \cdots + x + 1. \tag{7.1}$$

Then (β_n) is an increasing sequence which converges to $2^{-1}(1+\sqrt{5})$.

Throughout the chapter we will assume $\beta = \beta_n$ for some $n \geq 3$. In section 7.2, we will show that I can be identified with a full left shift. As a result it will be easy to find I-invariant measures, and to show that I is intrinsically ergodic (i.e. has a unique measure of maximal entropy). In the last section, we identify the dynamics of K with a topological Markov chain, and then use Parry's recipe to prove the following result.

Theorem 7.1.3. *For any $n \geq 3$, let $1 < \beta < 2$ be the largest positive real number satisfying the equation*

$$\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1.$$

Then the shrinking random β-transformation K and the induced transformation $I = K_{\Omega\times[a,b]}$ have intrinsically ergodic measures. Moreover, the induced transformation of K on $\Omega \times [a, b]$ does not yield the unique measure of maximal entropy.

7.2 Invariant measures for $I = K_{\Omega\times[a,b]}$

As above, β satisfies $\beta^n = \beta^{n-2} + \beta^{n-3} + \cdots + \beta + 1$, $n \geq 3$. It is easy to check that for $(\omega, x) \in \Omega \times [a, b]$, the first return time $\tau(\omega, x) \in \{2, 3, \cdots, n\}$.

Consider the space $\Omega \times \{2, 3, \cdots, n\}^{\mathbb{N}}$ equipped with the product σ-algebra, and the left shift σ'. Define the map $\phi : (\Omega \times [a, b]) \setminus (\cup_{i=0}^{\infty} K^{-i}(\Omega \times \{a\} \cup \Omega \times \{b\})) \to \Omega \times \{2, 3, \cdots, n\}^{\mathbb{N}}$ by

$$\phi(\omega, x) = (\omega, (n_1, n_2, \cdots, n_k, \cdots)),$$

where n_i is the i-th return time of (ω, x) to $(\Omega \times [a, b]) \setminus (\cup_{i=0}^{\infty} K^{-i}(\Omega \times \{a\} \cup \Omega \times \{b\})$, i.e. $n_i = \tau(I^{i-1}(\omega, x))$.

Given $(a_n) \in \{0, 1\}^{\mathbb{N}}$, we denote the value of the sequence (a_n) by $(a_n)_\beta = \sum_{n=1}^{\infty} a_n\beta^{-n}$.

Lemma 7.2.1. *The sequences*

$$(01)^{j_1}(1\underbrace{0\cdots0}_{n-1})^{j_2}(01)^{j_3}(1\underbrace{0\cdots0}_{n-1})^{j_4}\cdots$$

and

$$(10)^{j_1}(0\underbrace{1\cdots1}_{n-1})^{j_2}(10)^{j_3}(0\underbrace{1\cdots1}_{n-1})^{j_4}\cdots$$

are the possible β-expansions of a and b generated by the map K respectively, where $0 \leq j_k \leq \infty$.

Proof. The proof follows from the fact that $a = T_1(b) = T_0^{n-1}T_1(a)$ and $b = T_0(a) = T_1^{n-1}T_0(b)$. □

Lemma 7.2.2. *ϕ is a measurable bijection.*

Proof. Firstly we prove ϕ is one-to-one. Let $\phi(\omega, x_1) = \phi(\tau, x_2)$. Then we have $\omega = \tau$, and the first return time functions coincide. We denote the values of this function by $(n_i)_{i=1}^{\infty}$. One can easily check that

$$x_1 = (\omega_1\overline{\omega_1}^{n_1-1}\omega_2\overline{\omega_2}^{n_2-1}\cdots)_\beta,$$

and

$$x_2 = (\tau_1\overline{\tau_1}^{n_1-1}\tau_2\overline{\tau_2}^{n_2-1}\cdots)_\beta$$

where $\overline{\omega_i} = 1 - \omega_i$, and $(\omega_i)^k$ means k consecutive ω_i. As such we have $x_1 = x_2$. Now we prove ϕ is also a surjection. Given any $(\omega, (n_1, n_2, \cdots, n_k, \cdots))$, it is sufficient to show that

$$x = (\omega_1 \overline{\omega_1}^{n_1-1} \omega_2 \overline{\omega_2}^{n_2-1} \cdots)_\beta \in [a, b].$$

We decompose the sequence $(\omega_1 \overline{\omega_1}^{n_1-1} \omega_2 \overline{\omega_2}^{n_2-1} \cdots)$ into the blocks $\omega_i \overline{\omega_i}^{n_i-1}$. Note that for any $i \geq 1$, the value of the block can be classified in the following way:
If $\omega_i = 0$ and $n_i = 2$, then

$$(\omega_i \overline{\omega_i}^{n_i-1})_\beta = (01)_\beta.$$

If $\omega_i = 0$ and $n_i \geq 3$, then

$$(\omega_i \overline{\omega_i}^{n_i-1})_\beta \geq (1\underbrace{0\cdots0}_{n-1})_\beta.$$

If $\omega_i = 1$, then

$$(\omega_i \overline{\omega_i}^{n_i-1})_\beta \geq (1\underbrace{0\cdots0}_{n-1})_\beta.$$

Here we use the fact $1 < \beta < \dfrac{\sqrt{5}+1}{2}$, see Lemma 7.1.2. Hence, we have

$$x = (\omega_1 \overline{\omega_1}^{n_1-1} \omega_2 \overline{\omega_2}^{n_2-1} \cdots)_\beta \geq ((01)^{j_1}(1\underbrace{0\cdots0}_{n-1})^{j_2}(01)^{j_3}(1\underbrace{0\cdots0}_{n-1})^{j_4}\cdots)_\beta = a.$$

or

$$x = (\omega_1 \overline{\omega_1}^{n_1-1} \omega_2 \overline{\omega_2}^{n_2-1} \cdots)_\beta \geq ((1\underbrace{0\cdots0}_{n-1})^{j_1}(01)^{j_2}(1\underbrace{0\cdots0}_{n-1})^{j_3}(01)^{j_4}\cdots)_\beta = a.$$

Similarly, we prove by symmetry that

$$\bar{x} = (\beta-1)^{-1} - x = (\overline{\omega_1}\omega_1^{n_1-1}\overline{\omega_2}\omega_2^{n_2-1}\cdots)_\beta \geq ((01)^{j_1}(1\underbrace{0\cdots0}_{n-1})^{j_2}(01)^{j_3}(1\underbrace{0\cdots0}_{n-1})^{j_4}\cdots)_\beta = a$$

or

$$\bar{x} = (\beta-1)^{-1} - x = (\overline{\omega_1}\omega_1^{n_1-1}\overline{\omega_2}\omega_2^{n_2-1}\cdots)_\beta \geq ((1\underbrace{0\cdots0}_{n-1})^{j_1}(01)^{j_2}(1\underbrace{0\cdots0}_{n-1})^{j_3}(01)^{j_4}\cdots)_\beta = a.$$

Since $b = (\beta - 1)^{-1} - a$, we have $a \leq x \leq b$ and ϕ is surjective. It remains to show that ϕ is measurable. For any cylinders $C = \{\omega \in \Omega : \omega_1 = i_1, \cdots, \omega_m = i_m\}$ and $D = \{y \in \{2, 3, \cdots, n\}^{\mathbb{N}} : y_1 = n_1, \cdots, y_m = n_m\}$, we have

$$\phi^{-1}(C \times D) = \{(\omega, x) \in \Omega \times [a, b] : \tau(\omega, x) = n_1, \tau(I(\omega, x)) = n_2 \cdots, \tau(I^{m-1}(\omega, x)) = n_m\}$$

which is a measurable set, since τ and I are measurable. □

Lemma 7.2.3. *Let μ be any $\sigma\times\sigma'$-invariant measure on $\Omega\times\{2,3,\cdots,n\}^{\mathbb{N}}$. Then, the measure $\mu\circ\phi$ is I-invariant, and the dynamical systems $(\Omega\times[a,b], I, \mu\circ\phi)$, and $(\Omega\times\{2,3,\cdots,n\}^{\mathbb{N}}, \sigma\times\sigma', \mu)$ are isomorphic.*

Proof. It is easy to check that $(\sigma \times \sigma') \circ \phi = \phi \circ I$. Since ϕ is a measurable bijection, $\mu \circ \phi$ is I-invariant and the result follows. □

Corollary 7.2.4. *Let m_p be the $(p, 1-p)$ product measure on Ω, and μ_π the product measure on $\{2, 3, \cdots, n\}^{\mathbb{N}}$ induced by the probability vector $\pi = (\pi_2, \cdots, \pi_n)$, i.e. $\mu_\pi(\{(a_n) \in$*

$\{2,3,\cdots,n\}^{\mathbb{N}} : a_1 = i_i, \cdots, a_m = i_m\}) = \pi_{i_1}\cdots\pi_{i_m}$. *Then,* $(m_p \times \mu_\pi)\circ\phi$ *is an* I*-invariant ergodic measure on* $\Omega \times [a,b]$.

Proof. Note that μ_π is σ'-invariant, and since σ is weakly mixing, we have that $(m_p \times \mu_\pi)$ is $\sigma \times \sigma'$-invariant ergodic measure. By Lemma 7.2.3, it follows that $(m_p \times \mu_\pi)\circ\phi$ is an I-invariant ergodic measure on $\Omega \times [a,b]$. □

Note that for different probability vectors $\pi^{(1)}$ and $\pi^{(2)}$, the corresponding measures $(m_p \times \mu_{\pi^{(1)}})\circ\phi$ and $(m_p \times \mu_{\pi^{(2)}})\circ\phi$ are singular with respect to each other. It is natural to ask the following question: when do we have $(m_p \times \mu_\pi)\circ\phi = m_p \times \lambda$, where λ is the normalized Lebesgue measure on $[a,b]$?

To answer this question, we need to find an explicit expression for the induced transformation $K_{\Omega\times[a,b]}$ in terms of the first return time. We begin by partitioning $[a,b]$ using the greedy orbits, i.e. when $x \in [a,b]$ we implement T_1 on x. Define the greedy map $L_1(x) = \beta^{n-i+1}x - \beta^{n-i}$, where $x \in [c_i, c_{i+1})$, $c_1 = a, c_n = b$, $c_i = \beta^{i-1}a - \beta^{i-2} + \beta^{-1}$, $2 \le i \le n-1$. Similarly, we can define the lazy map $L_0(x)$ (we choose T_0 if the orbits fall into $[a,b]$) by

$$L_0(x) = \beta^{n-i+1}x - \beta^{n-i-1} - \beta^{n-i-2} - \beta^{n-i-3} - \cdots - \beta - 1,$$

if $x \in (d_i, d_{i+1}]$, where $d_1 = a, d_n = b$, $d_i = \beta^{n-i}b - \beta^{n-i-2} - \cdots \beta - 1 - \beta^{-1}$, $2 \le i \le n-1$. It is easy to see that L_1 and L_0 are Generalized Lüroth Series (GLS) maps [9]. Hence, the induced transformation $I = K_{\Omega\times[a,b]}$ is given by $I(\omega,x) = (\sigma(\omega), L_{\omega_1}(x))$. We now answer the question posed above.

Theorem 7.2.5. *Let* $P = \left(\frac{p_1}{b-a}, \frac{p_2}{b-a}, \cdots, \frac{p_{n-1}}{b-a}\right)$*, where* $p_i = \beta^i a - \beta^{i-1}a - (\beta^{i-1} - \beta^{i-2}), 1 \le i \le n-2$, $p_{n-1} = b - \beta^{n-2}a - \beta^{n-3}a + \beta^{-1}$. *Then,* $m_p \times \lambda$ *is an* I*-invariant ergodic measure and* $(m_p \times P)\circ\phi = m_p \times \lambda$.

