Lie Pseudogroups à la Cartan from a Modern Perspective

Thesis committee:
Prof. dr. Erik P. van den Ban, Utrecht University
Prof. dr. Christian Blohmann, Max Planck Institute for Mathematics
Prof. dr. Rui Loja Fernandes, University of Illinois at Urbana-Champaign
Prof. dr. Niky Kamran, McGill University
Prof. dr. Shlomo Sternberg, Harvard University

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Lie Pseudogroups à la Cartan from a Modern Perspective

Lie pseudogroepen à la Cartan vanuit een modern perspectief

(met een samenvatting in het Nederlands)

Proefschrift

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Ori Yudilevich

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Promotor: Prof. dr. M. N. Crainic

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Introduction

In the years 1904-05, Élie Cartan published two pioneering papers ([5, 6], and presented more concisely in [7, 8]) in which he introduced a structure theory for Lie pseudogroups. In this thesis, we present a modern formulation of Cartan's theory. This work is the result of our endeavor to read Cartan's writings from a modern perspective and understand his ideas and constructions in a more conceptual, global and coordinate-free fashion. In contrast to a great deal of the literature in this field, we have placed an emphasis here on remaining as close and faithful as possible to Cartan's original ideas.

In this introductory chapter, we present an overview of the thesis and state some of the main definitions and theorems (some in simplified form) with the aim of helping the reader navigate his way through. At the end of this chapter, we give a summary of the main results and an outline of the thesis.

Lie Pseudogroups: from Lie to Cartan In a three volume monograph [43][1] (1888-93), Sophus Lie, in collaboration with Friedrich Engel, laid the foundations of the theory of "continuous transformation groups". Lie's notion of a "transformation group" evolved into the modern notion of a *pseudogroup* (see Definition 3.1.1). A pseudogroup on a manifold is, roughly speaking, a set of local transformations of the manifold that is of a group-like and sheaf-like nature. Intuitively, one can think of a pseudogroup as a set of local symmetries. One of Lie's important insights, that served as the starting point of his theory, was that pseudogroups of local symmetries of geometric structures and differential equations have the property that they are characterized as the set of solutions of a *system of partial differential equations* (or a *PDE* in short). Lie referred to such pseudogroups as "continuous transformation groups", a notion that evolved into the modern notion of a *Lie pseudogroup* (see Definition 3.3.1).

The work of Lie, and a large part of the literature that followed (including Cartan), was restricted to the local study of Lie pseudogroups, i.e. Lie pseudogroups on open subsets of Euclidean spaces. In this case, given a Lie pseudogroup Γ on $\mathbb{R}^n$ or an open subset thereof, one typically introduces coordinates $(x, y, ...)$ on the copy of $\mathbb{R}^n$ on which the elements of Γ are applied, coordinates $(X, Y, ...)$ on the copy of $\mathbb{R}^n$ in which the elements of Γ take value, and, with respect to these coordinates, every element of Γ is represented by its component functions $X = X(x, y, ...),\ Y = Y(x, y, ...), ...$ Let us look at three examples cited from [7]:

Example 1. The diffeomorphisms of $\mathbb{R}$ of the form

$$X = ax + b,$$

parametrized by a pair of real numbers $a, b \in \mathbb{R}$, generate a pseudogroup (generate here means that one should impose the sheaf-like axioms of a pseudogroup, see Remark 3.1.3).

[1] the first volume was recently translated into English [52]

This pseudogroup is characterized as the set of local solutions of the ordinary differential equation

$$\frac{\partial^2 X}{\partial x^2} = 0.$$

Example 2. The diffeomorphisms of $\mathbb{R}^2 \backslash \{y = 0\}$ of the form

$$X = x + ay, \qquad Y = y,$$

parametrized by a real number $a \in \mathbb{R}$, generate a pseudogroup. This pseudogroup is characterized as the set of local solutions of the system of partial differential equations

$$\frac{\partial X}{\partial x} = 1, \quad \frac{\partial X}{\partial y} = \frac{X - x}{y}, \quad Y = y.$$

Example 3. The locally defined diffeomorphisms of $\mathbb{R}^2 \backslash \{y = 0\}$ of the form

$$X = f(x), \qquad Y = \frac{y}{f'(x)},$$

parametrized by a function $f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R})$ (locally defined diffeomorphisms of $\mathbb{R}$), generate a pseudogroup. This pseudogroup is characterized as the set of local solutions of the system of partial differential equations

$$\frac{\partial X}{\partial x} = \frac{y}{Y}, \quad \frac{\partial X}{\partial y} = 0, \quad \frac{\partial Y}{\partial y} = \frac{Y}{y}.$$

The above examples illustrate the general phenomenon that Lie observed – Lie pseudogroups are parametrized by a finite number of “continuous” parameters (hence the name “continuous transformation groups”), which he divided into two types: the “finite” parameters, i.e. real[2] variables; and the “infinite” parameters, i.e. real-valued functions. He divided the Lie pseudogroups accordingly into two classes: the “finite” Lie pseudogroups, those parameterized only by finite parameters; and the “infinite” Lie pseudogroups, those parametrized by at least one infinite parameter. Examples 1 and 2 are finite, whereas Example 3 is infinite.

In his celebrated three volume monograph [43], Lie developed an infinitesimal approach to the study of Lie pseudogroups in which he studied these objects by means of their infinitesimal transformations, i.e. their generating vector fields. Lie’s work is mainly concentrated on the case of finite Lie pseudogroups (although, it is worth mentioning that he also published two relatively short papers [41, 42] in 1891 that proposed a generalization of his theory to the infinite case). His work on the finite “case”, as is well known, formed the basis of what evolved into the modern notions of a Lie group and, at the infinitesimal level, a Lie algebra (see [2] for a historical account).

[2]Lie also considered complex parameters. Here we restrict ourselves to the real case.

In 1904-05, Élie Cartan, unsatisfied with the state of the theory of Lie pseudogroups in the infinite case, presented a new structure theory for Lie pseudogroups. Cartan writes [7]: *"The generalization of the structure theory of the finite pseudogroups of Lie to the infinite pseudogroups on the basis of infinitesimal transformations proved to be very difficult, if not to say impossible, despite the hard work of S. Lie, F. Engel, Medolaghi, etc. The theory that will be presented here starts from a very different principle; it is in the defining equations of the pseudogroup placed in a convenient form that we will find a starting point for the theory".*

Lie's Infinitesimal Approach Lie's idea was to associate an infinitesimal structure with each Lie pseudogroup and, consequently, to study Lie pseudogroups by means of their infinitesimal data. Cartan, in turn, generalized this idea, but approached it from a different angle. Before explaining Cartan's approach, let us briefly recall Lie's infinitesimal approach in the finite case. For simplicity and clarity, we recall Lie's construction in its modern form, i.e. in the abstract language of Lie groups and Lie algebras.

Given any Lie group G, we may consider its set of right invariant vector fields,

$$\mathfrak{g} := \mathfrak{X}_{\text{inv}}(G) \subset \mathfrak{X}(G).$$

We make the following three fundamental observations: 1) the set $\mathfrak{g}$ is a finite dimensional vector space. In other words, there exists a basis, i.e. a finite set of right invariant vector fields

$$X^1, ..., X^n \in \mathfrak{g},$$

spanning $\mathfrak{g}$; 2) $\mathfrak{g}$ is closed under the Lie bracket of vector fields and thus inherits a bracket, i.e. there exists a set of constants c_i^{jk}, the **structure constants** of $\mathfrak{g}$, such that

$$[X^j, X^k] = c_i^{jk} X^i; \tag{1}$$

3) the bracket of $\mathfrak{g}$ satisfies the Jacobi identity, a property inherited from the Lie bracket of vector fields. In terms of the structure constants, this translates into the equation

$$c_i^{mj} c_m^{kl} + c_i^{mk} c_m^{lj} + c_i^{ml} c_m^{jk} = 0. \tag{2}$$

To summarize, $\mathfrak{g}$ is a finite dimensional Lie subalgebra of the infinite dimensional Lie algebra of vector fields on G, and in this way one associates a finite dimensional Lie algebra $\mathfrak{g}$ with any Lie group G.

The importance of this construction is made clear in the following theorem, known as *Lie's third fundamental theorem*: any Lie algebra is isomorphic to the Lie algebra associated with some Lie group. Furthermore, restricting to the class of simply connected Lie groups, then this association defines a 1-1 correspondence between Lie groups and (finite dimensional) Lie algebras. Thus, at least in the simply connected case, a Lie group is fully encoded by its associated Lie algebra.

Cartan's Three Fundamental Theorems The immediate generalization of Lie's approach to infinite Lie pseudogroups, as Cartan pointed out, proved to be difficult. To circumvent this problem, Cartan chose a different strategy, which, among other things, involved passing from vector fields to the dual picture of differential forms. Following Lie's footsteps, Cartan exhibited that one can associate an infinitesimal structure, a Lie algebra-like object, with any Lie pseudogroup, and, in turn, that such an object (under certain conditions) comes from a Lie pseudogroup.

Cartan laid out this theory in what he called *the three fundamental theorems*. We give here preliminary versions of these theorems, which we phrased as a self contained story. The full statements will be given in the first section of each of Chapters 4 - 6. We only mention here that the notion of *equivalence* of pseudogroups, which appears in the first theorem, is Cartan's notion for when two pseudogroups should be called "the same". It is a rather flexible notion that allows for pseudogroups that act on spaces of different dimensions to be called "the same". See Definition 4.1.1 and Section 4.1 for more details.

Theorem. *(the first fundamental theorem) Any Lie pseudogroup* Γ *is equivalent to a pseudogroup* Γ' *on* $\mathbb{R}^N$*, for some* $N > 0$*, of the type*

$$\Gamma' = \{ \ \phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^N) \mid \phi^* I_a = I_a, \ \phi^*\omega_i = \omega_i \ \},$$

for some system

$$I_a = I_1, ..., I_n, \qquad\qquad \omega_i = \omega_1, ..., \omega_r \tag{3}$$

of linearly independent functions I_a *and linearly independent 1-forms* ω_i *on* $\mathbb{R}^N$.

Theorem. *(the second fundamental theorem) The system* (3) *in the first fundamental theorem can be chosen so that it satisfies the equations*

$$d\omega_i + \frac{1}{2} c_i^{jk} \omega_j \wedge \omega_k = a_i^{\lambda j} \pi_\lambda \wedge \omega_j, \tag{4}$$

for some set $\pi_\lambda = \pi_1, ..., \pi_p$ *of linearly independent 1-forms on* $\mathbb{R}^N$ *that completes the set* $\omega_1, ..., \omega_r$ *to a coframe, such that the coefficients*

$$c_i^{jk}, \ a_i^{\lambda j}, \tag{5}$$

a priori functions on $\mathbb{R}^N$*, only depend on* I_a *(and are, therefore, functions on an open subset of* $\mathbb{R}^n$*). Equations* (4) *are called the* ***structure equations*** *and the coefficients* (5) *the* ***structure functions***.

Theorem. *(the third fundamental theorem - necessary conditions) If a set of functions* c_i^{jk} *and* $a_i^{\lambda j}$ *on* $\mathbb{R}^n$ *arises as the set of structure functions of a set of structure equations* *of the type* (4)*, then they must satisfy the equations*

$$a_i^{\eta m} a_m^{\mu j} - a_i^{\mu m} a_m^{\eta j} = a_i^{\lambda j} \epsilon_\lambda^{\eta\mu}, \tag{C1}$$

$$c_i^{mj} c_m^{kl} + c_i^{mk} c_m^{lj} + c_i^{ml} c_m^{jk} + \left(\frac{\partial c_i^{kl}}{\partial x_j} + \frac{\partial c_i^{lj}}{\partial x_k} + \frac{\partial c_i^{jk}}{\partial x_l} \right) = a_i^{\lambda l} \nu_\lambda^{jk} + a_i^{\lambda k} \nu_\lambda^{lj} + a_i^{\lambda j} \nu_\lambda^{kl} \tag{C2}$$

$$a_m^{\lambda j} c_i^{mk} - a_m^{\lambda k} c_i^{mj} + a_i^{\lambda m} c_m^{jk} + \left(\frac{\partial a_i^{\lambda k}}{\partial x_j} - \frac{\partial a_i^{\lambda j}}{\partial x_k} \right) = a_i^{\mu k} \xi_\mu^{\lambda j} - a_i^{\mu j} \xi_\mu^{\lambda k}, \tag{C3}$$

for some set of functions $\epsilon_\lambda^{\eta\mu}, \nu_\lambda^{kl}, \xi_\mu^{\lambda j}$ on $\mathbb{R}^n$.

Theorem. *(the third fundamental theorem) If a set of analytic functions c_i^{jk} and $a_i^{\lambda j}$ on $\mathbb{R}^n$ satisfies (C1)-(C3) and if $a_i^{\lambda j}$ spans an involutive tableau, then, locally, it arises as the set of structure functions of structure equations of the type* (4).

To summarize: in the first and second fundamental theorems, by passing to an equivalent pseudogroup which is in a certain "normal form", Cartan associates a set of structure functions c_i^{jk} and $a_i^{\lambda j}$ with any Lie pseudogroup. These play the role of the infinitesimal structure, the analogue of a Lie algebra. Then, Cartan identifies the set of conditions (C1)-(C3) that these functions must satisfy, the analogue of the Jacobi identity, and proceeds to prove the analogue of Lie's third fundamental theorem.

The Third Fundamental Theorem and the Cartan-Kähler Theorem Let us add a few words of explanation on the third fundamental theorem, and, in particular, explain the underlined words that appear in the statement. In order to prove the third fundamental theorem, Cartan developed a new analytic tool in [5], which he called the theory of *Pfaffian Systems*, a theory that evolved into the modern day theory of Exterior Differential Systems and into the well known Cartan-Kähler theorem (e.g., see [3, 4]). The Cartan-Kähler theorem is a local existence theorem for systems of partial differential equations and, more generally, for exterior differential systems, and it is only valid in the analytic setting (since its proof relies on the Cauchy-Kowalewski theorem, see [3, 66]). The fact that Cartan's third fundamental theorem is restricted to the analytic setting and provides only local solutions is precisely due to these limitations of the Cartan-Kähler theorem. A smooth and global version of the theorem, as we have for Lie's third fundamental theorem, is yet to be proven (note that, in searching for such a proof, one cannot expect a straightforward generalization of the Cauchy-Kowalewski and the Cartan-Kähler theorems to the smooth category due to counter examples such as Hans Lewy's well known example of a linear partial differential equation with smooth coefficients that does not admit any solution, see [38]).

The involutivity condition that appears in the statement of the third fundamental theorem is a linear algebraic condition that appears naturally in the theory of PDEs and in the theory of Exterior Differential Systems as a sufficient (but not necessary) condition for the existence of local solutions in the analytic setting. Nowadays, this condition is understood as a cohomological condition in terms of the so called *Spencer cohomology* (see appendix in [25] and see also Section 1.5). The Spencer cohomology, as we will see, will play an important role in our modern formulation of Cartan's theory, one that goes beyond its role in the third fundamental theorem.

A Modern Formulation of the Fundamental Theorems Many parallels can be drawn between Cartan's approach (i.e. the three fundamental theorems) and Lie's approach as described above. For instance, the necessary conditions (C1)-(C3) are analogous to the Jacobi identity (2). In fact, the former reduces to the latter when $\mathbb{R}^n$ is a point (i.e. $n = 0$) and the $a_i^{\lambda j}$'s vanish. Thus, a Lie algebra is a particular case of the infinitesimal structure that appears in Cartan's third fundamental theorem. This observation suggests that, just as Lie's structure constants c_i^{jk} from (1) can be encoded in the notion of a Lie algebra, one could attempt to encode the data that appears in Cartan's fundamental theorems in globally defined objects that isolate the essential underlying geometric and algebraic structure. This is an important step towards understanding Cartan's work in a more conceptual, global and coordinate-free fashion, a program that we have undertaken and that will be presented in this thesis.

Let us introduce here the structures that will become the central players in the thesis. We begin with the infinitesimal data, the functions c_i^{jk} and $a_i^{\lambda j}$ on $\mathbb{R}^n$ that appear in the second fundamental theorem. These are encoded in the following structure (Definition 5.2.7):

Definition. *A* ***pre-Cartan algebroid*** *over a manifold N is a pair $(\mathcal{C}, \mathfrak{g})$ consisting of a transitive pre-Lie algebroid $(\mathcal{C}, [\cdot,\cdot], \rho)$ over N and a vector sub-bundle $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ such that $T(\mathcal{C}) \subset \mathrm{Ker}\ \rho$ for all $T \in \mathfrak{g}$.*

Here, a pre-Lie algebroid $(\mathcal{C}, [\cdot,\cdot], \rho)$ is like a Lie algebroid (where $[\cdot,\cdot]$ is the bracket on $\Gamma(\mathcal{C})$ and $\rho : \mathcal{C} \to TN$ is the anchor), but it is not required to satisfy the Jacobi identity (Definition 5.2.1). The functions c_i^{jk} are encoded in the bracket of the pre-Lie algebroid $\mathcal{C}$ and the functions $a_i^{\lambda j}$ are encoded in the vector bundle $\mathfrak{g}$, or rather in the inclusion $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$. Such structures, vector subbundles of a Hom-bundle, are called **tableau bundles** (see Definition 1.2.3) and they play an important role in the theory of PDEs (see Chapter 1), in the theory of Exterior Differential Systems and in Cartan's structure theory. We only mention here, for the purpose of this overview, that with any given tableau bundle $\mathfrak{g}$, one associates a sequence of tableau bundles $\mathfrak{g}, \mathfrak{g}^{(1)}, \mathfrak{g}^{(2)}, \ldots$ called the **prolongations** of $\mathfrak{g}$ and a cohomology theory $H^{l,m}(\mathfrak{g})$ called the **Spencer cohomology** of $\mathfrak{g}$ (see Section 1.5).

Next, a system as in (3) of functions I_a and 1-forms ω_i that are required to satisfy the structure equations (4) is encoded in the following structure (Definition 5.2.11):

Definition. *A* ***realization*** *of a pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ is a pair (P, Ω) consisting of a surjective submersion $I : P \to N$ and a pointwise surjective anchored 1-form $\Omega \in \Omega^1(P; I^*\mathcal{C})$ such that there exists a 1-form $\Pi \in \Omega^1(P; I^*\mathfrak{g})$ satisfying*

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega,$$

with the property that

$$(\Omega, \Pi) : TP \xrightarrow{\simeq} I^*(\mathcal{C} \oplus \mathfrak{g})$$

is vector bundle isomorphism.

The map $I : P \to N$ encodes the functions I_a, the vector bundle-valued 1-form Ω encodes the 1-forms ω_i, and the pre-Cartan algebroid structure of $(\mathcal{C}, \mathfrak{g})$ is precisely the structure that allows us to talk about structure equations in a global coordinate-free manner.

Finally, the last player in our game is the notion of a Cartan algebroid (Definition 6.2.1). A Cartan algebroid is the structure that encodes the necessary conditions (C1)-(C3) that a set of functions c_i^{jk} and $a_i^{\lambda j}$ must satisfy if they are to arise as the set of structure functions of a set of structure equations. By comparing the axioms in the following definition with conditions (C1)-(C3) above, one should already note many parallels.

Definition. *A **Cartan algebroid** is a pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N such that:*

1. $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ *is closed under the commutator bracket.*
2. *There exists a vector bundle map $t : \Lambda^2\mathcal{C} \to \mathfrak{g},\ (\alpha, \beta) \mapsto t_{\alpha,\beta}$, such that*
$$[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = t_{\alpha,\beta}(\gamma) + t_{\beta,\gamma}(\alpha) + t_{\gamma,\alpha}(\beta)$$
for all $\alpha, \beta, \gamma \in \Gamma(\mathcal{C})$.
3. *There exists a $\mathcal{C}$-connection ∇ on $\mathfrak{g}$ such that*
$$T([\alpha, \beta]) - [T(\alpha), \beta] - [\alpha, T(\beta)] = \nabla_\beta(T)(\alpha) - \nabla_\alpha(T)(\beta),$$
for all $\alpha, \beta \in \Gamma(\mathcal{C}),\ T \in \Gamma(\mathfrak{g})$.

The notions of a Pre-Cartan algebroid, a Cartan algebroid and a realization will be the central notions in this thesis. In terms of these notions, Cartan's three fundamental theorems take the following form (the following statements correspond to Theorems 4.3.1, 5.3.1, 6.3.1 and 6.4.2 in the thesis):

Theorem. *(the first fundamental theorem) Any Lie pseudogroup Γ is Cartan equivalent to a pseudogroup on a manifold P of the type*

$$\Gamma(P, \Omega) = \{\ \phi \in \mathrm{Diff}_{\mathrm{loc}}(P) \mid \phi^* I = I,\ \phi^*\Omega = \Omega\ \},$$

for some pair (P, Ω) consisting of a surjective submersion $I : P \to N$ into another manifold N and a pointwise surjective 1-form $\Omega \in \Omega^1(P; I^*\mathcal{C})$ with values in some vector bundle $\mathcal{C} \to N$.

Theorem. *(the second fundamental theorem) The pair (P, Ω) in the first fundamental theorem can be chosen so that it has the structure of a realization of a pre-Cartan algebroid.*

Theorem. *(the third fundamental theorem - necessary conditions) If a pre-Cartan algebroid admits a realization, then it is a Cartan algebroid.*

Theorem. *(the third fundamental theorem) If* $(\mathcal{C}, \mathfrak{g})$ *is an analytic Cartan algebroid* $(\mathcal{C}, \mathfrak{g})$ *such that the tableau bundle* $\mathfrak{g}$ *is involutive, then, locally, it admits a realization.*

Let us make two short remarks: 1) the notion of Cartan equivalence between pseudogroups, essentially a modern translation of Cartan's notion of equivalence that we saw earlier, will be defined and discussed in Section 4.2; 2) the three terms that are underlined in the statement of the last theorem were discussed earlier and correspond to the three terms that were underlined in Cartan's formulation of the theorem.

A New Approach to the Third Fundamental Theorem In Section 6.6, we will present an alternative way for encoding the structure of a Cartan algebroid, the global structure corresponding to the structure functions c_i^{jk} and $a_i^{\lambda j}$. This new structure, called a Cartan pair (Definition 6.6.3), slightly deviates from Cartan's point of view, but has the advantage of being more intuitive and closer to the better understood notion of a Lie algebroid.

Definition. *A* ***Cartan pair*** *over a manifold* N *is a pair* $(A, \mathfrak{g})$ *consisting of a transitive pre-Lie algebroid* $(A, [\cdot,\cdot], \rho)$ *over* N *and a vector subbundle* $\mathfrak{g} \subset A$*, such that*

$$[[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] \equiv 0 \mod \mathfrak{g},$$

for all $\alpha, \beta, \gamma \in \Gamma(A)$ *and* $\mathfrak{g} \subset \mathrm{Ker}\ \rho$*.*

Up to the correct notion of equivalence, Cartan pairs are in 1-1 correspondence with Cartan algebroids. Intuitively, one should think of a Cartan pair as a transitive Lie algebroid that satisfies the Jacobi identity modulo the subbundle $\mathfrak{g}$. With this alternative description, the notion of a realization takes the following form (Definition 6.6.10):

Definition. *A* ***realization*** *of a Cartan pair* $(A, \mathfrak{g})$ *over* N *is a pair* (P, Ω) *consisting of a surjective submersion* $I : P \to N$ *and an anchored 1-form* $\Omega \in \Omega^1(P; I^*A)$*, such that*

$$d\Omega + \frac{1}{2}[\Omega,\Omega] \equiv 0 \mod \mathfrak{g}$$

and Ω *is pointwise an isomorphism.*

Intuitively, a realization of a Cartan pair is a vector bundle-valued 1-form that satisfies the Maurer-Cartan equation modulo the subbundle $\mathfrak{g}$. A realization of a Cartan pair induces a realization of a Cartan algebroid, and vice versa. Thus, the question raised by the third fundamental theorem of whether a Cartan algebroid admits a realization, a question known as the **realization problem**, translates into the question of whether a Cartan pair admits a realization. The advantage of this formulation is that the problem becomes closer in spirit to the problem of integrating Lie algebras to Lie groups, or, more generally, to the problem of integrating Lie algebroids to Lie groupoids.

In Chapter 7, we exploit this alternative point of view and present a new method for tackling the realization problem. The method we present will allow us to solve the problem in the special case in which $\mathfrak{g} = 0$, i.e. where the Cartan pair is in fact a transitive Lie algebroid, and exhibit the precise role of the Jacobi identity in the process. The main theorem of the chapter, Theorem 7.4.6, clarifies the precise relation between the Jacobi identity expression and the Maurer-Cartan expression, indicating how one may attempt to proceed in solving the more general realization problem in which the Jacobi identity fails in a controlled way.

The Systatic Space and Reduction While there are many parallels between the theory of Lie groups and Cartan's theory of Lie pseudogroups, there is a fundamental difference between the two notions. While Lie groups are abstract geometric objects, Lie pseudogroups depend on the space on which they act, and, up to equivalence in the sense of Cartan, a Lie pseudogroup can appear in many incarnations. For example, the Lie pseudogroup generated by the action of a Lie group G on a manifold M is equivalent to the Lie pseudogroup generated by G acting diagonally on $M \times M$ by the same action. Thus, two pseudogroups can be equivalent even if they act on spaces of different dimensions. Intuitively, these two Lie pseudogroups have the same underlying "abstract object" and are hence "the same". One of the problems that Cartan deals with in his theory is that of finding the "smallest" Lie pseudogroup in a given equivalence class of Lie pseudogroups. If such a representative exists (one would, of course, have to make sense of "smallest"), then it would be a possible candidate for the "abstract object" underlying a Lie pseudogroup.

In the context of the problem of seeking "smaller representatives", Cartan notes that any given set of structure equations (4) on $\mathbb{R}^N$ encodes a system of equations,

$$a^i_{\lambda j}\omega^j = 0, \tag{6}$$

which he calls the *systatic system*, and shows that the set of solutions of this system defines an integrable distribution on $\mathbb{R}^N$ and, hence, a foliation. Cartan then proceeds to argue that the systatic system can be used to reduce a given Lie pseudogroup to an equivalent one which acts on a possibly smaller space, thus obtaining a "smaller" representative.

This reduction procedure that Cartan describes has remained quite mysterious throughout the years. In Chapter 8, by using our modern framework of Cartan algebroids and realizations, we hope to shed some light on this subject. We begin by clarifying the precise structure that underlies Cartan's systatic system and then proceed to show that this structure reveals a beautiful and intrinsic phenomenon in the form of a reduction procedure. The reduction procedure that we describe is canonical and natural, in the sense that it is guided by the geometric structure that is encoded in a Cartan algebroid and in its realizations. It is highly inspired by but goes a step beyond Cartan's reduction, a procedure which seems to be of a rather local nature and dependent on choices (see Section 8.1). We are, thus, breaking our self imposed rule of remaining faithful to Cartan. We do, however, believe that this phenomenon is "what Cartan would have discovered" had he the necessary tools at hand.

Let us give an outline of this story and refer the reader to Chapter 8 for the detailed exposition. To begin with, any Cartan algebroid carries the following structure (Definition 8.2.1):

Definition. *The* ***systatic space*** *of a Cartan algebroid* $(\mathcal{C}, \mathfrak{g})$ *over* N *is the set-theoretic vector subbundle* $\mathcal{S} \subset \mathcal{C}$ *defined by*

$$\mathcal{S} := \{ \ u \in \mathcal{C} \mid T(u) = 0 \ \ \forall \, T \in \mathfrak{g} \ \}.$$

In general, the systatic space may fail to be smooth. However, if it is, then one has the following implications of the axioms of a Cartan algebroid (Propositions 8.2.3 and 8.2.7):

Proposition. *Let* $(\mathcal{C}, \mathfrak{g})$ *be a Cartan algebroid. If the systatic space* $\mathcal{S} \subset \mathcal{C}$ *is of constant rank, then* $\mathcal{S}$*, as well as its extension*

$$\mathcal{S}^{(1)} := \mathcal{S} \oplus \mathfrak{g}$$

by $\mathfrak{g}$ *called the* ***1st systatic space****, are Lie algebroids.*

The most important property of the systatic space is the following (Proposition 8.3.1):

Proposition. *Let* (P, Ω) *be a realization of a Cartan algebroid* $(\mathcal{C}, \mathfrak{g})$*. There is a canonical Lie algebroid action of the 1st systatic space* $\mathcal{S}^{(1)}$ *on* $I : P \to N$*.*

Thus, every Cartan algebroid encodes a Lie algebroid, called the 1st systatic space. The 1st systatic space, in turn, acts on all realization of the Cartan algebroid. As we will show, the image of this action, a foliation in P, is precisely the foliation defined by (6). This clarifies the precise structure of Cartan's systatic system; namely, it comes from a Lie algebroid action. This fact already suggests that some type of reduction may be obtained by passing to the orbit space of the action. As it turns out, however, the correct reduction is more subtle and requires the introduction of two new ingredients into the theory.

The first ingredient is the notion of a **Pfaffian groupoid**. This notion was first introduced in [62] and will be reviewed in Section 2.7. Roughly speaking, a Pfaffian groupoid is a pair $(\mathcal{G}, \theta)$ consisting of a Lie groupoid $\mathcal{G}$ and a representation-valued multiplicative 1-form θ on $\mathcal{G}$. In addition to their role in the reduction procedure, Pfaffian groupoids will play a role in other parts of the theory.

The second ingredient is the notion of a **generalized pseudogroup** (Definition 3.6.1), which, as its name suggest, generalizes the notion of a pseudogroup. A generalized pseudogroup, roughly speaking, is a pseudogroup-like set of local bisections of a Lie groupoid. Generalized pseudogroups are natural objects to consider in the study of Pfaffian groupoids since every Pfaffian groupoid $(\mathcal{G}, \theta)$ induces a generalized pseudogroup of local bisections of $\mathcal{G}$, namely the set of "local solutions of θ". The notion of equivalence of pseudogroups extends naturally to the realm of generalized pseudogroups, allowing one to compare pseudogroups with generalized pseudogroups.

Pfaffian groupoids and generalized pseudogroups enter our story through the following proposition (Propositions 8.4.2 and 8.5.2):

Proposition. *Let (P, Ω) be a realization of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$. This data induces a Pfaffian groupoid $(\mathcal{G}, \theta)$. Moreover,*

1. *$\Gamma(P, \Omega)$ is equivalent to the generalized pseudogroup induced by $(\mathcal{G}, \theta)$,*
2. *the action of $\mathcal{S}^{(1)}$ on $I : P \to N$ extends to an action on $(\mathcal{G}, \theta)$.*

Thus, the proposition shows that a realization (and its associated pseudogroup) is encoded in a Pfaffian groupoid (and its induced generalized pseudogroup). It is this passage to the Pfaffian point of view that allows us to perform a canonical reduction by the action of the 1st systatic space (Theorem 8.6.1):

Theorem. *Let (P, Ω) be a realization of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N. If the Lie algebroid $\mathcal{S}^{(1)}$ integrates to a Lie groupoid Σ, and the action of $\mathcal{S}^{(1)}$ on $(\mathcal{G}, \theta)$ integrates to a "nice" action of Σ, then:*

1. *the Pfaffian groupoid $(\mathcal{G}(I), \theta)$ descends to a Lie-Pfaffian groupoid $(\mathcal{G}(I)_{red}, \theta_{red})$ by dividing out the action of Σ,*
2. *the pseudogroup $\Gamma(P, \Omega)$ associated with the realization (P, Ω) is Cartan equivalent to Γ_{red}, the generalized pseudogroup of the Lie-Pfaffian groupoid*

$$(\mathcal{G}(I)_{red}, \theta_{red}) \rightrightarrows P_{red}.$$

Thus, we have obtained a canonical reduction of a realization (and its associated pseudogroup) by the systatic space, but at the cost of passing to the more general picture of generalized pseudogroups. In Section 8.7, we will see two examples which exhibit the phenomenon. In these examples, the generalized pseudogroups that one obtains by reduction are indeed the underlying "abstract objects" that one expects.

Prolongations of Cartan Algebroids and Realizations Let us briefly mention one last topic that will be treated in this thesis, namely the notion of prolongation of structure equations. Cartan introduces the notion of prolongation as a tool for obtaining new Lie pseudogroups that are equivalent to a given one, and applies it to tackle various classification problems. In Chapter 9, we will describe Cartan's prolongation using our modern framework. We will define the notion of a **prolongation of a realization** and a **prolongation of a Cartan algebroid**. The former corresponds to Cartan's prolongation of structure equations, while the latter is its infinitesimal counterpart. Let us remark here that Cartan algebroids are generalizations of the *truncated Lie algebras* that were introduced by Singer and Sternberg [64], their prolongations are generalizations of the prolongations of truncated Lie algebras, and the following theorem (Theorem 9.3.1) is a generalization of a result in that same paper:

Theorem. *Let $(\mathcal{C}, \mathfrak{g})$ be a Cartan algebroid. If the Spencer cohomology group $H^{0,2}(\mathfrak{g})$ vanishes, then the* ***1st prolongation*** *of $(\mathcal{C}, \mathfrak{g})$, the pair $(\mathcal{C}^{(1)}, \mathfrak{g}^{(1)})$ with*

$$\mathcal{C}^{(1)} := \mathcal{C} \oplus \mathfrak{g},$$

is again a Cartan algebroid.

We see that the Spencer cohomology of the tableau bundle $\mathfrak{g}$ that was mentioned above appears as a sufficient condition for prolongation. This is a well known phenomenon that occurs in the theory of prolongations and formal integrability of geometric PDEs and in the theory of Exterior Differential Systems.

In Chapter 9, we will also prove an analogous prolongation theorem for realizations (Theorem 9.2.4).

The Modern Literature on Cartan's Theory of Lie Pseudogroups In the more than a century that has passed since Cartan's pioneering papers appeared, a great amount of work has been done and has brought us a long way towards a deeper understanding of Cartan's ideas, a task which has proven to be a very difficult one. Singer and Sternberg, in their work titled "The Infinite Groups of Lie and Cartan Part I (the Transitive Groups)", write ([64], pp. 57): *"We must confess that we find most of these papers extremely rough going and we certainly cannot follow all the arguments in detail. The best procedure is to guess at the theorems, then prove them, then go back to Cartan"*.

Cartan's work has drawn much attention throughout the years. A remarkable fact about Cartan's work on Lie pseudogroups, which explains its popularity and allure, is the large number of basic geometric concepts and analytic tools that were introduced by Cartan along the way, e.g. the theory of exterior differential systems, the theory of G-structures (a unifying theory for a large class of geometric structures), the "equivalence problem", and even the use of differential forms as an elementary tool in differential geometry. Of course, it took the work of many others for Cartan's ideas to take their modern shape (we only mention here Ehresmann, Chevalley, Weyl, Chern, Libermann, Spencer, Kodaira, Singer, Sternberg, Guillemin, Kuranishi, Malgrange, etc.). One thing, however, is clear: these basic concepts and tools that Cartan introduced in his study of Lie pseudogroups turned out to be much more influential than the original problem itself.

The first big step in modernizing Cartan's work is attributed to Charles Ehresmann. Some of Ehresmann's important contributions are: the modern definition of a pseudogroup, the theory of jet spaces, and the introduction of the notion of a Lie groupoid into the theory (see [40] for a historical account). Ehresmann's work marked the beginning of the "modern era" of Lie pseudogroups, a renewed interest in the subject that had its peak in the 1950's-1970's, but which continues until this very day. A full account of the literature that appeared on this subject over the years deserves a book of its own. Let us mention a small selection of the literature, with an emphasis on papers that have been influential in our work.

The notion of structure equations and the infinitesimal structure encoded in their structure functions has been studied from various perspectives in [39, 51, 37, 61, 34, 64, 24, 48, 49, 30, 45, 68, 57]. In particular, proofs of Cartan's first and second fundamental theorem can be found in [39, 37, 24, 30, 68] and a proof of the Cartan's third fundamental theorem can be found in [34]. The theory of Exterior Differential Systems has become a field of its own and several books have appeared on the subject, e.g. [35, 66, 3, 55, 29, 50, 4]. In particular, the notion of involutivity, which plays a role in Cartan's third fundamental theorem and in other parts of the theory, is discussed. The cohomological notion of involutivity in terms of the Spencer cohomology, which evolved from the work of Spencer [65], is studied in [25, 64, 17, 59, 19, 20, 69, 48, 49, 30, 50], and the proof of the equivalence between the two notions, a proof given by Serre, appears in [25, 64].

Throughout the years, people have also taken several new approaches in the study of Lie pseudogroups, many of which were directly inspired by Cartan's work. A few of these are: Kuranishi's formal theory of Lie (F)-groups and Lie (F)-algebras [36, 37]; the infinitesimal point of view of sheaves of Lie algebras of vector fields [60, 64] (see also Section 2.3); the study of Lie pseudogroups via the defining equations of their infinitesimal transformations, the so called Lie equations [48, 49]; an approach using Milnor's infinite dimensional Lie groups [31]; and the point of view of infinite jet bundles [56] (in [70], the approach in [56] is compared with Cartan's theory and the role of the systatic system is discussed).

While enormous progress has been made, it is also clear that there is still a long way ahead – one must only compare the state of this theory with the state of the theory in the "finite case", which has developed into the mature and thriving theory of Lie groups. Aside from the finite case, the case that was best understood of Cartan's work was the *transitive* case, i.e. pseudogroups with a single orbit such as Examples 1 and 3 above (in fact, this "case" gave rise to the modern theory of G-structures that was mentioned above). It is not that Cartan did not consider the general case, but rather that Cartan's writings were best understood in the transitive case. Our poor understanding of the general case goes in different directions:

1. Even the meaning of "intransitive" is rather restrictive. Instead of asking that the pseudogroup Γ have a single orbit, the orbits of Γ are required to be the fibers of a submersion (in particular, they must all have the same dimension). So even the existing literature that is devoted to the "intransitive case" is really about the "regular case".
2. Even in the regular case, much of the existing literature restricts to the local picture.
3. Much of the literature restricts to the analytic setting due to the fact that the main analytic tool, the Cartan-Kähler theorem, is only valid in this setting.

In our work, rather than trying new approaches, as is often done in this field (with wonderful results, we must add), we have taken Cartan's papers and tried to remain as faithful as possible to his original ideas. This thesis, roughly speaking, is a line by line translation of Cartan's work to the modern language of differential geometry, with some exceptions that will be pointed out along the way (such as the story of reduction and the systatic space that was discussed above).

A Summary of the Main Contributions of the Thesis Here is a summary of the main results and contributions of the thesis:

1. A modern and coordinate-free formulation of Cartan's structure theory of Lie pseudogroups (Chapters 4 - 6).

2. Global coordinate-free proofs of the first and second fundamental theorems (Theorems 4.3.1 and 5.3.1).
3. A new understanding of Cartan's systatic system and reduction that goes beyond Cartan (Chapter 8), with a novel reduction theorem, Theorem 8.6.1.
4. A global description of Cartan's notion of prolongation in terms of realizations (Section 9.2), including a prolongation theorem, Theorem 9.2.4.
5. A prolongation theorem for Cartan algebroids (Theorem 9.3.1), a generalization of Singer and Sternberg's theorem for truncated Lie algebras [64].
6. A new approach to Cartan's realization problem with applications to integration of Lie algebroids and symplectic realizations in Poisson geometry (Chapter 7).
7. A novel exposition of the background material: the geometric approach to PDEs, jet groupoids and algebroids, and Lie pseudogroups (Chapters 1 - 3); as well as new and simplified proofs of known results, e.g. of Goldschmidt's formal integrability criterion for non-linear PDEs from [20] (Section 1.6).

An Outline of the Chapters The first three chapters are introductory: Chapter 1 is an exposition of the geometric approach to PDEs; in Chapter 2, jet groupoids and jet algebroids are discussed, as well as the abstract notions of Pfaffian groupoids and algebroids; in Chapter 3, Lie pseudogroups are defined and their Pfaffian groupoid structure is explained.

The rest of the thesis, with the exception of Chapter 7, is dedicated to a modern exposition of Cartan's structure theory of Lie pseudogroups. Each chapter deals with a different aspect of the theory and begins with a section discussing Cartan's formulation, after which the modern approach is presented. Chapters 4, 5, and 6 treat the first, second, and third fundamental theorems, respectively; in Chapter 8, we study Cartan's notion of the systatic system, after which our novel reduction procedure is presented; Chapter 9 treats the notion of prolongation of structure equations, and, in the global picture, prolongations of Cartan algebroids and realizations.

In Chapter 7, an intermezzo in the story, our new approach to Cartan's realization problem is presented.

Notations Throughout the thesis we work in the smooth setting unless otherwise specified. We use the Einstein summation convention without further say.

Chapter 1

Partial Differential Equations

A Lie pseudogroup is, intuitively, a pseudogroup that is defined as the set of solutions of a *system of partial differential equations*, or *PDE* in short. In this first introductory chapter, we discuss the notion of a PDE from a geometric point of view, or what is often referred to as a *geometric PDE*. We will consider the two main types of PDEs:

1. Linear PDEs – PDEs imposed on local sections of a vector bundle $E \to M$ whose sets of solutions are required to be closed under linear operations.

2. Non-linear PDEs (or simply PDEs) – PDEs imposed on local sections of a surjective submersion $P \to M$.

We will spend the majority of the chapter discussing non-linear PDEs, which are more relevant to the main subject of this thesis. In the last two sections of the chapter, we will also briefly touch upon linear PDEs, which are, in a sense, a particular (and "easier") class of non-linear PDEs. In our presentation, we will place an emphasis on the fact that the structure of a PDE is encoded in an object called the *Cartan form* and the structure of a linear PDE is encoded in an object called the *Spencer operator*. As we will see later in the thesis, this approach goes hand in hand with Cartan's approach to the study of Lie pseudogroups. To motivate the various notions that will arise throughout the chapter, we will dedicate a small part of the chapter to the notion of *formal integrability* of PDEs. In this part, our presentation is inspired by the work of Quillen [59] and the subsequent work of Goldschmidt [19, 20], in which formal integrability theorems have been proven by means of the inductive procedure of *prolongation*, which we will explain in detail. By applying the same strategy as the one used by Quillen and Goldschmidt, but using a different language, we will give an alternative and simpler proof of Goldschmidt's existence theorem for non-linear PDEs [20] (which can be easily adapted to prove Quillen's existence theorem for linear PDEs [59]).

Our approach to the theory of PDEs is also highly influenced by the recent thesis of Salazar [62]. In her thesis, Salazar introduces abstract objects called *Relative Connections* and *Pfaffian bundles* with which she studies the notion of prolongation and the problem of formal integrability of PDEs from an abstract point of view. The two types of PDEs that we listed above are (the main) examples of these two abstract objects. In Section 1.9, we will briefly discuss this abstract point of view.

In addition to the works that have been cited so far, the literature on jets and PDEs from a geometric point of view is very rich. We only mention a few of the existing books on the subject for further reference (in chronological order): Pommaret [58], Gromov [23], Krasil'shchik et al. [33], Saunders [63], Bryant et al. [3], Olver [55], Bocharov et al. [1] and Stormarck [68].

1.1 Affine Bundles

We begin by recalling the notion of affine bundles. Intuitively, one can think of affine spaces as spaces equipped with a "subtraction operation" and affine bundles as bundles of affine spaces. Affine bundles play an important role in the geometry of PDEs for the following simple reason: given a k-th order Taylor polynomial at $y = (y_1, ..., y_m) \in \mathbb{R}^m$ of some smooth function $\sigma : \mathbb{R}^m \to \mathbb{R}^p,\ x \mapsto \sigma^\rho(x_1, ..., x_m)$,

$$\sum_{|\alpha| \le k} \frac{1}{\alpha!} \frac{\partial^{|\alpha|}\sigma^\rho}{\partial x^\alpha}(y_i)\ (x-y)^\alpha, \tag{1.1}$$

where α is a multi-index (see (1.4) for notation), then the space of all $k+1$-th order Taylor polynomials at $y = (y_1, ..., y_m)$ of smooth functions from $\mathbb{R}^m$ to $\mathbb{R}^p$ whose k-th order Taylor polynomial is (1.1) is an affine space modeled on the vector space of homogeneous polynomials of degree $k+1$.

Definition 1.1.1. *An* ***affine space*** *modeled on a vector space V is a set A together with a surjective map $A \times A \to V,\ (b, a) \mapsto b - a$, satisfying:*

1. $a - a = 0 \quad \forall a \in A$,
2. $(c - b) + (b - a) = c - a \quad \forall\, a, b, c \in A$,
3. *for all $a, b, c \in A$, if $c - a = b - a$ then $b = c$.*

We denote the affine space by (A, V).

Equivalently, an affine space on V is a set A equipped with a free and transitive action $A \times V \to A,\ (a, v) \mapsto a + v$, of the abelian group $(V, +)$ on A. The subtraction operation $A \times A \to V$ induces an action $A \times V \to A$ by setting $a + v \in A$ to be the unique element satisfying $(a + v) - a = v$, and vice versa. We will switch between the two points of view without further say.

The simplest example of an affine space is a vector space V. It is an affine space modeled on itself when equipped with its own subtraction map. Viewing a vector space as an affine space amounts to "forgetting" the zero element.

An important feature of affine spaces is that one can take affine combinations of elements: given $a_1, ..., a_n \in A$ and $\lambda_1, ..., \lambda_n \in [0, 1]$ such that $\lambda_1 + ... + \lambda_n = 1$,

$$\begin{aligned} \lambda_1 a_1 + ... + \lambda_n a_n &= (1 - \lambda_2 - ... - \lambda_n)a_1 + \lambda_2 a_2 + ... \lambda_n a_n \\ &= a_1 + \lambda_2(a_2 - a_1) + ... + \lambda_n(a_n - a_1). \end{aligned}$$

The last expression should be regarded as the definition of the first.

Affine maps are analogous to linear maps. An **affine map** from (A, V) to (B, W) is a pair (f, g) consisting of a map $f : A \to B$ and a linear map $g : V \to W$ such that $g(b - a) = f(b) - f(a)$ for all $a, b \in A$. An affine map is determined by g and by how f acts on a single element of A. If g is a linear isomorphism (and hence f is a bijection), then we say that (f, g) is an **affine isomorphism**. If g is injective (and hence f is injective), then we say that (A, V) is an **affine subspace** of (B, W). In this case, we identify (A, V) with its image in (B, W). Note that while the intersection of two vector subspaces V, V' of a given vector space W is non-empty (it contains at least the zero element) and is again a vector space, given two affine subspaces $(A, V), (A', V')$ of an affine space (B, W), the intersection $A \cap A'$ may be empty. If it is non-empty, then $A \cap A'$ is again an affine space modeled on $V \cap V'$.

Affine bundles are to affine spaces as vector bundles are to vector spaces.

Definition 1.1.2. *An **affine bundle** over a manifold M modeled on a vector bundle $E \to M$ is a surjective submersion $A \to M$ from a manifold A onto M such that each fiber A_x is an affine space modeled on the corresponding fiber E_x ($x \in M$) and such that: for every point x there exists an open neighborhood $U \subset M$ and a fiber-preserving diffeomorphism*

$$\varphi_U : A|_U \xrightarrow{\simeq} E|_U,$$

for which $(\varphi_U|_{A_y}, \mathrm{id}) : (A_y, E_y) \xrightarrow{\simeq} (E_y, E_y)$ is an affine isomorphism for all $y \in U$. We denote the affine bundle by (A, E).

Example 1.1.3. Any vector bundle is an affine bundle modeled on itself. $\diamondsuit$

Proposition 1.1.4. *Affine bundles have a global section.*

Proof. Local sections exist due to the local triviality property and these can be glued to a global section via a partition of unity because affine spaces are closed under affine combinations. $\square$

We can view a global section as a choice of a "zero section" of the affine bundle. In fact, a choice of a global section induces an isomorphism (of affine bundles) between the affine bundle and its underlying vector bundle.

Remark 1.1.5. All the definitions that have been given so far are also valid in the analytic category. However, while local sections of affine bundles always exist in the analytic category (by the very definition of an affine bundle), global sections may fail to exist. $\diamondsuit$

An **affine bundle map** from (A, E) to (B, F), both affine bundles over M, is a pair (f, g) consisting of a fiber-preserving smooth map $f : A \to B$ and a vector bundle map $g : E \to F$, both covering the identity map of M, such that $(f|_{A_x}, g|_{E_x})$ is an affine map for every $x \in M$. If g is a vector bundle isomorphism (and hence f is a diffeomorphism), then we say that (f, g) is an **isomorphism**. If g is injective (and hence f is injective), then we say that (A, V) is an **affine subbundle** of (B, W) and we identity (A, V) with its image in (B, W). As for vector bundles, one can also define the notion of an affine bundle map between affine bundles over different bases.

The intersection of two vector subbundles of a given vector bundle is again a vector bundle if and only if the intersection is of constant rank. The analogous statement for affine bundles is more subtle:

Proposition 1.1.6. *Let $(A, V), (A', V')$ be affine subbundles of an affine bundle (B, W) over M. Then $(A \cap A', V \cap V')$ is an affine bundle if and only if $V \cap V'$ is of constant rank and all the fibers of $A \cap A'$ are non empty.*

Proof. The forward direction follows directly from the definition of an affine bundle. In the reverse direction, one can reduce the problem to the analogous problem for vector bundles by the following "de-projectivization" trick[1]. We may assume that (B, W) is the vector bundle W. We include W as an affine subbundle of the vector bundle $W \oplus (M \times \mathbb{R})$ by the map $w \mapsto (w, 1)$, where $M \times \mathbb{R}$ is the trivial line bundle. This inclusion restricts to inclusions of A and A' as affine subbundles of $W \oplus (M \times \mathbb{R})$. Inside $W \oplus (M \times \mathbb{R})$ we consider the vector bundles $\tilde{A}$ and $\tilde{A}'$ spanned by A and A' by "de-projectivization". More precisely, each element of A spans a line in its fiber in $W \oplus (M \times \mathbb{R})$ and we take the union of these lines, and similarly for A'. The assumption that $V \cap V'$ is of constant rank and that the fibers of $A \cap A'$ are non empty implies that $\tilde{A} \cap \tilde{A}'$ is of constant rank and hence a vector bundle. Finally, one notes that W, as an affine bundle, is recovered from $W \oplus (M \times \mathbb{R})$ by projectivization of $W \oplus (M \times \mathbb{R}) \backslash W \oplus (M \times \{0\})$ (i.e., taking the space of lines through the origin in each fiber), and that $A \cap A'$ is recovered by restricting this projectivization map to $\tilde{A} \cap \tilde{A}'$ and is hence an affine bundle (to be more rigorous, one can explicitly construct local sections of $A \cap A'$ out of nowhere-zero local sections of $\tilde{A} \cap \tilde{A}'$). □

For more on affine bundles, see for example [20] (Chapter 3).

1.2 Jet Bundles

The language of jets allows us to formalize the notion of higher-order partial derivatives of smooth maps between manifolds and, consequently, to formalize the notion of a PDE in the differential geometric setting. In this section, we recall the notion of a jet bundle and discuss its geometric structure.

Let $\pi : P \to M$ be a surjective submersion with M an m-dimensional manifold and P an $(m+p)$-dimensional manifold (thus p is the dimension of the fibers of P). For every integer $k \geq 0$, one has the **k-th jet bundle** J^kP of P, which is constructed as follows. First, for every $x \in M$, one considers the set $\Gamma_x(P)$ of germs of local sections of P at x, where a germ at x of a local section σ is denoted by $\mathrm{germ}_x(\sigma) \in \Gamma_x(P)$. Next, one introduces an equivalence relation $\sim_x^k$ on $\Gamma_x(P)$, where $\mathrm{germ}_x(\sigma) \sim_x^k \mathrm{germ}_x(\sigma')$ if $\sigma(x) = \sigma'(x)$ and if the k-th Taylor polynomials of σ and σ' at x coincide (choose some coordinate charts around x and $\sigma(x)$ to express the k-th Taylor polynomial and then check that the definition is independent of the choice). Finally, one defines

$$(J^kP)_x := \Gamma_x(P)/\sim_x^k \qquad \text{and} \qquad J^kP := \sqcup_{x \in M}(J^kP)_x.$$

An element of J^kP represented by $\mathrm{germ}_x(\sigma)$ is denoted by $j_x^k\sigma \in J^kP$ and called the **k-jet** of σ at $x \in M$. The **source map** and **target map** of J^kP are defined by

$$s : J^kP \to M, \quad j_x^k\sigma \mapsto x \qquad \text{and} \qquad t : J^kP \to P, \quad j_x^k\sigma \mapsto \sigma(x).$$

[1] We thank Ioan Marcut for the idea of this proof.

Note that 0-jets are the same thing as points of P, i.e. there is a diffeomorphism

$$J^0P \xrightarrow{\simeq} P, \qquad j^0_x\sigma \mapsto \sigma(x).$$

Under this identification, $s : J^0P \to M$ coincides with $\pi : P \to M$. Also note that 1-jets are the same thing as pointwise splittings of the differential of the projection $d\pi : TP \to \pi^*TM$, i.e.

$$J^1P \xrightarrow{\simeq} \{\, \xi \in \mathrm{Hom}(\pi^*TM, TP) \mid d\pi \circ \xi = \mathrm{id} \,\}, \qquad j^1_x\sigma \mapsto (d\sigma)_x. \tag{1.2}$$

From now on, we will treat these isomorphisms as equalities.

The jet bundles fit into a sequence of projections

$$\ldots \xrightarrow{\pi} J^{k+1}P \xrightarrow{\pi} J^kP \xrightarrow{\pi} \ldots \xrightarrow{\pi} J^0P \xrightarrow{\simeq} P \xrightarrow{\pi} M, \tag{1.3}$$

where

$$\pi : J^{k+1}P \to J^kP, \qquad j^{k+1}_x\sigma \mapsto j^k_x\sigma.$$

Warning: to avoid cumbersome notations, we have chosen to use π for both $\pi : P \to M$ as well as the projections between the jet bundles. The meaning in each place should be clear from the context.

There is a natural smooth structure on J^kP in which a coordinate chart on P that is compatible with the submersion π (i.e., for which π is the projection onto the first m coordinates),

$$(x_i, y^\rho) = (x_1, ..., x_m, y^1, ..., y^n),$$

induces a coordinate chart on J^kP. To describe such a coordinate chart, we will use the standard multi-index notation, where

$$\alpha = (\alpha_1, ..., \alpha_m) \tag{1.4}$$

denotes a tuple of non-negative integers, $|\alpha| = \alpha_1 + ... + \alpha_m$, and given $i \in \{1, ..., m\}$ we write $\alpha + i = (\alpha_1, ..., \alpha_i + 1, ..., \alpha_m)$ and $\alpha - i = (\alpha_1, ..., \alpha_i - 1, ..., \alpha_m)$ when $\alpha_i > 0$. The induced coordinate chart on J^kP is

$$(x_i, y^\rho_\alpha)_{0 \le |\alpha| \le k}, \tag{1.5}$$

where a k-jet $j^k_x\sigma$, with $x = x_i$ and σ with components σ^ρ, is assigned the coordinates

$$\Big(x_i, \frac{\partial^{|\alpha|}\sigma^\rho}{\partial x_1^{\alpha_1} ... \partial x_m^{\alpha_m}}(x_i)\Big)_{0 \le |\alpha| \le k}.$$

With this smooth structure, the source, target and projection maps are all smooth and surjective submersions.

Remark 1.2.1. This intricate structure of two surjective submersions $s : J^kP \to M$ and $t : J^kP \to P$, together with the fact that the projection $\pi : J^kP \to J^{k-1}P$ has the structure of an affine bundle, as we will shortly see, motivates the word "bundle" in the term "jet bundle". $\diamondsuit$

Example 1.2.2. (A first glance at PDEs) To define the classical notion of a PDE, one takes $M = \mathbb{R}^m$, $P = \mathbb{R}^{m+p}$ and $\pi : P \to M$ to be the projection onto the first m coordinates. In this case, the coordinates (1.5) are globally defined and one defines a PDE to be a finite set of functions on J^kP,

$$F^\lambda(x_i, y^\rho_\alpha), \qquad \lambda = 1, ..., r. \tag{1.6}$$

A solution of the PDE is defined to be a section σ of $\pi : P \to M$ whose components, a set of smooth functions $\sigma^\rho : \mathbb{R}^m \to \mathbb{R}^p$, satisfy the equations

$$F^\lambda\Big(x_i, \frac{\partial^{|\alpha|}\sigma^\rho}{\partial x_1^{\alpha_1} ... \partial x_m^{\alpha_m}}(x_i)\Big) = 0. \tag{1.7}$$

The geometric notion of a PDE, as we will see, generalizes this classical notion; namely, one takes arbitrary manifolds M and P and a surjective submersion $\pi : P \to M$, and a PDE is defined to be a submanifold $R \subset J^kP$ (satisfying certain conditions). A solution of a PDE, in this geometric sense, is defined to be a section σ of $\pi : P \to M$ such that $j^k_x\sigma \in R$ for all $x \in \mathrm{Dom}(\sigma)$.

Note that, locally, any submanifold is the zero locus of a set of functions, a fact which relates the classical notion of a PDE with the geometric one. However, there is a subtle but important difference between the two notions that one should keep in mind: while a PDE in the classical sense is given by a set of functions of the type (1.6), a PDE in the geometric sense is (locally) given by the zero locus of such functions. $\diamondsuit$

Every (local) section σ of $\pi : P \to M$ gives rise to a (local) section $j^k\sigma$ of $s : J^kP \to M$,

$$j^k\sigma : \mathrm{Dom}(\sigma) \subset M \to J^kP, \qquad x \mapsto j^k_x\sigma,$$

called a **(local) holonomic section** of J^kP. The map $\sigma \mapsto j^k\sigma$ defines an isomorphism between the sheaf of sections of $\pi : P \to M$ and the sheaf of holonomic sections of J^kP. The 1-jet $j^1_x(j^k\sigma)$ of a local holonomic section $j^k\sigma$ at a point $x \in \mathrm{Dom}(\sigma)$, or, equivalently, its differential at x,

$$(d(j^k\sigma))_x : T_xM \to T_{j^k_x\sigma}J^kP, \tag{1.8}$$

is called an **integral element** of J^kP at $j^k_x\sigma$. Since $j^1_x(j^k\sigma)$ depends precisely on the $k+1$-th jet $j^{k+1}_x\sigma$, we have an inclusion

$$J^{k+1}P \hookrightarrow J^1(J^kP), \qquad j^{k+1}_x\sigma \mapsto j^1_x(j^k\sigma), \tag{1.9}$$

identifying $J^{k+1}P$ with the set of integral elements of J^kP. This identification plays an important role in the theory.

The vertical bundle of $\pi : J^kP \to J^{k-1}P$,

$$\mathfrak{g}^k := T^\pi J^kP = \mathrm{Ker}\,\big(d\pi : TJ^kP \to \pi^*TJ^{k-1}P\big) \subset TJ^kP, \tag{1.10}$$

is called the **symbol space** of J^kP. Elements of the symbol space $\mathfrak{g}^k$ are canonically identified with homogeneous Taylor polynomials of order k of sections of $\pi : P \to M$. More precisely, there is a canonical isomorphism

$$\mathfrak{g}^k \cong s^*(S^kT^*M) \otimes t^*T^\pi P. \tag{1.11}$$

The following diagram may be useful for keeping track of the various spaces involved in this isomorphism:

$$\begin{tikzcd}
 & \mathfrak{g}^k \arrow[d] & \\
S^kT^*M \arrow[d] & J^kP \arrow[dl, "s"'] \arrow[dr, "t"] & T^\pi P \arrow[d] \\
M & & P
\end{tikzcd}$$

In local coordinates, the symbol space is spanned by $\partial/\partial y^\rho_\alpha$, with $|\alpha| = k$, S^kT^*M is spanned by $dx^\alpha = dx_1^{\alpha_1}...dx_m^{\alpha_m}$, with $|\alpha| = k$, and the isomorphism is given by

$$\frac{\partial}{\partial y^\rho_\alpha} \mapsto dx^\alpha \otimes \frac{\partial}{\partial y^\rho}.$$

The isomorphism (1.11), together with the standard inclusion

$$S^kT^*M \subset T^*M \otimes S^{k-1}T^*M,$$

induces a canonical inclusion of vector bundles over J^kP

$$\mathfrak{g}^k \hookrightarrow \mathrm{Hom}(s^*TM, \pi^*\mathfrak{g}^{k-1}), \qquad T \mapsto \hat{T}. \tag{1.12}$$

In local coordinates, this inclusion is given by

$$\frac{\partial}{\partial y^\rho_\alpha} \mapsto \sum_{i=1}^m dx_i \otimes \frac{\partial}{\partial y^\rho_{\alpha-i}}, \qquad |\alpha| = k,\ \rho = 1, ..., p, \tag{1.13}$$

where we understand $\partial/\partial y^\rho_{\alpha-i}$ to be zero if $\alpha_i = 0$. This map can also be described in a coordinate-free fashion as follows: let $j^k_x\sigma \in J^kP$ and let $T \in (\mathfrak{g}^k)_{j^k_x\sigma}$, thus

$$T = \frac{d}{d\epsilon}\Big|_{\epsilon=0} j^k_x(\sigma_\epsilon)$$

for some curve $j^k_x(\sigma_\epsilon)$ in J^kP that satisfies $j^{k-1}_x(\sigma_\epsilon) = j^{k-1}_x\sigma$ for all ϵ and $j^k_x(\sigma_0) = j^k_x\sigma$. This implies that $(d(j^{k-1}\sigma_\epsilon))_x$ lands in $T_{j^{k-1}_x\sigma}J^{k-1}P$ for all ϵ, and hence the formula

$$\hat{T}(X) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} (d(j^{k-1}\sigma_\epsilon))_x(X), \quad \forall X \in T_xM, \tag{1.14}$$

is well defined. This formula defines a map $\hat{T} : T_x M \to (\mathfrak{g}^{k-1})_{j_x^{k-1}\sigma}$, which indeed takes values in $\mathfrak{g}^{k-1}$ because $d\pi \circ (d(j^{k-1}\sigma_\epsilon))_x = (d(j^{k-2}\sigma))_x$ for all ϵ. From now on we will identify $\mathfrak{g}^k$ with its image in $\mathrm{Hom}(s^*TM, \pi^*\mathfrak{g}^{k-1})$ and simply write T instead of $\hat{T}$.

With the inclusion (1.12), the symbol space $\mathfrak{g}^k$ has the structure of a tableau bundle in the following general sense. Recall that, given any two vector bundles E and F over M, $\mathrm{Hom}(E, F)$ is the vector bundle whose fiber at $x \in M$ consists of all linear maps from E_x to F_x.

Definition 1.2.3. *Let E, F be vector bundles over a manifold M. A **tableau bundle** relative to (E, F) is a vector subbundle*

$$\mathfrak{g} \subset \mathrm{Hom}(E, F).$$

Remark 1.2.4. In the literature, a *tableau* relative to two vector spaces V and W is a linear subspace of $\mathrm{Hom}(V, W)$. Thus, a tableau is a tableau bundle over a point. $\diamondsuit$

The structure of a tableau bundle is all one needs in order to introduce the notion of prolongations of the symbol space.

Definition 1.2.5. *Let $\mathfrak{g} \subset \mathrm{Hom}(E, F)$ be a tableau bundle. The **1st prolongation** of $\mathfrak{g}$ is the subset*

$$\mathfrak{g}^{(1)} \subset \mathrm{Hom}(E, \mathfrak{g})$$

whose fiber at $x \in M$ is

$$\mathfrak{g}_x^{(1)} := \{ \ \xi \in \mathrm{Hom}_x(E, \mathfrak{g}) \ \mid \ \xi(u)(v) = \xi(v)(u) \quad \forall\, u, v \in E_x \ \}.$$

Equivalently, one defines the vector bundle map

$$\delta : \mathrm{Hom}(E, \mathfrak{g}) \to \mathrm{Hom}(\Lambda^2 E, F), \qquad \delta(\xi)(u, v) = \xi(v)(u) - \xi(u)(v), \tag{1.15}$$

and then $\mathfrak{g}^{(1)} := \mathrm{Ker}\ \delta$. Being the kernel of a vector bundle map, $\mathfrak{g}^{(1)} \subset \mathrm{Hom}(E, \mathfrak{g})$ is a vector subbundle if and only if it is of constant rank (see [28], Theorem 8.2). Next, we set

$$\mathfrak{g}^{(0)} := \mathfrak{g}$$

and we define the higher prolongations of $\mathfrak{g}$ inductively as follows:

Definition 1.2.6. *Let $\mathfrak{g} \subset \mathrm{Hom}(E, F)$ be a tableau bundle, let $l > 0$ be an integer and assume that $\mathfrak{g}^{(l-1)}$ is of constant rank. The **l-th prolongation** of $\mathfrak{g}$ is the subset*

$$\mathfrak{g}^{(l)} \subset \mathrm{Hom}(E, \mathfrak{g}^{(l-1)})$$

defined by

$$\mathfrak{g}^{(l)} := (\mathfrak{g}^{(l-1)})^{(1)}.$$

Equivalently, $\mathfrak{g}^{(l)}$ is the kernel of the vector bundle map

$$\delta : \mathrm{Hom}(E, \mathfrak{g}^{(l-1)}) \to \mathrm{Hom}(\Lambda^2 E, \mathfrak{g}^{(l-2)}), \quad \delta(\xi)(u,v) = \xi(v)(u) - \xi(u)(v). \tag{1.16}$$

By setting $\mathfrak{g}^{(-1)} := F$, (1.15) can be regarded as a special case of (1.16).

Returning to our specific setting, the symbol space $\mathfrak{g}^k$ together with the inclusion (1.12) is a tableau bundle and we can talk about its 1st prolongation

$$(\mathfrak{g}^k)^{(1)} \subset \mathrm{Hom}(s^*TM, \mathfrak{g}^k).$$

Using (1.11), it is not difficult to see that there is a canonical isomorphism

$$\mathfrak{g}^{k+1} \cong \pi^*\big((\mathfrak{g}^k)^{(1)}\big). \tag{1.17}$$

Hence, each symbol space is obtained from the one below it by prolongation.

We conclude the section by returning to the opening remarks of Section 1.1, where we briefly discussed the fact that the $k+1$-th order Taylor polynomials that extend a given k-th order Taylor polynomial form an affine space. In the language of jets, this translates into an affine structure on the projection $\pi : J^{k+1}P \to J^kP$. The structure we have discussed so far allows us to describe this affine structure in a coordinate-free fashion:

Lemma 1.2.7. *Let $\pi : P \to M$ be a surjective submersion, and let J^kP and $J^{k+1}P$ be the associated jet bundles for some $k \geq 0$. The projection $\pi : J^{k+1}P \to J^kP$ is an affine bundle modeled on $(\mathfrak{g}^k)^{(1)}$, the 1st prolongation of the symbol space of J^kP. The affine operation is given by*

$$\begin{aligned} J^{k+1}P \times_{J^kP} J^{k+1}P &\to (\mathfrak{g}^k)^{(1)}, \\ (j^{k+1}_x\sigma, j^{k+1}_x\sigma') &\mapsto (d(j^k\sigma))_x - (d(j^k\sigma'))_x. \end{aligned} \tag{1.18}$$

The most pedagogical way to prove this lemma is to unravel the definitions and express the operation (1.18) in local coordinates. The lemma will immediately follow. In Lemma 1.4.8, we will also provide a coordinate-free argument involving the Cartan form. Another coordinate-free proof can be found in [20], Proposition 5.1.

1.3 The Cartan Form on Jet Bundles

Élie Cartan, in his work on Lie pseudogroups, discovered that the structure of a PDE can be fully encoded in a collection of differential 1-forms, known as the *contact forms*, that are canonically associated with the PDE. Globally, Cartan's contact forms are the components of a single canonical, vector bundle-valued differential 1-form known as the Cartan form (and under various other names). The idea of encoding the structure of a PDE in its Cartan form will be a key ingredient in this thesis.

Before presenting the definition, let us begin with a simple example. This example illustrates how the Cartan form, or rather the contact forms, appear in Cartan's work out of a systematic procedure of rewriting a PDE of any order as a 1st order PDE by adding "auxiliary variables", or what we now understand to be jet bundle coordinates.

Example 1.3.1. Let x be a coordinate on $\mathbb{R}$ and let $y(x)$ be a function on $\mathbb{R}$ that satisfies the equation

$$\frac{d^2y}{dx^2}(x) = -y(x). \tag{1.19}$$

Denoting by $p(x)$ the derivative of $y(x)$, this second order equation can be equivalently expressed as the set of two first order equations

$$\begin{aligned} \frac{dy}{dx}(x) &= p(x), \\ \frac{dp}{dx}(x) &= -y(x). \end{aligned} \tag{1.20}$$

Cartan's approach was to think of y and p as two new coordinates, the *auxiliary variables*, and to encode these two equations as a pair of differential forms on the larger space $\mathbb{R}^3$ with coordinates (x, y, p). Equations (1.20), expressed in differential form, become

$$\begin{aligned} dy - pdx &= 0, \\ dp + ydx &= 0. \end{aligned} \tag{1.21}$$

These equations should be interpreted as equations imposed on sections of the projection

$$s : \mathbb{R}^3 \to \mathbb{R}, \qquad (x, y, p) \mapsto x,$$

where a solution is a section that pulls back the differential forms on the left hand side of (1.21) to 0. Indeed, if we write $y(x)$ and $p(x)$ for the y and p components of such a section, we recover (1.20). The differential forms on the left hand side of (1.21) are called the *contact forms* of the PDE. $\diamondsuit$

The **Cartan form** on J^kP, with $k > 0$, is a vector bundle-valued 1-form

$$\omega \in \Omega^1(J^kP; E^{k-1}), \tag{1.22}$$

where

$$E^{k-1} := \pi^* T^s J^{k-1}P. \tag{1.23}$$

At a point $j^k_x\sigma$, ω is defined by

$$(\omega)_{j^k_x\sigma} = (d\pi - d(j^{k-1}\sigma) \circ ds)_{j^k_x\sigma}. \tag{1.24}$$

In this definition, we are using the identification of $j^k_x\sigma$ with $(d(j^{k-1}\sigma))_x$ (see (1.9) and the preceding discussion). Now, since $ds \circ (d\pi - d(j^{k-1}\sigma) \circ ds)_{j^k_x\sigma} = (ds - ds)_{j^k_x\sigma} = 0$, then, indeed, ω takes values in $T^sJ^{k-1}P$. Note that the Cartan form is pointwise surjective, which can be readily seen by restricting it to T^sJ^kP. The formula (1.24) should be somewhat reminiscent of the tautological form on the cotangent bundle of a manifold – project down and then "apply the point". Indeed, one reads the formula of the Cartan form as: project down to $J^{k-1}P$ by $d\pi$ and then use $j^k_x\sigma$, viewed as an integral element, to project onto the vertical bundle of $s : J^{k-1}P \to M$. The following diagram may be useful in unraveling the formula:

$$\begin{tikzcd}
J^kP \arrow[d, "\pi"'] \arrow[ddr, "s"] & \\
J^{k-1}P \arrow[dr, "s"] & \\
& M. \arrow[ul, bend left, "j^{k-1}\sigma"]
\end{tikzcd}$$

In (1.29) we give the local coordinate expression of ω, in which we recognize the components to be the well known contact forms on J^kP. As a simple exercise, we recommend revisiting Example 1.3.1 and verifying that the restriction of the Cartan form to the submanifold defined by (1.19) yields the differential forms on the left hand side of (1.21).

Note that the Cartan forms of two subsequent jet bundles are compatible with the projections, in the sense that

$$d\pi \circ \omega = \omega \circ d\pi. \tag{1.25}$$

This follows from (1.24) together with the fact that $d\pi \circ d(j^k\sigma) = d(j^{k-1}\sigma)$.

Remark 1.3.2. The Cartan form on a 1st jet bundle is “special” or “universal” in the sense that any other Cartan form is the restriction of such a Cartan form. Recalling that $J^kP \subset J^1(J^{k-1}P)$, then (1.24) is the restriction of the Cartan form

$$\omega_{j^1_x\eta} = (d\pi - (d\eta)_x \circ ds)_{j^1_x\eta}$$

on $J^1(J^{k-1}P)$. Indeed, set $\eta = j^{k-1}\sigma$ to recover (1.24). $\diamondsuit$

The kernel of the Cartan form,

$$C_\omega := \operatorname{Ker} \omega \subset TJ^kP,$$

is called the **Cartan distribution** of J^kP. Since ω is pointwise surjective, we have the following short exact sequence of vector bundles over J^kP:

$$0 \to C_\omega \to TJ^kP \xrightarrow{\omega_k} E^{k-1} \to 0.$$

The Cartan distribution is a non-involutive distribution and it contains the symbol space (1.10),

$$\mathfrak{g}^k = T^\pi J^kP = C_\omega \cap \operatorname{Ker} ds, \tag{1.26}$$

as an involutive subbundle. Thus, we also have the following short exact sequence of vector bundles over J^kP:

$$0 \to \mathfrak{g}^k \to C_\omega \xrightarrow{ds} s^*TM \to 0. \tag{1.27}$$

Pointwise splittings of this short exact sequence will play an important role in the problem of formal integrability. The space of such pointwise splittings,

$$J^1_\omega(J^kP) := \{\, j^1_x\eta \in J^1(J^kP) \mid (\eta^*\omega)_x = 0 \,\} \subset J^1(J^kP), \tag{1.28}$$

is called the **partial prolongation** of J^kP. We denote the projection to J^kP by

$$\pi : J^1_\omega(J^kP) \to J^kP, \qquad j^1_x\eta \mapsto \eta(x);$$

it has the structure of an affine bundle modeled on the vector bundle $\mathrm{Hom}(s^*TM, \mathfrak{g}^k)$.

An element $j^1_x\eta$ of $J^1_\omega(J^kP)$ projecting to $\eta(x) \in J^kP$ is called an **almost integral element** of J^kP at $\eta(x)$. By (1.24), an integral element of J^kP is also an almost integral element. We thus have the following inclusions of affine bundles:

$$J^{k+1}P \subset J^1_\omega(J^kP) \subset J^1(J^kP).$$

The most important property of the Cartan form on a jet bundle J^kP is that it detects the holonomic sections of J^kP, as the following proposition shows.

Proposition 1.3.3. *A (local) section η of $s : J^kP \to M$ is a (local) holonomic section if and only if $\eta^*\omega = 0$.*

Proof. We give two proofs. Both give insight into the workings of the Cartan form. A (local) section η of $s : J^kP \to M$ induces a (local) section $\sigma := \pi^k \circ \eta$ of $\pi : P \to M$, where π^k denotes the composition of the k projections $J^kP \xrightarrow{\pi} \dots \xrightarrow{\pi} J^0P = P$. Note that η is holonomic if and only if $\eta = j^k\sigma$.

First proof: in the local coordinates (1.5) on J^kP, the components of ω are

$$(dy^\rho_\alpha - \sum_{j=1}^m y^\rho_{\alpha+j}dx_j) \otimes \frac{\partial}{\partial y^\rho_\alpha}, \qquad |\alpha| < k,\ \rho = 1, \dots, p. \tag{1.29}$$

If η^ρ_α are the component functions of η, then $\eta^*\omega$ is

$$\Big(\frac{\partial \eta^\rho_\alpha}{\partial x_j} - \eta^\rho_{\alpha+j}\Big)dx_j \otimes \frac{\partial}{\partial y^\rho_\alpha}, \qquad |\alpha| < k,\ \rho = 1, \dots, p,\ j = 1, \dots, m.$$

These forms vanish precisely when $\eta = j^k\sigma$.

Second proof: starting with a (local) holonomic section $j^k\sigma$,

$$(j^k\sigma)^*\omega = \omega \circ d(j^k\sigma) = (d\pi - d(j^{k-1}\sigma)\circ ds)\circ d(j^k\sigma) = (d(j^{k-1}\sigma) - d(j^{k-1}\sigma)) = 0.$$

The converse is proven by induction. Let η be a (local) section of $s : J^kP \to M$ such that $\eta^*\omega = 0$ and recall that we write $\sigma = \pi^k \circ \eta$. We want to show that $\eta = j^k\sigma$. For the initial step, we note that $\pi^k \circ \eta = j^0\sigma$. For the inductive step, we assume that $\pi^l \circ \eta = j^{k-l}\sigma$ and prove that $\pi^{l-1} \circ \eta = j^{k-l+1}\sigma$. Let $x \in \mathrm{Dom}(\eta)$ and let σ' be a section of $s : P \to M$ such that $(\pi^{l-1}\circ\eta)(x) = j^{k-l+1}_x\sigma'$. By (1.25), $\eta^*\omega = 0$ implies that $(\pi^{l-1}\circ\eta)^*\omega^{k-l+1} = 0$ and hence

$$\begin{aligned}
0 &= ((\pi^{l-1}\circ\eta)^*\omega^{k-l+1})_x \\
&= (d\pi - d(j^{k-l}\sigma')\circ ds)\circ d(\pi^{l-1}\circ d\eta)_x \\
&= (d(\pi^l\circ\eta))_x - (d(j^{k-l}\sigma'))_x.
\end{aligned}$$

By the inductive assumption, $(d(\pi^l \circ \eta))_x = (d(j^{k-l}\sigma))_x$, and hence $(d(j^{k-l}\sigma'))_x = (d(j^{k-l}\sigma))_x$. By the identification (1.9), we conclude that $j_x^{k-l+1}\sigma' = j_x^{k-l+1}\sigma$. □

When we move on to discuss PDEs, we will see that the main problem in constructing formal solutions of a PDE is whether the PDE has integral elements or not. The relevant information to address this question is encoded in the differential of the Cartan form. In general, the differential "$d\omega$" of vector bundle-valued 1-forms is not canonically defined. However, its restriction to the kernel of the 1-form is canonical, and it is precisely this restriction of "$d\omega$" that encodes the relevant information. Let us recall the general construction. Given a vector bundle E over a manifold N, a choice of a connection ∇ on E induces a de-Rham like operator d_∇ on the module of E-valued differential forms on N by the usual Koszul type formula. In particular, if $\omega \in \Omega^1(N; E)$, then

$$d_\nabla\omega(X,Y) := \nabla_X(\omega(Y)) - \nabla_Y(\omega(X)) - \omega([X,Y]), \qquad \forall\, X, Y \in \mathfrak{X}(N).$$

From the formula we see that, indeed, the restriction of $d_\nabla\omega$ to Ker ω is independent of the choice of ∇. Applying this construction to the Cartan form, we obtain

$$d_\nabla\omega \in \Omega^2(J^kP; E^{k-1}),$$

whose restriction to $C_\omega = \text{Ker}\ \omega$,

$$\delta\omega := d_\nabla\omega\big|_{C_\omega} \in \Gamma(\text{Hom}(\Lambda^2 C_\omega, E^{k-1})), \tag{1.30}$$

is independent of the choice of ∇. In particular, we have an induced map

$$c_\omega : J^1_\omega(J^kP) \to \text{Hom}(s^*\Lambda^2 TM, E^{k-1}), \qquad j^1_x\eta \mapsto \delta\omega((d\eta)_x(\cdot), (d\eta)_x(\cdot)),$$

which is called the **curvature** of J^kP. The main properties of $\delta\omega$ and c_ω are collected in the following proposition:

Proposition 1.3.4. *Let ω be the Cartan form on J^kP. Then:*

1. *The curvature c_ω of J^kP takes values in $\mathfrak{g}^{k-1}$, i.e.*
$$c_\omega : J^1_\omega(J^kP) \to \text{Hom}(s^*\Lambda^2 TM, \pi^*\mathfrak{g}^{k-1}).$$
2. *Let $j^1_x\eta \in J^1_\omega(J^kP)$, then $j^1_x\eta$ is an integral element if and only if $c_\omega(j^1_x\eta) = 0$.*
3. *$\delta\omega((d\eta)_x(X), T) = T(X)$ for all $j^1_x\eta \in J^1_\omega(J^kP)_{j^k_x\sigma}$, $X \in T_xM$, $T \in (\mathfrak{g}^k)_{j^k_x\sigma}$.*
4. *$\delta\omega|_{\mathfrak{g}^k} = 0$.*

Remark 1.3.5. Thus, c_ω measures the failure of an almost integral element to be an integral element. This motivates the name curvature. ♢

Proof. We prove this in local coordinates. The components of $\delta\omega$ are computed by taking the differential of the components of ω, which are given by (1.29). The result is:

$$\sum_{j=1}^{m}(dy^\rho_{\alpha+j} \wedge dx_j) \otimes \frac{\partial}{\partial y^\rho_\alpha}, \qquad |\alpha| < k,\ \rho = 1, ..., p. \tag{1.31}$$

Now, a frame of $\mathfrak{g}^k$ is given by

$$\frac{\partial}{\partial y^\rho_\alpha}, \qquad |\alpha| = k,\ \rho = 1, ..., p,$$

and from this we se that $\delta\omega$ vanishes on $\mathfrak{g}^k$ (equivalently, this also follows from the fact that $\mathfrak{g}^k \subset C_\omega$ is involutive). Next, let Let $j^1_x\eta \in J^1_\omega(J^kP)$. A basis of $\mathrm{Im}\ (d\eta)_x \subset T_{\eta(x)}J^kP$ is given by

$$\frac{\partial}{\partial x_i} + \sum_{|\alpha|<k,\rho} y^\rho_{\alpha+i}\frac{\partial}{\partial y^\rho_\alpha} + \sum_{|\alpha|=k,\rho} a^\rho_{\alpha,i}\frac{\partial}{\partial y^\rho_\alpha}, \qquad i = 1, ..., m. \tag{1.32}$$

where $a^\rho_{\alpha,i}$ are real coefficients. Property 3 follows directly from (1.13). For properties 1 and 2 we apply (1.31) on a pair of vectors (1.32) with $i, j \in \{1, ..., m\}$. All the components with $|\alpha| < k - 1$ vanish, which proves property 1, and the remaining ones are

$$\left(a^\rho_{\alpha+j,i} - a^\rho_{\alpha+i,j}\right) \otimes \frac{\partial}{\partial y^\rho_\alpha}, \qquad |\alpha| = k - 1,\ \rho = 1, ..., p, \tag{1.33}$$

from which we deduce property 2 (since η is an integral element if and only if $a^\rho_{\alpha+j,i} = a^\rho_{\alpha+i,j}$ for all possible pairs). □

1.4 PDEs

A system of partial differential equations, or a PDE in short, is a "nice" submanifold of J^kP, for some $k > 0$. The meaning of "nice" depends on the applications one has in mind and various definitions can be found in the literature. Our definition contains regularity conditions that ensure that the Cartan form restricts "nicely" to the PDE. This will allow us to handle a PDE abstractly, i.e. independently of its ambient jet bundle.

Let $R \subset J^kP$ be a submanifold. We denote the restriction of the Cartan form of J^kP to R and its kernel also by

$$\omega := \omega|_R \in \Omega^1(R; E^{k-1}) \qquad \text{and} \qquad C_\omega := \mathrm{Ker}\ \omega \subset TR.$$

Definition 1.4.1. *A* ***PDE*** *of order $k > 0$ on $\pi : P \to M$ is a submanifold $R \subset J^kP$ satisfying:*

1. *$s = s|_R : R \to M$ is surjective,*
2. *ω is regular, in the sense that its kernel C_ω has constant rank,*
3. *$ds\big|_{C_\omega} : C_\omega \to s^*TM$ is pointwise surjective (in particular, $s : R \to M$ is a submersion).*

A ***(local) solution*** *of R is a (local) section σ of $\pi : P \to M$ satisfying: $j^k_x\sigma \in R$ for all $x \in \mathrm{Dom}(\sigma)$.*

Condition 3 of the definition can be equivalently rephrased as requiring that C_ω be transverse to the fibers of s, i.e.

$$C_\omega + \operatorname{Ker} ds = TR.$$

Remark 1.4.2. As was already remarked in Example 1.2.2, a PDE in this geometric sense is, strictly speaking, not the same thing as a PDE in the classical sense: whereas classically a PDE is a set of functions, geometrically it is the zero locus of the functions. Thus, rather than regarding R as an equation, one should regard R as the “solutions up to order k” of an equation. We will discuss this point further in Example 1.5.2. $\diamondsuit$

Example 1.4.3.

1. The simplest example of a PDE of order k is $J^k P$ itself, in which case the set of local solutions is the sheaf of sections of $\pi : P \to M$.

2. Here is a more “hands on” criteria for determining whether a given submanifold is a PDE. A submanifold $R \subset J^k P$ for which $\pi(R) \subset J^{k-1} P$ is a submanifold and such that

   (a) $\pi|_R : R \to \pi(R)$ is a submersion,

   (b) $s|_{\pi(R)} : \pi(R) \to M$ is a surjective submersion,

   is a PDE. The first regularity condition of a PDE is clearly satisfied. One shows that the other two are satisfied by choosing a splitting of $d\pi : TR \to \pi^* T(\pi(R))$. At each point $j^k_x \sigma \in R$, the induced integral element $(d(j^{k-1}\sigma))_x$ composed with the splitting is a pointwise splitting of $ds|_{C_\omega} : C_\omega \to s^*TM$, and these fit to give a global splitting. The existence of such a splitting implies the third regularity condition. The resulting splitting induces an isomorphism $TR \cong T^s R \oplus s^*TM$, for which ω kills the second component and restricts to $d\pi$ on the second. This implies the second regularity condition. $\diamondsuit$

Remark 1.4.4. Let us comment on the regularity conditions in the definition.

- Condition 3 is a necessary condition for having a solution through every point of R.

- Since it follows from condition 3 that $s : R \to M$ is a submersion, one may also remove condition 1 and replace M by the open subset $s(R) \subset M$.

- Relaxing conditions 2 and 3 will introduce singularities into the structure (e.g. the symbol space, to be defined shortly, will not be of constant rank). Since we are interested in the geometric picture, the technicalities arising from the presence of singularities are not so relevant. Here and throughout the thesis, we will assume strong enough regularity conditions to avoid singularities. $\diamondsuit$

Let $R \subset J^k P$ be a PDE. The regularity conditions ensure that the geometric structure of the ambient jet bundle restricts “nicely” to R, as we now explain.

A splitting of $ds|_{C_\omega} : C_\omega \to s^*TM$ induces a vector bundle isomorphism $TR \cong T^sR \oplus s^*TM$ for which ω kills the second component and its restriction to the first component is $d\pi$. Thus, $\mathrm{Im}\ d\pi|_{T^sR} = \mathrm{Im}\ \omega$ and

$$E := \mathrm{Im}\ d\pi|_{T^sR} = \mathrm{Im}\ \omega \subset E^{k-1} = \pi^*T^sJ^{k-1}P \tag{1.34}$$

is a vector bundle. The restriction of the Cartan form of J^kP to R,

$$\omega \in \Omega^1(R; E),$$

is called the **Cartan form** of R, and its kernel,

$$C_\omega = \mathrm{Ker}\ \omega \subset TR,$$

is called the **Cartan distribution** of R. Since the Cartan form is surjective onto E, we have the following short exact sequence of vector bundles over R:

$$0 \to C_\omega \to TR \xrightarrow{\omega} E \to 0.$$

The **symbol space** of R is defined to be the restriction of the symbol space of J^kP,

$$\mathfrak{g} := T^\pi R = TR \cap \mathfrak{g}^k, \tag{1.35}$$

or intrinsically,

$$\mathfrak{g} = C_\omega \cap \mathrm{Ker}\ ds. \tag{1.36}$$

Due to the regularity conditions, it is a vector bundle, and we also have the following short exact sequence of vector bundles:

$$0 \to \mathfrak{g} \to C_\omega \xrightarrow{ds} s^*TM \to 0. \tag{1.37}$$

The space of pointwise splittings of (1.37),

$$J^1_\omega R := \{\ j^1_x\eta \in J^1R \mid (\eta^*\omega)_x = 0\ \},$$

is called the **partial prolongation** of R. The projection

$$\pi : J^1_\omega R \to R, \qquad j^1_x\eta \mapsto \eta(x),$$

is an affine bundle modeled on the vector bundle $\mathrm{Hom}(s^*TM, \mathfrak{g})$. An element $j^1_x\eta$ of $J^1_\omega R$ is called an **almost integral element** of R at $\eta(x)$.

Recall that (local) sections of P are identified with (local) holonomic sections of J^kP via $\sigma \mapsto j^k\sigma$. This implies that (local) solutions of R are identified with (local) holonomic sections of R (i.e. (local) holonomic sections of J^kP with values in R). In general, a PDE may fail to have (local) solutions and the question of existence is the main problem in the theory of PDEs. As an immediate consequence of Proposition 1.3.3:

Proposition 1.4.5. *A (local) section η of $s : R \to M$ comes from a (local) solution if and only if $\eta^*\omega = 0$.*

Remark 1.4.6. Proposition 1.4.5 is the main evidence that the structure of a PDE is encoded in its Cartan form. This motivates the abstract point of view of thinking of a PDE as a manifold equipped with a vector bundle-valued 1-form that satisfies the essential properties of the Cartan form. We will discuss this point of view in more detail in Section 1.9. $\diamondsuit$

As for jet bundles, we construct the canonical part of "$d\omega$",

$$\delta\omega \in \Gamma(\mathrm{Hom}(\Lambda^2 C_\omega, E)),$$

by choosing a connection ∇ on E and setting $\delta\omega := d_\nabla \omega|_{C_\omega}$. Equivalently, we can define it as the restriction of $\delta\omega$ on $J^k P$ to C_ω. The map

$$c_\omega : J^1_\omega R \to \mathrm{Hom}(s^*\Lambda^2 TM, E), \qquad j^1_x\eta \mapsto \delta\omega((d\eta)_x(\cdot), (d\eta)_x(\cdot)),$$

is called the **curvature** of R.

The symbol space $\mathfrak{g}$ of R inherits the tableau bundle structure of $\mathfrak{g}^k$. More precisely, we have an inclusion

$$\mathfrak{g} \subset \mathrm{Hom}(s^*TM, E), \tag{1.38}$$

which, using Proposition 1.3.4, is described as follows: let $r \in R$ with $x = s(r) \in M$, then

$$T(X) = \delta\omega((d\eta)_x(X), T), \qquad \forall\, T \in \mathfrak{g}_r,\ X \in T_x M,$$

where $j^1_x\eta$ is some choice of an element in $J^1_\omega R$ with $\eta(x) = r$. Proposition 1.3.4 tells us that the map T is independent of the choice of $j^1_x\eta$, and, moreover, that it takes values in $E \cap \pi^*\mathfrak{g}^{k-1}$.

With (1.38), $\mathfrak{g}$ becomes a tableau bundle and we can talk about its 1st prolongation

$$\mathfrak{g}^{(1)} \subset \mathrm{Hom}(s^*TM, \mathfrak{g}).$$

In general, $\mathfrak{g}^{(1)}$ may fail to be of constant rank.

The **1st prolongation** of R is defined as

$$R^{(1)} := J^1(R) \cap J^{k+1}P, \tag{1.39}$$

where we recall that $J^{k+1}P \subset J^1(J^k P)$. By proposition 1.3.4, the 1st prolongation can also be described intrinsically as

$$R^{(1)} = \{\ j^1_x\eta \in J^1 R \mid (\eta^*\omega)_x = 0,\ c_\omega(j^1_x\eta) = 0\ \}. \tag{1.40}$$

We denote the source and projection maps by

$$s : R^{(1)} \to M, \quad j^1_x\eta \mapsto x \qquad \text{and} \qquad \pi : R^{(1)} \to R, \quad j^1_x\eta \mapsto \eta(x).$$

An element $j^1_x\eta$ of $R^{(1)}$ is called an **integral element** of R at $\eta(x)$. An integral element at a point $r \in R$, as we will explain in then next section, should be viewed as an extension of a "solution up to order k" to a "solution up to order $k+1$". Since any integral element is an almost integral element, we have the inclusions

$$R^{(1)} \subset J^1_\omega R \subset J^1 R.$$

Remark 1.4.7. In Sections 1.5 and 1.6, we discuss the notion of formal integrability of a PDE and prove a formal integrability theorem which will give an intuitive meaning to the notions of integral and almost integral elements. $\diamondsuit$

In the latter inclusions, we have seen that both J^1R and $J^1_\omega R$ are affine bundles. What structure does $R^{(1)}$ have? In general, as the following lemma shows, one can only say that the fibers of $\pi : R^{(1)} \to R$ are affine spaces. The question of whether these fibers glue smoothly to form an affine bundle is the central question in the problem of formal integrability.

Lemma 1.4.8. *The fibers of $\pi : R^{(1)} \to R$ are affine spaces modeled on the corresponding fibers of $\mathfrak{g}^{(1)}$. The affine operation is given by*

$$R^{(1)} \times_R \mathfrak{g}^{(1)} \to R^{(1)}, \qquad (j^1_x\eta, \xi) \mapsto (d\eta)_x + \xi.$$

Proof. Let $j^1_x\eta \in R^{(1)}_r$ and $\xi \in (\mathfrak{g}^{(1)})_r$, then

$$\omega(((d\eta)_x + \xi)(X)) = \omega((d\eta)_x(X)) = 0, \qquad \forall X \in T_{s(r)}M,$$

and

$$\delta\omega((\xi + \eta)(X), (\xi + \eta)(Y)) = \eta(Y)(X) - \eta(X)(Y) = 0, \qquad \forall X, Y \in T_{s(r)}M.$$

Hence, $(d\eta)_x + \xi$ corresponds to an integral element. Conversely, let $j^1_x\eta, j^1_x\eta' \in R^{(1)}_r$, then

$$\begin{aligned}((d\eta')_x - (d\eta)_x)(X)(Y) &= \delta\omega((d\eta)_x(Y), ((d\eta')_x - (d\eta)_x)(X)) \\ &= \delta\omega((d\eta')_x(X), ((d\eta')_x - (d\eta)_x)(Y)) \\ &= ((d\eta')_x - (d\eta)_x)(Y)(X)\end{aligned}$$

for all $X, Y \in T_xM$. Thus $(d\eta')_x - (d\eta)_x \in \mathfrak{g}^{(1)}_r$. $\square$

As the next proposition will show, two conditions must be satisfied for $\pi : R^{(1)} \to R$ to be an affine bundle: 1) $\mathfrak{g}^{(1)}$ must be a vector bundle, and 2) $\pi : R^{(1)} \to R$ must have a global section. Global sections of $\pi : R^{(1)} \to R$, in turn, can be viewed as special Ehresmann connections on R, as we now explain. A right splitting H of the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{g} \arrow[r] & C_\omega \arrow[r, "ds"] & s^*TM \arrow[r] \arrow[l, bend right, "H"'] & 0 \end{tikzcd} \tag{1.41}$$

is called a **Cartan-Ehresmann connection** on R. Thus, a Cartan-Ehresmann connection is a section of $\pi : J^1_\omega R \to R$, a choice of an almost integral element at each point of R. Cartan-Ehresmann connections always exist because splittings of (1.41) always exist. A Cartan-Ehresmann connection H is said to be **integral** if H is a section of $\pi : R^{(1)} \to R$, i.e. if the almost integral element $H_r \in J^1_\omega R$ is in fact an integral element for all $r \in R$.

Proposition 1.4.9. *Let $R \subset J^kP$ be a PDE with symbol space $\mathfrak{g}$ and assume that $\mathfrak{g}^{(1)}$ is of constant rank.*

1. *If $\pi : R^{(1)} \to R$ is surjective, then $\pi : R^{(1)} \to R$ is an affine bundle modeled on $\mathfrak{g}^{(1)}$.*
2. *$\pi : R^{(1)} \to R$ is surjective if and only if R admits an integral Cartan-Ehresmann connection.*
3. *If $\pi : R^{(1)} \to R$ is surjective, then $R^{(1)} \subset J^{k+1}P$ is a PDE*

Proof. Item 2 is clear. Item 1 follows directly from Proposition 1.1.6, since $R^{(1)}$ is the intersection of the affine bundles $J^1R \to R$ and $J^{k+1}P \to J^kP$ (restricted to R) and $\mathfrak{g}^{(1)}$ is the intersection of their modeling vector bundles. In item 3, if $\pi : R^{(1)} \to R$ is surjective, and hence an affine bundle, then $R^{(1)} \subset J^{k+1}P$ is a submanifold, $\pi : R^{(1)} \to R$ a submersion and the regularity conditions of a PDE are deduced by choosing a splitting of $d\pi : TR^{(1)} \to \pi^*TR$ (c.f. the second item in Example 1.4.3). □

Our working criterion for determining whether a Cartan-Ehresmann connection is integral or not will be:

Proposition 1.4.10. *A Cartan-Ehresmann connection H on a PDE $R \subset J^kP$ is integral if and only if*

$$\delta\omega(H(\cdot), H(\cdot)) = 0.$$

Proof. A direct consequence of Proposition 1.3.4. □

Thus, $\delta\omega(H(\cdot), H(\cdot))$ measures the failure of the Cartan-Ehresmann connection H on R to be integral. We call

$$c_H := \delta\omega(H(\cdot), H(\cdot)) \in \Gamma(\mathrm{Hom}(s^*\Lambda^2TM, E)) \tag{1.42}$$

the **weak curvature** of H. The term weak curvature is chosen to make the distinction with the usual notion of curvature of an Ehresmann connection. As the name suggests, the vanishing of the usual curvature $H([\cdot,\cdot]) - [H(\cdot), H(\cdot)]$ implies the vanishing the weak curvature c_H:

Lemma 1.4.11. *Let $R \subset J^kP$ be a PDE and H be a choice of a Cartan-Ehresmann connection on R. Then,*

$$c_H(X, Y) = d\pi\big(H([X, Y]) - [H(X), H(Y)]\big), \qquad \forall X, Y \in \mathfrak{X}(M).$$

Proof. Let $j^k_x\sigma \in R$,

$$\begin{aligned}
c_H(X, Y)|_{j^k_x\sigma} &= -\omega([H(X), H(Y)]_{j^k_x\sigma}) \\
&= -(\mathrm{id} - (d(j^{k-1}\sigma))_x \circ (ds)) \circ d\pi([H(X), H(Y)]_{j^k_x\sigma}) \\
&= d(j^{k-1}\sigma)([X, Y]_x) - d\pi([H(X), H(Y)]_{j^k_x\sigma}) \\
&= d\pi \circ H_{j^k_x\sigma}([X, Y]_x) - d\pi([H(X), H(Y)]_{j^k_x\sigma}) \\
&= d\pi(H([X, Y]) - [H(X), H(Y)])|_{j^k_x\sigma}.
\end{aligned}$$

In the third equality, we use that $ds \circ d\pi = ds$ together with the fact that $H(X)$ and $H(Y)$ are s-related to X and Y and hence $[H(X), H(Y)]$ is s-related to $[X, Y]$. In the fourth equality, we use the fact that $(d(j^{k-1}\sigma))_x = d\pi \circ H_{j^k_x\sigma}$, which follows from $\omega \circ H_{j^k_x\sigma} = 0$. $\square$

In summary, there are two tensors associated with a Cartan-Ehresmann connection: the weak curvature and the curvature. If the weak curvature vanishes, then the connection is integral, while if the curvature vanishes, then the connection is flat (in the usual sense of Ehresmann connections). The lemma shows that flat implies integral.

1.5 Formal Integrability of PDEs and the Spencer Cohomology

The elements of a PDE $R \subset J^kP$, as we mentioned in Remark 1.4.2, should be viewed as "solutions up to order k" of R. They are the k-th order Taylor polynomials of potential solutions of R. The problem of formal integrability is to determine whether an element of R can be extended to a "solution up to order $k+1$", then to a "solution up to order $k+2$", etc., until it is extended to a *formal solution* of R, i.e. to the Taylor series of a potential solution of R. In this section, we define and explain the notion of formal integrability and state a criterion for formal integrability due to Goldschmidt. In the next section, we will provide a new proof for Goldschmidt's theorem using the language that has been developed so far.

Recall that the 1st prolongation $R^{(1)}$ of R, given by (1.39), is a subset of $J^{k+1}P$ but not necessarily a smooth one. In the previous section, we discussed the structure of this subset. We saw that the fibers of $\pi : R^{(1)} \to R$ are in general affine spaces, but that $R^{(1)}$ may fail to be an affine bundle as a whole. Proposition 1.4.9 tells us that if $\mathfrak{g}^{(1)}$ is of constant rank and $\pi : R^{(1)} \to R$ is surjective (and hence an affine bundle modeled on $\mathfrak{g}^{(1)}$), then $R^{(1)} \subset J^{k+1}P$ is also a PDE. In that case, we may proceed and define the 1st prolongation of $R^{(1)}$. This observation leads us to the following inductive definition: set $R^{(0)} := R$, let $l > 1$ be an integer and assume that $\mathfrak{g}^{(l-1)}$ is of constant rank and $R^{(l-1)} \to R^{(l-2)}$ is surjective (and hence $R^{(l-1)} \subset J^{k+l-1}P$ is a PDE), then

$$R^{(l)} := (R^{(l-1)})^{(1)} \subset J^{k+l}P \tag{1.43}$$

is called the **l-th prolongation** of R. By Lemma 1.4.8, the fibers of $\pi : R^{(l)} \to R^{(l-1)}$ are affine spaces modeled on the corresponding fibers of $\pi^*\mathfrak{g}^{(l)}$, the pullback of $\mathfrak{g}^{(l)}$ by

$$\pi = \pi^{l-1} := \underbrace{\pi \circ \ldots \circ \pi}_{(l-1)\text{ - times}} : R^{(l-1)} \to R.$$

Definition 1.5.1. *A PDE $R \subset J^kP$ is **integrable up to order** $k+l$ if $\mathfrak{g}^{(j)}$ is of constant rank and $R^{(j)} \to R^{(j-1)}$ is surjective for all $0 < j \leq l$, and it is **formally integrable** if $\mathfrak{g}^{(l)}$ is of constant rank and $R^{(l)} \to R^{(l-1)}$ is surjective for all $l > 0$.*

If R is formally integrable, then we have an infinite sequence of affine bundles,

$$\ldots \xrightarrow{\pi} R^{(l+1)} \xrightarrow{\pi} R^{(l)} \xrightarrow{\pi} R^{(l-1)} \xrightarrow{\pi} \ldots \xrightarrow{\pi} R^{(1)} \xrightarrow{\pi} R, \tag{1.44}$$

where at each level $\pi : R^{(l+1)} \to R^{(l)}$ is an affine bundle modeled on the vector bundle $\pi^*\mathfrak{g}^{(l+1)}$. An element of the inverse limit of (1.44) is called a **formal solution** of R. To be more explicit, a formal solution is a sequence $(r_0, r_1, r_2, ...)$, with $r_i \in R^{(i)}$ (where $R^{(0)} = R$), such that $\pi(r_{i+1}) = r_i$ for all $i \geq 0$. Note that a solution σ of R induces a formal solution of R, while a formal solution may fail to come from a solution. In the following example, we explain the connection between this geometric notion of a formal solution and the classical notion.

Example 1.5.2 (Formal integrability in local coordinates)**.** Locally, we can choose coordinates (x_i, y^ρ) for P, coordinates $(x_i, y^\rho_\alpha)_{|\alpha|\leq k}$ for J^kP, coordinates $(x_i, y^\rho_\alpha)_{|\alpha|\leq k+1}$ for $J^{k+1}P$, etc. In these coordinates, a PDE $R \subset J^kP$ can be expressed as the zero locus of a collection of functions $F^\lambda = (F^1, ..., F^r)$ on J^kP, where r is the codimension of $R \subset J^kP$. Thus, R consists of all points $(x_i, y^\rho_\alpha)_{|\alpha|\leq k}$ of J^kP that satisfy,

$$F^\lambda(x_i, y^\rho_\alpha) = 0, \qquad \lambda = 1, ..., r.$$

On the other hand, a solution of R is a section σ of $\pi : P \to M$ whose component functions σ^ρ satisfy the equation:

$$F^\lambda\Big(x_i, \frac{\partial^{|\alpha|}\sigma^\rho}{\partial x_1^{\alpha_1}...\partial x_m^{\alpha_m}}(x_i)\Big) = 0, \qquad \lambda = 1, ..., r. \tag{1.45}$$

Therefore, elements of R are k-th order Taylor polynomials of potential solutions, or what we call "solutions up to order k".

If a solution exists, then it must also satisfy the first order partial derivatives of (1.45). Hence, by the chain rule, a solution σ^ρ must also satisfy

$$\begin{aligned}&\frac{\partial F^\lambda}{\partial x_j}\Big(x_i, \frac{\partial^{|\alpha|}\sigma^\rho}{\partial x_1^{\alpha_1}...\partial x_m^{\alpha_m}}(x_i)\Big)\\ &\qquad + \sum_{|\alpha|\leq k,\ \rho} \frac{\partial F^\lambda}{\partial y^\rho_\alpha}\Big(x_i, \frac{\partial^{|\alpha|}\sigma^\rho}{\partial x_1^{\alpha_1}...\partial x_m^{\alpha_m}}(x_i)\Big)\ \frac{\partial^{|\alpha|+1}\sigma^\rho}{\partial x_j\partial x_1^{\alpha_1}...\partial x_m^{\alpha_m}}(x_i) = 0,\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \lambda = 1, ..., r,\ j = 1, ..., m.\end{aligned}$$

On the other hand, by definition, the 1st prolongation $R^{(1)}$ of R consists of all points $(x_i, y^\rho_\alpha)_{|\alpha|\leq k+1}$ in $J^{k+1}P$ that satisfy

$$\frac{\partial F^\lambda}{\partial x_j}(x_i, y^\rho_\alpha) + \sum_{|\alpha|\leq k} \frac{\partial F^\lambda}{\partial y^\rho_\alpha}(x_i, y^\rho_\alpha)\ y^\rho_{\alpha+i} = 0, \qquad \lambda = 1, ..., r,\ j = 1, ..., m. \tag{1.46}$$

Thus, elements of $R^{(1)}$ are the $k+1$-th order Taylor polynomials of potential solutions, or what we call "solutions up to order $k+1$".

Continuing in this way, we see that the defining equations of $R^{(l)}$ are obtained as the l-th order differential consequences of (1.45), and that solving the equations at each stage amounts to extending the Taylor polynomial of a potential solution by one order. In the limit of this procedure, we obtain elements of the inverse limit of the sequence 1.44. Therefore, *elements of the inverse limit of*

$$\ldots \xrightarrow{\pi} R^{(l+1)} \xrightarrow{\pi} R^{(l)} \xrightarrow{\pi} R^{(l-1)} \xrightarrow{\pi} \ldots \xrightarrow{\pi} R^{(1)} \xrightarrow{\pi} R$$

are Taylor series of potential solutions, or what we defined to be formal solutions.

Upon constructing a formal solution, one may wonder whether it converges to a function or not, and if it converges, whether the function defines a solution. We just mention here that in the smooth category a formal solution that converges to a function may fail to be a solution, while in the analytic category a formal solution that converges to a function is always a solution. $\diamondsuit$

In [20], Goldschmidt gives a criterion for formal integrability of a PDE in terms of the Spencer cohomology associated with the symbol space of the PDE. Let us explain the notion of the Spencer cohomology associated with the symbol space of a PDE, or, more generally, of a tableau bundle, and then state Goldschmidt's theorem.

Recall from (1.38) that the symbol space $\mathfrak{g}$ of a PDE $R \subset J^k P$ has the structure of a tableau bundle. In general, given any pair of vector bundles A and B over M and a tableau bundle $\mathfrak{g} \subset \mathrm{Hom}(A, B)$ (Definition 1.2.3), one can construct the **Spencer complex** of $\mathfrak{g}$. It consists of the following sequence of cochain complexes:

$$\begin{array}{cllllll}
 & 0 & 1 & 2 & 3 & 4 \\
0 & \mathfrak{g} & \hookrightarrow \mathrm{Hom}(E,F) & & & \\
1 & \mathfrak{g}^{(1)} & \hookrightarrow \mathrm{Hom}(E,\mathfrak{g}) & \xrightarrow{\delta} \mathrm{Hom}(\Lambda^2 E,F) & & \\
2 & \mathfrak{g}^{(2)} & \hookrightarrow \mathrm{Hom}(E,\mathfrak{g}^{(1)}) & \xrightarrow{\delta} \mathrm{Hom}(\Lambda^2 E,\mathfrak{g}) & \xrightarrow{\delta} \mathrm{Hom}(\Lambda^3 E,F) & \\
3 & \mathfrak{g}^{(3)} & \hookrightarrow \mathrm{Hom}(E,\mathfrak{g}^{(2)}) & \xrightarrow{\delta} \mathrm{Hom}(\Lambda^2 E,\mathfrak{g}^{(1)}) & \xrightarrow{\delta} \mathrm{Hom}(\Lambda^3 E,\mathfrak{g}) & \xrightarrow{\delta} \mathrm{Hom}(\Lambda^4 E,F) \\
 & & & & \vdots &
\end{array} \tag{1.47}$$

The coboundary operator $\delta : \mathrm{Hom}(\Lambda^m E, \mathfrak{g}^{(l)}) \to \mathrm{Hom}(\Lambda^{m+1} E, \mathfrak{g}^{(l-1)})$ is defined by

$$\delta(\xi)(u_0, \ldots, u_m) := \sum_{i=0}^{m} (-1)^i \xi(u_0, \ldots, \hat{u}_i, \ldots, u_m)(u_i),$$

where $\hat{u}_i$ denotes the removal of the i-th term. Note that (1.16) is a special case of this formula. A simple computation shows that $\delta \circ \delta = 0$. The cocycles at $\mathrm{Hom}(\Lambda^m E, \mathfrak{g}^{(l)})$ are denoted by

$$Z^{l,m}(\mathfrak{g}) := \mathrm{Ker}\ \left(\delta : \mathrm{Hom}(\Lambda^m E, \mathfrak{g}^{(l)}) \to \mathrm{Hom}(\Lambda^{m+1} E, \mathfrak{g}^{(l-1)})\right).$$

By definition, $Z^{l,1}(\mathfrak{g}) = \mathfrak{g}^{(l+1)}$. The coboundaries at that same term are denoted by

$$B^{l,m}(\mathfrak{g}) := \mathrm{Im}\left(\delta : \mathrm{Hom}(\Lambda^{m-1}E, \mathfrak{g}^{(l+1)}) \to \mathrm{Hom}(\Lambda^m E, \mathfrak{g}^{(l)})\right),$$

and the induced cohomology group by

$$H^{l,m}(\mathfrak{g}) := Z^{l,m}(\mathfrak{g})/B^{l,m}(\mathfrak{g}).$$

By definition,

$$H^{l,1}(\mathfrak{g}) = 0 \quad \forall\, l \geq 0.$$

The resulting cohomology theory is called the **Spencer cohomology** of $\mathfrak{g}$. The indices l, m may be slightly confusing. It is convenient to remember that $Z^{l,m}(\mathfrak{g})$, $B^{l,m}(\mathfrak{g})$ and $H^{l,m}(\mathfrak{g})$ are located at row $l+m$ and column m.

Definition 1.5.3. *Let $r \geq 1$ be an integer. A tableau bundle $\mathfrak{g}$ is said to be r-**acyclic** if*

$$H^{l,m}(\mathfrak{g}) = 0 \qquad \forall\, 1 \leq m \leq r,\ l \geq 0.$$

*It is said to be **involutive** if it is r-acyclic for all $r \geq 1$.*

A fundamental fact, sometimes called the Cartan-Kuranishi prolongation theorem (although this usually refers to a similar theorem about prolongations in the theory of exterior differential systems), is the following theorem (see, for example, Lemma 2 in [21]):

Theorem 1.5.4. *Let $\mathfrak{g} \subset \mathrm{Hom}(E, F)$ be a tableau bundle. There exists l_0 such that $H^{l,m}(\mathfrak{g}) = 0$ for all $m \geq 1$ and $l \geq l_0$.*

In other words, this theorem says that, for large enough l_0, the tableau bundle $\mathfrak{g}^{l_0}$ is involutive.

Remark 1.5.5. The 2-acyclic condition, as the theorem below shows, appears as a sufficient condition for formal integrability of PDEs. As we will see later on in the thesis, the 3-acyclic condition plays a similar role in other formal integrability problems. The involutivity condition, on the other hand, appears as a sufficient condition for the existence of solutions of analytic PDEs. This explains the importance of Theorem 1.5.4 which says that after a sufficient number of prolongation steps any analytic PDE becomes integrable (i.e. admits solutions). The theorem, however, does not give a bound for the number of prolongation steps one must make. Let us also remark here that in the theory of exterior differential systems, a theory which deals with integrability problems in the analytic category, one typically uses an alternative definition for the involutivity condition in terms of a certain (non-canonical) sequence of numbers associated with a tableau that are known as *characters*. This alternative definition is, in fact, the original one that Cartan introduced in his theory of Pfaffian systems, the precursor of the theory of exterior differential systems. In [3] (p. 119), one can find the precise definition of involutivity in terms of characters, and in [64] (Proposition 4.6), one can find a proof due to Serre of the equivalence between the two definitions. $\diamondsuit$

Recall that given a tableau bundle $\mathfrak{g} \subset \mathrm{Hom}(A, B)$, its prolongations may fail to be vector bundles. One of the roles of the 2-acyclic condition is to ensure that the prolongations are indeed vector bundles:

Lemma 1.5.6. *Let $\mathfrak{g} \subset \mathrm{Hom}(E, F)$ be a tableau bundle over a connected manifold M. If $\mathfrak{g}$ is 2-acyclic and $\mathfrak{g}^{(1)} \subset \mathrm{Hom}(E, \mathfrak{g})$ is of constant rank, then $\mathfrak{g}^{(l)} \subset \mathrm{Hom}(E, \mathfrak{g}^{(l-1)})$ is of constant rank for all $l > 1$.*

Proof. The lemma is a consequence of the observation that for any exact sequence of vector bundles $E \xrightarrow{f} E' \xrightarrow{f'} E''$ over a connected manifold M, both $\mathrm{Ker}\ f$ and Coker f' are of constant rank. Indeed, because $\mathrm{Im}\ f = \mathrm{Ker}\ f'$ and because the rank of $\mathrm{Ker}\ f'$ is an upper semi-continuous function on M while the rank of $\mathrm{Im}\ f$ is a lower semi-continuous function on M, then both $\mathrm{Im}\ f$ and $\mathrm{Ker}\ f'$ are of constant rank on each connected component of M (and there is only one such component by assumption). The observation then follows from the fact that, by a dimension count, Rk$(\mathrm{Ker}\ f) =$ Rk$(E) -$ Rk$(\mathrm{Im}\ f)$ and Rk$(\text{Coker } f') =$ Rk$(E'') -$ Rk$(E') +$ Rk$(\mathrm{Ker}\ f')$.

In our case, $\mathfrak{g}^{(l)}$ is the kernel of the left map in the exact sequence

$$\mathrm{Hom}(E, \mathfrak{g}^{(l-1)}) \xrightarrow{\delta} \mathrm{Hom}(\Lambda^2 E, \mathfrak{g}^{(l-2)}) \xrightarrow{\delta} \mathrm{Hom}(\Lambda^3 E, \mathfrak{g}^{(l-3)}).$$

The lemma now follows by induction on l, starting with our assumption that $F = \mathfrak{g}^{(-1)}$, $\mathfrak{g} = \mathfrak{g}^{(0)}$ and $\mathfrak{g}^{(1)}$ are vector bundles. □

Remark 1.5.7. This proof is a simplification of the proof of Lemma 6.5 in [20], whose statement is slightly more general in that it allows $\mathfrak{g}$ to be the kernel of a vector bundle map that need not be of constant rank. ♢

The following theorem is a criterion for formal integrability given by Goldschmidt in [20] (Theorem 8.1). In the next section, we will present an alternative proof of this theorem.

Theorem 1.5.8. *Let $R \subset J^k P$ be a PDE. If*

1. *$\mathfrak{g}$ is 2-acyclic,*

2. *$\mathfrak{g}^{(1)}$ is of constant rank,*

3. *$\pi : R^{(1)} \to R$ is surjective,*

then R is formally integrable.

Remark 1.5.9. Strictly speaking, this theorem is weaker than Goldschmidt's theorem, although only slightly weaker, as we explain in Remark 1.6.3. ♢

1.6 A Proof of Goldschmidt's Formal Integrability Criterion

In this section, we will present a simple proof of Theorem 1.5.8 which relies solely on the properties of the Cartan form.

Let $R \subset J^k P$ be a PDE and recall that R is formally integrable if $\mathfrak{g}^{(l)}$ is of constant rank and $R^{(l)} \to R^{(l-1)}$ is surjective for all $l > 0$, which, by Proposition 1.4.9, is equivalent to $\mathfrak{g}^{(l)}$ being of constant rank and $R^{(l-1)}$ admitting an integral Cartan-Ehresmann connection for all $l > 0$. By Proposition 1.4.10, in turn, the problem of formal integrability reduces to the following problem: assuming that R is integrable up to order $k + l$, hence $R^{(l)} \subset J^{k+l}P$ is a PDE, prove that $\mathfrak{g}^{(l+1)}$ is of constant rank and prove that $R^{(l)}$ admits a Cartan-Ehresmann connection H whose weak curvature vanishes, i.e.

$$c_H = \delta\omega(H(\cdot), H(\cdot)) = 0.$$

What kind of object is c_H? Proposition 1.3.4 implies that $\delta\omega(H(\cdot), H(\cdot))$ takes values in $\mathfrak{g}^{(l-1)}$, thus

$$c_H \in \Gamma(\pi^*\mathrm{Hom}(s^*\Lambda^2 TM, \mathfrak{g}^{(l-1)})). \tag{1.48}$$

Now note that c_H lives in the Spencer complex associated with the symbol space $\mathfrak{g} \subset \mathrm{Hom}(s^*TM, E)$. Explicitly, the vector bundle $\mathrm{Hom}(s^*\Lambda^2 TM, \mathfrak{g}^{(l-1)})$ fits in the following sequence of vector bundles over R:

$$\mathrm{Hom}(s^*TM, \mathfrak{g}^{(l)}) \xrightarrow{\delta} \mathrm{Hom}(s^*\Lambda^2 TM, \mathfrak{g}^{(l-1)}) \xrightarrow{\delta} \mathrm{Hom}(s^*\Lambda^3 TM, \mathfrak{g}^{(l-2)}).$$

This is a piece of the Spencer complex of $\mathfrak{g}$, where, for $\xi \in \Gamma(\mathrm{Hom}(s^*TM, \mathfrak{g}^{(l)}))$, the coboundary operator δ is given by

$$\delta\xi(X, Y) = \xi(Y)(X) - \xi(X)(Y), \qquad \forall\, X, Y \in \mathfrak{X}(M),$$

while, for $\xi \in \Gamma(\mathrm{Hom}(s^*\Lambda^2 TM, \mathfrak{g}^{(l-1)}))$, it is given by

$$\delta\xi(X, Y, Z) = \xi(X, Y)(Z) + \xi(Y, Z)(X) + \xi(Z, X)(Y), \qquad \forall\, X, Y, Z \in \mathfrak{X}(M).$$

Recall that the resulting cohomology group at $\mathrm{Hom}(\Lambda^2 s^*TM, \mathfrak{g}^{(l-1)})$ is denoted by $H^{l-1,2}(\mathfrak{g})$ and that $\mathfrak{g}$ is 2-acyclic if $H^{l,2}(\mathfrak{g}) = 0$ for all $l \geq 0$.

Lemma 1.6.1. *Let $R \subset J^k P$ be a PDE with symbol space $\mathfrak{g}$ and assume that it is integrable up to order $k + l$. Then the weak curvature c_H of a Cartan-Ehresmann connection H on $R^{(l)}$ is a cocycle in the Spencer complex and its cohomology class $[c_H] \in H^{l-1,2}(\mathfrak{g})$ is independent of the choice of H.*

Proof. To show that the weak curvature is a cocycle, it is enough to prove the case $R = J^k P$. We must show that,

$$\delta\omega(H(s^*X), H(s^*Y))(s^*Z) + (\text{c.p. of } X, Y, Z) = 0,$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where "c.p." is short for "cyclic permutations". In the proof of Proposition 1.3.4, we explicitly computed $\delta\omega(H(\cdot), H(\cdot))$ by applying it to a pair of vector fields $\partial/\partial x_i$ and $\partial/\partial x_j$. Applying this on a third vector field $\partial/\partial x_l$ and taking the cyclic sum on i, j and l, we get

$$\Big(\big(a^\rho_{\alpha+j+l,i} - a^\rho_{\alpha+i+l,j}\big) + \big(a^\rho_{\alpha+i+j,l} - a^\rho_{\alpha+l+j,i}\big) + \big(a^\rho_{\alpha+l+i,j} - a^\rho_{\alpha+j+i,l}\big)\Big) \otimes \frac{\partial}{\partial y^\rho_\alpha},$$

where the components are indexed by $|\alpha| = k-2$, $\rho = 1, ..., p$. The terms vanish pairwise.

For the second claim, let H and H' be two Cartan Ehresmann connections. Then $H' - H \in \Gamma(\pi^*\mathrm{Hom}(s^*TM, \mathfrak{g}^{(1)}))$. Applying the coboundary operator δ,

$$\begin{aligned} \delta(H'-H)(X,Y) &= (H'-H)(Y)(X) - (H'-H)(X)(Y) \\ &= \delta\omega(H'(X), (H'-H)(Y)) - \delta\omega(H(Y), (H'-H)(X)) \\ &= \delta\omega(H'(X), H'(Y)) - \delta\omega(H(X), H(Y)) \\ &= c_H(X,Y) - c_{H'}(X,Y). \end{aligned}$$

□

With the aid of the Cartan form and its properties, the proof of Theorem 1.5.8 becomes a "simple computation". The strategy is as follows: at each step, start with some Cartan-Ehresmann connection and show that it can be "perturbed" to an integral Cartan-Ehresmann connection. At a point, this means perturbing an almost integral element of $R^{(l)}$ to an integral element.

Remark 1.6.2. In (1.46) we wrote down in local coordinates the condition for being an integral element of R. An illuminating exercise is to write down the condition for being an almost integral element and write down the equations which determine whether an almost integral element is an integral element. In doing this, one can "re-discover" the following proof. ♢

Proof of Theorem 1.5.8. By Lemma 1.5.6, the fact that $\mathfrak{g}^{(l)}$ is a vector bundle for all $l > 1$ is a consequence of the assumption that $\mathfrak{g}^{(1)}$ is a vector bundle and that $\mathfrak{g}$ is 2-acyclic.

We are left with showing that if R is formally integrable up to order $k+l$, with $l > 0$, then $R^{(l)} \subset J^{k+l}P$ admits an integral Cartan-Ehresmann connection. Let $H : s^*TM \to \mathrm{Ker}\ \omega$ be some Cartan-Ehresmann connection, not necessarily integral. We will choose a vector bundle map (over $R^{(l)}$)

$$\eta : s^*TM \to \pi^*\mathfrak{g}^{(l)}$$

so that the Cartan-Ehresmann connection $H + \eta$ is integral, or equivalently, so that, $\delta\omega((H+\eta)(\cdot), (H+\eta)(\cdot)) = 0$. Let $X, Y \in \Gamma(s^*TM)$, then

$$\delta\omega((H+\eta)(X), (H+\eta)(Y)) = \delta\omega(H(X), H(Y)) - \eta(X)(Y) + \eta(Y)(X), \quad (1.49)$$

by using the properties in Proposition 1.3.4. By Lemma 1.6.1, $\delta\omega(H(\cdot), H(\cdot))$ is a section of $H^{l-1,2}(\mathfrak{g})$, which we are assuming to vanish, and so we can find an

$$\eta \in \Gamma(\pi^*\mathrm{Hom}(s^*TM, \mathfrak{g}^{(l)}))$$

such that

$$\delta\eta = \delta\omega(H(\cdot), H(\cdot)).$$

For this choice of η, the right hand side of (1.49) vanishes. $\square$

Remark 1.6.3. Strictly speaking, the above theorem is slightly weaker than Theorem 8.1 in [20] because the very notion of a PDE in [20] is weaker than ours, namely it is defined as a submanifold $R \subset J^kP$ such that $s|_R : R \to M$ is a surjective submersion. The main implication of this weaker definition is that the symbol space $\mathfrak{g}$ may fail to be of constant rank. However, even with such a singular tableau bundle, one can still define the prolongations and the Spencer complex of $\mathfrak{g}$ in the same way by noting that the definitions are all pointwise in nature. In this more singular setting, one can reread our proof of Theorem 1.5.8 as is, with the only difference being that the application of Lemma 1.5.6 in the first paragraph of the proof should be replaced by an application of the more general Lemma 6.5 in [20] to prove that $\mathfrak{g}^{(l)}$ is a vector bundle for all $l > 1$. We have chosen our "smoother" definition of a PDE in order to avoid having to deal with issues that arise due to the presence of singularities. These often have the effect of hiding the nice geometry behind a messier language that one is forced to use in order to accommodate for singular objects. As we see, at least in the case of this theorem, none of the essential ideas are lost by adding the extra regularity conditions that we chose to impose in our definition of a PDE. $\diamondsuit$

1.7 PDEs of Finite Type

PDEs of finite type can be thought of as "Frobenius type" PDEs, that is, the ones for which the Frobenius theorem applies. In this short section, we briefly discuss such PDEs.

Definition 1.7.1. *A PDE $R \subset J^kP$ with symbol space $\mathfrak{g}$ is said to be of* ***finite type*** *l if there exists an integer $l > 0$ such that $\mathfrak{g}^{(l)} = 0$.*

Note that if a PDE $R \subset J^kP$ is of finite type l, then $\mathfrak{g}^{(l')} = 0$ for all $l' \geq l$. If the PDE is also integrable up to order $k + l$, then the projection $\pi : R^{(l)} \to R^{(l-1)}$ is an isomorphism.

Proposition 1.7.2. *Let $R \subset J^kP$ be a PDE of finite type l and integrable up to order $k+l$, then there exists a unique integral Cartan-Ehresmann connection on $R^{(l)}$. Moreover, it is flat.*

Proof. Uniqueness is clear, because $\pi : R^{(l)} \to R^{(l-1)}$, which is an isomorphism, only has a single section. Flatness follows directly from Lemma 1.4.11 (see also the discussion following the lemma). $\square$

By the Frobenius theorem, the existence of a flat Ehresmann connection implies the existence of local sections, which are, furthermore, maximal and unique, in the sense of Frobenius. Therefore:

Theorem 1.7.3 (Frobenius)**.** *Let $R \subset J^kP$ be a PDE of finite type l and integrable up to order $k+l$. For every $\xi \in R^{(l)}$ there exists a unique maximal local solution σ of R such that $j^{k+l}_x\sigma = \xi$.*

Remark 1.7.4. Here, maximal means that if there exists another local solution σ' of R around x satisfying $j^{k+l}_x\sigma' = \xi$, then $\mathrm{Dom}(\sigma') \subset \mathrm{Dom}(\sigma)$ and the two coincide on $\mathrm{Dom}(\sigma')$. $\diamondsuit$

Needless to say that, in the finite case, formal integrability up to order l implies formal integrability.

1.8 Linear PDEs and the Spencer Operator

A linear PDE is, roughly, a PDE $R \subset J^kE$ where $E \to M$ is a vector bundle and for which any linear combination of solutions is again a solution. In theory, we could regard a linear PDE as a special case of a PDE equipped with extra linear structure. That would be rather unnatural, however, like studying a vector space as a special type of manifold. Nevertheless, as we will see in this section, the two theories are completely analogous.

Linear Jet Bundles and the Spencer Operator Let $\pi : E \to M$ be a vector bundle. For each $k \geq 0$, we will call the k-th jet bundle J^kE associated with the surjective submersion $\pi : E \to M$ the **k-th linear jet bundle** of E. The vector bundle structure of E induces a vector bundle structure on $s : J^kE \to M$, namely for all $j^k_x\sigma, j^k_x\sigma' \in J^kE$ and $\lambda \in \mathbb{R}$,

$$j^k_x\sigma + j^k_x\sigma' = j^k_x(\sigma + \sigma') \qquad \text{and} \qquad \lambda\, j^k_x\sigma = j^k_x(\lambda\,\sigma).$$

As with any jet bundle, every (local) section $\sigma \in \Gamma(E)$ induces a (local) section $j^k\sigma \in \Gamma(J^kE)$,

$$j^k\sigma : x \mapsto j^k_x\sigma,$$

called a **(local) holonomic section** of J^kE, and an **integral element** of J^kE at $j^k_x\sigma$ is an element in the image of the inclusion

$$J^{k+1}E \hookrightarrow J^1(J^kE), \qquad j^{k+1}_x\sigma \mapsto j^1_x(j^k\sigma), \tag{1.50}$$

that projects to $j^k_x\sigma$.

For each $k > 0$, the projection

$$\pi : J^kE \to J^{k-1}E, \qquad j^k_x\sigma \mapsto j^{k-1}_x\sigma, \tag{1.51}$$

is a vector bundle map. Its kernel,

$$\mathfrak{g}^k := \mathrm{Ker}\,(\pi : J^kE \to J^{k-1}E) \subset J^kE,$$

is called the **symbol space** of J^kE (if we regard a linear jet bundle as a special case of a jet bundle, then this definition coincides with definition (1.26), since, as for any vector bundle, the vertical tangent spaces of $\mathfrak{g}^k$ are canonically isomorphic to its fibers). Thus, $\mathfrak{g}^k$ consists of all k-jets $j^k_x\sigma$ for which $j^{k-1}_x\sigma = 0_x$ (the zero $k-1$ jet at x), i.e. homogeneous Taylor polynomials of degree k of sections of E.

The inclusion (1.12) in the case of linear jet bundles simplifies and becomes an inclusion of vector bundles over M,

$$\mathfrak{g}^k \hookrightarrow \mathrm{Hom}(TM, \mathfrak{g}^{k-1}), \qquad T \mapsto \hat{T},$$

endowing the symbol space with the structure of a tableau bundle. The inclusion is described as follows: let $T \in \mathfrak{g}^k$, thus $T = j^k_x\sigma$ for some σ such that $j^{k-1}_x\sigma = 0$, then

$$\hat{T}(X) := (d(j^{k-1}\sigma))_x(X), \qquad \forall\, X \in T_xM.$$

The resulting vector $(d(j^{k-1}\sigma))_x(X)$ is an element of $T_{0_x}J^{k-1}E$ and it is killed by $d\pi$, and hence an element of $\mathfrak{g}^{k-1}$. From now on we will simply write $T = \hat{T}$.

In analogy to jet bundles,

$$\mathfrak{g}^k = (\mathfrak{g}^{k-1})^{(1)}$$

and the projection (1.51) is an affine bundle modeled on the vector bundle $s^*\mathfrak{g}^k$. This can be verified in local coordinates, or, after we introduce the Spencer operator, by a proof analogous to the proof of Lemma 1.4.8. Indeed, the role of the Cartan form of a jet bundle is played by the Spencer operator in the linear setting.

Proposition 1.8.1. *There is a unique bilinear map*

$$D : \mathfrak{X}(M) \times \Gamma(J^kE) \to \Gamma(J^{k-1}E), \qquad (X, \sigma) \mapsto D_X(\sigma),$$

*called the **Spencer operator** of J^kE, such that*

$$D_{fX}(\sigma) = fD_X(\sigma), \qquad D_X(f\sigma) = fD_X(\sigma) + X(f)\,\pi(\sigma), \tag{1.52}$$

for all $\sigma \in \Gamma(J^kE),\ f \in C^\infty(M)$, and

$$D_X(j^k\sigma) = 0 \qquad \forall\, X \in \mathfrak{X}(M),\ \sigma \in \Gamma(E).$$

Furthermore, D satisfies the property: if $\sigma \in \Gamma(J^kE)$, then σ is holonomic if and only $D_X(\sigma) = 0$ for all $X \in \mathfrak{X}(M)$.

Remark 1.8.2. Due to the $C^\infty(M)$-linearity in the first slot, D can also be viewed as a map

$$D : \Gamma(J^kE) \to \Omega^1(M; J^{k-1}E);$$

the two points of view are related by $D(s)(X) = D_X(s)$. $\diamondsuit$

Proof. Let $(x_1, ..., x_m)$ be the coordinates on M and $\{e_\rho\}$, with $\rho = 1, ..., p$, a local frame of E. Thus, locally, any vector in E at x_i can be expanded as $\sum_\rho y^\rho(e_\rho)_{x_i}$, where y^ρ are real coefficients. This induces a coordinate chart (x_i, y^ρ) on E, which induces a coordinate chart $(x_i, y^\rho_\alpha)_{|\alpha|\leq k-1}$ on $J^{k-1}E$ and $(x_i, y^\rho_\alpha)_{|\alpha|\leq k}$ on J^kE (see (1.5)). In turn, this induces a local frame $\{e_\rho^\alpha\}_{|\alpha|\leq k-1}$ on $J^{k-1}E$ and $\{e_\rho^\alpha\}_{|\alpha|\leq k}$ on J^kE. By the connection-like properties, D is determined by how it acts on this frame of J^kE. Acting on holonomic sections of J^kE induced by "monomial" sections of E of increasing order, i.e. $j^1(x^\alpha e_\rho)$ with $|\alpha| = 0$, then $|\alpha| = 1$, etc., we readily find that

$$\begin{aligned} &D(e_\rho) = 0 && \forall\ \rho, \\ &D(e_\rho^\alpha) = -\sum_i e_\rho^{\alpha-i} \otimes dx_i, && \forall\ \rho,\ 1 \leq |\alpha| \leq k, \end{aligned} \tag{1.53}$$

where we understand $e_\rho^{\alpha-i}$ to be zero if $\alpha_i = 0$. This proves the first assertion. The second assertion follows from this explicit formula. □

Remark 1.8.3. Regarding a linear PDE as a special case of a PDE, the Spencer operator on J^kE is induced by the Cartan form on J^kE. Indeed, since the vertical tangent spaces of $s : J^{k-1}E \to M$ are identified with the fibers, then the Cartan form ω induces the map $\Gamma(J^kE) \to \Omega(M; J^{k-1}E)$, $\sigma \mapsto \sigma^*\omega$, and one checks that this map satisfies the defining properties of D. ♢

The Spencer operators on two subsequent linear jet bundles commute with the projection $\pi : J^kE \to J^{k-1}E$, i.e.

$$D \circ \pi = \pi \circ D. \tag{1.54}$$

This is evident from the local expression (1.53), but it can also by proven by noting that: 1) both sides vanish on $\mathfrak{g}^k = \mathrm{Ker}\ \pi$, and 2) $D = \pi \circ D \circ \xi$, with ξ being some splitting of $\pi : J^kE \to J^{k-1}E$ (simply verify that the right hand side satisfies the defining properties of D).

Just like the Cartan form, any Spencer operator is the restriction of a Spencer operator on a 1st linear jet bundle by means of the inclusion $J^kE \subset J^1(J^{k-1}E)$ (c.f. Remark 1.3.2). This "universal" Spencer operator can also be described as follows: consider the short exact sequence of vector bundles

$$0 \to T^*M \otimes E \to J^1E \to E \to 0,$$

where the map on the right is the projection and the map on the left, at the level of sections, is defined by

$$df \otimes \sigma \mapsto f j^1\sigma - j^1(f\ \sigma).$$

This short exact sequence of vector bundles induces the following short exact sequence of $C^\infty(M)$-modules:

$$0 \to \Omega^1(M; E) \to \Gamma(J^1E) \to \Gamma(E) \to 0.$$

A right splitting of this short exact sequence is given by the 1st prolongation map $\sigma \mapsto j^1\sigma$, and the induced left splitting is precisely the Spencer operator.

In the linear case, the role of the Cartan distribution is played by the partial prolongation. Because of the second condition in (1.52), $D(\eta)(x) \in \mathrm{Hom}(TM, J^{k-1}E)$ depends only on $j^1_x\eta$ (i.e. it is a 1st order differential operator), and we have an induced vector bundle map

$$j^1D : J^1(J^kE) \to \mathrm{Hom}(TM, J^{k-1}E), \qquad j^1_x\sigma \mapsto D(\eta)(x).$$

Its kernel, the vector bundle

$$J^1_D(J^kE) := \{ \, j^1_x\eta \in J^1(J^kE) \mid D(\eta)(x) = 0 \, \} \subset J^1(J^kE)$$

over M is called the **partial prolongation** of J^kE. The projection

$$\pi : J^1_D(J^kE) \to J^kE, \qquad j^1_x\eta \mapsto \eta(x),$$

is an affine bundle modeled on $s^*\mathrm{Hom}(TM, \mathfrak{g})$, where $s : J^kE \to M$ is the source map. An element of $J^1_D(J^kE)$ projecting to $j^k_x\sigma$ is called an **almost integral element** of J^kE at $j^k_x\sigma$. From Proposition 1.8.1, we see that an integral element is an almost integral element. Thus, we have inclusions of affine bundles

$$J^{k+1}E \subset J^1_D(J^kE) \subset J^1(J^kE).$$

Regarding a linear PDE as a PDE, the Cartan distribution of J^kE would be the distribution in J^kE spanned by the image of all linear maps of the type $(d\eta)_x : T_xM \to T_{\eta(x)}J^kE$, with $j^1_x\eta \in J^1_D(J^kE)$.

As for jet bundles, we have the notion of the curvature of a linear jet bundle. Let $X, Y \in \mathfrak{X}(M)$ and $\eta \in \Gamma(J^kE)$. Due to conditions (1.52), the expression

$$(D_X \circ D_Y - D_Y \circ D_X - \pi \circ D_{[X,Y]})(j^1\eta)(x)$$

depends only on $j^1_x\eta, X_x$ and Y_x. Note that in this expression, there are two different Spencer operators involved: the one on $J^1(J^kE)$ and the one on J^kE. The expression above defines a vector bundle map

$$c_D : J^1_D(J^kE) \to \mathrm{Hom}(\Lambda^2TM, J^{k-1}E)$$

called the **curvature** of J^kE.

Proposition 1.8.4. *The Spencer operator D on J^kE satisfies the following properties:*

1. *c_D takes values in $\mathfrak{g}^{k-1}$, i.e.*

$$c_D : J^1_D(J^kE) \to \mathrm{Hom}(\Lambda^2TM, \mathfrak{g}^{k-1}).$$

2. *$j^1_x\eta \in J^1(J^kE)$ is an integral element if and only*

$$D(\eta)(x) = 0 \qquad \textit{and} \qquad c_D(j^1_x\eta)(X, Y) = 0.$$

3. *$T(X) = -D_X(T)$ for all pairs $(X, T) \in TM \times_M \mathfrak{g}^k$.*

The proof is by means of the local coordinate expression (1.53) of D and is along the same lines as the proof of Proposition (1.3.4) for the Cartan form.

Linear PDEs Classically, a linear PDE is a PDE satisfying the property that any linear combination of solutions is again a solution. Geometrically, this means that the PDE should have the structure of a vector bundle. As in the definition of a PDE, we will impose regularity conditions that will allow us to study linear PDEs intrinsically.

Let $\pi : E \to M$ be a vector bundle and $R \subset J^k E$ a vector subbundle. Let us write

$$\mathfrak{g} := \mathrm{Ker}\ (\pi|_R : R \to J^{k-1}E) = R \cap \mathfrak{g}^k \quad \text{and} \quad F := \mathrm{Im}\ (\pi|_R : R \to J^{k-1}E).$$

Furthermore, we have the restriction of the Spencer operator of $J^k E$,

$$D : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(J^{k-1}E),$$

and the restriction of the partial prolongation of $J^k P$,

$$J^1_D R := \{\ j^1_x \eta \in J^1 R \mid D(\eta)(x) = 0\ \} \subset J^1 R, \tag{1.55}$$

together with the projection

$$\pi : J^1_D R \to R, \qquad j^1_x \eta \mapsto \eta(x).$$

Definition 1.8.5. *A **linear PDE** of order $k > 0$ on a vector bundle $\pi : E \to M$ is a vector subbundle $R \subset J^k E$ such that:*

1. *$\mathfrak{g}$ is a vector bundle (and hence F is a vector bundle).*

2. *$\pi : J^1_D R \to R$ is surjective.*

*A **(local) solution** of R is a (local) section σ of $\pi : E \to M$ satisfying: $j^k_x \sigma \in R$ for all $x \in \mathrm{Dom}(\sigma)$.*

As a consequence of the definition:

Lemma 1.8.6. *Let $R \subset J^k E$ be a linear PDE. Then*

1. *D takes values in $\Gamma(F)$ and hence we have an induced operator on R,*

$$D : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(F).$$

2. *$J^1_D R$ is a vector bundle.*

Proof. Fixing an element $j^1_x \eta \in J^1 R$, any other element in the same fiber of $\pi : J^1 R \to R$ can be realized as $j^1_x(f\eta)$ for some $f \in C^\infty(M)$. Because of condition 2 in the definition of a linear PDE, any fiber of $\pi : J^1 R \to R$ contains an element of $J^1_D R$, and, thus, any element in $J^1 R$ can be written as $j^1_x(f\eta)$ with $D(\eta)(x) = 0$. This implies, by (1.52), that $D_X(f\eta)(x) \in F$ for all $X \in \mathfrak{X}(M)$.

For the second item, note that $J^1_D R$ is the kernel of the map

$$j^1 D : J^1 R \to \mathrm{Hom}(TM, F), \qquad j^1_x \eta \mapsto D(\eta)(x). \tag{1.56}$$

We will show that this vector bundle map is surjective. Let $x \in M$, choose coordinates $(x_1, ..., x_m)$ on M around x, choose a local frame $\{f^j\}$ of F around x and let ξ be a splitting of $\pi : R \to F$. Now, if $A^i_j \; dx_i \otimes f^j|_x$ is an element of $\mathrm{Hom}(TM, F)_x$, then using (1.52),

$$D(x_i A^i_j (\xi \circ f^j))(x) = A^i_j \; dx_i \otimes f^j|_x.$$

$\square$

Our definition of a linear jet bundle together with the lemma ensure that the structure of the linear jet bundle restricts "nicely" to the linear PDE. Let $R \subset J^k E$ be a linear PDE. Recall that

$$F = \mathrm{Im}\ (\pi|_R : R \to J^{k-1}E).$$

The **source map** of R and the projection are denoted by

$$s := s|_R : R \to M \qquad \text{and} \qquad \pi = \pi|_R : R \to F.$$

The kernel of the projection,

$$\mathfrak{g} = \mathrm{Ker}\ (\pi : R \to F) = R \cap \mathfrak{g}^k,$$

is called the **symbol space** of R. By definition, both $\mathfrak{g}$ and F are vector bundles. The restriction of the Spencer operator of the ambient linear jet bundle,

$$D : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(F)$$

is called the **Spencer operator** of R. The symbol space $\mathfrak{g}$ inherits the tableau bundle structure of $\mathfrak{g}^k$,

$$\mathfrak{g} \subset \mathrm{Hom}(TM, F),$$

which, by Proposition 1.8.4, is encoded in the Spencer operator:

$$T(X) = -D_X(T), \qquad \forall x \in M,\ T \in \mathfrak{g}_x,\ X \in T_xM.$$

The space

$$J^1_D R = \{\ j^1_x \eta \in J^1 R \mid D(\eta)(x) = 0\ \} \subset J^1 R$$

is called the **partial prolongation** of R. It is an affine bundle modeled on the vector bundle $s^*\mathrm{Hom}(TM, \mathfrak{g})$. An element $j^1_x\eta \in J^1_D R$ projecting to $\eta(x) \in R$ is called an **almost integral element** of R at $\eta(x)$. The space

$$R^{(1)} := \{\ j^1_x \eta \in J^1_D R \mid c_D(j^1_x\eta) = 0\ \} \subset J^1_D R$$

is called the **1st prolongation** of R. By Proposition 1.8.4, $R^{(1)} = J^1(R) \cap J^{k+1}E$. We denote the projection by

$$\pi : R^{(1)} \to R, \qquad j^1_x\eta \mapsto \eta(x).$$

Its fibers are affine spaces modeled on $s^*\mathfrak{g}^{(1)}$. An element $j^1_x\eta \in R^{(1)}$ projecting to $\eta(x) \in R$ is called an **integral element** of R at $\eta(x)$.

A **Cartan linear connection** on R is a splitting $H : R \to J^1_D R$ of the surjective vector bundle map

$$\pi : J^1_D R \to R.$$

The composition of H with the Spencer operator of $J^1_D R$ induces a linear connection on R,

$$\nabla^H : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(R), \qquad \nabla^H_X(\eta) := D_X(H \circ \eta).$$

Cartan linear connections are the linear versions of Cartan-Ehresmann connections on PDEs and they play the same key role in the problem of formal integrability. The main question of formal integrability is whether there exists a Cartan linear connection on R that takes values in $R^{(1)}$. By Proposition 1.8.4, the failure of H to take values in $R^{(1)}$ is measured by the projection of the curvature of ∇^H,

$$\pi \circ (\nabla^H_X \circ \nabla^H_Y - \nabla^H_Y \circ \nabla^H_X - \nabla^H_{[X,Y]}).$$

This expression defines a vector bundle map

$$c_H : R \to \mathrm{Hom}(\Lambda^2 TM, F) \tag{1.57}$$

called the **weak curvature** of H (which is clearly weaker than the usual curvature of ∇^H). Thus, H takes values in $R^{(1)}$ if and only if $c_H = 0$.

At this point, precisely as for PDEs, we can define the notion of higher prolongations of R and the notion of formal integrability of R. The proof of the formal integrability theorem analogous to Theorem 1.5.8 will be along the same lines, resulting in an alternative proof to Quillen's formal integrability theorem, Proposition 14.5 in [59]. We invite the reader to fill in the details.

1.9 The Abstract Approach to Linear/Non-Linear PDEs: Pfaffian Bundles

The Cartan form on a PDE, as we saw in Proposition 1.4.5, and the Spencer operator on a linear PDE, as we saw in Proposition 1.8.1, satisfy the property that they detect solutions, suggesting that the essential structures of a PDE and of a linear PDE is encoded in the Cartan form and in the Spencer operator, respectively. Further pointing in this direction is the fact that the proof of the formal integrability theorem, Theorem 1.5.8, relies solely on the properties of the Cartan form. These observations motivate the point of view of studying PDEs abstractly, i.e. not as subspaces of a jet bundle but rather as spaces equipped with a "PDE structure". Such a program was carried out in [62]. In this section, we briefly recall this abstract point of view.

Pfaffian Bundles Given a surjective submersion $s : R \to M$, a vector bundle $E \to R$ and a 1-form $\omega \in \Omega^1(R; E)$, we set:

$$C_\omega := \mathrm{Ker}\,\omega \subset TR.$$

Definition 1.9.1. *A* ***Pfaffian bundle*** *over a manifold M consists of a surjective submersion $s : R \to M$ and a pointwise surjective 1-form $\omega \in \Omega^1(R; E)$, with values in a vector bundle $E \to R$, such that:*

1. $ds\big|_{C_\omega} : C_\omega \to s^*TM$ *is pointwise surjective (equivalently, $C_\omega + \mathrm{Ker}\; ds = TR$)*,
2. $\mathfrak{g} := C_\omega \cap \mathrm{Ker}\; ds$ *is an involutive distribution.*

A ***(local) solution*** *of R is a (local) section η of $s : R \to M$ that satisfies $\eta^*\omega = 0$.*

Example 1.9.2. A PDE $R \subset J^kP$ can be interpreted as a Pfaffian bundle, where we take ω to be the Cartan form on R. ♢

Indeed, most (if not all) of the theory presented in this chapter can be developed using solely the data of a Pfaffian bundle. Let us look at some of the immediate consequences of the definition. Let (R, ω) be a Pfaffian bundle. The vector bundle

$$\mathfrak{g} = C_\omega \cap \mathrm{Ker}\; ds \tag{1.58}$$

is called the **symbol space** of (R, ω). As for PDEs,

$$\delta\omega \in \Gamma(\mathrm{Hom}(\Lambda^2 C_\omega, E))$$

will denote the canonical restriction of $d_\nabla \omega$ to the kernel C_ω of ω, where ∇ is some connection on E. A **Cartan-Ehresmann connection** H on R is a splitting of the vector bundle map $ds|_{C_\omega} : C_\omega \to s^*TM$. By the first axiom of the definition, Cartan-Ehresmann connections always exist. Choosing a Cartan-Ehresmann connection H, the **symbol map** of ω is defined by

$$\partial_\omega : \mathfrak{g} \to \mathrm{Hom}(s^*TM, E), \qquad T \mapsto \delta\omega(H(\cdot), T).$$

It is independent of the choice of H due to the second axiom of a Pfaffian bundle. In the example of PDEs, this map is injective and defines the tableau bundle structure of $\mathfrak{g}$.

Definition 1.9.3. *A Pfaffian bundle R is* ***standard*** *if ∂_ω is injective.*

For general Pfaffian bundles, this map may fail to be injective and then $\mathfrak{g}$ fails to be a tableau bundle in the usual sense. However, with some slight modifications, the constructions one encounters in the theory of PDEs generalize to this setting in which one has a map $\mathfrak{g} \to \mathrm{Hom}(s^*TM, E)$ rather than an inclusion. For example, one defines the 1st prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ as the subspace of $\mathrm{Hom}(s^*TM, \mathfrak{g})$ whose fiber at $y \in R$ is

$$\mathfrak{g}^{(1)}_y := \{\ \xi \in \mathrm{Hom}_y(s^*TM, \mathfrak{g}) \ \mid\ \partial_\omega(\xi(X))(Y) = \partial_\omega(\xi(Y))(X) \quad \forall X, Y \in (s^*TM)_y\ \}.$$

The definitions of the higher prolongations $\mathfrak{g}^{(l)}$ and of the Spencer complex of $\mathfrak{g}$ are adapted in a similar fashion by adding ∂_ω in the appropriate places.

The data of a Pfaffian bundle R also allows us to talk about prolongations of R. Although we do not have the $k+1$-jet bundle at our disposal, one can still make sense of the notion of an integral element. An **integral element** of R is an element $j^1_x\eta \in J^1R$ that satisfies

$$\omega((d\eta)_x(\cdot)) = 0 \qquad \text{and} \qquad \delta\omega((d\eta)_x(\cdot), (d\eta)_x(\cdot)) = 0.$$

The **1st prolongation** $R^{(1)}$ of R is the space of all integral elements of R. Thus,

$$R^{(1)} := \{\ \xi \in J^1(R) \mid \omega(\xi(\cdot)) = 0,\ \delta\omega(\xi(\cdot), \xi(\cdot)) = 0\ \} \subset J^1R,$$

where recall that we view J^1R as sitting inside $\mathrm{Hom}(s^*TM, TR)$ by (1.2). One may now proceed to show that $\pi : R^{(1)} \to R$ is an affine bundle modeled on the fibers of $\mathfrak{g}^{(1)}$, as in Lemma 1.4.8. We say that a Cartan-Ehresmann connection H on R is **integral** if $\delta\omega(H(\cdot), H(\cdot)) = 0$. As for PDEs, if $\mathfrak{g}^{(1)}$ is a vector bundle, then $\pi : R^{(1)} \to R$ is an affine bundle modeled on $\mathfrak{g}^{(1)}$ if and only if R admits an integral Cartan-Ehresmann connection. If $R^{(1)}$ is an affine bundle, then it inherits the Cartan form of J^1R with which it becomes a Pfaffian bundle. In this way, we can inductively define the higher prolongations of R and consider the problem of formal integrability in the abstract setting of Pfaffian bundles.