Proof. By [9] [Theorems 1], the GLS maps L_0 and L_1 preserve the normalized Lebesgue measure. Since the induced transformation I is a skew product, it follows that $m_p \times \lambda$ is an I-invariant measure. To show $(m_p \times P)\circ\phi = m_p \times \lambda$, it is enough to show that $(m_p \times P) = (m_p \times \lambda)\circ\phi^{-1}$. Let $C = \{\omega \in \Omega : \omega_1 = i_1, \cdots, \omega_m = i_m\}$ and $D = \{y \in \{2,3,\cdots,n\}^{\mathbb{N}} : y_1 = n_1, \cdots, y_m = n_m\}$. Then,

$$\phi^{-1}(C \times D) = \{(\omega,x) \in \Omega \times [a,b] : \tau(\omega,x) = n_1, \cdots, \tau(I^{m-1}(\omega,x)) = n_m\} = C \times J,$$

where

$$J = D_{n_1} \cap L_{i_1}^{-1}(D_{n_2}) \cap (L_{i_2}\circ L_{i_1})^{-1}(D_{n_3}) \cap \cdots \cap (L_{i_{m-1}}\circ\cdots\circ L_{i_1})^{-1}(D_{n_m})$$

with $D_{n_j} = [c_{n-n_j+1}, c_{n-n_j+2})$ if $i_j = 1$ and $D_{n_j} = (d_{n_j-1}, d_{n_j}]$ if $i_j = 0$. Since the maps L_0 and L_1 are piecewise linear and surjective, an easy calculation shows that $\lambda(J) = \frac{p_{n_1}\cdots p_{n_m}}{(b-a)^m} = P(D)$. Thus,

$$(m_p \times \lambda)\Big(\phi^{-1}(C \times D)\Big) = m_p(C)P(D) = (m_p \times P)(C \times D).$$

□

Now, we turn our attention in finding the intrinsically ergodic measure for $I = K|_{\Omega\times[a,b]}$, i.e. the unique measure of maximal entropy. For this, we will identify the dynamics of I with a full left shift. Consider the space

$$\Lambda = \{(0,2),(0,3),\cdots,(0,n),(1,2),(1,3),\cdots,(1,n)\}^{\mathbb{N}}.$$

Here the first coordinate denotes the outcome of the coin toss (heads=0 or tails=1), and the second denotes the return time to $\Omega \times [a,b]$. Let S be the left shift on Λ, i.e. $S((i,j)_n) = (i^{'},j^{'})_n$, where $((i^{'},j^{'})_n) = (i,j)_{n+1}$. We define the following map

$$\rho : \Omega \times \{2,3,\cdots,n\}^{\mathbb{N}} \to \Lambda$$

by

$$\rho((\omega,(n_1,n_2,\cdots))) = ((\omega_1,n_1),(\omega_2,n_2),(\omega_3,n_3),\cdots).$$

Evidently, ρ is a bijection and $\rho \circ (\sigma \times \sigma^{'}) = S \circ \rho$. This leads to the following theorem.

Theorem 7.2.6. *The induced transformation I is intrinsically ergodic with maximal maximal entropy* $\log(2n-2)$.

Proof. Let m be the product $\left(\frac{1}{2n-2},\frac{1}{2n-2},\cdots,\frac{1}{2n-2}\right)$ measure on Λ. Note that m is shift invariant, and is intrinsically ergodic. Since ρ is a commuting bijection, the measure $m \circ \rho$ is $\sigma \times \sigma'$-invariant and is intrinsically ergodic. By Lemma 7.2.3, $m \circ \rho \circ \phi$ is the unique measure of maximal entropy for I. Since entropy is preserved under an isomorphism, the maximal entropy is $\log(2n-2)$. □

7.3 Invariant measures for K

It is a classical fact that if ν is an I-invariant probability measure on $\Omega \times [a,b]$, then the probability measure μ defined on $\Omega \times [T_1(a), T_0(b)]$ by

$$\mu(E) = \frac{1}{\int \tau \, d\nu} \sum_{n \geq 0} \nu(\{(\omega,x) \in \Omega \times [a,b] : \tau(\omega,x) > n\} \cap K^{-n}(E))$$

is a K-invariant probability measure. So to any measure ν as defined in the previous section corresponds a K-invariant measure.

Now we consider the intrinsically ergodic measure of K. It can be found via Parry's work, see [85]. For the sake of convenience, we give a brief introduction to Parry's result. Given any one-dimensional subshift of finite type with irreducibility condition, the Parry measure given by a probability vector $(p_0,p_1,\cdots,p_{k-1})$ and stochastic matrix (p_{ij}) is constructed as follows. If λ is the largest positive eigenvalue of A ($A = (a_{ij})$ is the adjacency matrix of the subshift of finite type) and $(u_0,u_1,\cdots,u_{k-1})$ is a strictly positive left eigenvector and $(v_0,v_1,\cdots,v_{k-1})$ is a strictly positive right eigenvector with $\sum_{i=0}^{k-1} u_i v_i = 1$, then $p_i = u_i v_i$ and $p_{ij} = \frac{a_{ij} v_j}{\lambda v_i}$. We state the following classical result.

Theorem 7.3.1. *Given any one-dimensional subshift of finite type with irreducibility condition, then the Parry measure is the intrinsically ergodic measure for this subshift of finite type. The maximal entropy is* $\log \lambda$.

Recall the definition of β, given $n \geq 3$, let β be the largest positive root of the following equation:

$$\beta^n = \sum_{i=0}^{n-2} \beta^i.$$

We can partition $[T_1(a), T_0(b)]$ in terms of the image of $[a, b]$ under K. More precisely, let

$$\{[T_0^k T_1(a), T_0^{k+1} T_1(a)], 0 \leq k \leq n-2, [a, b], [T_1^i T_0(b), T_1^{i+1} T_0(b)], 1 \leq i \leq n-1\}$$

be a Markov partition of $[T_1(a), T_0(b)]$, where $T_j^0 = id, j = 0, 1$. It is easy to see that the image of each set of the Markov partition is the union of some sets of this partition. For instance, when $n = 3$, let

$$A = [T_1(a), T_0T_1(a)], B = [T_0T_1(a), a], C = [a, b], D = [b, T_1T_0(b)], E = [T_1T_0(b), T_0(b)].$$

Evidently,

$$T_0(A) = B, T_0(B) = C, T_0(C) = D \cup E, T_1(C) = A \cup B, T_1(D) = C, T_1(E) = D.$$

Hence the associated adjacency matrix for this Markov partition is

$$S_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

This matrix can generate a subshift of finite type, denoted by Σ_3, i.e.

$$\Sigma_3 = \{(i_n) \in \{1, 2, 3, 4, 5\}^{\mathbb{N}} : S_{3_{i_n, i_{n+1}}} = 1\}.$$

Similarly, for general n, we can find the adjacency matrix S_n and its corresponding subshift of finite type Σ_n. It is easy to see that the matrix S_n is irreducible. Hence, we can make use of Parry's idea to find the unique measure of maximal entropy.

Denote $a_n = \det(\lambda I - S_n)$. The following lemma is doing some trivial calculation in linear algebra.

Lemma 7.3.2. *$a_{n+1} = \lambda^2 a_n - 2\lambda^n$ for any $n \geq 3$, and $a_3 = \lambda^2(\lambda^3 - 2\lambda - 2)$. By induction, we have*

$$a_n = \lambda^{n-1}(\lambda^n - 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2})).$$

The right eigenvector of S_n is

$$\vec{v} = (v_0, v_1, \cdots, v_{2n-2}) = (c, \lambda c, \lambda^2 c, \cdots, \lambda^{n-2} c, \lambda^{n-1} c, \lambda^{n-2} c, \lambda^{n-3} c, \cdots, \lambda c, c)$$

where $c > 0$.

The left eigenvector of S_n, denoted by $\vec{u} = (u_0, u_1, u_2, \cdots, u_{2n-2})$, is

$$\left(d, \frac{1+\lambda}{\lambda} d, \frac{1+\lambda+\lambda^2}{\lambda^2} d, \cdots, \frac{1+\lambda+\cdots+\lambda^{n-2}}{\lambda^{n-2}} d, \lambda d, \frac{1+\lambda+\cdots+\lambda^{n-2}}{\lambda^{n-2}} d, \cdots, \frac{1+\lambda}{\lambda} d, d\right)$$

where $d > 0$.

Now we can find the Parry measure as follows, given any $(a_1 a_2 \cdots a_k) \in \{1, \cdots 2n-2\}^k$, the Parry measure defined on the cylinder $[a_1 a_2 \cdots a_k]$ is

$$\mu([a_1 a_2 \cdots a_k]) = p_{a_1} p_{a_1 a_2} \cdots p_{a_{k-1} a_k}.$$

Let ν be the induced measure of μ on $\Omega \times [a, b]$, that is

$$\nu(E) = \frac{\mu(E)}{\mu(\Omega \times [a, b])},$$

for E a measurable subset of $\Omega \times [a, b]$. By Abramov formula,

$$h(K, \mu) = h(I, \nu) \times \mu(\Omega \times [a, b]),$$

where h denotes the entropy of the underlying system, and $I = K_{\Omega\times[a,b]}$. By the construction of the Parry measure,

$$h(I, \nu) = \frac{\log \lambda}{u_n v_n} = \frac{\log \lambda}{cd\lambda^n}.$$

To prove the remaining part of Theorem 7.1.3, we need to compare $h(I, \nu) = \dfrac{\log \lambda}{cd\lambda^n}$ with $\log(2n - 2)$, the maximal entropy of I.

Lemma 7.3.3. *For any $n \geq 3$,*

$$\log(2n - 2) > \frac{\log \lambda}{cd\lambda^n}.$$

Proof. It is easy to check that for $n = 3$ or 4, the inequality is correct. Hence, it suffices to prove this lemma when $n \geq 5$. Note that λ is the largest positive root of the following equation

$$\lambda^n - 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2}) = 0.$$

Since S_n is irreducible, it follows by Perron-Frobenius Theorem that such a λ exists, and furthermore $1 < \lambda < 2$. By the construction of the Parry measure, we have $\vec{u} \cdot \vec{v} = 1$, which implies that

$$\frac{1}{cd} = \frac{2}{\lambda - 1}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) + \lambda^n.$$

Hence, in order to prove

$$\frac{\log \lambda}{cd\lambda^n} = \left[\frac{2}{\lambda - 1}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) + \lambda^n\right] \frac{\log \lambda}{\lambda^n} < \log(2n - 2),$$

it suffices to prove that

$$\lambda^{\frac{2}{(\lambda-1)\lambda^n}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right)} < \frac{2n - 2}{\lambda}.$$

Since $n \geq 5$ and $1 < \lambda < 2$, it follows that $\dfrac{2n - 2}{\lambda} \geq \dfrac{8}{\lambda} \geq \lambda^2$. Hence it remains to show that

$$\frac{2}{(\lambda - 1)\lambda^n}\left(\lambda^{n-1} - n + \frac{\lambda^n}{2}\right) < 2.$$

However, this inequality immediately follows from

$$\lambda^n - 2(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-2}) = 0$$

and $1 < \lambda < 2$. □

Proof of Theorem 7.1.3. By Lemma 7.3.3, Theorem 7.2.6 and Theorem 7.3.1, we finish the proof of Theorem 7.1.3. □

7.4 Some remarks

The shrinking random β-transformation we defined is very special. For a general sub switch region, i.e. $(a, b) \subset [\beta^{-1}, \beta^{-1}(\beta - 1)^{-1}]$, does the intrinsically ergodic measure exist? For general $1 < \beta < 2^{-1}(1 + \sqrt{5})$, how can we find an invariant measure (or intrinsically ergodic measure) for the shrinking random β-transformation? In the setting of classical random beta transformation, similar questions can be considered, see [5].

Chapter 8

Sum of self-similar sets

Abstract

Let $\beta > 1$. We define a class of similitudes

$$S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.$$

Taking any finite collection of similitudes $\{f_i(x)\}_{i=1}^m$ from S, it is well known that there is a unique self-similar set K_1 satisfying $K_1 = \cup_{i=1}^m f_i(K_1)$. Similarly, another self-similar set K_2 can be generated via the finite contractive maps of S. We call $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$ the arithmetic sum of two self-similar sets. In this chapter, we prove that $K_1 + K_2$ is either a self-similar set or a unique attractor of some infinite iterated function system. Using this result we can calculate the exact Hausdorff dimension of K_1+K_2 under some conditions, which partially provides the dimensional result of $K_1 + K_2$ if the IFS's of K_1 and K_2 fail the irrationality assumption, see Peres and Shmerkin [72].

8.1 Introduction

Let $\{g_j\}_{j=1}^m$ be an iterated function system (IFS) of similitudes which are defined on $\mathbb{R}$ by

$$g_j(x) = r_j x + a_j,$$

where the similarity ratios satisfy $0 < r_j < 1$ and the translation parameter $a_j \in \mathbb{R}$. It is well known that there exists a unique non-empty compact set $K \subset \mathbb{R}$ such that

$$K = \bigcup_{j=1}^m g_j(K). \tag{8.1}$$

We call K the self-similar set or attractor for the IFS $\{g_j\}_{j=1}^m$, see [46] for further details. The IFS $\{g_j\}_{j=1}^m$ is called homogeneous if all the similarity ratios are equal. We say that $\{g_j\}_{j=1}^m$ satisfies the open set condition(OSC) [46] if there exists a non-empty bounded open set $V \subseteq \mathbb{R}$ such that

$$g_i(V) \cap g_j(V) = \varnothing,\ i \neq j$$

and $g_j(V) \subseteq V$ for all $1 \leq j \leq m$. Under the open set condition, the Hausdorff dimension of K coincides with the similarity dimension which is the unique solution s of the equation $\sum_{j=1}^m r_j^s = 1$.

Let F_1 and F_2 be the self-similar sets with IFS's $\{r_i x + a_i\}_{i=1}^n$ and $\{r_j^{'} x + b_j^{'}\}_{j=1}^m$ respectively. We call $F_1 + F_2 = \{x + y : x \in F_1, y \in F_2\}$ the arithmetic sum of self-similar sets. The arithmetic sum of Cantor sets appears naturally in dynamical systems. Palis

[69] posed the following problem which is currently known as the Palis' conjecture. Whether it is true (at least generically) that the arithmetic sum of dynamically defined Cantor sets either has measure zero or contains an interval. This conjecture was solved in [1]. However, for the general self-similar sets this conjecture is still open. Dekking et.al. discussed the Palis' conjecture and related questions for the random version, see [23, 21, 24, 22].

In [63], Mendes and Oliveira proved that for the homogeneous Cantor sets, there are five possible structures for the sum. For the fractal structure, i.e. the similarity of the sum of self-similar sets, there are few results regarding this aspect. This is the first reason why we study the sum of self-similar sets. Another natural question concerning with the sum of self-similar sets is to consider the Hausdorff dimension or Hausdorff measure of $F_1 + F_2$. Many papers have been devoted to this aspect. Let C_a be the central Cantor set generated by removing a central interval of length $1 - 2a$ from $[0, 1]$, and then continuing this process inductively on each remaining two intervals. Denote $\gamma(a) = \dim_H(C_a) = \dfrac{\log 2}{-\log a}$. Peres and Solomyak [74] proved that

Theorem 8.1.1. *Given a fixed compact set $K \subset \mathbb{R}$, the following two statements hold for almost every $a \in (0, \frac{1}{2})$:*

if $\gamma(a) + \dim_H(K) \leq 1$, then $\dim_H(K + C_a) = \gamma(a) + \dim_H(K)$;

if $\gamma(a) + \dim_H(K) > 1$, then the Lebesgue measure of $C_a + K$ is positive.

Motivated by this result Eroglu [29] considered the Hausdorff measure of the arithmetic sum of two Cantor sets, and gave a necessary and sufficient condition such that the Hausdorff measure of the sum of Cantor sets is positive. Peres and Solomyak's main idea is using the potential theory. This is the main reason why their result is the almost-type result. An important progress of the dimensional problem is due to Peres and Shmerkin. In [72], Peres and Shmerkin showed that

Theorem 8.1.2. *If there exist i, j satisfying $\dfrac{\log r_i}{\log r_j^{'}} \notin \mathbb{Q}$, then*

$$\dim_H(F_1 + F_2) = \min\{1, \dim_H(F_1) + \dim_H(F_2)\}.$$

The hypothesis of this theorem is called the irrationality assumption. It is easy to see that many pairs of iterated function systems satisfy this assumption. Peres and Shmerkin's formula gives a sufficient condition under which the expected dimension of the sum of self-similar sets can be obtained. Their main idea is to project the product of two one-dimensional self-similar sets into the real line and to show that under the irrationality assumption the expected dimension of $F_1 + F_2$ can be achieved. Later, Nazarov et al. [66] investigated similar problem for the convolutions of Cantor measures without resonance.

Motivated by Peres and Shmerkin's result and Palis' conjecture, we consider the IFS's of F_1 and F_2 failing the irrationality assumption. With a little effort, it may be shown that the IFS's $\{r_i x + a_i\}_{i=1}^n$ and $\{r_j^{'} x + b_j^{'}\}_{j=1}^m$ do not satisfy the irrationality assumption if and only if there exist $\beta > 1$, n_i and $m_j \in \mathbb{N}$ such that $r_i = \dfrac{1}{\beta^{n_i}}$, $1 \leq i \leq n$ and $r_j^{'} = \dfrac{1}{\beta^{m_j}}$, $1 \leq j \leq m$. Unless stated otherwise, in what follows we always assume that the similitudes of K_1 and K_2 are from

$$S := \left\{ f_i(x) = \frac{x}{\beta^{n_i}} + a_i : n_i \in \mathbb{N}^+, a_i \in \mathbb{R} \right\}.$$

We suppose without loss of generality that the IFS's of K_1 and K_2 are $\{f_i(x) = \frac{x}{\beta^{n_i}} + a_i\}_{i=1}^n$ and $\{g_j(x) = \frac{x}{\beta^{m_j}} + b_j\}_{j=1}^m$, respectively.

We shall prove that $K_1 + K_2$ is either a self-similar set or an attractor of some infinite iterated function system (IIFS) [61, 35]. Therefore, to calculate the Hausdorff dimension of $K_1 + K_2$ is reduced to considering the dimension of the attractor of some IFS (IIFS). It is well known that generally it is difficult to calculate the Hausdorff dimension of a self-similar set, especially when overlaps occur. It is much more difficult to find the dimension of the attractor of some IIFS even if the IIFS satisfies certain separation condition. Here the attractor of the IIFS is in the sense of Definition 8.2.1, we will introduce this definition in the next section. In fact, Peres and Shmerkin's dimensional formula implies that we may not find the exact Hausdorff dimension of $F_1 + F_2$ generally. In this chapter, we shall consider some cases which allow us to calculate the dimension of $K_1 + K_2$ explicitly. An important difference between our main result and Peres and Shmerkin's formula is that we may not obtain the expected dimension for the sum of self-similar sets, see the first example in section 4. Peres and Shmerkin gave a uniform formula while we emphasize on the individual example. In other words, our method is analyzing a single example rather than giving a uniform formula for the dimension of the sum of self-similar sets. When $K_1 + K_2$ is a self-similar set with overlapping IFS, the techniques of the paper [45] could be useful. However, this is beyond our discussion, and we do not give further details.

For the topological structure of $K_1 + K_2$, e.g. connected property and so on, generally we may not easily get further information. The main reasons are that the IFS (IIFS) of $K_1 + K_2$ may vary from each other and that discussing these two cases needs different techniques.

The structure of the chapter is as follows. In section 8.2, we introduce some basic results of the infinite iterated function systems and define some necessary terminology. Next, we prove the similarity of $K_1 + K_2$. In section 8.3, we concentrate on the Hausdorff dimension of $K_1 + K_2$. We consider both cases, i.e. $K_1 + K_2$ is a self-similar set or a unique attractor of some IIFS, and give some dimensional results. In section 8.4, we offer some examples for which we can explicitly calculate the Hausdorff dimension of $K_1 + K_2$. Finally, we give some further remarks.

8.2 Preliminaries and Main results

8.2.1 Infinite iterated function systems

Before stating our main results, we introduce some definitions and results of infinite iterated function systems (IIFS). Infinite iterated function systems behave differently from IFS's [61], [35]. There are two definitions of the invariant set of IIFS, see for example, [35], [61] and [44]. We adopt Fernau's definition [35].

Definition 8.2.1. *Let $\mathcal{A} = \{\phi_i(x) = r_i x + a_i : i \in \mathbb{N}, 0 < r_i < 1, a_i \in \mathbb{R}\}$. If there exists $0 < s < 1$ such that for every $\phi_i \in \mathcal{A}$, $|\phi_i(x) - \phi_i(y)| \leq s|x - y|$, then $\mathcal{A}$ is called an infinite iterated function system, abbreviated as IIFS. A unique non-empty compact set J is called the attractor of $\mathcal{A}$ if*

$$J = \overline{\bigcup_{i \in \mathbb{N}} \phi_i(J)},$$

where $\overline{A}$ denotes the closure of A.

Remark 8.2.2. *The existence and uniqueness of J can be found in [35]. In [61], Mauldin and Urbanski gave another definition of the attractor of IIFS, i.e.* $J_0 = \bigcup_{i\in\mathbb{N}} \phi_i(J_0)$. *However, for their definition the attractor J_0 may not be unique or compact, see example 1.3 from [35]. Evidently,* $\overline{J_0} = J$.

An infinite iterated function system $\mathcal{A} = \{\phi_i : i \in \mathbb{N}\}$ satisfies the open set condition if there exists a non-empty bounded open set $O \subseteq \mathbb{R}$ such that

$$\phi_i(O) \cap \phi_j(O) = \varnothing,\ i \neq j,$$

and $\phi_j(O) \subseteq O$ for all $j \in \mathbb{N}$. Under this separation condition, we can find the Hausdorff dimension of J_0. The following result can be found in [61], [64] or [44].

Theorem 8.2.3. *For any IIFS satisfying the open set condition, we have*

$$\dim_H(J_0) = \inf\left\{t : \sum_{i\in\mathbb{N}} r_i^t \leq 1\right\}.$$

On the other hand, generally the Hausdorff dimension of J is more complicated. One of the difficulties is to analyze $J \setminus J_0$, see [44, Corollary 2]. For the most cases, we shall prove that $J = K_1 + K_2$ is an attractor of some IIFS in the sense of Definition 8.2.1. This makes the dimension of $K_1 + K_2$ complicated. We mentioned above that $\overline{J_0} = J$. If J_0 and J coincide except for a countable set, then by the countable stability of the Hausdorff dimension we have that $\dim_H(J_0) = \dim_H(J)$. We will give a sufficient condition under which we can identify J_0 with J apart from a countable set. This is the main idea we will implement, provided $K_1 + K_2$ is the unique attractor of some IIFS.