Remark 1.9.4. The main advantage of this abstract picture is not in the generalization that it provides for the notion of a PDE but rather in that it highlights the essential properties of a PDE. For example, it provides us with extra insight on the classical problem of formal integrability of PDEs. $\diamondsuit$

Linear Pfaffian Bundles Similarly, one defines the abstract notion of a linear PDE. To start off, we need an object that will play the role of the Spencer operator, a connection-like operator.

Definition 1.9.5. *Let R and F be vector bundles over M and let $\pi : R \to F$ be a surjective vector bundle map. A π-**connection** on R is a bilinear map*

$$D : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(F) \tag{1.59}$$

such that

$$D_{fX}(\eta) = fD_X(\eta), \qquad D_X(f\eta) = fD_X(\eta) + X(f)\ (\pi \circ \eta), \tag{1.60}$$

for all $X \in \mathfrak{X}(M), \eta \in \Gamma(R)$ and $f \in C^\infty(M)$

Let $D : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(F)$ be a π-connection. The surjectivity of π implies that the vector bundle map

$$j^1D : J^1R \to \mathrm{Hom}(TM, F), \qquad j^1_x\eta \mapsto D(\eta)(x),$$

is surjective (a simple exercise involving the second condition in (1.60), see also the proof of lemma 1.8.6). Hence,

$$J^1_DR := \{j^1_x\eta \in J^1R \mid D(\eta)(x) = 0\} \subset J^1R, \tag{1.61}$$

the **partial prolongation** of D, is a vector bundle. We denote the projection by

$$\pi : J^1_DR \to R, \qquad j^1_x\eta \mapsto \eta(x).$$

Definition 1.9.6. *A* ***linear Pfaffian bundle*** *over a manifold M is a vector bundle R over M equipped with a surjective vector bundle map $\pi : R \to F$ onto a vector bundle F over M and a π-connection*

$$D : \mathfrak{X}(M) \times \Gamma(R) \to \Gamma(F).$$

A ***(local) solution*** *of R is a (local) section η of R that satisfies $D(\eta) = 0$.*

Let (R, D) be a linear Pfaffian bundle. We call the operator D the **Spencer operator**. The vector bundle

$$\mathfrak{g} := \mathrm{Ker}\ (\pi : R \to F)$$

is called the **symbol space** of $\mathfrak{g}$, and the vector bundle map

$$\partial_D : \mathfrak{g} \to \mathrm{Hom}(TM, F), \qquad T \mapsto \big(X \mapsto -D_X(T)\big), \tag{1.62}$$

is called the **symbol map**.

Definition 1.9.7. *A linear Pfaffian bundle R is* ***standard*** *if ∂_D is injective.*

Example 1.9.8. A linear PDE $R \subset J^kE$, as in Definition 1.8.5, is a standard linear Pfaffian bundle. $\diamondsuit$

As in the case of Pfaffian bundles, this data is all one needs in order to study the notions of prolongations and formal integrability of linear PDEs. We refer the reader to [62] for more details. In particular, linear Pfaffian bundles are discussed in Chapter 2 and general Pfaffian bundles are discussed in Chapter 3.

Chapter 2

Jet Groupoids and Jet Algebroids

PDEs that define Lie pseudogroups have a yet richer structure than the general PDEs that were discussed in the previous chapter; namely, they are multiplicative, in the sense that they have the structure of a Lie groupoid, and their Cartan forms are compatible with the Lie groupoid structure, i.e. they are multiplicative forms.

In this second introductory chapter, continuing to build up towards the definition of a Lie pseudogroup, we begin by recalling the definitions of a Lie groupoid and a Lie algebroid and discuss some important notions and examples. We present the definition of a pseudogroup, and then move on to discuss the notion of a jet groupoid (the multiplicative version of a jet bundle) and the notion of a jet algebroid (the multiplicative version of a linear jet bundle and the infinitesimal counterpart of a jet groupoid). At the end of the chapter, we briefly discuss the notions of a Pfaffian groupoid and a Pfaffian algebroid (the multiplicative versions of a Pfaffian bundle and a linear Pfaffian bundle that we saw in Section 1.9).

2.1 Lie Groupoids and Lie Algebroids

Let us begin with a review of Lie groupoids and Lie algebroids, key ingredients in the theory of Lie pseudogroups. The purpose of the chapter is to both fix notation as well as to recall some notions and examples that will be used throughout the chapter and later on in the thesis. We refer the reader to [11, 53, 47, 15] for more on this subject.

Lie Groupoids A **groupoid** is a small category whose arrows are all invertible. It is denoted by $\mathcal{G} \rightrightarrows M$ or simply $\mathcal{G}$, where M is the space of objects and $\mathcal{G}$ the space of arrows. Its structure is encoded by the following structure maps (which satisfy the axioms of a category):

$s : \mathcal{G} \to M$		(source)
$t : \mathcal{G} \to M$		(target)
$m : \mathcal{G}_2 \to \mathcal{G}$	$(g, h) \mapsto g \cdot h$	(multiplication)
$i : \mathcal{G} \to \mathcal{G}$	$g \mapsto g^{-1}$	(inverse)
$u : M \to \mathcal{G}$	$x \mapsto 1_x$	(unit)

Here,

$$\mathcal{G}_2 := \mathcal{G}_s \times_t \mathcal{G} \subset \mathcal{G} \times \mathcal{G}$$

is the **space of composable arrows** of $\mathcal{G}$. It is the subset of $\mathcal{G} \times \mathcal{G}$ consisting of all pairs (g, h) that satisfy $t(h) = s(g)$.

A **Lie groupoid** is a groupoid $\mathcal{G} \rightrightarrows M$ for which M and $\mathcal{G}$ are manifolds, s and t are smooth submersions (from which it follows that the space of composable arrows $\mathcal{G}_2 \subset \mathcal{G} \times \mathcal{G}$ is a submanifold), and m, i, u are smooth maps. In general, one does not require that $\mathcal{G}$ be Hausdorff. Indeed, in some natural examples this fails to be the case (e.g. the Lie groupoid of germs of local diffeomorphisms of a manifold, see Section 4.2). However, one does require that the fibers of the source map be Hausdorff (this is all that is needed in order to make sense of the induced Lie algebroid, flows of sections, etc.).

The axioms of a Lie groupoid imply that the unit map $u : M \to \mathcal{G}$ is an embedding. When there is no source of confusion, we will denote the image of u, the submanifold of units of $\mathcal{G}$, by M. For example, $T\mathcal{G}|_M$ will denote the restriction of the vector bundle $T\mathcal{G}$ to $u(M)$. The subset $s^{-1}(x) \subset \mathcal{G}$ is called the s**-fiber** at $x \in M$, and the subset $t^{-1}(y) \subset \mathcal{G}$ is called the t**-fiber** at $y \in M$; both are embedded submanifolds. We say that $\mathcal{G}$ is s**-connected** if all of its s-fibers are connected and t**-connected** if all of its t-fibers are connected. The embedded submanifold $\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x) \subset \mathcal{G}$ is called the **isotropy group** at $x \in M$. Equipped with the restrictions of the multiplication and inverse maps, the isotropy group $\mathcal{G}_x$ has the structure of a Lie group, and equipped with the right action of $\mathcal{G}_x$ on $s^{-1}(x)$ induced by the multiplication of $\mathcal{G}$, the s-fiber $s^{-1}(x)$ has the structure of a right principal $\mathcal{G}_x$-bundle. Similarly, each t-fiber has the structure of a left principal bundle ([53], Theorem 5.4). For each $x \in M$, the subset $\mathcal{O}_x := t(s^{-1}(x)) \subset M$ is an immersed submanifold called the **orbit** through x. It consists of all points that are connected to x by some arrow in $\mathcal{G}$. The orbits partition M into a union of immersed submanifolds. The space of orbits $M/\mathcal{G}$ is a topological space equipped with the quotient topology called the **orbit space** of $\mathcal{G}$. A Lie groupoid is said to be **transitive** if it has a single orbit, i.e. if its orbit space consists of a single point.

A **Lie groupoid map** (F, f) from $\mathcal{G} \rightrightarrows M$ to $\mathcal{H} \rightrightarrows N$ is a pair of smooth maps $F : \mathcal{G} \to \mathcal{H}$ and $f : M \to N$ that preserve the groupoid structures, or categorically, a functor. A **Lie subgroupoid** of $\mathcal{G} \rightrightarrows M$ is a Lie groupoid $\mathcal{H} \rightrightarrows N$ together with a Lie groupoid map (F, f) such that F (and hence f) is an injective immersion. As an example, given a Lie groupoid $\mathcal{G} \rightrightarrows M$ and an orbit $\mathcal{O} \subset M$, we can restrict $\mathcal{G}$ to $\mathcal{O}$ by taking all arrows that begin (and hence end) in $\mathcal{O}$. The resulting Lie groupoid, denoted by $\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{O}$, is a Lie subgroupoid of $\mathcal{G}$ with the inclusion maps. Finally, a Lie subgroupoid $\mathcal{H} \subset \mathcal{G}$ is said to be **wide** if it has the same units as $\mathcal{G}$, i.e. if f is surjective.

Here are some basic (but important) examples of Lie groupoids:

Example 2.1.1.

1. Given a manifold M, we have the **pair groupoid** $M \times M \rightrightarrows M$ of M. An arrow (y, x) has source x and target y, the product is given by

$$(z, y) \cdot (y, x) = (z, x),$$

the inverse by $(y, x)^{-1} = (x, y)$ and the unit at x is (x, x).

2. Given a surjective submersion $\pi : P \to N$, we have the **submersion groupoid** $P \times_N P \to P$. It is the (wide) Lie subgroupoid of $P \times P \rightrightarrows P$ consisting of all arrows $(p, q) \in P \times P$ for which $\pi(p) = \pi(q)$.

3. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we have the **tangent groupoid** $T\mathcal{G} \rightrightarrows TM$ of $\mathcal{G}$ whose structure maps are the differentials of the structure maps of $\mathcal{G}$. It is the Lie groupoid obtained by applying the so called "tangent functor".

4. Any Lie group G can be regarded as a Lie groupoid $G \rightrightarrows \{*\}$ over a point $\{*\}$ with the obvious structure maps.

5. Given a Lie group G and an action on a manifold M,

$$G \times M \to M, \qquad (g, x) = g \cdot x,$$

we construct the **action groupoid** $G \ltimes M \rightrightarrows M$. Its space of arrows is the product $G \ltimes M = G \times M$, with $s(g, x) = x$ and $t(g, x) = g \cdot x$. The product is given by $(h, y) \cdot (g, x) = (hg, x)$, the inverse by $(g, x)^{-1} = (g^{-1}, g \cdot x)$ and the unit at x is (e, x), where e is the identity of G.

$\diamondsuit$

Bisections and Local Bisections A Lie groupoid $\mathcal{G}$ has the simultaneous structure of two fibrations (i.e., surjective submersions) over M, one by the source map and one by the target map, and one can talk about "simultaneous" sections of both maps in the following sense: a **bisection** of $\mathcal{G}$ is a section σ of $s : \mathcal{G} \to M$ such that $t \circ \sigma$ is a diffeomorphism of M. We denote the set of bisections of $\mathcal{G}$ by $\mathrm{Bis}(\mathcal{G})$. The set of bisections forms a group. Given two bisections, the product $\sigma \cdot \sigma'$ is the bisection defined by composing the arrows at the image, i.e.

$$(\sigma \cdot \sigma')(x) = \sigma(t \circ \sigma'(x)) \cdot \sigma'(x), \qquad \forall\, x \in M. \tag{2.1}$$

The unit map u serves as the identity bisection for this product, and the inverse of a bisection σ is the bisection σ^{-1} given by

$$\sigma^{-1}(x) = \sigma((t \circ \sigma)^{-1}(x))^{-1}, \qquad \forall\, x \in M. \tag{2.2}$$

A **local bisection** of $\mathcal{G}$ is a local section σ of $s : \mathcal{G} \to M$ such that $t \circ \sigma$ is a diffeomorphism between its domain and image (what we will call a *local diffeomorphism*, see next section). Given any $g \in \mathcal{G}$, there exists a local bisection σ of $\mathcal{G}$ through g, i.e. for which $\sigma(s(g)) = g$ (Proposition 1.4.9 in [47]). Using the same formulas as for bisection, we can define the product and inverse of local bisections, with the difference that the product $\sigma \cdot \sigma'$ of two local bisections is only defined if $\mathrm{Im}(t \circ \sigma') \subset \mathrm{Dom}(\sigma)$. We denote the set of all local bisections of $\mathcal{G}$ by $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$.

Remark 2.1.2. With this partially defined product operation, inverse operation and identity bisection, $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ has the structure of what we will later define to be a *generalized pseudogroup* (see Definition 3.6.1). $\diamondsuit$

Example 2.1.3. Consider the pair groupoid $M \times M \rightrightarrows M$ from Example 2.1.1. We denote the diffeomorphism group of M by Diff(M). Bisections of $M \times M$ are the same thing as diffeomorphisms of M. More precisely, there is a canonical bijection

$$\mathrm{Bis}(M \times M) \xrightarrow{\simeq} \mathrm{Diff}(M), \qquad \sigma \mapsto t \circ \sigma,$$

whose inverse maps a diffeomorphism ϕ to its graph, i.e. the bisection $M \to M \times M,\ x \mapsto (\phi(x), x)$.

We denote by $\mathrm{Diff}_{\mathrm{loc}}(M)$ the set of all local diffeomorphisms of M (by which we mean diffeomorphisms between open subsets of M, see Section 3.1). Local bisections of $M \times M$ are the same thing as local diffeomorphisms of M, i.e. there is a canonical bijection

$$\mathrm{Bis}_{\mathrm{loc}}(M \times M) \xrightarrow{\simeq} \mathrm{Diff}_{\mathrm{loc}}(M), \qquad \sigma \mapsto t \circ \sigma. \qquad \diamondsuit$$

Lie Algebroids A **Lie algebroid** is a vector bundle $A \to M$ equipped with an anti-symmetric bilinear map $[\cdot,\cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ (the **bracket**) and a vector bundle map $\rho : A \to TM$ (the **anchor**) that satisfy:

$$[\alpha, f\beta] = f[\alpha,\beta] + \rho(\alpha)(f)\beta \quad \text{and} \quad [[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] = 0,$$

for all $\alpha, \beta, \gamma \in \Gamma(A)$ and $f \in C^\infty(M)$. We denote a Lie algebroid by $A \to M$ or simply A. A Lie algebroid is said to be **transitive** if ρ is surjective. For every $x \in M$, the vector space $\mathfrak{g}_x(A) := \mathrm{Ker}\ \rho|_{A_x} \subset A_x$, equipped with the restriction of the bracket of A, has the structure of a Lie algebra and is called the **isotropy algebra** at x.

Given two Lie algebroids A and A' over the same base M, a vector bundle map that covers the identity map of M is called a **Lie algebroid map** if it induces a Lie algebra homomorphism at the level of sections and commutes with the anchors. The notion of a Lie algebroid map that does not cover the identity map or, more generally, between two Lie algebroids over different bases is more subtle and we refer the reader to [11] for more details.

Here are some basic examples of Lie algebroids:

Example 2.1.4.

1. Given a manifold M, its tangent bundle TM is a (transitive) Lie algebroid when equipped with the bracket of vector fields and the identity map as the anchor.

2. Given a surjective submersion $\pi : P \to M$, the vector bundle $T^\pi P := \mathrm{Ker}\ (d\pi : TP \to \pi^*TM)$, equipped with the bracket of vector fields and the inclusion $T^\pi P \hookrightarrow TP$ as the anchor, is a Lie algebroid.

3. Let $\pi : A \to M$ be a Lie algebroid. The tangent bundle TA has the structure of a Lie algebroid over TM. It is called the **tangent algebroid** of A (see [47], section 9.7, for details).

4. Any Lie algebra $\mathfrak{g}$ can be regarded as a Lie algebroid over a point $\{*\}$ (with the zero anchor map).

5. Given a Lie algebra $\mathfrak{g}$ together with an infinitesimal action of $\mathfrak{g}$ on a manifold M, i.e. a Lie algebra homomorphism

$$\rho : \mathfrak{g} \to \mathfrak{X}(M),$$

we construct the **action algebroid** $\mathfrak{g} \ltimes M \to M$. It is the trivial vector bundle $M \times \mathfrak{g} \to M$ equipped with the anchor induced by the action ρ and a bracket determined by the bracket of $\mathfrak{g}$ (which determines what happens on constant sections and extended by the Leibniz identity). ♢

The Lie Algebroid of a Lie Groupoid As with Lie groups and Lie algebras, any Lie groupoid $\mathcal{G} \rightrightarrows M$ induces a Lie algebroid $A = A(\mathcal{G})$ over M by "linearization". The construction is analogous to the case of Lie groups with two main differences to keep in mind: a Lie groupoid has many units, and right translation maps,

$$R_g : s^{-1}(t(g)) \to s^{-1}(s(g)), \qquad R_g(h) := hg,$$

are only defined along s-fibers. Starting with a Lie groupoid $\mathcal{G} \rightrightarrows M$, one defines $A \to M$ to be the vector bundle whose fiber at $x \in M$ is

$$A_x = T_{1_x}(s^{-1}(x)) \quad \text{(or globally, } A = \ker(ds)|_M = T^s\mathcal{G}|_M),$$

where recall that $1_x \in \mathcal{G}$ denotes the unit arrow at x. A vector field $X \in \Gamma(T^s\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$ is said to be **right-invariant** if

$$dR_h(X_g) = X_{gh}, \qquad \forall\, (g, h) \in \mathcal{G}_2.$$

The set of right-invariant vector fields is denoted by $\mathfrak{X}^{\mathrm{inv}}_s(\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$. Each section of A induces a right-invariant vector field of $\mathcal{G}$ via the map

$$\alpha \in \Gamma(A) \;\mapsto\; \widetilde{\alpha} \in \mathfrak{X}^{\mathrm{inv}}_s(\mathcal{G}), \qquad (\widetilde{\alpha})_g := (dR_g)_{1_{t(g)}} \alpha_{t(g)}, \tag{2.3}$$

inducing a 1-1 correspondence between $\Gamma(A)$ and $\mathfrak{X}^{\mathrm{inv}}_s(\mathcal{G})$. One proves that the subspace $\mathfrak{X}^{\mathrm{inv}}_s(\mathcal{G}) \subset \mathfrak{X}(\mathcal{G})$ is a Lie subalgebra (a consequence of the associativity axiom of the multiplication of $\mathcal{G}$), and the bracket of A is defined by imposing the relation

$$\widetilde{[\alpha, \beta]} = [\widetilde{\alpha}, \widetilde{\beta}], \quad \forall\, \alpha, \beta \in \Gamma(A).$$

Finally, one equips A with the anchor given by the restriction of dt to $A \subset T\mathcal{G}$. With this structure, A becomes a Lie algebroid. Note that with this construction each isotropy algebra $\mathfrak{g}_x(A)$ is the Lie algebra of the isotropy group $\mathcal{G}_x$, and the (possibly singular) distribution $\rho(A) \subset TM$ is precisely the distribution tangent to the (possibly singular) foliation of M by the orbits of $\mathcal{G}$. As the reader might have suspected, Examples 2.1.4 are the Lie algebroids of Examples 2.1.1, respectively.

Flows of Bisections Let $A \to M$ be the Lie algebroid of a Lie groupoid $\mathcal{G} \rightrightarrows M$. One often thinks of a Lie algebroid as a "generalized tangent bundle". From this point of view, sections of A are "generalized vector fields". What are then "generalized flows"? Given a section $\alpha \in \Gamma(A)$, let $\varphi^\epsilon_{\widetilde{\alpha}}$ be the flow along the induced right invariant vector field $\widetilde{\alpha} \in \mathfrak{X}(\mathcal{G})$. The flow of α is defined to be the one-parameter family of local bisections (whose domain of definition at each time ϵ may vary, depending on the domain of definition of $\varphi^\epsilon_{\widetilde{\alpha}}$)

$$\varphi^\epsilon_\alpha := \varphi^\epsilon_{\widetilde{\alpha}}\big|_M,$$

where recall that M is identified with the submanifold of units $u(M) \subset \mathcal{G}$. Indeed, one readily verifies that $t \circ \varphi^\epsilon_\alpha$ is the flow of the vector field $\rho(\alpha)$ and hence a local diffeomorphism (a diffeomorphism between its domain and image), and its domain of definition coincides with the domain of definition of φ^ϵ_α for each ϵ. The fact that φ^ϵ_α is a section of s for each ϵ follows from the fact that $\widetilde{\alpha}$ is tangent to the s-fibers. The following lemma shows that $\varphi^\epsilon_{\widetilde{\alpha}}$ can be recovered from φ^ϵ_α.

Lemma 2.1.5. *Let $\mathcal{G}$ be a Lie groupoid. For all $g \in \mathcal{G}$ and $\alpha \in \Gamma(A)$,*

$$\varphi^\epsilon_{\widetilde{\alpha}} \circ R_g = R_g \circ \varphi^\epsilon_{\widetilde{\alpha}},$$

in the domain where both sides are defined. In particular, setting $y = t(g)$, then

$$\varphi^\epsilon_{\widetilde{\alpha}}(g) = \varphi^\epsilon_\alpha(y) \cdot g. \tag{2.4}$$

Proof. Define $\psi^\epsilon := R_g \circ \varphi^\epsilon_{\widetilde{\alpha}}$ and $\bar{\psi}^\epsilon := \varphi^\epsilon_{\widetilde{\alpha}} \circ R_g$. Both are one parameter families of diffeomorphisms. Evaluate both on an element $h \in \mathcal{G}$ and take the derivative with respect to ϵ,

$$\begin{aligned}
\frac{d}{d\epsilon}\psi^\epsilon(h) &= dR_g \frac{d}{d\epsilon}\varphi^\epsilon_{\tilde{\alpha}}(h) = dR_g \tilde{\alpha}_{\varphi^\epsilon_{\tilde{\alpha}}(h)} = dR_g\ dR_{\varphi^\epsilon_{\tilde{\alpha}}(h)}\ \alpha_{t(\varphi^\epsilon_{\tilde{\alpha}}(h))} \\
&\underset{t(\psi^\epsilon(h))=t(\varphi^\epsilon(h))}{\overset{}{\underset{\uparrow}{=}}} dR_{\psi^\epsilon(h)}\ \alpha_{t(\psi^\epsilon(h))} = \tilde{\alpha}_{\psi^\epsilon(h)}, \\
\frac{d}{d\epsilon}\bar{\psi}^\epsilon(h) &= \frac{d}{d\epsilon}(\varphi^\epsilon_{\tilde{\alpha}}(hg)) = \tilde{\alpha}_{\varphi^\epsilon_{\tilde{\alpha}}(hg)} = dR_{\varphi^\epsilon_{\tilde{\alpha}}(hg)}\alpha_{t(\varphi^\epsilon_{\tilde{\alpha}}(hg))} \\
&= dR_{\bar{\psi}^\epsilon(h)}\alpha_{t(\psi'^\epsilon(h))} = \tilde{\alpha}_{\bar{\psi}^\epsilon(h)}.
\end{aligned}$$

From uniqueness of integral curves of vector fields, we conclude that $\psi = \bar{\psi}$. □

Actions and Representations The notions of an action and a representation of a Lie group and a Lie algebra generalize naturally to the setting of Lie groupoids and Lie algebroids. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\pi : P \to M$ a surjective submersion. A **(left) action** of $\mathcal{G}$ on P (along π) is a smooth map

$$\mathcal{G}_s\times_\pi P \to P, \qquad (g,p) \mapsto g \cdot p, \tag{2.5}$$

such that $\pi(g \cdot p) = t(g)$ for all $(g,p) \in \mathcal{G}_s\times_\pi P$ and such that the following action axioms are satisfied: $(gh)\cdot p = g \cdot (h \cdot p)$ and $1_x \cdot p = p$ for all $p \in P$, $(g,h) \in \mathcal{G}_2$

and $x \in M$ for which the conditions make sense. In particular, a left action of $\mathcal{G}$ on P associates with each arrow $g \in \mathcal{G}$, say with $s(g) = x$ and $t(g) = y$, a diffeomorphism

$$\pi^{-1}(x) \xrightarrow{\simeq} \pi^{-1}(y), \qquad p \mapsto g \cdot p.$$

The subset $\mathcal{O}_p := \{g \cdot p \mid g \in s^{-1}(\pi(p))\} \subset P$ is called the **orbit** of the action through p. The action induces an equivalence relation on P, namely $p \sim q$ if $q \in \mathcal{O}_p$, and the resulting quotient space $P/\mathcal{G}$, endowed with the quotient topology, is called the **orbit space**.

An action of $\mathcal{G}$ on P is said to be **free** if $g \cdot p = p$ implies that g is the unit at $\pi(p)$ for all $(g, p) \in \mathcal{G}_s\times_\pi P$. It is said to be **proper** if the map $\mathcal{G}_s\times_\pi P \to P \times P,\ (g, p) \mapsto (g \cdot p, p)$, is a proper map. As for Lie group actions, if an action is free and proper, then the orbit space $P/\mathcal{G}$ has a unique smooth structure for which the projection $P \to P/\mathcal{G}$ is a surjective submersion (see [15], Section 3.3).

A left action of $\mathcal{G}$ on P induces (and is encoded by) the **action groupoid** $\mathcal{G} \ltimes P \rightrightarrows P$. It is the Lie groupoid whose space of arrows is the domain of the action map $\mathcal{G} \ltimes P := \mathcal{G}_s \times_\pi P$, and its structure maps are

$$s(g,p) = p, \qquad t(g,p) = g \cdot p,$$
$$(g,p) \cdot (h,q) = (g \cdot h, q), \qquad (g,p)^{-1} = (g^{-1}, g \cdot p), \qquad 1_p = (1_{s(p)}, p).$$

Note that the target map of the Lie groupoids is precisely the action map, the orbit $\mathcal{O}_p$ through p of the action groupoid coincides with the orbit through p of the action, the orbit space coincides with the orbit space of the action, and the action is free if and only if all of the isotropy groups of the action groupoid are trivial.

A **representation** of a Lie groupoid $\mathcal{G} \rightrightarrows M$ consists of a vector bundle $\pi : E \to M$ together with an action of $\mathcal{G}$ on E,

$$\mathcal{G}_s \times_\pi E \to E, \qquad (g, v) \mapsto g \cdot v, \tag{2.6}$$

for which the map $E_{s(g)} \to E_{t(g)},\ v \mapsto g \cdot v$, is linear (and hence a linear isomorphism) for all $g \in \mathcal{G}$.

Example 2.1.6. (Normal Representation) Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $\mathcal{O} \subset M$ be an orbit of $\mathcal{G}$. The restriction of $\mathcal{G}$ to the orbit $\mathcal{O}$,

$$\mathcal{G}_\mathcal{O} := \{\, g \in \mathcal{G} \mid s(g) \in \mathcal{O},\ t(g) \in \mathcal{O} \,\},$$

is a transitive Lie groupoid over $\mathcal{O}$ when equipped with the restriction of all the structure maps. We denote the normal bundle of $\mathcal{O}$ by $N\mathcal{O} \to \mathcal{O}$, i.e. the quotient of $TM|_\mathcal{O}$ by $T\mathcal{O}$,

$$0 \to T\mathcal{O} \to TM|_\mathcal{O} \to N\mathcal{O} \to 0.$$

Any arrow $g \in \mathcal{G}_\mathcal{O}$, say with $s(g) = x$ and $t(g) = y$, induces a linear map $N\mathcal{O}_x \to N\mathcal{O}_y,\ [X] \mapsto g \cdot [X]$, as follows: let $X \in TM|_\mathcal{O}$ be a representative of $[X] \in N\mathcal{O}_x$ and let $\widetilde{X} \in T_g\mathcal{G}$ such that $ds(\widetilde{X}) = X$, then $g \cdot [X] = [dt(\widetilde{X})]$. This definition is readily verified to be independent of the choices and to define an action with which $N\mathcal{O}$ becomes a representation of $\mathcal{G}_{\mathcal{O}} \rightrightarrows \mathcal{O}$. This representation is called the **normal representation** of $\mathcal{G}$ at $\mathcal{O}$. See [14] (Section 1.2) or [15] (Section 3.4) for more details. $\diamondsuit$

At the infinitesimal level, let $A \to M$ be a Lie algebroid and $\pi : P \to M$ a surjective submersion. An **action** of A on P is a vector bundle map

$$a : \pi^*A \to TP, \tag{2.7}$$

such that $d\pi \circ a = \rho$ and for which the induced map of sections $a : \Gamma(A) \to \mathfrak{X}(P),\ \alpha \mapsto a(\pi^*\alpha)$, is a Lie algebra homomorphism, i.e.

$$a([\alpha, \beta]) = [a(\alpha), a(\beta)], \qquad \forall\, \alpha, \beta \in \Gamma(A).$$

An action (2.5) of a Lie groupoid $\mathcal{G}$ on P induces an action of its Lie algebroid A on P. Explicitly, for every $p \in P$ with $x = \pi(p) \in M$, $a_p : A_x \to T_pP$ is the differential of the map $s^{-1}(x) \to P,\ g \mapsto g \cdot p$, at 1_x.

An action of a Lie algebroid $A \to M$ on a surjective submersion $\pi : P \to M$ induces (and is encoded by) the **action groupoid** $\pi^*A \to P$. As a vector bundle, it is simply the pull-back π^*A, the anchor is the action map (2.7) and the bracket is uniquely defined by the condition $[\pi^*\alpha, \pi^*\beta] = \pi^*[\alpha, \beta]$ for all $\alpha, \beta \in \Gamma(A)$. As one expects, the Lie algebroid of the action groupoid induced by the action of a Lie groupoid $\mathcal{G}$ is the action algebroid of the induced action of the Lie algebroid A of $\mathcal{G}$.

To talk about representations of A, one has to first introduce the notion of an A-connection. Let $A \to M$ be a Lie algebroid and $E \to M$ a vector bundle. An **A-connection** on E is a bilinear map

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E),$$

satisfying the connection-like properties

$$\nabla_{f\alpha}(\sigma) = f\nabla_\alpha(\sigma), \qquad \nabla_\alpha(f\sigma) = f\nabla_\alpha(\sigma) + \rho(\alpha)(f)\,\sigma,$$

for all $\alpha \in \Gamma(A),\ \sigma \in \Gamma(E)$ and $f \in C^\infty(M)$. An A-connection ∇ on E is said to be **flat** if

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha - \nabla_{[\alpha,\beta]})(\sigma) = 0, \qquad \forall \alpha, \beta \in \Gamma(A),\ \sigma \in \Gamma(E).$$

A **representation** of a Lie algebroid $A \to M$ is a vector bundle $E \to M$ equipped with a flat A-connection ∇. A representation $E \to M$ of $A \to M$ is in particular an action of $A \to M$ on $E \to M$, but an action which is "linear".

A representation $E \to M$ of a Lie groupoid $\mathcal{G}$ induces a representation of the Lie algebroid. Explicitly, the induced A-connection is obtained by differentiation as follows:

$$\nabla_\alpha(\sigma)_x := \frac{d}{d\epsilon}\Big|_{\epsilon=0} g(\epsilon)^{-1} \cdot \sigma_{t(g(\epsilon))},$$

where $x \in M,\ \alpha \in \Gamma(A)$ and $\sigma \in \Gamma(E)$, and $g(\epsilon)$ is a curve in $s^{-1}(x)$ representing $\alpha_x \in T_{1_x}(s^{-1}(x))$.

2.2 Jet Groupoids

To define the notion of a Lie pseudogroup on a manifold M, we will need to consider PDEs imposed on the set of local diffeomorphisms $\mathrm{Diff}_{\mathrm{loc}}(M)$ of M. To define such PDEs in the geometric sense of Chapter 1, we will need to consider jet spaces associated with $\mathrm{Diff}_{\mathrm{loc}}(M)$. As we will see in this section, due to the group-like nature of $\mathrm{Diff}_{\mathrm{loc}}(M)$, these jet spaces are themselves of a group-like nature, or more precisely, they have the structure of a Lie groupoid. More generally, as we saw in Example 2.1.3, we may regard local diffeomorphisms of M as local bisections of the pair groupoid $M\times M \rightrightarrows M$. This point of view motivates considering more general jet spaces, namely the jet spaces associated with the set of local bisections $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ of a general Lie groupoid $\mathcal{G} \rightrightarrows M$. These also have the structure of a Lie groupoid due to the group-like nature of $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$, and, in fact, they capture the essential structure that one needs in order to study the special case of $\mathcal{G} = M \times M$.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Since a local bisection σ of $\mathcal{G}$ is, in particular, a local section of the source map $s : \mathcal{G} \to M$, we can talk about k-jets of local bisections as defined in Chapter 1. Let $k \geq 0$ be an integer. The **k-th jet groupoid** of $\mathcal{G}$ is the Lie groupoid $J^k\mathcal{G} \rightrightarrows M$ with

$$J^k\mathcal{G} := \{\ j^k_x\sigma \mid \sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}),\ x \in \mathrm{Dom}(\sigma)\ \}. \tag{2.8}$$

The source and target maps are

$$s : J^k\mathcal{G} \to M,\quad j^k_x\sigma \mapsto x, \qquad\qquad t : J^k\mathcal{G} \to M,\quad j^k_x\sigma \mapsto t(\sigma(x)).$$

The unit at $x \in M$ is $j^k_x u$, where u is the unit map of $\mathcal{G}$, the inverse is given by $(j^k_x\sigma)^{-1} = j^k_x(\sigma^{-1})$ and the multiplication by

$$j^k_y\sigma' \cdot j^k_x\sigma = j^k_x(\sigma' \cdot \sigma),$$

when $t(j^k_x\sigma) = s(j^k_y\sigma')$. The projection between any subsequent jet groupoids will be denoted by

$$\pi : J^{k+1}\mathcal{G} \to J^k\mathcal{G}, \qquad\qquad j^{k+1}_x\sigma \mapsto j^k_x\sigma. \tag{2.9}$$

By its very definition, a jet groupoid $J^k\mathcal{G}$ is a subset of the k-th jet bundle associated with the source map $s : \mathcal{G} \to M$, since local bisections of $\mathcal{G}$ are, in particular, local sections of the source map. Let us look more carefully into this inclusion order by order:

- $k = 0$: there is a bijection

  $$J^0\mathcal{G} \cong \mathcal{G}, \qquad\qquad j^0_x\sigma \mapsto \sigma(x),$$

  and, hence, the jet groupoid coincides with the jet bundle.

- $k = 1$: the 1-jet $j^1_x\sigma$ of a local section of $s : \mathcal{G} \to M$ belongs to $J^1\mathcal{G}$ if and only if $(dt \circ \sigma)_x : T_xM \to T_{t\circ\sigma(x)}M$ is invertible. Since this is an open condition, $J^1\mathcal{G}$ is an open subset of the 1st jet bundle. Note, however, that the projection $\pi : J^1\mathcal{G} \to J^0\mathcal{G}$ is no longer an affine bundle.

- $k > 1$: $J^k\mathcal{G}$ consists of all k-jets of local sections of $s : \mathcal{G} \to M$ that project to $J^1\mathcal{G}$ (the inverse function theorem ensures that any such k-jet is represented by a local bisection). Thus, $J^k\mathcal{G}$ is an open subset of the k-th jet bundle and $\pi : J^k\mathcal{G} \to J^{k-1}\mathcal{G}$ is an affine bundle.

The essential difference between the jet groupoids and their ambient jet bundles, as we see, lies in $k = 1$ and then "propagates" up to the higher orders. To summarize: jet groupoids inherit the smooth structure from their ambient jet bundle and they fit in the following diagram:

$$\underbrace{\underbrace{\dots \xrightarrow{\pi} J^{k+1}\mathcal{G} \xrightarrow{\pi} J^k\mathcal{G} \xrightarrow{\pi} \dots \xrightarrow{\pi} J^1\mathcal{G}}_{\text{tower of affine bundles}} \xrightarrow{\pi} J^0\mathcal{G} \cong \mathcal{G}}_{\text{sequence of Lie groupoids}}\,. \tag{2.10}$$

With this last observation, we can take the theory of PDEs that was developed in Chapter 1 as our starting point. In fact, it is worth remarking that $J^1\mathcal{G}$ is a PDE in the sense of Definition 1.4.1 and that the higher jet groupoids are its prolongations. It is the PDE that has any local bisection as a solution and whose solutions, when restricted to small enough domains, are guaranteed to be local bisections (by the inverse function theorem).

In the realm of jet groupoids, we talk about holonomic bisections rather than holonomic sections. Any (local) bisection σ of $\mathcal{G}$ gives rise to a (local) bisection $j^k\sigma$ of $J^k\mathcal{G}$ with the same domain,

$$j^k\sigma : x \mapsto j^k_x\sigma,$$

called a **(local) holonomic bisection**. The map

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}) \to \mathrm{Bis}_{\mathrm{loc}}(J^k\mathcal{G}), \qquad \sigma \mapsto j^k\sigma, \tag{2.11}$$

preserves the group-like structure. For example, given two composable local bisections σ and σ', then $j^k\sigma' \cdot j^k\sigma = j^k(\sigma' \cdot \sigma)$.

Example 2.2.1. Our main examples of interest are the jet groupoids

$$J^kM := J^k(M \times M)$$

of M, i.e. the jet groupoids associated with the pair groupoid $M \times M \rightrightarrows M$. Unraveling the identification of local diffeomorphisms with local bisections, the k-th jet groupoid of M is the Lie groupoid $J^kM \rightrightarrows M$ with

$$J^kM = \{\ j^k_x\phi \mid \phi \in \mathrm{Diff}_{\mathrm{loc}}(M),\ x \in \mathrm{Dom}(\phi)\ \}.$$

The source and target maps are

$$s : J^kM \to M, \quad j^k_x\phi \mapsto x, \qquad t : J^kM \to M, \quad j^k_x\phi \mapsto \phi(x).$$

The unit at $x \in M$ is $j^k_x(\mathrm{id}_M)$, the inverse is given by $(j^k_x\phi)^{-1} = j^k_x(\phi^{-1})$ and the multiplication by $j^k_y\phi' \cdot j^k_x\phi = j^k_x(\phi' \circ \phi)$, when $t(j^k_t\phi') = s(j^k_x\phi)$. In Examples 2.6.1 and 2.6.2, we will describe the smooth structure on J^kM in the cases $k = 0$ and $k = 1$. $\diamondsuit$

2.3 Jet Algebroids

Let us turn to the infinitesimal picture.

Let M be a manifold and let us denote the sheaf of vector fields on M by $\mathfrak{X}_{\text{loc}}(M)$. A subsheaf $L \subset \mathfrak{X}_{\text{loc}}(M)$ is said to be a **sheaf of Lie algebras (of vector fields)** on M if it is closed under the Lie bracket of vector fields. If we think of a pseudogroup $\Gamma \subset \text{Diff}_{\text{loc}}(M)$ as a set of local symmetries, then a sheaf of Lie algebras $L \subset \mathfrak{X}_{\text{loc}}(M)$ would correspond to a set of infinitesimal local symmetries. For instance, the set of local Killing vector fields on a Riemannian manifold is an example of such a sheaf. Continuing with this line of thought, the infinitesimal counterpart of a Lie pseudogroup would then be a sheaf of Lie algebras of vector fields that is defined as the set of solutions of a linear PDE. In turn, to define such PDEs, one needs to consider the linear jet bundles J^kTM associated with $\mathfrak{X}_{\text{loc}}(M)$. These, due to the Lie algebra structure of $\mathfrak{X}_{\text{loc}}(M)$, have the structure of a Lie algebroid and are called the *jet algebroids* of M. They are the infinitesimal counterpart of the jet groupoids J^kM.

More generally, given a Lie groupoid $\mathcal{G} \rightrightarrows M$ whose Lie algebroid is $A \to M$, we will construct the jet algebroids J^kA associated with the sheaf of sections of A. These are the infinitesimal counterparts of the jet groupoids $J^k\mathcal{G}$.

Remark 2.3.1. The study of sheaves of Lie algebras of vector fields, in itself very interesting and very much related to the topic of this thesis, goes beyond the scope of this work, and we mention them here mainly to motivate the notion of a jet algebroid. Such sheaves were studied in the so called *transitive case* by Singer and Sternberg in [64] under the name of *Lie algebra sheaves (LAS)*. The aim of their work, as is ours, was to gain a better understanding of Cartan's theory of Lie pseudogroups and it has been one of our main sources of inspiration. Those who are familiar with that paper will see many parallels with our modern formulation of Cartan's structure theory in Chapter 4, and in particular with the notion of a Cartan algebroid that will be introduced in Section 6.2. $\diamondsuit$

Let A be a Lie algebroid over M. The k**-the jet algebroid** of A, with $k \geq 0$, is the linear jet bundle $J^kA \to M$ associated with the vector bundle $A \to M$ (see Section 1.8). It is endowed with the Lie algebroid structure induced by the Lie algebroid structure of A. The anchor map is given by

$$\rho : J^kA \to TM, \qquad j^k_x\alpha \mapsto \rho(\alpha_x),$$

where $\rho : A \to M$ is the anchor of A, and the bracket,

$$[\cdot,\cdot] : \Gamma(J^kA) \times \Gamma(J^kA) \to \Gamma(J^kA),$$

is uniquely defined by how it acts on holonomic sections:

$$[j^k\alpha, j^k\alpha'] = j^k[\alpha,\alpha'], \qquad \forall \alpha,\alpha' \in \Gamma(A).$$

While this description of the bracket is simple and natural, it has the disadvantage of not being manifestly well defined. Alternatively, one can also write down a formula for how the bracket acts on any two sections of J^kA. See [13] (Remark 3.3) for details. Note that for $k = 0$ there is a canonical isomorphism of Lie algebroids

$$J^0A \xrightarrow{\simeq} A, \qquad j^0_x\alpha \mapsto \alpha_x.$$

Denoting the projections by $\pi : J^kA \to J^{k-1}A$, we obtain a sequence of Lie algebroids associated with M,

$$\ldots \xrightarrow{\pi} J^kA \xrightarrow{\pi} J^{k-1}A \xrightarrow{\pi} \ldots \xrightarrow{\pi} J^0A \cong A.$$

Recall that the kernel of the projection at each level,

$$\mathfrak{g}^k = \mathfrak{g}^k(A) := \text{Ker } (\pi : J^kA \to J^{k-1}A),$$

is called the **symbol space** of J^kA. Since the symbol space is the kernel of a surjective Lie algebroid map, it has the structure of a totally non-transitive Lie algebroid (i.e. with zero anchor), also known as a bundle of Lie algebras.

Example 2.3.2. Our main examples of interest are the jet algebroids J^kTM associated with the tangent bundle of a manifold M. As we discussed in the introduction to this section, these are the linear jet bundles associated with the sheaf of vector fields $\mathfrak{X}_{\text{loc}}(M)$. Their Lie algebroid structure is induced by the Lie bracket of vector fields on M. $\diamondsuit$

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The k-th jet algebroid J^kA of the Lie algebroid $A = A(\mathcal{G})$ of $\mathcal{G}$ is the infinitesimal counterpart of the jet groupoid $J^k\mathcal{G}$. Let us explain what we mean. Denote the Lie algebroid of $J^k\mathcal{G}$ by

$$A^k = A^k(\mathcal{G}) := A(J^k\mathcal{G}).$$

The projection (2.9) induces a projection $d\pi : A^{k+1} \to A^k$, and we obtain an induced sequence of Lie algebroids maps

$$\ldots \xrightarrow{d\pi} A^{k+1} \xrightarrow{d\pi} A^k \xrightarrow{d\pi} \ldots \xrightarrow{d\pi} A^0 \xrightarrow{\simeq} TM.$$

The kernel of the projection at each level,

$$\mathfrak{g}^k = \mathfrak{g}^k(\mathcal{G}) := \text{Ker } (d\pi : A^k \to A^{k-1}), \tag{2.12}$$

is called the **symbol space** of the jet groupoid $J^k\mathcal{G}$.

A priori, at each level, there are two Lie algebroids around: the k-th jet algebroid and the Lie algebroid of the k-th jet groupoid. However, the two are canonically isomorphic. The isomorphism is given by

$$J^kA \xrightarrow{\simeq} A^k(\mathcal{G}), \quad j^k_x\alpha \mapsto \frac{d}{dt}\Big|_{t=0} j^k_x(\varphi^t_\alpha), \tag{2.13}$$

Here, φ^t_α is the flow of local bisections associated with the section $\alpha \in \Gamma(A)$ (see Section 2.1). One now readily verifies that this is indeed a Lie algebroid isomorphism, and that the isomorphisms at all levels commute with the projections. In Examples 2.6.1 and 2.6.2, we will describe this map explicitly for the cases of $\mathcal{G} = M \times M$ and $k = 0, 1$. Note that (2.13) restricts to an isomorphism of the symbol spaces

$$\mathfrak{g}^k(A) \cong \mathfrak{g}^k(\mathcal{G}).$$

Keeping both descriptions in mind is important when working with jet groupoids. On the one hand the infinitesimal jet structure is directly accessible through $J^k A$ (e.g., the Spencer operator, as we will see), and on the other hand $A^k(\mathcal{G})$ is closer to the jet groupoid. In particular, we have an isomorphism of vector bundles over $J^k\mathcal{G}$

$$T^s J^k\mathcal{G} \cong t^* A^k(\mathcal{G}) \tag{2.14}$$

induced by the right translation map

$$R_{j^k_x\sigma} : s^{-1}(t \circ \sigma(x)) \to s^{-1}(x), \qquad j^k_y\sigma' \mapsto j^k_y\sigma' \cdot j^k_x\sigma.$$

This isomorphism (which one has for any Lie groupoid, it is not special for jet groupoids) is often viewed as an element of $\Omega^1_s(J^k\mathcal{G}; t^*A^k)$, the space of source-foliated A^k-valued 1-forms on $J^k\mathcal{G}$, called the **Maurer-Cartan form** of $J^k\mathcal{G}$ (in the case of a Lie group, viewed as a Lie groupoid over a point, it coincides with the usual right-invariant Maurer-Cartan form). Note that (2.14) restricts to the following isomorphism

$$T^\pi J^k\mathcal{G} \xrightarrow{\cong} t^*\mathfrak{g}^k.$$

Hence, up to a canonical isomorphism, the symbol space of $J^k\mathcal{G}$ viewed as a jet bundle (see (1.10)) coincides with the symbol space of $J^k\mathcal{G}$ as defined in (2.12).

2.4 The Cartan Form on a Jet Groupoid

As for jet bundles, the essential jet structure of a jet groupoid is encoded in its Cartan form. The jet structure, in turn, is compatible with the groupoid structure, a fact which is clear from the very definition of a jet groupoid. This compatibility is captured by a single property of the Cartan form known as multiplicativity.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let A be its Lie algebroid. The Cartan form of the jet groupoid $J^k\mathcal{G}$ is simply the restriction of the Cartan form on the ambient jet bundle (1.24), up to the isomorphism (2.14). Explicitly, the **Cartan form** of $J^k\mathcal{G}$ is the 1-form

$$\omega \in \Omega^1(J^k\mathcal{G}; t^*A^{k-1}) \tag{2.15}$$

that is defined at a point $j^k_x\sigma$ by

$$\omega_{j^k_x\sigma} := dR_{(j^{k-1}_x\sigma)^{-1}} \circ (d\pi - (d(j^{k-1}\sigma))_x \circ ds)_{j^k_x\sigma}. \tag{2.16}$$

As for jet bundles, the **Cartan distribution** is the kernel of the Cartan form,

$$C_\omega := \mathrm{Ker}\,\omega \subset TJ^k\mathcal{G}.$$

Since the Cartan form on a jet groupoid is the restriction of the Cartan form on the ambient jet bundle (up to the isomorphism (2.14)), all of the properties of the latter hold. In particular, Proposition 1.3.3 tells us that the Cartan form detects (local) holonomic bisections, and Proposition 1.3.4 enumerates the properties of $\delta\omega = d_\nabla \omega|_{C_\omega}$.

Multiplicative Forms The groupoid structure of the jet groupoid is reflected in the structure of the Cartan form. In [13], the authors identify a single property of the Cartan form which captures this structure, a property known as multiplicativity. As the authors explain, this fact places jet groupoids in a broader context of Lie groupoids equipped with a multiplicative form. Another example of such a structure is a symplectic groupoid, a structure which plays an important role in the field of Poisson geometry. Let us briefly recall the general notion of a multiplicative form on a Lie groupoid before specializing to our specific case. We refer the reader to [13] for more details.

Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, recall that the multiplication map is denoted by $m : \mathcal{G}_2 \to \mathcal{G}$ and let us denote the projections onto the first and second components of $\mathcal{G}_2 = \mathcal{G}_t \times_s \mathcal{G}$ by pr_1 and pr_2. Recall that a representation of $\mathcal{G}$ consists of a vector bundle together with a linear action (2.6) of $\mathcal{G}$.

Definition 2.4.1. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $E \to M$ be a representation of $\mathcal{G}$. An E-valued k-form $\omega \in \Omega^k(\mathcal{G}; t^*E)$ is said to be **multiplicative** if*

$$(m^*\omega)_{(g,h)} = \left(\mathrm{pr}_1^*\,\omega\right)_{(g,h)} + g \cdot \left(\mathrm{pr}_2^*\,\omega\right)_{(g,h)}, \tag{2.17}$$

for all composable arrows $(g, h) \in \mathcal{G}_2$.

Intuitively, multiplicative forms are "invariant" under the action of $\mathcal{G}$. As a simple example, equipping a Lie groupoid with the trivial representation $E = M \times \mathbb{R}$, a function $f \in C^\infty(\mathcal{G}) = \Omega^0(\mathcal{G}; t^*E)$ is multiplicative if

$$f(gh) = f(g) + f(h).$$

Multiplicativity of the Cartan Form Returning to our particular case, there is a natural linear action of the jet groupoid $J^k\mathcal{G}$ on the jet algebroid A^{k-1},

$$J^k\mathcal{G}_s \times_\pi A^{k-1} \to A^{k-1}, \qquad (j^k_x\sigma, \alpha_x) \mapsto j^k_x\sigma \cdot \alpha_x, \tag{2.18}$$

with which A^{k-1} becomes a representation of $J^k\mathcal{G}$ called the **adjoint representation** of $J^k\mathcal{G}$. It is defined as follows: let $(j^{k-1}_x\sigma'_\epsilon)$ be a path representing α_x, thus $\alpha_x = \frac{d}{d\epsilon}\big|_{\epsilon=0} j^{k-1}_x\sigma'_\epsilon$, and let γ_ϵ be the path in M given by $\gamma_\epsilon = t(j^{k-1}_x\sigma'_\epsilon)$. Then,

$$j^k_x\sigma \cdot \alpha_x := \frac{d}{d\epsilon}\bigg|_{\epsilon=0} (j^{k-1}_{\gamma_\epsilon}\sigma) \cdot (j^{k-1}_x\sigma'_\epsilon) \cdot (j^{k-1}_x\sigma)^{-1}. \tag{2.19}$$

The resulting vector is indeed an element of $A^{k-1}_{t(\sigma(x))}$ and depends only on $j^k_x\sigma$ and on α_x. This representation is reminiscent of the adjoint representation of a Lie group. Equivalently, the action (2.19) can also be expressed more economically in terms of the Cartan form, albeit at the expense of the expression (but not its result) being non-canonical:

Lemma 2.4.2. *Let $j^k_x\sigma \in J^k\mathcal{G}$ and $\alpha_x \in A^{k-1}_x$. The action* (2.19) *defining the adjoint representation of $J^k\mathcal{G}$ is given by*

$$j^k_x\sigma \cdot \alpha_x = \omega(dm(Y, \widehat{\alpha}_x)),$$

where $\widehat{\alpha}_x \in A^k_x$ satisfies $d\pi(\widehat{\alpha}_x) = \alpha_x$ and $Y \in (C_\omega)_{j^k_x\sigma}$ satisfies $ds(Y) = \rho(\alpha_x)$.

Proof.

$$\begin{aligned}
\omega(dm(Y, \widehat{\alpha}_x)) &= dR_{(j^{k-1}_x\sigma)^{-1}} \circ (d\pi - d(j^k\sigma) \circ ds) \circ dm(Y, \widehat{\alpha}_x) \\
&= dR_{(j^{k-1}_x\sigma)^{-1}} \circ dm(d\pi(Y), d\pi(\widehat{\alpha}_x)) \\
&= dR_{(j^{k-1}_x\sigma)^{-1}} \circ dm\Big(d(j^{k-1}\sigma) \circ \rho(\alpha_x), \alpha_x\Big) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} (j^{k-1}_{\gamma_\epsilon}\sigma) \cdot (j^{k-1}_x\sigma'_\epsilon) \cdot (j^{k-1}_x\sigma)^{-1}.
\end{aligned}$$

In the second equality we used the fact that $ds(dm(Y, \widehat{\alpha}_x)) = ds(\widehat{\alpha}_x) = 0$ and that π is a Lie groupoid map. $\square$

With respect to this representation, the Cartan form on the jet groupoid is a multiplicative form. This fact was proven in [13] for the case $k = 1$. For the general case, we can either use the fact that $J^k\mathcal{G} \subset J^1(J^{k-1}\mathcal{G})$, or, alternatively, give a direct proof:

Proposition 2.4.3. *The Cartan form $\omega \in \Omega^1(J^k\mathcal{G}; t^*A^{k-1})$ is multiplicative with respect to the adjoint representation* (2.18)*.*

Proof. Let $(g, g') \in (J^k\mathcal{G})_2$ be a pair of composable k-jets ($s(g) = t(g')$). Thus, for some local bisections σ and σ' of $\mathcal{G}$,

$$g = j^k_x\sigma \qquad \text{and} \qquad g' = j^k_{x'}\sigma',$$

and hence $\pi(g) = j^{k-1}_x\sigma$ and $\pi(g') = j^{k-1}_{x'}\sigma'$. For the choice σ and σ' of representative bisections we write

$$b = j^{k-1}\sigma \qquad \text{and} \qquad b' = j^{k-1}\sigma'.$$

Let us write $\Delta : M \to M \times M,\ x \to (x, x)$, for the diagonal map. Note that the product of two (composable) local bisections is given by (see (2.1))

$$b \cdot b' = m \circ (b \circ t \circ b', b') \circ \Delta. \tag{2.20}$$

Let $(X, X') \in T_{(g,g')}(J^k\mathcal{G})_2$. Thus,

$$X \in T_gJ^k\mathcal{G},\ X' \in T_{g'}J^k\mathcal{G} \qquad \text{are such that} \qquad (ds)_g(X) = (dt)_{g'}(X').$$

We start with the left hand side of (2.17) and make our way to the right hand side:

$$
\begin{aligned}
&\omega \circ (dm)_{(g,g')}(X, X') \\
&= dR_{(\pi(g)\cdot\pi(g'))^{-1}} \circ \big(d\pi - d(b\cdot b')\circ ds\big) \circ (dm)_{(g,g')}(X, X') \\
&= dR_{(\pi(g)\cdot\pi(g'))^{-1}} \circ \big(\underbrace{dm \circ \big((d\pi)_g(X), (d\pi)_{g'}(X')\big)}_{\pi \text{ is a Lie groupoid map}} - d(b\cdot b')\circ \underbrace{(ds)_g(X')}_{s(hh') = s(h') \text{ for all } h, h' \in J^k\mathcal{G}}\big) \\
&= dR_{(\pi(g)\cdot\pi(g'))^{-1}} \circ dm\big(\underbrace{(d\pi)_g(X) - db\circ d(t\circ b'\circ s)_{g'}(X'), (d\pi - db'\circ ds)_{g'}(X')}_{\text{use (2.20)}}\big) \\
&= dR_{(\pi(g)\cdot\pi(g'))^{-1}} \circ dm\big(\underbrace{(d\pi - db\circ ds)_g(X) + db\circ dt\circ(d\pi - b'\circ s)_{g'}(X')}_{\text{add and subtract } (db\circ ds)_g(X) \text{ and use } (ds)_g(X) = (dt)_{g'}(X')}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (d\pi - db'\circ ds)_{g'}(X')\big) \\
&= dR_{(\pi(g)\cdot\pi(g'))^{-1}}\big(dm((d\pi - db\circ ds)_g(X), 0_{\pi(g')}) + \\
&\qquad\qquad dm\big(db\circ dt\circ(d\pi - b'\circ s)_{g'}(X'), (d\pi - db'\circ ds)_{g'}(X')\big)\big) \\
&= dR_{\pi(g)} \circ (d\pi - db\circ ds)_g(X) + dR_{(\pi(g))^{-1}} \circ dm\big(db\circ dt\circ \omega_{g'}(X'), \omega_{g'}(X')\big) \\
&= \omega_g(X) + \underbrace{g\cdot\omega_{g'}(X')}_{\text{by (2.19)}}.
\end{aligned}
$$

□

A direct consequence of the multiplicativity property of the Cartan form is that its kernel, the Cartan distribution, is also multiplicative in a sense that is made precise in the following proposition. In fact, in a certain sense, the multiplicativity of the Cartan distribution is equivalent to the multiplicativity of the Cartan form. We refer the reader to [13] (Lemma 3.6) for more details.

Proposition 2.4.4. *The Cartan distribution $C_\omega \subset TJ^k\mathcal{G}$ of $J^k\mathcal{G}$ is a (wide) Lie subgroupoid of the tangent groupoid $TJ^k\mathcal{G} \rightrightarrows TM$ (see Example 2.1.1).*

Proof. To begin with, note that the restriction of the source map ds of $TJ^k\mathcal{G}$ to $C_\omega \subset TJ^k\mathcal{G}$ is a surjective submersion (see (1.27)). We show that the multiplicativity of ω implies that C_ω is closed under the multiplication map dm, the unit map du and the inverse map di. Choosing a pair $X, X' \in C_\omega$ of composable arrows, i.e. $X \in T_gJ^k\mathcal{G},\ X' \in T_{g'}J^k\mathcal{G}$ such that $s(g) = t(g')$, $ds(X) = dt(X')$ and $\omega(X) = \omega(X') = 0$, then

$$\omega(dm(X, X')) = \omega(X) + g\cdot\omega(X') = 0.$$

Next, for all $X \in T_xM$,

$$\omega(du(X)) = \omega(dm(du(X), du(X))) = \omega(du(X)) + 1_x\cdot\omega(du(X)) = 2\omega(du(X)),$$

and hence $\omega(du(X)) = 0$. Finally, let $X \in T_gJ^k\mathcal{G}$ such that $\omega(X) = 0$, and note that

$$0 = \omega(dm(di(X), X)) = \omega(di(X)) + g\cdot\omega(X).$$

□

Remark 2.4.5. Let us comment on the importance of the multiplicativity property of the Cartan form. In Chapter 1, we isolated a set of properties of the Cartan form on a jet bundle (Proposition 1.3.4) and claimed that these are the essential ones. We motivated this by showing that the construction of the prolongation and formal integrability of a PDE depend precisely on these properties (Section 1.6). These properties allowed us at each step to construct the prolongation and determine its affine structure. In the groupoid setting, we are interested in studying PDEs whose set of solutions forms a pseudogroup (Definition 3.1.1), and, more generally, a generalized pseudogroup (Definition 3.6.1). This means studying PDEs that live inside jet groupoids and have the property that they are themselves subgroupoids. If we were to prove formal integrability of such PDEs, as in Section 1.6, we would see that the multiplicativity of the Cartan form is precisely the property that ensures that the prolongations of such PDEs are themselves groupoids. $\diamondsuit$

2.5 The Spencer Operator on a Jet Algebroid

Returning to the infinitesimal picture, in the global vs. infinitesimal correspondence, under which Lie groupoid correspond to Lie algebroids and jet groupoids correspond to jet algebroids, the Cartan form on a jet groupoid corresponds to the so called Spencer operator on a jet algebroid. Accordingly, if A is the Lie algebroid of a Lie groupoid $\mathcal{G}$, then, as we will see, the Spencer operator on J^kA is obtained by linearizing the Cartan form on $J^k\mathcal{G}$.

Let A be a Lie algebroid over M. Recall that any linear jet bundle, and in particular the jet algebroid J^kA, is equipped with a Spencer operator

$$D : \mathfrak{X}(M) \times \Gamma(J^kA) \to \Gamma(J^{k-1}A)$$

as defined in Proposition 1.8.1. The Spencer operator, as we recall, can also be regarded as a map

$$D : \Gamma(J^kA) \to \Omega^1(M; J^{k-1}A). \tag{2.21}$$

The essential properties of the Spencer operator of a linear jet bundle were collected in Proposition 1.8.4. In addition to these properties, the Spencer operator on a jet algebroid is also compatible with the Lie algebroid structure of the jet algebroid, and, as for the Cartan form, this compatibility can be expressed as a single property which we call multiplicativity and which we now explain.

Recall from Section 2.1 that a representation of a Lie algebroid $A \to M$ consists of a vector bundle $E \to M$ together with a flat A-connection on E. A jet algebroid J^kA has a canonical representation called the **adjoint representation**, the infinitesimal counterpart of the adjoint representation at the Lie groupoid level. The vector bundle is the jet algebroid $J^{k-1}A$ and the J^kA-connection

$$\nabla : \Gamma(J^kA) \times \Gamma(J^{k-1}A) \to \Gamma(J^{k-1}A)$$

is defined uniquely by the condition

$$\nabla_{j^k\alpha}(j^{k-1}\beta) = j^{k-1}[\alpha, \beta] \qquad \forall\, \alpha, \beta \in \Gamma(A).$$

One readily checks that this connection is flat. While this definition is natural, it is not obviously well defined. Using the Spencer operator, we also have the following equivalent description:

$$\nabla_\alpha(\beta) = [\pi(\alpha), \beta] + D_{\rho(\beta)}(\alpha), \qquad \forall \alpha \in \Gamma(J^k A), \beta \in \Gamma(J^{k-1}A).$$

Next, having a J^kA-connection, one defines a Lie derivative operation

$$\mathcal{L}_\alpha : \Omega^1(M; J^{k-1}A) \to \Omega^1(M; J^{k-1}A),$$

for any $\alpha \in \Gamma(J^kA)$. It is defined by

$$\mathcal{L}_\alpha\omega(X) = \nabla_\alpha(\omega(X)) - \omega([\rho(\alpha), X]).$$

Of course, one can also extend this Lie derivative operation to higher degree forms. We refer the reader to [13] (Section 2.2) for details. Finally, the Spencer operator on J^kA satisfies the following property:

$$D([\alpha, \beta]) = \mathcal{L}_\alpha D(\beta) - \mathcal{L}_\beta D(\alpha), \qquad \forall\ \alpha, \beta \in \Gamma(J^kA). \tag{2.22}$$

We will call (2.22) the multiplicativity property, and say that the Spencer operator is **multiplicative**. In the global vs. infinitesimal correspondence, (2.22) is the infinitesimal counterpart of property (2.17). A simple exercise in unraveling the definitions shows that the multiplicativity property (2.22) can also be expressed in the following straightforward (but less conceptual) form:

$$D_X[\alpha, \beta] - [D_X\alpha, \pi(\beta)] - [\pi(\alpha), D_X\beta] = D_{\rho D_X\beta + [\rho(\beta), X]}\alpha - D_{\rho D_X\alpha + [\rho(\alpha), X]}\beta.$$

We omit the proof that the Spencer operator on J^kA is indeed multiplicative, the analogue of Proposition 2.4.3. For the case $k = 1$, from which one can deduce the general case, see [62] (Proposition 5.1.18).

Remark 2.5.1. Just as one can define the notion of multiplicativity for representation-valued forms of arbitrary degrees on a Lie groupoid (Definition 2.4.1), one can also define Spencer operators of arbitrary degree on Lie algebroids with values in a representation. This notion was introduced in [13], where the authors also explain how multiplicative forms linearize to Spencer operators and prove an integrability result. $\diamondsuit$

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The Spencer operator of the jet algebroid J^kA of the Lie algebroid $A = A(\mathcal{G})$ of $\mathcal{G}$ is obtained by linearizing the Cartan form on $J^k\mathcal{G}$ via the following formula:

$$D_X(\alpha)_x = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi^\epsilon_\alpha(x)^{-1} \cdot \omega(d\varphi^\epsilon_\alpha(X_x)), \tag{2.23}$$

for all $x \in M,\ X \in \mathfrak{X}(M)$ and $\alpha \in \Gamma(J^kA)$, where φ^ϵ_α is the flow of bisections of $J^k\mathcal{G}$ induced by α. To check that this formula is correct, one verifies that D satisfies the defining properties of Proposition 1.8.1. The fact that $D(j^k\alpha) = 0$ for all $\alpha \in \Gamma(A)$ follows from the fact that the Cartan form kills holonomic sections of $J^k\mathcal{G}$ and because $\varphi^\epsilon_{j^k\alpha}(x) = j^k_x\varphi^\epsilon_\alpha$ (a simple exercise). The fact that D satisfies the connection-like properties (1.52) becomes clear from (2.24) of the following lemma.

Lemma 2.5.2. *Let ω be the Cartan form on $J^k\mathcal{G}$ and D the Spencer operator on J^kA. Then*

$$D_X(\alpha)_{t(g)} = \omega([\widehat{X}, \widetilde{\alpha}]_g), \tag{2.24}$$

for all $g \in J^k\mathcal{G}$, $X \in \mathfrak{X}(M)$ and $\alpha \in \Gamma(J^kA)$. Here, $\widetilde{\alpha} \in \mathfrak{X}(J^k\mathcal{G})$ is the right invariant vector field induced by α and $\widehat{X} \in \mathfrak{X}(J^k\mathcal{G})$ is a choice of a lift of X that satisfies

$$dt(\widehat{X}) = X \qquad \text{and} \qquad \omega(\widehat{X}) = 0.$$

Proof. First note that a lift $\widehat{X}$ always exist because $dt : C_\omega \to t^*TM$ is surjective. Let us write $x = t(g)$. By Lemma 2.1.5 we know that $\varphi^\epsilon_{\widetilde{\alpha}}(g) = \varphi^\epsilon_\alpha(x) \cdot g$ and, by replacing g with a curve representing $\widehat{X}_g$, that $d\varphi^\epsilon_{\widetilde{\alpha}}(\widehat{X}_g) = dm(d\varphi^\epsilon_\alpha(X_x), \widehat{X}_g)$. Applying ω on both sides of the latter expression and using the multiplicativity of ω,

$$\omega(d\varphi^\epsilon_{\widetilde{\alpha}}(\widehat{X}_g)) = \omega(d\varphi^\epsilon_\alpha(X_x)) + \varphi^\epsilon_\alpha(x) \cdot \cancel{\omega(\widehat{X}_g)}.$$

With these identities, the right hand side of (2.23) can be re-expressed as

$$\varphi^\epsilon_\alpha(x)^{-1} \cdot \omega(d\varphi^\epsilon_\alpha(X_x)) = g \cdot \varphi^\epsilon_{\widetilde{\alpha}}(g)^{-1} \cdot \omega(d\varphi^\epsilon_{\widetilde{\alpha}}(\widehat{X}_g)).$$

One now derives the desired expression as follows:

$$\begin{aligned}
D_X(\alpha)_x &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi^\epsilon_\alpha(x)^{-1} \cdot \omega(d\varphi^\epsilon_\alpha(X_x)) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} g \cdot \varphi^\epsilon_{\widetilde{\alpha}}(g)^{-1} \cdot \omega(d\varphi^\epsilon_{\widetilde{\alpha}}(\widehat{X}_g)) \\
&= \lim_{\epsilon\to 0} \frac{g \cdot \varphi^\epsilon_{\widetilde{\alpha}}(g)^{-1} \cdot \omega(d\varphi^\epsilon_{\widetilde{\alpha}}(\widehat{X}_g)) - \omega(\widehat{X}_g)}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{g \cdot \varphi^\epsilon_{\widetilde{\alpha}}(g)^{-1} \cdot \omega(d\varphi^\epsilon_{\widetilde{\alpha}}(d\varphi^{-\epsilon}_{\widetilde{\alpha}}\, \widehat{X}_{\varphi^\epsilon_{\widetilde{\alpha}}(x)})) - \omega(d\varphi^{-\epsilon}_{\widetilde{\alpha}}\, \widehat{X}_{\varphi^\epsilon_{\widetilde{\alpha}}(g)})}{\epsilon} \\
&= \lim_{\epsilon\to 0} \frac{g \cdot \varphi^\epsilon_{\widetilde{\alpha}}(g)^{-1} \cdot \cancel{\omega(\widehat{X}_{\varphi^\epsilon_{\widetilde{\alpha}}(g)})} - \omega(d\varphi^{-\epsilon}_{\widetilde{\alpha}}\, \widehat{X}_{\varphi^\epsilon_{\widetilde{\alpha}}(g)})}{\epsilon} \\
&= -\omega\Big(\frac{d}{d\epsilon}\Big|_{\epsilon=0} d\varphi^{-\epsilon}_{\widetilde{\alpha}}\, \widehat{X}_{\varphi^\epsilon_{\widetilde{\alpha}}(g)}\Big) \\
&= \omega([\widehat{X}, \widetilde{\alpha}]_g)
\end{aligned}$$

The only tricky step is in the fourth equality. There, $\hat{X}_g$ can be replaced by $d\varphi^{-\epsilon}_{\widetilde{\alpha}}\, \widehat{X}_{\varphi^\epsilon_{\widetilde{\alpha}}(g)}$ because the two are equal in the limit. □

2.6 Jet Groupoids and Jet Algebroids in Local Coordinates

It is instructive to get our hands dirty and compute some explicit examples of jet groupoids and jet algebroids in local coordinates. In this section, we will describe the coordinate charts of the jet groupoids J^kM and the jet algebroids $J^kTM \cong A^k(M)$ in the cases $k = 0$ and $k = 1$, and we will compute the Cartan form and Spencer operator in the case $k = 1$. The formulas that we will obtain here will serve us when computing explicit examples of Lie pseudogroups in later chapters.

Example 2.6.1. For $k = 0$,

$$J^0M = M \times M \qquad \text{and} \qquad A^0(M) = TM.$$

Let $M = \mathbb{R}^n$. Choose coordinates $x = (x_1, ..., x_n)$ on M, which induce coordinates $(X, x) = (X_1, ..., X_n, x_1, ..., x_n)$ on $M \times M$. Thus,

$$J^0M = \{(X, x) = (X_1, ..., X_n, x_1, ..., x_n) \mid x \in \mathbb{R}^n, X \in \mathbb{R}^n\},$$

where a representative ϕ of a 0-jet $j^0_x\phi \in J^0M$ with coordinates (X, x) satisfies $\phi(x) = X$. The structure maps are accordingly

$$s(X, x) = x, \quad t(X, x) = X,$$
$$(\bar{X}, X) \cdot (X, x) = (\bar{X}, x), \quad 1_x = (x, x), \quad (X, x)^{-1} = (x, X).$$

On the infinitesimal side, the fiber of the Lie algebroid $A^0(M)$ at a point $x \in M$ is

$$A^0(M)_x = T_{1_x}\big(s^{-1}(x)\big) = T_{(x,x)}\{(X, x) \mid X \in \mathbb{R}^n\},$$

and hence $A^0(M)$ has a frame

$$\partial_{X_1}, ..., \partial_{X_n} \in \Gamma(A^0(M)) \quad \text{where} \quad \partial_{X_i}(x) := \frac{\partial}{\partial X_i}(x, x).$$

Using the right translation map

$$R_{(X,x)} : s^{-1}(X) \to s^{-1}(x), \quad (\bar{X}, X) \mapsto (\bar{X}, x),$$

we compute the following induced right invariant vector fields on J^0M:

$$\widetilde{\partial}_{X_1}, ..., \widetilde{\partial}_{X_n} \in \mathfrak{X}(J^0M) \quad \text{where} \quad \widetilde{\partial}_{X^i}(X, x) = \frac{\partial}{\partial X_i}(X, x).$$

The bracket of $A^0(M)$ is then determined by

$$[\partial_{X_i}, \partial_{X_j}] = 0,$$

and the anchor, which provides an identification between $A^0(M)$ and TM, is

$$\rho : A^0(M) \to TM, \qquad \partial_{X_i} \mapsto \frac{\partial}{\partial x_i}.$$

On the other hand, the jet algebroid J^0TM has a frame

$$j^0(\frac{\partial}{\partial x_i}) \in \Gamma(J^0TM),$$

and the isomorphism (2.13) between J^0TM and $A^0(M)$ is given by

$$\Gamma(J^0TM) \ni j^0(\frac{\partial}{\partial x_i}) \quad \mapsto \quad \partial_{X_i} \in \Gamma(A^0(M)). \qquad \diamondsuit$$

Example 2.6.2. Let $M = \mathbb{R}^n$, with coordinates as in the previous example. For $k = 1$,

$$J^1M = \{(X, x, p) \mid x \in \mathbb{R}^n, X \in \mathbb{R}^n,\ p = (p_i^j)_{1\le i,j\le n} \in GL_n\}, \tag{2.25}$$

where a representative ϕ with components $\phi_1, ..., \phi_n$ of an element $j_x^1\phi \in J^1M$ with coordinates (X, x, p) satisfies $\phi(x) = X$ and $\frac{\partial \phi_i}{\partial x_j}(x) = p_i^j$. The structure maps are

$$s(X, x, p) = x, \quad t(X, x, p) = X, \quad (\bar{X}, X, q)\cdot(X, x, p) = (\bar{X}, x, qp),$$
$$1_x = (x, x, 1_{n\times n}), \quad (X, x, p)^{-1} = (x, X, p^{-1}),$$

where $(qp)_i^j = q_i^l p_l^j$. The fiber of $A^1(M)$ at x is

$$A^1(M)_x = T_{1_x}(s^{-1}(x)) = T_{(x,x,1_{n\times n})}\{(X, x, p) \mid X \in \mathbb{R}^n,\ p \in GL_n\},$$

and $A^1(M)$ has a frame

$$\partial_{X_i}(x) := \frac{\partial}{\partial X_i}(x, x, 1_{n\times n}), \qquad \partial_{p_i^j}(x) := \frac{\partial}{\partial p_i^j}(x, x, 1_{n\times n}). \tag{2.26}$$

Using the right translation map

$$R_{(X,x,p)} : s^{-1}(X) \to s^{-1}(x), \quad (\bar{X}, X, q) \mapsto (\bar{X}, x, qp),$$

we compute the induced right invariant vector fields

$$\widetilde{\partial}_{X_i}(X, x, p) = \frac{\partial}{\partial X_i}(X, x, p), \quad \widetilde{\partial}_{p_i^j}(X, x, p) = \sum_l p_j^l \frac{\partial}{\partial p_i^l}(X, x, p).$$

The bracket of $A^1(M)$ is then

$$[\partial_{X_i}, \partial_{X_j}] = 0, \quad [\partial_{X_i}, \partial_{p_j^l}] = 0, \quad [\partial_{p_i^j}, \partial_{p_l^m}] = \delta_m^i p_j^n \partial_{p_l^n} - \delta_j^l p_m^n \partial_{p_i^n}, \tag{2.27}$$

where δ is the Kronecker symbol, and the anchor is

$$\rho : A^1(M) \to TM, \qquad \partial_{X_i} \mapsto \frac{\partial}{\partial x_i}, \quad \partial_{p_i^j} \mapsto 0.$$

On the other hand, J^1TM has a frame

$$j^1(\frac{\partial}{\partial x_i}),\ j^1(x_j\frac{\partial}{\partial x_i}) \in \Gamma(J^1TM), \tag{2.28}$$

and the isomorphism between J^1TM and $A^1(M)$ is given by

$$j^1(\frac{\partial}{\partial x_i}) \ \mapsto\ \partial_{X_i}, \qquad j^1(x_j\frac{\partial}{\partial x_i}) \ \mapsto\ x_j\partial_{X_i} + \partial_{p_i^j}. \tag{2.29}$$

The Cartan form $\omega \in \Omega^1(J^1M; t^*A^0(M))$ on J^1M is computed using (2.16):

$$\omega = (dX_i - p_i^j dx_j)\, t^*\partial_{X_i}.$$

The Spencer operator $D : \Gamma(J^1TM) \to \Omega^1(M; J^0TM)$ is uniquely determined by the connection-like properties (1.52) and by the condition that it kills the frame (2.28). Finally, one computes the Spencer operator $D : \Gamma(A^1(M)) \to \Omega^1(M; A^0(M))$ either from ω by using (2.24) or from the Spencer operator on J^1TM via the isomorphism (2.29). In either case,

$$D:\ \partial_{X_i} \mapsto 0, \quad \partial_{p_i^j} \mapsto -\delta_j^l dx_l \otimes \partial_{X_i}. \tag{2.30}$$

$\diamondsuit$

2.7 Pfaffian Groupoids and Pfaffian Algebroids

In Section 1.9, we discussed the point of view of studying PDEs abstractly as spaces equipped with a "PDE structure". The analogous program was carried out in the "multiplicative world" in [62]. In the multiplicative world, the notion of a Pfaffian bundle becomes a *Pfaffian groupoid*, and on the linear or infinitesimal side, the notion of a linear Pfaffian bundle becomes a *Pfaffian algebroid*. The notion of a Pfaffian groupoid will play a role in our presentation of Cartan's theory of Lie pseudogroups, where we will see that some of Cartan's key constructions and concepts can be understood more conceptually in the language of Pfaffian groupoids. In this section, we recall the definitions of a Pfaffian groupoid and a Pfaffian algebroid and some immediate consequences. We refer the reader to [62] (Chapters 5 and 6) for further details.