8.2.2 Some definitions

In this section, we introduce some definitions which make our discussion far more succinct. Given any finite reals $s_1, s_2, s_3, \cdots, s_n$. Let $\sum = \{s_1,\ s_2,\ \cdots,\ s_n\}^{\mathbb{N}}$ be a symbolic space. We say $c_1c_2\cdots c_m \in \{s_1,\ s_2,\ \cdots,\ s_n\}^m$ is a block with length m, and we use capital letters with hats to denote the finite blocks of Σ. For instance, we denote $c_1c_2\cdots c_m$ by $\hat{P}$, i.e. $\hat{P} = c_1c_2\cdots c_m$.

Definition 8.2.4. *Let $\hat{P}_1 = d_1d_2\cdots d_m$ and $\hat{P}_2 = c_1c_2\cdots c_m$ be two blocks of $\{s_1,\ s_2,\ \cdots,\ s_n\}^m$. We define the concatenation of $\hat{P}_1$ and $\hat{P}_2$ by $\hat{P}_1 * \hat{P}_2 = d_1d_2\cdots d_mc_1c_2\cdots c_m$. The sum of $\hat{P}_1$ and $\hat{P}_2$ is defined by $\hat{P}_1 + \hat{P}_2 = (d_1 + c_1)(d_2 + c_2)\cdots(d_m + c_m)$. Concatenating $k \in \mathbb{N}$ blocks of $\hat{P}_1$ is denoted by*

$$\hat{P}_1^k = \underbrace{\hat{P}_1 * \hat{P}_1 * \cdots * \hat{P}_1}_{k \text{ times}}.$$

The value of the block $\hat{P}_1 = d_1d_2\cdots d_m$ with respect to $\beta > 1$ is

$$(d_1d_2\cdots d_m)_\beta = \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots + \frac{d_m}{\beta^m}.$$

Similarly, we can define the value of an infinite sequence $(d_n) \in \sum$ by $(d_n)_\beta = \sum\limits_{n=1}^{\infty} \frac{d_n}{\beta^n}$.

Remark 8.2.5. *In this definition, when we define the summation of two blocks, we assume that these two blocks have the same length. However, in some cases we may need to consider the* *concatenation of infinite blocks. For instance, let $\{\hat{P}_i\}_{i=1}^{\infty}$ and $\{\hat{Q}_i\}_{i=1}^{\infty}$ be two block sets, the concatenations of $\hat{P}_1 * \hat{P}_2 * \cdots$ and $\hat{Q}_1 * \hat{Q}_2 * \cdots$ are two infinite sequences in Σ, we denote them by (a_n) and (b_n) respectively. The summation of $\hat{P}_1 * \hat{P}_2 * \cdots$ and $\hat{Q}_1 * \hat{Q}_2 * \cdots$ is $(a_n + b_n)_{n=1}^{\infty}$. We shall emphasize this case in the proofs of some results.*

Now we give the definition of the codings of the points in the self-similar sets. It is sightly different from the usual way. Recall the IFS's of K_1 and K_2 are $\{f_i(x) = \frac{x}{\beta^{n_i}} + a_i\}_{i=1}^{n}$ and $\{g_j(x) = \frac{x}{\beta^{m_j}} + b_j\}_{j=1}^{m}$, where n_i, a_i, m_j, b_j are determined by the IFS's of K_1 and K_2. It is well known that for any $x \in K_1$, there exists $(i_k)_{k=1}^{\infty}$ such that

$$x = \lim_{k\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}(0).$$

Usually, $(i_k)_{k=1}^{\infty}$ is called a coding of x. Nevertheless, we may make use of another representation.

Note that

$$f_i(x) = \frac{x}{\beta^{n_i}} + a_i = \frac{x + \beta^{n_i} a_i}{\beta^{n_i}} = \frac{x}{\beta^{n_i}} + \frac{0}{\beta} + \frac{0}{\beta^2} + \cdots + \frac{0}{\beta^{n_i - 1}} + \frac{\beta^{n_i} a_i}{\beta^{n_i}},$$

therefore, we can identify $f_i(x)$ with a block $(\underbrace{000\cdots 0}_{n_i - 1} a_i^{'})$, where $a_i^{'} = \beta^{n_i} a_i$. In fact, $f_i(x)$ and $(\underbrace{000\cdots 0}_{n_i - 1} a_i^{'})$ can be determined mutually. Given $(\underbrace{000\cdots 0}_{n_i - 1} a_i^{'})$ with length n_i and $a_i^{'} = \beta^{n_i} a_i$, we can find a similitude

$$f_i(x) = \frac{x}{\beta^{n_i}} + \frac{0}{\beta} + \frac{0}{\beta^2} + \cdots + \frac{0}{\beta^{n_i - 1}} + \frac{\beta^{n_i} a_i}{\beta^{n_i}} = \frac{x + \beta^{n_i} a_i}{\beta^{n_i}} = \frac{x}{\beta^{n_i}} + a_i.$$

For simplicity we denote this block by $\hat{P}_i = (\underbrace{000\cdots 0}_{n_i - 1} a_i^{'})$ if there is no fear of ambiguity. We identify f_i with $f_{\hat{P}_i}$. The only difference between f_i and $f_{\hat{P}_i}$ is the symbol as both of them represent the map $f_i(x) = f_{\hat{P}_i}(x) = \frac{x}{\beta^{n_i}} + a_i$. Similarly, we may define blocks in terms of the IFS of K_2. Let $D_1 = \{\hat{P}_1, \hat{P}_2, \cdots, \hat{P}_n\}$ and $D_2 = \{\hat{Q}_1, \hat{Q}_2, \cdots, \hat{Q}_m\}$, where $\hat{P}_i = (\underbrace{000\cdots 0}_{n_i - 1} a_i^{'})$, $a_i^{'} = \beta^{n_i} a_i$, $\hat{Q}_j = (\underbrace{000\cdots 0}_{m_j - 1} b_j^{'})$ and $b_j^{'} = \beta^{m_j} b_j$. We say D_1 and D_2 are the digital sets of K_1 and K_2 respectively. The elements of D_i are called the blocks. We emphasize that different blocks may stand for the same similitude, for example let $\hat{R}_1 = (08)$ and $\hat{R}_2 = (22)$ be two blocks with respect to base 3, since their associated similitudes coincide, i.e. $\varphi_{\hat{R}}(x) = \dfrac{x}{3^2} + \dfrac{0}{3} + \dfrac{8}{3^2} = \dfrac{x}{3^2} + \dfrac{2}{3} + \dfrac{2}{3^2}$, we can choose either of them if we want to find the digital sets of $K_i, 1 \le i \le 2$. This replacement does not affect our main result. Usually, we pick the simpler blocks which facilitate our calculation. Once we choose the blocks, we fix them. With this new representation, we have the following simple lemma.

Lemma 8.2.6.

$$K_1 = \{x = \lim_{n\to\infty} f_{\hat{P}_{i_1}} \circ f_{\hat{P}_{i_2}} \circ \cdots \circ f_{\hat{P}_{i_n}}(0) : \hat{P}_{i_j} \in D_1\}.$$

$$K_2 = \{y = \lim_{n\to\infty} g_{\hat{Q}_{i_1}} \circ g_{\hat{Q}_{i_2}} \circ \cdots \circ g_{\hat{Q}_{i_n}}(0) : \hat{Q}_{i_j} \in D_2\}.$$

We call the concatenation $\hat{P}_{i_1} * \hat{P}_{i_2} * \cdots$ $(\hat{Q}_{i_1} * \hat{Q}_{i_2} * \cdots)$ a coding of x (y).

Proof. For any $x \in K_1$, we know that there exists $(i_n)_{n=1}^{\infty}$ such that

$$x = \lim_{n\to\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(0).$$

The lemma is a restatement of this fact. □

Remark 8.2.7. *Although the lemma above is very simple, the significance of this lemma is that we can translate over the problem, i.e. in order to study the sum of two numbers from K_1 and K_2 respectively, it is sufficient to consider the sum of the blocks from D_1 and D_2.*

Motivated by this lemma, we define a crucial definition of this paper.

Definition 8.2.8. *Take s blocks*

$$\hat{P}_{i_1},\ \hat{P}_{i_2},\ \hat{P}_{i_3},\ \cdots,\ \hat{P}_{i_s}$$

from D_1 with lengths $p_1,\ p_2,\ p_3,\ \cdots,\ p_s$, t blocks

$$\hat{Q}_{j_1},\ \hat{Q}_{j_2},\ \hat{Q}_{j_3},\ \cdots,\ \hat{Q}_{j_t}$$

from D_2 with lengths $q_1,\ q_2,\ q_3,\ \cdots,\ q_t$. If there exist integers $k_1,\ k_2,\ k_3, \cdots, k_s$, $l_1,\ l_2,\ l_3, \cdots, l_t$ such that

$$\sum_{i=1}^{s} k_i p_i = \sum_{j=1}^{t} l_i q_i,$$

then the block $(\hat{P}_{i_1}^{k_1} * \hat{P}_{i_2}^{k_2} * \cdots * \hat{P}_{i_s}^{k_s}) + (\hat{Q}_{j_1}^{l_1} * \hat{Q}_{j_2}^{l_2} * \cdots * \hat{Q}_{j_t}^{l_t})$ is called a Matching with respect to β.

Remark 8.2.9. *Let A and B be two concatenations of some blocks from D_1 and D_2 respectively. If A and B have the same length, then the summation of A and B is a Matching, i.e. $A + B$ is a Matching. We call the elements of D_i blocks. However, a Matching, in fact, is also a block which is the sum of concatenated blocks from D_1 and D_2 respectively. In what follows, we still call a Matching a block if there is no fear of ambiguity. Clearly, in this definition the blocks $\hat{P}_{i_j}$ and $\hat{P}_{i_k}$ $(j \neq k)$ could coincide. Given a Matching we may find its associated similitude. For instance, let (abc) be a Matching with respect to β, then the corresponding similitude is $\varphi(x) = \frac{x}{\beta^3} + \frac{a}{\beta} + \frac{b}{\beta^2} + \frac{c}{\beta^3}$.*

We show that D_1 and D_2 generate countably many Matchings.

Lemma 8.2.10. *The cardinality of Matchings which are generated by D_1 and D_2 is at most countable.*

Proof. The proof is constructive. Firstly, we find out all the possible Matchings which have length 1. The cardinality of Matchings with length 1 is finite due to the finite cardinalities of D_1 and D_2. If there are no such Matchings (see Example 8.2.14), we then consider the Matchings with length 2. Similarly, we can find finite Matchings which are of length 2. If there do not exist such Matchings, then we may consider the Matchings with length 3. We continue this procedure and prove the lemma. However, the following adjustment is helpful to reduce some unnecessary Matchings, i.e. if the new born Matchings can be concatenated by the old Matchings, then we do not choose these new Matchings. In the remaining chapter we always abide by this rule. In some cases, after some steps, all the new Matchings can be concatenated by the former old Matchings (see Example 8.2.13), then we stop the procedure. For this case, the cardinality of

Matchings is finite. If the procedure can be continued for infinitely many times, then the cardinality of Matchings is infinitely countable. Hence, the cardinality of Matchings is either finite or countably infinite. □

Remark 8.2.11. *We shall prove that if the cardinality of Matchings is finite, then $K_1 + K_2$ is a self-similar set while $K_1 + K_2$ is the unique attractor of some IIFS if the cardinality of Matchings is infinitely countable.*

Example 8.2.12. *Let $K_1 = K_2$ be the attractor of the IFS $\{g_1(x) = \frac{x}{3},\ g_2(x) = \frac{x+8}{9}\}$. All the possible Matchings are*

$$\{(0),\ (22),\ (44),\ (242),\ (2442),\ (24442),\ (244442),\ (2444442),\ (24444442),\cdots\},$$

where $D_1 = D_2 = \{(0),\ (08) = (22)\}$. Here, for simplicity we assume that $\hat{R} = (08) = (22)$ as their corresponding similitudes are the same, i.e. $\varphi_{\hat{R}}(x) = \dfrac{x}{3^2} + \dfrac{0}{3} + \dfrac{8}{3^2} = \dfrac{x}{3^2} + \dfrac{2}{3} + \dfrac{2}{3^2}$.