Pfaffian Groupoids Given a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a 1-form $\omega \in \Omega^1(\mathcal{G}; t^*E)$ with values in a vector bundle $E \to M$, we write

$$C_\omega := \mathrm{Ker}\ \omega \subset T\mathcal{G}.$$

Definition 2.7.1. *A **Pfaffian groupoid** over a manifold M consists of a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a pointwise surjective multiplicative form*

$$\omega \in \Omega^1(\mathcal{G}; t^*E)$$

with values in a representation $E \to M$ of $\mathcal{G}$, such that $(\mathcal{G}, \omega)$ is a Pfaffian bundle over M (Definition 1.9.1), i.e.

1. $ds\big|_{C_\omega} : C_\omega \to s^*TM$ *is pointwise surjective (equivalently,* $C_\omega + \mathrm{Ker}\ ds = T\mathcal{G}$*),*
2. $C_\omega \cap \mathrm{Ker}\ ds$ *is an involutive distribution.*

We say that a Pfaffian groupoid $(\mathcal{G}, \omega)$ *is a* ***Lie-Pfaffian groupoid*** *if*

3. $C_\omega \cap \mathrm{Ker}\ dt = C_\omega \cap \mathrm{Ker}\ ds$.

A ***(local) solution*** *or a* ***(local) holonomic bisection*** *of* $(\mathcal{G}, \omega)$ *is a (local) bisection* η *of* $\mathcal{G}$ *satisfying* $\eta^*\omega = 0$*. We denote the set of local holonomic bisections by* $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$.

Remark 2.7.2. In [62], the notion of a Pfaffian groupoid is defined as a Lie groupoid equipped with a multiplicative distribution that satisfies the same conditions as in the definition above. To go from our definition to their definition, simply take C_ω as the multiplicative distribution. For the opposite direction, see Section 6.1.2 in [62]. $\diamondsuit$

Let $(\mathcal{G}, \omega)$ be a Pfaffian groupoid and let $A = A(\mathcal{G})$ be the Lie algebroid of $\mathcal{G}$. The vector bundle

$$\mathfrak{g} = C_\omega\big|_M \cap A \tag{2.31}$$

is called the **symbol space** of $(\mathcal{G}, \omega)$. The identification $t^*A \cong T^s\mathcal{G}$ given by right translation restricts to the identification $t^*\mathfrak{g} \cong C_\omega \cap \mathrm{Ker}\ ds$. This relates the definition (1.58) of the symbol space of a Pfaffian bundle with (2.31). Note that condition 2 is equivalent to saying that $\mathfrak{g} \subset A$ is a Lie subalgebroid.

As for Pfaffian bundles, we can talk about **Cartan-Ehresmann connections** on $(\mathcal{G}, \omega)$, i.e. splittings H of the map $ds|_{C_\omega} : C_\omega \to s^*TM$, and we have the **symbol map** of ω, which simplifies to the vector bundle map over M,

$$\partial_\omega : \mathfrak{g} \to \mathrm{Hom}(TM, E), \qquad T \mapsto \delta\omega(H(\cdot), T),$$

where $H : s^*TM \to C_\omega$ is some choice of a Cartan-Ehresmann connection. A Pfaffian groupoid is said to be **standard** if its symbol map is injective.

Example 2.7.3. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. The jet groupoid $J^k\mathcal{G}$, with $k > 0$, equipped with the Cartan form is a standard Lie-Pfaffian groupoid. $\diamondsuit$

Condition 3, with which a Pfaffian groupoid becomes a Lie-Pfaffian groupoid, is special to the multiplicative setting and has important implications. First, the vector bundle E underlying the representation has a unique Lie algebroid structure with which $\omega|_A : A \to E$ becomes a surjective Lie algebroid map ([62], Proposition 6.1.8). Its kernel is precisely $\mathfrak{g}$. Secondly:

Lemma 2.7.4. *Let* $(\mathcal{G}, \omega)$ *be a Pfaffian groupoid. There exists a unique isomorphism* $s^*TM \cong t^*TM$ *with which the following diagram commutes and the rows are exact:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & t^*\mathfrak{g} & \longrightarrow & C_\omega & \xrightarrow{ds} & s^*TM & \longrightarrow & 0 \\ & & \downarrow\| & & \downarrow\| & & \downarrow\wr & & \\ 0 & \longrightarrow & t^*\mathfrak{g} & \longrightarrow & C_\omega & \xrightarrow{dt} & t^*TM & \longrightarrow & 0. \end{array}$$

Proof. The top short exact sequence is condition 1 of the definition of a Pfaffian groupoid. The bottom one then follows from condition 3 by a dimension count. The isomorphism $s^*TM \xrightarrow{\simeq} t^*TM$ is given by choosing a splitting $H : s^*TM \to C_\omega$ of $ds|_{C_\omega} : C_\omega \to s^*TM$ and composing it with dt. Condition 3 implies that the isomorphism is independent of the choice of splitting, since the difference of any such splittings H and H' takes values in $C_\omega \cap \mathrm{Ker}\ ds$ and is, hence, killed by dt. □

Pfaffian Algebroids Turning to the infinitesimal side, let A and E be Lie algebroids over M and let $\pi : A \to E$ be a surjective Lie algebroid map. Any π-connection D on A (see Definition 1.9.5) induces a connection $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ defined by

$$\nabla_\alpha(\beta) = [\pi(\alpha), \beta] + D_{\rho(\beta)}(\alpha), \qquad \forall \alpha \in \Gamma(A), \beta \in \Gamma(E).$$

The connection, in turn, induces the Lie derivative operation

$$\mathcal{L}_\alpha : \Omega^1(M; A) \to \Omega^1(M; A),$$

for every $\alpha \in \Gamma(A)$, that is defined by

$$\mathcal{L}_\alpha \omega(X) = \nabla_\alpha(\omega(X)) - \omega([\rho(\alpha), X]).$$

Definition 2.7.5. *A* ***Lie-Pfaffian algebroid*** *over a manifold M consists of a pair of Lie algebroids A and E over M, a surjective Lie algebroid map $\pi : A \to E$, and a π-connection*

$$D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E),$$

such that

$$D([\alpha, \beta]) = \mathcal{L}_\alpha(D(\beta)) - \mathcal{L}_\beta(D(\alpha)), \qquad \forall\ \alpha, \beta \in \Gamma(A).$$

Remark 2.7.6. Here we are only defining the infinitesimal counterpart of a Lie-Pfaffian groupoid. For the infinitesimal counterpart of a Pfaffian groupoid, see [62]. ♢

Example 2.7.7. Let $A \to M$ be a Lie algebroid. The jet algebroid J^kA, with $k > 0$, equipped with the Spencer operator is a standard Lie-Pfaffian algebroid. ♢

Let (A, D) be a Pfaffian algebroid. We call the operator D the **Spencer operator**. A Pfaffian algebroid is, in particular, a linear Pfaffian bundle. We thus have the **symbol space** $\mathfrak{g} = \mathrm{Ker}\ (\pi : A \to E)$, the **symbol map** $\partial_D : \mathfrak{g} \to \mathrm{Hom}(TM, E)$ defined in (1.62), and we say that a Pfaffian algebroid is **standard** if its symbol map is injective.

To complete the story, we state the following theorem that relates the global and infinitesimal pictures. For the proof, see [62] (Theorem 6.1.23 and Proposition 6.1.25).

Theorem 2.7.8. *If $(\mathcal{G}, \omega)$ is a Lie-Pfaffian groupoid, then (A, D_ω) is a Lie-Pfaffian algebroid, where $A = A(\mathcal{G})$ and D_ω is the linearization of ω (see* (2.23)*). Conversely, if (A, D) is a Lie-Pfaffian algebroid and $\mathcal{G}$ is a simply-connected integration of A, then there is a unique Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$ integrating (A, D).*

Chapter 3

Lie Pseudogroups

Until now we have sufficed with the intuitive idea that a Lie pseudogroup is a group-like set of local transformations of a manifold that is defined as the set of solutions of a PDE. In this chapter, we give the rigorous definition of this notion and show that the defining equations of a Lie pseudogroup have the structure of a Lie-Pfaffian groupoid, a notion that was discussed in Section 2.7. This latter observation will lead us to a natural generalization of pseudogroups on manifolds to pseudogroup-like structures on Lie groupoids called *generalized pseudogroups*. These will play an important role in our modern formulation of Cartan's structure theory.

3.1 Pseudogroups

A pseudogroup is, intuitively, a set of *local symmetries* (think for example of the set of all local isometries of a Riemannian manifold). The word "symmetries" refers to the group-like structure of a pseudogroup, while the word "local" refers to the sheaf-like structure of a pseudogroup.

Let M be a manifold. A **local diffeomorphism** of M is a diffeomorphism $\phi : U \to V$ between two open subsets $U, V \subset M$. We denote the set of all local diffeomorphisms of M by $\mathrm{Diff}_{\mathrm{loc}}(M)$. Warning: do not confuse this meaning of the term local diffeomorphism with the more common meaning of the term, namely a smooth map that is locally, around every point, a diffeomorphism (to avoid confusion, one may also use the slightly longer term "locally defined diffeomorphism").

Definition 3.1.1. *A **pseudogroup** on a manifold M is a subset $\Gamma \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ that satisfies the following axioms:*

1. *Group-like axioms:*

   1) *if $\phi, \phi' \in \Gamma$ and $\mathrm{Im}(\phi') \subset \mathrm{Dom}(\phi)$, then $\phi \circ \phi' \in \Gamma$,*

   2) *if $\phi \in \Gamma$, then $\phi^{-1} \in \Gamma$,*

   3) $\mathrm{id}_M \in \Gamma$.

2. *Sheaf-like axioms:*

   1) *if $\phi \in \Gamma$ and $U \subset \mathrm{Dom}(\phi)$ is an open subset, then $\phi|_U \in \Gamma$,*

   2) *if $\phi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ and $\{U_i\}_{i \in I}$ is an open cover of $\mathrm{Dom}(\phi)$ such that $\phi|_{U_i} \in \Gamma$ for all $i \in I$, then $\phi \in \Gamma$.*

Given a pseudogroup Γ on M, an **orbit** of Γ is an equivalence class of points of M given by the equivalence relation

$$x \sim y \quad \text{if and only if} \quad \exists\, \phi \in \Gamma \text{ such that } \phi(x) = y.$$

A pseudogroup is said to be **transitive** if it has a single orbit, and otherwise it is said to be **intransitive**.

Remark 3.1.2. There is an important distinction between the "sheaf-like" axioms of a pseudogroup and the usual axioms of a sheaf. For a sheaf, if a collection of sections agree on all pairwise intersections, then they are required to fit together to form a single section of the sheaf. For a pseudogroup, if a collection of elements agree on all pairwise intersections, then they are required to fit together to form an element only if they first fit together to form a local diffeomorphism. ♢

Remark 3.1.3. Any subset $\Gamma_0 \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ that satisfies the three group-like properties generates a pseudogroup $\langle\Gamma_0\rangle$ by "imposing" the sheaf-like axioms, i.e. by adding all possible restrictions and gluings of elements of Γ_0. In other words, $\langle\Gamma_0\rangle$ is the smallest pseudogroup on M containining Γ_0. Many examples of pseudogroups arise in this way. ♢

3.2 Haefliger's Approach to Pseudogroups

In this thesis, we are mainly interested in pseudogroups of symmetries of geometric structures, or, more generally, pseudogroups that are defined by a PDE. Pseudogroups which are not of this type, however, appear naturally when studying the transversal geometry of foliations (the so called pseudogroups of holonomy transformations). In that context, Haefliger recasted the notion of pseudogroup in the framework of étale Lie groupoids. The étale point of view allows for more elegant formulations, a fact that we will use later when making Cartan's notion of equivalence of pseudogroups precise.

Haefliger's observation is that pseudogroups are the same thing as *effective étale groupoids* ([26]). Let us briefly outline this point of view. A Lie groupoid $\mathcal{G} \rightrightarrows M$, with source map s and target map t, is called **étale** if around each arrow $g \in \mathcal{G}$ there exists an open neighborhood U such that the restriction of s to U is a diffeomorphism onto its image. An étale groupoid is called **effective** if for any pair of local bisections b and b' with a common domain, $t \circ b = t \circ b'$ implies $b = b'$.

With any pseudogroup Γ on M, we associate the groupoid

$$\mathcal{G}\mathrm{erm}(\Gamma) \rightrightarrows M$$

of germs of Γ. The space of arrows of $\mathcal{G}\mathrm{erm}(\Gamma)$ consists of germs of elements of Γ, where a germ at $x \in \mathrm{Dom}(\phi)$ with representative $\phi \in \Gamma$ is denoted by $\mathrm{germ}_x\phi$. The structure maps of the groupoid are

$$s(\mathrm{germ}_x\phi) = x, \qquad t(\mathrm{germ}_x\phi) = \phi(x), \qquad 1_x = \mathrm{germ}_x\mathrm{id},$$
$$\mathrm{germ}_{\phi(x)}\phi' \cdot \mathrm{germ}_x\phi = \mathrm{germ}_x(\phi' \circ \phi), \qquad (\mathrm{germ}_x\phi)^{-1} = \mathrm{germ}_{\phi(x)}\phi^{-1}.$$

Every element $\phi \in \Gamma$ gives rise to a local bisection b_ϕ of $\mathcal{G}\mathrm{erm}(\Gamma)$ defined by $b_\phi(x) = \mathrm{germ}_x\phi$, for all $x \in \mathrm{Dom}(\phi)$. The groupoid $\mathcal{G}\mathrm{erm}(\Gamma)$ has a natural smooth structure (with a non-Hausdorff topology in general) that is uniquely defined by the requirement that each such local bisection must be a diffeomorphism onto its image. With this smooth structure, $\mathcal{G}\mathrm{erm}(\Gamma) \rightrightarrows M$ becomes an effective étale Lie groupoid. Conversely, any effective étale Lie groupoid $\mathcal{G} \rightrightarrows M$ induces the pseudogroup

$$\Gamma(\mathcal{G}) = \{\ t \circ b \mid b \text{ local bisection of } \mathcal{G}\ \} \subset \mathrm{Diff}_{\mathrm{loc}}(M).$$

The maps $\Gamma \mapsto \mathcal{G}\mathrm{erm}(\Gamma)$ and $\mathcal{G} \mapsto \Gamma(\mathcal{G})$ between the set of pseudogroups on M and the set of effective étale Lie groupoids with base M are inverse to each other. We thus have:

Proposition 3.2.1. *Let M be a manifold. There is a 1-1 correspondence*

$$\{\ \textit{pseudogroups}\ \Gamma \textit{ on } M\ \} \quad \longleftrightarrow \quad \{\ \textit{effective étale Lie groupoids}\ \mathcal{G} \rightrightarrows M\ \},$$

given by $\Gamma \mapsto \mathcal{G}\mathrm{erm}(\Gamma)$ in the one direction and $\mathcal{G} \mapsto \Gamma(\mathcal{G})$ in the other.

Proving this proposition is an exercise in unraveling the definitions. See also [26] (Chapter I, Section 6) or [53] (Example 5.23).

3.3 Lie Pseudogroups

We are now ready to present the definition of a Lie pseudogroup, a definition which relies on the language of jets (Chapter 1) and on the notion of a jet groupoid (Chapter 2). Note that, in the literature, there are slight variations in the precise definition of a Lie pseudogroup, the differences being mainly in the precise regularity conditions that are imposed (e.g., compare axioms 1 and 2 in Section 3 of [24], Definition IV.1 in [37] and Definition 3.1 in [57]). In our definition, we impose enough regularity conditions that will allow us to detach the essential structure of a Lie pseudogroup (in particular, the Cartan form) from its ambient jet groupoid and to study it as a stand-alone geometric object.

Let Γ be a pseudogroup on a manifold M and consider the jet groupoids J^kM (see Section 2.2 and, in particular, Example 2.2.1). For each integer $k \geq 0$, Γ induces a subgroupoid

$$J^k\Gamma := \{\ j^k_x\phi \mid \phi \in \Gamma,\ x \in \mathrm{Dom}(\phi)\ \} \subset J^kM$$

called the **k-th jet groupoid** of Γ. The fact that it is a subgroupoid follows directly from the group-like axioms of a pseudogroup. We denote the source and target maps of $J^k\Gamma$ by $s : J^k\Gamma \to M$ and $t : J^k\Gamma \to M$. Recall that these are defined by $s(j^k_x\phi) = x$ and $t(j^k_x\phi) = \phi(x)$ for all $j^k_x\phi \in J^k\Gamma$. The projection between two consecutive jet groupoids of a given pseudogroup Γ,

$$\pi : J^{k+1}\Gamma \to J^k\Gamma, \qquad j^{k+1}_x\phi \mapsto j^k_x\phi,$$

is a groupoid map (it preserves the groupoid structure). Depending on Γ, some or all of the subgroupoids $J^k\Gamma$ may fail to be smooth in the sense that they may fail to be Lie subgroupoids of J^kM.

Definition 3.3.1. *A **Lie pseudogroup of (at least) order** $k > 0$ on a manifold M is a pseudogroup Γ on M that satisfies the following properties:*

1. *$J^k\Gamma \subset J^kM$ and $J^{k-1}\Gamma \subset J^{k-1}M$ are Lie subgroupoids and $\pi : J^k\Gamma \to J^{k-1}\Gamma$ is a submersion (π is automatically surjective).*

2. *If $\phi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ and if $j^k_x\phi \in J^k\Gamma$ for all $x \in \mathrm{Dom}(\phi)$, then $\phi \in \Gamma$.*

Remark 3.3.2. The regularity conditions in the definition ensure that $J^k\Gamma$ is a PDE of order k in the sense of Definition 1.4.1 (c.f. Example 1.4.3). It is, in fact, a "very nice" PDE in the sense that it comes with a large set of solutions, namely each $j^k_x\phi \in J^k\Gamma$ is guaranteed to be represented by at least one solution $\phi \in \Gamma$. $\diamondsuit$

Remark 3.3.3. If one reads Cartan's writings, one sees that Cartan always adds the extra requirement in the definition of a Lie pseudogroup that the defining equations be involutive. In the modern description, this would correspond to requiring that the symbol space of the defining PDE $J^k\Gamma$ of a Lie pseudogroup Γ of order k be involutive as a tableau bundle (Definition 1.5.3). In the analytic setting, which Cartan always works in (albeit implicitly), this condition guarantees that the defining PDE has sufficient solutions (that there is in fact a pseudogroup underlying the equations). In turn, the fact that Cartan allows himself to impose the condition of involutivity is justified by Theorem 1.5.4, which says that, after sufficient prolongation steps, any PDE becomes involutive (see also Remark 1.5.5). $\diamondsuit$

In the final part of this chapter, we will look at some examples of Lie pseudogroups. At this point, let us look at the following non-example due to Sophus Lie:

Example 3.3.4. Let $M = \mathbb{R}^2$ and let (x, y) be coordinates on M. Consider the pseudogroup Γ on M generated by all local diffeomorphisms of the form

$$\phi : (x, y) \mapsto (f(x), f(y)), \qquad\qquad f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}).$$

One readily verifies that the fibers of $s : J^k\Gamma \to M$ and $s : J^kM \to M$ coincide outside of the diagonal $\{(x, x) \mid x \in \mathbb{R}\} \subset M$. This implies that any local diffeomorphism of M that does not intersect the diagonal is a solution of $J^k\Gamma$. Thus, condition 2 of Definition 3.3.1 fails. Of course, in this example, also the regularity condition 1 fails. Hence Γ is a pseudogroup which is not Lie. $\diamondsuit$

3.4 The Pfaffian Groupoids of a Lie Pseudogroup Γ

We now exhibit the essential structure present on the jet groupoids $J^k\Gamma$ associated with a Lie pseudogroup Γ.

Let Γ be a Lie pseudogroup of order k on M. At the infinitesimal level we have the Lie algebroid of $J^k\Gamma$,

$$A^k = A^k(\Gamma) := A(J^k\Gamma).$$

Similarly, the Lie algebroid of $J^{k-1}\Gamma$ is denoted by $A^{k-1} = A^{k-1}(\Gamma)$. The assumption that $\pi : J^k\Gamma \to J^{k-1}\Gamma$ is a submersion implies that $d\pi : A^k \to A^{k-1}$ is surjective. We call its kernel, the vector bundle

$$\mathfrak{g}^k = \mathfrak{g}^k(\Gamma) := \mathrm{Ker}\,(d\pi : A^k \to A^{k-1}) = \mathfrak{g}^k(M) \cap A^k,$$

the **k-th symbol space** of Γ. We thus have the short exact sequence

$$0 \to \mathfrak{g}^k \to A^k \xrightarrow{d\pi} A^{k-1} \to 0.$$

Intuitively, $\mathfrak{g}^k$ is a linear object that measure the difference between $J^k\Gamma$ and $J^{k-1}\Gamma$. Note that the canonical vector bundle isomorphism $t^*A^k \cong T^sJ^k\Gamma$ given by right translation (c.f. (2.14)) restricts to a vector bundle isomorphism

$$t^*\mathfrak{g}^k \cong T^\pi J^k\Gamma$$

between the (pullback of the) k-th symbol space of Γ and the symbol space of $J^k\Gamma$ as a PDE (see (1.35)). We will also refer to $\mathfrak{g}^k$ as the symbol space of $J^k\Gamma$.

We may restrict the Cartan form of J^kM (see (2.15)) to the PDE $J^k\Gamma$. One can check that the restriction takes values in $A^{k-1}(\Gamma)$, which we denote by A^{k-1}. The resulting restriction,

$$\omega \in \Omega^1(J^k\Gamma; t^*A^{k-1}), \tag{3.1}$$

will be called the **Cartan form** of $J^k\Gamma$. Note that due to the assumption that $\pi : J^k\Gamma \to J^{k-1}\Gamma$ is a submersion, the Cartan form on $J^k\Gamma$ is pointwise surjective. The kernel of ω, called the **Cartan distribution** of $J^k\Gamma$, is denoted by

$$C_\omega = \mathrm{Ker}\,\omega \subset TJ^k\Gamma.$$

As for any PDE, we have the following short exact sequence (see (1.37)):

$$0 \to t^*\mathfrak{g}^k \cong T^\pi J^k\Gamma \to C_\omega \xrightarrow{ds} s^*TM \to 0. \tag{3.2}$$

Recall that a right splitting of this short exact sequence is called a **Cartan-Ehresmann connection** on $J^k\Gamma$.

On the infinitesimal side, we may restrict the Spencer operator of $A(J^kM)$ to A^k,

$$D : \mathfrak{X}(M) \times \Gamma(A^k) \to \Gamma(A^{k-1}).$$

The fact that it takes values in A^{k-1} follows from the fact that it is the linearization of ω. We call D the **Spencer operator** of A^k.

The main structure of $J^k\Gamma$ is summarized as follows:

Proposition 3.4.1. *Let Γ be a Lie pseudogroup of order k. Then, $(J^k\Gamma, \omega)$ is a standard Lie-Pfaffian groupoid (Definition 2.7.1) whose associated Lie-Pfaffian algebroid is $(A^k(\Gamma), D)$.*

Moreover, the elements of Γ are in a 1-1 correspondence with the holonomic bisections of $(J^k\Gamma, \omega)$:

$$\Gamma \xrightarrow{\simeq} \mathrm{Bis}_{\mathrm{loc}}(J^k\Gamma, \omega), \qquad \phi \mapsto j^k\phi.$$

Remark 3.4.2. As we will see in the next chapter, Cartan's key construction of structure equations for a given Lie pseudogroup (what we will call a *realization*) relies solely on the Lie-Pfaffian groupoid structure underlying a Lie pseudogroup. In this sense, the notion of a Lie-Pfaffian groupoid encodes the essential structure of a Lie pseudogroup. $\diamondsuit$

Proof. Since $J^k\Gamma$ is a PDE, we know that $(J^k\Gamma, \omega)$ is a Pfaffian bundle. Furthermore, we have two exact sequences that fit into the following commutative diagram (c.f. Lemma 2.7.4) with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & t^*\mathfrak{g}^k \cong T^\pi J^k\Gamma \arrow[r] \arrow[d, "\parallel"] & C_\omega \arrow[r, "ds"] \arrow[d, "\parallel"] & s^*TM \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & t^*\mathfrak{g}^k \cong T^\pi J^k\Gamma \arrow[r] & C_\omega \arrow[r, "dt"] & t^*TM \arrow[r] & 0.
\end{tikzcd}$$

This shows that

$$C_\omega \cap \text{Ker } ds = C_\omega \cap \text{Ker } dt,$$

which is the third property in the definition of a Lie-Pfaffian groupoid. Note that the isomorphism

$$s^*TM \xrightarrow{\simeq} t^*TM \tag{3.3}$$

in the diagram is canonical and is given by $(d\phi)_x$ at a point $j^k_x\phi \in J^k\Gamma$. Globally, given any Cartan-Ehresmann connection H, the isomorphism is equal to $dt \circ H$.

To show that ω is multiplicative, it is enough to show that A^{k-1} is a representation of $J^k\Gamma$, where the action is the restriction of (2.18). We have to verify that the restriction of the action map takes values in A^{k-1}. To show this, one simply rewrites the formula from Lemma 2.4.2 as follows: let $j^k_x\sigma \in J^k\Gamma,\ \alpha_x \in A^{k-1}_x$, then

$$\begin{aligned}
\omega(dm(Y, \widehat{\alpha}_x)) &= dR_{(j^{k-1}_x\sigma)^{-1}} \circ (d\pi) \circ dm(Y, \widehat{\alpha}_x) \\
&= (d\pi) \circ dR_{(j^k_x\sigma)^{-1}} \circ dm(Y, \widehat{\alpha}_x) \\
&= \omega_{(u\circ t)(j^k_x\sigma)}\big(dR_{(j^k_x\sigma)^{-1}} \circ dm(Y, \widehat{\alpha}_x)\big).
\end{aligned}$$

The multiplicativity of the Cartan form on $J^k\Gamma$ follows from the multiplicativity of the one on J^kM.

The last assertion follows from the definition of a Lie pseudogroup together with the fact that ω detects the holonomic bisections of $J^k\Gamma$, i.e. the solutions. $\square$

Remark 3.4.3. Definition 3.3.1 is, strictly speaking, stronger than what is needed in order to obtain a Lie-Pfaffian groupoid structure and, accordingly, in order to study Cartan's structure theory. It would have been sufficient to replace condition 1 with the requirement that $J^k\Gamma$ be a Lie subgroupoid and its symbol space $\mathfrak{g}^k$ be of constant rank. Even though these properties do not guarantee that $J^{k-1}\Gamma$ is smooth, one could still make sense of its Lie algebroid by setting $A^{k-1} = d\pi(A^k)$, and $J^k\Gamma$ would still be a PDE in the sense of

Definition 1.4.1. This, in turn, would imply that there is still an intrinsic Cartan form on $J^k\Gamma$. However, to interpret Cartan's work, it is convenient to have actual coordinates at the $k-1$ level. $\diamondsuit$

Some Examples In Section 5.1, we will present several examples cited from Cartan's work ([7], pp. 1344-1347) in which he derives the structure equations of a selection of Lie pseudogroups. In this section, we have chosen two of these examples and for each of the Lie pseudogroups in those examples, say Γ of order k, we will compute the following:

1. the jet groupoids $J^k\Gamma$ and $J^{k-1}\Gamma$,
2. the associated Lie algebroids $A^k(\Gamma)$ and $A^{k-1}(\Gamma)$,
3. the Cartan form on $J^k\Gamma$,
4. the Spencer operator on $A^k(\Gamma)$.

The computations we present here will exemplify the Lie-Pfaffian structure of a Lie pseudogroup. In addition, they will serve as a preparation for the examples of Section 5.4, where our modern approach is used to compute structure equations explicitly.

Example 3.4.4. Let $M = \mathbb{R}^2\backslash\{y=0\}$ and let (x,y) be coordinates on M. We consider the pseudogroup Γ on M generated by the local diffeomorphisms

$$\phi : (x,y) \mapsto (\phi_x(x,y), \phi_y(x,y)) = (f(x), \frac{y}{f'(x)}), \qquad f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}). \tag{3.4}$$

The domain of each generating element is determined by the domain of f. Note that although this pseudogroup also makes sense on all of $\mathbb{R}^2$, it is not a Lie pseudogroup in this case because the induced jet groupoids fail to be smooth.

The pseudogroup is transitive and, hence, $J^0\Gamma = J^0M$ and $A^0(\Gamma) = A^0(M)$. These were computed in Example 2.6.1. We denote the coordinates on $J^0\Gamma$ by

$$J^0\Gamma = \{(X,Y,x,y) \mid y \neq 0 \text{ and } Y \neq 0\}$$

and $A^0(\Gamma)$ has a frame

$$\partial_X, \partial_Y \in \Gamma(A^0(\Gamma)).$$

To compute $J^1\Gamma$, we begin by calculating the first derivatives of the elements of Γ,

$$\begin{aligned} \frac{\partial\phi_x}{\partial x} &= \frac{y}{\phi_y}, & \frac{\partial\phi_x}{\partial y} &= 0, \\ \frac{\partial\phi_y}{\partial x} &= -\frac{f''(x)y}{(f'(x))^2}, & \frac{\partial\phi_y}{\partial y} &= \frac{\phi_y}{y}. \end{aligned} \tag{3.5}$$

The first, second and fourth relations select a subset of J^1M, while from the third equation we see that any point in this subset is the jet of some element of Γ, since we can always find an f with any prescribed second derivative at a given point x. We can choose global coordinates (X, Y, x, y, u) on $J^1\Gamma$, where a jet $j^1_{(x,y)}\phi$ is assigned to the coordinates $(\phi_x(x,y), \phi_y(x,y), x, y, \frac{\partial \phi_y}{\partial x}(x,y))$. Thus,

$$J^1\Gamma = \{(X, Y, x, y, u) \mid y \neq 0 \text{ and } Y \neq 0\}$$

is a 5-dimensional submanifold of J^1M. The embedding is given by

$$J^1\Gamma \hookrightarrow J^1M, \qquad (X, Y, x, y, u) \mapsto \left(X, Y, x, y, \begin{pmatrix} \frac{y}{Y} & 0 \\ u & \frac{Y}{y} \end{pmatrix}\right). \tag{3.6}$$

The structure maps of $J^1\Gamma$ can be calculated either directly or via the embedding in J^1M. Either way,

$$s(X, Y, x, y, u) = (x, y), \quad t(X, Y, x, y, u) = (X, Y), \quad 1_{(x,y)} = (x, y, x, y, 0),$$
$$(\bar{X}, \bar{Y}, X, Y, \bar{u}) \cdot (X, Y, x, y, u) = (\bar{X}, \bar{Y}, x, y, \frac{\bar{u}y + \bar{Y}u}{Y}),$$
$$(X, Y, x, y, u)^{-1} = (x, y, X, Y, -u).$$

We also have the projection

$$J^1\Gamma \to J^0\Gamma, \qquad (X, Y, x, y, u) \mapsto (X, Y, x, y).$$

One readily verifies that Γ is indeed a Lie pseudogroup of order 1: the regularity conditions are clearly satisfied, and one is left with checking that Γ consists of all the solutions to $J^1\Gamma$ by explicitly solving the first, second and fourth equations in (3.5), which is a simple exercise in this case. By computing the higher jet groupoids of Γ, it is also not hard to see that the symbol space of $J^l\Gamma$ is non-zero for all $l > 0$, and hence that Γ is a Lie pseudogroup of infinite type (see Definition 3.5.3).

The Lie algebroid $A^1(\Gamma)$ has a global frame e_X, e_Y, e_u given by

$$e_X(x,y) = \frac{\partial}{\partial X}(x, y, x, y, 0), \quad e_Y(x,y) = \frac{\partial}{\partial Y}(x, y, x, y, 0),$$
$$e_u(x,y) = \frac{\partial}{\partial u}(x, y, x, y, 0).$$

The projection is given by

$$d\pi : A^1(\Gamma) \to A^0(\Gamma), \qquad e_X \mapsto \partial_X, \ e_y \mapsto \partial_Y, \ e_u \mapsto 0.$$

To compute the bracket of $A^1(\Gamma)$, we could either use the right translation map

$$R_{(X,Y,x,y,u)} : s^{-1}(X, Y) \to s^{-1}(x, y), \quad (\bar{X}, \bar{Y}, X, Y, v) \mapsto (\bar{X}, \bar{Y}, x, y, \frac{vy + \bar{y}u}{Y}),$$

to derive the induced right invariant vector fields

$$\widetilde{e}_X = \frac{\partial}{\partial X}, \quad \widetilde{e}_Y = \frac{\partial}{\partial Y} + \frac{u}{Y}\frac{\partial}{\partial u}, \quad \widetilde{e}_u = \frac{y}{Y}\frac{\partial}{\partial u},$$

on $J^1\Gamma$, or embed $A^1(\Gamma)$ into $A^1(M)$ by differentiating (3.6),

$$A^1(\Gamma) \hookrightarrow A^1(M), \qquad e_X \mapsto \partial_X, \quad e_Y \mapsto \partial_Y - \frac{1}{y}\partial_{p^x_X} + \frac{1}{y}\partial_{p^y_Y}, \quad e_u \mapsto \partial_{p^x_Y},$$

and then use the bracket (2.27) of $A^1(M)$. Either way,

$$[e_X, e_Y] = 0, \quad [e_X, e_u] = 0, \quad [e_Y, e_u] = -\frac{2}{y}e_u.$$

Similarly, we compute the anchor of $A^1(\Gamma)$,

$$\rho : A^1(\Gamma) \to TM, \qquad e_X \mapsto \frac{\partial}{\partial x}, \;\; e_Y \mapsto \frac{\partial}{\partial y}, \;\; e_u \mapsto 0.$$

We now compute the Cartan form $\omega \in \Omega^1(J^1\Gamma; t^*A^0(\Gamma))$. At a point $j^1_x\phi$ with coordinates (X, Y, x, y, u), by (2.16),

$$\omega_{(X,Y,x,y,u)} = dR_{(X,Y,x,y)^{-1}} \circ (d\pi - d(j^0\phi) \circ ds)_{(X,Y,x,y,u)}.$$

For the computation, we need to differentiate the holonomic bisection

$$j^0\phi : (x, y) \mapsto (\phi_x(x, y), \phi_y(x, y), x, y),$$

from which we find that

$$d(j^0\phi)_{(x,y)} : \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{y}{Y}\frac{\partial}{\partial X} + u\frac{\partial}{\partial Y}, \qquad \frac{\partial}{\partial y} \mapsto \frac{\partial}{\partial y} + \frac{Y}{y}\frac{\partial}{\partial Y},$$

and we need to differentiate the right translation map

$$R_{(X,Y,x,y)^{-1}} : (\bar{X}, \bar{Y}, x, y) \mapsto (\bar{X}, \bar{Y}, X, Y).$$

Computing the various cases, we find that

$$\omega : \frac{\partial}{\partial x} \mapsto -\frac{y}{Y}\partial_X - u\partial_Y, \quad \frac{\partial}{\partial y} \mapsto -\frac{Y}{y}\partial_Y, \quad \frac{\partial}{\partial X} \mapsto \partial_X, \quad \frac{\partial}{\partial Y} \mapsto \partial_Y, \quad \frac{\partial}{\partial u} \mapsto 0.$$

Thus,

$$\omega = (dX - \frac{y}{Y}dx)\, t^*\partial_X + (dY - udx - \frac{Y}{y}dy)\, t^*\partial_Y.$$

The Spencer operator $D : \Gamma(A^1(\Gamma)) \to \Omega^1(M; A^0(\Gamma))$ is computed by restricting the Spencer operator (2.30) on $A^1(M)$ to $A^1(\Gamma)$,

$$D : e_X \mapsto 0, \quad e_Y \mapsto \frac{1}{y}(dx \otimes \partial_X - dy \otimes \partial_Y), \quad e_u \mapsto -dx \otimes \partial_Y.$$

One readily verifies that the relation (2.24) between ω and D is satisfied. $\diamondsuit$

Example 3.4.5. Let $M = \mathbb{R}$ and let x be the coordinate on $\mathbb{R}$. We consider the pseudogroup Γ on M generated by the following local diffeomorphisms (that are defined where $cx + d \neq 0$):

$$\phi : x \mapsto \frac{ax+b}{cx+d}, \qquad a, b, c, d \in \mathbb{R} \quad \text{with} \quad ad - bc \neq 0.$$

The first three derivatives of ϕ are

$$\frac{\partial \phi}{\partial x} = \frac{ad-bc}{(cx+d)^2}, \qquad \frac{\partial^2 \phi}{\partial x^2} = -2c\frac{ad-bc}{(cx+d)^3}, \qquad \frac{\partial^3 \phi}{\partial x^3} = \frac{3}{2}\left(\frac{\partial^2 \phi}{\partial x^2}\right)^2 \left(\frac{\partial \phi}{\partial x}\right)^{-1}, \quad (3.7)$$

from which we see that Γ is defined by equations of order 3. We compute $J^2\Gamma$ and $J^3\Gamma$.

We denote the coordinates on J^2M by

$$J^2M = \{ \ (X, x, u, v) \mid X, x, u, v \in \mathbb{R}, \ u \neq 0 \ \}.$$

Thus, $j^2_x\phi$ with coordinates (X, x, u, v) satisfies $\phi(x) = X, \ \frac{\partial \phi}{\partial x}(x) = u$ and $\frac{\partial^2 \phi}{\partial x^2}(x) = v$. The structure maps are given by

$$s(X, x, u, v) = x, \quad t(X, x, u, v) = X, \quad 1_x = (x, x, 1, 0),$$
$$(\bar{X}, X, \bar{u}, \bar{v}) \cdot (X, x, u, v) = (\bar{X}, x, \bar{u}u, \bar{v}u^2 + \bar{u}v), \quad (X, x, u, v)^{-1} = (x, X, \frac{1}{u}, -\frac{v}{u^3}).$$

By finding a solution (a, b, c, d) satisfying $ad - bc \neq 0$ to the equations

$$X = \frac{ax+b}{cx+d}, \quad u = \frac{ad-bc}{(cx+d)^2}, \quad v = -2c\frac{ad-bc}{(cx+d)^3},$$

for each tuple (X, x, u, v), one shows that $J^2\Gamma = J^2M$ (these equations become simple to solve when one imposes the stronger condition $ad - bc = \pm 1$). The Lie algebroid $A^2(\Gamma) = A^2(M)$ has a frame

$$\partial_X(x) := \frac{\partial}{\partial X}(x, x, 1, 0), \quad \partial_u(x) := \frac{\partial}{\partial u}(x, x, 1, 0), \quad \partial_v(x) := \frac{\partial}{\partial v}(x, x, 1, 0).$$

The induced right invariant vector fields on $J^2\Gamma = J^2M$ are

$$\widetilde{\partial}_X = \frac{\partial}{\partial X}, \quad \widetilde{\partial}_u = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \quad \widetilde{\partial}_v = u^2\frac{\partial}{\partial v}.$$

The bracket on $A^2(M)$ is given by

$$[\partial_X, \partial_u] = 0, \quad [\partial_X, \partial_v] = 0, \quad [\partial_u, \partial_v] = \partial_v,$$

and the anchor is given by

$$\rho : A^2(M) \to TM, \qquad \partial_X \mapsto \frac{\partial}{\partial x}, \ \ \partial_u \mapsto 0, \ \ \partial_v \mapsto 0.$$

Next, we turn to $J^3\Gamma$. The third equation in 3.7 shows that each element of $J^2\Gamma$ uniquely extends to an element of $J^3\Gamma$. This implies that there is an isomorphism of Lie groupoids given by the projection $\pi : J^3\Gamma \xrightarrow{\simeq} J^2\Gamma$. More precisely, we introduce the coordinates

$$J^3M = \{ (X, x, u, v, w) \mid X, x, u, v, w \in \mathbb{R},\ u \neq 0 \},$$

such that $\phi(x) = X,\ \frac{\partial\phi}{\partial x}(x) = u,\ \frac{\partial^2\phi}{\partial x^2}(x) = v$ and $\frac{\partial^3\phi}{\partial x^3}(x) = w$. With the coordinates

$$J^3\Gamma = \{ (x, X, u, v) \mid X, x, u, v \in \mathbb{R},\ u \neq 0 \},$$

we have an embedding

$$J^3\Gamma \hookrightarrow J^3M, \qquad (x, X, u, v) \mapsto (x, X, u, v, \frac{3}{2}\frac{v^2}{u}).$$

In these coordinates, the projection becomes the identity,

$$\pi : J^3\Gamma \xrightarrow{\simeq} J^2\Gamma, \qquad (X, x, u, v) \mapsto (X, x, u, v).$$

Clearly, $J^3\Gamma = (J^2\Gamma)^{(1)}$ (since differentiating the second order equation in (3.7) gives the third order equation), and hence Γ is a Lie pseudogroup of finite type $l = 3$ (see Definition 3.5.3). We also deduce that $A^3(\Gamma) \cong A^2(\Gamma)$.

One readily computes the Cartan form $\omega \in \Omega^1(J^3\Gamma; t^*A^2(\Gamma))$ using (2.16),

$$\omega = (dX - udx)\, t^*\partial_X + \frac{1}{u}(du - vdx)\, t^*\partial_u + \frac{1}{u^2}(dv - \frac{v}{u}du - \frac{1}{2}\frac{v^2}{u}dx)\, t^*\partial_v.$$

We would like to point out here that it is remarkable that the formulas for the components of the Cartan form precisely coincide with formulas that Cartan obtains using various tricks and manipulations (e.g., see (5.8)). This indicates that, while Cartan's writings are local in nature, he really had an intrinsic and geometric picture in his mind.

From the latter formula, we compute $D : \Gamma(A^3(\Gamma)) \to \Omega^1(M; A^2(\Gamma))$, the Spencer operator, using (2.24):

$$D : \partial_X \mapsto 0, \quad \partial_u \mapsto -dx \otimes \partial_X, \quad \partial_v \mapsto -dx \otimes \partial_u.$$

Alternatively, we could compute the embedding $A^3(\Gamma) \hookrightarrow J^3(TM)$ and use that to derive D. $\diamondsuit$

3.5 Towers of Jet Groupoids of Γ

There are various prolongations associated with a single Lie pseudogroup that one should keep in mind.

Let Γ be a Lie pseudogroup of order k on M. To simplify the discussion, let us also assume that $J^l\Gamma$ is smooth and that $J^l\Gamma \to J^{l-1}\Gamma$ is a submersion for all $l > k$. In fact, people that study Lie pseudogroup from the point of view of infinite jets often impose this stronger regularity condition in the definition of a Lie pseudogroup (e.g., see Definition 3.1 in [57]).

Thus, we have an infinite tower of Lie-Pfaffian groupoids associated with Γ,

$$... \xrightarrow{\pi} (J^{k+2}\Gamma, \omega) \xrightarrow{\pi} (J^{k+1}\Gamma, \omega) \xrightarrow{\pi} (J^k\Gamma, \omega), \tag{3.8}$$

where the Cartan form on each $J^l\Gamma$ takes values in the Lie algebroid A^{l-1} of the next groupoid. For completeness, we mention that each two consecutive Cartan forms are related by a Maurer-Cartan equation, in the sense of Definition 6.2.15 in [62]. Similarly, one has a tower of Lie-Pfaffian algebroids

$$... \xrightarrow{\pi} (A^{k+2}, D) \xrightarrow{\pi} (A^{k+1}, D) \xrightarrow{\pi} (A^k, D). \tag{3.9}$$

At the infinitesimal level, the compatibility of each two consecutive Spencer operators is simply $D^2 = 0$ (see [62], Section 2.2.1). One also obtains a sequence of symbol spaces,

$$..., \mathfrak{g}^{k+2}, \mathfrak{g}^{k+1}, \mathfrak{g}^k, \tag{3.10}$$

where for each l (c.f. (1.11)),

$$\mathfrak{g}^l \subset \mathrm{Hom}(S^l TM, TM).$$

The sequence (3.8), in addition to being a sequence of Lie groupoids, is a tower of affine bundles, with $\pi : J^{k+l}\Gamma \to J^{k+l-1}\Gamma$ an affine bundle modeled on $\mathfrak{g}^{k+l}$.

Since each $J^l\Gamma$ in (3.8) is a PDE (even a Lie-Pfaffian groupoid), we can also consider its prolongations. If we further assume that $J^l\Gamma$ is formally integrable (Definition 1.5.1), e.g. if it satisfies the conditions of Theorem 1.5.8, then we also have an infinite tower of prolongations associated with this specific $(J^l\Gamma, \omega)$, namely

$$... \xrightarrow{\pi} (J^l\Gamma)^{(2)} \xrightarrow{\pi} (J^l\Gamma)^{(1)} \xrightarrow{\pi} J^l\Gamma. \tag{3.11}$$

Note that each of these prolongations is itself a Lie-Pfaffian groupoid. To see that it is a groupoid, one uses the fact that the Cartan form ω on $J^l\Gamma$ is multiplicative (see also Remark 2.4.5). To obtain the Lie-Pfaffian structure, one embeds the prolongations into $J^m M$ for the appropriate m.

As for any PDE, each projection in (3.11) is an affine bundle modeled on the appropriate prolongation of $\mathfrak{g}^l$. Thus, for each $l \geq k$, we have a sequence of prolongations of the symbol space $\mathfrak{g}^l$,

$$..., (\mathfrak{g}^l)^{(2)}, (\mathfrak{g}^l)^{(1)}, \mathfrak{g}^l.$$

The following explains the relations amongst this “zoo” of prolongations:

Proposition 3.5.1. *Let Γ be a pseudogroup of order k on M. Assume that $J^l\Gamma$ and $J^{l+1}\Gamma$ are smooth for some $l \geq k$ and $\pi : J^{l+1}\Gamma \to J^{l+1}\Gamma$ is a submersion. Then,*

$$J^{l+1}\Gamma \subset (J^l\Gamma)^{(1)} \qquad \textit{and} \qquad \mathfrak{g}^{l+1} \subset (\mathfrak{g}^l)^{(1)}. \tag{3.12}$$

Proof. Any solution of $J^{l+1}\Gamma$ is a solution of $J^l\Gamma$ (the solutions are simply the elements of Γ), while $J^l\Gamma$ may have a solution up to order $l+1$ (i.e. an integral element) that does not come from an actual solution in Γ (see Example 1.5.2). The inclusion at the level of jet groupoids implies the inclusion of the associated symbol spaces. □

The importance of inclusions of type (3.12) is made clear in the following stabilization lemma (see the proof in [64], Section 4.6):

Lemma 3.5.2. *Let V, W be vector spaces and let $\mathfrak{g}^k \subset \mathrm{Hom}(S^kV, W)$ be a sequence of vector subspaces, with $k = 0, 1, 2, ...$, such that $\mathfrak{g}^{k+1} \subset (\mathfrak{g}^k)^{(1)}$ for all $k \geq 0$. Then, there exists $k_0 \geq 0$ such that $\mathfrak{g}^{k+1} = (\mathfrak{g}^k)^{(1)}$ for all $k > k_0$.*

The sequence of symbol spaces 3.10, at each point, is precisely a sequence of the type described in the lemma. This implies, that, for each point, there exists $k_0 \geq k$ (possibly varying with the point) such that we have the equalities $\mathfrak{g}^{l+1} = (\mathfrak{g}^l)^{(1)}$ for all $l \geq k_0$ at that point. Unfortunately, we are not aware of a version of this lemma for vector bundles. We believe, however, that such a k_0 can be chosen uniformly. If this is the case, then we can directly conclude that, for all $l \geq k_0$ and $m > 0$, the inclusions (3.12) become equalities. Thus, if we take this uniform k_0 (which we conjecture to exist) to be the order of our Lie pseudogroup, then the sequence (3.8) of jet groupoids of Γ and the sequences (3.11) coincide (i.e. $(J^l\Gamma)^{(m)} = J^{k+l}\Gamma$ for all $l \geq k_0$ and $m > 0$).

An important notion in the theory of Lie pseudogroups is the notion of Lie pseudogroups of finite and infinite type. As we explained in the introduction, Sophus Lie's work was mainly dedicated to the finite case, which evolved into the theory of Lie groups, while Cartan aimed at extending the theory to the infinite case. Here are the modern definitions of these notions (see the introduction for the classical definitions).

Definition 3.5.3. *A Lie pseudogroup Γ is of **finite type** l if Γ is of order l and $\mathfrak{g}^l = 0$. Otherwise it is of **infinite type**.*

If a Lie pseudogroup Γ on M is of finite type l, then there is a regular foliation on $J^l\Gamma$ whose leaves are of the same dimension as M and such that the restriction of the source map $s : J^l\Gamma \to M$ to each leaf is a diffeomorphism onto its image (this follows from the Frobenius theorem, c.f. Theorem 1.7.3). Each leaf in this foliation corresponds to an element of Γ, and the collection of all such elements generate Γ (see Remark 3.1.3). Locally, these generators (the leaves) are parametrized by a transversal to the foliation, by which we mean a finite dimensional submanifold transversal to the leaves at each point and whose dimension is equal to the codimension of the foliation. This explains the fact that, classically, people regarded a Lie pseudogroup of finite type as a Lie pseudogroup that is parametrized by a finite number of real parameters. This is indeed the case locally, if we disregard the subtleties related to the sheaf-like axioms of a pseudogroup.

Here is a simple consequence of Proposition 3.5.1:

Corollary 3.5.4. *Let Γ be a Lie pseudogroup of order k. If Γ is of infinite type as a Lie pseudogroup, then $J^k\Gamma$ is of infinite type as a PDE.*

Proof. Since $\mathfrak{g}^{k+l} \subset (\mathfrak{g}^k)^{(l)}$, then also $(\mathfrak{g}^k)^{(l)} \neq 0$ for all $l > 0$ (see Definitions 3.5.3 and 1.7.1). □

3.6 Generalized Pseudogroups

A notion that will play an important role in our modern description of Cartan's structure theory is the notion of a generalized pseudogroup. The idea is very simple: while elements $\phi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ can be interpreted as (local) bisections of the pair groupoid $M \times M$, we can also allow for bisections of more general Lie groupoids. In fact, bisections of more general groupoids have already appeared naturally in our story (e.g., see Proposition 3.4.1). But more importantly, generalized pseudogroups will be essential for understanding the reduction procedure of Cartan that will be discussed in Chapter 8 (see Theorem 8.6.1).

Recall that for a Lie groupoid $\mathcal{G}$, the set $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ of local bisections comes with group-like structure (see Section 2.1).

Definition 3.6.1. *A **generalized pseudogroup** on a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a subset $\Gamma \subset \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ that satisfies the following axioms:*

A) Group-like axioms:

1) if $\sigma, \sigma' \in \Gamma$ and $\mathrm{Im}(t \circ \sigma') \subset \mathrm{Dom}(\sigma)$, then $\sigma \cdot \sigma' \in \Gamma$,

2) if $\sigma \in \Gamma$, then $\sigma^{-1} \in \Gamma$,

3) $1 \in \Gamma$.

B) Sheaf-like axioms:

1) if $\sigma \in \Gamma$ and $U \subset \mathrm{Dom}(\sigma)$ is an open subset, then $\sigma|_U \in \Gamma$,

2) if $\sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ and $\{U_i\}_{i \in I}$ is an open cover of $\mathrm{Dom}(\sigma)$ such that $\sigma|_{U_i} \in \Gamma$ for all $i \in I$, then $\sigma \in \Gamma$.

*We say that a generalized pseudogroup is **transitive** if for any $g \in \mathcal{G}$, there exists $\sigma \in \Gamma$ such that $g = \sigma(s(g))$.*

Remark 3.6.2. $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ has the structure of a groupoid over $\mathrm{Open}(M)$, the set of open subsets of M. Using this observation, the first three axioms can be rephrased as saying that $\Gamma \subset \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$ is a wide subgroupoid. ♢

Example 3.6.3.

1. A **pseudogroup** on M is the same thing as a generalized pseudogroup on the pair groupoid $M \times M \rightrightarrows M$.

2. The smallest generalized pseudogroup on $\mathcal{G} \rightrightarrows M$ is the one consisting of the identity bisection and its restrictions to all opens subsets of M.

3. The largest generalized pseudogroup on $\mathcal{G} \rightrightarrows M$ is $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G})$. ♢

Example 3.6.4. (Lie groups as generalized pseudogroups) While realizing a Lie group G as a Lie pseudogroup depends on the choice of a space M and an action of G on M, the generalized point of view allows us to make sense of a Lie group as a (generalized) pseudogroup in a canonical way: take as the Lie groupoid $G \rightrightarrows \{*\}$. Its group of local bisections coincides with G as a group. This is a trivial yet conceptually important interpretation. For instance, using Cartan's reduction procedure mentioned above (e.g., see Example 8.7.1), it is this point of view that allows us to reduce some Lie pseudogroups to Lie groups. $\diamondsuit$

Example 3.6.5. (The generalized pseudogroup of a Pfaffian groupoid) A conceptually important class of examples comes from Pfaffian groupoids (Definition 2.7.1):

Proposition 3.6.6. *For any Pfaffian groupoid* $(\mathcal{G}, \omega)$*,* $\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}, \omega)$ *of local holonomic bisections is a generalized pseudogroup.*

The proof is a consequence of the multiplicativity of ω. $\diamondsuit$

Example 3.6.7. (the classical shadow) Any generalized pseudogroup Γ on a Lie groupoid $\mathcal{G} \rightrightarrows M$ induces a pseudogroup Γ_{cl} on the base M by "projecting" the elements, i.e.

$$\Gamma_{\mathrm{cl}} := \{\ t \circ \sigma \mid \sigma \in \Gamma\ \} \subset \mathrm{Diff}_{\mathrm{loc}}(M).$$

We call Γ_{cl} the **classical shadow** of Γ. "Classical" pseudogroups often arise as the classical shadows of generalized pseudogroups that are more natural than the actual pseudogroups one is interested in studying. Consider, for instance, the pseudogroup from Cartan's example 5.1.7, the pseudogroup on $\mathbb{R}$ generated by the set of diffeomorphisms

$$\phi : \mathbb{R} \to \mathbb{R}, \qquad \phi(x) = (ax + b),$$

parametrized by $a \in \mathbb{R}\backslash\{0\}$ and $b \in \mathbb{R}$. It clearly comes from the action of a Lie group. Indeed, we take the Lie groups $(\mathbb{R}, +)$ and $(\mathbb{R}\backslash\{0\}, \times)$, and the map $\varphi : \mathbb{R}\backslash\{0\} \to \mathrm{Aut}(\mathbb{R}),\ a \mapsto (b \mapsto ab)$. From this data we construct the semi direct product

$$\mathbb{R}\backslash\{0\}_{\varphi}\ltimes \mathbb{R},$$

where the product of two elements (a, b) and (a', b') is given by

$$(a', b') \cdot (a, b) = (a'a, b' + a'b).$$

We then consider the action groupoid induced the action of $\mathbb{R}\backslash\{0\}_{\varphi} \ltimes \mathbb{R}$ on $\mathbb{R}$, where the action is given by $(a, b) \cdot x = ax + b$. Cartan's example is the classical shadow of the generalized pseudogroup of "constant" bisections of this action groupoid. Similarly, Cartan's examples 5.1.8 and 5.1.9 are of this type. $\diamondsuit$

Example 3.6.8. (classical pseudogroups made nicer) Cartan's approach to Lie pseudogroups works well under suitable regularity conditions (such as in Definition 3.3.1) and is best understood in the transitive case. Generalized pseudogroups often allow one to handle ill-behaved or non-transitive pseudogroups by applying Cartan's ideas to well-behaved or even transitive generalized pseudogroups. Here is an illustration of this phenomenon. Consider the pseudogroup Γ_{cl} of rotations on $\mathbb{R}^2$, i.e. the one generated by the set of diffeomorphisms

$$\phi : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \phi(x, y) = (x \cos 2\pi\theta + y \sin 2\pi\theta, -x \sin 2\pi\theta + y \cos 2\pi\theta),$$

parametrized by $\theta \in \mathbb{R}/\mathbb{Z}$. Since it is has a singular orbit, it is not a Lie pseudogroup, but it is the classical shadow of the generalized pseudogroup Γ_{gen} of "constant" bisections of the action groupoid $\mathbb{R}/\mathbb{Z} \ltimes \mathbb{R}^2$ (see Example 2.1.1) associated with the action of the Lie group $\mathbb{R}/\mathbb{Z}$ on $\mathbb{R}^2$ by rotations, i.e.,

$$\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \theta \cdot (x, y) = (x \cos 2\pi\theta + y \sin 2\pi\theta, -x \sin 2\pi\theta + y \cos 2\pi\theta).$$

Note also that although removing the origin does make Γ_{cl} into a Lie pseudogroup, it is not transitive, while Γ_{gen} is as a generalized pseudogroup. $\diamondsuit$

Passing from Pseudogroups to Generalized Pseudogroups All the basic definitions and constructions we presented for pseudogroups extend to the generalized context in a rather straightforward manner. For instance, for a generalized pseudogroup $\Gamma \subset \text{Bis}_{\text{loc}}(\mathcal{G})$, the k-jets of elements of Γ form a subgroupoid of the jet groupoid $J^k\mathcal{G}$ that we have already discussed (see Section 2.2):

$$J^k\Gamma \subset J^k\mathcal{G}.$$

Similarly, the algebroid A^k of $J^k\Gamma$ is a subalgebroid of the algebroid $A^k(\mathcal{G})$ of $J^k\mathcal{G}$; the symbol space is handled similarly, etc.

Also the germ groupoid is defined as before, giving rise to a (possibly non-effective) étale groupoid

$$\mathcal{G}\text{erm}(\Gamma) \rightrightarrows M. \tag{3.13}$$

This étale groupoid should be viewed as being closer to the abstract object underlying the generalized pseudogroup, since it "decouples" Γ from its "hosting" Lie groupoid $\mathcal{G}$. The relationship with $\mathcal{G}$ is encoded in a morphism of Lie groupoids $\mathcal{G}\text{erm}(\Gamma) \to \mathcal{G}$, $\text{germ}_x(\sigma) \mapsto \sigma(x)$. The transitivity of Γ is equivalent to the surjectivity of this map, and the same map can be used to make sense of effectiveness in the generalized sense.

Filling in the remaining details in passing to the generalized context is an instructive exercise which we leave to the reader.

Chapter 4

The First Fundamental Theorem

At the foundation of Cartan's work on Lie pseudogroups are the three fundamental theorems with which Cartan lays out a structure theory for Lie pseudogroups. Cartan begins by defining the notion of *equivalence* between two pseudogroups, which he regards as the correct notion of isomorphism in the realm of pseudogroups. Then, in the first and second fundamental theorems, Cartan introduces a *normal form* for Lie pseudogroups and shows that any Lie pseudogroup is equivalent to one in normal form. Hence, if one were to classify Lie pseudogroups (a problem that Cartan addresses), then, up to equivalence, it suffices to restrict one's attention to the special class of pseudogroups in normal form. In the third fundamental theorem, Cartan identifies the infinitesimal structure that is associated with a pseudogroup in normal form, in analogy with how one associates a Lie algebra with a Lie group. As in the case of Lie groups, Cartan proves that such an infinitesimal structure integrates to a pseudogroup in normal form.

Cartan presented this structure theory in his pioneering papers [5, 6] (1904-1905) and then later in a more concise form in [7, 8] (1937). Cartan's presentation is very local in nature (i.e. local coordinates), a fact which makes his writing difficult to access and often hides beautiful geometric structures and phenomena that are present in his work. The rich differential geometric language that has been developed since Cartan's time, part of which was directly inspired by this very work of Cartan, presents us with the possibility of placing this theory on more solid ground. In the coming chapters, Chapters 4 - 9 (with the exception of Chapter 7), we revisit Cartan's work and translate his structure theory for Lie pseudogroups into modern language with the aim of gaining a deeper understanding of his approach and ideas. Each chapter will deal with a different aspect of the theory, and, in each chapter, we will begin by recalling Cartan's formulation of the topic at hand and then proceed to present our modern formulation.

The topic of this chapter is the first fundamental theorem, the first step towards the construction of a pseudogroup in normal form. In this theorem, Cartan shows that any Lie pseudogroup is equivalent to one that is defined as the set of local symmetries of a certain system of functions and 1-forms.

The chapter is organized as follows: we begin by recalling Cartan's formulation of the first fundamental theorem. Moving on to the modern picture, we discuss the modern definition of equivalence of pseudogroups (or *Cartan equivalence*, as we call it) and then state and prove the first fundamental theorem.

4.1 Cartan's Formulation

At the starting point of Cartan's structure theory of Lie pseudogroups are Cartan's definition of equivalence between pseudogroups and what Cartan calls the first fundamental theorem for Lie pseudogroups. Let us recall these in Cartan's own words.

Equivalence of Pseudogroups If one is to study any mathematical structure, one must begin by clarifying when two instances of the structure are "the same", i.e. isomorphic. The correct notion for an isomorphism between two pseudogroups is not at all evident. Cartan writes ([7], p. 1336): *"The notion of an 'abstract group' does not lend itself to the theory of infinite Lie pseudogroups with the same level of purity as it does in the finite case, and it is for this reason that it has been proven difficult to find a simple analytic characterization for the notion of isomorphism. It is remarkable that M. Vessiot and I were simultaneously led to the same definition of an isomorphism of two Lie pseudogroups."*

Let us consider a simple example. Consider the pseudogroup on $\mathbb{R}$ generated by the diffeomorphisms

$$X = x + a, \qquad a \in \mathbb{R},$$

and the pseudogroup on $\mathbb{R}^2$ generated by the diffeomorphisms

$$X = x + a, \quad Y = y, \qquad a \in \mathbb{R}.$$

Morally, these two pseudogroups should be declared isomorphic because, forgetting the sheaf-like nature of a pseudogroup for a moment, there is a bijection between the two which preserves the group-like structure. But how does one make sense of an isomorphism between pseudogroups that act on spaces of different dimension? Cartan proposes the following solution:

Definition 4.1.1. *Let Γ be a pseudogroup acting on n variables $x_1, x_2, ..., x_n$. A pseudogroup $\widetilde{\Gamma}$ is said to be a **prolongation** of Γ if it acts on the same variables $x_1, x_2, ..., x_n$, but at the same time on another set of variables $y_1, y_2, ..., y_p$, in such a way that it transforms the x variables between themselves in the same way as the pseudogroup Γ does. Hence, to each transformation of Γ there corresponds at least one transformation of $\widetilde{\Gamma}$; the prolongation is said to be an **isomorphic prolongation** if there corresponds precisely one. Two pseudogroups Γ and Γ' are said to be **equivalent** if they admit two similar isomorphic prolongations (that is to say, with the same number of variables and one transforms to the other by a change of variables)*

Remark 4.1.2. Cartan makes a distinction between two types of prolongations: a *holoédrique* prolongation and a *mériédrique* prolongation. The first is what we have translated to *isomorphic* prolongation, while in the latter he does not require the correspondence between Γ and $\widetilde{\Gamma}$ to be one-to-one. In a sense, $\widetilde{\Gamma}$ is allowed to be "larger". Accordingly, he talks about a *holoédrique* isomorphism, which is what we have translated to an *equivalence*, and a *mériédrique* isomorphism. See [7] (p. 1336) for more details. $\diamondsuit$

The First Fundamental Theorem The first fundamental theorem is the first of two steps in which Cartan defines the notion of a pseudogroup in normal form and shows that any Lie pseudogroup is equivalent to one in normal form. The second step, the second fundamental theorem, will be discussed in the next chapter. Cartan writes ([7], 1336): "The theorem that lies at the foundation of the theory of Lie pseudogroups is the following:"

Theorem 4.1.3. *(the first fundamental theorem) Any Lie pseudogroup Γ admits an isomorphic prolongation $\widetilde{\Gamma}$ acting on a certain number r of variables $x_i = (x_1, ..., x_r)$, whose elements are the transformations that leave invariant*

1. *a certain number of functions of the variables x_i,*
2. *r linearly independent 1-forms $\omega_i = \sum_{j=1}^{r} a_i^j(x,u)dx_j$, where the coefficients $a_i^j(x,u)$ may depend on another set of auxiliary variables u_ρ.*

4.2 Cartan's Notion of Equivalence

Cartan's notions of isomorphic prolongation and equivalence can be made rigorous in terms of the 1-1 correspondence of Haefliger between pseudogroups and effective étale groupoids (Proposition 3.2.1), which identifies a pseudogroup Γ on M with the effective étale Lie groupoid $\mathcal{G}\mathrm{erm}(\Gamma) \rightrightarrows M$ consisting of all germs of elements of Γ. Recall first from Section 2.1 that given a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a surjective submersion $\pi : P \to M$, one can talk about actions of $\mathcal{G}$ on P, and that such an action gives rise to the action groupoid $\mathcal{G} \ltimes P \rightrightarrows P$.

Definition 4.2.1. *Let $\pi : P \to M$ be a surjective submersion, Γ a pseudogroup on M and $\widetilde{\Gamma}$ a pseudogroup on P. We say that $\widetilde{\Gamma}$ is an **isomorphic prolongation** of Γ (along π) if there exist an action of $\mathcal{G}\mathrm{erm}(\Gamma) \rightrightarrows M$ on P and an isomorphism of Lie groupoids*

$$\mathcal{G}\mathrm{erm}(\widetilde{\Gamma}) \cong \mathcal{G}\mathrm{erm}(\Gamma) \ltimes P.$$

Similarly for generalized pseudogroups (see (3.13)*).*

Remark 4.2.2. Representing germs by local diffeomorphisms, we can write down the condition of isomorphic prolongation in more "down to earth" terms and recover Cartan's definition: for any element $\phi \in \Gamma$ there exists a unique element $\widetilde{\phi} \in \widetilde{\Gamma}$ such that $\mathrm{Dom}(\widetilde{\phi}) = \pi^{-1}(\mathrm{Dom}(\phi))$ and such that $\widetilde{\phi}$ projects to ϕ, i.e. $\pi \circ \widetilde{\phi} = \phi \circ \pi|_{\mathrm{Dom}(\widetilde{\phi})}$, and any element of $\widetilde{\Gamma}$ descends locally to an element of Γ (see also [45, 44]). However, one has to take care of the subtleties related with the sheaf-like properties of a pseudogroup. $\diamondsuit$

Definition 4.2.3. *A pseudogroup Γ on M is said to be **Cartan equivalent** to a pseudogroup Γ' on M' if they admit a common isomorphic prolongation. We write $\Gamma \sim \Gamma'$ when Γ is Cartan equivalent to Γ'. Similarly for generalized pseudogroups.*

Proposition 4.2.4. *Cartan equivalence defines an equivalence relation on the set of all generalized pseudogroups.*

Proof. Symmetry and reflexivity are clear. We are left with proving transitivity. Assume that we have three pseudogroups $\Gamma, \Gamma', \Gamma''$ on M, M', M'', respectively, such that $\Gamma \sim \Gamma'$ and $\Gamma' \sim \Gamma''$. We need to show that $\Gamma \sim \Gamma''$. We are thus given the following zig-zag of maps in the lower two levels of the following diagram:

$$\begin{tikzcd}
 & & P \times_{M'} P' \arrow[dl] \arrow[dr] & & \\
 & P \arrow[dl] \arrow[dr] & & P' \arrow[dl] \arrow[dr] & \\
M & & M' & & M''
\end{tikzcd}$$

The idea is to construct an isomorphic prolongation of both Γ and Γ' on the fibered product $Q := P \times_{M'} P'$ by using the fact that we have an action of $\mathcal{G}\mathrm{erm}(\Gamma')$ on both P and P' and isomorphisms $\mathcal{G}\mathrm{erm}(\Gamma) \ltimes P \cong \mathcal{G}\mathrm{erm}(\Gamma') \ltimes P$ and $\mathcal{G}\mathrm{erm}(\Gamma') \ltimes P' \cong \mathcal{G}\mathrm{erm}(\Gamma'') \ltimes P'$. We first define an action of $\mathcal{G}\mathrm{erm}(\Gamma)$ on Q as follows: let $\phi \in \Gamma$, $x \in \mathrm{Dom}(\phi)$ and $(p, p') \in Q$ such that p projects to x. The isomorphism $\mathcal{G}\mathrm{erm}(\Gamma) \ltimes P \xrightarrow{\cong} \mathcal{G}\mathrm{erm}(\Gamma') \ltimes P$ maps the pair $(\mathrm{germ}_x(\phi), p)$ to a pair $(\mathrm{germ}_x(\phi'), p)$. We then define the action by $\mathrm{germ}_x(\phi) \cdot (p, p') := (\mathrm{germ}_x(\phi') \cdot p, \mathrm{germ}_x(\phi') \cdot p')$. Similarly, one defines an action of $\mathcal{G}\mathrm{erm}(\Gamma'')$ on Q. It is a straightforward exercise to verify that the pseudogroup associated with $\mathcal{G}\mathrm{erm}(\Gamma) \ltimes Q$ is an isomorphic prolongation of Γ, and similarly for Γ'', and to verify that $\mathcal{G}\mathrm{erm}(\Gamma'') \ltimes Q \cong \mathcal{G}\mathrm{erm}(\Gamma) \ltimes Q$, from which the conclusion follows. $\square$

Note also that Cartan equivalence allows one to compare classical pseudogroups with generalized ones. For instance, the last part of Proposition 3.4.1 translates to the following:

Corollary 4.2.5. *Given any Lie pseudogroup Γ of order k, Γ is Cartan equivalent to the generalized pseudogroup of holonomic bisections of $(J^k\Gamma, \omega)$.*

Also the reduction procedure discussed in Chapter 8 will allow us to replace a Lie pseudogroup by a smaller generalized one, which is Cartan equivalent to the original one.

4.3 The First Fundamental Theorem

In the first fundamental theorem, Theorem 4.1.3, Cartan shows that any Lie pseudogroup can be isomorphically prolonged to the pseudogroup of local symmetries of a system of functions and 1-forms. Globally, the theorem can be phrased as follows:

Theorem 4.3.1. *(the first fundamental theorem) Any Lie pseudogroup Γ is Cartan equivalent to a pseudogroup Γ' on a manifold P of type*

$$\Gamma' = \{ \phi \in \mathrm{Diff}_{\mathrm{loc}}(P) \mid \phi^* I = I,\ \phi^*\Omega = \Omega \},$$

where $I : P \to N$ is a surjective submersion into another manifold N, and $\Omega \in \Omega^1(P; I^*\mathcal{C})$ is a pointwise surjective 1-form for some vector bundle $\mathcal{C} \to N$.

Prolonging a Lie Pseudogroup The proof of the theorem is by the well known construction of a prolongation of a Lie pseudogroup. Let $\Gamma \subset \mathrm{Diff}_{\mathrm{loc}}(M)$ be a Lie pseudogroup of order k on M. Throughout the section, we use the notation that was introduced in Section 3.3 for the structure induced by the Lie pseudogroup Γ. Any $\phi \in \Gamma$, with

$$\phi : U \to V, \qquad U, V \subset M,$$

induces a local diffeomorphism $\phi^k \in \mathrm{Diff}_{\mathrm{loc}}(J^k\Gamma)$ given by

$$\phi^k : s^{-1}(U) \to s^{-1}(V), \qquad j^k_x\phi' \mapsto j^k_x\phi' \cdot (j^k_x\phi)^{-1}, \tag{4.1}$$

called the k**-th prolongation** of ϕ. Note that $\phi^k \circ \phi'^k = (\phi \circ \phi')^k$ for any composable pair $\phi, \phi' \in \Gamma$; $(\phi^{-1})^k = (\phi^k)^{-1}$ for any $\phi \in \Gamma$; and the identity is mapped to the identity. This fact almost implies that the subset $\{\phi^k \mid \phi \in \Gamma\} \subset \mathrm{Diff}_{\mathrm{loc}}(J^k\Gamma)$ defines a pseudogroup. One, however, still needs to "impose" the sheaf-like axioms. The pseudogroup generated by this subset,

$$\Gamma^k := \langle \{\phi^k \mid \phi \in \Gamma\} \rangle \subset \mathrm{Diff}_{\mathrm{loc}}(J^k\Gamma),$$

is called the k**-th prolongation** of Γ.

The first claim is that Γ^k is an isomorphic prolongation of Γ along the source map $s : J^k\Gamma \to M$ (Definition 4.2.1). Proving this amounts to showing that there exist an action of the Lie groupoid of germs $\mathcal{G}\mathrm{erm}(\Gamma) \rightrightarrows M$ of Γ on $s : J^k\Gamma \to M$ and an isomorphism $\mathcal{G}\mathrm{erm}(\Gamma) \ltimes J^k\Gamma \cong \mathcal{G}\mathrm{erm}(\Gamma^k)$ between the induced action groupoid and the Lie groupoid of germs $\mathcal{G}\mathrm{erm}(\Gamma^k) \rightrightarrows J^k\Gamma$ of Γ^k. The action here is given by

$$\mathcal{G}\mathrm{erm}(\Gamma)\,{}_s\!\times_s J^k\Gamma \to J^k\Gamma, \qquad (\mathrm{germ}_x\phi, j^k_x\phi') \mapsto j^k_x\phi' \cdot (j^k_x\phi)^{-1}. \tag{4.2}$$

One readily verifies that:

Proposition 4.3.2. *Let Γ be a Lie pseudogroup of order k on M. Then*

$$\mathcal{G}\mathrm{erm}(\Gamma) \ltimes J^k\Gamma \xrightarrow{\cong} \mathcal{G}\mathrm{erm}(J^k\Gamma), \qquad (\mathrm{germ}_x\phi, j^k_x\phi') \mapsto \mathrm{germ}_{j^k_x\phi'}\phi^k,$$

is an isomorphism of Lie groupoids. Thus, Γ^k is an isomorphic prolongation of Γ along $s : J^k\Gamma \to M$.

We should, thus, think of Γ and its k-th prolongation Γ^k as "isomorphic". The remarkable fact about Γ^k is that, however complicated the defining equations of Γ may be, Γ^k is always defined by a simple invariance property. This, at the cost of passing to a larger space. Indeed, Theorem 4.3.1 is a direct corollary of the following proposition:

Proposition 4.3.3. *Let Γ be a Lie pseudogroup of order k on M. Then,*

$$\Gamma^k = \{\ \phi \in \mathrm{Diff}_{\mathrm{loc}}(J^k\Gamma) \mid \phi^*t = t,\ \phi^*\omega = \omega.\ \}.$$

Remark 4.3.4. Proposition 4.3.3 is just one of many possible constructions that prove the first fundamental theorem. Another option would be to take a higher k (if the spaces are smooth). And yet another option, as we will see later in this section, would be to simplify Γ^k by *restricting to a transversal.* $\diamondsuit$

Before turning to the proof of the proposition, let us make the following simple but important observation: if $\psi^*t = t$ then $\psi^*dt = dt$. Thus, (4.3.3) is equivalent to the invariance condition

$$\psi^*t = t \qquad \text{and} \qquad \psi^*\Omega = \Omega, \tag{4.3}$$

where

$$\Omega := (dt, \omega) \in \Omega^1(J^k\Gamma; t^*\mathcal{C}) \qquad \text{and} \qquad \mathcal{C} := TM \oplus A^{k-1}. \tag{4.4}$$

We call Ω the **extended Cartan form** of $J^k\Gamma$. The pairs (t, ω) and (t, Ω) contain the same information. Cartan uses the latter pair in his structure theory. For the proof of the proposition, we will need the following lemma.

Lemma 4.3.5. *Let Γ be a Lie pseudogroup of order k on M. The extended Cartan form Ω on $J^k\Gamma$ is pointwise surjective. Furthermore,*

$$\text{Ker } \Omega = T^\pi J^k\Gamma \cong t^*\mathfrak{g}^k \qquad \textit{and} \qquad \Omega^{-1}(t^*\mathcal{C}_\rho) = T^s J^k\Gamma \cong t^*A^k,$$

where

$$\mathcal{C}_\rho := \{\ (\rho(\alpha), \alpha) \in TM \oplus A^{k-1} \mid \alpha \in A^{k-1}\ \} \subset \mathcal{C}.$$

Proof. Choose a splitting $\xi : s^*TM \to C_\omega$ of $ds|_{C_\omega} : C_\omega \to s^*TM$. This induces an isomorphism

$$TJ^k\Gamma \cong s^*TM \oplus T^sJ^k\Gamma.$$

Let us compute how Ω acts on each component of the splitting, from which the lemma will follow immediately. Let $X \in (s^*TM)_{j^k_x\phi} = T_xM$. Then $\omega(X) = 0$, which implies that $d\pi(\xi(X)) = d(j^{k-1}\phi) \circ ds(\xi(X))$ by definition (2.16) of ω, and hence

$$dt \circ \xi(X) = dt \circ d\pi \circ \xi(X) = dt \circ (d(j^{k-1}\phi))_x(X) = (d\phi)_x(X).$$

Thus,

$$\Omega_{j^k_x\phi}(X) = \big((d\phi)_x(X), 0\big).$$

For the second component, it is enough to check how Ω acts on right invariant vector fields, i.e. $\widetilde{\alpha} \in \Gamma(T^sJ^k\Gamma)$ induced by $\alpha \in \Gamma(A^k)$ by right translation. Using the explicit formula for $\Omega = (dt, \omega)$,

$$\Omega(\widetilde{\alpha}) = t^*(\rho(\alpha), d\pi(\alpha)),$$

where $\rho : A^k \to TM$ is the anchor map and $d\pi : A^k \to A^{k-1}$ the projection. $\square$

Proof of Proposition 4.3.3. We begin by proving the forward direction. Assume that $\psi = \phi^k$. First, since

$$t(\phi^k(j^k_x\phi')) = t(j^k_x\phi' \cdot (j^k_x\phi)^{-1}) = t(j^k_x\phi')$$

for any $j^k_x\phi' \in \mathrm{Dom}(\psi)$, then $(\phi^k)^*t = t$. Second, to prove that $(\phi^k)^*\omega = \omega$, say at a point $j^k_x\phi'$, we apply both sides of the equality on vectors that are vertical with respect to the source map $s : J^k\Gamma \to M$, on the one hand, and on a choice of a horizontal subspace at $j^k_x\phi'$, on the other. Let $X \in (T^sJ^k\Gamma)_{j^k_x\phi'}$. By the definition of ω,

$$\omega((d\phi^k)(X)) = dR_{j^{k-1}_x\phi\cdot(j^{k-1}_x\phi')^{-1}}\circ d\pi\circ dR_{(j^k_x\phi)^{-1}}(X) = dR_{(j^{k-1}_x\phi')}\circ d\pi(X) = \omega(X).$$

Next, choose a representative $\phi' \in \Gamma$ of $j^k_x\phi'$. Since $(j^k\phi')^*\omega = 0$, then

$$(j^k\phi')^*(\phi^k)^*\omega = (\phi^k \circ j^k\phi')^*\omega = (j^k(\phi' \cdot \phi^{-1}) \circ \phi^{-1})^*\omega = 0.$$

Thus, both $(\phi^k)^*\omega$ and ω vanish on the image of $(d(j^k\phi'))_x$, which is a horizontal subspace at $j^k_x\phi'$.

For the reverse direction, let $\psi \in \mathrm{Diff}_{\mathrm{loc}}(J^k\Gamma)$ and assume that it satisfies (4.3.3) or, equivalently, (4.3). First, we show that, locally, ψ descends to a local diffeomorphism of M via the source map $s : J^k\Gamma \to M$, i.e. there exists $\phi \in \mathrm{Diff}_{\mathrm{loc}}(M)$ such that $s \circ \psi = \phi \circ s$. It is sufficient to show that ψ preserves the foliation by the s-fibers infinitesimally, i.e. that $d\psi(T^sJ^k\Gamma) \subset T^sJ^k\Gamma$. Using Lemma 4.3.5 and $\phi^*\Omega = \Omega$, we see that for any $Y \in T^sJ^k\Gamma$,

$$\Omega\big(d\psi(Y)\big) = \Omega(Y) \in \mathcal{C}_\rho,$$

and hence $d\psi(Y) \in T^sJ^k\Gamma$.

Next, we prove that $\phi \in \Gamma$ and that, locally, $\psi = \phi^k$. Let $j^k_x\phi' \in \mathrm{Dom}(\psi)$ and let $\phi' \in \Gamma$ be a representative of $j^k_x\phi'$. We claim that $\psi \circ j^k\phi' \circ \phi^{-1}$ is a holonomic section of $J^k\Gamma$. Indeed, because $\psi^*\omega = \omega$, then $(\psi \circ j^k\phi' \circ \phi^{-1})^*\omega = 0$, and thus $\psi \circ j^k\phi' \circ \phi^{-1} = j^k\eta$ for some $\eta \in \Gamma$. Finally, because $\psi^*t = t$, we have $\phi' \circ \phi^{-1} = \eta$, which implies that $\phi \in \Gamma$ and that

$$\psi(j^k_x\phi') = j^k_{\phi(x)}(\phi' \circ \phi^{-1}) = j^k_x\phi' \cdot (j^k_x\phi)^{-1}.$$

$\square$

Remark 4.3.6. Cartan's proof of the first fundamental theorem appears in [5] and [7]. A modern presentation of the proof in the case of transitive Lie pseudogroups, one which also uses the language of jet groupoids, can be found in [24] (Theorem 4.1). $\diamondsuit$

Restricting to a Transversal As we noted in Remark (4.3.4), Γ^k is just one possibility for replacing a given Lie pseudogroup by a pseudogroup that is characterized as the set of local symmetries of a system of functions and 1-forms. Another possibility, as Cartan himself shows (see [7], 3rd paragraph), is to simplify the pseudogroup Γ^k by removing certain "irrelevant" variables, what we call *restricting to a transversal*. In Section 5.1, we will see some explicit instances of how Cartan applies this simplification (Examples 5.1.8 and 5.1.11). Here, we briefly describe Cartan's simplification in global terms.

Let Γ be a Lie pseudogroup of order k on M. Given any submanifold $N \subset M$, because the orbits of the k-th prolongation Γ^k are precisely the t-fibers of $J^k\Gamma$ (by its very definition), then any element of Γ^k can be restricted to $J^k\Gamma|_N := t^{-1}(N) \subset J^k\Gamma$. The resulting pseudogroup on $J^k\Gamma|_N$ that is obtained by restricting all elements of Γ^k in this fashion is denoted by $\Gamma^k|_N$. This is depicted in the following diagram:

$$\begin{tikzcd}
 & \Gamma^k \curvearrowright J^k\Gamma \arrow[dl, "s"'] \arrow[d, "t"] & \supset & J^k\Gamma|_N \curvearrowleft \Gamma^k|_N \arrow[d, "t"] \arrow[dr, "s"] & \\
\Gamma \curvearrowright M & M & \supset & N & M \curvearrowleft \Gamma
\end{tikzcd}$$

Clearly, $t : J^k\Gamma|_N \to N$ is again a surjective submersion. Additionally, we would like to ensure that $s : J^k\Gamma|_N \to M$ is a surjective submersion, which would immediately imply that $\Gamma^k|_N$ is an isomorphic prolongation of Γ along $s : J^k\Gamma|_N \to M$ (by the restriction of the action (4.2) to $J^k\Gamma|_N$). This is achieved by requiring N to satisfy the following properties:

Definition 4.3.7. *Let Γ be a Lie pseudogroup of order k on M. A submanifold $N \subset M$ is a **transversal** if*

$$TM|_N = TN + \rho(A^k)|_N$$

and if N intersects each orbit at least once.

Since the orbits of Γ are precisely the orbits of the Lie groupoid $J^k\Gamma$, then $\rho(A^k)$ is the tangent distribution to the orbits of Γ. Thus, a transversal N is a transversal to the (possibly singular) foliation by orbits.

Thus, by restricting to a transversal N, we obtain a smaller isomorphic prolongation $\Gamma^k|_N$ and we claim that this prolongation is also characterized by an invariance property

$$\psi^* t = t \qquad \text{and} \qquad \psi^*\Omega_N = \Omega_N,$$

only now $t : J^k\Gamma|_N \to N$ is the restriction of the target map, $\Omega_N \in \Omega^1(J^k\Gamma|_N; t^*\mathcal{C}_N)$ is the restriction of Ω and $\mathcal{C}_N := TN \oplus A^{k-1}|_N$.

Ideally, to obtain the optimal simplification, one would like to choose a transversal N that crosses each orbit precisely once. In this case, N can be regarded as the orbit space of Γ, but one which is obtained by choosing a slice rather than by taking a quotient. In fact, this is what Cartan always does, but in the global picture this is only possible if the orbits are “nice enough”. For example, if the Lie pseudogroup is transitive, one takes N to be a point in M.

Remark 4.3.8. In [24], the authors study Cartan’s structure theory in the case of transitive Lie pseudogroups and develop the whole theory by restricting to a point in the sense that was described in this section. ♢

Comparing with Cartan To conclude, let us show that, in local coordinates, Proposition 4.3.3 produces a Cartan equivalent pseudogroup that satisfies the properties as stated in Cartan's first fundamental theorem, Theorem 4.1.3. Let Γ be a Lie pseudogroup of order k on M. Locally, we may choose coordinates as follows:

- $x_a = (x_1, ..., x_m)$ on M,
- $x_i = (x_1, ..., x_m, x_{m+1}, ..., x_r)$ on $J^{k-1}\Gamma$,
- $(x_i, u_\rho) = (x_1, ..., x_m, x_{m+1}, ..., x_r, u_1, ..., u_p)$ on $J^k\Gamma$,

such that $t : J^{k-1}\Gamma \to M$ is the projection onto the first m coordinates and $\pi : J^k\Gamma \to J^{k-1}\Gamma$ is the projection onto the first r coordinates. Since $\mathrm{Ker}\ \Omega = \mathrm{Ker}\ d\pi$, then, locally, the components of Ω are r linearly independent 1-forms of the form

- $\omega_i = \sum_{j=1}^{r} a_i^j(x,u)dx_j$, $1 \leq i \leq r$, with $\omega_1 = dx_1, ..., \omega_m = dx_m$.

One can interpret the extended Cartan form as a "moving coframe" of $J^{k-1}\Gamma$ that is parametrized by the "auxiliary" variables u_ρ. In this local description, Proposition 4.3.3 says that the elements $\psi \in \Gamma^k$ of the k-th prolongation are characterized by the invariance property

$$\psi^* x_a = x_a \qquad \text{and} \qquad \psi^*\omega_i = \omega_i, \tag{4.5}$$

with $1 \leq a \leq m$ and $1 \leq i \leq r$, which coincides with Cartan's statement.

Note that, in his statement of the theorem, Cartan views x_i as the "actual" variables and u_ρ as "auxiliary" variables. In his proof of the theorem, Cartan obtains the k-th prolongation Γ^k by an inductive procedure. Starting with Γ, he constructs $\Gamma^0,\ \Gamma^1,\ \Gamma^2, ...,$ $\Gamma^{k-1},\ \Gamma^k$. In each step, Γ^l is obtained as an isomorphic prolongation of Γ^{l-1} along the projection $\pi : J^l\Gamma \to J^{l-1}\Gamma$. Accordingly, in each step, Cartan views elements of Γ^l as transformations of the "actual" variables of $J^{l-1}\Gamma$ that act on the new "auxiliary" variables that parametrize the fibers of $\pi : J^l\Gamma \to J^{l-1}\Gamma$.

Chapter 5

The Second Fundamental Theorem

In the first fundamental theorem, Cartan showed that any Lie pseudogroup is equivalent to one that is defined as the set of local symmetries of a system of functions and 1-forms. In the second fundamental theorem, Cartan identifies the essential properties of this system. Namely, he proves that it satisfies a set of equations known as the *structure equations*. We say that a pseudogroup defined as the set of local symmetries of such a system is in *normal form*, and, hence, the second fundamental theorem states that any Lie pseudogroup is equivalent to one in normal form.

The aim of this chapter is to formulate Cartan's notion of structure equations in a global and coordinate-free fashion and to present a modern proof of the second fundamental theorem. To this end, we introduce the notion of a *pre-Cartan algebroid* and its *realizations*, geometric structures that, in local coordinates, recover Cartan's notion of structure equations. In these modern terms, a pseudogroup is said to be in *normal form* if it is defined as the set of local symmetries of some realization of a pre-Cartan algebroid. We then prove the global version of the second fundamental theorem, namely that any Lie pseudogroup (under a mild regularity condition) is equivalent to a pseudogroup in normal form. As in the local picture, this theorem builds upon the first fundamental theorem (Theorem 4.3.1), where we proved that any Lie pseudogroup (under a mild regularity condition) is defined as the set of local symmetries of a certain pair denoted by (t, Ω). To prove the second fundamental theorem, we are left to show that this pair defines a realization, and, in particular, that it satisfies the structure equations in their global and coordinate-free form. In the last part of the chapter, we compute the pre-Cartan algebroids and realizations in two examples taken from Cartan's work, and show that the resulting structures coincide with those of Cartan.

5.1 Cartan's Formulation

In the second fundamental theorem, Cartan proves that the system of functions and 1-forms that he derives in the first fundamental theorem (Theorem 4.1.3) satisfy a set of equations which he calls the structure equations. In order to state Cartan's theorem clearly and concisely, let us encode the notion of structure equations in the following structure (the definition is based on [7], pp. 1341-1344, and see also [34], pp. 1-2):

Definition 5.1.1. *Let $x = (x_1, ..., x_N)$ be coordinates on $\mathbb{R}^N$. A* ***normal form data*** *on $\mathbb{R}^N$ consists of:*

1. *the projection*

$$I = (I_1, ..., I_n) : \mathbb{R}^N \to \mathbb{R}^n, \qquad I_a(x) = x_a, \tag{5.1}$$

onto the first n coordinates, for some $0 \leq n \leq N$,

2. *a collection of $r \geq n$ linearly independent differential 1-forms $\omega_1, ..., \omega_r \in \Omega^1(\mathbb{R}^N)$ with*
$$\omega_1 = dI_1, ..., \omega_n = dI_n, \tag{5.2}$$

such that the 1-forms $\omega_1, ..., \omega_r$ can be completed to a coframe of $\mathbb{R}^N$ by adding another set of p linearly independent 1-forms $\pi_1, ..., \pi_p \in \Omega^1(\mathbb{R}^N)$ (thus $r + p = N$), and these satisfy the set of equations

$$d\omega_i + \frac{1}{2} c_i^{jk}\, \omega_j \wedge \omega_k = a_i^{\lambda j}\, \pi_\lambda \wedge \omega_j, \tag{5.3}$$

where c_i^{jk} and $a_i^{\lambda j}$ $(1 \leq i, j, k \leq r,\ 1 \leq \lambda \leq p)$ are some smooth functions on $\mathbb{R}^N$ (necessarily unique) that satisfy the following properties:

a) $c_i^{jk} = -c_i^{kj}$,

b) $c_i^{jk}, a_i^{\lambda j}$ *depend only on* $x_1, ..., x_n$,

c) at each point, the matrices $A^\lambda = (a_i^{\lambda j})$ are linearly independent.

Remark 5.1.2. Let us add a short grammatical remark concerning our use of the word "data". In recent years, it has become the rule rather than the exception to use "data" as a singular noun, e.g. *the data is incorrect* rather than *the data are incorrect*. This is comparable to the word "agenda", which has gone through a similar transformation. However, while saying *an agenda* is common and correct, saying *a data* still sounds wrong. Since we would like to treat "a normal form data" as a single mathematical object (and later also "a pre-Cartan data" and "a Cartan data"), we will allow ourselves this slight deviation from the common grammatical usage. $\diamondsuit$

We denote a normal form data by (I_a, ω_i), indicating that the forms π_λ are not part of the structure, but rather appear as a condition. The functions c_i^{jk} and $a_i^{\lambda j}$, on the other hand, should be viewed as part of the structure. Any normal form data (I_a, ω_i) on $\mathbb{R}^N$ induces a pseudogroup on $\mathbb{R}^N$ that is defined by

$$\Gamma(I_a, \omega_i) := \{\ \phi \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^N) \mid \phi^* I_a = I_a,\ \phi^*\omega_i = \omega_i\ \}. \tag{5.4}$$

We view $\Gamma(I_a, \omega_i)$ as the pseudogroup of "local symmetries" of the (I_a, ω_i).

In general, the pseudogroup $\Gamma(I_a, \omega_i)$ may be "too small", in the sense that the set of equations

$$\phi^* I_a = I_a \qquad \text{and} \qquad \phi^*\omega_i = \omega_i, \tag{5.5}$$

may fail to have sufficient solutions. A priori, we are only certain that the identity map of $\mathbb{R}^N$ and its restriction to open subsets solve (5.5). We will say that $\Gamma(I_a, \omega_i)$ is in normal form if it has "sufficient" solutions. More precisely:

Definition 5.1.3. *A pseudogroup Γ on $\mathbb{R}^N$ is in **normal form** if there exists a normal form data (I_a, ω_i) on $\mathbb{R}^N$ such that $\Gamma = \Gamma(I_a, \omega_i)$ and such that the orbits of Γ are precisely the fibers of* (5.1).

With these definitions in place, Cartan's second fundamental theorem can be phrased as follows ([7], p. 1342):

Theorem 5.1.4. *(the second fundamental theorem) Any Lie pseudogroup on an open subset of a Euclidean space is equivalent to one in normal form.*

The second fundamental theorem builds upon the first fundamental theorem. Given a Lie pseudogroup Γ, the first fundamental theorem already tells us that there exist a set of functions I_a and 1-forms ω_i such that $\Gamma(I_a, \omega_i)$ is equivalent to Γ and such that the orbits of $\Gamma(I_a, \omega_i)$ coincide with the fibers of (5.1). In the second fundamental theorem, Cartan proceeds to prove that this data indeed satisfies Equations (5.3) and that the coefficients that arise from these equations satisfy properties (a)-(c) of Definition 5.1.1.

Equations (5.3) are called the **structure equations**, and the functions c_i^{jk} and $a_i^{\lambda j}$ that arise as the coefficients in the equations are called the **structure functions**. Note that as an immediate consequence of (5.2) and (5.3), some of the structure functions are trivial, namely

$$c_i^{jk} = 0 \quad \text{and} \quad a_i^{\lambda j} = 0 \ \text{ for } \ 1 \leq i \leq n. \tag{5.6}$$

One should regard the structure functions as the infinitesimal data associated with the normal form data. Here one should keep in mind the following analogy with Lie groups: given a Lie group G, the coframe $\omega_1, ..., \omega_m, \pi_1, ..., \pi_p$ is analogous to the Maurer-Cartan form on G, the structure equations are analogous to the Maurer-Cartan equation, and the structure coefficients are analogous to the structure constants of the Lie algebra $\mathfrak{g}$ of G that arise as the coefficients of the Maurer-Cartan equation.

We package the infinitesimal data that is associated with a normal form data in the following definition:

Definition 5.1.5. *A **pre-Cartan data** on $\mathbb{R}^n$ consists of a collection of smooth functions*

$$c_i^{jk},\ a_i^{\lambda j} \in C^\infty(\mathbb{R}^n), \qquad 1 \leq i, j, k \leq r,\ 1 \leq \lambda \leq p,$$

for some integers $r \geq n$ and $p \geq 0$, such that:

1. $c_i^{jk} = -c_i^{kj}$,
2. *at each point, the matrices $A^\lambda = (a_i^{\lambda j})$ are linearly independent,*
3. $c_i^{jk} = 0$ *and* $a_i^{\lambda j} = 0$ *for all* $1 \leq i \leq n$.

The prefix "pre-" in the name pre-Cartan data is meant to indicate that this definition does not yet capture the full infinitesimal structure that underlies the notion of a normal form data. We will return to this point in Section 6.1, where we will see that the structure equations have several additional implications on the type of functions that can appear as structure functions. Returning to the analogy with Lie groups, think that an arbitrary collection of constants c_i^{jk} that is anti-symmetric in the upper indices does not yet define a Lie algebra, since the bracket they define must further satisfy the Jacobi identity.

The Converse to the Second Fundamental Theorem and Involutivity The second fundamental theorem shows that, up to equivalence, any Lie pseudogroup is encoded in the structure of a normal form data. A natural follow-up question that arises is whether an arbitrary normal form data, not necessarily one coming from a Lie pseudogroup, induces a pseudogroup in normal form. If the answer is positive, then studying Lie pseudogroups up to equivalence amounts to studying the space of normal form data. This question, in turn, boils down to the problem of solving the PDE given by (5.5) for a given normal form data. This problem is one of two integrability problems that arise in Cartan's structure theory, the second being the realization problem that will be discussed in the next section in the context of the third fundamental theorem.

To solve these integrability problems, Cartan developed a powerful analytic tool, the *theory of Pfaffian systems*, which evolved into the modern day theory of exterior differential systems and to the Cartan-Kähler theorem (see also the introduction chapter). Using this theory, Cartan proves the following theorem, which he calls the converse to the second fundamental theorem (see [7], p. 1343, for Cartan's proof and see also the last paragraph of [34]):

Theorem 5.1.6. *Let (I_a, ω_i) be an analytic normal form data on $\mathbb{R}^N$ (i.e. all functions and forms are analytic). If the tableau spanned by the matrices $A^\lambda = (a_i^{\lambda j})$ at each point is involutive, then the induced pseudogroup $\Gamma(I_a, \omega_i)$ is in normal form.*

Let us comment on the two conditions in the statement: analyticity and involutivity. As we explained in the introduction, the Cartan-Kähler theorem is a *local* existence theorem for exterior differential systems in the *analytic* setting, and, in particular, for *analytic* PDEs. Thus, to begin with, the theorem is only applicable in the analytic category, and hence the condition of analyticity. Now, given an analytic PDE that satisfies a certain set of conditions, the Cartan Kähler theorem guarantees the existence of local solutions, which, in this case, is what we are after, namely local diffeomorphisms that solve (5.5). Cartan proves that, in this specific problem, these conditions are equivalent to the requirement that the tableau spanned by the matrices $A^\lambda = (a_i^{\lambda j})$ at each point is involutive (see Definition 1.5.3 for the notion of an involutive tableau, where a tableau is a tableau bundle over a point, and see also Remark 1.5.5).

Thus, because of the fact that one must require both analyticity and involutivity, Cartan's theorem is only a partial answer to the above question. Whether this theorem can be extended to more general situations, e.g. replacing "analytic" with "smooth" and/or relaxing the involutivity condition, remains an open problem.

Examples In this final part of the section, we cite five examples given by Cartan in [7] (pp. 1344-1347). In each example, Cartan derives the normal form data and, in particular, the structure equations that are induced by a given Lie pseudogroup. Starting from a Lie pseudogroup Γ_0 on $\mathbb{R}^{n_0}$, Cartan derives a collection of functions $I_1, ..., I_n$ and linearly independent 1-forms $\omega_1, ..., \omega_r$ and $\pi_1, ..., \pi_p$ on $\mathbb{R}^N$ for some N (with $n \leq r$ and $r + p = N$), and shows that they form a normal form data (I_a, ω_i). In particular, he shows that they satisfy the structure equations. As we saw earlier in this section, this data induces the pseudogroup $\Gamma = \Gamma(I_a, \omega_i)$ on $\mathbb{R}^N$ that is defined as the set of local symmetries of the system (I_a, ω_i). Cartan does not give the explicit formulas for Γ, and so, in addition to citing the text translated from Cartan's work in each example, we also give explicit formulas for the induced pseudogroup Γ that show that Γ is indeed an isomorphic prolongation of Γ_0.

The examples are all different in nature and give an idea of how transitivity / intransitivity, finiteness / infiniteness and the order of the defining equations of Γ are reflected in the resulting normal form data. The following table will help in keeping track of these properties.

Example	Transitive	Finite	Order	n_0	N	n	r	p
5.1.7	yes	yes	2nd	1	2	0	2	0
5.1.8	no	yes	1st	2	2	1	2	0
5.1.9	yes	yes	3rd	1	3	0	3	0
5.1.10	no	no	1st	2	3	1	2	1
5.1.11	yes	no	1st	2	3	0	2	1

Before moving on to the examples, we only mention that in Section 5.4 we will revisit two of these examples and rederive Cartan's formulas by using the theory that will be presented in the upcoming sections.

Example 5.1.7. Cartan: *"We start with the finite pseudogroup* Γ_0

$$X = ax + b,$$

where the defining equations are clearly

$$dX = \omega_1 = udx, \quad du = 0$$

(the equation $\frac{d^2X}{dx^2} = 0$ *is of second order). We have*

$$d\omega_1 = du \wedge dx = \frac{du}{u} \wedge \omega_1;$$

the isomorphic prolongation Γ *of* Γ_0 *acting on the variables* x *and* u *is thus defined by the invariance of the forms*

$$\omega_1 = udx, \quad \omega_2 = \frac{du}{u}.$$

We directly obtain the structure equations

$$d\omega_1 = \omega_2 \wedge \omega_1, \quad d\omega_2 = 0."$$

Let us compute the isomorphic prolongation Γ on $\mathbb{R}^2$ for this example. We denote the source variables by (x, u) and the target variables by $(\bar{x}, \bar{u})$. The prolongation Γ consists of all local diffeomorphisms ψ of $\mathbb{R}^2$, with $\psi : (x,u) \mapsto (\bar{x}(x,u), \bar{u}(x,u))$, that satisfy the invariance condition

$$\psi^*\omega_1 = \omega_1 \qquad \text{and} \qquad \psi^*\omega_2 = \omega_2.$$

Solving these equations, we see that Γ consists of local diffeomorphisms of the form

$$\bar{x}(x,u) = ax + b, \quad \bar{u}(x,u) = au,$$

parametrized by $a \in \mathbb{R}_{\neq 0},\ b \in \mathbb{R}$. This is indeed an isomorphic prolongation of Γ_0. ♢

Example 5.1.8. Cartan: *"Let Γ_0 be the pseudogroup in two variables x, y and one invariant y, where the finite equations are*

$$X = x + ay, \quad Y = y; \tag{5.7}$$

the defining equations are

$$Y = y, \quad dX = \omega_1 = dx + \frac{X - x}{y} dy;$$

the pseudogroup Γ coincides here with Γ_0. To obtain the structure equations, we can set X in the form ω_1 to have a fixed value, say $X = 0$; we have, therefore,

$$\omega_1 = dx - \frac{x}{y} dy \quad \text{and} \quad d\omega_1 = -\omega_1 \wedge \frac{dy}{y}$$

or

$$\omega_1 = dx - \frac{x}{y} dy, \qquad \omega_2 = dy,$$

with

$$d\omega_1 = \frac{1}{y} \omega_2 \wedge \omega_1, \qquad d\omega_2 = 0."$$

The resulting isomorphic prolongation Γ on $\mathbb{R}^2$ with source variables (x, y) and target variables (X, Y), defined by the invariance of

$$I_1(x, y) = y$$

and ω_1, ω_2 as above, is in this case Γ_0 itself, as Cartan points out. ♢

Note that in the last example, the 1-forms are not defined at $y = 0$. In fact, Cartan disregards issues concerning domains of definition in his writing. Although it is nowhere mentioned explicitly, it is quite evident that Cartan implicitly assumes suitable regularity conditions when those are needed.

Example 5.1.9. Cartan: *"Let Γ_0 be the pseudogroup of homographic transformations in one variable*

$$X = \frac{ax+b}{cx+d} \quad a,b,c,d \in \mathbb{R}, \quad ad-bc \neq 0.$$

We know that the defining equation of the pseudogroup is

$$X'X''' - \frac{3}{2}(X'')^2 = 0.$$

We set

$$X' = u, \quad X'' = v,$$

and we have the system

$$dX = \omega_1 = udx, \quad du = vdx, \quad dv = \frac{3}{2}\frac{v^2}{u}dx.$$

We have

$$d\omega_1 = du \wedge dx = \frac{du - vdx}{u} \wedge udx = \frac{du - vdx}{u} \wedge \omega_1.$$

The form $\frac{du-vdx}{u}$ is thus invariant, we denote it by ω_2,

$$\omega_2 = \frac{du}{u} - \frac{v}{u}dx.$$

We compute

$$\begin{aligned} d\omega_2 &= -\frac{1}{u}dv \wedge dx + \frac{v}{u^2}du \wedge dx = \big(-\frac{1}{u^2}dv + \frac{v}{u^3}du \big) \wedge \omega_1 \\ &= \Big(-\frac{1}{u^2}\big(dv - \frac{3}{2}\frac{v^2}{u}dx\big) + \frac{v}{u^3}(du - vdx) \Big) \wedge \omega_1, \end{aligned}$$

from which we obtain the new invariant form

$$\omega_3 = -\frac{1}{u^2}dv + \frac{v}{u^3}du + \frac{1}{2}\frac{v^2}{u^3}dx.$$

We compute

$$d\omega_3 = \frac{1}{u^3}du \wedge dv + \frac{v}{u^3}dv \wedge dx - \frac{3}{2}\frac{v^2}{u^4}du \wedge dx = \omega_3 \wedge \omega_2.$$

The structure equations are

$$d\omega_1 = \omega_2 \wedge \omega_1, \quad d\omega_2 = \omega_3 \wedge \omega_1, \quad d\omega_3 = \omega_3 \wedge \omega_2.\text{"} \tag{5.8}$$

A rather long but worthwhile computation yields the isomorphic prolongation Γ on $\mathbb{R}^3 \backslash \{u = 0\}$

$$\bar{x} = \frac{ax+b}{cx+d}, \quad \bar{u} = u\frac{(cx+d)^2}{ad-bc}, \quad \bar{v} = \frac{v(cx+d)^4 + 2uc(cx+d)^3}{(ad-bc)^2},$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$. $\diamondsuit$

Example 5.1.10. Cartan: *"Let Γ_0 be the pseudogroup on $\mathbb{R}^2$ defined by*

$$X = x + f(y), \quad Y = y,$$

where f is an arbitrary function of y. The defining equations are

$$Y = y, \quad \frac{\partial X}{\partial x} = 1,$$

and thus

$$Y = y, \quad dX = dx + udy.$$

We, therefore, have

$$\omega_1 = dx + udy, \quad \omega_2 = dy,$$

with the structure equations

$$d\omega_1 = \pi_1 \wedge dy = \pi_1 \wedge \omega_2, \quad d\omega_2 = 0,$$

where $\pi_1 = du \pmod{dy}$."

The resulting isomorphic prolongation Γ on $\mathbb{R}^2$, induced by

$$I_1(x, y) = y$$

and ω_1, ω_2 as above, is given by

$$\bar{x} = x + f(y), \quad \bar{y} = y, \quad \bar{u} = u - f'(y), \qquad f \in C^\infty(\mathbb{R}). \qquad \diamondsuit$$

Example 5.1.11. Cartan: *"Let Γ_0 be the pseudogroup on $\mathbb{R}^2$ whose elements are given by*

$$X = f(x), \quad Y = \frac{y}{f'(x)}, \tag{5.9}$$

where f is an arbitrary function of x and f' its derivative (nowhere vanishing). The defining equations are

$$dX = \frac{y}{Y} dx, \quad dY = udx + \frac{Y}{y} dy =: \omega_2,$$

they are of 1st order. We set $Y = 1$ on the right hand side of both equation, and obtain

$$\omega_1 = ydx, \quad \omega_2 = udx + \frac{1}{y} dy,$$

with the structure equations

$$d\omega_1 = \omega_2 \wedge \omega_1, \quad d\omega_2 = \pi \wedge \omega_1,$$

where $\pi = \frac{1}{y}du \pmod{dx}$*. We remark here that the pseudogroup* Γ *is the isomorphic prolongation of the pseudogroup* $X = f(x)$*, where the defining equation is* $dX = udx$*, with*

$$\omega_1 = udx, \quad d\omega_1 = \pi \wedge \omega_1.\text{"} \tag{5.10}$$

The resulting isomorphic prolongation Γ on $\mathbb{R}^3$ is given by

$$\bar{x} = f(x), \quad \bar{y} = \frac{y}{f'(x)}, \quad \bar{u} = \frac{uf'(x) + f''(x)}{(f'(x))^2}, \qquad f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}). \qquad \diamondsuit$$

Note that in the last example, Cartan simplifies the expression by setting the target variable Y to the fixed value 1. Similarly, in Example 5.1.8, Cartan sets the variable X to zero. These are two instances of the simplification of *restricting to a transversal* that was discussed in Section 4.3. Cartan uses this simplification to reduce the dimension of the space on which the isomorphic prolongation acts, thus obtaining a smaller isomorphic prolongation. In the next example we show that one may choose not to apply this simplification, thus obtaining a larger but canonical isomorphic prolongation.

Example 5.1.12. Consider again the pseudogroup from example 5.1.11. Prior to the simplification of setting $Y = 1$, we had the 1-forms

$$\omega_1 = \frac{y}{Y}dx, \quad \omega_2 = udx + \frac{Y}{y}dy.$$

Adding to this data the projection functions

$$I_1 = X, \quad I_2 = Y,$$

and their differentials

$$\omega_3 = dX, \quad \omega_4 = dY,$$

the structure equations are

$$d\omega_1 = \frac{1}{Y}(\omega_2 - \omega_4) \wedge \omega_1, \quad d\omega_2 = \frac{1}{Y}\omega_4 \wedge \omega_2 + \pi \wedge \omega_1, \quad d\omega_3 = 0, \quad d\omega_4 = 0,$$

with

$$\pi = \frac{Y}{y}du - \frac{u}{y}dY \mod dx.$$

The resulting isomorphic prolongation on $\mathbb{R}^5$, with coordinates (x, y, X, Y, u), is

$$\bar{x} = f(x), \;\; \bar{y} = \frac{y}{f'(x)}, \;\; \bar{X} = X, \;\; \bar{Y} = Y, \;\; \bar{u} = \frac{uf'(x) + Yf''(x)}{(f'(x))^2}, \quad f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}).$$

The restriction to the orbit $\{X = 0, \; Y = 1\}$ is precisely Cartan's isomorphic prolongation. $\diamondsuit$

The general idea behind Cartan's algorithm for constructing a prolongation becomes clear after working through these examples. Given a Lie pseudogroup Γ_0 on $\mathbb{R}^{n_0}$, we pass to a larger space by adding variables that correspond to "free derivatives", i.e. derivatives that are not determined by the defining equations of Γ_0, and express the defining equations in "differential form". Globally, the space of free derivatives is, of course, the k-th jet groupoid $J^k\Gamma_0$ of Γ_0, where k is the order of the defining equations, and the ω_i's that Cartan obtains are the components of the extended Cartan form Ω of $J^k\Gamma_0$ that we saw in Section 4.3. However, how does one makes sense of Cartan's structure equations globally, i.e. of an expression of the type "$d\Omega = ...$"? And how does one prove that such structure equations are satisfied in general? We now turn to address these questions.

5.2 The Modern Formulation of Structure Equations: pre-Cartan Algebroids and their Realizations

In this section, we present a global and coordinate-free formulation of Cartan's notion of structure equations and normal form data (Definition 5.1.1).

Pre-Cartan Algebroids Let us begin with the very basic idea of Cartan: pseudogroups realized as local diffeomorphisms preserving a system of functions and 1-forms. We start with a fibered manifold $I : P \to N$, a vector bundle $\mathcal{C} \to N$ and a $\mathcal{C}$-valued 1-form on P, $\Omega \in \Omega^1(P; I^*\mathcal{C})$. Such data induces a pseudogroup $\Gamma(P, \Omega)$ on P consisting of all local diffeomorphisms $\psi \in \mathrm{Diff}_{\mathrm{loc}}(P)$ satisfying

$$\psi^* I = I \qquad \text{and} \qquad \psi^*\Omega = \Omega.$$

Note that the first condition ensures that the second condition makes sense. One would like to understand the first order consequences of the previous equations (e.g. "$\psi^*(d\Omega) = d\Omega$"). This becomes easier when $\mathcal{C}$ is endowed with extra structure. This extra structure is specified in the next definition and it is hidden in all the considerations of Cartan.

Definition 5.2.1. *A **pre-Lie algebroid** over a manifold N is a vector bundle $\mathcal{C} \to N$ equipped with a vector bundle map $\rho : \mathcal{C} \to TN$ ('the anchor') and a bilinear antisymmetric map $[\cdot,\cdot] : \Gamma(\mathcal{C}) \times \Gamma(\mathcal{C}) \to \Gamma(\mathcal{C})$ ('the bracket') satisfying the Leibniz identity*

$$[\alpha, f\beta] = f[\alpha, \beta] + L_{\rho(\alpha)}(f)\beta, \qquad \forall\, \alpha, \beta \in \Gamma(\mathcal{C}),\ f \in C^\infty(N),$$

and

$$\rho([\alpha, \beta]) = [\rho(\alpha), \rho(\beta)], \qquad \forall\, \alpha, \beta \in \Gamma(\mathcal{C}).$$

*A pre-Lie algebroid $\mathcal{C}$ is said to be **transitive** if $\rho : \mathcal{C} \to TN$ is surjective.*

Example 5.2.2. The best known example of a pre-Lie algebroid is a Lie algebroid: a pre-Lie algebroid whose bracket satisfies the Jacobi identity. ♢

Definition 5.2.3. *Given a pre-Lie algebroid $\mathcal{C}$ over N and a surjective submersion $I : P \to N$, a 1-form $\Omega \in \Omega^1(P; I^*\mathcal{C})$ is called* ***anchored*** *if*

$$\rho \circ \Omega = dI.$$

The anchored condition on Ω ensures that, although the expression $d\Omega$ does not make sense globally, the Maurer-Cartan type expression

$$\text{“}d\Omega + \frac{1}{2}[\Omega, \Omega]\text{”}$$

does. The construction is the same as for Lie algebroids: let $I : P \to N$ be a fibered manifold, $\mathcal{C} \to N$ a pre-Lie algebroid and $\Omega^*(P; I^*\mathcal{C})$ the graded vector space of $\mathcal{C}$-valued forms on P. Choose a vector bundle connection ∇ on $\mathcal{C}$. On the one hand, the connection induces a de Rham type operator $d_\nabla : \Omega^*(P; I^*\mathcal{C}) \to \Omega^{*+1}(P; I^*\mathcal{C})$ defined by the usual formula

$$\begin{aligned}(d_\nabla \phi)(X_0, ..., X_p) = &\sum_{i=0}^{p} (-1)^i (I^*\nabla)_{X_i}(\phi(X_0, ..., \hat{X}_i, ..., X_p)) \\ &+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \phi([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ..., X_p),\end{aligned}$$

where $\phi \in \Omega^p(P; I^*\mathcal{C})$ and $X_0, ..., X_p \in \mathfrak{X}(P)$. The operator d_∇ squares to zero if and only if ∇ is a flat connection. On the other hand, we have the $\mathcal{C}$-torsion of ∇, $[\cdot, \cdot]_\nabla \in \mathrm{Hom}(\Lambda^2\mathcal{C}, \mathcal{C})$, which is defined at the level of sections by

$$[\alpha, \beta]_\nabla = [\alpha, \beta] - \nabla_{\rho(\alpha)}\beta + \nabla_{\rho(\beta)}\alpha, \quad \forall \alpha, \beta \in \Gamma(\mathcal{C}),$$

and easily checked to be $C^\infty(N)$-linear in both entries. The torsion induces a graded bracket,

$$[\cdot, \cdot]_\nabla : \Omega^p(P; I^*\mathcal{C}) \times \Omega^q(P; I^*\mathcal{C}) \to \Omega^{p+q}(P; I^*\mathcal{C}), \tag{5.11}$$

which generalizes the wedge product and which is defined by the analogous formula,

$$\begin{aligned}&[\phi, \phi']_\nabla(X_1, ..., X_{p+q}) = \\ &\qquad \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma)\ [\phi(X_{\sigma(1)}, ..., X_{\sigma(p)}), \phi'(X_{\sigma(p+1)}, ..., X_{\sigma(p+q)})]_\nabla,\end{aligned}$$

where $S_{p,q}$ is the set of (p, q)-shuffles.

Proposition 5.2.4. *If $\Omega \in \Omega^1(P; I^*\mathcal{C})$ is anchored, then the Maurer-Cartan 2-form*

$$MC_\Omega := d_\nabla\Omega + \frac{1}{2}[\Omega, \Omega]_\nabla \in \Omega^2(P; I^*\mathcal{C})$$

is independent of the choice of connection.

Remark 5.2.5. Hence, from now on we will suppress ∇ from the notation and simply write $d\Omega + \frac{1}{2}[\Omega, \Omega]$ when Ω is anchored. $\diamondsuit$

Proof. Let ∇ and ∇' be two connections on $\mathcal{C}$, then $\eta := \nabla - \nabla' \in \Omega^1(N; \mathrm{Hom}(\mathcal{C}, \mathcal{C}))$. Let $p \in P$ and $X, Y \in T_pP$, then the sum of the following two equations vanishes if Ω is anchored:

$$(d_\nabla \Omega - d_{\nabla'}\Omega)(X, Y) = \eta(dI(X))(\Omega(Y)) - \eta(dI(Y))(\Omega(X))$$
$$([\Omega, \Omega]_\nabla - [\Omega, \Omega]_{\nabla'})(X, Y) = -\eta(\rho \circ \Omega(X))(\Omega(Y)) + \eta(\rho \circ \Omega(Y))(\Omega(X))$$

$\square$

Intuitively, MC_Ω measures the failure of $\Omega : TP \to \mathcal{C}$ to be a morphism of pre-Lie algebroids (c.f. Remark 7.4.3). For example, when $P = N$ and I is the identity,

$$\mathrm{MC}_\Omega(X, Y) = -\Omega([X, Y]) + [\Omega(X), \Omega(Y)], \quad \forall X, Y \in \mathfrak{X}(N).$$

When Ω is anchored and pointwise surjective, we have the following:

Lemma 5.2.6. *Let $\Omega \in \Omega^1(P; I^*\mathcal{C})$ be anchored and pointwise surjective. Given any $\alpha \in \Gamma(\mathcal{C})$, there exists $X_\alpha \in \mathfrak{X}(P)$ such that*

$$\Omega(X_\alpha) = I^*\alpha.$$

Any pair $X_\alpha, X_\beta \in \mathfrak{X}(P)$, with $\alpha, \beta \in \Gamma(\mathcal{C})$, that satisfies the above equation satisfies

$$MC_\Omega(X_\alpha, X_\beta) = -\Omega([X_\alpha, X_\beta]) + I^*[\alpha, \beta].$$

Proof. An X_α as in the statement can be obtained by choosing a splitting of the short exact sequence of vector bundles $0 \to \ker(\Omega) \to TP \xrightarrow{\Omega} I^*\mathcal{C} \to 0$. By the anchored condition:

$$\begin{aligned}(d\Omega + \frac{1}{2}[\Omega, \Omega])(X_\alpha, X_\beta) = \cancel{(I^*\nabla)_{X_\alpha}(\Omega(X_\beta))} - \cancel{(I^*\nabla)_{X_\beta}(\Omega(X_\alpha))} - \Omega([X_\alpha, X_\beta]) \\ + I^*[\alpha, \beta] - \cancel{I^*(\nabla_{\rho(\alpha)}\beta)} + \cancel{I^*(\nabla_{\rho(\beta)}\alpha)}.\end{aligned}$$

$\square$

In order to make sense of structure equations globally, one needs a little more than a pre-Lie algebroid. The necessary structure is encoded in the following definition:

Definition 5.2.7. *A **pre-Cartan algebroid** over a manifold N is a pair $(\mathcal{C}, \mathfrak{g})$ consisting of a transitive pre-Lie algebroid $\mathcal{C} \to N$ and a vector sub-bundle $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ such that $T(\mathcal{C}) \subset \mathrm{Ker}\ \rho$ for all $T \in \mathfrak{g}$.*

Thus, $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\ \rho)$, where ρ is the anchor of $\mathcal{C}$. Note that $\mathfrak{g}$ is a tableau bundle, in the sense of Definition 1.2.3, and hence we may talk about the prolongations and the Spencer cohomology of $\mathfrak{g}$. This will play an important role in the theory.

Example 5.2.8. Locally, we are back to Cartan: pre-Cartan algebroids correspond to the notion of pre-Cartan data (Definition 5.1.5) and are encoded by functions c_i^{jk} and $a_i^{\lambda j}$. More precisely, following the notation from Section 5.1,

- $N = \mathbb{R}^n$
- $\mathcal{C} \to N$ is the trivial vector bundle of rank r (with $r \geq n$) with trivializing frame $\{e^1, ..., e^r\}$ and endowed with the pre-Lie algebroid structure given by

$$\rho(e^i) = \frac{\partial}{\partial x_i} \text{ for } 1 \leq i \leq n, \quad \rho(e^i) = 0 \text{ for } i > n,$$

  and

$$[e^j, e^k] = c_i^{jk} e^i,$$

  where the bracket is extended to all sections of $\mathcal{C}$ by the Leibniz identity. The fact that ρ is a Lie algebra homomorphism is equivalent to the condition $c_i^{jk} = 0$ for $i \leq n$.

- $\mathfrak{g} \to N$ is the trivial vector bundle of rank p with trivializing frame denoted by $\{t^1, ..., t^p\}$. Each element of the frame acts on $\mathcal{C}$ by

$$t^\lambda(e^j) = a_i^{\lambda j} e^i,$$

  and, extending by linearity, we obtain a map $\mathfrak{g} \to \mathrm{Hom}(\mathcal{C}, \mathcal{C})$. The injectivity of this map is equivalent to Cartan's condition that, at each point of $\mathbb{R}^n$, the matrices $A^\lambda = (a_i^{\lambda j})$ are linearly independent. The condition that $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\ \rho)$, i.e. that the elements of $\mathfrak{g}$ actually take values in the kernel of ρ, is equivalent to the condition $a_{j\lambda}^i = 0$ for $i \leq n$. $\diamondsuit$

Gauge Equivalence In Section 5.3, as part of the proof of the second fundamental theorem, we will construct a pre-Cartan algebroid out of a given Lie pseudogroup. As we will see, while the construction of the vector bundle $\mathcal{C}$ and its anchor are canonical, as well as the inclusion $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$, the construction of the bracket of $\mathcal{C}$ will depend on a choice. However, up to *gauge equivalence*, a notion that we now introduce, the bracket will also become canonical (locally, this notion already appears in [34]).

Let $(\mathcal{C}, \mathfrak{g})$ be a pre-Cartan algebroid over N. A choice of a section $\eta \in \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}))$ induces a new bracket $[\cdot, \cdot]^\eta$ on $\mathcal{C}$ defined by

$$[\alpha, \beta]^\eta := [\alpha, \beta] + \eta(\alpha)(\beta) - \eta(\beta)(\alpha), \qquad \forall\ \alpha, \beta \in \Gamma(\mathcal{C}).$$

We denote by $\mathcal{C}^\eta$ the vector bundle $\mathcal{C}$ equipped with the new bracket $[\cdot, \cdot]^\eta$ and with the same anchor ρ as $\mathcal{C}$.

Lemma 5.2.9. *Let $(\mathcal{C}, \mathfrak{g})$ be a pre-Cartan algebroid over N and let $\eta \in \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}))$. Then $(\mathcal{C}^\eta, \mathfrak{g})$ is a pre-Cartan algebroid over N.*

Proof. We only need to verify that $\mathcal{C}^\eta$ is a pre-Lie algebroid. The Leibniz identity is clear, and

$$\rho[\alpha,\beta]^\eta = [\rho(\alpha),\rho(\beta)] + \cancel{\rho(\eta(\alpha)(\beta))} - \cancel{\rho(\eta(\beta)(\alpha))}, \qquad \forall\, \alpha,\beta \in \Gamma(\mathcal{C}),$$

because $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\ \rho)$. □

Definition 5.2.10. *Two pre-Cartan algebroids $(\mathcal{C},\mathfrak{g})$ and $(\mathcal{C}',\mathfrak{g})$ over N are* ***gauge equivalent*** *if there exists $\eta \in \Gamma(\mathrm{Hom}(\mathcal{C},\mathfrak{g}))$ s.t. $\mathcal{C}' = \mathcal{C}^\eta$.*

Gauge equivalence defines an equivalence relation on the set of pre-Cartan algebroids, which is straightforward to see.

Realizations The structure of a pre-Cartan algebroid is what we need in order to formulate the notion of structure equations in a coordinate-free fashion. Given a pre-Cartan algebroid $(\mathcal{C},\mathfrak{g})$ and a fibered manifold $I : P \to N$, in addition to the pairing (5.11), the extra data coming from $\mathfrak{g}$ and its inclusion in $\mathrm{Hom}(\mathcal{C},\mathcal{C})$ induces a graded pairing $\wedge : \Omega^p(P; I^*\mathfrak{g}) \times \Omega^q(P; I^*\mathcal{C}) \to \Omega^{p+q}(P; I^*\mathcal{C})$, generalizing the wedge product and defined by

$$(\eta \wedge \phi)(X_1, ..., X_{p+q}) = \sum_{\sigma \in S_{p,q}} \mathrm{sgn}(\sigma)\ \eta(X_{\sigma(1)}, ..., X_{\sigma(p)})(\phi(X_{\sigma(p+1)}, ..., X_{\sigma(p+q)})).$$

The following definition captures Cartan's notion of a normal form data:

Definition 5.2.11. *A* ***realization*** *of a pre-Cartan algebroid $(\mathcal{C},\mathfrak{g})$ is a pair (P,Ω) consisting of a surjective submersion $I : P \to N$ and a pointwise surjective anchored 1-form $\Omega \in \Omega^1(P; I^*\mathcal{C})$ such that there exists a 1-form $\Pi \in \Omega^1(P; I^*\mathfrak{g})$ satisfying*

$$d\Omega + \frac{1}{2}[\Omega,\Omega] = \Pi \wedge \Omega, \tag{5.12}$$

with the property that

$$(\Omega,\Pi) : TP \xrightarrow{\simeq} I^*(\mathcal{C} \oplus \mathfrak{g}) \tag{5.13}$$

is vector bundle isomorphism.

We call (5.12) the **structure equation** and (5.13) the **coframe condition** (see Example 5.2.14). The data of a realization induces a pseudogroup $\Gamma(P,\Omega)$ on the total space P characterized by the following invariance property: a local diffeomorphism $\psi \in \mathrm{Diff}_{\mathrm{loc}}(P)$ belongs to $\Gamma(P,\Omega)$ if and only if

$$\psi^* I = I \qquad \text{and} \qquad \psi^*\Omega = \Omega. \tag{5.14}$$

Definition 5.2.12. *A pseudogroup Γ on P is in* ***normal form*** *if there exists a realization (P,Ω) of some pre-Cartan algebroid $(\mathcal{C},\mathfrak{g})$ over N such that $\Gamma = \Gamma(P,\Omega)$ and such that the orbits of Γ coincide with the fibers of $I : P \to N$.*

Example 5.2.13. (Lie groups) The most familiar example of a realization is one coming from a Lie group. Let H be a Lie group with Lie algebra $\mathfrak{h}$. The pair

$$(\mathcal{C} = \mathfrak{h}, \mathfrak{g} = 0)$$

is a pre-Cartan algebroid, and the pair

$$(P = H, \Omega = \Omega_{\mathrm{MC}}),$$

where $\Omega_{\mathrm{MC}} \in \Omega^1(H; \mathfrak{h})$ is the canonical right-invariant Maurer-Cartan form of H, is a realization of $(\mathfrak{h}, 0)$. Indeed, this follows from the basic properties of the Maurer-Cartan form, namely that Ω_{MC} is pointwise an isomorphism and satisfies the Maurer-Cartan structure equation

$$d\Omega_{\mathrm{MC}} + \frac{1}{2}[\Omega_{\mathrm{MC}}, \Omega_{\mathrm{MC}}] = 0.$$

The induced pseudogroup $\Gamma(H, \Omega_{\mathrm{MC}})$ is precisely the pseudogroup generated by the group of right translations

$$\{\, R_{h^{-1}} : H \to H \mid h \in H \,\}$$

of the Lie group H, where $R_{h^{-1}}(h') = h'h^{-1}$. $\diamondsuit$

Locally, a realization corresponds to Cartan's notion of a normal form data:

Example 5.2.14. Continuing example 5.2.8,

- $P = \mathbb{R}^{r+p}$ with coordinates $(x_1, ..., x_{r+p})$ and $N = \mathbb{R}^n$ with $(x_1, ..., x_n)$ such that $I : P \to N$ is the projection onto the first n coordinates (recall that $n \le r$).
- The forms Ω and Π can be expressed as

  $$\Omega = \omega_i\, I^*e^i, \qquad \Pi = \pi_\lambda\, I^*t^\lambda,$$

  with $\omega_i, \pi_\lambda \in \Omega^1(P)$. The anchored condition on Ω is equivalent to

  $$\omega_1 = dx_1, ..., \omega_n = dx_n.$$

  Equation (5.12) becomes

  $$d\omega_i + \frac{1}{2}c_i^{jk}\, \omega_j \wedge \omega_k = a_i^{\lambda j}\, \pi_\lambda \wedge \omega_j, \tag{5.15}$$

  where c_i^{jk} and $a_i^{\lambda j}$ are functions on $\mathbb{R}^n$ viewed as functions on $\mathbb{R}^{r+p}$ that are constant along the fibers of I. Condition (5.13) is equivalent to $\{\omega_1, ..., \omega_r, \pi_1, ..., \pi_p\}$ being a coframe.

- The induced pseudogroup $\Gamma(P,\Omega)$ is the pseudogroup on $\mathbb{R}^{r+p}$ given by

$$\Gamma(P,\Omega)=\{\ \phi\in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}^{r+p})\mid \phi^*I_a=I_a,\ \phi^*\omega_i=\omega_i\ \},\tag{5.16}$$

which is precisely (5.4). This pseudogroup is in normal form when its orbits are the fibers of the projection

$$I=(I_1,...,I_n):\mathbb{R}^{r+p}\to\mathbb{R}^n,\qquad I_a(x)=x_a.\qquad\diamondsuit$$

In Theorem 5.1.6, Cartan proved, in the analytic setting and under the condition of involutivity, that any normal form data induces a pseudogroup in normal form. In this global setting, the theorem can be rephrased as follows:

Theorem 5.2.15. *Let (P,Ω) be an analytic realization of an analytic Cartan algebroid $(\mathcal{C},\mathfrak{g})$ (i.e., all manifolds and maps are analytic). If the tableau bundle $\mathfrak{g}$ is involutive, then the induced pseudogroup $\Gamma(P,\Omega)$ is in normal form.*

It is useful to keep in mind the following "dual" point of view of the above definition, which is, geometrically speaking, somewhat more intuitive. Let (P,Ω) be a realization of a pre-Cartan algebroid $(\mathcal{C},\mathfrak{g})$ over N. Given a choice of Π, as in definition 5.2.11, we consider the inverse of the vector bundle isomorphism (5.13),

$$(\Omega,\Pi)^{-1}:I^*(\mathcal{C}\oplus\mathfrak{g})\xrightarrow{\simeq}TP.$$

This map decomposes into two injective vector bundle maps,

$$\Psi_{\mathcal{C},\Pi}:=(\Omega,\Pi)^{-1}\big|_{I^*\mathcal{C}}:I^*C\to TP,$$
$$\Psi_{\mathfrak{g},\Pi}:=(\Omega,\Pi)^{-1}\big|_{I^*\mathfrak{g}}:I^*\mathfrak{g}\to TP.$$

These induce maps at the level of sections,

$$\begin{aligned}\Psi_{\mathcal{C},\Pi}&:\Gamma(\mathcal{C})\to\mathfrak{X}(P), & \alpha&\mapsto X_\alpha=\Psi_{\mathcal{C},\Pi}(\alpha),\\ \Psi_{\mathfrak{g},\Pi}&:\Gamma(\mathfrak{g})\to\mathfrak{X}(P), & S&\mapsto X_S=\Psi_{\mathfrak{g},\Pi}(S).\end{aligned}\tag{5.17}$$

Thus, $X_\alpha,X_S\in\mathfrak{X}(P)$ are the unique vector fields satisfying

$$\begin{aligned}\Omega(X_\alpha)&=I^*\alpha, & \Pi(X_\alpha)&=0,\\ \Omega(X_S)&=0, & \Pi(X_S)&=I^*S.\end{aligned}$$

Lemma 5.2.16. *Let (P,Ω) be a realization of a pre-Cartan algebroid $(\mathcal{C},\mathfrak{g})$ and fix a choice of* Π. *Then,*

$$\begin{aligned}\Omega([X_\alpha,X_{\alpha'}])&=I^*[\alpha,\alpha'],\\ \Omega([X_\alpha,X_S])&=I^*S(\alpha),\\ \Omega([X_S,X_{S'}])&=0,\end{aligned}$$

for all $\alpha,\alpha'\in\Gamma(\mathcal{C})$ and $S,S'\in\Gamma(\mathfrak{g})$. In particular, Ker $\Omega\subset TP$ *is an involutive distribution.*

Proof. Follows directly from the structure equation (5.12) together with lemma 5.2.6. $\square$

The fact that Ker Ω is an involutive distribution is one first consequence of the structure equations. Another important consequence is the following lemma:

Lemma 5.2.17. *Let (P, Ω) be a realization of a pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$. The map*

$$\Psi_{\mathfrak{g}} = \Psi_{\mathfrak{g},\Pi} : I^*\mathfrak{g} \to TP \tag{5.18}$$

is independent of the choice of Π. *Thus, there is a canonical isomorphism*

$$\Psi_{\mathfrak{g}} : I^*\mathfrak{g} \xrightarrow{\simeq} \mathrm{Ker}\ \Omega.$$

Proof. Fix a choice of Π. We must show that if Π' is another such choice, then $\Pi'(X_S) = I^*S$, or equivalently, that $\Pi'(X_S)(I^*\alpha) = \Pi(X_S)(I^*\alpha)$ for any $\alpha \in \Gamma(C)$. Subtracting the structure equations for Π and Π' from each other, we see that

$$(\Pi' - \Pi) \wedge \Omega = 0.$$

Thus, for any $\alpha \in \Gamma(\mathcal{C})$,

$$\begin{aligned} 0 &= \big((\Pi' - \Pi) \wedge \Omega\big)(X_S, X_\alpha) \\ &= \Pi'(X_S)(\Omega(X_\alpha)) - \Pi'(X_\alpha)(\cancel{\Omega(X_S)}) - \Pi(X_S)(\Omega(X_\alpha)) + \Pi(X_\alpha)(\cancel{\Omega(X_S)}) \\ &= \Pi'(X_S)(I^*\alpha) - \Pi(X_S)(I^*\alpha). \end{aligned}$$

$\square$

Remark 5.2.18. While condition 5.12 in the definition of a realization is rather natural, condition 5.13, the "coframe condition" is less so. In the examples of realizations coming from Lie pseudogroups, this condition is always satisfied. A natural direction of investigation "beyond Cartan" would be to look at structures in which this condition is relaxed. This condition can be relaxed in two directions. In one direction, one may simply not require that (Ω, Π) be an isomorphism, or require some weaker condition. In another direction, one can consider the dual point of view we discussed above. The maps $\Psi_{C,\Pi}$ and $\Psi_{\mathfrak{g}}$ have the flavor of an infinitesimal action of the object $(\mathcal{C}, \mathfrak{g})$ on the surjective submersion $I : P \to N$, an action which is transitive in the sense that the "action map" $(\Psi_{\mathcal{C},\Pi}, \Psi_{\mathfrak{g}}) : I^*(\mathcal{C}, \mathfrak{g}) \to P$ is a vector bundle isomorphism. In this direction, one may relax the transitivity condition. $\diamondsuit$

Realizations of pre-Cartan algebroids behave well under gauge equivalence of the underlying pre-Cartan algebroids in the following sense:

Proposition 5.2.19. *If (P, Ω) is a realization of a pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$, then it is also a realization of any gauge equivalent pre-Cartan algebroid $(\mathcal{C}^\eta, \mathfrak{g})$, where $\eta \in \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}))$. Moreover, if $\Pi \in \Omega^1(P; I^*\mathfrak{g})$ is a choice for the realization of $(\mathcal{C}, \mathfrak{g})$, as in Definition 5.2.11, then $\Pi^\eta \in \Omega^1(P; I^*\mathfrak{g})$ defined by*

$$\Pi^\eta(X) = \Pi(X) + (I^*\eta)(\Omega(X)), \qquad \forall\ X \in \mathfrak{X}(P),$$

is a choice for the realization of $(\mathcal{C}^\eta, \mathfrak{g})$.

Proof. Recall that given $\alpha \in \Gamma(\mathcal{C})$ and $S \in \Gamma(\mathfrak{g})$, we denote by $X_\alpha, X_S \in \mathfrak{X}(P)$ the unique vector fields that satisfy $(\Omega, \Pi)(X_\alpha) = I^*\alpha$ and $(\Omega, \Pi)(X_S) = I^*S$. One now easily checks that

$$d\Omega + \frac{1}{2}[\Omega, \Omega]^\eta = \Pi^\eta \wedge \Omega$$

is satisfied by applying both sides of the equation on all pairs of the type $(X_\alpha, X_{\alpha'})$, (X_α, X_S), $(X_S, X_{S'})$.

Next, it follows from the formula for Π^η that the vector fields $X^\eta_\alpha := X_\alpha - X_{\eta(\alpha)}$ and $X^\eta_S := X_S \in \mathfrak{X}(P)$ satisfy $(\Omega, \Pi^\eta)(X^\eta_\alpha) = I^*\alpha$ and $(\Omega, \Pi^\eta)(X^\eta_S) = I^*S$, from which it follows that $(\Omega, \Pi^\eta) : TP \to I^*(\mathcal{C} \oplus \mathfrak{g})$ is an isomorphism. □

In the definition of a realization, while the form Ω is a fixed part of the data, the form Π is only required to exist and, in general, may be non-unique. One may wonder as to how much freedom there is in the choice of Π. This ambiguity can be be described in terms of the 1st prolongation of $\mathfrak{g}$. Recall that given a tableau bundle $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$, its 1st prolongation is the subspace

$$\mathfrak{g}^{(1)} = \{\ \xi \in \mathrm{Hom}(\mathcal{C}, \mathfrak{g}) \mid \xi(\alpha)(\beta) = \xi(\beta)(\alpha)\ \} \ \subset\ \mathrm{Hom}(\mathcal{C}, \mathfrak{g}), \tag{5.19}$$

whose fibers are vector spaces that may vary in dimension. Let us assume that $\mathfrak{g}^{(1)}$ is of constant rank and let us fix a $\Pi_0 \in \Omega^1(P; I^*\mathfrak{g})$ as in Definition 5.2.11 as a "reference point" (it will serve as the choice of a "zero section" of an affine bundle). The choice of Π_0 fixes a choice of the maps (5.17). With such a choice, we have an induced vector bundle map

$$I^*\mathrm{Hom}(\mathcal{C}, \mathfrak{g}) \to \mathrm{Hom}(TP, I^*\mathfrak{g}), \qquad \xi \mapsto \hat{\xi}, \tag{5.20}$$

where $\hat{\xi}$ is uniquely determined by the conditions

$$\hat{\xi}(X_\alpha) = \xi(I^*\alpha) \qquad \forall\ \alpha \in \Gamma(\mathcal{C}), \tag{5.21}$$

$$\hat{\xi}(X_S) = 0 \qquad \forall\ S \in \Gamma(\mathfrak{g}). \tag{5.22}$$

The map (5.20) defines an isomorphism of vector bundles

$$I^*\mathrm{Hom}(\mathcal{C}, \mathfrak{g}) \cong \{\ \hat{\xi} \in \mathrm{Hom}(TP; I^*\mathfrak{g}) \mid \hat{\xi}(X_s) = 0\ \ \forall S \in \Gamma(\mathfrak{g})\ \},$$

and it restricts to an isomorphism of vector bundles

$$I^*\mathfrak{g}^{(1)} \cong \{\ \hat{\xi} \in \mathrm{Hom}(TP; I^*\mathfrak{g}) \mid \hat{\xi}(X_s) = 0\ \ \forall S \in \Gamma(\mathfrak{g}) \text{ and } \hat{\xi} \wedge \Omega = 0\ \}. \tag{5.23}$$

Thus, at the level of sections, we have a linear isomorphism between sections $\xi \in \Gamma(I^*\mathfrak{g}^{(1)})$ and 1-forms $\hat{\xi} \in \Omega^1(P; I^*\mathfrak{g})$ that satisfy both (5.22) and

$$\hat{\xi} \wedge \Omega = 0. \tag{5.24}$$

From now on, we write $\xi = \hat{\xi}$.

Proposition 5.2.20. *Let (P, Ω) be a realization of the pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ and assume that $\mathfrak{g}^{(1)}$ is of constant rank. The subspace of $\Omega^1(P; I^*\mathfrak{g})$ consisting of elements Π satisfying* (5.12) *and* (5.13) *is an affine space modeled on $\Gamma(I^*\mathfrak{g}^{(1)})$.*

Proof. Fix a choice of $\Pi \in \Omega^1(P; I^*\mathfrak{g})$ satisfying (5.12) and (5.13). Given any other choice Π', Lemma 5.2.17 implies that the difference $\Pi' - \Pi \in \Omega^1(P; I^*\mathfrak{g})$ satisfies the property $(\Pi' - \Pi)(X_S) = 0$ for all $S \in \Gamma(\mathfrak{g})$, and the structure equations imply that

$$(\Pi' - \Pi) \wedge \Omega = 0.$$

Hence, $\Pi' - \Pi \in \Gamma(I^*\mathfrak{g}^{(1)})$.

Conversely, let $\xi \in \Gamma(I^*\mathfrak{g}^{(1)})$. We claim that $\Pi + \xi$ satisfies conditions (5.12) and (5.13). Because ξ satisfies (5.24), it follows directly that $\Pi + \xi \in \Omega^1(P; I^*\mathfrak{g})$ satisfies (5.12). Moreover, the composition

$$\big((\Omega, \Pi + \xi) \circ (\Omega, \Pi)^{-1}\big)(I^*\alpha, I^*S) = (I^*\alpha, I^*S + \xi(I^*\alpha))$$

is clearly a vector bundle isomorphism of $I^*(\mathcal{C} \oplus \mathfrak{g})$ with itself, which implies that $(\Omega, \Pi + \xi)$ satisfies (5.13). $\square$

5.3 The Second Fundamental Theorem

Having globally formulated Cartan's notion of a normal form data, the global version of Cartan's second fundamental theorem, Theorem 5.1.4, takes the following form:

Theorem 5.3.1. *(the second fundamental theorem) Let Γ be a Lie pseudogroup of order k on M and assume that $(\mathfrak{g}^k)^{(1)}$ is of constant rank. Then Γ is Cartan equivalent to a pseudogroup in normal form (see Definition 5.2.12).*

Remark 5.3.2. In this section, we will present a global and coordinate-free proof of the second fundamental theorem. In addition to Cartan's proof of this theorem, or rather of the local version, Theorem 5.1.4, one can find other proofs in the literature. The ones we are aware familiar with were mentioned in the introduction to the thesis. We would like to particularly point out the proof and presentation by Kamran in [30] (see Theorem in Section 4), a work which has been an important source of inspiration for us. $\diamondsuit$

The proof of the second fundamental theorem builds upon the first fundamental theorem. In Section 4.3, we saw that any Lie pseudogroup Γ of order k on M admits an isomorphic prolongation Γ^k on $J^k\Gamma$ that is defined by the invariance property

$$\psi^* t = t \qquad \text{and} \qquad \psi^*\Omega = \Omega,$$

where $t : J^k\Gamma \to M$ is the target map, $\Omega = (dt, \omega) \in \Omega^1(J^k\Gamma; t^*\mathcal{C})$ the extended Cartan form and $\mathcal{C} = TM \oplus A^{k-1}$. Clearly the orbits of Γ^k coincide with the fibers of t. Let us write $\mathfrak{g} := \mathfrak{g}^k$. Thus, the the second fundamental theorem directly follows from the following proposition:

Proposition 5.3.3. *Let Γ be a Lie pseudogroup of order k on M and assume that $(\mathfrak{g}^k)^{(1)}$ is of constant rank. Then $(\mathcal{C}, \mathfrak{g})$ has the structure of a pre-Cartan algebroid and $(J^k\Gamma, \Omega)$ is a realization of $(\mathcal{C}, \mathfrak{g})$.*

Remark 5.3.4. We would like to emphasize that the pseudogroup in normal form from the second fundamental theorem is not unique. While using Γ^k is one possible choice, one may also use Γ^{k+1} or higher, or restrict Γ^k to a transversal (see Section 4.3). $\diamondsuit$

Our goal in this section is to prove Proposition 5.3.3. The proof involves two tasks. The first is to specify the pre-Cartan algebroid structure of the pair $(\mathcal{C}, \mathfrak{g})$ underlying the realization $(J^k\Gamma, \Omega)$. The pre-Cartan algebroid encodes the infinitesimal structure of Γ. In the previous section, we saw that the pre-Cartan algebroid corresponds to Cartan's structure functions $c_i^{jk}, a_i^{\lambda j}$ that arise out of the structure equations (5.3). If we were to imitate Cartan, we would go about "discovering" the pre-Cartan algebroid structure by computing "$d\Omega$" and making sense of the resulting coefficients globally. We will not take that path, but rather directly spell out the pre-Cartan algebroid structure out of the infinitesimal structure of Γ. Reading this section "backwards", one will see that this is indeed the structure that arises out of the structure equations.

The second task is to show that $(J^k\Gamma, \Omega)$ is a realization of $(\mathcal{C}, \mathfrak{g})$. To do so, we first show that the existence of Π as in Definition 5.2.11 of a realization is equivalent to the existence of an integral Cartan-Ehresmann connection on the PDE $J^k\Gamma$, a notion that was introduced in Section 1.4. In a sense, this clarifies the precise role of Π and bridges between Cartan's structure equations and known concepts from the theory of PDEs. As we will see, the second fundamental theorem is then a simple corollary of this alternative characterization of Π.

In carrying out both of these tasks, it is instructive to identify the precise structure and properties that one needs in order to construct the pre-Cartan algebroid and prove the theorem. In doing so, things become both simpler in terms of language and more conceptual. In Section 2.7, we discussed the notion of a Lie-Pfaffian groupoid and saw that the defining PDE of a Lie pseudogroup is an example of a standard Lie-Pfaffian groupoid. As we will see in the course of this section, the constructions and proofs underlying Cartan's second fundamental theorem rely solely on the properties of a Lie pseudogroup that are encoded in the notion of a Lie-Pfaffian groupoid. Therefore, rather than taking a Lie pseudogroup Γ of order k on M as our starting point, let us take a standard Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$ (Definition 2.7.1) as our starting point while keeping the following principal example in mind:

standard Lie-Pfaffian groupoid		**standard Lie-Pfaffian algebroid**
$\mathcal{G} = J^k\Gamma$		$A = A^k(\Gamma)$
$E = A^{k-1}(\Gamma)$	$\Longrightarrow$	$\pi = d\pi : A^k(\Gamma) \to A^{k-1}(\Gamma)$
$\omega =$ the Cartan form on $J^k\Gamma$		$D =$ Spencer operator on $A^k(\Gamma)$.

The pre-Cartan algebroid We begin by constructing a pre-Cartan algebroid out of the data of a standard Lie-Pfaffian algebroid (A, D), the infinitesimal structure induced by the standard Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$. The main objects we have at hand are: a pair of Lie algebroids A and E over M, a short exact sequence of Lie algebroids

$$0 \to \mathfrak{g} \to A \xrightarrow{\pi} E \to 0, \tag{5.25}$$

where $\mathfrak{g}$ is the symbol space of A, and a π-connection

$$D : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(E).$$

The operator D is obtained by linearization of the form ω on $\mathcal{G}$, namely by (2.23). By Lemma 2.5.2, the two objects are also related by

$$t^*(D_X(\alpha)) = \omega([\widehat{X}, \widetilde{\alpha}]), \qquad \forall\, X \in \mathfrak{X}(M),\ \alpha \in \Gamma(A), \tag{5.26}$$

where $\widetilde{\alpha} \in \mathfrak{X}(\mathcal{G})$ is the right invariant vector field induced by α and $\widehat{X} \in \mathfrak{X}(\mathcal{G})$ is a choice of a lift of X that satisfies

$$dt(\widehat{X}) = X \qquad \text{and} \qquad \omega(\widehat{X}) = 0.$$

We also recall that there is a map

$$\mathfrak{g} \to \mathrm{Hom}(TM, E), \qquad T \mapsto (X \mapsto -D_X(T)), \tag{5.27}$$

and that (A, D) being standard means that this map is injective.

From this data, we will construct a pre-Cartan algebroid. We set

$$\mathcal{C} := TM \oplus E. \tag{5.28}$$

The bracket of $\mathcal{C}$ depends on a choice of splitting

$$\xi : E \to A \tag{5.29}$$

of the short exact sequence (5.25). Such a splitting, which we called a Cartan linear connection, induces a linear connection on E defined by

$$\nabla^\xi : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \qquad \nabla^\xi_X(\alpha) := D_X(\xi \circ \alpha). \tag{5.30}$$

Note that the weak curvature (1.57) of ∇^ξ vanishes and hence ξ is in fact an integral Cartan-linear connection. Associated with ∇^ξ, we have the torsion tensor

$$c^\xi \in \Gamma(\mathrm{Hom}(\Lambda^2 E, E)), \qquad c^\xi(\alpha, \beta) := [\alpha, \beta] - \nabla^\xi_{\rho(\alpha)}\beta - \nabla^\xi_{\rho(\beta)}\alpha, \tag{5.31}$$

where ρ is the anchor of E. The bracket of $\mathcal{C}$,

$$[\cdot, \cdot] : \Gamma(\mathcal{C}) \times \Gamma(\mathcal{C}) \to \Gamma(\mathcal{C}), \tag{5.32}$$

is defined by

$$[(X, \alpha), (Y, \beta)] := \big([X, Y],\ c^\xi(\alpha, \beta) + \nabla^\xi_X(\beta) - \nabla^\xi_Y(\alpha)\big). \tag{5.33}$$

For the anchor, we take the projection

$$\rho : \mathcal{C} \to TM, \qquad (X, \alpha) \mapsto X. \tag{5.34}$$

To indicate that the bracket on $\mathcal{C}$ depends on ξ, we write $\mathcal{C}_\xi$. It is straightforward to verify that $\mathcal{C}_\xi$ is a transitive pre-Lie algebroid.