Example 8.2.13. *Let $\{f_1(x) = \frac{x}{3},\ f_2(x) = \frac{x+2}{3}\}$ be the IFS of K_1, K_2 is generated by $\{g_1(x) = \frac{x}{3},\ g_2(x) = \frac{x+8}{9}\}$. Then the Matchings generated by D_1 and D_2 are*

$$\{(0),\ (2),\ (24),\ (42),\ (44)\},$$

where $D_1 = \{(0),\ (2)\}$ and $D_2 = \{(0),\ (22)\}$.

Example 8.2.14. *Let $\{f_1(x) = \frac{x}{9},\ f_2(x) = \frac{x}{3^3} + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3}\}$ be the IFS of $K_1 = K_2$, where $D_1 = D_2 = \{(00),\ (222)\}$. For this example, there is no Matching with length 1.*

After we find all the possible Matchings, we denote this set by

$$D = \{\hat{R}_1,\ \hat{R}_2,\ \cdots,\ \hat{R}_{n-1},\ \hat{R}_n,\cdots\},$$

the lengths of these Matchings are increasing. By Remark 8.2.9, D uniquely determines a set of similitudes $\Phi^\infty \triangleq \{\phi_1,\ \phi_2,\ \phi_3,\ \phi_4,\cdots\}$. We define $E \triangleq \bigcup_{\{\phi_n\}\in\Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \cdots \circ \phi_n([0,1])$ and have $E = \bigcup_{i\in\mathbb{N}} \phi_i(E)$, see section 2 from [61].

Now we state the first main result.

Theorem 8.2.15. *$K_1 + K_2$ is either a self-similar set or an attractor of some infinite iterated function system. More precisely, if the cardinality of Matchings is finite, then $K_1 + K_2$ is a self-similar set. When the cardinality is infinitely countable, we have*

$$K_1 + K_2 = \overline{\bigcup_{\phi_i\in\Phi^\infty} \phi_i(K_1 + K_2)}.$$

Remark 8.2.16. *A minor modification enables us to prove the following stronger result: for any $n \in \mathbb{N}^+$ and any $\{K_i\}_{i=1}^n$, $K_1 + K_2 + \cdots + K_n = \{\sum_{i=1}^n x_i : x_i \in K_i\}$ is either a self-similar set or a unique attractor of some IIFS, where $\{K_i\}_{i=1}^n$ are generated by the similitudes of S. In [63], Mendes and Oliveira proved that for the homogeneous Cantor sets, there are five possible structures for the sum. However, in our setting we may find only two structures, i.e. $K_1 + K_2$ is either a self-similar set or an attractor of some IIFS.*

We have an interesting corollary of Theorem 8.2.15.

Corollary 8.2.17. *Let F_1 and F_2 be the self-similar sets with IFS's $\{r_i x + a_i\}_{i=1}^n$ and $\{r_j' x + b_j'\}_{j=1}^m$, if $0 < r_i, r_j' < 1$ for any $1 \le i \le n$ and $1 \le j \le m$, then*

$$\dim_P(F_1 + F_2) = \overline{\dim}_B(F_1 + F_2).$$

8.2.3 Proofs of Theorem 8.2.15 and Corollary 8.2.17

To begin with we assume that the cardinality of all Matchings is infinitely countable. Before we prove the main results, we need some preliminaries. In Lemma 8.2.6 we give the definition of the codings of K_i, $1 \le i \le 2$. Here we define the coding of $x + y \in K_1 + K_2$ in a natural way, i.e. we denote the coding of $x + y$ by $(x_n + y_n)_{n=1}^\infty$, where (x_n) and (y_n) are the codings of x and y respectively.

We know that (x_n) ((y_n)) can be decomposed into infinite blocks from D_1(D_2), see the following figure

X_1	X_2	X_3	$\cdots$
Y_1	Y_2	Y_3	$\cdots$

There are two floors in this figure. By Remark 8.2.5, the concatenation of $X_1 * X_2 * \cdots$ ($Y_1 * Y_2 * \cdots$) is (x_n) ((y_n)), and we can define the summation of the concatenated infinite blocks.

We call the top floor (bottom floor) the x-floor (y-floor). In other words, in the x-floor the concatenation of each block X_i is the coding of x. We shall use this diagram representing the blocks in the proofs of some lemmas. Let $(a_n)_{n=1}^\infty$ be a coding of some point $x + y \in K_1 + K_2$, i.e., $(a_n) = (x_n + y_n)$, where $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are the codings of $x \in K_1$ and $y \in K_2$ respectively. Given $k > 0$, we say $(c_{i_1} c_{i_2} \cdots c_{i_k})$ is a segment of $(a_i)_{i=1}^\infty$ with length k if there exists $j > 0$ such that $c_{i_1} c_{i_2} \cdots c_{i_k} = a_{j+1} \cdots a_{j+k}$. We define

$$C = \Big\{ (a_n) = (x_n + y_n) : \text{there exists } N \in \mathbb{N}^+ \text{ such that any segment of } (a_{N+i})_{i=1}^\infty \text{ is not a Matching} \Big\}.$$

Lemma 8.2.18. *Let $(a_n) \in C$, for any $\epsilon > 0$ we can find a coding $(b_n)_{n=1}^\infty$ which is the concatenation of infinite Matchings such that*

$$|(a_n)_\beta - (b_n)_\beta| < \epsilon$$

Proof. Let $(a_n) \in C$ and $\epsilon > 0$, then there exists $n_0 \in \mathbb{N}$ satisfying $\beta^{-n_0} < \epsilon$. Now, we choose $(b_n)_{n=1}^\infty$ such that its value in base β is a point of E. Let $b_1 b_2 b_3 \cdots b_{n_0} = a_1 a_2 a_3 \cdots a_{n_0}$. If $a_1 a_2 a_3 \cdots a_{n_0}$ is a Matching or a concatenation of some Matchings, then we can choose arbitrary tail $(b_{n_0+i})_{i=1}^\infty$ which is the concatenation of infinite Matchings. Subsequently we have that

$$\begin{aligned} |(a_n)_\beta - (b_n)_\beta| &= |(a_{n_0+1} a_{n_0+2} a_{n_0+3} \cdots)_\beta - (b_{n_0+1} b_{n_0+2} b_{n_0+3} \cdots)_\beta| \\ &\le M \sum_{i=n_0+1}^{\infty} \beta^{-i} < M(\beta - 1)^{-1} \epsilon, \end{aligned}$$

where M is a positive constant which depends on β and the translations of the IFS's of K_1 and K_2. Hence we prove that there exists a point $b \in E$, i.e. $b = (b_n)_\beta$, such that

$$|(a_n)_\beta - (b_n)_\beta| < \epsilon.$$

If $a_1a_2a_3\cdots a_{n_0}$ is not a concatenation of some Matchings, by virtue of the definition of (a_n), $(a_n) = (x_n + y_n)$, where $(x_n), (y_n)$ are the codings of some points in K_1 and K_2, respectively. However, (x_n) ((y_n)) can be decomposed into the concatenation of $X_1 * X_2 * \cdots$ ($Y_1 * Y_2 * \cdots$). We use the following diagram to represent this.

X_1	X_2	X_3	$\cdots$
Y_1	Y_2	Y_3	$\cdots$

From this figure, we know that the summation of $X_1 * X_2 * \cdots$ and $Y_1 * Y_2 * \cdots$ is precisely the coding (a_n). Suppose that there exist p, q such that $a_1a_2a_3\cdots a_{n_0}$ is a prefix of $(X_1 * X_2 * \cdots * X_p) + (Y_1 * Y_2 * \cdots * Y_q)$, here we should emphasize that the lengths of $X_1 * X_2 * \cdots * X_p$ and $Y_1 * Y_2 * \cdots * Y_q$ may not coincide. However, we can still define the summation of their prefixes. Since $X_1 * X_2 * \cdots * X_p$ and $Y_1 * Y_2 * \cdots * Y_q$ do not have the same length, we assume that $\sum_{i=1}^{p} |X_i| < \sum_{i=1}^{q} |Y_i|$, where $|X_i|$ denotes the length of the block X_i, then the first n_0 digits of the "summation" $(X_1 * X_2 * \cdots * X_p) + (Y_1 * Y_2 * \cdots * Y_q)$ are $a_1a_2a_3\cdots a_{n_0}$. Let $k_1 = \sum_{i=1}^{p} |X_i|$ and $k_2 = \sum_{i=1}^{q} |Y_i|$. Then $(X_1 * X_2 * \cdots * X_p)^{k_2} + (Y_1 * Y_2 * \cdots * Y_q)^{k_1}$ is a Matching or a concatenation of some Matchings as $(X_1 * X_2 * \cdots * X_p)^{k_2}$ and $(Y_1 * Y_2 * \cdots * Y_q)^{k_1}$ have the same length. Moreover, the initial n_0 digits of $(X_1 * X_2 * \cdots * X_p)^{k_2} + (Y_1 * Y_2 * \cdots * Y_q)^{k_1}$ are $a_1a_2a_3\cdots a_{n_0}$. Now the remaining proof is the same as the first case. □

Remark 8.2.19. *The main idea of this lemma is that any $(a_n) \in C$ can be approximated by a sequence (c_n) which is the concatenation of infinite Matchings.*

Lemma 8.2.20. $\overline{E} = K_1 + K_2$.

Proof. For every $\epsilon > 0$ and $x + y \in K_1 + K_2$, we can find a coding (a_n) satisfying $x + y = \sum_{n=1}^{\infty} a_n\beta^{-n}$. If there exists a subsequence of integers $n_k \to \infty$ such that $(a_1, a_2, a_3, \cdots, a_{n_k})$ is a concatenation of some Matchings, then by the definition of $E \triangleq \bigcup_{\{\phi_n\}\in\Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \cdots \phi_n([0, 1])$ we have $x + y \in E$. If $(a_n) \in C$, by Lemma 8.2.18 there exists $b \in E$ such that $|b - x - y| < \epsilon$. □

Lemma 8.2.21. $\overline{\bigcup_{i\in\mathbb{N}} \phi_i(K_1 + K_2)} = K_1 + K_2$.