Next, for the vector subbundle $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$, we take the symbol space $\mathfrak{g}$ of A together with the sequence of inclusions

$$\mathfrak{g} \hookrightarrow \mathrm{Hom}(TM, E) \hookrightarrow \mathrm{Hom}(\mathcal{C}, \mathcal{C}), \tag{5.35}$$

where the first inclusion is (5.27) (which is an inclusion rather than just a map because we require of the Pfaffian algebroid to be standard) and the second is given by

$$T \mapsto \hat{T}, \qquad \hat{T}(X, \alpha) = \big(0, T(\rho(\alpha) - X)\big).$$

Indeed, $\hat{T}$ takes values in $\mathrm{Ker}\ \rho$, and hence $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\ \rho)$. Note that if we endow $\mathrm{Hom}(TM, E)$ with the bracket

$$[T, S] := T \circ \rho \circ S - S \circ \rho \circ T,$$

then the second inclusion in (5.35) becomes a Lie algebra map, where $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ is equipped with the commutator bracket.

Proposition 5.3.5. *The pair $(\mathcal{C}_\xi, \mathfrak{g})$ defined above is a pre-Cartan algebroid, and, up to gauge equivalence, it is independent of the choice of splitting ξ.*

Proof. We have already seen that $(\mathcal{C}_\xi, \mathfrak{g})$ is a pre-Cartan algebroid. We are left with showing that, for any two choices of splittings $\xi : E \to A$ and $\xi' : E \to A$, the resulting pre-Cartan algebroids are gauge equivalent. Taking the difference, we get a map $(\xi - \xi') : E \to \mathfrak{g}$, which we can interpret as a gauge equivalence by letting it act trivially on the first component, i.e.

$$(\xi - \xi') : \mathcal{C} = TM \oplus E \to \mathfrak{g}, \qquad (X, \alpha) \mapsto (\xi - \xi')(\alpha).$$

It is now straightforward to verify that gauge transforming the pre-Cartan algebroid $(\mathcal{C}_\xi, \mathfrak{g})$ by $\xi - \xi'$ yields the pre-Cartan algebroid $(\mathcal{C}_{\xi'}, \mathfrak{g})$. □

Remark 5.3.6. The fact that we require the Pfaffian algebroid we begin with to be standard is because the vector bundle $\mathfrak{g}$ of the pre-Cartan algebroid is required to be a vector subbundle of $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$. This construction suggests relaxing the definition of a Cartan algebroid and only requiring for there to be a map $\mathfrak{g} \to \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ rather than an inclusion. ♢

Propositions 5.2.19 and 5.3.5 together imply that if we manage to construct a realization of $(\mathcal{C}_\xi, \mathfrak{g})$, then it will be independent of the choice of splitting ξ. Thus, from now on we fix a splitting ξ and simply write $(\mathcal{C}, \mathfrak{g})$ for the pre-Cartan algebroid, omitting ξ from the notation.

The Realization We turn to the realization. Given the standard Lie-Pfaffian groupoid $(\mathcal{G}, \omega)$, the data for the realization will consist of the target map $t : \mathcal{G} \to M$ and the 1-form $\Omega = (dt, \omega) \in \Omega^1(\mathcal{G}; t^*\mathcal{C})$. The question we must answer is whether $(\mathcal{G}, \Omega)$ is a realization of the pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$, or, explicitly, whether there exists a $\Pi \in \Omega^1(\mathcal{G}; t^*\mathfrak{g})$ that satisfies both the structure equation (5.12) and the coframe condition (5.13). We will show that the existence of such a Π is equivalent to the existence of an integral Cartan-Ehresmann connection on $\mathcal{G}$.

Recall that for any Lie-Pfaffian groupoid, we have a short exact sequence

$$0 \to t^*\mathfrak{g} \to C_\omega \xrightarrow{ds} s^*TM \to 0, \tag{5.36}$$

that a right splitting $H : s^*TM \to C_\omega$ of this sequence is called a Cartan-Ehresmann connection, and that a Cartan-Ehresmann H is said to be integral if its weak curvature

$$c_H = \delta\omega(H(\cdot), H(\cdot))$$

vanishes. Cartan-Ehresmann connections are in particular Ehresmann connections, i.e. right splittings of the short exact sequence

$$0 \to t^*A \cong T^s\mathcal{G} \to T\mathcal{G} \xrightarrow{ds} s^*TM \to 0, \tag{5.37}$$

or, equivalently, left splittings known as connection 1-forms. Explicitly, these are elements of $\Omega^1(\mathcal{G}; t^*A)$ that restrict to the Maurer-Cartan form (2.14) on $T^s\mathcal{G}$. Our choice of splitting ξ induces an isomorphism $A \cong E \oplus \mathfrak{g}$, which induces a (linear) isomorphism

$$\Omega^1(\mathcal{G}; t^*A) \cong \Omega^1(\mathcal{G}; t^*E) \oplus \Omega^1(\mathcal{G}; t^*\mathfrak{g}). \tag{5.38}$$

Cartan-Ehresmann connections correspond precisely to those connection 1-forms whose first component in this decomposition is the Cartan form ω. This defines a map

$$H \mapsto \Pi \tag{5.39}$$

associating with a Cartan-Ehresmann connection H on $\mathcal{G}$ an element $\Pi \in \Omega^1(\mathcal{G}; t^*\mathfrak{g})$, the second component in the decomposition (5.38).

Proposition 5.3.7. *Let $(\mathcal{G}, \omega)$ be a standard Lie-Pfaffian groupoid. The map* (5.39) *defines a 1-1 correspondence*

$$\begin{array}{c} \textit{Cartan-Ehresmann} \\ \textit{connections } H \textit{ on } \mathcal{G} \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \Pi \in \Omega^1(\mathcal{G}; t^*\mathfrak{g}) \\ \textit{satisfying } (5.13), \end{array} \tag{5.40}$$

that restricts to the 1-1 correspondence

$$\begin{array}{c} \textit{integral} \\ \textit{Cartan-Ehresmann} \\ \textit{connections } H \textit{ on } \mathcal{G} \end{array} \quad \longleftrightarrow \quad \begin{array}{c} \Pi \in \Omega^1(\mathcal{G}; t^*\mathfrak{g}) \\ \textit{satisfying } (5.13) \\ \textit{and } (5.12). \end{array} \tag{5.41}$$

In particular, $(\mathcal{G}, \Omega)$ is a realization if and only if $(\mathcal{G}, \omega)$ admits an integral Cartan-Ehresmann connection.

Proof. Let us begin with (5.40). For the forward direction, let H be a Cartan-Ehresmann connection. Indeed,

$$(\Omega, \Pi) = (dt, \omega, \Pi) : T\mathcal{G} \xrightarrow{\simeq} t^*(TM \oplus E \oplus \mathfrak{g}) \tag{5.42}$$

is an isomorphism because dt is surjective onto t^*TM and (ω, Π) restricts to the Maurer-Cartan form on $T^s\mathcal{G}$, and is, hence, surjective onto $t^*(E \oplus \mathfrak{g}) \cong t^*A$. Thus (5.13) is satisfied.

In fact, we can explicitly describe the inverse of (5.42), and this will serve us in the second part of the proof. Let us denote the map at the level of sections that is induced by the inverse of (5.42) by (c.f. the discussion following Definition 5.2.11 of a realization):

$$\mathfrak{X}(M) \to \mathfrak{X}(\mathcal{G}),\ X \mapsto Y_X; \qquad \Gamma(E) \to \mathfrak{X}(\mathcal{G}),\ \alpha \mapsto Y_\alpha; \qquad \Gamma(\mathfrak{g}) \to \mathfrak{X}(\mathcal{G}),\ S \mapsto Y_S.$$

Thus, $Y_X, Y_\alpha, Y_S \in \mathfrak{X}(\mathcal{G})$ are the unique vector fields that satisfy:

$$\begin{array}{lll} dt(Y_X) = t^*X, & \omega(Y_X) = 0, & \Pi(Y_X) = 0, \\ dt(Y_\alpha) = 0, & \omega(Y_\alpha) = t^*\alpha, & \Pi(Y_\alpha) = 0, \\ dt(Y_S) = 0, & \omega(Y_S) = 0, & \Pi(Y_S) = t^*S. \end{array} \tag{5.43}$$

Given a section $\alpha \in \Gamma(A)$, we denote the induced right invariant vector field by $\widetilde{\alpha} \in \mathfrak{X}(\mathcal{G})$. Also recall that there is canonical isomorphism

$$\psi : s^*TM \xrightarrow{\simeq} t^*TM \tag{5.44}$$

of vector bundles over $J^k\Gamma$, which is given by $dt \circ H$ (and independent of the choice of H). We define the following lift:

$$\mathfrak{X}(M) \to \mathfrak{X}(J^k\Gamma), \qquad X \mapsto X^H = H \circ (dt \circ H)^{-1}(t^*X).$$

One now readily verifies that:

$$Y_X = X^H, \qquad Y_\alpha = \widetilde{\xi(\alpha)} - \rho(\alpha)^H, \qquad Y_S = \widetilde{S}. \tag{5.45}$$

For the reverse direction of (5.40), choose $\Pi \in \Omega^1(\mathcal{G}; t^*\mathfrak{g})$ that satisfies (5.13). Thus, we assume that we have an isomorphism (5.42). This induces a Cartan-Ehresmann connection $H : s^*TM \to T\mathcal{G}$ as follows: denote the restriction of the inverse of (5.42) to t^*TM by $H' : t^*TM \to T\mathcal{G}$ and set $H = H' \circ \psi$, where ψ is the isomorphism (5.44). Indeed,

$$ds \circ H = \underset{=\psi^{-1}\circ dt}{\underset{\uparrow}{ds}} \circ H' \circ \psi = \psi^{-1} \circ \underbrace{dt \circ H'}_{=\mathrm{id}} \circ \psi = \psi^{-1} \circ \psi = \mathrm{id}.$$

We immediately see that this construction is inverse to (5.39).

Next, we move on to (5.41). For the forward direction, let H be an integral Cartan-Ehresmann connection. We must show that the induced Π satisfies the structure equation (5.12). For this, it is enough to verify that the expression

$$d\Omega + \frac{1}{2}[\Omega, \Omega] - \Pi \wedge \Omega = d(dt, \omega) + \frac{1}{2}[(dt, \omega), (dt, \omega)] - \Pi \wedge (dt, \omega) \tag{5.46}$$

vanishes when applied to all possible pairs of the type (5.45). In the following computations, we use Lemma 5.2.6 to evaluate the Maurer-Cartan expression $\mathrm{MC}_\Omega = d\Omega + \frac{1}{2}[\Omega,\Omega]$.

1. $$(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(Y_X, Y_{X'})$$
$$= \big(d(dt,\omega) + \frac{1}{2}[(dt,\omega),(dt,\omega)] - \Pi\wedge(dt,\omega)\big)(Y_X, Y_{X'})$$
$$= -\big(dt([X^H, X'^H]), \underbrace{\cancel{\omega([X^H, X'^H])}}_{H \text{ is integral}}\big) + t^*[(X,0),(X',0)]$$
$$= -\big(\underbrace{t^*[X,X']}_{Y_X,\, Y_{X'} \text{ are } t\text{-related to } X,\, X'}, 0\big) + t^*([X,X'],0) = 0$$

2. $$(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(Y_X, Y_\alpha)$$
$$= -\big(\underbrace{\cancel{dt([Y_X, Y_\alpha])}}_{Y_\alpha \text{ is } t\text{-related to } 0}, \omega([Y_X, Y_\alpha])\big) + t^*[(X,0),(0,\alpha)]$$
$$= -\Big(0, \underbrace{\omega([X^H, \widetilde{\xi(\alpha)}])}_{t^*\nabla^\xi_X(\alpha) \text{ by (5.26)}} - \underbrace{\cancel{\omega([X^H, \rho(\alpha)^H])}}_{H \text{ is integral}}\Big) + t^*(0, \nabla^\xi_X(\alpha)) = 0$$

3. $$(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(Y_X, Y_S)$$
$$= -\big(dt([Y_X, Y_S]), \omega([Y_X, Y_S])\big) + \Pi(Y_S)((dt,\omega)(Y_X))$$
$$= -\big(0, \underbrace{\omega([X^H, \widetilde{S}])}_{t^* D_X(S) = -t^* S(X)}\big) + t^*(0, S(-X)) = 0$$

4. $$(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(Y_\alpha, Y_{\alpha'})$$
$$= -\big(dt([Y_\alpha, Y_{\alpha'}]), \omega([Y_\alpha, Y_{\alpha'}])\big) + t^*[(0,\alpha),(0,\alpha')]$$
$$= -\big(0, \underbrace{\omega([\widetilde{\xi(\alpha)}, \widetilde{\xi(\alpha')}])}_{=t^*[\alpha,\alpha']} - \underbrace{\omega([\rho(\alpha)^H, \widetilde{\xi(\alpha')}])}_{=t^*\nabla^\xi_{\rho(\alpha)}(\alpha)} + \underbrace{\omega([\rho(\alpha')^H, \widetilde{\xi(\alpha)}])}_{=t^*\nabla^\xi_{\rho(\alpha')}(\alpha)}\big)$$
$$+ \big(0, \underbrace{\cancel{\omega([\rho(\alpha)^H, \rho(\alpha')^H])}}_{H \text{ is integral}}\big) - t^*(0, c^\xi(\alpha,\alpha')) = 0$$

5. $$(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(Y_\alpha, Y_S)$$
$$= -\big(dt([Y_\alpha, Y_S]), \omega([Y_\alpha, Y_S])\big) + \Pi(Y_S)((dt,\omega)(Y_\alpha))$$
$$= -\big(0, \underbrace{\cancel{\omega([\widetilde{\xi(\alpha)}, \widetilde{S}])}}_{= d\pi([\xi(\alpha), S]) = 0} - \underbrace{\omega([\rho(\alpha)^H, \widetilde{S}])}_{=-t^*S(\rho(\alpha))}\big) + t^*(0, S(\rho(\alpha)) = 0$$

6. $$\begin{aligned}&(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(Y_S, Y_{S'})\\ &= -\big(dt([Y_S, Y_{S'}]), \omega([Y_S, Y_{S'}])\big)\\ &= -\big(0, \underbrace{\cancel{\omega([\widetilde{S}, \widetilde{S'}])}}_{= d\pi([S,S'])=0}\big) = 0\end{aligned}$$

For the reverse direction, let $\Pi \in \Omega^1(\mathcal{G}; t^*\mathfrak{g})$ satisfy (5.12) and (5.13). Thus, Π induces a Cartan-Ehresmann connection H on $\mathcal{G}$. Let $X, X' \in \mathfrak{X}(M)$ and set $X^H := H \circ \psi^{-1}(t^*X)$ and $X'^H := H \circ \psi^{-1}(t^*X')$. Using (5.12),

$$\begin{aligned}0 &= (d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega)(X^H, X'^H)\\ &= -\big(dt([X^H, X'^H]), \omega([X^H, X'^H])\big) + t^*[(X,0),(X',0)]\\ &= -\big(0, \delta\omega(H\circ\psi^{-1}(X), H\circ\psi^{-1}(X'))\big).\end{aligned}$$

We conclude that $c_H = \delta\omega(H(\cdot), H(\cdot)) = 0$ and, hence, that H is integral. $\square$

With the characterization given to us by Proposition 5.3.7, the proof of Proposition 5.3.3 becomes a simple argument to show that the regularity conditions we are imposing suffice to ensure the existence of an integral Cartan-Ehresmann connection.

Proof of Proposition 5.3.3. Recall that an integral Cartan-Ehresmann connection is a section of the projection $\pi : (J^k\Gamma)^{(1)} \to J^k\Gamma$, where $(J^k\Gamma)^{(1)}$ is the 1st prolongation of $J^k\Gamma$. Thus, by Proposition 5.3.7, we must show that such a section exists, and, in turn, it is sufficient to show that $\pi : (J^k\Gamma)^{(1)} \to J^k\Gamma$ is an affine bundle. To this end, we apply Proposition 1.1.6. Indeed, $(J^k\Gamma)^{(1)}$ is, by definition, the intersection of two affine bundles, namely the restriction of $\pi : J^{k+1}M \to J^kM$ to $J^k\Gamma$ and $J^1(J^k\Gamma) \to J^k\Gamma$. All of the fibers of the intersection are non-empty because any $j^k_x\phi$ has a representative $\phi \in \Gamma$, and $j^1(j^k\phi) \in (J^k\Gamma)^{(1)}$ lies in the fiber over $j^k_x\phi$. Furthermore, the intersection of the modeling vector bundles is precisely $(\mathfrak{g}^k)^{(1)}$, which is by assumption of constant rank. Thus, $\pi : (J^k\Gamma)^{(1)} \to J^k\Gamma$ is an affine bundle by Proposition 1.1.6. The final assertion follows from the very definition of $J^k\Gamma$. $\square$

5.4 Cartan's Examples Revisited

In this section, we revisit two of Cartan's examples from Section 5.1 and show that our global construction precisely recovers Cartan's formulas. In each example, we compute the pre-Cartan algebroid and the realization induced by the given Lie pseudogroups. The computations build on the examples from Section 3.3, where we already did the groundwork and computed the jet groupoids, the Cartan form and the Spencer operator associated with the Lie pseudogroups from these two examples.

Example 5.4.1. We revisit Cartan's Example 5.1.11, the 1st order transitive Lie pseudogroup Γ on $M = \mathbb{R}^2 \backslash \{y = 0\}$ generated by the local diffeomorphisms

$$\phi : (x, y) \mapsto (\phi_x(x, y), \phi_y(x, y)) = (f(x), \frac{y}{f'(x)}), \qquad f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}). \tag{5.47}$$

Let us recall from Example 3.4.4 the pieces of the structure that we need in order to construct the realization associated with Γ. Recall that

$$J^1\Gamma = \{(X, Y, x, y, u) \mid y \neq 0 \text{ and } Y \neq 0\},$$

with $t : J^1\Gamma \to M,\ (X, Y, x, y) \mapsto (X, Y)$. At the infinitesimal level we have $A^1(\Gamma)$, for which we choose the frame

$$e_X = \frac{\partial}{\partial X}\Big|_M, \quad e_Y = \frac{\partial}{\partial Y}\Big|_M, \quad e_u = \frac{\partial}{\partial u}\Big|_M,$$

and $A^0(\Gamma) = A^0(M)$, for which we choose the frame

$$\partial_X = \frac{\partial}{\partial X}\Big|_M, \quad \partial_Y = \frac{\partial}{\partial Y}\Big|_M.$$

The projection between the two is given by

$$d\pi : A^1(\Gamma) \to A^0(\Gamma), \qquad e_X \mapsto \partial_X,\ e_y \mapsto \partial_Y,\ e_u \mapsto 0. \tag{5.48}$$

The bracket and anchor of $A^0(\Gamma)$ are

$$[\partial_X, \partial_Y] = 0 \qquad \text{and} \qquad \rho : A^0(\Gamma) \to TM, \quad \partial_X \mapsto \frac{\partial}{\partial x},\ \partial_Y \mapsto \frac{\partial}{\partial y}.$$

Finally, the Cartan form $\omega \in \Omega^1(J^1\Gamma; t^*A^0(\Gamma))$ is

$$\omega = (dX - \frac{y}{Y}dx)\, t^*\partial_X + (dY - u dx - \frac{Y}{y}dy)\, t^*\partial_Y,$$

and the Spencer operator $D : \Gamma(A^1(\Gamma)) \to \Omega^1(M; A^0(\Gamma))$ is

$$D : e_X \mapsto 0, \quad e_Y \mapsto \frac{1}{y}(dx \otimes \partial_X - dy \otimes \partial_Y), \quad e_u \mapsto -dx \otimes \partial_Y.$$

We start by describing the pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over M associated with Γ. The pre-Lie algebroid is

$$\mathcal{C} = TM \oplus A^0(\Gamma).$$

One natural frame for this vector bundle would be $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \partial_X, \partial_Y$. We will, however, choose the frame

$$e^1 = -\partial_X, \;\; e^2 = -\partial_Y, \;\; e^3 = \partial_X + \frac{\partial}{\partial x}, \;\; e^4 = \partial_Y + \frac{\partial}{\partial y},$$

which, as we will see, conforms to the choices Cartan makes. Furthermore, for

$$\mathfrak{g} = \mathrm{Ker}\ (d\pi : A^1(\Gamma) \to A^0(\Gamma)),$$

we choose the frame $t = e_u$. The bracket of $\mathcal{C}$ depends on a choice of a splitting of (5.48), for which we choose

$$\xi : A^0(\Gamma) \to A^1(\Gamma), \qquad \partial_X \mapsto e_X,\ \partial_Y \mapsto e_Y.$$

The induced connection ∇^ξ on $A^0(\Gamma)$ defined (5.30) is given by

$$\nabla^\xi_{\partial/\partial x}(\partial_X) = 0, \quad \nabla^\xi_{\partial/\partial y}(\partial_X) = 0, \quad \nabla^\xi_{\partial/\partial x}(\partial_Y) = \frac{1}{y}\partial_X, \quad \nabla^\xi_{\partial/\partial y}(\partial_Y) = -\frac{1}{y}\partial_Y,$$

and the torsion c^ξ of ∇^ξ defined in (5.31) is given by

$$c^\xi(\partial_X, \partial_Y) = -\frac{1}{y}\partial_X.$$

The bracket defined in 5.33 is now easily computed and, in terms of Cartan's frame, is given by

$$\begin{aligned} &[e^1, e^2] = \frac{1}{y}e^1, && [e^1, e^3] = 0, && [e^1, e^4] = -\frac{1}{y}e^1, \\ &[e^2, e^3] = 0, && [e^2, e^4] = \frac{1}{y}e^2, && [e^3, e^4] = 0. \end{aligned}$$

The anchor (5.34) is given by

$$\rho : \mathcal{C} \to TM, \qquad e^1 \mapsto 0,\ \ e^2 \mapsto 0,\ \ e^3 \to \frac{\partial}{\partial x},\ \ e^4 \mapsto \frac{\partial}{\partial y},$$

and the action of $\mathfrak{g}$ on $\mathcal{C}$ defined in (5.35) is given by

$$t(e^1) = e^2, \quad t(e^2) = 0, \quad t(e^3) = 0, \quad t(e^4) = 0.$$

Thus, writing $[e^j, e^k] = \sum_{i=1}^4 c_i^{jk} e^i$ and $t(e^j) = \sum_{i=1}^4 a_i^j e^i$, the non-zero structure functions are

$$c_1^{12} = \frac{1}{y}, \ \ c_1^{14} = -\frac{1}{y}, \ \ c_2^{23} = \frac{1}{y}, \ \ a_2^1 = 1.$$

Next, we describe the realization $(J^1\Gamma, \Omega)$ of $(\mathcal{C}, \mathfrak{g})$. The realization consists of the target map $t : J^1\Gamma \to M$ and the extended Cartan form $\Omega = (dt, \omega)$. In terms of the frame of $\mathcal{C}$, Ω decomposes as

$$\Omega = \omega_1\, t^*e_1 + \omega_2\, t^*e_2 + \omega_3\, t^*e_3 + \omega_4\, t^*e_4,$$

with

$$\omega_1 = \frac{y}{Y}dx, \quad \omega_2 = udx + \frac{Y}{y}dy, \quad \omega_3 = dX, \quad \omega_4 = dY.$$

These are precisely the forms from Example 5.1.12, and restricting to the orbit $X = 0, Y = 1$, these are precisely Cartan's forms from Example 5.1.11.

Let us check explicitly that this indeed defines a realization. We show that there is a 1-form $\Pi \in \Omega^1(J^1M; t^*\mathfrak{g})$ as in Definition 5.2.11. By Proposition 5.3.7, such a 1-form is the same thing as an integral Cartan-Ehresmann connection on $J^1\Gamma$, i.e. a smooth choice of integral elements of $J^1\Gamma$. We will use the fact that we have "enough" solutions of $J^1\Gamma$, namely the elements of Γ, to construct such a connection. Consider an element $j^1_{(x,y)}\phi$ with coordinates (X, Y, x, y, u) and represented by $\phi \in \Gamma$. The induced holonomic bisection is

$$j^1\phi : (x, y) \mapsto (\phi_x, \phi_y, x, y, \frac{\partial \phi_y}{\partial x}).$$

Its differential at the point (x, y) is the integral element at (X, Y, x, y, u) given by

$$\begin{aligned}(d(j^1\phi))_x : \frac{\partial}{\partial x} &\mapsto \frac{\partial}{\partial x} + \frac{y}{Y}\frac{\partial}{\partial X} + u\frac{\partial}{\partial Y} + \frac{\partial^2 \phi_y}{\partial x^2}(x, y)\frac{\partial}{\partial u},\\ \frac{\partial}{\partial y} &\mapsto \frac{\partial}{\partial y} + \frac{Y}{y}\frac{\partial}{\partial Y} + \frac{u}{y}\frac{\partial}{\partial u}.\end{aligned}$$

From (5.47) we see that if ϕ is induced by a function $f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R})$, then

$$\frac{\partial^2 \phi_y}{\partial x^2}(x, y) = -y\frac{f'''(x)f'(x) - 2(f''(x))^2}{(f'(x))^3},$$

and from this expression we see that any possible value of $\frac{\partial^2 \phi_y}{\partial x^2}(x, y)$ can be achieved by choosing a suitable f. In particular, we can choose the value 0 at all points of $J^1\Gamma$, thus obtaining an integral Cartan-Ehresmann connection $H : s^*TM \to TJ^1\Gamma$ given by

$$H : \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial x} + \frac{y}{Y}\frac{\partial}{\partial X} + u\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial y} \mapsto \frac{\partial}{\partial y} + \frac{Y}{y}\frac{\partial}{\partial Y} + \frac{u}{y}\frac{\partial}{\partial u}.$$

Alternatively, we could have chosen any function of $J^1\Gamma$ instead of the constant function 0. The connection H induces a 1-form Π, as needed, by the map (5.39). Unraveling the definition of this map, we arrive at

$$\Pi = \pi \otimes t \quad \text{with} \quad \pi = \frac{1}{y}(u^2 dx - u dy + Y du).$$

With this choice of Π, we see that $(J^1\Gamma, \Omega)$ defines a realization, i.e. it satisfies the two conditions of Definition 5.2.11,

$$(\Omega, \Pi) : TJ^1\Gamma \xrightarrow{\simeq} t^*(\mathcal{C} \oplus \mathfrak{g}) \qquad \text{and} \qquad d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega.$$

In terms of the local data (see Example 5.2.14), this translates to the property that the 1-forms $\omega_1, \ldots, \omega_4$, π form a coframe of $J^1\Gamma$ (as one directly sees) and to the structure equations

$$\begin{aligned}
&d\omega_1 + \frac{1}{Y}\,\omega_1 \wedge (\omega_2 - \omega_4) = 0,\\
&d\omega_2 + \frac{1}{Y}\,\omega_2 \wedge \omega_4 = \pi \wedge \omega_1,\\
&d\omega_3 = 0,\\
&d\omega_4 = 0.
\end{aligned}$$

Note that the factor $1/2$ in (5.15) disappears since each term appears twice in the sum. By differentiating ω_i, one can directly check that the equations are satisfied. Finally, we note that the resulting structure equations are precisely those of Example 5.1.12 that were derived by Cartan's method, and if we restrict to the transversal $X = 0, Y = 1$ (see Section 4.3), then we recover Cartan's structure equations of Example 5.1.11. $\diamondsuit$

Example 5.4.2. We turn to our second example, Cartan's Example 5.1.9, the 3rd order Lie pseudogroups Γ on $M = \mathbb{R}$ generated by

$$\phi : x \mapsto \frac{ax+b}{cx+d}, \qquad a, b, c, d \in \mathbb{R} \quad \text{with} \quad ad - bc \neq 0.$$

Recall from Example 3.4.5 that

$$J^3\Gamma = \{\ (x, X, u, v) \mid X, x, u, v \in \mathbb{R},\ u \neq 0\ \},$$

with $t : (x, X, u, v) \mapsto X$. In this example, $A^3(\Gamma) \cong A^2(\Gamma) = A^2(M)$. For both vector bundles we choose the frame

$$\partial_X = \frac{\partial}{\partial X}\Big|_M, \quad \partial_u = \frac{\partial}{\partial u}\Big|_M, \quad \partial_v = \frac{\partial}{\partial v}\Big|_M,$$

and then the projection $d\pi : A^3(\Gamma) \to A^2(\Gamma)$ is given by the identity. We will need the bracket of $A^2(\Gamma)$,

$$[\partial_X, \partial_u] = 0, \quad [\partial_X, \partial_v] = 0, \quad [\partial_u, \partial_v] = \partial_v,$$

its anchor

$$\rho : A^2(M) \to TM, \qquad \partial_X \mapsto \frac{\partial}{\partial x}, \ \ \partial_u \mapsto 0, \ \ \partial_v \mapsto 0,$$

the Cartan form $\omega \in \Omega^1(J^3\Gamma; t^*A^2(\Gamma))$,

$$\omega = (dX - u dx)\, t^*\partial_X + \frac{1}{u}(du - v dx)\, t^*\partial_u + \frac{1}{u^2}(dv - \frac{v}{u}du - \frac{1}{2}\frac{v^2}{u}dx)\, t^*\partial_v,$$

and the Spencer operator $D : \Gamma(A^3(\Gamma)) \to \Omega^1(M; A^2(\Gamma))$,

$$D : \partial_X \mapsto 0, \quad \partial_u \mapsto -dx \otimes \partial_X, \quad \partial_v \mapsto -dx \otimes \partial_u.$$

The pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ associated with Γ consists of

$$\mathcal{C} = TM \oplus A^2(\Gamma),$$

for which we take the frame (as before, we make these choices to conform with Cartan's choices)

$$e^1 = -\partial_X, \quad e^2 = \partial_u, \quad e^3 = -\partial_v, \quad e^4 = \frac{\partial}{\partial x} + \partial_X,$$

and, in this case, $\mathfrak{g} = 0$. The bracket on $\mathcal{C}$ is canonical, since there is no choice in splitting the projection from $A^3(\Gamma)$ to $A^2(\Gamma)$. Thus, the connection (5.30) coincides with the Spencer operator D and the associated torsion (5.31) is determined by

$$c(\partial_X, \partial_u) = \partial_X, \quad c(\partial_X, \partial_v) = \partial_u, \quad c(\partial_u, \partial_v) = \partial_v.$$

From this, we compute the bracket on $\mathcal{C}$,

$$\begin{aligned} &[e^1, e^2] = e^1, \qquad &&[e^1, e^3] = e^2, \qquad &&[e^1, e^4] = 0, \\ &[e^2, e^3] = e^3, \qquad &&[e^2, e^4] = 0, \qquad &&[e^3, e^4] = 0. \end{aligned}$$

The anchor on $\mathcal{C}$ is

$$\rho : \mathcal{C} \to TM, \qquad e^1 \mapsto 0, \;\; e^2 \mapsto 0, \;\; e^3 \mapsto 0, \;\; e^4 \mapsto \frac{\partial}{\partial x}.$$

The induced realization $(J^3\Gamma, t)$ of $(\mathcal{C}, 0)$ consists of the target map $t : J^3\Gamma \to M$ and the extended Cartan form $\Omega = (dt, \omega)$, which, in terms of our choice of a frame, decomposes as

$$\Omega = \omega_1\, t^*e^1 + \omega_2\, t^*e^2 + \omega_3\, t^*e^3 + \omega_4\, t^*e^4,$$

with

$$\omega_1 = u dx, \quad \omega_2 = \frac{1}{u}(du - v dx), \quad \omega_3 = -\frac{1}{u^2}(dv - \frac{v}{u} du - \frac{1}{2}\frac{v^2}{u} dx), \quad \omega_4 = dX.$$

In this case, $\Pi = 0$, and

$$\Omega : J^3\Gamma \xrightarrow{\simeq} t^*\mathcal{C} \qquad \text{and} \qquad d\Omega + \frac{1}{2}[\Omega, \Omega] = 0,$$

or, in terms of components, $\omega_1, \omega_2, \omega_3, \omega_4$ is a coframe of $J^3\Gamma$ and

$$\begin{aligned} &d\omega_1 + \omega_1 \wedge \omega_2 = 0, \\ &d\omega_2 + \omega_1 \wedge \omega_3 = 0, \\ &d\omega_3 + \omega_2 \wedge \omega_3 = 0, \\ &d\omega_4 = 0. \end{aligned}$$

Restricting to the transversal $X = 0$, we have that $\omega_4 = 0$ and we recover Cartan's forms and structure equations from Example 5.1.9. $\diamondsuit$

Chapter 6

The Third Fundamental Theorem

In the third fundamental theorem, Cartan clarifies the precise infinitesimal structure that underlies the structure equations of a Lie pseudogroup, or, more precisely, the notion of a normal form data. In Section 5.1, we encoded the structure functions – the coefficients of the structure equations – in the notion of a pre-Cartan data, which plays the role of the infinitesimal structure associated with a normal form data. Cartan now poses the following question known as the *realization problem*: given an arbitrary set of structure functions, i.e. a pre-Cartan data, does it arise as the set of coefficients of some set of structure equations, i.e. as the pre-Cartan data associated with a normal form data? As a first step towards answering this question, Cartan identifies a set of necessary conditions that a pre-Cartan data must satisfy for this to be the case. These conditions, phrased in Cartan's language, will be encoded in the notion of a *Cartan data*. Then in the third fundamental theorem, Cartan proves that – locally, in the analytic setting and under an assumption known as involutivity – these are also sufficient. Cartan's third fundamental theorem is the analogue of Lie's third fundamental theorem in the theory of Lie groups.

In this chapter, we will incorporate the notion of a Cartan data into the modern formulation that was presented in the previous chapter. We will introduce the notion of a *Cartan algebroid*, the global object which, in local coordinates, recovers the notion of a Cartan data. Following Cartan, we will prove that if a pre-Cartan algebroid admits a realization, then it must be a Cartan algebroid. We will then formulate the realization problem in these global terms. Its solution, when one removes the word "local" and/or replaces the word "analytic" with "smooth", is an open problem. We will conclude the chapter by discussing several examples and constructions of Cartan algebroids, the most important ones being those coming from Lie pseudogroups via the second fundamental theorem.

6.1 Cartan's Formulation

In the first and second fundamental theorems, Cartan proves that, up to equivalence, Lie pseudogroups are encoded in the notion of a normal form data. Motivated by Lie's infinitesimal approach to the study of Lie groups, namely the idea of studying a Lie group by means of its associated Lie algebra, Cartan seeks to identify the precise infinitesimal structure that underlies the notion of a normal form data. To this end, Cartan poses the following problem known as the **realization problem** ([7], p. 1347):

Problem 6.1.1. *(Cartan's realization problem) Given a pre-Carta data* $(c_i^{jk}, a_i^{\lambda j})$ *on* $\mathbb{R}^n$ *(with* $1 \leq i, j, k \leq r$ *and* $1 \leq \lambda \leq p$*, see Definition 5.1.5), find a normal form data that has* $(c_i^{jk}, a_i^{\lambda j})$ *as its associated pre-Cartan data. Explicitly, let* $N := r + p$

and let $(x_1, ..., x_N)$ *be coordinates on* $\mathbb{R}^N$*, find a set of linearly independent 1-forms* $\omega_1, ...\omega_r, \pi_1, ..., \pi_p$ *on* $\mathbb{R}^N$ *such that*

$$\omega_1 = dx_1, ..., \omega_n = dx_n \tag{6.1}$$

and such that

$$d\omega_i + \frac{1}{2} c_i^{jk} \ \omega_j \wedge \omega_k = a_i^{\lambda j} \ \pi_\lambda \wedge \omega_j. \tag{6.2}$$

As a first step in solving this problem, Cartan derives a set of necessary conditions for the existence of solutions ([5], pp. 186-191):

Theorem 6.1.2. *(the third fundamental theorem - necessary conditions) Let* $(c_i^{jk}, a_i^{\lambda j})$ *be a pre-Cartan data on* $\mathbb{R}^n$ *(with* $1 \le i, j, k \le r,\ 1 \le \lambda \le p$*). If the realization problem admits a solution, then there exist smooth functions* $\nu_\lambda^{jk}, \xi_\lambda^{\mu j}, \epsilon_\lambda^{\eta\mu}$ *(with* $1 \le j, k \le r$ *and* $1 \le \lambda, \eta, \mu \le p$*) on* $\mathbb{R}^n$*, with* $\nu_\lambda^{jk} = -\nu_\lambda^{kj}$ *and* $\epsilon_\lambda^{\eta\mu} = -\epsilon_\lambda^{\mu\eta}$*, such that*

$$a_i^{\eta m} a_m^{\mu j} - a_i^{\mu m} a_m^{\eta j} = a_i^{\lambda j} \epsilon_\lambda^{\eta\mu}, \tag{C1}$$

$$c_i^{mj} c_m^{kl} + c_i^{mk} c_m^{lj} + c_i^{ml} c_m^{jk} + \Big(\frac{\partial c_i^{kl}}{\partial x_j} + \frac{\partial c_i^{lj}}{\partial x_k} + \frac{\partial c_i^{jk}}{\partial x_l} \Big) = a_i^{\lambda l} \nu_\lambda^{jk} + a_i^{\lambda k} \nu_\lambda^{lj} + a_i^{\lambda j} \nu_\lambda^{kl} \tag{C2}$$

$$a_m^{\lambda j} c_i^{mk} - a_m^{\lambda k} c_i^{mj} + a_i^{\lambda m} c_m^{jk} + \Big(\frac{\partial a_i^{\lambda k}}{\partial x_j} - \frac{\partial a_i^{\lambda j}}{\partial x_k} \Big) = a_i^{\mu k} \xi_\mu^{\lambda j} - a_i^{\mu j} \xi_\mu^{\lambda k}, \tag{C3}$$

where a partial derivative of a function with respect to x_j *with* $j > n$ *should be regarded as zero.*

Proof. Let ω_i, π_λ be a solution to the realization problem, i.e. a coframe that satisfies the structure equations (6.2). Decompose the differential of π_λ with respect to the coframe,

$$d\pi_\lambda = \frac{1}{2} \nu_\lambda^{jk} \ \omega_j \wedge \omega_k + \xi_\lambda^{\mu j} \ \pi_\mu \wedge \omega_j + \frac{1}{2} \epsilon_\lambda^{\eta\mu} \ \pi^\eta \wedge \pi_\mu.$$

Next, by differentiating the structure equations (6.2), replacing any appearance of $d\pi_\lambda$ with the latter expression and collecting like terms, one obtains the three equations in the statement as coefficients that are required to vanish. To obtain functions on $\mathbb{R}^n$ rather than on $\mathbb{R}^N$, simply set $x_{n+1} = ... = x_N = 0$ (and note that the formulas remain unchanged). □

Thus, conditions (C1)-(C3) appear as differential consequences of the structure equations. These necessary conditions for having a solution to the realization problem are encoded in the following structure:

Definition 6.1.3. *A pre-Cartan data* $(c_i^{jk}, a_i^{\lambda j})$ *on* $\mathbb{R}^n$ *is called a* ***Cartan data*** *if conditions (C1)-(C3) are satisfied for some set of smooth functions* $\nu_\lambda^{jk}, \xi_\lambda^{\mu j}, \epsilon_\lambda^{\eta\mu}$ *on* $\mathbb{R}^n$.

Using the theory of Pfaffian systems (see introduction and Section 5.1), the theory that has evolved into the theory of exterior differential systems and the Cartan-Kähler theorem, Cartan provides the following solution to the realization problem (see pp. 191-197 in [5] and pp. 1347-1351 in [7]):

Theorem 6.1.4. *(the third fundamental theorem) Let* $(c_i^{jk}, a_i^{\lambda j})$ *be an analytic Cartan data on* $\mathbb{R}^n$ *(i.e. the functions are analytic). If the tableau spanned by the matrices* $A^\lambda = (a_i^{\lambda j})$ *at each point is involutive, then there exists a local solution to the realization problem , i.e. for every* $p \in \mathbb{R}^N$ *there exists an open neighborhood* $U \subset \mathbb{R}^N$ *of* p *and there exists linearly independent* 1*-forms* $\omega_1, ..., \omega_r, \pi_1, ..., \pi_p$ *on* U *that satisfy* (6.1) *and* (6.2).

As we mentioned before (see introduction and Section 5.1), the analytic tool that Cartan uses to solve the problem is only applicable in the analytic setting and it only provides local solutions. It is interesting to note that the obstruction of involutivity that appears in this problem is precisely the same obstruction that appears in Theorem 5.1.6, although the two problems are seemingly very different in nature.

Remark 6.1.5. In addition to Cartan's proof, a nicely written proof of the theorem which also using the Cartan-Kähler theorem is given in [34]. $\diamondsuit$

Example 6.1.6. The most familiar example of a Cartan data is the case $n = 0$ (i.e. $\mathbb{R}^n$ is a point) and $p = 0$. In this case, a Cartan data defines a Lie algebra structure on $\mathbb{R}^r$ and the third fundamental theorem reduces to the local version of Lie's third fundamental theorem in the theory of Lie groups. Indeed, writing $\mathfrak{g} = \mathbb{R}^r$ and denoting the canonical basis of $\mathfrak{g}$ by $e^1, ..., e^r$, define a bracket on $\mathfrak{g}$ by setting $[e^j, e^k] := c_i^{jk} e^i$. The only non-vacuous condition that one has in this case is (C2), which is equivalent to requiring that $[\cdot, \cdot]$ satisfy the Jacobi identity. The third fundamental theorem then states that there exists an open neighborhood $U \subset \mathbb{R}^r$ of 0 and a coframe $\omega_1, ..., \omega_r$ on U that satisfies the structure equations $d\omega_i + \frac{1}{2} c_i^{jk} \omega_j \wedge \omega_k = 0$. Such a coframe can be interpreted as the components of a 1-form $\Omega \in \Omega^1(U; \mathfrak{g})$, which is pointwise a linear isomorphism, and the structure equations become the usual Maurer-Cartan structure equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$. Finally, one remarks that having a form Ω with these properties implies that U has the induced structure of a local Lie group whose Lie algebra is precisely $\mathfrak{g}$ (see Chapter 7 for more details). $\diamondsuit$

6.2 Cartan Algebroids

In the previous chapter, we introduced the notion of a pre-Cartan algebroid (Definition 5.2.7) and its realizations (Definition 5.2.11) as the global versions of a pre-Cartan data and a normal form data. While the notion of a pre-Cartan algebroid already allows us to talk about structure equations, the correct infinitesimal structure that underlies the structure equations is more subtle. This is already clear in Cartan's local picture, where the coefficients c_{ij}^k and $a_{i\lambda}^k$ suffice to write down the structure equations, but the fact that they come from structure equations implies that they should form a Cartan data (Definition 6.1.3). In the global description, these extra conditions are encoded in the notion of a Cartan algebroid.

Before presenting the definition, let us recall the following notion, which is standard for Lie algebroids (see Section 2.1) but which also makes sense for pre-Lie algebroids. Let $\mathcal{C}$ be a pre-Lie algebroid over N. A **$\mathcal{C}$-connection** on a vector bundle $\mathfrak{g}$ over N is a bilinear operation

$$\nabla : \Gamma(\mathcal{C}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g}), \qquad (\alpha, T) \to \nabla_\alpha(T)$$

satisfying

$$\nabla_{f\alpha}(T) = f\nabla_\alpha(T), \qquad \nabla_\alpha(fT) = f\nabla_\alpha(T) + L_{\rho(\alpha)}(f)T,$$

for all $\alpha \in \Gamma(\mathcal{C}),\ T \in \Gamma(\mathfrak{g})$ and $f \in C^\infty(N)$. Note also that $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$ is a bundle of Lie algebras with the fiberwise commutator bracket $[T, S] = T \circ S - S \circ T$.

Definition 6.2.1. *A* ***Cartan algebroid*** *is a pre-Cartan algebroid* $(\mathcal{C}, \mathfrak{g})$ *over* N *such that:*

1. $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ *is closed under the commutator bracket.*
2. *There exists a vector bundle map* $t : \Lambda^2\mathcal{C} \to \mathfrak{g},\ (\alpha, \beta) \mapsto t_{\alpha,\beta}$*, such that*
$$[[\alpha, \beta], \gamma] + [[\beta, \gamma], \alpha] + [[\gamma, \alpha], \beta] = t_{\alpha,\beta}(\gamma) + t_{\beta,\gamma}(\alpha) + t_{\gamma,\alpha}(\beta) \qquad (6.3)$$
for all $\alpha, \beta, \gamma \in \Gamma(\mathcal{C})$.
3. *There exists a* $\mathcal{C}$*-connection* ∇ *on* $\mathfrak{g}$ *such that*
$$T([\alpha, \beta]) - [T(\alpha), \beta] - [\alpha, T(\beta)] = \nabla_\beta(T)(\alpha) - \nabla_\alpha(T)(\beta), \qquad (6.4)$$
for all $\alpha, \beta \in \Gamma(\mathcal{C}),\ T \in \Gamma(\mathfrak{g})$.

Intuitively, t controls the failure of the Jacobi identity and ∇ controls the failure of the action of $\mathfrak{g}$ on $\mathcal{C}$ to be compatible with the bracket of $\mathcal{C}$. Condition 1 in the definition can be rephrased as saying that $\mathfrak{g}$ is a bundle of Lie algebras or a totally intransitive Lie algebroid, if equipped with the zero anchor. It is interesting to note that if we were to relax the definition of a pre-Cartan algebroid and only require that $\mathfrak{g}$ be a vector subbundle of $\mathrm{Hom}(\mathcal{C}, \mathcal{C})$, and not necessarily of $\mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\ \rho)$, then the fact that $\mathfrak{g}$ actually lies in $\mathrm{Hom}(\mathcal{C}, \mathrm{Ker}\ \rho)$ would follow from condition 3 in the above definition. Indeed, replacing β by $f\beta$ in this condition, where $f \in C^\infty(N)$, one sees that

$$L_{\rho(T(\alpha))}(f)\beta = 0,$$

which implies that $\rho \circ T$ must vanish for all $T \in \Gamma(\mathfrak{g})$.

Example 6.2.2. Locally, a Cartan algebroid is the same thing as a Cartan data. Continuing from examples 5.2.8 and 5.2.14:

- Condition (1) is equivalent to the existence of functions $\epsilon^{\lambda}_{\eta\mu}$ on N such that

$$[t_\eta, t_\mu] = \epsilon^{\lambda}_{\eta\mu} t_\lambda.$$

  This is precisely (C1).

- The bundle map $t : \Lambda^2\mathcal{C} \to \mathfrak{g}$ can be written as

$$t(e_i, e_j) = \nu^{\lambda}_{ij} t_\lambda,$$

  and a straightforward computation shows that condition (2) is equivalent to (C2).

- A $\mathcal{C}$-connection on $\mathfrak{g}$ is determined by

$$\nabla_{e_i}(t_\mu) = \xi^{\lambda}_{\mu i} t_\lambda,$$

  and we readily verify that condition (3) is equivalent to (C3). $\diamondsuit$

Let us discuss some of the consequences of the definition of a Cartan algebroid. A first consequence is that Cartan algebroids are preserved under gauge equivalences in the following sense:

Lemma 6.2.3. *Let $(\mathcal{C}, \mathfrak{g})$ be a Cartan algebroid over N and let $\xi \in \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}))$ be a gauge equivalence. Then $(\mathcal{C}^\xi, \mathfrak{g})$ is again a Cartan algebroid over N. Furthermore, if a pre-Cartan algebroid is gauge equivalent to a Cartan algebroid, then it is itself a Cartan algebroid.*

Proof. We know that $(\mathcal{C}^\xi, \mathfrak{g})$ is a pre-Cartan algebroid and we must verify that it is a Cartan algebroid by checking the three conditions of Definition 6.2.1. The first condition is immediately satisfied. For the other two, choose t and ∇ for the Cartan algebroid $(\mathcal{C}, \mathfrak{g})$. A straightforward computation shows that the remaining conditions are satisfied with $t^\xi : \Lambda^2\mathcal{C}^\xi \to \mathfrak{g}$ and $\nabla^\xi : \Gamma(\mathcal{C}^\xi) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ defined by

$$\begin{aligned} t^\xi_{\alpha,\beta} &:= t_{\alpha,\beta} - \nabla_\alpha(\xi(\beta)) + \nabla_\beta(\xi(\alpha)) - [\xi(\alpha), \xi(\beta)] + \xi([\alpha, \beta]) \\ &\qquad - \xi(\xi(\beta)(\alpha)) + \xi(\xi(\alpha)(\beta)), \\ \nabla^\xi_\alpha(S) &:= \nabla_\alpha(S) + [\xi(\alpha), S] + \xi(S(\alpha)). \end{aligned}$$

The second assertion follows from the first assertion together with the fact that gauge equivalence defines an equivalence relation on the set of pre-Cartan algebroids. $\square$

Next, as for realizations, one may wonder as to how much freedom there is in the choice of t and ∇ in the definition of a Cartan algebroid. This freedom can be expressed in terms of the Spencer complexes (1.47) associated with the tableau bundle $\mathfrak{g}$. Recall that the the coboundary operator

$$\delta : \mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}) \to \mathrm{Hom}(\Lambda^3\mathcal{C}, \mathcal{C}) \tag{6.5}$$

is defined by

$$\delta(\xi)(\alpha,\beta,\gamma) = \xi(\alpha,\beta)(\gamma) + \xi(\beta,\gamma)(\alpha) + \xi(\gamma,\alpha)(\beta),$$

the cocycles at $\mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g})$ are denoted by

$$Z^{0,2}(\mathfrak{g}) = \{\ \xi \in \mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g}) \mid \delta(\xi) = 0\ \} \subset \mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g}), \tag{6.6}$$

and the 1st prolongation $\mathfrak{g}^{(1)} \subset \mathrm{Hom}(\mathcal{C},\mathfrak{g})$ of $\mathfrak{g}$ is the kernel of the coboundary operator

$$\delta : \mathrm{Hom}(\mathcal{C},\mathfrak{g}) \to \mathrm{Hom}(\Lambda^2\mathcal{C},\mathcal{C}), \qquad \delta(\xi)(\alpha,\beta) = \xi(\beta)(\alpha) - \xi(\alpha)(\beta). \tag{6.7}$$

Proposition 6.2.4. *Let* $(\mathcal{C},\mathfrak{g})$ *be a Cartan algebroid over* N.

1. *The subspace of* $\Gamma(\mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g}))$ *consisting of elements* t *satisfying* (6.3) *is an affine space modeled on* $\Gamma(Z^{0,2}(\mathfrak{g}))$.
2. *For each* $S \in \Gamma(\mathfrak{g})$, *the subspace of* $\Gamma(\mathrm{Hom}(\mathcal{C},\mathfrak{g}))$ *consisting of elements* $\nabla(S)$ *satisfying* (6.4) *is an affine space modeled on* $\Gamma(\mathfrak{g}^{(1)})$.

Proof. By (6.3), the difference of two choices t and t' satisfies

$$\delta(t' - t) = 0,$$

where δ is the coboundary operator (6.5). Conversely, given a choice of a t and $\xi \in \Gamma(Z^{0,2}(\mathfrak{g}))$, clearly (6.3) is satisfied when replacing t by $t + \xi$. This proves items 1. Similarly, in item 2, given two choices ∇ and ∇' and $S \in \Gamma(\mathfrak{g})$,

$$\delta(\nabla(S)) = 0$$

by (6.4), where δ is the coboundary operator (6.7). Conversely, given a choice of ∇ and $\xi \in \Gamma(\mathfrak{g}^{(1)})$, (6.4) is satisfied when replacing $\nabla(S)$ by $\nabla(S) + \xi$. □

6.3 The Need of Cartan Algebroids for the Existence of Realizations

In the local picture, we saw that if a pre-Cartan data is induced by a normal form data, then it must be a Cartan data. This was, as we recall, the way the conditions one imposes on a Cartan data were derived in the first place. As one expects, this piece of the puzzle extends to the global picture:

Theorem 6.3.1. *(the third fundamental theorem - necessary conditions) If a pre-Cartan algebroid admits a realization, then it is a Cartan algebroid.*

Proof. The proof is essentially the global version of the proof of Theorem 6.1.2. Choose $\Pi \in \Omega^1(P; I^*\mathcal{C})$ as in definition 5.2.11. The 2-form $d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega \in \Omega^2(P; I^*\mathcal{C})$ vanishes, as well as its "differential consequence" $d_\nabla(d\Omega + \frac{1}{2}[\Omega,\Omega] - \Pi\wedge\Omega) \in \Omega^3(P; I^*\mathcal{C})$, for any choice of ∇. Applying $d_\nabla(d\Omega + \frac{1}{2}[\Omega, \Omega] - \Pi \wedge \Omega) = 0$ on a triple of vector fields of type $X_\alpha, X_{\alpha'}, X_{\alpha''} \in \mathfrak{X}(P)$ and using Lemmas 5.2.6 and 5.2.16 implies the identity

$$\begin{aligned} &I^*([[\alpha, \alpha'], \alpha'']] + [[\alpha', \alpha''], \alpha]] + [[\alpha'', \alpha], \alpha']]) = \\ &\qquad \Pi([X_\alpha, X_{\alpha'}])(I^*\alpha'') + \Pi([X_{\alpha'}, X_{\alpha''}])(I^*\alpha) + \Pi([X_{\alpha''}, X_\alpha])(I^*\alpha'), \end{aligned}$$

applying it on $X_\alpha, X_{\alpha'}, X_S$ implies

$$I^*(S([\alpha, \alpha']) - [S(\alpha), \alpha']] - [\alpha, S(\alpha')]]) = \Pi([X_{\alpha'}, X_S])(I^*\alpha) - \Pi([X_\alpha, X_S])(I^*\alpha'),$$

and applying it on $X_\alpha, X_S, X_{S'}$ implies

$$I^*(S' \circ S(\alpha) - S \circ S'(\alpha)) = \Pi([X_S, X_{S'}])(I^*\alpha). \tag{6.8}$$

The three latter equations are equalities in $\Gamma(I^*\mathcal{C})$. Choosing a local section σ of $I : P \to N$ with domain $U \subset N$ and precomposing each of the equations with σ precisely produces the three conditions in definition 6.2.1, but restricted to U. More precisely, the maps t and ∇ at a point $x \in U$ are given by

$$\begin{aligned} (t_{\alpha,\alpha'})_x &= \Pi([X_\alpha, X_{\alpha'}])_{\sigma(x)}, \\ (\nabla_\alpha(S))_x &= \Pi([X_\alpha, X_S])_{\sigma(x)}. \end{aligned} \tag{6.9}$$

The fact that $\Pi(X_\alpha) = \Pi(X_{\alpha'}) = 0$ and $\Pi(X_S) = I^*S$ implies that t defines a tensor and ∇ a connection. A standard partition of unity argument produces a global t and ∇. □

Remark 6.3.2. From now on we will talk about realizations of Cartan algebroids rather than realization of pre-Cartan algebroids, just as we talk about integrating Lie algebras and Lie algebroids, rather than integrating pre-Lie algebras or pre-Lie algebroids (see also the discussion in the introduction to Chapter 7). ♢

Corollary 6.3.3. *If (P, Ω) is a realization of $(\mathcal{C}, \mathfrak{g})$, then for any choice of Π, t and ∇,*

$$\begin{aligned} &(\alpha, \alpha') \mapsto \Pi([X_\alpha, X_{\alpha'}]) - I^*t_{\alpha,\alpha'} \quad \in \Gamma(Z^{0,2}(\mathfrak{g})), \\ &\alpha \mapsto \Pi([X_\alpha, X_S]) - I^*\nabla_\alpha(S) \quad \in \Gamma(\mathfrak{g}^{(1)}), \\ &\Pi([X_S, X_{S'}]) + I^*[S, S'] = 0 \end{aligned}$$

for all $S, S' \in \mathfrak{g}$.

Proof. This corollary is a consequence of the proof of Theorem 6.3.1. The third equation in the statement of the corollary is precisely (6.8). The first two equations in the statement follow from Proposition (6.2.4) together with (6.9), first locally by taking a local section of $I : P \to N$ (which is a submersion), and then globally by a standard partition of unity argument. □

The interaction between Cartan algebroids and their realizations is quite intricate. The following proposition is a first hint of this. Recalling the notion of a Lie algebroid action on a surjective submersion (Section 2.1), Lemma 5.2.17 together with the third identity in 5.2.16 imply that the bundle of Lie algebras $\mathfrak{g}$ of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ acts canonically on all realization of the Cartan algebroid:

Proposition 6.3.4. *Let (I, Ω) be a realization of the Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N. The canonical vector bundle map*

$$\Psi_{\mathfrak{g}} : I^*\mathfrak{g} \to TP$$

defines a Lie algebroid action of $\mathfrak{g}$ on $I : P \to N$. Moreover, the action is injective and its image is $\mathrm{Ker}\ \Omega \subset TP$.

In Chapter 8, we will see that this action of $\mathfrak{g}$ is just part of a larger action of a certain Lie algebroid associated with the Cartan algebroid called the 1st systatic space. This larger action will have rather surprising consequences.

6.4 The Realization Problem

With our modern framework of Cartan's structure theory in place, we may now formulate the modern and global version of Problem 6.1.1, Cartan's realization problem:

Problem 6.4.1. *(Cartan's realization Problem) Given a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N, find a realization (P, Ω) of $(\mathcal{C}, \mathfrak{g})$.*

In the third fundamental theorem (Theorem 6.1.4), Cartan proves the existence of local solutions to the realization problem in the analytic category. Noting that a Cartan algebroid $(C, \mathfrak{g})$ over N can be restricted to an open subset $U \subset N$, namely $(C, \mathfrak{g})|_U := (C|_U, \mathfrak{g}|_U)$ and it is equipped with the restrictions of the structure maps, Theorem 6.1.4 can be rephrased as follows:

Theorem 6.4.2. *(the third fundamental theorem, local and analytic) Let $(\mathcal{C}, \mathfrak{g})$ be an analytic Cartan algebroid over N (i.e. all manifolds and maps are analytic). If the tableau bundle $\mathfrak{g}$ is involutive, then every $x \in N$ has a neighborhood $U \subset N$ such that the Cartan algebroid $(\mathcal{C}, \mathfrak{g})|_U$ over U admits a realization.*

Remark 6.4.3. Note that the existence of local solutions to the realization problem trivially implies the existence of a global solution, since realizations of $(\mathcal{C}, \mathfrak{g})|_U$ and $(\mathcal{C}, \mathfrak{g})|_V$, with $U, V \subset N$ open subsets, induce a realization of $(\mathcal{C}, \mathfrak{g})|_{U \cup V}$ by simply taking the disjoint union of the two realizations. More interesting is the question of whether there exists a global realization (P, Ω) with P connected. This global problem is still open in the analytic case, while in the smooth case both the local and global problems are open. These problems have proven to be very difficult ones, and, at least in the smooth category, they may require new ideas and possibly new analytic tools, such as an analogue of the Cartan-Kähler theorem in the smooth setting. We hope that this modern formulation of the problem will provide new insights into this interesting problem. In Chapter 7, we propose one possible new approach for tackling the realization problem, one which is based on a reformulation of the problem that will be discussed in Section 6.6. $\diamondsuit$

On the Proof of Theorem 6.4.2 In [34], Kumpera presents a proof of Theorem 6.4.2 which uses the theory of exterior differential systems and the Cartan-Kähler theorem. It is along the lines of Cartan's proofs ([5, 7]), but presented in a rigorous and clear fashion. Let us explain the general idea of the proof using our modern framework.

Let $(\mathcal{C}, \mathfrak{g})$ be a Cartan algebroid over N. Since we are looking for local solutions, we may assume that N is an open subset of $\mathbb{R}^n$. Let r and p be the ranks of $\mathcal{C}$ and $\mathfrak{g}$, respectively, and let $\mathrm{pr} : \mathbb{R}^{r+p} \to \mathbb{R}^n$ be the projection onto the first n coordinates. Set $P := \mathrm{pr}^{-1}(N) \subset \mathbb{R}^{r+p}$ and denote the restriction of pr to P by $I : P \to N$. Given $p \in P$, we would like to find a 1-form $\Omega^1(P; I^*\mathcal{C})$ such that (P, Ω) is a realization of $(\mathcal{C}, \mathfrak{g})$. Again, since we are interested in local solutions, we may shrink P to an arbitrarily small open neighborhood of p, and consequently shrink N to $I(P)$.

The main idea is to consider the bundle of "anchored frames" of P. More precisely, recall that $\rho : \mathcal{C} \to TN$ is the anchor of the pre-Lie algebroid $\mathcal{C}$ and let us also denote by $\rho : \mathcal{C} \oplus \mathfrak{g} \to TN$ the map $(\alpha, T) \mapsto \rho(\alpha)$. Consider the following space of linear isomorphisms, "frames", that are compatible with the anchor:

$$\mathrm{Fr}(P) := \{\ \xi : T_pP \xrightarrow{\simeq} (\mathcal{C} \oplus \mathfrak{g})_{I(p)} \mid p \in P \text{ and } \rho \circ \xi = dI|_{T_pP}\ \}.$$

We denote the natural projection from $\mathrm{Fr}(P)$ to P by π and the composition $I \circ \pi$ also by I, thus:

$$\begin{tikzcd} \mathrm{Fr}(P) \arrow[d, "\pi"] \arrow[dd, bend right, "I"'] \\ P \arrow[d, "I"] \\ N. \end{tikzcd}$$

The bundle of "anchored frames" $\mathrm{Fr}(P)$ comes equipped with two tautological 1-forms, one with values in $\mathcal{C}$ and one with values in $\mathfrak{g}$:

$$\begin{aligned} &\overline{\Omega} \in \Omega^1(\mathrm{Fr}(P); I^*\mathcal{C}), \qquad \overline{\Omega}_\xi = \xi^{\mathcal{C}} \circ d\pi, \\ &\overline{\Pi} \in \Omega^1(\mathrm{Fr}(P); I^*\mathfrak{g}), \qquad \overline{\Pi}_\xi = \xi^{\mathfrak{g}} \circ d\pi. \end{aligned}$$

Here $\xi^{\mathcal{C}}$ and $\xi^{\mathfrak{g}}$ denote the $\mathcal{C}$ and $\mathfrak{g}$ components of $\xi \in \mathrm{Fr}(P)$, respectively. On $\mathrm{Fr}(P)$, we may write down the following 2-form, which is constructed precisely as we constructed the structure equation in the definition of a realization:

$$d\overline{\Omega} + \frac{1}{2}[\overline{\Omega}, \overline{\Omega}] - \overline{\Pi} \wedge \overline{\Omega} \in \Omega^2(\mathrm{Fr}(P); I^*\mathcal{C}). \tag{6.10}$$

The key observation is that a local solution to the realization problem is the same thing as a local section σ of $\pi : \mathrm{Fr}(P) \to P$ that pulls-back (6.10) to zero. Indeed, if this is the case, then $\sigma^*\overline{\Omega} \in \Omega^1(P; I^*\mathcal{C})$ satisfies the structure equation as well as the coframe condition, and is, hence, a solution.

The main challenge is to construct such a local section σ. The strategy taken in [34] (and by Cartan) is to consider the exterior differential system on $\mathrm{Fr}(P)$ spanned by the components of the vector bundle-valued 2-form (6.10). Integral manifolds of dimension $r + p$ of this exterior differential system that project diffeomorphically to P correspond to the desired local sections. One proves that if $\mathfrak{g}$ is involutive, then the assumptions of the Cartan-Kähler theorem are satisfied, and, hence, such integral manifolds exist. The three integrability conditions in the definition of a Cartan algebroid play a crucial role in the proof. We refer the reader to [34] for the details of the proof. Possible references for the theory of exterior differential systems are [66, 3, 4].

6.5 Examples of Cartan Algebroids

In this section, we present some examples of Cartan algebroids.

Example 6.5.1. (Cartan algebroids associated with Lie pseudogroups) The most important source of examples is the second fundamental theorem. Since any realization induces a Cartan algebroid (Theorem 6.3.1), then any realization arising from a Lie pseudogroup via the second fundamental theorem gives rise to a Cartan algebroid. Thus:

Theorem 6.5.2. *If Γ is a Lie pseudogroup of order k on M such that $(\mathfrak{g}^k)^{(1)}$ is of constant rank, then $(TM \oplus A^{k-1}(\Gamma), \mathfrak{g}^k)$ is a Cartan algebroid.*

Proof. Starting with a Lie pseudogroup Γ on M of order k, the induced pair $(TM \oplus A^{k-1}(\Gamma), \mathfrak{g}^k)$ is a pre-Cartan algebroid by Proposition 5.3.5. Then, Proposition 5.3.3 states that this pre-Cartan algebroid admits a realization if we assume that $(\mathfrak{g}^k)^{(1)}$ is of constant rank. Consequently, Theorem 6.3.1 implies that $(TM \oplus A^{k-1}(\Gamma), \mathfrak{g}^k)$ is in fact a Cartan algebroid. $\diamondsuit$

In particular, the pre-Cartan algebroids that were constructed in Examples 5.4.1 and 5.4.2 are Cartan algebroids. $\diamondsuit$

Example 6.5.3. (Cartan algebroids associated with Pfaffian groupoids) More generally, in Section 5.3 we constructed a pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ and a pair $(\mathcal{G}, \Omega)$ out of any given Pfaffian groupoid $(\mathcal{G}, \omega)$, and in Proposition 5.3.7 we identified the precise condition under which the pair $(\mathcal{G}, \Omega)$ is a realization, namely under the condition that $\mathcal{G}$ admits an integral Cartan-Ehresmann connection. By Theorem 6.3.1, under this same condition, $(\mathcal{C}, \mathfrak{g})$ is a Cartan algebroid. Thus:

Theorem 6.5.4. *If $(\mathcal{G}, \omega)$ is a Pfaffian groupoid which admits an integral Cartan-Ehresmann connection, then the induced pre-Cartan algebroid $(E, \mathfrak{g})$ is a Cartan algebroid.* $\diamondsuit$

Going beyond Lie pseudogroups, let us now look at a few other examples and general constructions of Cartan algebroids.

Example 6.5.5 (Abstract Atiyah sequences)**.** Transitive Lie algebroids A over a manifold N are the same thing as Cartan algebroids $(A, 0)$ over N. Transitive Lie algebroids are also known as "abstract Atiyah sequences" for the following reason. Given a principal G-bundle $\pi : P \to N$ (where G is a Lie group acting from the left), one has an associated exact sequence of vector bundles over N known as the "Atiyah sequence of P",

$$0 \to P[\mathfrak{g}] \to TP/G \xrightarrow{d\pi} TN \to 0,$$

where $\mathfrak{g}$ is the Lie algebra of G and $P[\mathfrak{g}] = (P \times \mathfrak{g})/G$. The middle term

$$A(P) := TP/G$$

has the structure of a transitive Lie algebroid: the anchor is induced by $d\pi$, while the bracket comes from the Lie bracket of vector fields on P and the identification $\Gamma(A(P)) = \mathfrak{X}(P)^G$ (see [47], Section 3.2, for more details).

The relevance of this sequence comes from the fact that connections on P are the same thing as splittings of the sequence, while the curvature of a connection appears as the failure of the splitting to preserve the Lie brackets.

Note also that the quotient map $TP \to TP/G$ induces a tautological form

$$\Omega \in \Omega^1(P; A(P)), \tag{6.11}$$

with which (π, Ω) becomes a realization of the Cartan algebroid $(A, 0)$.

Similarly, given a general transitive Lie algebroid A over N, there is an exact sequence

$$0 \to \mathrm{Ker}\ (\rho) \to A \xrightarrow{\rho} TN \to 0$$

called an "abstract Atiyah sequence". The question of the existence of a principal bundle P so that A is isomorphic to $A(P)$ is equivalent to the integrability of A as a Lie algebroid. Hence, the integrability of the Lie algebroid A is also closely related to the existence of a realization of the Cartan algebroid $(A, 0)$; the only difference is that a general realization P might have an induced action of $\mathfrak{g}$, but this action may fail to integrate to an action of G. ♢

Example 6.5.6 (Lie groups as pseudogroups)**.** Changing a bit the point of view of the previous example, we see that any Lie group G can be realized as a pseudogroup in normal form by making it act freely and properly on a space P. To be more precise, assume that $\pi : P \to N$ is a principal G-bundle, then the left multiplication by elements in G induces a pseudogroup $\Gamma_{G,P}$ on P. To see that this is a pseudogroup in normal form, we just use the Lie algebroid $A(P)$ from the previous example and the associated tautological form Ω from (6.11). It is not difficult to see that $\Gamma_{G,P}$ is characterized by the invariance of π and Ω.

Note that up to Cartan equivalences, the choice of P is not so important. If Q is another principal G-bundle, then $\Gamma_{G,P}$ and $\Gamma_{G,Q}$ admit a common isomorphic prolongation, namely $\Gamma_{G,P\times Q}$ (along the canonical projections from $Q \times P$ to P and Q, respectively).

The simplest choice for P would be $P = G$ with the left action of G. Then N is a point, $A(P)$ is the Lie algebra $\mathfrak{g}$ of G and

$$\Omega \in \Omega^1(G; \mathfrak{g})$$

is the Maurer-Cartan form. The resulting pseudogroup on G (preserving Ω) is the pseudogroup generated by left translations. $\diamondsuit$

Example 6.5.7 (Truncated Lie algebras)**.** When N is a point, a Cartan algebroid modulo gauge equivalence is the same thing as the truncated Lie algebras of Singer and Sternberg ([64], Definition 4.1). $\diamondsuit$

Example 6.5.8. Here is a general construction of Cartan algebroids that underlies Cartan's proof of the second fundamental theorem (c.f. the proof of the second fundamental theorem in Section 5.3). Start with a Lie algebroid A over N, and define

$$\mathcal{C} = \mathcal{C}(A) := TM \oplus A, \qquad \mathfrak{g} := \mathrm{Hom}(TM, A).$$

Note that $\mathfrak{g}$ is a bundle of Lie algebras with the Lie bracket

$$[T, S] := T \circ \rho \circ S - S \circ \rho \circ T,$$

where ρ is the anchor of A. Also, it can be realized as a subbundle

$$\mathfrak{g} \hookrightarrow \mathrm{Hom}(\mathcal{C}, \mathcal{C}), \qquad T \mapsto \hat{T},$$

with

$$\hat{T}(\alpha, X) = (0, T(\rho(\alpha) - X)), \qquad \forall\, \alpha \in \Gamma(\mathcal{C}),\ X \in \mathfrak{X}(M).$$

The bracket of $\mathcal{C}$ depends on the choice of a connection ∇ on A, and is defined by the formula

$$[(X, \alpha), (Y, \beta)]_{\nabla} := ([X, Y], [\alpha, \beta]_{\nabla} + \nabla_X(\beta) - \nabla_Y(\alpha)),$$

which uses the A-torsion of ∇,

$$[\alpha, \beta]_{\nabla} = [\alpha, \beta] - \nabla_{\rho(\alpha)}\beta + \nabla_{\rho(\beta)}\alpha.$$

The anchor of $\mathcal{C}$ is just the projection onto the first component.

Proposition 6.5.9. *The pair $(\mathcal{C}(A), \mathfrak{g})$ is a Cartan algebroid which, up to gauge equivalence, is independent of the choice of a connection ∇.*

Proof. First, given two connections ∇ and ∇' as in the example, we would like to show that their induced pre-Cartan algebroids are gauge equivalent. Indeed, the difference of the two connections induces a vector bundle map $(\nabla' - \nabla) : A \to \mathrm{Hom}(TM, A)$, which, in turn, induces a vector bundle map from $\mathcal{C} = TM \oplus A$ to $\mathfrak{g} = \mathrm{Hom}(TM, A)$ by acting trivially on TM. This is the desired gauge equivalence (simple computation).

We now describe two ways to prove that $(\mathcal{C}, \mathfrak{g})$ is a Cartan algebroid – an indirect proof and a direct proof. We start with the indirect proof. We assume that A integrates to a Lie groupoid $\mathcal{G}$ (in fact, it suffices to have an integration to a local groupoid, and such an integration always exists). In this case, $(J^1\mathcal{G}, \omega)$ is a Lie-Pfaffian groupoid, where $\omega \in \Omega^1(J^1\mathcal{G}; t^*A)$ is the Cartan form of the jet groupoid $J^1\mathcal{G}$. We apply the recipe described in Section 5.3 for producing a pre-Cartan algebroid from a given Lie-Pfaffian groupoid. The pre-Cartan algebroid one obtains is precisely the one described in the above example, with the connection being the one obtained from the choice of a splitting (5.29). Next, one notes that the Lie-Pfaffian groupoid $(J^1\mathcal{G}, \omega)$ admits an integral Cartan-Ehresmann connection, since an integral Cartan-Ehresmann connection in this case is the same thing as a section of the projection $\mathrm{pr} : J^2\mathcal{G} \to J^1\mathcal{G}$, which is an affine bundle. Finally, the fact that $(\mathcal{C}, \mathfrak{g})$ is a Cartan algebroid follows from the fact that the pair $(J^1\mathcal{G}, (dt, \omega))$ is a realization of $(\mathcal{C}, \mathfrak{g})$ (Proposition 5.3.7) together with the fact that a pre-Cartan algebroid that admits realizations is a Cartan algebroid (Theorem 6.3.1).

Alternatively, we can directly prove that $(\mathcal{C}, \mathfrak{g})$ is a Cartan algebroid by verifying the axioms of Definition 6.2.1. A straightforward computation shows that axiom 1 is satisfied. We are left with finding a vector bundle map $\bar{t} : \Lambda^2\mathcal{C} \to \mathfrak{g}$ and a connection $\bar{\nabla} : \Gamma(\mathcal{C}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ with which axioms 2 and 3 of are satisfied. The idea is straightforward (but the computations are a bit tedious): compute the left hand side of the equations in both axioms and try to decompose the resulting expressions so as to obtain a suitable $\bar{t}$ and $\bar{\nabla}$. We will write down explicit solutions, i.e. expressions for $\bar{t}$ and $\bar{\nabla}$ that are obtained in this way, and leave it as an exercise to check that these satisfy axioms 2 and 3.

To write down $\bar{t}$, we first define the tensors $R : \Lambda^2TM \to \mathrm{Hom}(A, A)$ and $\bar{R} : \Lambda^2 A \to \mathrm{Hom}(TM, A)$ by

$$\begin{aligned} R_{X,Y}(\alpha) &:= \nabla_{[X,Y]}\alpha - \nabla_X\nabla_Y\alpha + \nabla_Y\nabla_X\alpha, \\ \bar{R}_{\alpha,\beta}(X) &:= \nabla_X[\alpha,\beta] - [\nabla_X\alpha,\beta] - [\alpha,\nabla_X\beta] + \nabla_{\rho(\nabla_X\alpha)}\beta - \nabla_{\rho(\nabla_X\beta)}\alpha \\ &\qquad - \nabla_{[X,\rho(\alpha)]}\beta + \nabla_{[X,\rho(\beta)]}\alpha, \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$ and $\alpha, \beta \in \Gamma(A)$. Then,

$$\begin{aligned} &\bar{t}_{(X,\alpha),(Y,\beta)}(Z,\gamma) := \\ &\qquad (0, \bar{R}_{\alpha,\beta}(\rho(\gamma) - Z) + \frac{1}{2}R_{\rho(\gamma)-Z,\rho(\alpha)-X}(\beta) + \frac{1}{2}R_{\rho(\beta)-Y,\rho(\gamma)-Z}(\alpha)), \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and $\alpha, \beta, \gamma \in \Gamma(A)$.

To write down $\bar{\nabla}$, we need to choose a torsion-free connection on TM, which (by abuse of notation) we denote by $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. Thus, $[X, Y] = \nabla_X Y - \nabla_Y X$ for all $X, Y \in \mathfrak{X}(M)$. Then,

$$\begin{aligned} &(\bar{\nabla}_{(X,\alpha)}\hat{T})(Y,\beta) := \\ &\qquad [(X,\alpha), \hat{T}(Y,\beta)]_\nabla + \big(0, T\big(\rho(\nabla_{\rho(\alpha)-X}\beta) - \nabla_{\rho(\alpha)}\rho(\beta) + \nabla_X Y\big)\big), \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$, $\alpha, \beta \in \Gamma(A)$ and $T \in \Gamma(\mathfrak{g})$. ♢

♢

Example 6.5.10. (Restrictions of Cartan Algebroids) A Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N can be restricted to any submanifold $S \subset N$ giving rise to a Cartan algebroid $(\mathcal{C}_S, \mathfrak{g}_S)$ over S. Here, $\mathfrak{g}_S := \mathfrak{g}|_S$ (the restriction of the vector bundle to S), $\mathcal{C}_S := \{ \alpha \in \mathcal{C} \mid \rho(\alpha) \in TS \}$ and the bracket is uniquely determined by

$$[\alpha|_S, \beta|_S] = [\alpha, \beta]|_S,$$

for all $\alpha, \beta \in \Gamma(\mathcal{C})$.

A realization (I, Ω) of $(\mathcal{C}, \mathfrak{g})$ induces a realization (I_S, Ω_S) of $(\mathcal{C}_S, \mathfrak{g}_S)$ by taking the restrictions $P_S := I^{-1}(S)$, $I_S := I|_{P_S}$ and $\Omega_S := \Omega|_S$. This restriction operation is the operation underlying Cartan's restriction to a transversal (see Section 4.3). ♢

6.6 An Alternative Approach to Cartan Algebroids: Cartan Pairs

In this section, we present an alternative way for encoding the structure of a Cartan algebroid, a way which deviates from Cartan's point of view, but which is more intuitive and, in a sense, a more natural relaxation of the axioms of a Lie algebroid. The resulting objects are called Cartan pairs. In turn, the realization problem for Cartan algebroids can be reformulated in terms of Cartan pairs, a formulation that points towards new directions for tackling the classical realization problem. This will be exploited in the next chapter.

From Cartan Algebroids to Cartan Pairs Let $(\mathcal{C}, \mathfrak{g})$ be a pre-Cartan algebroid over N (Definition 5.2.7), $t : \Lambda^2\mathcal{C} \to \mathfrak{g}$ a vector bundle map, and $\nabla : \Gamma(\mathcal{C}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ a $\mathcal{C}$-connection. From this data, we construct a new pre-Lie algebroid $A \to N$. For the vector bundle, we take the direct sum

$$A := \mathcal{C} \oplus \mathfrak{g},$$

which we then equip with the anchor induced by the anchor of $\mathcal{C}$,

$$\rho : A \to TN, \qquad \rho(\alpha, S) := \rho(\alpha), \tag{6.12}$$

and the bracket

$$[\cdot, \cdot] : \Gamma(A) \times \Gamma(A) \to \Gamma(A) \tag{6.13}$$

defined by

$$[(\alpha, S), (\beta, T)] := ([\alpha, \beta] + T(\alpha) - S(\beta),\ t_{\alpha,\beta} + \nabla_\alpha T - \nabla_\beta S - [S, T]),$$

for all $\alpha, \beta \in \Gamma(\mathcal{C}),\ S, T \in \Gamma(\mathfrak{g})$. One easily verifies that A is indeed a pre-Lie algebroid (Definition 5.2.1).

Remark 6.6.1. This construction is reminiscent of the construction of an non-abelian extension of a Lie algebroid (see [46], Chapter 4, Section 3, and see also the construction of the *1st systatic space* in Section 8.2). We can think of A as an extension of the pre-Lie algebroid $\mathcal{C}$ by the non-abelian (and totally non-transitive) Lie algebroid $\mathfrak{g}$. This same construction will be applied in Section 9.2 in the construction of the 1st prolongation of a Cartan algebroid. In fact, studying Cartan's notion of prolongation led us to this alternative point of view on Cartan algebroids. $\diamondsuit$

In terms of this induced pre-Lie algebroid A, the three rather cumbersome axioms of a Cartan algebroid (see Definition 6.2.1) can be neatly packaged in a single compact and elegant property of A. Let us denote the **Jacobiator** tensor associated with the bracket of A by

$$\mathrm{Jac}_A \in \Gamma(\mathrm{Hom}(\Lambda^3 A, A)).$$

At the level of sections, it is defined by

$$\mathrm{Jac}_A(\alpha,\beta,\gamma) := [[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta], \qquad \forall\, \alpha,\beta,\gamma \in \Gamma(A).$$

Lemma 6.6.2. *Let $(\mathcal{C},\mathfrak{g})$ be a pre-Cartan algebroid over N. Let $t : \Lambda^2\mathcal{C} \to \mathfrak{g}$ be some vector bundle map and $\nabla : \Gamma(\mathcal{C}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ some $\mathcal{C}$-connection, and let $A = \mathcal{C} \oplus \mathfrak{g}$ be the pre-Lie algebroid induced by this choice of t and ∇. Then, the axioms 1,2 and 3 of a Cartan algebroid (Definition 6.2.1) are satisfied with this choice of t and ∇ if and only if $Jac_A \equiv 0 \pmod{\mathfrak{g}}$ (i.e. $Jac_A(\alpha,\beta,\gamma) \in \Gamma(\mathfrak{g})$ for all $\alpha,\beta,\gamma \in \Gamma(A)$).*

Proof. We compute the $\mathcal{C}$-component of Jac_A. For any $\alpha,\beta,\gamma \in \Gamma(\mathcal{C})$ and $S,T,U \in \Gamma(\mathfrak{g})$, we have that

$$\begin{aligned}
&\mathrm{Jac}_A\big((\alpha,0),(\beta,0),(\gamma,0)\big)\\
&\qquad = \big([[\alpha,\beta],\gamma] + [[\beta,\gamma],\alpha] + [[\gamma,\alpha],\beta] - t_{\alpha,\beta}(\gamma) - t_{\beta,\gamma}(\alpha) - t_{\gamma,\alpha}(\beta),\ \dots\big),\\
&\mathrm{Jac}_A\big((\alpha,0),(\beta,0),(0,T)\big)\\
&\qquad = \big(T([\alpha,\beta]) - [T(\alpha),\beta] - [\alpha,T(\beta)] - \nabla_\beta(T)(\alpha) + \nabla_\alpha(T)(\beta),\ \dots\big),\\
&\mathrm{Jac}_A\big((\alpha,0),(0,S),(0,T)\big) = \big([S,T](\alpha) - S(T(\alpha)) + T(S(\alpha)),\ \dots\big),\\
&\mathrm{Jac}_A\big((0,S),(0,T),(0,U)\big) = \big(0,\ \dots\big).
\end{aligned}$$

The proposition directly follows. $\square$

The above construction and lemma motivate the following definition:

Definition 6.6.3. *A **Cartan pair** over a manifold N is a pair $(A,\mathfrak{g})$ consisting of a transitive pre-Lie algebroid $(A,[\cdot,\cdot],\rho)$ over N and a vector subbundle $\mathfrak{g} \subset A$ such that $Jac_A \equiv 0 \pmod{\mathfrak{g}}$ and $\mathfrak{g} \subset \mathrm{Ker}\,\rho$.*

The basic example of a Cartan pair is a transitive Lie algebroid: given a transitive Lie algebroid $A \to N$, $(A,0)$ is a Cartan pair over N. General Cartan pairs can be thought of as "almost" transitive Lie algebroids, "almost" in the sense that the Jacobi identity is satisfied modulo $\mathfrak{g}$.

From the above lemma, we conclude that any Cartan algebroid induces a Cartan pair. The type of Cartan pair one obtains from this construction is of a particular type, in a sense which we will now explain. Given a Cartan pair $(A, \mathfrak{g})$, consider the vector bundle map

$$\mathfrak{g} \to \mathrm{Hom}(A/\mathfrak{g}, A/\mathfrak{g}), \qquad T \mapsto \hat{T}, \tag{6.14}$$

which, at the level of sections, is defined by

$$\hat{T}(\mathrm{pr}(\alpha)) := \mathrm{pr}([\alpha, T]), \qquad \forall\, T \in \Gamma(\mathfrak{g}),\ \alpha \in \Gamma(A).$$

One easily checks that the right hand side of this definition is $C^\infty(N)$-linear in both the α-slot and the T-slot due to the fact that $\mathfrak{g}$ is killed by both ρ and pr.

Definition 6.6.4. *A Cartan pair $(A, \mathfrak{g})$ is called **standard** if the map* (6.14) *is injective, or, equivalently, if the following property is satisfied:*

$$\forall\, T \in \Gamma(\mathfrak{g}) : \quad [T, \Gamma(A)] \subset \Gamma(\mathfrak{g}) \ \Rightarrow\ T = 0.$$

In the case of the construction of a Cartan pair out of a Cartan algebroid, as described above, the map (6.14) corresponds precisely to the inclusion $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ that is part of the structure of a Cartan algebroid. Thus, the Cartan pair one obtains from a Cartan algebroid is standard. We can summarize the construction as follows:

Proposition 6.6.5. *Let $(\mathcal{C}, \mathfrak{g})$ be a Cartan algebroid equipped with a choice of a t and ∇ as in Definition 6.2.1. Then, the induced pair $(\mathcal{C} \oplus \mathfrak{g}, \mathfrak{g})$, equipped with the structure defined above, is a standard Cartan pair.*

Remark 6.6.6. In Remark 5.3.6, we discussed the idea of relaxing the notion of a Cartan algebroid: rather than requiring for there to be a vector subbundle $\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$, requiring for there to be a vector bundle $\mathfrak{g}$ together with a map $\mathfrak{g} \to \mathrm{Hom}(\mathcal{C}, \mathcal{C})$. Such a generalized notion of a Cartan algebroid will give rise to Cartan pairs that are not necessarily standard. In the case of Cartan pairs, it is more natural to add the "standard" property as separate property rather than embed it in the definition itself. In the case of a Cartan algebroid, we chose to embed this property in the definition (by requiring for there to be an inclusion rather than just a map) because it is consistent with Cartan's local definition. $\diamondsuit$

Example 6.6.7. In Section 5.3, we described the construction of a pre-Cartan algebroid out of a standard Pfaffian algebroid. Under certain conditions, these pre-Cartan algebroids are in fact a Cartan algebroid (e.g., when they admit realizations). In turn, a Cartan algebroid obtained in this way gives rise to a standard Cartan pair. Alternatively, one could directly construct the Cartan pair out of the standard Pfaffian algebroid. In fact, one can go a step further and reformulate the entire structure theory for Lie pseudogroups in terms of Cartan pairs. In general, it is useful to keep both points of view in mind. We should also remark that, in this construction, one could also start with a Pfaffian algebroid that is not standard, in which case the Cartan pair that one obtains will also fail to be standard. $\diamondsuit$

From Cartan Pairs to Cartan Algebroids We have described how one constructs a Cartan pair (that is standard) out of a Cartan algebroid. Conversely, starting with a Cartan pair $(A, \mathfrak{g})$ that is standard, one constructs a Cartan algebroid given by the pair $(A/\mathfrak{g}, \mathfrak{g})$. The construction depends on a choice of a splitting

$$\xi : A/\mathfrak{g} \to A$$

of the short exact sequence

$$0 \to \mathfrak{g} \hookrightarrow A \xrightarrow{\mathrm{pr}} A/\mathfrak{g} \to 0, \tag{6.15}$$

where $\mathrm{pr} : A \to A/\mathfrak{g}$ is the quotient map. We equip the vector bundle $A/\mathfrak{g}$ with the bracket

$$[\cdot,\cdot] : \Gamma(A/\mathfrak{g}) \times \Gamma(A/\mathfrak{g}) \to \Gamma(A/\mathfrak{g}), \qquad [\alpha, \beta] := \mathrm{pr}([\xi(\alpha), \xi(\beta)]),$$

and the anchor

$$\rho : A/\mathfrak{g} \to TN, \qquad \rho(\alpha) := \rho(\xi(\alpha)).$$

With this structure, $A/\mathfrak{g}$ becomes a pre-Lie algebroid. Note that while the bracket of A depends on the choice of ξ, the anchor is independent of this choice. The inclusion $\mathfrak{g} \hookrightarrow \mathrm{Hom}(A/\mathfrak{g}, A/\mathfrak{g})$ is given by (6.14). We require of the Cartan pair to be standard precisely so that this map will be an inclusion.

Proposition 6.6.8. *Let $(A, \mathfrak{g})$ be a standard Cartan pair and let $\xi : A/\mathfrak{g} \to A$ be a a choice of a splitting of* (6.15)*. Then, the pair $(A/\mathfrak{g}, \mathfrak{g})$ equipped with the structure defined above is a Cartan algebroid. Moreover, up to gauge equivalence, the resulting Cartan algebroid is independent of the choice of a splitting.*

Proof. We must show the existence of a vector bundle map $t : \Lambda^2(A/\mathfrak{g}) \to \mathfrak{g}$ and an $A/\mathfrak{g}$-connection $\nabla : \Gamma(A/\mathfrak{g}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ for which the axioms of a Cartan algebroid (Definition 6.2.1) are satisfied. Let us denote by $\eta : A \to \mathfrak{g}$ the right splitting of (6.15) that is induced by the left splitting ξ. We define t by

$$t_{\alpha,\beta}(\gamma) := \eta([\xi(\alpha), \xi(\beta)])$$

and ∇ by

$$\nabla_\alpha(T) := \eta([\xi(\alpha), T]).$$

By the definition of a Cartan pair, $\mathrm{pr} \circ \mathrm{Jac}_A = 0$. Applying this equality to the three types of triples $(\xi(\alpha), \xi(\beta), \xi(\gamma))$, $(\xi(\alpha), \xi(\beta), T)$ and $(\xi(\alpha), S, T)$, where $\alpha, \beta, \gamma \in \Gamma(A/\mathfrak{g})$ and $S, T \in \Gamma(\mathfrak{g})$, a simple computation shows that the three axioms of a Cartan algebroid are satisfied. For the final assertion, one notes that given two splittings ξ and ξ' of (6.15), their difference $(\xi' - \xi) : A/\mathfrak{g} \to \mathfrak{g}$ defines the desired gauge equivalence. $\square$

The two constructions defined above are inverse to each other in the following sense: starting with a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ and applying the first construction, we obtain the Cartan pair $(\mathcal{C} \oplus \mathfrak{g}, \mathfrak{g})$. Applying the second construction with the canonical splitting of $\mathcal{C} \oplus \mathfrak{g}$ in $\mathcal{C}$ and $\mathfrak{g}$, one recovers the original Cartan algebroid.

Remark 6.6.9. With a suitable notion of equivalence of Cartan pairs, the two constructions define a 1-1 correspondence between Cartan algebroids up to gauge equivalence and standard Cartan pairs up to equivalence. In this sense, the picture of Cartan pairs, or, more precisely, standard Cartan pairs, forms an alternative approach to the notion of Cartan algebroids. $\diamondsuit$

Realizations of Cartan Pairs The notion of a realization of a Cartan algebroid has a counterpart in the Cartan pair approach, and just as the axioms of a Cartan algebroid become rather simple, so do the axioms of a realization.

Definition 6.6.10. *A **realization** of a Cartan pair $(A, \mathfrak{g})$ over N is a pair (P, Ω) consisting of a surjective submersion $I : P \to N$ and an anchored 1-form $\Omega \in \Omega^1(P; I^*A)$, such that*

$$d\Omega + \frac{1}{2}[\Omega, \Omega] \equiv 0 \ (\mathrm{mod}\ \mathfrak{g}) \tag{6.16}$$

and such that Ω is pointwise an isomorphism.

The meaning of anchored is as before, namely $dI = \rho \circ \Omega$. Equation 6.16 should be read as: $(d\Omega + \frac{1}{2}[\Omega, \Omega])(X, Y) \in \Gamma(\mathfrak{g})$ for all $X, Y \in \mathfrak{X}(P)$. Comparing this notion of a realization with Definition 5.2.11 in the case of Cartan algebroids, condition 6.16 is the analogue of the structure equation 5.12 while the requirement of Ω to be pointwise an isomorphism is the analogue of 5.13.

A realization of a Cartan pair can be thought of as an "almost" Maurer-Cartan form on P with values in A, "almost" in the sense that the 1-form Ω satisfies the Maurer-Cartan equation modulo $\mathfrak{g}$.

We leave it as a simple exercise to check that given a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$, a realization (P, Ω) of $(\mathcal{C}, \mathfrak{g})$ induces a realization of the induced standard Cartan pair $(\mathcal{C} \oplus \mathfrak{g}, \mathfrak{g})$. The idea is to choose a Π as in Definition 5.2.11 and take $(P, (\Omega, \Pi))$ for the realization of $(\mathcal{C} \oplus \mathfrak{g}, \mathfrak{g})$. Clearly, (Ω, Π) is pointwise an isomorphism, and one is left to verify that 6.16 is equivalent to the structure equations 5.12. Similarly, it is not hard to show that a realization of a standard Cartan pair $(A, \mathfrak{g})$ induces a realization of the induced Cartan algebroid $(A/\mathfrak{g}, \mathfrak{g})$.

The Realization Problem Reformulated This new point of view of Cartan pairs leads us to the following reformulation of the realization problem:

Problem 6.6.11. *(Realization Problem) Given a Cartan pair $(A, \mathfrak{g})$ over N, find a realization (P, Ω) of $(A, \mathfrak{g})$.*

From the discussion in the previous paragraph, we conclude that a solution to this realization problem will also give a solution to the the realization problem for Cartan algebroids, problem 6.4.1.

In the basic example of Cartan pairs, namely $(A, 0)$ where A is a transitive Lie algebroid, one can obtain a solution to the realization problem by integrating the Lie algebroid to a Lie groupoid (when the Lie algebroid is integrable), in which case the Maurer-Cartan form on any source fiber of the Lie groupoid defines a solution (see also Section 7.5 in the next chapter). In [10], the authors present a method for integrating Lie algebroids to Lie groupoids by constructing a Lie groupoid out of the space of so called A-paths of the Lie algebroid. In the same paper, the authors identify the precise obstructions for integration. A large part of the construction in that paper does not rely on the fact that the Lie algebroid one starts with satisfies the Jacobi identity. Our point of view of Cartan pairs – "transitive Lie algebroids that fail to satisfy the Jacobi identity modulo $\mathfrak{g}$" – suggests a new method for tackling the realization problem, namely by imitating the construction in [10] and pinpointing the precise role of the Jacobi identity in that construction, in the hope that the same approach will lead to a solution of this more general problem.

In the next chapter, we will make a pause in our task of translating Cartan's structure theory and discuss yet another possible approach for tackling the realization problem. We will present a new method for solving the realization problem in the basic example of a transitive Lie algebroid, and pinpoint the precise role of the Jacobi identity in the procedure. We will explain until what point our method can be used to tackle the general realization problem, as phrased in terms of Cartan pairs, and discuss the steps that are yet to be understood.

Chapter 7

The Role of the Jacobi Identity in the Realization Problem

The current chapter is an intermezzo in our modern presentation of Cartan's theory for Lie pseudogroups in which we present a new method for tackling the realization problem. In Chapter 6, we discussed Cartan's realization problem, first phrased locally in Cartan's language (Problem 6.1.1), then phrased globally in terms of Cartan algebroids and their realizations (Problem 6.4.1), and then rephrased globally in terms of the alternative picture of Cartan pairs and their realizations (Problem 6.6.11). In the latter formulation, the realization problem amounts to finding a realization (P,Ω) of a given a Cartan pair $(A,\mathfrak{g})$. A Cartan pair, as we recall, is an "almost" transitive Lie algebroid A that satisfies the Jacobi identity modulo a vector subbundle $\mathfrak{g}\subset A$ (see Definition 6.6.3). A realization of a Cartan pair is an A-valued 1-form Ω on P that "almost" satisfies the Maurer-Cartan equation, in the sense that the Maurer-Cartan expression $d\Omega+\frac{1}{2}[\Omega,\Omega]$ vanishes modulo $\mathfrak{g}$, and is pointwise an isomorphism (Definition 6.6.10). From this formulation, it becomes evident that a key step in solving the realization problem is to clarify the role of the Jacobi identity in the problem.

In this chapter, we present a method for solving the realization problem in the simple case $\mathfrak{g}=0$, i.e. when the Jacobi identity *is* satisfied and the Cartan pair reduces to a transitive Lie algebroid. The main advantage of this method is that it clarifies the precise role of the Jacobi identity in the solution process, and, in particular, it produces an explicit formula, Equation (7.32), which relates the Jacobi identity with the Maurer-Cartan expression. Using this equation, we believe that our method can be extended to solve the general realization problem, the case $\mathfrak{g}\neq 0$ (more details in Section 7.4). Another nice feature of the method is that we obtain an explicit formula for a solution to the problem. In the simplest case of a Lie algebra (where the base manifold is a point), we recover the well known formula (7.2).

We begin our presentation by describing the method in the case of a Lie algebra. In Section 7.4, we extend the method to the case of a transitive Lie algebroid. In fact, we go a step further and solve a generalization of this problem, the realization problem for (possibly non-transitive) Lie algebroids. This problem, as we explain, is related to the problem of integrating Lie algebroids to local Lie groupoids. As in the case of a Lie algebra, we obtain an explicit solution, Equation (7.28). In Section 7.3, as another application, we show that the method can be adapted to solve the problem of existence of local symplectic realizations in Poisson geometry, shedding light on the role of the Poisson equation, the analogue of the Jacobi identity, as an obstruction.

This chapter was published in [72] (with some changes, mainly in the introduction and motivating remarks).

7.1 An Outline of our Approach to the Realization Problem

In this introductory section, we describe the realization problem in the case of Lie algebras, describe our method of solution and make the connection between this problem and the more general problems of the existence of local symplectic realizations of Poisson structures and the realization problem for Lie algebroids.

Realization Problem for Lie Algebras Any Lie group G carries a canonical 1-form with values in the tangent space to the identity $\mathfrak{g}$,

$$\phi \in \Omega^1(G; \mathfrak{g}),$$

known as the *Maurer-Cartan form* of G. Actually, the Lie group structure is encoded, in some sense, in the 1-form and its properties; this is in fact Cartan's approach to Lie's infinitesimal theory. The two main properties of the Maurer-Cartan form are: it satisfies the so-called *Maurer-Cartan equation* and it is pointwise an isomorphism (the latter is often phrased as the property that the components of the 1-form with respect to some basis form a coframe). The Maurer-Cartan equation reveals a Lie algebra structure on $\mathfrak{g}$. Of course, the resulting Lie algebra is the same one obtained in the more common approach of using invariant vector fields.

Conversely, if we begin with an n-dimensional Lie algebra $\mathfrak{g}$, we can formulate the following *realization problem for Lie algebras*: find a $\mathfrak{g}$-valued 1-form $\phi \in \Omega^1(U; \mathfrak{g})$ defined on some open neighborhood $U \subset \mathfrak{g}$ of the origin such that ϕ is pointwise an isomorphism and satisfies the Maurer-Cartan equation

$$d\phi + \frac{1}{2}[\phi, \phi] = 0. \tag{7.1}$$

A solution to the problem induces a local Lie group structure on some open subset of U (see [22], p. 368-369), and we can, therefore, think of this realization problem as the problem of locally integrating Lie algebras.

A solution to this problem can be obtained by supposing that the Lie algebra integrates to a Lie group and pulling back the canonical Maurer-Cartan form on the Lie group by the exponential map. This produces the following $\mathfrak{g}$-valued 1-form $\phi \in \Omega^1(\mathfrak{g}; \mathfrak{g})$ whose defining formula refers only to data coming from the Lie algebra and not from the Lie group:

$$\phi_x(y) = \int_0^1 e^{-t\,\mathrm{ad}_x} y\, dt, \qquad x \in \mathfrak{g},\ y \in T_x\mathfrak{g}. \tag{7.2}$$

This formula defines a solution to (7.1), as can be verified directly, and since it is equal to the identity at the origin, it is pointwise an isomorphism in a neighborhood of the origin. See [16, 67] for more details.

We now make the following observation: neither Equation (7.1) nor Formula (7.2) rely on the Jacobi identity; they make perfect sense if we replace the Lie algebra with the weaker notion of a *pre-Lie algebra*, namely a vector space $\mathfrak{g}$ equipped with an antisymmetric bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. However, (7.2) is a solution of (7.1) if and only if

$\mathfrak{g}$ is a Lie algebra, which is not difficult to show. This leads to the natural question: what is the precise role of the Jacobi identity? Put differently, at what point in the integration process does the Jacobi identity appear?

In Section 7.2 we present a 2-step method for solving the realization problem for Lie algebras which answers this question. The method can be outlined as follows:

- Step 1 (Theorem 7.2.2): we formulate a weaker version of the realization problem which admits a unique solution for any given pre-Lie algebra.
- Step 2 (Theorem 7.2.4): we show that the solution of the weak realization problem is a solution of the complete realization problem if and only if the Jacobi identity is satisfied.

Two nice features of the method are:

- Step 1 produces an explicit formula for the solution.
- Step 2 gives an explicit relation between the Maurer-Cartan equation and the Jacobi identity. Loosely speaking, one is the derivative of the other.

Similar Phenomenon: Poisson Realizations There is a striking similarity between the phenomenon we just observed and a phenomenon that occurs in the story of symplectic realizations of Poisson manifolds. Recall that a Poisson manifold (M, π) is a manifold M equipped with a bivector π which satisfies the Poisson equation $[\pi, \pi] = 0$ (of course, the Poisson equation is equivalent to the condition that the induced Poisson bracket satisfies the Jacobi identity). A symplectic realization of a Poisson manifold (M, π) is a symplectic manifold (S, ω) together with a surjective submersion $p : S \to M$ that satisfy the equation

$$dp(\omega^{-1}) = \pi. \tag{7.3}$$

It was shown in [12] that for any Poisson manifold (M, π), a symplectic realization is explicitly given by the cotangent bundle T^*M equipped with the symplectic form

$$\omega = \int_0^1 (\varphi_t)^* \omega_{\text{can}} dt \tag{7.4}$$

together with the projection $p : T^*M \to M$. Here, ω_{can} is the canonical symplectic form and φ_t is the flow associated with a choice of a contravariant spray on T^*M. See [12] for more details.

As in the realization problem of Lie algebras, we make the following observation: neither Equation (7.3) nor Formula (7.4) depend on the Poisson equation; they make perfect sense when replacing π with any bivector. And as before, there is the natural question as to the precise role of the Poisson equation in the existence of symplectic realizations, a question which was raised in [12] (see last paragraph of the paper).

An explicit relation between the symplectic realization equation and the Maurer-Cartan equation was observed by Alan Weinstein [71] in his pioneering work on Poisson manifolds. Weinstein showed that, locally, (7.3) is equivalent to a Maurer-Cartan equation associated with an infinite dimensional Lie algebra, and exploited this fact to prove the existence of local symplectic realizations by using a heuristic argument to solve this Maurer-Cartan equation, producing an explicit local solution of the type (7.4).

In Section 7.4, we apply our method to solve the Maurer-Cartan equation which Weinstein formulated. As with Lie algebras, we do this by identifying a weaker version of the equation that admits a unique solution given any bivector, not necessarily Poisson, and proceed to show that the solution is a local symplectic realization if and only if the bivector satisfies the Poisson equation. We obtain an explicit relation between the Poisson equation and the symplectic realization condition, thus pinpointing the role of the Poisson equation in the problem of existence of local symplectic realizations.

The Lie Algebroid Case In addition to local symplectic realizations, we believe that our method can be adapted to various other situations which generalize or resemble the classical Lie algebra case. One important generalization, which we treat in Section 7.4, is the realization problem of a Lie algebroid. Although extra difficulties do arise, it is remarkable that the procedure continues to work in this case, despite the fact that the simple to handle bilinear bracket of a Lie algebra is replaced by a more cumbersome bi-differential operator. This is largely facilitated by the presence of certain flows known as infinitesimal flows that are associated with time-dependent sections of the Lie algebroid.

As we noted in "Step 1" above, our method produces an explicit solution. In the Lie algebra case, this is the well known Formula (7.2), whereas the formula we obtain in the Lie algebroid case does not appear in the literature to the best of our knowledge (see Theorem 7.4.4). Having this explicit formula at hand can prove to be useful; in particular, one can attempt to use it to explicitly integrate Lie algebroids locally (as an indication of feasibility, we refer the reader to [9] where a symplectic realization of a Poisson manifold was used to integrate the associated Lie algebroid to a local symplectic groupoid; see also the discussion in last paragraph of Section 7.4).

7.2 The Lie Algebra Case

In this section, we present the 2-step method for solving the realization problem for a Lie algebra which was outlined in the introduction. Let us first recall the necessary definitions.

Definition 7.2.1. *A **pre-Lie algebra** is a vector space* $\mathfrak{g}$ *equipped with an antisymmetric bilinear map* $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$*. A **Lie algebra** is a pre-Lie algebra that satisfies the Jacobi identity:*

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0, \qquad \forall\, x,y,z\in\mathfrak{g}.$$

Associated with a pre-Lie algebra is the **adjoint map** $\mathrm{ad}:\mathfrak{g}\to\mathrm{End}(\mathfrak{g}),\ \mathrm{ad}_x(y)=[x,y]$, and the **Jacobiator**

$$\mathrm{Jac}\in\mathrm{Hom}(\Lambda^3\mathfrak{g},\mathfrak{g}), \qquad \mathrm{Jac}(x,y,z)=[[x,y],z]+[[y,z],x]+[[z,x],y]. \tag{7.5}$$

The space of $\mathfrak{g}$-valued differential forms on $\mathfrak{g}$ is denoted by $\Omega^*(\mathfrak{g};\mathfrak{g})$. This space is equipped with the de Rham differential $d:\Omega^*(\mathfrak{g};\mathfrak{g})\to\Omega^{*+1}(\mathfrak{g};\mathfrak{g})$ and with a bracket, $[\cdot,\cdot]:\Omega^p(\mathfrak{g};\mathfrak{g})\times\Omega^q(\mathfrak{g};\mathfrak{g})\to\Omega^{p+q}(\mathfrak{g};\mathfrak{g})$, that plays the role of the wedge product on $\mathfrak{g}$-valued forms and is defined by the analogous formula:

$$[\omega,\eta](X_1,...,X_{p+q})=\sum_{\sigma\in S_{p,q}}\mathrm{sgn}(\sigma)[\omega(X_{\sigma(1)},...,X_{\sigma(p)}),\eta(X_{\sigma(p+1)},...,X_{\sigma(p+q)})],\tag{7.6}$$

where $S_{p,q}$ is the set of (p,q)-shuffles.

Given any open subset $U\subset\mathfrak{g}$, we also have the space of $\mathfrak{g}$-valued forms $\Omega^*(U;\mathfrak{g})$ on U equipped with a differential and a bracket, defined in the same manner. Given any $\phi\in\Omega^1(U;\mathfrak{g})$, the **Maurer-Cartan 2-form** associated with ϕ is defined by:

$$\mathrm{MC}_\phi:=d\phi+\frac{1}{2}[\phi,\phi]\in\Omega^2(U;\mathfrak{g}),$$

and the **Maurer-Cartan equation** is

$$\mathrm{MC}_\phi=0,$$

or more explicitly,

$$(\mathrm{MC}_\phi)_x(y,z)=0,\qquad\forall\, x\in U,\ y,z\in\mathfrak{g}.$$

Note that in the last equation, and throughout the paper, we identify the tangent spaces of a vector space with the vector space itself without further mention.

Recall the **realization problem for Lie algebras**: find a 1-form $\phi\in\Omega^1(U;\mathfrak{g})$ on some open neighborhood $U\subset\mathfrak{g}$ of the origin such that ϕ is pointwise an isomorphism and satisfies the Maurer-Cartan equation.

We now present our method for solving this realization problem.

Step 1: We show that a weaker version of the realization problem admits a solution for any given pre-Lie algebra. We accomplish this by imposing a boundary condition which transforms the equation into a simple ODE that can be easily solved.

Theorem 7.2.2. *Given any pre-Lie algebra $\mathfrak{g}$, the equation*

$$(MC_\phi)_x(x,y)=0,\qquad\forall\, x,y\in\mathfrak{g}\tag{7.7}$$

admits a solution in $\Omega^1(\mathfrak{g};\mathfrak{g})$ which is pointwise an isomorphism at the origin (and thus on some open neighborhood of the origin). Moreover, if we impose the boundary condition

$$\phi_x(x)=x,\qquad\forall x\in\mathfrak{g},\tag{7.8}$$

then the solution is unique and is given by the following formula:

$$\phi_x(y)=\int_0^1 e^{-tad_x}\,y\,dt.\tag{7.9}$$

Remark 7.2.3. To get a "geometric feel" of the equations, note that (7.7) is the restriction of the Maurer-Cartan equation to all two dimensional subspaces of $\mathfrak{g}$, and (7.8) is the condition that ϕ restricts to the identity on all one-dimensional subspaces. $\diamondsuit$

Proof. First note that (7.8) implies that $\phi_0 = \mathrm{id}$. In particular, ϕ is pointwise an isomorphism at the origin.

Let $\phi \in \Omega^1(\mathfrak{g}; \mathfrak{g})$ be a solution of (7.7) and (7.8). We will show that ϕ must be of the form (7.9), which implies uniqueness. Conversely, as we explain at the end of the proof, reading the steps in the reverse direction implies that (7.9) is a solution, thus proving existence.

Note that by linearity, (7.7) and (7.8) are equivalent to $(\mathrm{MC}_\phi)_{tx}(x, y) = 0$ and $\phi_{tx}(x) = x$ for all $t \in (0, 1)$ and $x, y \in \mathfrak{g}$. In particular, by continuity, this implies that

$$(\mathrm{MC}_\phi)_0(x, y) = 0 \quad \text{and} \quad \phi_0(x) = x \quad \forall\, x, y \in \mathfrak{g}. \tag{7.10}$$

Fix $x, y \in \mathfrak{g}$. The solution ϕ satisfies

$$(d\phi + \frac{1}{2}[\phi, \phi])_{tx}(x, ty) = 0, \tag{7.11}$$

for all $t \in (0, 1)$.

To compute $(d\phi)_{tx}(x, ty)$, consider the map $f : (0, 1) \times (-\delta, \delta) \to \mathfrak{g},\ f(t, \epsilon) = t(x + \epsilon y)$.

$$\begin{aligned}
(d\phi)_{tx}(x, ty) &= (f^* d\phi)_{(t,0)}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \epsilon}) = (df^*\phi)_{(t,0)}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \epsilon}) \\
&= \frac{\partial}{\partial t}\Big((f^*\phi)(\frac{\partial}{\partial \epsilon})\Big)\Big|_{(t,0)} - \frac{\partial}{\partial \epsilon}\Big((f^*\phi)(\frac{\partial}{\partial t})\Big)\Big|_{(t,0)} \\
&= \frac{\partial}{\partial t}\big(\phi_{t(x+\epsilon y)}(ty)\big)\big|_{(t,0)} - \frac{\partial}{\partial \epsilon}\big(\phi_{t(x+\epsilon y)}(x + \epsilon y)\big)\big|_{(t,0)} \\
&= \frac{\partial}{\partial t}\big(\phi_{tx}(ty)\big) - y,
\end{aligned}$$

where, in the last equality, we have used that (7.8) implies $\phi_{t(x+\epsilon y)}(x + \epsilon y) = x + \epsilon y$.

To compute $(\frac{1}{2}[\phi, \phi])_{tx}(x, ty)$, we use (7.8) again:

$$(\frac{1}{2}[\phi, \phi])_{tx}(x, ty) = [\phi_{tx}(x), \phi_{tx}(ty)] = [x, \phi_{tx}(ty)] = \mathrm{ad}_x(\phi_{tx}(ty)).$$

Thus for a ϕ that satisfies (7.8), (7.11) is equivalent to

$$\frac{\partial}{\partial t}(\phi_{tx}(ty)) - y + \mathrm{ad}_x(\phi_{tx}(ty)) = 0,$$

which is equivalent to

$$\frac{\partial}{\partial t}\big(e^{t\ \mathrm{ad}_x}\ \phi_{tx}(ty)\big) = e^{t\ \mathrm{ad}_x}\ y. \tag{7.12}$$

Integrating from 0 to t', we obtain

$$\phi_{t'x}(t'y) = \int_0^{t'} e^{(t-t')\ \mathrm{ad}_x}\ y\ dt = \int_0^1 e^{-t\ \mathrm{ad}_{t'x}}\ (t'y)\ dt. \tag{7.13}$$

Setting $t' = 1$ proves that ϕ coincides with (7.9) .

Next, we show that (7.9) defines a solution. Note that $\phi_x(x) = \int_0^1 e^{-t\ ad_x}\ x\ dt = \int_0^1 x\ dt = x$, and thus (7.8) is satisfied. Equation (7.9) is equivalent to (7.13), which is a solution of (7.12), and since ϕ satisfies (7.8), it is a solution of (7.11). In particular, setting $t = 1$ implies that $(\mathrm{MC}_\phi)_x(x, y) = 0$, and thus (7.7) is satisfied. □

Step 2: By obtaining explicit equations relating the Maurer-Cartan 2-form with the Jacobiator, we show that the solution obtained in the previous step is a solution of the Maurer-Cartan equation if and only if the Jacobiator vanishes.

Theorem 7.2.4. *Let $\mathfrak{g}$ be a pre-Lie algebra and let $\phi \in \Omega^1(\mathfrak{g}; \mathfrak{g})$ be the solution of* (7.7) *and* (7.8)*. Then,*

$$MC_\phi = 0 \quad \Longleftrightarrow \quad \mathit{Jac} = 0,$$

or, more precisely,

$$\mathit{Jac}(x, y, z) = -3\tfrac{d}{dt}(MC_\phi)_{tx}(y, z)\big|_{t=0}, \tag{7.14}$$

$$(MC_\phi)_x(y, z) = -\int_0^1 e^{(t-1)ad_x}\ \mathit{Jac}(x, \phi_{tx}(ty), \phi_{tx}(tz))\ dt. \tag{7.15}$$

Proof. Equations (7.14) and (7.15) imply that $MC_\phi = 0$ if and only if Jac $= 0$. Let us derive these equations. Fix $x, y, z \in \mathfrak{g}$. We will compute

$$d(\mathrm{MC}_\phi)_{tx}(x, ty, tz), \tag{7.16}$$

with $t \in (0, 1)$, in two different ways.

1) Consider the map $f : (0, 1) \times (-\delta, \delta)^2 \to \mathfrak{g},\ f(t, \epsilon, \epsilon') = t(x + \epsilon y + \epsilon' z)$, then

$$\begin{aligned}(d\mathrm{MC}_\phi)_{tx}(x, ty, tz) &= (f^* d\mathrm{MC}_\phi)_{(t,0,0)}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial \epsilon'}) \\ &= (df^*\mathrm{MC}_\phi)_{(t,0,0)}(\frac{\partial}{\partial t}, \frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial \epsilon'}) \\ &= \frac{d}{dt}(\mathrm{MC}_\phi)_{tx}(ty, tz).\end{aligned}$$

In the last equality, terms containing $(\mathrm{MC}_\phi)_{tx}(x, ty)$ and $(\mathrm{MC}_\phi)_{tx}(x, tz)$ vanish by (7.7).

2) On the other hand,

$$\begin{aligned}
&(d\mathrm{MC}_\phi)_{tx}(x,ty,tz)\\
&= (d\frac{1}{2}[\phi,\phi])_{tx}(x,ty,tz)\\
&= ([d\phi,\phi])_{tx}(x,ty,tz)\\
&= [(d\phi)_{tx}(x,ty),\phi_{tx}(tz)] + [(d\phi)_{tx}(tz,x),\phi_{tx}(ty)] + [(d\phi)_{tx}(ty,tz),\phi_{tx}(x)]\\
&= -[[x,\phi_{tx}(ty)],\phi_{tx}(tz)] - [[\phi_{tx}(tz),x],\phi_{tx}(ty)]\\
&\qquad\qquad\qquad + [(\mathrm{MC}_\phi)_{tx}(ty,tz) - [\phi_{tx}(ty),\phi_{tx}(tz)],x]\\
&= -[x,(\mathrm{MC}_\phi)_{tx}(ty,tz)] - \mathrm{Jac}(x,\phi_{tx}(ty),\phi_{tx}(tz)).
\end{aligned}$$

In the fourth equality, we have used (7.7) and (7.8). In particular, (7.8) implies that $(d\phi)_{tx}(x,y) + [\phi_{tx}(x),\phi_{tx}(y)] = 0$ and $(d\phi)_{tx}(x,z) + [\phi_{tx}(x),\phi_{tx}(z)] = 0$.

Equation (7.16) becomes

$$\mathrm{Jac}(x,\phi_{tx}(ty),\phi_{tx}(tz)) = -(\frac{d}{dt} + \mathrm{ad}_x)(\mathrm{MC}_\phi)_{tx}(ty,tz),$$

or equivalently,

$$e^{t\ \mathrm{ad}_x}\ \mathrm{Jac}(x,\phi_{tx}(ty),\phi_{tx}(tz)) = -\frac{d}{dt}\left(e^{t\ \mathrm{ad}_x}\ (\mathrm{MC}_\phi)_{tx}(ty,tz)\right). \tag{7.17}$$

Integrating from 0 to 1 produces (7.15), while multiplying both sides of the equation by $\frac{1}{t^2}$, taking the limit of t to 0 and using the fact that $(\mathrm{MC}_\phi)_0(y,z) = 0$ (see (7.10)) produces (7.14). □

Remark 7.2.5. This 2-step method was inspired by the method used in [67] (see sections 1.3-1.5) to compute the differential of the exponential map of a Lie group and to derive the Baker-Campbell-Hausdorff formula of a Lie algebra. ♢

7.3 The Case of Local Symplectic Realizations of Poisson Structures

In this section, we apply the method from the previous section to the problem of existence of symplectic realizations of Poisson manifolds. The role of the Poisson equation becomes manifest in the same way that the role of the Jacobi identity was made manifest in the Lie algebra case.

Definition 7.3.1. *A* ***pre-Poisson manifold*** (M,π) *is a manifold* M *together with a choice of a bivector field* $\pi \in \mathfrak{X}^2(M)$*. A* ***Poisson manifold*** (M,π) *is a pre-Poisson manifold with the extra condition that* π *satisfies the* ***Poisson equation*** $[\pi,\pi] = 0$ *(where* $[\cdot,\cdot]$ *is the Schouten-Nijenhuis bracket).*

Equivalently, a pre-Poisson manifold is a manifold M equipped with an $\mathbb{R}$-bilinear antisymmetric operation $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ (called “the Poisson bracket") that satisfies the Leibniz identity, $\{fg,h\} = f\{g,h\} + \{f,h\}g$ for all $f,g,h \in C^\infty(M)$. A bivector π induces a bracket by $\{f,g\}(m) = \pi_m(df,dg)$ for all $m \in M$, $f,g \in C^\infty(M)$, and vice versa. The Poisson equation is equivalent to the Jacobi identity, i.e. to the condition Jac $= 0$, where Jac is the Jacobiator associated with $\{,\}$ (defined as in the previous section).

By the Leibniz identity, a function $f \in C^\infty(M)$ induces a vector field $X_f \in \mathfrak{X}(M)$, the **Hamiltonian vector field** associated with f, by the condition $X_f(g) = \{f,g\}$ for all $g \in C^\infty(M)$, or equivalently, $X_f(g) = \pi(df,dg)$ for all $g \in C^\infty(M)$.

Poisson manifolds can be localized, i.e. if (M,π) is a Poisson manifold and $U \subset M$ is an open subset, then $(U, \pi|_U)$ is a Poisson manifold.

A **symplectic realization** of a Poisson manifold (M,π) is a symplectic manifold (S,ω) together with a surjective submersion $p : S \to M$ such that p is a Poisson map, i.e. the bivector ω^{-1} induced by the symplectic form ω is p-projectable to the bivector π, that is to say,

$$dp(\omega^{-1}) = \pi.$$

A **local symplectic realization** of (M,π) around a point $m \in M$ is a symplectic realization of $(U, \pi|_U)$, where U is some open neighborhood of m.

In the problem of existence of local symplectic realizations it is enough to consider Poisson manifolds of the type $(\mathcal{O},\pi)$, where

$$\mathcal{O} \subset V$$

is an open subset of a vector space V. The following proposition was proven by Alan Weinstein in Section 9 of [71] (to be more precise, Weinstein proved the proposition for the case that $(\mathcal{O},\pi)$ is a Poisson manifold; however, the arguments do not rely on the Jacobi identity and the proposition also holds for the case that $(\mathcal{O},\pi)$ is a pre-Poisson manifold).

Proposition 7.3.2. *Let $(\mathcal{O},\pi)$ be a pre-Poisson manifold. Let $\phi \in \Omega^1(V^*; C^\infty(\mathcal{O}))$ be defined by*

$$\phi_\xi(\zeta) = \int_0^1 (\varphi_{X_\xi}^{-t})^* \zeta \, dt, \qquad \forall \xi, \zeta \in V^*. \tag{7.18}$$

Here ξ and ζ are interpreted as linear functionals on V, X_ξ is the corresponding Hamiltonian vector field and φ_{X_ξ} its flow.

Let $\tilde{\phi} \in \Omega^1(\mathcal{O} \times V^*)$ be the induced 1-form on $\mathcal{O} \times V^* = T^*\mathcal{O}$ defined by $\tilde{\phi}_{(x,\xi)}(y,\zeta) := \phi_\xi(\zeta)(x)$.

Then, the 2-form $d\tilde{\phi}$ is symplectic on some neighborhood $U \subset \mathcal{O} \times V^*$ of the zero-section, and, writing $p : \mathcal{O} \times V^* \to \mathcal{O}$ for the projection,

$$p\big|_U : (U, d\tilde{\phi}) \to (\mathcal{O},\pi) \ \text{ is a symplectic realization} \quad \Longleftrightarrow \quad d\phi + \frac{1}{2}\{\phi,\phi\} = 0.$$

Remark 7.3.3. The 1-form ϕ defined by (7.18) and the induced 1-form $\tilde{\phi}$ are only well-defined on some open neighborhood of the zero section of $\mathcal{O} \times V^*$, namely on all points (x, ξ) such that $\varphi_{X_\xi}(x)$ is defined up to time 1. This does not pose a problem, since, in the end, we are only interested in the symplectic form $d\tilde{\phi}$ in some neighborhood of the zero section. $\diamondsuit$

Weinstein's remarkable observation was that the symplectic realization condition can be locally rephrased as a Maurer-Cartan equation. This equation lives in the space

$$\Omega^*(V^*; C^\infty(\mathcal{O})),$$

consisting of differential forms with values in $C^\infty(\mathcal{O})$, where $\phi \in \Omega^1(V^*; C^\infty(\mathcal{O}))$ is smooth if the map $\mathcal{O} \times V^* \to \mathbb{R}$, $(x, \xi) \mapsto \phi_\xi(\zeta)(x)$, is smooth for all $\zeta \in V^*$, and similarly for higher degree forms. This space is equipped with the de Rham differential d defined as usual, and a bracket $\{,\}$ defined as in (7.6) (with the Lie bracket replaced by the Poisson bracket); thus, one can make sense of the Maurer-Cartan 2-form associated with a 1-form $\phi \in \Omega^1(V^*; C^\infty(\mathcal{O}))$:

$$\mathrm{MC}_\phi := d\phi + \frac{1}{2}\{\phi, \phi\} \in \Omega^2(V^*; C^\infty(\mathcal{O})).$$

Weinstein proceeded to show that if $(\mathcal{O}, \pi)$ is a Poisson manifold, then the 1-form given by (7.18) satisfies the Maurer-Cartan equation, thus proving the existence of local symplectic realizations. Of course, the fact that the Poisson bracket satisfies the Jacobi identity is used in the proof, but its precise role is somewhat obscure, appearing as a "mere step" in the calculation (see [71], p. 547).

The following two theorems shed further light on the role of the Jacobi identity as an obstruction in this problem. The first of the two theorems, an analog of "Step 1" of the previous section, demonstrates how close the 2-form $d\tilde{\phi}$, induced by (7.18), is from being a symplectic realization, regardless of the Jacobi identity.

Theorem 7.3.4. *Let $(\mathcal{O}, \pi)$ be a pre-Poisson manifold. The 1-form $\phi \in \Omega^1(V^*; C^\infty(\mathcal{O}))$ defined by* (7.18) *satisfies the equation*

$$(MC_\phi)_\xi(\xi, \zeta) = 0, \qquad \forall\, \xi, \zeta \in V^*. \tag{7.19}$$

Moreover, it is the unique solution of (7.19) *together with the boundary condition*

$$\phi_\xi(\xi) = \xi, \quad \forall\, \xi \in V^*. \tag{7.20}$$

Proof. The proof is essentially the same as the proof of Theorem 7.2.2. One must only make the following replacements:

- $\mathfrak{g}$ with V^* (and acccordingly x, y with ξ, ζ),

- the Lie bracket $[,]$ with the Poisson bracket $\{,\}$,
- $e^{t\ \mathrm{ad}_\xi}$ with $(\varphi^t_{X_\xi})^*$.

While making the last of the three replacements, one notes that derivatives of matrix valued functions of t become derivatives of flows. □

The next theorem, an analog of "Step 2" of the previous section, gives an explicit relation between Jac and MC_ϕ. This translates to a precise relation between the failure of the Poisson equation and the failure of $d\tilde{\phi}$ from being a symplectic realization. Of course, it follows that if the Poisson equation is satisfied, then $d\tilde{\phi}$ is a symplectic realization.

Theorem 7.3.5. *Let $\phi \in \Omega^1(V^*; C^\infty(\mathcal{O}))$ be a solution to* (7.19) *and* (7.20)*, then*

$$\mathit{Jac} = 0 \iff MC_\phi = 0,$$

or more precisely,

$$\begin{aligned}\mathit{Jac}(\xi,\zeta,\eta) &= -3\frac{d}{dt}(\mathit{MC}_\phi)_{t\xi}(\zeta,\eta)\big|_{t=0},\\ (\mathit{MC}_\phi)_\xi(\zeta,\eta) &= -\int_0^1 (\varphi^{t-1}_{X_\xi})^*\ \mathit{Jac}(\xi,\phi_{t\xi}(t\zeta),\phi_{t\xi}(t\eta))\ dt.\end{aligned}$$

Proof. The proof is essentially the same as the proof of Theorem 7.2.4 after making the necessary adjustments as in the proof of the previous theorem and using the fact that, by the Leibniz identity, the vanishing of the Jacobiator on linear functions implies that it vanishes. □

Remark 7.3.6. Theorems 7.2.2 and 7.2.4 are in fact special cases of theorems 7.3.4 and 7.3.5. Recall that a linear Poisson structure on the vector space $\mathfrak{g}^*$ is a Poisson bracket on $C^\infty(\mathfrak{g}^*)$ satisfying the property that it restricts to a Lie bracket on the linear functions $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$. This defines a one-to-one correspondence between linear Poisson structures on $\mathfrak{g}^*$ and Lie algebra structures on $\mathfrak{g}$. In the case of linear Poisson structures, the Hamiltonian vector field on $\mathfrak{g}^*$ associated with an element $x \in \mathfrak{g} = (\mathfrak{g}^*)^*$ is simply the transpose $(\mathrm{ad}_x)^*$ of the linear map $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$. The flow of $(\mathrm{ad}_x)^*$ is the transpose of the linear map $e^{t\ \mathrm{ad}_x}$, and the pullback by the flow is precisely $e^{t\ \mathrm{ad}_x}$. This implies that the solution (7.18) takes values in $\mathfrak{g} \subset C^\infty(\mathfrak{g}^*)$, and it follows that theorems 7.3.4 and 7.3.5 for linear Poisson structures coincide with theorems 7.2.2 and 7.2.4. ♢

7.4 The Lie Algebroid Case

In this section, we generalize our method from the Lie algebra case to the Lie algebroid case. We begin by recalling Cartan's realization problem for Cartan pairs in the case where the Cartan pair is a transitive Lie algebroid, and generalizing it to arbitrary Lie algebroids. We then recall some basic tools that will be necessary in the Lie algebroid setting, after which we state and prove theorems 7.4.4 and 7.4.6, which generalize theorems 7.2.2 and 7.2.4. We end the section by remarking on the possibility of extending this method to solve Cartan's realization problem and relating the Lie algebroid case with the Poisson case that was discussed in the previous section.

The Realization Problem for Lie Algebroids In Section 6.6, we stated the realization problem for Cartan pairs $(A, \mathfrak{g})$, Problem 6.6.11. In this section, we consider the case $\mathfrak{g} = 0$, the case in which the Cartan pair reduces to a transitive Lie algebroid A.

Problem 7.4.1. *(realization problem for transitive Lie algebroids) Let $\pi : A \to M$ be a transitive Lie algebroid. Find a manifold P, a surjective submersion $I : P \to M$ and an anchored 1-form $\Omega \in \Omega^1(P; I^*A)$ (i.e. $dI = \rho \circ \Omega$) such that*

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

and Ω is pointwise an isomorphism.

This problem can be reformulated in the following concise and conceptual manner. Rather than looking at the map $\Omega : TP \xrightarrow{\simeq} I^*A$, we look at its inverse, which we denote by

$$a : I^*A \xrightarrow{\simeq} TP.$$

From this point of view, the anchored condition together with the Maurer-Cartan equation precisely translate into the condition that the map a must define an action of the Lie algebroid A on the surjective submersion $I : P \to N$ (see Section 2.1). Moreover, requiring that a be an isomorphism should be understood as requiring that the action be both infinitesimally free (a be injective) and infinitesimally transitive (a be surjective). This perspective will also play a role in Chapter 8, where we discuss the notion of systatic space; in particular, see Proposition 8.3.1.

The realization problem for transitive Lie algebroids, phrased in terms of Lie algebroid actions, has a natural generalization to arbitrary Lie algebroids. Namely, given a Lie algebroid $\pi : A \to M$, one looks for a Lie algebroid action $a : I^*A \to TP$ on a surjective submersion. Only now, since A is not necessarily transitive, we can only require that the action be infinitesimally free. In this case, there is an induced foliation on P whose dimension is the rank of A. This problem, in turn, can be stated as a realization problem (by looking again at the "inverse" of a) as follows:

Problem 7.4.2. *(realization problem for Lie algebroids) Let $\pi : A \to M$ be a Lie algebroid. Find a manifold P together with a foliation $\mathcal{F}$ on P (an involutive distribution), a surjective submersion $I : P \to N$ and a foliated anchored 1-form $\Omega \in \Omega^1_{\mathcal{F}}(P; I^*A)$ (i.e. Ω is only defined on vectors tangent to $\mathcal{F}$) such that*

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

and Ω is pointwise an isomorphism.

Here we must be a bit careful in our claim that this is a generalization. If we solve this problem in the case that A is transitive, then we get slightly more, namely we will get a family of solutions of Problem 7.4.1, where the restriction of Ω to each leaf of $\mathcal{F}$ is a solution. However, there is the following subtlety: while $I : P \to M$ is required to be surjective, its restriction to a leaf may fail to be surjective onto M, and so, in general, we only obtain a family of local solutions to Problem 7.4.1. To obtain a global solution, we must also require for there to be at least one leaf on which the restriction of $I : P \to M$ is surjective. Typically, this occurs if suitable completeness conditions are satisfied.

In this section, we solve this realization problem for Lie algebroids, Problem 7.4.2. To further motivate the problem, we also explain its relation with the problem of integration of Lie algebroids to local Lie groupoids.

Basic Definitions Let us recall some basic definition and tools that will be essential. Recall from Definition 5.2.1 the notion of a pre-Lie algebroid. The notions of A-connections, A-paths, geodesics and infinitesimal flows that appear in the context of Lie algebroids remain unchanged when we give up on the Jacobi identity and pass to pre-Lie algebroids. These notions are recalled in Appendix 7.5.

Let $A \to M$ be a pre-Lie algebroid equipped with an A-connection $\overline{\nabla}$. Having fixed an A-connection, there is a unique maximal geodesic starting at every point $a \in A$, i.e. a maximal curve

$$g_a : I \to A \quad \text{ satisfying } \quad \overline{\nabla}_{g_a} g_a = 0, \;\; g_a(0) = a.$$

We denote its base curve by $\gamma_a : I \to M$. Let $A_0 \subset A$ be a neighborhood of the zero-section such that g_a is defined up to at least time 1 for all $a \in A_0$. On A_0 we have the **exponential map**

$$\exp = \exp_{\overline{\nabla}} : A_0 \to A, \qquad a \mapsto g_a(1),$$

and the **target map**

$$\tau = \tau_{\overline{\nabla}} := \pi \circ \exp : A_0 \to M.$$

Let $\Omega^*_\pi(A_0; \tau^* A)$ be the space of foliated differential forms (foliated with respect to the foliation by π-fibers) with values in $\tau^* A$. Throughout this section we will use the canonical identification between the vertical bundle of A_0 and the pullback of A to A_0, i.e. $T_a A_0 \cong A_x$ for all $a \in (A_0)_x$. Thus, given a 1-form $\phi \in \Omega^1_\pi(A_0; \tau^* A)$, we will write $\phi_a(b)$ with $a \in (A_0)_x,\ b \in A_x$.

A 1-form $\phi \in \Omega^1_\pi(A_0; \tau^* A)$ is said to be **anchored** if $\rho \circ \phi = d\tau$. Given a vector bundle connection $\nabla : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$, we can write the **Maurer-Cartan 2-form** associated with an anchored 1-form $\phi \in \Omega^1_\pi(A_0; \tau^* A)$,

$$\mathrm{MC}_\phi := d_{\tau^*\nabla}\phi + \frac{1}{2}[\phi, \phi]_\nabla \in \Omega^2_\pi(A_0; \tau^* A).$$

The differential-like map $d_{\tau^*\nabla}$ and bracket on $\Omega^*_\pi(A_0; \tau^* A)$ are defined in the usual way (see Section 7.5). The anchored condition implies that MC_ϕ is independent of the choice of connection (Proposition 7.5.2). The auxiliary connection ∇ should not be confused with the A-connection $\overline{\nabla}$, which is part of the data we fix.

Given any open subset $U \subset A_0$, we can also consider the space of forms $\Omega^*_\pi(U;\tau^*A)$ on U, defined in the same manner. We will solve the realization problem for Lie algebroids, Problem 7.4.2, by finding an anchored 1-form $\phi \in \Omega^1_\pi(U;\tau^*A)$ on some open neighborhood of the zero-section of A_0 such that ϕ is pointwise an isomorphism and satisfies the Maurer-Cartan equation:

$$\mathrm{MC}_\phi = 0. \tag{7.21}$$

Remark 7.4.3. We note that a solution of the Maurer-Cartan equation can also be interpreted as a Lie algebroid map in the following sense. A 1-form $\phi \in \Omega^1_\pi(A_0;\tau^*A)$ can be viewed as a vector bundle map from the Lie algebroid $T^\pi A_0 \to A_0$ (the vertical bundle, a Lie subalgebroid of $TA_0 \to A_0$) to the Lie algebroid $A \to M$ covering τ, the anchored condition on ϕ is equivalent to the vector bundle map commuting with the anchors, and ϕ satisfies the Maurer-Cartan equation if and only if the vector bundle map is a Lie algebroid map (see [11] or [18] for more details). From this point of view, the Maurer-Cartan equation is a special case of the *generalized Maurer-Cartan equation* for vector bundle maps between Lie algebroids which commute with the anchors studied in [18] (section 3.2). $\diamondsuit$

Solving the Realization Problem for Lie Algebroids As in the case of Lie algebras (see Section 7.1), one can find a solution to the realization problem by assuming that the Lie algebroid integrates to a Lie groupoid and pulling back the canonical Maurer-Cartan 1-form on the Lie groupoid by the exponential map. The resulting formula will not depend on the Lie groupoid, and one can verify directly that the formula is indeed a solution and, therefore, not have to require that the Lie algebroid be integrable.

Let us explain this in more detail. Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with source/target map s/t. The canonical Maurer-Cartan 1-form $\phi_{\mathrm{MC}} \in \Omega^1_s(\mathcal{G};t^*A)$ is a foliated differential 1-form on $\mathcal{G}$ (foliated with respect to the foliation by s-fibers) with values in t^*A. It is defined precisely as in the case of Lie groups:

$$(\phi_{\mathrm{MC}})_g = (dR_{g^{-1}})_g, \quad \forall\, g \in \mathcal{G}, \tag{7.22}$$

the difference being that the right multiplication map $R_{g^{-1}}$ is only defined on $s^{-1}(s(g))$. For this reason, the resulting form is foliated. The Maurer-Cartan form satisfies the anchored property $\rho((\phi_{\mathrm{MC}})_g(X)) = (dt)_g(X)$ and the Maurer-Cartan equation $d_{t^*\nabla}\phi_{\mathrm{MC}} + \frac{1}{2}[\phi_{\mathrm{MC}},\phi_{\mathrm{MC}}]_\nabla = 0$ (for more details, see [18], section 4).

The exponential map $\mathrm{Exp} := \mathrm{Exp}_{\overline{\nabla}} : A_0 \to \mathcal{G}$ on a Lie groupoid requires a choice of an A-connection $\overline{\nabla}$ on A, where A_0 is as above. Such a choice induces a normal connection on each s-fiber and the exponential map is then defined in the usual way. This choice of an A-connection also gives rise to an exponential on the Lie algebroid, as we saw above, and the two satisfy the following relations:

$$\exp(a) = (dR_{\mathrm{Exp}(a)^{-1}})_{\mathrm{Exp}(a)} \frac{d}{dt} \mathrm{Exp}(ta)\big|_{t=1}, \tag{7.23}$$

$$\pi \circ \exp = t \circ \mathrm{Exp}, \tag{7.24}$$

$$\pi = s \circ \mathrm{Exp}.$$

If we pull back the Maurer-Cartan form by the exponential map, the resulting form will be an element of $\Omega^1_\pi(A_0; \tau^* A)$. It will be anchored as a result of (7.24). It is now not difficult to verify that the fact that ϕ_{MC} satisfies the Maurer-Cartan equation on the Lie groupoid implies that $\mathrm{Exp}^* \phi_{\mathrm{MC}}$ satisfies the Maurer-Cartan equation on the Lie algebroid, i.e. satisfies (7.21).

In the following two theorems, we obtain a solution by taking a different path, namely by generalizing our method from section 7.2. The first theorem is the generalization of "Step 1": a weaker version of the realization problem which admits a unique solution for any pre-Lie algebroid. The theorem gives an explicit formula for a solution to the realization problem of Lie algebroids. In Corollary 7.4.5, we show that our solution coincides with $\mathrm{Exp}^* \phi_{\mathrm{MC}}$.

Theorem 7.4.4. *Let $A \xrightarrow{\pi} M$ be a pre-Lie algebroid equipped with an A-connection $\overline{\nabla}$, and let A_0 be a neighborhood of the zero section of A on which the exponential map $exp : A_0 \to A$ is defined. The equations*

$$(MC_\phi)_a(a, b) = 0, \qquad \forall x \in M,\ a \in (A_0)_x,\ b \in A_x, \tag{7.25}$$

$$\rho \circ \phi = d\tau, \tag{7.26}$$

admit a solution in $\Omega^1_\pi(A_0; \tau^* A)$ which is pointwise an isomorphism on a small enough neighborhood of the zero section of A_0. Moreover, if we impose the boundary condition

$$\phi_a(a) = exp(a), \qquad \forall a \in A_0, \tag{7.27}$$

then the solution is unique and can be described as follows: let $\xi : [0, 1] \times (-\delta, \delta) \times M \to A$ be a smooth map such that $\xi^t_\epsilon = \xi(t, \epsilon, \cdot)$ is a section of A and $\xi^t_\epsilon(\gamma_{a+\epsilon b}(t)) = g_{a+\epsilon b}(t)$ for all $(t, \epsilon) \in [0, 1] \times (-\delta, \delta)$, where g_a is the geodesic starting at $a \in A$ and γ_a its base path, and let ψ_{ξ_0} be the infinitesimal flow associated with the time dependent section ξ_0 (see Section 7.5). The solution is given by

$$\phi_a(b) = \int_0^1 \psi^{1,t}_{\xi_0} \frac{d}{d\epsilon}\Big|_{\epsilon=0} \xi^t_\epsilon(\gamma_a(t))\, dt. \tag{7.28}$$

Proof. Equation (7.27) implies that a solution ϕ is equal to the identity on the zero section of A and thus pointwise an isomorphism on a small enough neighborhood of the zero section.

Let $\phi \in \Omega^1_\pi(A_0; \tau^* A)$ be a solution of (7.25), (7.26) and (7.27). In this proof we show that ϕ must be given by (7.28). The remaining arguments are precisely as in the proof of Theorem 7.2.2.

By (7.27), $\phi_a(a) = \exp(a) = g_a(1)$ for all $a \in A_0$. This implies that $\phi_{ta}(ta) = g_{ta}(1) = tg_a(t)$, by using (7.39), and by linearity,

$$\phi_{ta}(a) = g_a(t), \tag{7.29}$$

for all $t \in (0,1)$. Equation (7.29) is thus equivalent to (7.27).

Fix $x \in M,\ a \in (A_0)_x$ and $b \in A_x$. Let ∇ be a vector bundle connection on A. Equation (7.25) implies that

$$(d_{\tau^*\nabla}\phi + \frac{1}{2}[\phi,\phi]_\nabla)_{ta}(a, tb) = 0, \tag{7.30}$$

for all $t \in (0,1)$. We will compute this equation for a fixed $t' \in (0,1)$.

To compute $(d_{\tau^*\nabla}\phi)_{t'a}(a, t'b)$, we consider the map $f : (0,1) \times (-\delta,\delta) \to (A_0)_x$, $f(t,\epsilon) = t(a+\epsilon b)$. The composition $\tau \circ f$ restricted to $\epsilon = 0$ is the curve $t \mapsto \tau(ta)$, which is precisely γ_a, the base curve of the geodesic g_a, and $\tau \circ f$ restricted to $t = t'$ is the curve $\gamma_\epsilon : (-\delta,\delta) \to M,\ \epsilon \mapsto \tau(t'(a+\epsilon b))$.

$$\begin{aligned}
(d_{\tau^*\nabla}\phi)_{t'a}(a,t'b) &= (f^* d_{\tau^*\nabla}\phi)_{(t',0)}(\frac{\partial}{\partial t},\frac{\partial}{\partial \epsilon}) = (d_{f^*\tau^*\nabla} f^*\phi)_{(t',0)}(\frac{\partial}{\partial t},\frac{\partial}{\partial \epsilon}) \\
&= (f^*\tau^*\nabla)_{\frac{\partial}{\partial t}}(f^*\phi)(\frac{\partial}{\partial \epsilon})\Big|_{(t',0)} - (f^*\tau^*\nabla)_{\frac{\partial}{\partial \epsilon}}(f^*\phi)(\frac{\partial}{\partial t})\Big|_{(t',0)} \\
&= (\nabla)_{\dot\gamma_a}\phi_{ta}(tb)\Big|_{t=t'} - (\nabla)_{\dot\gamma_\epsilon} g_{a+\epsilon b}(t')\Big|_{\epsilon=0}
\end{aligned}$$

In the second equality, we used Lemma 7.5.1 to commute the pullback with $d_{\tau^*\nabla}$, and in the last equality, we used (7.29) which is equivalent to (7.27). The two terms in the final expression are covariant derivatives of paths, which make sense because γ_a is the base curve of the curve $t \mapsto \phi_{ta}(tb)$ and γ_ϵ is the base curve of $\epsilon \mapsto g_{a+\epsilon b}(t')$.

To compute $(\frac{1}{2}[\phi,\phi]_\nabla)_{t'a}(a,t'b)$, let ξ be the map as in the statement of the theorem and let η be a time dependent section of A satisfying $\eta^t(\gamma_a(t)) = \phi_{ta}(tb)$.

$$\begin{aligned}
(\frac{1}{2}[\phi,\phi]_\nabla)_{t'a}(a,t'b) &= [\xi_0^{t'}, \eta^{t'}]_\nabla(\gamma_a(t')) \\
&= [\xi_0^{t'}, \eta^{t'}](\gamma_a(t')) - \nabla_{\rho(\xi_0^{t'})}\eta^{t'}(\gamma_a(t')) + \nabla_{\rho(\eta^{t'})}\xi_0^{t'}(\gamma_a(t')) \\
&= \frac{d}{dt}\Big|_{t=t'} \psi_{\xi_0}^{t',t}\eta^{t'}(\gamma_a(t)) - \nabla_{\dot\gamma_a}\eta^{t'}(\gamma_a(t')) + \nabla_{\dot\gamma_\epsilon}\xi_0^{t'}(\gamma_a(t'))
\end{aligned}$$

In the last equality, we used the defining property (7.38) of the infinitesimal flow for the first term, $\rho(\xi_0^{t'}(\gamma_a(t'))) = \rho(g_a(t')) = \dot\gamma_a(t')$ for the second term, and

$$\begin{aligned}
\rho(\eta^{t'}(\gamma_a(t'))) &= \rho(\phi_{t'a}(t'b)) = (d\tau)_{t'a}(t'b) = d(\pi \circ \exp)_{t'a}(t'b) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(\pi(exp(t'a + \epsilon t'b))) = \frac{d}{d\epsilon}\Big|_{\epsilon=0}(\pi(g_{t'(a+\epsilon b)}(1))) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0}(\pi(t' g_{(a+\epsilon b)}(t'))) = \dot\gamma_\epsilon(0)
\end{aligned}$$

for the third term, where the anchored property (7.26) was used in the second equality.

Thus for ϕ that satisfies (7.27), (7.30) is equivalent to

$$\frac{d}{dt}\Big|_{t=t'}\psi^{t',t}_{\xi_0}\eta^{t'}(\gamma_a(t)) + \frac{d}{dt}\Big|_{t=t'}\eta^t(\gamma_a(t')) = \frac{d}{d\epsilon}\Big|_{\epsilon=0}\xi^{t'}_\epsilon(\gamma_a(t')),$$

where we used the characterization (7.37) of covariant derivatives of curves.

Applying $\psi^{1,t'}_{\xi_0}$ to both sides and using the product rule, the latter equation is equivalent to

$$\frac{d}{dt}\psi^{1,t}_{\xi_0}\eta^t(\gamma_a(t)) = \psi^{1,t}_{\xi_0}\frac{d}{d\epsilon}\Big|_{\epsilon=0}\xi^t_\epsilon(\gamma_a(t)).$$

Integrating t' from 0 to 1, and using the definition of η and the property $\psi^{1,1}_{\xi_0} = \mathrm{id}$, we obtain (7.28). □

Corollary 7.4.5. *The pullback of the canonical Maurer-Cartan form of a Lie groupoid by the exponential map Exp$^*\phi_{MC}$ is equal to the 1-form defined by* (7.28).

Proof. As we saw already in the text preceding the last theorem, the form $\mathrm{Exp}^*\phi_{\mathrm{MC}} \in \Omega^1_\pi(A_0;\tau^*A)$ is anchored and satisfies the Maurer-Cartan equation, and, in particular, it satisfies (7.25). Moreover, the initial condition (7.27) is satisfied since it is precisely the relation (7.23) when written out explicitly. The corollary now follows from the uniqueness assertion in the theorem. □

The second theorem is the generalization of "Step 2" from section 7.2. It shows that the solution from the previous theorem is indeed a solution of the realization problem.

Theorem 7.4.6. *Let $A \xrightarrow{\pi} M$ be a pre-Lie algebroid equipped with an A-connection $\overline{\nabla}$ and let $\phi \in \Omega^1_\pi(A_0;\tau^*A)$ be the solution of* (7.25), (7.26) *and* (7.27), *where A_0 is a neighborhood of the zero section of A on which the exponential map is defined. Choose A_0 to be small enough so that ϕ is pointwise an isomorphism. Then $MC_\phi = 0$ if and only if Jac $= 0$. Moreover,*

$$Jac(a,b,c) = -3\tfrac{d}{dt}\left(\psi^{0,t}_\xi(MC_\phi)_{ta}(b,c)\right)\big|_{t=0}, \tag{7.31}$$

$$(MC_\phi)_a(b,c) = -\int_0^1 \psi^{1,t}_\xi Jac(\tfrac{1}{t}g_a(t),\phi_{ta}(tb),\phi_{ta}(tc))\,dt, \tag{7.32}$$

where g_a is the geodesic starting at $a \in A$ and γ_a its base path, ξ is a time dependent section of A satisfying $\xi^t(\gamma_a(t)) = g_a(t)$ for all $t \in (0,1)$, and $\psi^{1,t}_\xi$ is the infinitesimal flow associated with the time dependent section ξ (see Section 7.5).

Remark 7.4.7. As we mentioned earlier, if A is transitive, then the restriction of the solution ϕ to any fiber $(A_0)_x$ ($x \in M$) gives a local solution to the realization problem for transitive Lie algebroids, Problem 7.4.1. The domain of the solution is the image of $\tau|_{(A_0)_x} : (A_0)_x \to M$, an open subset of M. To obtain a global solution, two things must be satisfied: 1) there must exist a fiber of A_0 for which the restriction of τ to the fiber is surjective, and 2) ϕ must be pointwise an isomorphism on the entire fiber. ◊

Proof. The proof goes along the same line as the proof of Theorem 7.2.4. As in Theorem 7.2.4, we compute

$$d_{\tau^*\nabla}(\mathrm{MC}_\phi)_{ta}(a, tb, tc) \tag{7.33}$$

in two different ways, where $t \in (0,1),\ x \in M,\ a \in (A_0)_x,\ y, z \in A_x$ and ∇ is some vector bundle connection on A:

1) Consider the map $f : (0,1) \times (-\delta, \delta)^2 \to \mathfrak{g},\ f(t, \epsilon, \epsilon') = t(a + \epsilon b + \epsilon' c)$. Recall that γ_a is the base curve of the geodesic g_a that satisfies $\gamma_a(t) = \tau(ta)$.

$$\begin{aligned}
(d_{\tau^*\nabla}\mathrm{MC}_\phi)_{ta}(a, tb, tc) &= (f^* d_{\tau^*\nabla}\mathrm{MC}_\phi)_{(t,0,0)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial \epsilon'}\right) \\
&= (d_{f^*\tau^*\nabla} f^*\mathrm{MC}_\phi)_{(t,0,0)}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial \epsilon}, \frac{\partial}{\partial \epsilon'}\right) \\
&= \nabla_{\dot\gamma_a}(\mathrm{MC}_\phi)_{ta}(tb, tc),
\end{aligned}$$

where the final expression is the covariant derivative of the curve $t \mapsto (\mathrm{MC}_\phi)_{ta}(tb, tc)$ covering γ_a.