Proof. On the one hand, $E = \bigcup_{i\in\mathbb{N}} \phi_i(E)$, this equality implies that

$$\overline{E} = \overline{\bigcup_{i\in\mathbb{N}} \phi_i(E)} = \overline{\overline{\bigcup_{i\in\mathbb{N}} \phi_i(E)}} \supseteq \overline{\bigcup_{i\in\mathbb{N}} \overline{\phi_i(E)}} = \overline{\bigcup_{i\in\mathbb{N}} \phi_i(K_1 + K_2)},$$

i.e. we have

$$\overline{\bigcup_{i\in\mathbb{N}} \phi_i(K_1 + K_2)} \subseteq K_1 + K_2.$$

On the other hand, $E = \bigcup_{i\in\mathbb{N}} \phi_i(E) \subseteq \bigcup_{i\in\mathbb{N}} \phi_i(K_1 + K_2)$, therefore we prove the converse inclusion in terms of Lemma 8.2.20. □

Proof of Theorem 8.2.15: Using Lemma 8.2.10, we know that there are at most countably many Matchings generated by D_1 and D_2. If the cardinality of Matchings is infinitely countable, then by Lemma 8.2.21, $K_1 + K_2$ is an attractor of Φ^∞. If the cardinality is finite, then $K_1 + K_2$ is a self-similar set. The proof is similar with Lemmas 8.2.20

and 8.2.18. The only difference is that it is not necessary to approximate the coding of $x+y \in K_1+K_2$. In fact, we can directly find a coding which is the concatenation of infinite Matchings such that the value of this infinite coding is $x+y$. In other words, we have $E = K_1 + K_2$. □

Now, we can prove Corollary 8.2.17. When the IFS's of F_1 and F_2 satisfy the irrationality assumption, it is easy to prove Corollary 8.2.17 due to Peres and Shmerkin [72]. In fact, we can prove a stronger result. Let us recall their main result.

Theorem 8.2.22. *Let F_1 and F_2 be the attractors of $\{r_i x + a_i\}_{i=1}^{n}$, $\{r_j^{'} x + b_j\}_{j=1}^{m}$ respectively. If there exist i, j such that $\frac{\log r_i}{\log r_j^{'}} \notin \mathbb{Q}$, then $\dim_H(F_1+F_2) = \min\{\dim_H F_1 + \dim_H F_2,\, 1\}$.*

Proof of Corollary 8.2.17. Firstly, we prove under the irrationality assumption that

$$\dim_H(F_1+F_2) = \dim_P(F_1+F_2) = \dim_B(F_1+F_2) = \min\{\dim_H F_1 + \dim_H F_2,\, 1\}.$$

Using the theorem above, if $\dim_H(F_1+F_2) = 1$, then

$$1 = \dim_H(F_1+F_2) \le \dim_P(F_1+F_2) \le \overline{\dim}_B(F_1+F_2) \le 1.$$

Suppose $\dim_H(F_1+F_2) = \dim_H(F_1) + \dim_H(F_2)$. We note that for any $A,\, B \subseteq \mathbb{R}$, we have $B - A = P_{\frac{\pi}{4}}(A \times B)$, where $P_{\frac{\pi}{4}}(A \times B)$ denotes the projection of $A \times B$ on the y axis along lines having 45° angle with the x axis. Therefore,

$$\begin{aligned} \dim_H(F_1+F_2) &\le \overline{\dim}_B(F_1+F_2) \\ &\le \overline{\dim}_B((-F_2) \times F_1) \\ &\le \overline{\dim}_B(F_1) + \overline{\dim}_B(F_2) \\ &= \dim_H(F_1) + \dim_H(F_2) \end{aligned}$$

The second inequality holds as the projection is a Lipschitz map, the third inequality is due to the property of product of fractal sets, see the product formula 7.5, page 102, [34]. For the last equality, we use the fact that for any self-similar set, its Hausdorff dimension and the Box dimension coincide.

If K_1 and K_2 are generated by the similitudes of S and the cardinality of Matchings is infinitely countable, then we have $\dim_P(K_1+K_2) = \overline{\dim}_B(K_1+K_2) = \dim_P(E) = \overline{\dim}_B(E)$ due to Lemma 8.2.20 and Theorem 3.1 from [61]. By Theorem 8.2.15, we know that K_1+K_2 is a self-similar set if the cardinality of Matchings is finite. Hence, whether the irrationality assumption holds or not we always have $\dim_P(K_1+K_2) = \overline{\dim}_B(K_1+K_2)$.

□

8.3 Dimension of $K_1 + K_2$

8.3.1 IFS case

Let $\sharp D$ be the cardinality of all Matchings generated by D_1 and D_2. In this section we give a necessary and sufficient condition for the finiteness of $\sharp D$. We know that $K_1 + K_2$ is a self-similar set if $\sharp D$ is finite. Hence, in this case we may make use of various techniques finding the Hausdorff dimension of $K_1 + K_2$.

We say that $D_i, 1 \le i \le 2$, is homogeneous if the length of all the blocks is equal. For simplicity we may identify the blocks with the lengths of the blocks. There is one point we should keep in mind, namely different blocks of D_i may have the same length. Hence we should count the multiplicity when some blocks have the same length, see the following example.

Example 8.3.1. *Let $\{f_1(x) = \frac{x}{3},\ f_2(x) = \frac{x+2}{3}\}$ be the IFS of K_1, K_2 is generated by $\{g_1(x) = \frac{x}{3},\ g_2(x) = \frac{x+8}{9}\}$. The digital sets are $D_1 = \{(0),\ (2)\}$ and $D_2 = \{(0),\ (22)\}$. We can denote D_1 by $D_1^{'} = \{1, 1\}$. For simplicity we still use D_1. Similarly, $D_2 = \{1, 2\}$. It is clear that D_1 is homogeneous and that two 1's in the set refer to different similitudes.*

It is easy to find that the digits in D_i stand for the length of the blocks and the similarity ratios, see the following example.

Example 8.3.2. *Let $\{f_1(x) = \frac{x}{\beta^6} + a_1,\ f_2(x) = \frac{x}{\beta^{10}} + a_2\}$ be the IFS of K_1. We know that $D_1 = \{6, 10\}$. 6 represents the length of the block $(00000\,(a_1\beta^6))$ and stands for the similarity ratios $\frac{1}{\beta^6}$.*

For this example, by the definition of K_1 we have $K_1 = f_1(K_1) \cup f_2(K_1)$. Iterating this equation, then we have that

$$K_1 = f_1 \circ f_1(K_1) \cup f_1 \circ f_2(K_1) \cup f_2 \circ f_1(K_1) \cup f_2 \circ f_2(K_1).$$

Hence we obtain 4 similitudes $\{f_1 \circ f_1,\ f_1 \circ f_2,\ f_1 \circ f_2,\ f_2 \circ f_2\}$. Their associated digital set which consists of some blocks can also be denoted by a simpler set $D^{''} = \{12, 16, 16, 20\}$. Similarly, we can iterate the original IFS for any finite times. For the sake of convenience, we still use the set of the lengths of the blocks as it not only stands for the new iterated blocks but also refers to the similarity ratios under new IFS.

Definition 8.3.3. *Let $D_1 = \{k, k, \cdots, k\}$ be a homogeneous set with l digits. We say D_2 is a multiplier set of D_1 if we iterate the IFS of K_2 for finite times, all the numbers of the new digital set $D^{'}$ are the multipliers of k,i.e., $D^{'} = \{l_1k, l_2k, \cdots, l_tk\}$, where $l_i \in \mathbb{N}^+$. Similarly, if D_2 is homogeneous, we can also define D_1 as the multiplier set of D_2 if D_1 satisfies similar property.*

Theorem 8.3.4. *$\sharp D$ is finite if and only if D_1 (D_2) is homogeneous and D_2 is a multiplier set of D_1 (D_1 is a multiplier set of D_2).*

We partition the proof of this theorem into several lemmas.

Lemma 8.3.5. *If D_1 is homogeneous and D_2 is a multiplier set of D_1, then $\sharp D$ is finite.*

Proof. Let $D_1 = \{k, k, \cdots, k\}$ be a homogeneous set and D_2 be a multiplier set of D_1. By the definition of multiplier set, after finite iterations of the IFS of K_2, say t times, $D_2^{'} = \{l_1k, l_2k, \cdots, l_mk\}$, where $l_i \in \mathbb{N}^+$. Now we prove that $\sharp D$ is finite. Let $D_2 = \{s_1, s_2, \cdots s_p\}$, where $s_p \in \mathbb{N}^+$. If we take any t digits from D_2, each time we can pick any numbers, which means we can pick s_i for any $1 \leq k \leq t$ times, then by the definition of multiplier set, $s_{i_1} + s_{i_2} + \cdots + s_{i_t}$ is a multiplier of k. Since $D_1 = \{k, k, \cdots, k\}$ is homogeneous and the cardinality of $D_2^{'} = \{l_1k, l_2k, \cdots, l_mk\}$ is finite, it follows that $\sharp D$ is finite. □

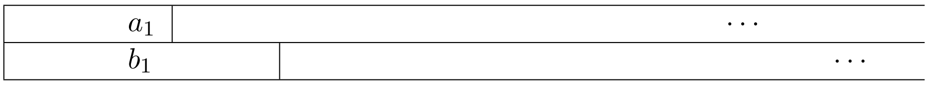
Lemma 8.3.6. *If $\sharp D$ is finite, then either D_1 or D_2 is homogeneous.*

Proof. We have proved that if $\sharp D$ is finite, then $K_1 + K_2$ is a self-similar set. This fact implies that for any coding of $x + y \in K_1 + K_2$, say $(a_n) = (x_n + y_n)$, its associated value in base β is $x + y$, where (x_n) and (y_n) are the codings of x and y respectively. Moreover, (a_n) is the infinite concatenation of some Matchings. In other words, there exists a sequence $N_k \to \infty$ such that $(a_1a_2 \cdots a_{N_k})$ is a concatenation of some Matchings.

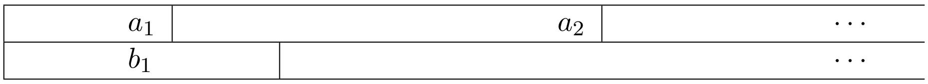
If neither D_1 nor D_2 is homogeneous, we may find a coding of some point in K_1+K_2 which does not contain any Matchings in its arbitrary long prefix. This contradicts with the assumption that $\sharp D$ is finite.

Now we find a coding which satisfies the property we mentioned above. Without loss of generality, we assume that $D_1 = \{a_1, a_2, \cdots, a_p\}$ and $D_2 = \{b_1, b_2, \cdots, b_q\}$, where $a_1 \neq a_2$ and $b_1 \neq b_2$.

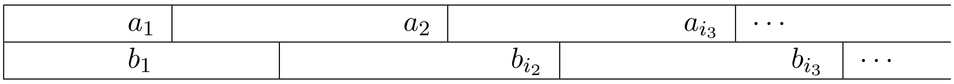
We demonstrate how we can construct the coding we need. Recall the definition of x-floor and y-floor, we know that summation of the concatenation of the blocks of x-floor and y-floor is the coding of some point of $K_1 + K_2$. Since $a_1 \neq a_2$ and $b_1 \neq b_2$, we may suppose $a_1 \neq b_1$ and put them in the x-floor and y-floor respectively, see the following figure



Here we identify the block with its length. Since $a_1 \neq b_1$, it follows that no Matching appears. Next, for the x-floor, we pick a_2 which satisfies that $a_1+a_2 \neq b_1$. If $a_1+a_2 = b_1$, then we pick a_1 again. The Matching cannot appear as $a_1 \neq a_2$ and $a_1 + a_2 = b_1$ imply that $a_1 + a_1 \neq b_1$. Now the x-floor and y-floor become the following:



For the y-floor, we repeat the same procedure. Finally we have



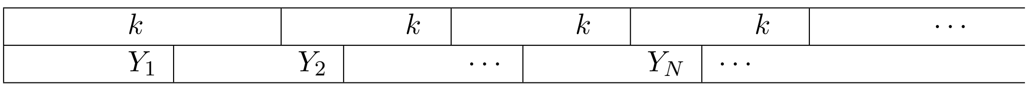
For each step, the Matching does not appear as the length of the concatenations of blocks from x and y-floor is not matched. The summation of the infinite concatenated blocks from x and y-floor is the coding we need. □

Now we may set $D_1 = \{k, k, \cdots, k\}$, if D_2 is not a multiplier set of D_1, we implement similar idea constructing a coding such that its arbitrary long prefix is not a concatenation of some Matchings.

Hence, in order to prove Theorem 8.3.4, it remains to prove the following lemma.

Lemma 8.3.7. *Let $D_1 = \{k, k, \cdots, k\}$, if D_2 is not a multiplier set of D_1, then $\sharp D$ is not finite.*

Proof. If $\sharp D$ is finite, then any coding of $x + y \in K_1 + K_2$, say $(a_n) = (x_n + y_n)$, is the infinite concatenation of some Matchings. Namely there exists a sequence $N_k \to \infty$ such that $(a_1 a_2 \cdots a_{N_k})$ is a concatenation of some Matchings. If we can find a coding (a_n) such that for any n $(a_1 a_2 \cdots a_n)$ is not a concatenation of some Matchings, then we prove this lemma. Since D_2 is not a multiplier set of D_1, it follows that for any finite iterations of the IFS of K_2, there always exists one block which is the concatenation of some blocks from D_2 such that its length is not a multiplier of k. We let this block be $(b_1 b_2 \cdots b_t)$, see the following figure:



We may assume that $(b_1 b_2 \cdots b_t) = Y_1 * Y_2 * \cdots * Y_N$ for some N, where each Y_i is some block from D_2. By the assumption we know that its length is not a multiplier of k. Hence we can find such coding (a_n) (sum of the x and y-floor) satisfying that for any n, $(a_1 a_2 \cdots a_n)$ is not a concatenation of some Matchings. □

Remark 8.3.8. *When $K_1 + K_2$ is a self-similar set, we do not know whether $\sharp D$ is finite or not.*

If $\sharp D$ is finite, then $K_1 + K_2$ is a self-similar set. In this case, we can explicitly find all the similitudes of the IFS. Therefore we can implement many ideas calculating $\dim_H(K_1 + K_2)$. We do not discuss this problem in detail.

8.3.2 IIFS case

Comparing with IFS case, it is much more complicated when $K_1 + K_2$ is a unique attractor of some IIFS. We have mentioned the main reasons in the second section.

By Lemma 8.2.20, we know that when $\sharp D$ is infinitely countable, $\overline{E} = K_1 + K_2$. If $\overline{E} \setminus E$ is uncountable, we may not calculate the dimension of $K_1 + K_2$ in terms of the dimensional theory of IIFS. Hence, we need to find some class that can guarantee $\dim_H(E) = \dim_H(K_1 + K_2)$. In fact, even for calculating $\dim_H(E)$, it is not easy to find $\dim_H(E)$ when the IIFS has some overlaps [67, 44].

Let $(a_n)_{n=1}^{\infty}$ be the coding of some point $x + y \in K_1 + K_2$, i.e., $(a_n) = (x_n + y_n)$, where (x_n) and (y_n) are the codings of x and y respectively. Recall the definition of C,

$$C = \{(a_n) : \text{there exists } N \in \mathbb{N}^+ \text{ such that any segment of } (a_{N+i})_{i=1}^{\infty} \text{ is not a Matching}\}.$$

We have

Lemma 8.3.9. *If C is countable, then we have that $E = K_1 + K_2$ apart from a countable set.*

Proof. By Lemma 8.2.20, $\overline{E} = K_1 + K_2$. It remains to prove that there are only countably many limit points of E which are not in E. For any $x + y \in K_1 + K_2 = \overline{E}$, there is a coding (a_n) such that the value of this coding is $x + y$. If there exists $n_k \to \infty$ satisfying that $(a_1 a_2 \cdots a_{n_k})$ is a Matching or a concatenation of some Matchings, by the definition of $E \triangleq \bigcup_{\{\phi_n\} \in \Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n([0, 1])$, we know that $x + y \in E$. If $(a_n) \in C$, then $\overline{E} \setminus E$ is countable as C and the cardinality of all the Matchings are countable. □

The following lemma gives a sufficient condition which implies that C is countable.

Lemma 8.3.10. *C is countable if there exists k such that $D_1 = \{k, k, \cdots, k, 2k\}$ and $D_2 = \{k, k, \cdots, k, 2k\}$, i.e. both D_1 and D_2 have only blocks with length k apart from the last block with length $2k$.*

Proof. If $D_1 = \{k, k, \cdots, k, 2k\}$ and $D_2 = \{k, k, \cdots, k, 2k\}$, we need to find all possible sequences of C. Without loss of generality, we assume that the prefix of the summation of the x and y-floor does not contain any Matchings. Firstly, we choose two blocks from D_1 and D_2 and stack on the x-floor and y-floor respectively. We can pick only k from D_1 and $2k$ from D_2(or $2k$ from D_1 and k from D_2). Otherwise, a Matching will appear, see the following figure

k	$\cdots$
$2k$	$\cdots$

Then at the second step for the x-floor we cannot take any block of D_1 with length k as $k + k = 2k$ and a new Matching appears. Hence for the x-floor we can pick only the block with length $2k$. Similarly, for the y-floor we cannot take a block of D_2 with length k as $k + 2k = 2k + k$, which can generate a new Matching. Therefore, we must take a block with length $2k$ for the y-floor if we do not want a new Matching to appear. The figure now is

k	$2k$	$\cdots$
$2k$	$2k$	$\cdots$

It is easy to see that if we want to avoid the new Matchings in the summed blocks of two floors we cannot choose blocks freely from the second step on. The figure below illustrates this idea.

k	$2k$	$2k$	$\cdots$
$2k$	$2k$	$2k$	$\cdots$

From the analysis above, we see that the sequences in C are eventually periodic. Thus, we prove that C is countable. □

Remark 8.3.11. *The condition of the lemma is not necessary, for instance, let* $D_1 = \{k, 2k\}$ *and* $D_2 = \{k, 3k\}$*. We can similarly prove that in this case* C *is countable. Generally it is not easy to find all the Matchings. However, for the case in this lemma we can find all possible Matchings without much calculation.*

This lemma enables us to define following IFS.

For any $k \in \mathbb{N}^+$, let the IFS's of K_1 and K_2 be

$$\left\{ f_i(x) = \frac{x}{\beta^k} + a_i, 1 \le i \le n-1, f_n(x) = \frac{x}{\beta^{2k}} + a_n \right\} \tag{8.2}$$

and

$$\left\{ g_j(x) = \frac{x}{\beta^k} + b_j, 1 \le j \le n-1, g_n(x) = \frac{x}{\beta^{2k}} + b_n \right\}, \tag{8.3}$$

where $a_i, b_j \in \mathbb{R}^+ \cup \{0\}$. We denote their attractors by K_1 and K_2 respectively. Without loss of generality, we let the convex hull of K_i be $[0, B_i]$, $0 \le i \le 2$. This assumption yields that $f_i([0, B_1]) \subset [0, B_1]$, $1 \le i \le n$ and $g_j([0, B_2]) \subset [0, B_2]$, $1 \le j \le n$.

Let $D = \{\hat{R}_1, \hat{R}_2, \cdots, \hat{R}_{n-1}, \hat{R}_n \cdots\}$ be all the Matchings generated by $D_1 = \{k, k, \cdots, 2k\}$ and $D_2 = \{k, k, \cdots, 2k\}$ and its associated IIFS be $\Phi^\infty \triangleq \{\phi_1, \phi_2, \phi_3, \phi_4, \cdots\}$. Define $E \triangleq \bigcup_{\{\phi_n\} \in \Phi^\infty} \bigcap_{n=1}^{\infty} \phi_1 \circ \phi_2 \circ \cdots \circ \phi_n([0, B_1 + B_2])$. We know that a Matching $\hat{R}_i$ is a block. Suppose $\hat{R}_i = (c_1 c_2, \cdots, c_p)$ for some $p \in \mathbb{N}$. We call each c_i the digit of $\hat{R}_i$. Since D_1 and D_2 have a finite number of blocks, it follows that the range of every possible digit c_j in each Matching $\hat{R}_i$ is finite, i.e. c_j can take only finite numbers. Let c be the positive constant defined as follows:

$$c = \min\{|c_i - c_j| : c_i \text{ and } c_j \text{ are any digits which are from two Matchings}\}.$$

Similarly, we let A and B be the largest digits of the blocks of D_1 and D_2 respectively.

Theorem 8.3.12. *Let* K_1 *and* K_2 *be two self-similar sets with IFS's* (8.2) *and* (8.3)*, respectively. Then* $E = K_1 + K_2$ *up to a countable set. If*

$$A + B + B_1 + B_2 < c(\beta - 1),$$

then Φ^∞ *satisfies the open set condition and* $\dim_H(K_1 + K_2)$ *is computable.*

Proof of Theorem 8.3.12. By Lemmas 8.3.9 and 8.3.10, we prove the first statement. For the second statement, given any two Matchings $(s_1 s_2 \cdots s_p), (t_1 t_2 \cdots t_q)$ with $p < q$, their associated similitudes are $\phi_{s_1s_2\cdots s_p}(x) = \beta^{-p}x + \sum_{i=1}^{p} s_i\beta^{-i}$ and $\phi_{t_1t_2\cdots t_q}(x) = \beta^{-q}x + \sum_{i=1}^{q} t_i\beta^{-i}$ respectively. Let $V = (0, B_1 + B_2)$, simple calculation implies that

$$\phi_{s_1s_2\cdots s_p}(V) = \left(\sum_{i=1}^{p} s_i\beta^{-i}, \sum_{i=1}^{p} s_i\beta^{-i} + (B_1 + B_2)\beta^{-p}\right)$$

$$\phi_{t_1t_2\cdots t_q}(V) = \left(\sum_{i=1}^{q} t_i\beta^{-i}, \sum_{i=1}^{q} t_i\beta^{-i} + (B_1 + B_2)\beta^{-q}\right).$$

We assume that $(s_1s_2\cdots s_p) < (t_1t_2\cdots t_q)$, i.e, there exists $1 \leq i_0 \leq p$ such that $s_k = t_k$ for any $1 \leq k \leq i_0 - 1$ and $s_{i_0} < t_{i_0}$. By the definition of c, we can check that the two intervals above do not overlap, namely $\phi_{s_1s_2\cdots s_p}(V) \cap \phi_{t_1t_2\cdots t_q}(V) = \emptyset$. It remains to prove that $\phi(V) \subset V$ for any $\phi \in \Phi^\infty$. Let ϕ be generated by the Matching $\hat{R}_1 * \hat{R}_2 + \hat{T}_1 * \hat{T}_2$, the associated similitudes of $\hat{R}_i$ and $\hat{T}_i$ are $H_i(x)$ and $I_i(x)$ respectively. Let the length of $\hat{R}_1 * \hat{R}_2 + \hat{T}_1 * \hat{T}_2$ be k_0. It is easy to find that

$$\phi(x) = H_1 \circ H_2(x) + I_1 \circ I_2(0)$$

Hence,

$$\phi(V) = \left(H_1 \circ H_2(0) + I_1 \circ I_2(0), H_1 \circ H_2(0) + I_1 \circ I_2(0) + \frac{B_1 + B_2}{\beta^{k_0}}\right).$$

Recall the assumption of K_1 and K_2, the convex hull of K_i is $[0, B_i]$, $1 \leq i \leq 2$, i.e., $H_s([0, B_1]) \subset [0, B_1], 1 \leq s \leq 2$ and $I_t([0, B_2]) \subset [0, B_2], 1 \leq t \leq 2$. Therefore $0 < \phi(x) < B_1 + B_2$. Similarly, we can prove that $\phi(V) \subset V$ for any $\phi \in \Phi^\infty$. As such Φ^∞ satisfies the open set condition. The calculation of $\dim_H(K_1 + K_2)$ now is a straightforward application of Theorem 8.2.3. □

Generally we do not know how to calculate $\dim_P(K_1 + K_2)$ or when do we have following equality

$$\dim_H(K_1 + K_2) = \dim_P(K_1 + K_2) = \dim_B(K_1 + K_2).$$

We finish this section by making some remarks on these two problems. Let F_n be the attractor of the first n similitudes of Φ^∞, i.e., F_n is the attractor of the IFS $\{\phi_i\}_{i=1}^{n}$. Clearly

$$F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots.$$

Recall the definition of Hausdorff metric [34]. Given two compact sets $J_1, J_2 \subset \mathbb{R}$, then the Hausdorff metric of J_1 and J_2 is defined by

$$\mathcal{H}(J_1, J_2) = \inf\{s : J_1 \subset (J_2)_s, J_2 \subset (J_1)_s\},$$

where $(A)_s = \{x : \text{there exists } y \in A \text{ such that } |x - y| \leq s\}$.

We have

Lemma 8.3.13. $\overline{\cup_{n=1}^{\infty} F_n} = K_1 + K_2$.

Proof. $0 \leq \mathcal{H}(\overline{\cup_{n=1}^{\infty} F_n}, K_1 + K_2) \leq \mathcal{H}(F_n, K_1 + K_2) \to 0$ as $n \to \infty$. Here $\mathcal{H}(F_n, K_1 + K_2) \to 0$ can be found in [35]. □

Proposition 8.3.14. *If* $(\overline{\cup_{n=1}^{\infty} F_n}) \setminus (\cup_{n=1}^{\infty} F_n)$ *is a countable set, then*

$$\dim_H(K_1+K_2) = \dim_P(K_1+K_2) = \dim_B(K_1+K_2).$$

Proof. Since $(\overline{\cup_{n=1}^{\infty} F_n}) \setminus (\cup_{n=1}^{\infty} F_n)$ is countable, it follows by Lemma 8.3.13 that

$$\begin{aligned}\dim_P(K_1+K_2) &= \dim_P(\cup_{n=1}^{\infty} F_n) = \lim_{n\to\infty} \dim_P(F_n) \\ &= \dim_H(\cup_{n=1}^{\infty} F_n) = \dim_H(\overline{\cup_{n=1}^{\infty} F_n}) = \dim_H(K_1+K_2).\end{aligned}$$

We finish the proof by Corollary 8.2.17. □

8.4 Examples

In this section, we give some examples for which Theorem 8.2.22 cannot calculate $\dim_H(K_1+K_2)$.

Example 8.4.1. *Let* $K_1 = K_2$ *be the self-similar sets with IFS* $\{g_1(x) = \frac{x}{3},\, g_2(x) = \frac{x+8}{3^2}\}$*, then* $\dim_H(K_1+K_2) = \frac{\ln t_0}{-\ln 3}$*, where* t_0 *is the smallest positive root of* $t^3 - t^2 - 2t + 1 = 0$*.*

We know that $D_1 = D_2 = \{(0),\ (22)\}$, all the Matchings which are generated by D_1 and D_2 are

$$D = \{(0), (22), (44),\ (242),\ (2442),\ (24442),\ (244442)\cdots\}.$$

The corresponding IIFS of D is

$$\Phi^{\infty} = \{\varphi_1 = f_0,\, \varphi_2 = f_2 \circ f_2,\, \varphi_3 = f_4 \circ f_4,\, \varphi_4 = f_2 \circ f_4 \circ f_2,\, \cdots\},$$

where $f_0(x) = \frac{x}{3},\ f_2(x) = \frac{x+2}{3},\ f_4(x) = \frac{x+4}{3}$.

By Theorem 8.3.12, $\dim_H(K_1+K_2) = \dim_H(E)$. Obviously this IIFS satisfies the OSC, i.e.

$$\varphi_i((0,2)) \cap \varphi_j((0,2)) = \varnothing$$

for any $i \neq j$ and $\varphi_i((0,2)) \subseteq (0,2)$ for any $i \in \mathbb{N}$. Now we can use Theorem 8.2.3 to calculate the dimension. It is easy to check that

$$\dim_H(K_1+K_2) < \min\{1, \dim_H(K_1) + \dim_H(K_2)\}.$$

This example illustrates that without the irrationality assumption, the expected dimension of K_1+K_2 may not be achieved. This differs from Peres and Shmerkin's result [72].

Example 8.4.2. *Let* $\{f_1(x) = \frac{x}{\beta},\, f_2(x) = \frac{x+2}{\beta}\}$ *and* $\{g_1(x) = \frac{x}{\beta},\, g_2(x) = \frac{x}{\beta^2} + \frac{2}{\beta} + \frac{2}{\beta^2}\}$ *be the IFS's of* K_1 *and* K_2 *respectively. Then* K_1+K_2 *is a self-similar set, the IFS is* $\{\varphi_1(x) = \frac{x}{\beta},\, \varphi_2(x) = \frac{x+2}{\beta},\, \varphi_3(x) = \frac{x}{\beta^2} + \frac{2}{\beta} + \frac{4}{\beta^2},\, \varphi_4(x) = \frac{x}{\beta} + \frac{4}{\beta} + \frac{2}{\beta^2},\, \varphi_5(x) = \frac{x}{\beta^2} + \frac{4}{\beta} + \frac{4}{\beta^2}\}$*. This IFS does not satisfy the OSC generally, in fact it is of finite type if* β *is a Pisot number, see [68, Theorem 2.5]. Hence, we can calculate the Hausdorff dimension of* K_1+K_2 *in terms of the main result of [68]. We omit the details.*

8.5 Final remarks

The main result of this chapter is that $K_1 + K_2$ is either a self-similar set or a unique attractor of some IIFS. However, to calculate the dimension of $K_1 + K_2$ is difficult, especially the IIFS case. As in this case, we should consider the limit points of E as well as the separation condition. Ignoring either of them may hinder the calculation of the dimension of $K_1 + K_2$. In fact, even finding all the Matchings is not a trivial task. On the other hand, we may implement the Vitali process if the IIFS has overlaps, see [65, Theorem 3.1], this process is complicated. Ngai and Tong [67] gave a dimensional formula of J_0 under the so-called weak separation condition, but it is still not easy to check this condition generally. Some techniques of [45] are useful to analyze the Hausdorff dimension of self-similar sets. This chapter is the extended work of my master thesis, I corrected some mistakes and improved some results.

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Summary

In Chapter 1, we introduce the main results of this thesis.

In Chapter 2, we consider the homogeneous iterated function system generating codings representing simultaneously two different points in two different bases. We show that for β's sufficiently close to 1, the attractor contains a neighbourhood of the origin.

In Chapter 3, given any IFS, we investigate the points of the attractor with unique codings. Such set is called the univoque set. When the attractor is an interval, we prove under mild condition that the univoque set can be identified with a subshift of finite type. This result enables us to find the Hausdorff dimension of the univoque set. Similar idea can be implemented in higher dimensions. Our main result generalizes some results of de Vries and Komornik.

In Chapter 4, when the attractor is not an interval, we also give another approach such that the attractor can be identified with a subshift of finite type. With this identification, we can find the dimension of univoque set as well. Moreover, we give some applications of our results.

In Chapter 5, we investigate β-expansions in base β over digit set $\{0, 1, \beta\}$. We prove that for any $\beta > 1$ and any finite integer k, the Hausdorff dimension of the set of points with exactly k different expansions coincides with the Hausdorff dimension of the univoque set. Various related results are proved.

In Chapter 6, we study the multiple codings for self-similar sets with overlaps. In particular, for a self-similar set $E \in \mathcal{E}$ and $k = 1, 2, \cdots, \aleph_0$ we investigate the Hausdorff dimension of the subset $U_k(E)$ which contains all points $x \in E$ having exactly k different codings. This generalizes many results obtained in Chapters 4 and 5.

In Chapter 7, we define a new type of random β-transformation, i.e. the shrinking random β-transformation. This transformation is motivated by the analog-to-digit conversion. We study when does the induced transformation have a unique measure of maximal entropy.

In Chapter 8, we study the arithmetic sum of self-similar sets, and prove that the sum of two self-similar sets is either a self-similar set or a unique attractor of some infinite iterated function systems.

Samenvatting

In hoofdstuk 1 worden de hoofdresultaten van dit proefschrift besproken.

In hoofdstuk 2 bestuderen we het homegene *geïtereerde functie systeem* (IFS) die tegelijkertijd de ontwikkeling (codering) geeft van twee verschillende punten in twee verschillende bases. We tonen aan dat de attractor een omgeving van de oorsprong bevat voor twee β's voldoende dicht bij 1.

In hoofdstuk 3 onderzoeken we voor een willekeurige IFS de punten van de attractor met een unieke ontwikkeling (codering). De verzameling van dit soort punten is de *univoque set*. Als de attractor een interval is, dan bewijzen we onder milde condities dat de univoque set geidentificeert kan worden met een subshift of finite type. Dankzij dit resultaat kunnen we de Hausdorff dimensie van de univoque set bepalen. Een vergelijkbare aanpak kan ook in hogere dimensies toegepast worden. Ons hoofdresultaat is een generalisatie van resultaten van De Vries en Komornik.

In hoofdstuk 4 laten we met behulp van een nadere methode zien, dat als de attractor niet een interval is, zij toch geïdentificeerd kan worden met een subshift of finite type. Opnieuw geeft ons dit de mogelijkheid om de dimensie van de univoque set te bepalen. Verder geven worden er een aantal toepassingen van dit resultaat gegeven.

In hoofdstuk 5 onderzoeken we β-expansies in basis β met $\{0, 1, \beta\}$ als mogelijke digits. We bewijzen dat voor elke $\beta > 1$ en elk natuurlijk getal k er geldt dat de Hausdorf dimensie van de verzameling van de verzameling van punten met precies k verschillende ontwikkelingen hetzelfde is als de Hausdorf dimensie van de univoque set. Verschillende hiermee gerelateerde resultaten worden vervolgens bewezen.

In hoofdstuk 6 studeren we meervoudige ontwikkelingen (codings) voor zelfgelijkvormige verzamelingen met overlap. In het bijzonder onderzoeken we voor een zelfgelijkvormige verzameling $E \in \mathcal{E}$ en $k = 1, 2, \ldots, \aleph_0$ de Hausdorff dimensie van de deelverzameling $U_k(E)$ van E die alle punten $x \in E$ bevat met precies k verschillende ontwikkelingen. Dit generaliseerd resultaten uit hoofstukken 4 en 5.

In hoofdstuk 7 geven we een nieuwe definitie van *random* β-transformaties; de *krimpende* random β-transformatie. De motivatie om deze random β-transformatie te bestuderen komt van resultaten in analog-to-digital conversion. We bestuderen onder welke omstandigheden de geinduceerde transformatie een unieke maat van maximale entropie heeft.

In hoofdstuk 8 wordt de arithmetische som van zelfgelijkvormige verzamelingen bestudeerd, en bewijzen we dat de som van twee zelfgelijkvormige verzamelingen òf een zelfgelijkvormige verzameling is, òf de unieke attractor is van een oneindige IFS.

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Curriculum Vitae

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