2) Since $\phi \in \Omega^1_\pi(A_0; \tau^*A)$ is a pointwise isomorphism, it induces a linear map $\phi^{-1} : \Gamma(A) \to \mathfrak{X}(A_0)$. Let ξ be as in the statement of the theorem, let η_b and η_c be time dependent sections of A satisfying $\eta_b^t(\gamma_a(t)) = \phi_{ta}(tb)$ and $\eta_c^t(\gamma_a(t)) = \phi_{ta}(tc)$, and let σ be a time dependent section of A satisfying $\sigma^t(\gamma_a(t)) = (\mathrm{MC}_\phi)_{ta}(tb, tc)$. Let $\tilde a, \tilde b, \tilde c$ be time dependent vector fields on A_0 defined by $\tilde a^t = \phi^{-1}(\xi^t),\ \tilde b^t = \phi^{-1}(\eta_b^t),\ \tilde c^t = \phi^{-1}(\eta_c^t)$.

$$\begin{aligned}
&(d_{\tau^*\nabla}\mathrm{MC}_\phi)_{ta}(a, tb, tc) = (d_{\tau^*\nabla}\mathrm{MC}_\phi)(\tilde a, t\tilde b, t\tilde c)_{ta} \\
&= \nabla_{\dot\gamma_a}\sigma^t(\gamma_a(t)) - [[\xi^t, \eta_b^t], \eta_c^t]_{ta} - [[\eta_c^t, \xi^t], \eta_b^t]_{ta} + [\sigma^t - [\eta_b^t, \eta_c^t], \xi^t]_{ta} \\
&= \nabla_{\dot\gamma_a}\sigma^t(\gamma_a(t)) - \frac{d}{ds}\Big|_{s=t}\psi_\xi^{t,s}\sigma^t(\gamma_a(s)) - \mathrm{Jac}\left(\frac{1}{t}\exp(ta), \phi_{ta}(tb), \phi_{ta}(tc)\right).
\end{aligned}$$

The second equality is a slightly messy yet straightforward computation. It involves expanding MC_ϕ with respect to the chosen connection, using the choices we made above of time dependent sections, and using (7.25), (7.27) and (7.26). In particular, it is used that (7.25) implies that: $\phi_{ta}([\tilde a, \tilde b]) = [\xi^t, \eta_b^t]_{\gamma_a(t)},\ \phi_{ta}([\tilde a, \tilde c]) = [\xi^t, \eta_c^t]_{\gamma_a(t)}$. Furthermore, $(\mathrm{MC}_\phi)_{ta}(b, c) = -\phi_{ta}([\tilde b, \tilde c]) + [\eta_b^t, \eta_c^t]_{\gamma_a(t)}$. In the last equality, we express the bracket $[\xi^t, \sigma^t]$ using the infinitesimal flow, see (7.38).

After equating the two expressions obtained, using the characterization (7.37) of covariant derivatives of curves and applying $\psi_\xi^{1,t}$, (7.33) becomes

$$\psi_\xi^{1,t}\mathrm{Jac}\left(\frac{1}{t}\exp(ta), \phi_{ta}(tb), \phi_{ta}(tc)\right) = -\frac{d}{dt}\left(\psi_\xi^{1,t}(\mathrm{MC}_\phi)_{ta}(tb, tc)\right). \tag{7.34}$$

The remaining arguments are identical to Theorem 7.2.4. □

The Relation of our Method to Cartan's Realization Problem Let us comment on the possible application of our method to the general realization problem for Cartan pairs, Problem 6.6.11.

Our method consists of two steps. The first step, Theorem 7.4.4, does not rely on the Jacobi identity and can be applied to any Cartan pair. Namely, given a Cartan pair $(A, \mathfrak{g})$, one can apply the theorem on the pre-Lie algebroid A, thus obtaining a potential solution ϕ given by (7.28) that solves the partial Maurer-Cartan equation (7.25).

The second step, Theorem 7.4.6, tells us precisely how the Maurer-Cartan equation fails in terms of the failure of the Jacobi identity. Starting with a Cartan pair $(A, \mathfrak{g})$, we know that the Jacobiator of A vanishes modulo the vector subbundle $\mathfrak{g} \subset A$ and we need to show that the Maurer-Cartan expression MC_ϕ associated with the solution ϕ also vanishes modulo $\mathfrak{g}$. From (7.32), we see that this is the case if the infinitesimal flow $\psi_\xi^{1,t}$ preserves the vector subbundle $\mathfrak{g}$. The infinitesimal flow, in turn, is induced by our choice of an A-connection $\overline{\nabla}$ on A. Our method, therefore, reduces the realization problem to the geometric problem of finding an A-connection $\overline{\nabla}$ on A whose induced infinitesimal flow preserves the vector subbundle $\mathfrak{g}$, a problem which can be tackled with Lie theoretic methods. Let us point out one case in which this method works without any problem and for any connection: when

$$[\Gamma(\mathfrak{g}), \Gamma(A)] \subset \Gamma(\mathfrak{g}). \tag{7.35}$$

Indeed in this case the flows $\psi_\xi^{1,t}$ preserves $\mathfrak{g}$ and, hence, (7.32) implies:

Corollary 7.4.8. *Any Cartan pair satisfying* (7.35) *locally admits a realization.*

The Poisson Case vs. the Lie Algebroid Case Given the well known relations between Poisson manifolds and Lie algebroids, it is natural to wonder as to the relation between the instances of the Maurer-Cartan equation associated with these structures, i.e. as to the relation between Section 7.3 and Section 7.4 of this paper. Let us briefly touch upon this.

In one direction, any Lie algebroid $A \to M$ induces a Poisson structure on the total space of the dual vector bundle $A^* \to M$ known as a linear Poisson structure (see [47]). This generalizes the construction of a linear Poisson structure on the dual of a Lie algebra. At the level of the associated Maurer-Cartan equations, it is not hard to verify that, locally and under obvious identifications, the Maurer-Cartan equations as well as the solutions are one and the same on both sides of this correspondence. In particular, trivializing A and computing the 1-form (7.28) will produce the same result as obtained by computing the 1-form (7.18) associated with the induced trivialization of A^*. This is, of course, a generalization of the case of a Lie algebra which was discussed in Remark 7.3.6.

In the opposite direction, any Poisson manifold (M, π) induces a Lie algebroid structure on the cotangent bundle $T^*M \to M$, as originally shown in [9]. In that same paper, the authors prove that the local symplectic realization constructed by Weinstein in [71] (and discussed in section 7.3 above) has a canonically induced local symplectic groupoid structure on its total space whose associated Lie algebroid is (the restriction of) $T^*M \to M$. This same phenomenon occurs at the level of the Maurer-Cartan equations. Using the notation of Section 7.3, the local solution of the Maurer-Cartan equation associated with the Poisson manifold $(\mathcal{O}, \pi)$, with $\mathcal{O} \subset V$, induces a local solution to the Maurer-Cartan equation associated with the Lie algebroid $T^*\mathcal{O} = \mathcal{O} \times V^* \xrightarrow{\pi} \mathcal{O}$ by differentiation of the coefficients, or more precisely, by the map

$$\Omega^1(V^*; C^\infty(\mathcal{O})) \to \Omega^1_\pi((T^*\mathcal{O})_0; \tau^*(T^*\mathcal{O})), \quad \phi \mapsto \hat{\phi}, \tag{7.36}$$

with $\hat{\phi}_{x,\xi}(\zeta) = d(\phi_\xi(\zeta))_{\tau(x)}$ for all $x \in \mathcal{O},\ \xi, \zeta \in V^*$.

Note that while in the Lie algebroid case we are able to obtain a "wide" solution, i.e. on an open neighborhood of the zero section of $T^*M \to M$, in the Poisson case we only obtain a local one around a point in M. It would be interesting to further investigate the relation given by (7.36) to see if a "wide" solution of the Lie algebroid case induces a "wide" solution of the Poisson case, thus producing yet another proof for the existence of global symplectic realizations.

7.5 Appendix: The Maurer-Cartan Equation on a Lie Algebroid

In this section, various notions are recalled which were used in section 7.4 for the formulation of the Maurer-Cartan equation on a Lie algebroid and its solution. For more details, the reader is referred to [10]. Note that all the notions that appear here and that are presented in [10] do not require the Jacobi identity and are therefore valid for pre-Lie algebroids as they are for Lie algebroids.

Let $A \to M$ be a pre-Lie algebroid (see section 7.4 for the definition). An **A-connection** on a vector bundle $E \to M$ is an $\mathbb{R}$-bilinear map $\overline{\nabla} : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ satisfying the connection-like properties

$$\overline{\nabla}_{f\alpha} s = f\overline{\nabla}_\alpha s,\ \ \overline{\nabla}_\alpha(fs) = f\overline{\nabla}_\alpha s + \mathcal{L}_{\rho(\alpha)}(f)s,\ \ \forall \alpha \in \Gamma(A),\ s \in \Gamma(E),\ f \in C^\infty(M).$$

For the remainder of this section, let $A \to M$ be a pre-Lie algebroid equipped with an A-connection $\overline{\nabla}$. Note that there will be two different connections that will play a role in this section (and in section 7.4): an A-connection $\overline{\nabla}$ on A that is part of the data, and an auxiliary vector bundle connection ∇ on A that is used to write down the Maurer-Cartan equation globally, and which is not part of the data.

Time Dependent Sections A **time dependent section** ξ of A is a map $\xi : I \times M \to A,\ (t, x) \mapsto \xi^t(x)$ (with I some open interval), such that ξ^t is a section of A for all $t \in I$.

If $\nabla : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ is a vector bundle connection, then given a base curve $\gamma : I \to M$ and a curve $u : I \to A$ covering γ, the covariant derivative $(\nabla_{\dot\gamma} u)(t) = ((\gamma^*\nabla)_{\frac{\partial}{\partial t}} u)(t)$ can be characterized using time dependent sections as follows: choose a time dependent section ξ of A satisfying $\xi^t(\gamma(t)) = u(t)$ for all $t \in I$, then

$$(\nabla_{\dot\gamma} u)(t) = (\nabla_{\dot\gamma}\xi^t)(x) + \frac{d\xi^t}{dt}(x), \tag{7.37}$$

where $x = \gamma(t)$.

We will also use time dependent sections to express the bracket of a pre-Lie algebroid in a Lie derivative-like fashion, as one does for the bracket of vector fields. This involves the notiÖůÖůon of an infinitesimal flow. Let ξ be a time dependent section of A and $\rho(\xi)$ the corresponding time dependent vector field on M. Let $\varphi^{t,s}_{\rho(\xi)}$ denote the flow of $\rho(\xi)$ from time s to t. The **infinitesimal flow**,

$$\psi^{t,s}_{\xi} : A_x \to A_{\varphi^{t,s}_{\rho(\xi)}}, \qquad x \in M,$$

is the unique linear map satisfying the properties $\psi^{u,t}_{\xi} \circ \psi^{t,s}_{\xi} = \psi^{u,s}_{\xi}$, $\psi^{s,s}_{\xi} = \mathrm{id}$ and

$$\frac{d}{dt}\Big|_{t=s} \psi^{s,t}_{\xi}\alpha(\varphi^{t,s}_{\rho(\xi)}(x)) = [\xi^s, \alpha]_x, \qquad \forall \alpha \in \Gamma(A),\ x \in M.$$

If we define the pullback of sections by the infinitesimal flow as

$$(\psi^{t,s}_{\xi})^*(\alpha)(x) = \psi^{s,t}_{\xi}\alpha(\varphi^{t,s}_{\rho(\xi)}(x))$$

for all $\alpha \in \Gamma(A),\ x \in M$, then the previous equation can be expressed in the more familiar form

$$\frac{d}{dt}\Big|_{t=s} (\psi^{t,s}_{\xi})^*\alpha = [\xi^s, \alpha], \qquad \forall \alpha \in \Gamma(A). \tag{7.38}$$

For more on infinitesimal flows and their global counterparts, flows along invariant time dependent vector fields on Lie groupoids, see [10].

Geodesics An A**-path** is a curve $g : I \to A$ with base curve $\gamma : I \to M, \gamma(t) = \pi(g(t))$, such that

$$\rho(g(t)) = \dot{\gamma}(t), \qquad \forall t \in I.$$

Let g be an A-path with base curve γ, and let $u : I \to A$ be another curve covering γ. The **covariant derivative** of u with respect to g is the curve $\overline{\nabla}_g u : I \to A$, which is defined in analogy to the usual covariant derivative described above: choose a time dependent section ξ of A satisfying $\xi^t(\gamma(t)) = u(t)$ for all $t \in I$, then

$$(\overline{\nabla}_g u)(t) = (\overline{\nabla}_g \xi^t)(x) + \frac{d\xi^t}{dt}(x),$$

where $x = \gamma(t)$.

A **geodesic** is a curve $g : I \to A$ satisfying the geodesic equation $\overline{\nabla}_g g = 0$. Geodesics are A-paths. Given any point $a \in A$, there is a unique maximal geodesic $g_a : I_a \to A$ satisfying $g_a(0) = a$ with domain I_a. The base curve of g_a will be denoted by γ_a. Geodesics satisfy the following basic property:

$$g_{sa}(t) = s g_a(st), \qquad \forall\, a \in A,\ s, t \in \mathbb{R},\ t \in I_{sa}, \tag{7.39}$$

which can be easily verified by checking that the curve $t \mapsto s g_a(st)$ satisfies the geodesic equation and then by noting that by uniqueness it must be equal to g_{sa} since at time 0 it takes the value sa.

Let $A_0 \subset A$ be a neighborhood of the zero section such that g_a is defined up to time 1 for all $a \in A_0$. The **exponential map** is defined as $\exp : A_0 \to A,\ a \mapsto g_a(1)$. The point $\pi(\exp(a)) \in M$ will be called the **target** of a and $\tau = \pi \circ \exp : A_0 \to M$ the **target map**.

The Maurer-Cartan 2-Form Let $\Omega^*_\pi(A_0;\tau^*A)$ denote the space of foliated differential forms on A_0 (foliated with respect to the foliation by π-fibers) which take values in τ^*A.

Let $\nabla : \mathfrak{X}(M)\times\Gamma(A)\to\Gamma(A)$ be a vector bundle connection on A. The torsion $[\cdot,\cdot]_\nabla\in\mathrm{Hom}(\Lambda^2A,A)$ of ∇ is defined at the level of sections by

$$[\alpha,\beta]_\nabla=[\alpha,\beta]-\nabla_{\rho(\alpha)}\beta+\nabla_{\rho(\beta)}\alpha,\qquad\forall\alpha,\beta\in\Gamma(A),$$

and easily checked to be $C^\infty(M)$-linear in both slots. The torsion induces a bracket $[\cdot,\cdot]_\nabla : \Omega^p_\pi(A_0;\tau^*A)\times\Omega^q_\pi(A_0;\tau^*A)\to\Omega^{p+q}_\pi(A_0;\tau^*A)$ which plays the role of the wedge product on A-valued forms, and similarly to the wedge product, it is defined by the following formula

$$[\omega,\eta]_\nabla(X_1,...,X_{p+q})_a=\sum_{\sigma\in S_{p,q}}\mathrm{sgn}(\sigma)[\omega(X_{\sigma(1)},...,X_{\sigma(p)})_a,\eta(X_{\sigma(p+1)},...,X_{\sigma(p+q)})_a]_\nabla,$$

for all $a\in A_0$, where $S_{p,q}$ is the set of (p,q)-shuffles.

In general, a connection ∇ on a vector bundle $E\to M$ induces a differential-like map $d_\nabla : \Omega^*(M;E)\to\Omega^{*+1}(M;E)$ by the usual Koszul-type formula. For example, if $\phi\in\Omega^1(M;E)$,

$$d_\nabla\phi(X,Y)=\nabla_X\phi(Y)-\nabla_Y\phi(X)-\phi([X,Y]),\qquad\forall X,Y\in\mathfrak{X}(M).$$

The map d_∇ squares to zero if and only if the connection is flat. If M has a foliation $\mathcal{F}$ and $\Omega^*_{\mathcal{F}}(M;E)$ are the foliated forms, then the map d_∇ descends to a map of foliated forms $d_\nabla : \Omega^*_{\mathcal{F}}(M;E)\to\Omega^{*+1}_{\mathcal{F}}(M;E)$. The proof of the following property is elementary and will be left out:

Lemma 7.5.1. *Let $E\to M$ be a vector bundle equipped with a connection ∇ and let $f : N\hookrightarrow M$ be a submanifold. Then the following property holds:*

$$f^*d_\nabla\phi=d_{f^*\nabla}f^*\phi,$$

for any $\phi\in\Omega^*(M;E)$. If N and M are foliated and f is a foliated map, then the property holds for $\phi\in\Omega^*_{\mathcal{F}}(M;E)$.

In our particular case, the induced pull-back connection $\tau^*\nabla$ on the vector bundle $\tau^*A\to A_0$ induces a differential-like map $d_{\tau^*\nabla} : \Omega^*_\pi(A_0;\tau^*A)\to\Omega^{*+1}_\pi(A_0;\tau^*A)$.

A 1-form $\phi\in\Omega^1_\pi(A_0;\tau^*A)$ is said to be **anchored** if $\rho\circ\phi=d\tau$, or more explicitly, if $\rho(\phi_a(b))=(d\tau)_a(b)$ for all $a\in(A_0)_x,\ b\in A_x$ (where we are using the canonical identification $T_aA_0\cong A_x$).

Proposition 7.5.2. *Let $\phi\in\Omega^1_\pi(A_0;\tau^*A)$. If ϕ is anchored, then the 2-form*

$$d_{\tau^*\nabla}\phi+\frac{1}{2}[\phi,\phi]_\nabla\in\Omega^2_\pi(A_0;\tau^*A)\tag{7.40}$$

is independent of the choice of connection ∇.

Proof. Let ∇ and ∇' be two connections, then by the defining properties of a connection, $\omega := \nabla - \nabla' \in \Omega^1(M; \mathrm{Hom}(A, A))$. Let $a \in A_0$ and $X, Y \in T_a A_0$ s.t. $d\pi(X) = d\pi(Y) = 0$. On the one hand,

$$(d_{\tau^*\nabla}\phi - d_{\tau^*\nabla'}\phi)(X, Y) = \omega_{\tau(a)}((d\tau)_a(X))(\phi_a(Y)) - \omega_{\tau(a)}((d\tau)_a(Y))(\phi_a(X)),$$

and on the other hand,

$$\begin{aligned}(\frac{1}{2}[\phi, \phi]_\nabla - \frac{1}{2}[\phi, \phi]_{\nabla'})(X, Y) \\ = -\omega_{\tau(a)}(\rho(\phi_a(X)))(\phi_a(Y)) + \omega_{\tau(a)}(\rho(\phi_a(Y)))(\phi_a(X)).\end{aligned}$$

The sum of these two equations vanishes if ϕ is anchored. $\square$

We call the 2-form given by (7.40) the **Maurer-Cartan 2-form** and denote it by MC_ϕ.

Chapter 8

The Systatic Space and Reduction

Having completed the formulation of the structure theory for Lie pseudogroups by means of the three fundamental theorems, Cartan turns to discuss the notions of *systatic system* and *essential invariants*. Cartan observes that the structure equations associated with any given Lie pseudogroup encode a certain involutive distribution, which he calls the *systatic system*, and shows that the systatic system allows one to split the set of invariants of a Lie pseudogroup into those which are *essential* and those which are *inessential*. In turn, he shows that if a given Lie pseudogroup possesses inessential invariants, then it can be simplified by removing these invariants and obtaining an equivalent pseudogroup that acts on a space of smaller dimension. We will call this procedure the *reduction by the systatic system*, or simply *reduction*.

Cartan's procedure is rather local in nature, as we will see in examples, but at the same time it points towards an interesting global phenomenon that is occurring in the theory. While in the previous chapters we made a point of remaining as faithful as possible to Cartan, in this section we feel that this global phenomenon is best understood if we allow ourselves the freedom to "go beyond Cartan". In the first part of the chapter, after recalling Cartan's formulation, we show that Cartan's systatic system already appears at the infinitesimal level as a certain Lie algebroid that sits canonically within any given Cartan algebroid. We call this Lie algebroid the *1st systatic space* of the Cartan algebroid. We then show that the structure equations encode an infinitesimal action of the 1st systatic space on any realization of the Cartan algebroid, and that the distribution at the image of the action is precisely Cartan's systatic system. Up until this point, our global description does not deviate from Cartan's ideas, but it gives extra insight on the precise structure underlying the systatic system. However, the next step, the reduction by the systatic system, is best understood by taking a step away from Cartan.

Having recognized a Lie algebroid action on a realization, the question that comes to mind is what happens at the level of the quotient, the orbit space of the action. We study this quotient by shifting our point of view and introducing two new notions into the picture. The first is the notion of a Pfaffian groupoid, which was discussed in Section 2.7 and already appeared in our modern presentation in the context of the second fundamental theorem (Section 5.3). The second is the notion of a generalized pseudogroup that will be introduced here. Roughly speaking, a generalized pseudogroup is a subset of the set of local bisections of a Lie groupoid that satisfies both group-like and sheaf-like axioms analogous to those of a pseudogroup. As for pseudogroups, we can talk about *Cartan equivalence* between two generalized pseudogroups. We show that by regarding a realization as a certain Pfaffian groupoid and by regarding the pseudogroup induced by the realization as a certain generalized pseudogroup on the Pfaffian groupoid, one obtains a canonically *reduced* Pfaffian groupoid and a canonically *reduced* generalized pseudogroup at the level of the orbit space (under suitable smoothness conditions), and that the reduced generalized pseudogroup is Cartan equivalent to the original pseudogroup one starts with. Thus, while the storyline of our reduction procedure is very much reminiscent of Cartan's reduction, the specifics require some new "technology".

8.1 Cartan's Formulation

Let us briefly recall the notions of *systatic system* and *essential invariants* in Cartan's words (see [7], pp. 1352-1358).

Recall that in the second fundamental theorem, given an initial Lie pseudogroup, Cartan constructs a normal form data (I_a, ω_i) on some $\mathbb{R}^N$ (Definition 5.1.1) and its associated pseudogroup in normal $\Gamma(I_a, \omega_i)$. In particular, one has the structure equations

$$d\omega_i + \frac{1}{2}c_i^{jk}\omega_j \wedge \omega_k = a_i^{\lambda j}\pi_\lambda \wedge \omega_j$$

associated with the normal form data. Cartan defines the **systatic system** associated with (I_a, ω_i) as the system of equations

$$a^i_{\lambda j}\omega^j = 0, \qquad i = 1, ..., m,\ \lambda = 1, ..., p. \tag{8.1}$$

This should be understood as the distribution

$$\{\ X \in T\mathbb{R}^N \mid a^i_{\lambda j}\omega^j(X) = 0 \quad \forall\, i = 1, ..., m,\ \lambda = 1, ..., p\ \} \subset T\mathbb{R}^N, \tag{8.2}$$

which we call the **systatic distribution**. Cartan then proves, by means of the structure equations, that the systatic distribution is an integrable distribution.

The importance of the systatic system in Cartan's theory is that it allows detecting the existence of what Cartan calls the *essential invariants* of a pseudogroup in normal form, and, consequently, to simplify the pseudogroup by removing the *inessential invariants*, thus obtaining an equivalent pseudogroup that acts on a space of lower dimension. We call this procedure *reduction*. Let us get acquainted with Cartan's procedure by considering two examples cited from his work (p. 1357 of [7]):

Example 8.1.1. Let (x, y) be coordinates on $\mathbb{R}^2$. Consider the Lie pseudogroup on $\mathbb{R}^2\backslash\{y = 0\}$ from Example 5.1.8 generated by the transformations

$$(x, y) \mapsto (x + ay, y),$$

with $a \in \mathbb{R}$. It is defined by the invariance of the y coordinate function and the 1-forms

$$\omega_1 = dx - \frac{x}{y}dy, \qquad \omega_2 = dy,$$

which, in turn, satisfy the structure equations

$$d\omega_1 = \frac{1}{y}\omega_2 \wedge \omega_1, \qquad d\omega_2 = 0.$$

From the structure equations we see that there is no systatic system, since there are no "π's", and hence the systatic distribution is simply the full tangent bundle $T\mathbb{R}^2$. Cartan deduces from this that there are no essential invariants, meaning that the only invariant of the pseudogroup, the coordinate function y, is an inessential invariant. Cartan then shows that if we perform the change of coordinates $(x, y) \mapsto (\frac{x}{y}, y)$, then, denoting the new coordinates by (x', y'), the pseudogroup takes the simplified form

$$(x', y') \mapsto (x' + a, y').$$

Thus, the systatic system indicates that we can simplify the pseudogroup by splitting it to the product of a transitive pseudogroup, namely $x' \mapsto x' + a$, and a trivial pseudogroup $y' \mapsto y'$ acting on the inessential invariant. In other words, the original pseudogroup on $\mathbb{R}^2 \backslash \{y = 0\}$ is reduced to an equivalent pseudogroup on $\mathbb{R}$. $\diamondsuit$

Example 8.1.2. Let (x, y, z) be coordinates on $\mathbb{R}^3$. Consider the Lie pseudogroup on $\mathbb{R}^3 \backslash \{y = 0\} \cup \{z = 0\}$ generated by the transformations

$$(x, y, z) \mapsto (x + ay + bz, y, z),$$

with $a, b \in \mathbb{R}$. Introducing a new variable u, the pseudogroup admits an isomorphic prolongation defined by the invariance of the coordinate functions y and z and the 1-forms

$$\omega_1 = dx - x\frac{dz}{z} + u\Big(dy - y\frac{dz}{z}\Big), \qquad \omega_2 = dy, \qquad \omega_3 = dz,$$

which, in turn, satisfy the structure equations

$$d\omega_1 = -\frac{1}{z}\omega_1 \wedge \omega_3 + \pi_1 \wedge (\omega_2 - \frac{y}{z}\omega_3), \qquad d\omega_2 = 0, \qquad d\omega_3 = 0, \tag{8.3}$$

with $\pi_1 = du$ which satisfies

$$d\pi_1 = 0.$$

The systatic system (8.1), which we read off the structure equations, consists of the single equation

$$\omega^2 - \frac{y}{z}\omega^3 = dy - \frac{y}{z}dz = 0.$$

Cartan then concludes that the first integral $\frac{y}{z}$ of the integrable systatic distribution is an essential invariant, and notes that if one applies the change of coordinates $(x, y, z) \mapsto (\frac{x}{z}, \frac{y}{z}, z)$, then, denoting the new coordinates by (x', y', z), the pseudogroup takes the simplified form

$$(x', y', z') \mapsto (x' + ay' + b, y', z').$$

We see that, in these new coordinates, of the two invariants $y' = \frac{y}{z}$ and $z' = z$, only the first one, the integral of the systatic distribution, is essential, since the pseudogroup splits into the product of two pseudogroups: one acting on $\mathbb{R}^2$ by $(x', y') \mapsto (x' + ay' + b, y')$ and one acting trivially on the inessential invariant by the identity map $z' \mapsto z'$. In this way, the Lie pseudogroup on $\mathbb{R}^3$ we started with is reduced by the systatic system to an equivalent one on $\mathbb{R}^2$ by removing the inessential invariant z'. $\diamondsuit$

The purpose of these two examples was only to give an idea of how Cartan uses the systatic system to reduce a given Lie pseudogroup. We refer the reader to Cartan's own writings for more details on his general scheme (see also [68], Sections 14.4-14.5).

There are two problems that one encounters when attempting to describe Cartan's reduction procedure in global terms. First, the very notion of invariant coordinate functions is of a purely local nature, and second, the simplifications of the pseudogroups that Cartan introduces is also of a local nature and depends largely on a smart choice of some special coordinate chart (which may fail to exist globally). In this chapter, after discussing the role of the systatic system in our modern framework, we propose an alternative approach to reduction which has the advantage of being global, canonical, and conceptual.

8.2 The Systatic Space

In the global picture, the systatic system and distribution have a rather elegant geometric meaning, as we will see in the current section.

Given a pre-Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N, while the conditions that one imposes for it to be a Cartan algebroid (Definition 6.2.1) may seem rather obscure, as they require the existence of t and ∇ which themselves are not part of the structure, there are several interesting consequences that are of an intrinsic nature and that turn out to be intimately related to Cartan's systatic system.

Definition 8.2.1. *The* ***systatic space*** *of a Cartan algebroid* $(\mathcal{C}, \mathfrak{g})$ *over* N *is the set-theoretical vector subbundle* $\mathcal{S} \subset \mathcal{C}$ *whose fiber at* $x \in N$ *is*

$$\mathcal{S}_x := \{ \, u \in \mathcal{C}_x \mid T(u) = 0 \;\; \forall\, T \in \mathfrak{g}_x \, \}. \tag{8.4}$$

Because $\mathcal{S}$ can be expressed as the kernel of the vector bundle map

$$\mathcal{C} \to \mathrm{Hom}(\mathfrak{g}, \mathcal{C}), \qquad u \mapsto (T \mapsto T(u)),$$

it is a smooth vector subbundle if and only if it is of constant rank.

Assumption 8.2.2. *We will assume that the systatic spaces of all Cartan pairs appearing from now on are of constant rank.*

As a consequence of Definition 6.2.1 of a Cartan algebroid, or more specifically, as a consequence of conditions 2 and 3 of the definition:

Proposition 8.2.3. *The systatic space* $\mathcal{S}$ *of a Cartan algebroid* $(\mathcal{C}, \mathfrak{g})$*, equipped with the bracket and anchor inherited from* $\mathcal{C}$*, is a Lie algebroid.*

Proof. Using the fact that all elements of $\mathcal{S}$ are killed by all elements of $\mathfrak{g}$, (6.4) implies that $\mathcal{S}$ is closed under the restriction of the bracket of $\mathcal{C}$ and (6.3) reduces to the Jacobi identity. □

Given $u, v \in \Gamma(\mathcal{S})$, we consider the vector bundle map

$$\mathrm{J}_{u,v} : \mathcal{C} \to \mathcal{C},$$

which is defined at the level of sections by

$$\mathrm{J}_{u,v}(\alpha) = [[u, v], \alpha] + [[v, \alpha], u] + [[\alpha, u], v], \qquad \forall \alpha \in \Gamma(\mathcal{C}).$$

Given $u \in \Gamma(\mathcal{S})$ and $T \in \Gamma(\mathfrak{g})$, we consider the vector bundle map

$$\mathrm{Ad}_u(T) : \mathcal{C} \to \mathcal{C},$$

which is defined at the level of sections by

$$\mathrm{Ad}_u(T)(\alpha) := [u, T(\alpha)] - T([u, \alpha]), \qquad \forall \alpha \in \Gamma(\mathcal{C}).$$

As a second consequence of conditions 2 and 3 of Definition 6.2.1:

Proposition 8.2.4. *Let $(\mathcal{C}, \mathfrak{g})$ be a Cartan algebroid. For all $u, v \in \Gamma(\mathcal{S})$ and $T \in \mathfrak{g}$,*

$$J_{u,v} \in \Gamma(\mathfrak{g}) \qquad \textit{and} \qquad Ad_u(T) \in \Gamma(\mathfrak{g}).$$

Proof. Choose t and ∇ as in definition 6.2.1 of a Cartan algebroid. By 6.3,

$$\mathrm{J}_{u,v}(\alpha) = t_{u,v}(\alpha),$$

and by definition 6.4,

$$\mathrm{Ad}_u(T) = \nabla_u(T).$$

□

Thus, while the $\mathcal{C}$-connection $\nabla : \Gamma(\mathcal{C}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ and the vector bundle map $t : \Lambda^2\mathcal{C} \to \mathfrak{g}$ on a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ are non-canonical, their restrictions to $\mathcal{S}$, the $\mathcal{S}$-connection

$$\mathrm{Ad} : \Gamma(\mathcal{S}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$$

and the vector bundle map

$$\mathrm{J} : \Lambda^2\mathcal{S} \to \mathfrak{g},$$

are canonical.

The two previous propositions can be neatly packaged in a single object by using the following construction of a non-abelian extension of a Lie algebroid (see also [46], Chapter 4, Section 3).

Definition 8.2.5. *The* 1***st systatic space*** *of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ is the vector bundle*

$$\mathcal{S}^{(1)} := \mathcal{S} \oplus \mathfrak{g} \ \subset \ \mathcal{C} \oplus \mathfrak{g}.$$

We equip $\mathcal{S}^{(1)}$ with the structure of a Lie algebroid. The anchor $\rho : \mathcal{S}^{(1)} \to TN$ is defined by

$$\rho(u, T) := \rho(u), \qquad \forall\, u \in \Gamma(\mathcal{S}),\ T \in \Gamma(\mathfrak{g}),$$

and the bracket $[\cdot, \cdot] : \Gamma(\mathcal{S}^{(1)}) \times \Gamma(\mathcal{S}^{(1)}) \to \Gamma(\mathcal{S}^{(1)})$ by the formula

$$[(u, S), (v, T)] := ([u, v], \mathrm{J}_{u,v} + \mathrm{Ad}_u(T) - \mathrm{Ad}_v(S) - [S, T]).$$

To prove that this indeed defines a Lie algebroid structure on $\mathcal{S}^{(1)}$, we first prove the following lemma. We denote the de Rham like operator induced by the connection Ad on the graded module of forms $\Omega^*(\mathcal{S}; \mathfrak{g}) = \mathrm{Hom}(\Lambda^*\mathcal{S}, \mathfrak{g})$ by d_{Ad}.

Lemma 8.2.6. *For all* $u, v \in \Gamma(\mathcal{S}),\ S, T \in \Gamma(\mathfrak{g})$*,*

1. $d_{Ad} J = 0$*,*
2. $Ad_u(Ad_v(S)) - Ad_v(Ad_u(S)) - Ad_{[u,v]}(S) = [S, J_{u,v}]$*,*
3. $Ad_u([S,T]) = [Ad_u(S), T] + [S, Ad_u(T)]$*.*

Proof. We explicitly compute the first identity. Let $u, v, w \in \Gamma(\mathcal{S})$ and $\alpha \in \Gamma(\mathcal{C})$,

$$\begin{aligned} d_{\mathrm{Ad}}\mathrm{J}(u,v,w)(\alpha) &= \mathrm{Ad}_u(J_{v,w})(\alpha) - J_{[u,v],w}(\alpha) + \text{ cyclic permutations of } u, v, w \\ &= [u, [[v,w],\alpha]] + [u, [[w,\alpha],v]] + [u, [[\alpha,v],w]] \\ &\quad - [[v,w],[u,\alpha]] - [[[u,\alpha],v],w] - [[w,[u,\alpha]],v] \\ &\quad - [[[u,v],w],\alpha] - [[w,\alpha],[u,v]] - [[\alpha,[u,v]],w] \\ &\quad + \text{ cyclic permutations of } u, v, w. \end{aligned}$$

The 7th term (together with its cyclic permutations) vanishes by the Jacobi identity of $\mathcal{S}$ and all other terms cancel pairwise. The other identities are dealt with similarly. We only point out that the remaining two identities do not rely on the Jacobi identity of $\mathcal{S}$. ☐

Proposition 8.2.7. *The 1st systatic space $\mathcal{S}^{(1)}$ of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ is a Lie algebroid.*

Proof. The proof involves computing the Jacobiator of $\mathcal{S}^{(1)}$ and showing that the Jacobi identity holds if and only if the three identities in Lemma 8.2.6 are satisfied. This is a straightforward computation (see also [46], Theorem 3.20). ☐

8.3 The Systatic Action on Realizations

Now, the most important property of the 1st systatic space $\mathcal{S}^{(1)}$ is that it acts canonically on all realizations of $(\mathcal{C}, \mathfrak{g})$, extending the action of $\mathfrak{g}$ (see Proposition 6.3.4). This is made precise in the following proposition.

Proposition 8.3.1. *Let (P, Ω) be a realization of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$. The map*

$$a := (\Omega, \Pi)^{-1}|_{I^*\mathcal{S}^{(1)}} : I^*\mathcal{S}^{(1)} \to TP \tag{8.5}$$

defines a canonical Lie algebroid action of $\mathcal{S}^{(1)}$ on $I : P \to N$ (thus, independent of the choice of Π) and its image is the involutive distribution

$$\{\, X \in TP \mid \Omega(X) \in \mathcal{S} \,\}. \tag{8.6}$$

Proof. By Lemma 5.2.17, we already know that the map $S \in \Gamma(\mathfrak{g}) \mapsto X_S \in \mathfrak{X}(P)$ is canonical. Fix a choice of $\Pi \in \Omega^1(P; I^*\mathfrak{g})$ as in Definition 5.2.11 and consider the induced map $\alpha \in \Gamma(\mathcal{C}) \mapsto X_\alpha \in \mathfrak{X}(P)$. We would like to show that its restriction

$u \in \Gamma(\mathcal{S}) \mapsto X_u \in \mathfrak{X}(P)$ is canonical, or equivalently, that $\Pi'(X_u) = 0$ for any other choice of Π' and $u \in \Gamma(\mathcal{S})$. This follows from the structure equations (c.f. the proof of Lemma 5.2.17): for any $\alpha \in \Gamma(\mathcal{C})$,

$$\begin{aligned} 0 &= \big((\Pi' - \Pi) \wedge \Omega\big)(X_u, X_\alpha) \\ &= \Pi'(X_u)(\Omega(X_\alpha)) - \Pi'(X_\alpha)(\Omega(X_u)) \\ &= \Pi'(X_u)(I^*\alpha) - \cancel{\Pi'(X_\alpha)(I^*u)}. \end{aligned}$$

Next, we prove that (8.5) defines an action. Let $(u, S), (v, T) \in \Gamma(\mathcal{S}^{(1)})$ and let us write $X_{(u,S)} := X_u + X_S$ and $X_{(v,T)} := X_v + X_T$. We claim that $[X_{(u,S)}, X_{(v,T)}] = X_{[(u,S),(v,T)]}$. On the one hand, by the definition of the bracket of $\mathcal{S}^{(1)}$,

$$\begin{aligned} &\Omega\big(X_{[(u,S),(v,T)]}\big) = I^*[u, v], \\ &\Pi\big(X_{[(u,S),(v,T)]}\big) = I^*\big(\mathrm{J}_{u,v} + \mathrm{Ad}_u(T) - \mathrm{Ad}_v(S) - [S, T]\big). \end{aligned}$$

On the other hand, by Lemma 5.2.16 (which follows from the structure equations),

$$\Omega\big([X_{(u,S)}, X_{(v,T)}]\big) = I^*[u, v],$$

Furthermore, note that for any $\xi \in \Gamma(\mathfrak{g}^{(1)})$ and $\xi' \in \Gamma(Z^{0,2}(\mathfrak{g}))$ (see (6.6)),

$$\xi(u)(\alpha) = \xi(\alpha)(u) = 0, \qquad \xi'(u, v)(\alpha) = -\xi'(v, \alpha)(u) - \xi'(\alpha, u)(v) = 0,$$

for all $u, v \in \Gamma(\mathcal{S})$ and $\alpha \in \Gamma(\mathcal{C})$. So, by Corollary 6.3.3 (which also uses the structure equations),

$$\Pi\big([X_{(u,S)}, X_{(v,T)}]\big) = I^*\big(\mathrm{J}_{u,v} + \mathrm{Ad}_u(T) - \mathrm{Ad}_v(S) - [S, T]\big).$$

The last assertion concerning the image of the action is immediate. □

Definition 8.3.2. *Given a realization (P, Ω) of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$, the action of the 1st systatic $\mathcal{S}^{(1)}$ of $(\mathcal{C}, \mathfrak{g})$ on $I : P \to N$ is called the* ***systatic action****.*

Example 8.3.3. (Lie groups) Let us consider the example of a realization coming from a Lie group H with Lie algebra $\mathfrak{h}$. We continue from Example 5.2.13, the case in which the systatic space, the 1st systatic space and $\mathcal{C}$ itself all coincide with the Lie algebra,

$$\mathcal{S}^{(1)} = \mathcal{S} = \mathcal{C} = \mathfrak{h}.$$

The systatic action in this case, the inverse of the Maurer-Cartan form, is given at the level of sections by the map

$$\mathfrak{h} \to \mathfrak{X}(H), \qquad X \mapsto X^R, \tag{8.7}$$

which sends an element of the Lie algebra $X \in \mathfrak{h}$ to the induced right invariant vector field $X^R \in \mathfrak{X}(H)$, which is defined by $(X^R)_h := (dR_h)_e(X)$.

In this example, the pseudogroup $\Gamma(H, \Omega_{\mathrm{MC}})$ induced by the realization, which is the pseudogroup generated by right translations, can be also characterized as the pseudogroup on H consisting of all local diffeomorphisms that are equivariant w.r.t. the systatic action. By this we mean that $\psi \in \mathrm{Diff}_{\mathrm{loc}}(H)$ belongs to $\Gamma(H, \Omega_{\mathrm{MC}})$ if and only if

$$d\psi(X^R) = X^R, \qquad \forall\, X \in \mathfrak{h}. \qquad \diamondsuit$$

In the previous example, we saw a special case in which the pseudogroup of local diffeomorphisms that are equivariant w.r.t. the systatic action coincides with the pseudogroup induced by the realization. In general, this is not the case, however:

Proposition 8.3.4. *The elements of the pseudogroup $\Gamma(P, \Omega)$ are equivariant w.r.t. the Lie algebroid action of $\mathcal{S}^{(1)}$ on P, i.e.*

$$d\psi(X_u) = X_u, \qquad d\psi(X_S) = X_S,$$

for all $u \in \Gamma(\mathcal{S}),\ S \in \Gamma(\mathfrak{g}),\ \psi \in \Gamma(P, \Omega)$.

Proof. We need to show that

$$\begin{aligned} &\Omega(d\psi(X_u)) = I^*u, &\qquad &\Pi(d\psi(X_u)) = 0,\\ &\Omega(d\psi(X_S)) = 0, &\qquad &\Pi(d\psi(X_S)) = I^*S. \end{aligned}$$

First,

$$\begin{aligned} \Omega(d\psi(X_u)) &= \Omega(X_u) = I^*u,\\ \Omega(d\psi(X_S)) &= \Omega(X_S) = 0, \end{aligned}$$

because $\psi^*\Omega = \Omega$. Next, we note that the structure equation $d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega$ combined with the invariance condition $\psi^*\Omega = \Omega$ imply that $(\psi^*\Pi - \Pi) \wedge \Omega = 0$. Applying this equation on pairs (X_u, X_α) and (X_S, X_α) and using the fact that $T(u) = 0$ for all $T \in \Gamma(\mathfrak{g})$, we see that

$$\begin{aligned} (\psi^*\Pi - \Pi)(X_u)(I^*\alpha) = 0,\\ (\psi^*\Pi - \Pi)(X_S)(I^*\alpha) = 0. \end{aligned}$$

Since this is true for all $\alpha \in \Gamma(\mathcal{C})$, then

$$\begin{aligned} \Pi(d\psi(X_u)) &= \Pi(X_u) = 0,\\ \Pi(d\psi(X_S)) &= \Pi(X_S) = I^*S. \end{aligned}$$

$\square$

To conclude this section, let us compare the global picture with Cartan's local picture:

Example 8.3.5. Recall from Example 5.2.14 the local coordinate description of a realization. Unraveling the defining equations of the distribution (8.6), one sees that, in local coordinates, it consists of all tangent vectors $X \in T\mathbb{R}^N$ that satisfy

$$a^i_{\lambda j}\omega^j(X) = 0, \qquad \forall\, i = 1, ..., m,\ \lambda = 1, ..., p.$$

This is precisely Cartan's systatic distribution (8.2). In particular, since the systatic distribution is the image of a Lie algebroid action map, it is an involutive distribution, confirming Cartan's claim that the systatic distribution is integrable (recall that we are assuming that $\mathcal{S}$ is of constant rank). $\diamondsuit$

8.4 Intermezzo: The Pfaffian Groupoid of a Realization

Our goal in the remainder of the chapter is to make use of the systatic action in order to reduce the pseudogroup $\Gamma(P,\Omega)$ acting on P to a pseudogroup acting on a smaller manifold (without changing the Cartan equivalence class). As an intermediate step in this process, and independently of the systatic action, it will prove useful to realize $\Gamma(P,\Omega)$ as the holonomic bisections of a Pfaffian groupoid over P, which is the subject of this section. The Lie groupoid underlying this Pfaffian groupoid will simply be the submersion groupoid associated with $I : P \to N$:

$$\mathcal{G}(I) := P \times_N P = \{\ (p,q) \in P \times P \mid I(p) = I(q)\ \} \rightrightarrows P.$$

The main problem, therefore, is to exhibit the Pfaffian form θ on $\mathcal{G}(I)$.

Note that since the local bisections σ of $\mathcal{G}(I)$ are of type $\sigma(p) = (p, \psi(p))$, they are in a 1-1 correspondence with local diffeomorphisms ψ of P that satisfy $\psi^*I = I$:

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}(I)) \xrightarrow{\simeq} \{\ \psi \in \mathrm{Diff}_{\mathrm{loc}}(P) \mid \psi^*I = I\ \}, \qquad \sigma \mapsto t \circ \sigma. \tag{8.8}$$

Hence, we are looking for a Pfaffian form θ with which this isomorphism restricts to

$$\mathrm{Bis}_{\mathrm{loc}}(\mathcal{G}(I), \theta) \xrightarrow{\simeq} \{\ \psi \in \mathrm{Diff}_{\mathrm{loc}}(P) \mid \psi^*I = I,\ \psi^*\theta = \theta\ \}.$$

We will define θ as a 1-form with coefficients in E_P, where

$$E := \mathrm{Ker}\ \rho \subset \mathcal{C},$$

and $E_P := I^*E$ and $\mathcal{C}_P := I^*\mathcal{C}$ are the pullbacks to P. Both of these are trivially representations of $\mathcal{G}(I)$: any $(p,q) \in \mathcal{G}(I)$ acts on a vector $\alpha \in (\mathcal{C}_P)_{s(p,q)}$ by the identity map, thus $(p,q) \cdot \alpha = \alpha$, which makes sense because

$$(\mathcal{C}_P)_{s(p,q)} = \mathcal{C}_{I(p)} = \mathcal{C}_{I(q)} = (\mathcal{C}_P)_{t(p,q)}.$$

We have the following general lemma:

Lemma 8.4.1. *Given a Lie algebroid $\mathcal{G} \rightrightarrows M$, a representation $\mathcal{C} \to M$ of $\mathcal{G}$ and an $\mathcal{C}$-valued 1-form $\Omega \in \Omega^1(M;\mathcal{C})$, $\theta = s^*\Omega - t^*\Omega \in \Omega^1(\mathcal{G}; t^*\mathcal{C})$ is a multiplicative form.*

Proof. Writing pr_1 and pr_2 for the projections onto the first and second components of $\mathcal{G}_2 = \mathcal{G}\,{}_s\!\times_t \mathcal{G}$, we must show that

$$(m^*\theta)_{(g,h)} = (\mathrm{pr}_1^*\theta)_{(g,h)} + g \cdot (\mathrm{pr}_2^*\theta)_{(g,h)}, \tag{8.9}$$

for any pair of composable arrows $(g,h) \in \mathcal{G}_2$. A vector in $\mathcal{G}_2$ at (g,h) consists of a pair (X,Y) with $X \in T_g\mathcal{G}, Y \in T_h\mathcal{G}$ that satisfy $ds(X) = dt(Y)$, and using the fact that $ds \circ dm(X,Y) = ds(Y)$ and $dt \circ dm(X,Y) = dt(X)$, we see that indeed

$$\theta_g(X)+g\cdot\theta_h(Y) = \cancel{\Omega(ds(X))}-\Omega(dt(X))+\Omega(ds(Y))-\cancel{\Omega(dt(Y))} = \theta_{g\cdot h}(dm(X,Y)).$$

□

Returning to our problem, define

$$\theta := s^*\Omega - t^*\Omega \in \Omega^1(\mathcal{G}(I); t^*\mathcal{C}_P).$$

Proposition 8.4.2. *Let (P,Ω) be a realization of a Cartan algebroid $(\mathcal{C},\mathfrak{g})$. The 1-form θ is t^*E_P-valued,*

$$\theta \in \Omega^1(\mathcal{G}(I); t^*E_P),$$

and the pair $(\mathcal{G}(I),\theta)$ is a Pfaffian groupoid whose local holonomic bisections are in 1-1 correspondence with the elements of $\Gamma(P,\Omega)$.

Proof. First, we show that θ is E-valued. A vector in $\mathcal{G}(I)$ at a point (p,q) is a pair (X,Y) consisting of $X \in T_pP$ and $Y \in T_qP$ that satisfy $dI(X) = dI(Y)$, and because Ω is anchored ($dI = \rho\circ\Omega$), we have that $\rho\circ\theta(X,Y) = \rho\circ\Omega(X) - \rho\circ\Omega(Y) = dI(X) - dI(Y) = 0$.

By the previous lemma, θ is multiplicative.

We verify the remaining axioms of Definition 2.7.1 of a Pfaffian groupoid. To this end, we fix a Π for the realization (P,Ω), which induces the map (5.17). We first look at the tangent spaces to $\mathcal{G}(I)$. Any pair of vector fields $X,Y \in \mathfrak{X}(P)$ induces a vector field on the product $(X,Y) \in \mathfrak{X}(P\times P)$, where $(X,Y)_{(p,q)} = (X_p,Y_q) \in T_pP \oplus T_qP = T_{(p,q)}(P\times P)$, and vector fields on $P\times P$ are generated as a $C^\infty(P\times P)$-module by pairs of the type

$$(X_\alpha + X_S, X_\beta + X_T), \qquad \alpha,\beta \in \Gamma(\mathcal{C}),\ S,T \in \Gamma(\mathfrak{g}).$$

Since $dI = \rho\circ\Omega$, vector fields on $\mathcal{G}(I) = P\times_N P$ are generated as a $C^\infty(\mathcal{G}(I))$-module by pairs of the type

$$(X_\alpha + X_S, X_\beta + X_T), \qquad \alpha,\beta \in \Gamma(\mathcal{C}) \text{ with } \rho(\alpha)=\rho(\beta),\ S,T \in \Gamma(\mathfrak{g}).$$

Here, by $(X_\alpha + X_S, X_\beta + X_T)$ we mean the restriction to $\mathcal{G}(I)$ of the corresponding vector field on $P\times P$. Applying θ on such a vector field, we see that

$$\theta(X_\alpha + X_S, X_\beta + X_T) = (I\circ t)^*(\beta-\alpha). \tag{8.10}$$

Denoting the kernel of θ by $C_\theta := \mathrm{Ker}\ \theta \subset T\mathcal{G}(I)$, we see from (8.10) that sections of C_θ are generated by pairs of the type

$$(X_\alpha + X_S, X_\alpha + X_T), \qquad \alpha \in \Gamma(\mathcal{C}),\ S,T \in \Gamma(\mathfrak{g}). \tag{8.11}$$

Let us check the axioms of a Pfaffian groupoid: if $\alpha \in \Gamma(E)$ ($E = \text{Ker } \rho$), then $\theta(X_\alpha, 0) = (I \circ t)^*\alpha$ by (8.10), and hence θ is pointwise surjective. Applying ds on vector fields of the type (8.11) is the same as projecting onto the first component and we conclude that $ds : \text{Ker } \theta \to s^*TP$ is surjective. Finally, note that $\text{Ker } dI \subset TP$ is the Lie algebroid of $\mathcal{G}(I)$, and from (8.11) it follows that the intersection of the Lie algebroid with C_θ is precisely $\text{Ker } \Omega \subset dI$, which is involutive and, hence, a Lie subalgebroid.

The last assertion follows from the definition of $\Gamma(P, \Omega)$. □

8.5 The Systatic Action on the Pfaffian Groupoid

Having constructed the Pfaffian groupoid $(\mathcal{G}(I), \theta)$ of a realization (P, Ω), we now show that the systatic action on P (Section 8.3) extends to an action on the Pfaffian groupoid $(\mathcal{G}(I), \theta)$ with which θ becomes a basic form (in the sense of Appendix 8.8). This fact will be essential for the reduction.

The action on $\mathcal{G}(I)$ as a manifold is simply the diagonal action. This is an action along the surjective submersion $I : \mathcal{G}(I) \to N$, which denotes the composition of the source (or target) map of $\mathcal{G}(I)$ with $I : P \to N$. With the previous notations, it is given by

$$\Gamma(\mathcal{S}^{(1)}) \to \mathfrak{X}(\mathcal{G}(I)), \qquad (u, S) \mapsto (X_u + X_S, X_u + X_S). \tag{8.12}$$

Note that this is well defined, i.e. takes values in $\mathfrak{X}(\mathcal{G}(I))$, because $dI(X_\alpha + X_S) = \rho \circ \Omega(X_\alpha + X_S) = \rho(\alpha)$. Thus, the 1st systatic space acts on both the base and the space of arrows of $\mathcal{G}(I)$. In fact, the action of $\mathcal{S}^{(1)}$ on $\mathcal{G}(I) \rightrightarrows P$ can be interpreted as the action of a Lie algebroid on a Lie groupoid, the linearization of the notion of a Lie groupoid action on a Lie groupoid ([27], Definition 3.1).

For simplicity, let us introduce the simplified notation $E = E_P$.

To explain the compatibility of θ with the systatic action, we need to make sense of the systatic action on the representation E.

Lemma 8.5.1. *The following map defines a representation of $\mathcal{S}^{(1)}$:*

$$\nabla : \Gamma(\mathcal{S}^{(1)}) \times \Gamma(E) \to \Gamma(E), \qquad \nabla_{(u,S)}(\alpha) = [u, \alpha] - S(\alpha), \tag{8.13}$$

Proof. Clearly, ∇ is an $\mathcal{S}^{(1)}$-connection. We prove that ∇ is flat. Using the defining formula of ∇,

$$\begin{aligned}
(\nabla_{(u,S)}\nabla_{(v,T)} - \nabla_{(v,T)}\nabla_{(u,S)} - \nabla_{[(u,S),(v,T)]})(\alpha) \\
= S(T(\alpha)) - T(S(\alpha)) - [S, T](\alpha) \\
+J_{u,v}(\alpha) + [u, [v, \alpha]] - [[u, v], \alpha] - [v, [u, \alpha]] \\
+T([u, \alpha]) - T([u, \alpha]) + S([v, \alpha]) - S([v, \alpha]) \\
+[v, S(\alpha)] - [v, S(\alpha)] + [u, T(\alpha)] - [u, T(\alpha)] \\
= 0.
\end{aligned}$$

□

The main conclusion of this section is stated in the following proposition, namely that θ is a basic form with respect to the systatic actions. Here we use the rather straightforward generalization of basic forms from Lie group and Lie algebra actions to the "-oid" setting. This is reviewed in Appendix 8.8. As in the classical setting, the basic condition implies that a form descends to the quotient.

Proposition 8.5.2. *The 1-form $\theta \in \Omega^1(\mathcal{G}(I), I^*E)$ is $\mathcal{S}^{(1)}$-basic.*

Proof. Let us place ourselves in the setting of Appendix 8.8 on basic forms. We have an action of a Lie algebroid $\mathcal{S}^{(1)} \to N$ on the surjective submersion $I : \mathcal{G}(I) \to N$, a representation $\nabla : \Gamma(\mathcal{S}^{(1)}) \times \Gamma(E) \to \Gamma(E)$ and the 1-form $\theta = s^*\Omega - t^*\Omega \in \Omega^1(\mathcal{G}(I), I^*E)$, where s/t is the source/target map of $\mathcal{G}(I) \rightrightarrows P$. First, from (8.11) we see that all the vector fields at the image of (8.12) lie in the kernel of θ and hence θ is horizontal. Furthermore, θ satisfies the equivariance condition (8.20):

$$
\begin{aligned}
&(I^*\nabla)_{I^*(u,W)}\theta(X_\alpha + X_S, X_\beta + X_T) \\
&\qquad - \theta([(X_u + X_W, X_u + X_W), (X_\alpha + X_S, X_\beta + X_T)]) \\
&\underset{\substack{\uparrow \\ \theta = s^*\Omega - t^*\Omega}}{=} I^*\big(\nabla_{(u,W)}(\beta - \alpha)\big) - s^*\big(\Omega([X_u + X_W, X_\beta + X_T])\big) \\
&\qquad + t^*\big(\Omega([X_u + X_W, X_\alpha + X_S])\big) \\
&\underset{\substack{\uparrow \\ \text{Lemma 5.2.16}}}{=} I^*\big([u, \beta - \alpha] - W(\beta - \alpha) - [u, \beta] - W(\beta) - [u, \alpha] + W(\alpha)\big) = 0.
\end{aligned}
$$

□

8.6 The Reduction Theorem

Let us review the picture that has been developed so far. Starting with a realization (P, Ω) of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N, we constructed a Pfaffian groupoid $(\mathcal{G}(I), \theta)$. We then proceeded to show that the 1st systatic space $\mathcal{S}^{(1)}$ of $(\mathcal{C}, \mathfrak{g})$ acts on $\mathcal{G}(I)$ (by the diagonal action) and that $E = \ker \rho \subset \mathcal{C}$ is a representation of $\mathcal{S}^{(1)}$. Finally, we proved that $\theta \in \Omega^1(\mathcal{G}(I); t^*E)$ is basic with respect to this action. In this section, we study the quotient that is obtained by, loosely speaking, dividing out the Pfaffian groupoid $(\mathcal{G}(I), \theta)$ by the action of $\mathcal{S}^{(1)}$. More precisely, our goal in this section is to explain and prove the following theorem:

Theorem 8.6.1. *Let (P, Ω) be a realization of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N, and let $(\mathcal{G}(I), \theta)$ be the induced Pfaffian groupoid. Assume that there exists an s-connected Lie groupoid $\Sigma \rightrightarrows N$ integrating the Lie algebroid $\mathcal{S}^{(1)} \to N$, such that*

(a) the action of $\mathcal{S}^{(1)}$ on P integrates to a free and proper action of Σ,

(b) the representation E of $\mathcal{S}^{(1)}$ integrates to a representation of Σ.

Then:

1. *the Pfaffian groupoid* $(\mathcal{G}(I), \theta)$ *descends to a Lie-Pfaffian groupoid* $(\mathcal{G}(I)_{red}, \theta_{red})$ *by dividing out the action of* Σ,
2. *the original pseudogroup* $\Gamma(P, \Omega)$ *on* P *is Cartan equivalent to* Γ_{red}*, the generalized pseudogroup of the Lie-Pfaffian groupoid*

$$(\mathcal{G}(I)_{red}, \theta_{red}) \rightrightarrows P_{red}.$$

Note that while in some examples Γ_{red} may be a classical pseudogroup, in others $P_{\text{red}} = \{*\}$ and Γ_{red} is just a Lie group (e.g., see Example 8.7.1 in the next section). In contrast, Cartan's reduction is a partial reduction that always produces classical pseudogroups (see Section 8.1).

We begin with few remarks on the assumptions. Note, first, that the freeness assumption is a rather weak one, since the action of $\mathcal{S}^{(1)}$ on P is already infinitesimally free in the sense that the action map (8.5) is injective. While the purpose of all the assumptions is to ensure that the reduced objects are smooth, one can, of course, vary the assumptions. For example, if one assumes that the integration Σ is the source-simply connected integration, then it follows that the representation E integrates to a representation of Σ and one can give up on the third assumption. If one further assumes that the map $I : P \to N$ is proper, then the action of $\mathcal{S}^{(1)}$ on P integrates to an action of Σ, but one would still need to assume that the resulting action is free and proper. One can also relax the assumptions at the cost of working with local groupoids and local actions.

With these assumptions, one can talk about the quotient of the Lie groupoid $\mathcal{G}(I) \rightrightarrows P$ by the action of Σ. Namely, the action of Σ on P induces a diagonal action on $\mathcal{G}(I) = P \times_N P$ which integrates the diagonal action of $\mathcal{S}^{(1)}$, and one has:

Lemma 8.6.2. *Let* $\Sigma \rightrightarrows N$ *be a Lie groupoid and assume that there is a free and proper action of* Σ *on a surjective submersion* $I : P \to N$*. Then the quotient*

$$\begin{array}{ccc} \mathcal{G}(I)_{\text{red}} & = & \mathcal{G}(I)/\Sigma \\ \downdownarrows & & \\ P_{\text{red}} & = & P/\Sigma \end{array}$$

has the unique structure of a Lie groupoid for which the quotient maps

$$\begin{array}{ccc} \mathcal{G}(I) & \xrightarrow{pr} & \mathcal{G}(I)_{\text{red}} \\ \downdownarrows & & \downdownarrows \\ P & \xrightarrow{pr} & P_{\text{red}} \end{array}$$

are surjective submersions and form a Lie groupoid map. Furthermore, the induced quotient map of the space of composable arrows $\text{pr} : \mathcal{G}(I)_2 \to (\mathcal{G}(I)_{\text{red}})_2$ *is a surjective submersion.*

Proof. We first remark that this situation is a special case of an action of a Lie groupoid on a Lie groupoid (see [54]), namely the action of $\Sigma \rightrightarrows N$ on $\mathcal{G}(I) \rightrightarrows P$. One can then apply Lemma 2.1 of [54] to conclude that $\mathcal{G}(I)_{\mathrm{red}}$ is a Lie groupoid.

However, since the action of Σ on $\mathcal{G}(I)$ is the diagonal action induced by the action on P, our situation is rather simple and more can be said. One first notes that the freeness and properness of the action on P implies freeness and properness of the action on $\mathcal{G}(I)$, and thus $\mathcal{G}(I)_{\mathrm{red}}$ has a unique smooth structure with which the projection is a surjective submersion. The structure maps of $\mathcal{G}(I)_{\mathrm{red}}$ are uniquely determined by the condition that the projection should be a Lie groupoid map. For example, the source map must satisfy $s([p,q]) = [s(p,q)] = [q]$, which uniquely determines it. Similarly, the product must satisfy $[p,q]\cdot[q,r] = [(p,q)\cdot(q,r)] = [p,r]$, which defines it. One easily verifies that these maps are well defined, smooth and that s and t are submersions. For example, s is well defined, since if $g \in \Sigma$ can act on p, then $s([g\cdot p, g\cdot q]) = [g\cdot p] = [p]$.

For the last assertion, consider a vector (X,Y) in $(\mathcal{G}(I)_{\mathrm{red}})_2$ at the composable pair $([p,q],[q,r])$. Thus, $X \in T_{[p,q]}(\mathcal{G}(I)_{\mathrm{red}})$ and $Y \in T_{[q,r]}(\mathcal{G}(I)_{\mathrm{red}})$, and they satisfy $dt(Y) = ds(X) \in T_{[q]}P_{\mathrm{red}}$. Because $\mathrm{pr} : \mathcal{G}(I) \to \mathcal{G}(I)_{\mathrm{red}}$ is a submersion, then there exist $\widetilde{X} \in T_{(p,q)}(\mathcal{G}(I))$ and $\widetilde{Y} \in T_{(q,r)}(\mathcal{G}(I))$ that project to X and Y, respectively. The problem is that $Z := ds(\widetilde{X}) - dt(\widetilde{Y}) \in T_qP$ may be non-zero. However, the projection $P \to P_{\mathrm{red}}$ maps Z to zero, and, hence, we may construct a $\widetilde{Z} \in T_{(q,r)}(\mathcal{G}(I))$ that projects to zero under $\mathrm{pr} : \mathcal{G}(I) \to \mathcal{G}(I)_{\mathrm{red}}$ by means of the action on P. Thus, the pair $(\widetilde{X}, \widetilde{Y}+\widetilde{Z})$ is a vector in $\mathcal{G}(I)_2$ at (p,q) that projects to (X,Y). □

Given a free and proper action of $\Sigma \rightrightarrows N$ on $I : P \to N$ and a representation $E \to N$ of Σ, we obtain the associated vector bundle $E_{\mathrm{red}} \to P_{\mathrm{red}}$ that is obtained as the quotient of $I^*E \to P$ by the action of Σ (see Proposition 8.8.5 and the preceding text). In particular, E_{red} is of the same rank as E. One easily verifies that the trivial action of $\mathcal{G}(I) \rightrightarrows P$ on I^*E descends to an action of $\mathcal{G}(I)_{\mathrm{red}} \rightrightarrows P_{\mathrm{red}}$ on $E_{\mathrm{red}} \to P_{\mathrm{red}}$. Similarly, since we have an action of Σ on $I : \mathcal{G}(I) \to N$ and a representation $E \to N$, then we obtain an associated vector bundle, which, in this case, is simply the pull-back $t^*E_{\mathrm{red}} \to \mathcal{G}(I)$. Now, since $\theta \in \Omega^1(\mathcal{G}(I); \pi^*E)$ is basic with respect to Σ, then, together with the assumption that Σ is s-connected and Proposition 8.8.3, there is a unique 1-form $\theta_{\mathrm{red}} \in \Omega^1(\mathcal{G}(I)_{\mathrm{red}}; t^*E_{\mathrm{red}})$ such that $\theta = \mathrm{pr}^*\theta_{\mathrm{red}}$. This follows from Proposition 8.8.5.

This clarifies what we mean by the quotient $(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$ of the Pfaffian groupoid $(\mathcal{G}(I), \theta)$ by the action of Σ. We now turn to the proof of the theorem:

Proof of Theorem 8.6.1. We begin by proving item 1. Let us denote the Lie algebroid of $\mathcal{G}(I)$ by $A = T^IP$ and the Lie algebroid of $\mathcal{G}(I)_{\mathrm{red}}$ by A_{red}. We must verify the axioms of a Lie-Pfaffian groupoid (Definition 2.7.1).

1) Multiplicativity of θ_{red}: one notes that the multiplicativity expression

$$(m^*\theta_{\mathrm{red}})_{(g,h)} - (\mathrm{pr}_1^*\theta_{\mathrm{red}})_{(g,h)} - g\cdot(\mathrm{pr}_2^*\theta_{\mathrm{red}})_{(g,h)},$$

with $(g,h) \in (\mathcal{G}(I)_{\mathrm{red}})_2$, defines an E_{red}-valued 1-form on $(\mathcal{G}(I)_{\mathrm{red}})_2$, and that the pull-back of this form by the surjective submersion $\mathrm{pr} : \mathcal{G}(I)_2 \to (\mathcal{G}(I)_{\mathrm{red}})_2$ is precisely the corresponding expression for θ (using the fact that $\theta = \mathrm{pr}^*\theta_{\mathrm{red}}$ and that pr is a Lie groupoid morphism). Since the pull-back by a submersion is an injective map, the multiplicativity of θ implies the multiplicativity of θ_{red}.

2) Surjectivity of θ: the fact that θ is surjective together with $\theta = \mathrm{pr}^*\theta_{\mathrm{red}}$ implies that θ_{red} is surjective, since I^*E (the space in which θ takes values) is just the pullback of E_{red} (the space in which θ_{red} takes values).

3) $ds : C_{\theta_{\mathrm{red}}} \to s^*TP_{\mathrm{red}}$ is surjective: recall that $C_\theta = \mathrm{Ker}\ \theta \subset T\mathcal{G}(I)$ and $C_{\theta_{\mathrm{red}}} = \mathrm{Ker}\ \theta_{\mathrm{red}} \subset T\mathcal{G}(I)_{\mathrm{red}}$. The fact that $\theta = \mathrm{pr}^*\theta_{\mathrm{red}}$ directly implies that $d\mathrm{pr}(C_\theta) = C_{\theta_{\mathrm{red}}}$, and because $ds : C_\theta \to s^*TP$ is surjective and $\mathrm{pr} : P \to P_{\mathrm{red}}$ is a submersion, it follows that $ds : C_{\theta_{\mathrm{red}}} \to s^*TP_{\mathrm{red}}$ is surjective.

4) $A_{\mathrm{red}} \cap C_{\theta_{\mathrm{red}}}$ is a Lie subalgebroid of A_{red}: we prove this by showing that $A_{\mathrm{red}} \cap C_{\theta_{\mathrm{red}}}$ is the image of a Lie subalgebroid under a Lie algebroid map, namely the image of $A \cap C_\theta$ under the induced map $d\mathrm{pr} : A \to A_{\mathrm{red}}$, and that it is of constant rank. The tangent space of $\mathcal{G}(I)$ at (p, q) is spanned by vectors of the type $(X_\alpha + X_S, X_\beta + X_T)$, where $\alpha, \beta \in \mathcal{C}_{I(p)} = \mathcal{C}_{I(q)}$ such that $\rho(\alpha) = \rho(\beta)$ and $S, T \in \mathfrak{g}_{I(p)} = \mathfrak{g}_{I(q)}$. The kernel of the differential of $\mathrm{pr} : \mathcal{G}(I) \to \mathcal{G}(I)_{\mathrm{red}}$ at (p, q) is spanned by vectors of the type $(X_u + X_S, X_u + X_S)$, where $u \in \mathcal{S}_{I(p)}, S \in \mathfrak{g}_{I(p)}$. Thus, the tangent space of $\mathcal{G}(I)_{\mathrm{red}}$ at $[p, q]$ is spanned by equivalence classes of vectors represented by $(X_\alpha + X_S, X_\beta + X_T)$. Similarly, the tangent space of P at p is spanned by $(X_\alpha + X_S)$, the kernel of the differential of $\mathrm{pr} : P \to P_{\mathrm{red}}$ at p is spanned by $(X_u + X_S)$, and the tangent space of P_{red} at $[p]$ is spanned by equivalence classes of vectors represented by $(X_\alpha + X_S)$. Now, using (8.11), $C_{\theta_{\mathrm{red}}}$ at $[p, q]$ is spanned by vectors that are represented by $(X_\alpha + X_S, X_\alpha + X_T)$ and $C_{\theta_{\mathrm{red}}} \cap \mathrm{Ker}\ ds$ is spanned by vectors that are represented by $(X_u + X_S, X_u + X_T)$. It is of constant rank, namely the rank of $\mathfrak{g}$, and it is precisely the image of $C_\theta \cap \mathrm{Ker}\ ds$ under $d\mathrm{pr} : T\mathcal{G}(I) \to pr^*T\mathcal{G}(I)_{\mathrm{red}}$. Restricting the latter map to the units, we conclude that $A_{\mathrm{red}} \cap C_{\theta_{\mathrm{red}}}$ is the image of $A \cap C_\theta$ under $d\mathrm{pr}$ and that it is of constant rank. This concludes the proof that $(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$ is a Pfaffian groupoid.

5) $C_{\theta_{\mathrm{red}}} \cap \mathrm{Ker}\ dt = C_{\theta_{\mathrm{red}}} \cap \mathrm{Ker}\ ds$: from the previous item, we also see that $C_{\theta_{\mathrm{red}}} \cap \mathrm{Ker}\ dt$ is spanned by vectors represented by $(X_u + X_S, X_u + X_T)$, and hence $C_{\theta_{\mathrm{red}}} \cap \mathrm{Ker}\ dt = C_{\theta_{\mathrm{red}}} \cap \mathrm{Ker}\ ds$ and $(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$ is a Lie-Pfaffian groupoid.

We move on to item 2. The setting is depicted in the following diagram:

$$\begin{tikzcd}
& & \mathcal{G}(I) \arrow[dl, "pr"'] \arrow[dd, shift left] \arrow[dd, shift right] & \\
& \mathcal{G}(I)_{\mathrm{red}} \arrow[dd, shift left] \arrow[dd, shift right] & & \curvearrowleft \Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}}) \\
\Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}}) \curvearrowright & & P \arrow[dl, "pr"'] & \\
& P_{\mathrm{red}} & &
\end{tikzcd}$$

It is sufficient to prove that $\Gamma(\mathcal{G}(I), \theta)$ is an isomorphic prolongation of $\Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$, from which it follows that the two are Cartan equivalent.

There is a canonical action of $\mathcal{G}(I)_{\mathrm{red}}$ on $\mathrm{pr} : P \to P_{\mathrm{red}}$ given by $[p, q] \cdot q = p$, where p is the first component of the unique representative of $[p, q]$ that has q as the second component. Using this action, any element $\sigma \in \Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$ with domain

$U = \mathrm{Dom}(\sigma)$ induces a bisection $\widetilde{\sigma}$ of $\mathcal{G}(I)$ with domain $\mathrm{pr}^{-1}(U) \subset P$ given by

$$\widetilde{\sigma}(p) = (\sigma(\mathrm{pr}(p)) \cdot p, p). \tag{8.14}$$

We show that the induced generalized pseudogroup on $\mathcal{G}(I) \rightrightarrows P$ is precisely $\Gamma(\mathcal{G}(I), \theta)$. Unraveling the definition of an isomorphic prolongation, this shows that $\Gamma(\mathcal{G}(I), \theta)$ is an isomorphic prolongation of $\Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$. There are two things to verify:

a) Given $\widetilde{\sigma}$ induced by an element $\sigma \in \Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$ by (8.14), then $\widetilde{\sigma} \in \Gamma(\mathcal{G}(I), \theta)$, i.e. $\widetilde{\sigma}^*\theta = 0$.

b) Locally, any $\widetilde{\sigma} \in \Gamma(\mathcal{G}(I), \theta)$ arises from some $\sigma \in \Gamma(\mathcal{G}(I)_{\mathrm{red}}, \theta_{\mathrm{red}})$, i.e. for every $p \in \mathrm{Dom}(\widetilde{\sigma})$ there exists a neighborhood $U \subset \mathrm{Dom}(\widetilde{\sigma})$ of p such that $\widetilde{\sigma}|_U$ is induced by some σ via (8.14).

We first prove a). Let $p \in P$ and let $q \in P$ be the element satisfying $\widetilde{\sigma}(p) = (q, p)$. Choose a local section η of $\mathrm{pr} : P \to P_{\mathrm{red}}$ such that $p \in \mathrm{Im}\ \eta$. By (8.14), $\sigma = \mathrm{pr} \circ \widetilde{\sigma} \circ \eta$, and so

$$0 = \sigma^*\theta_{\mathrm{red}} = (\mathrm{pr} \circ \widetilde{\sigma} \circ \eta)^*\theta_{\mathrm{red}} = \eta^*(\widetilde{\sigma}^*\theta). \tag{8.15}$$

Hence, $\widetilde{\sigma}^*\theta$ vanishes on a horizontal subspace of T_pP with respect to the projection $\mathrm{pr} : P \to P_{\mathrm{red}}$. Now, the vertical subbundle of TP is spanned by vector fields of the type $X_u + X_s \in \mathfrak{X}(P)$, with $u \in \Gamma(\mathcal{S})$ and $S \in \Gamma(\mathfrak{g})$ at the image of the action map of $\mathcal{S}^{(1)}$ on P. Choose a curve g_ϵ in Σ such that $g_0 = 1_{I(p)}$ and $\frac{d}{d\epsilon}\big|_{\epsilon=0}\, g_\epsilon \cdot p = (X_u + X_S)_p$, and hence $\frac{d}{d\epsilon}\big|_{\epsilon=0}\, g_\epsilon \cdot q = (X_u + X_S)_q$. Then,

$$\begin{aligned}
(d\widetilde{\sigma})_p(X_u + X_S) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \widetilde{\sigma}(g_\epsilon \cdot p) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\sigma(\mathrm{pr}(g_\epsilon \cdot p))) \cdot (g_\epsilon \cdot p), g_\epsilon \cdot p) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\sigma(\mathrm{pr}(p))) \cdot (g_\epsilon \cdot p), g_\epsilon \cdot p) \\
&= \frac{d}{d\epsilon}\Big|_{\epsilon=0} (g_\epsilon \cdot q, g_\epsilon \cdot p) \\
&= (X_u + X_S, X_u + X_S)_{(q,p)}.
\end{aligned} \tag{8.16}$$

And so, by the definition of θ,

$$\begin{aligned}
(\widetilde{\sigma}^*\theta)_p(X_u + X_S) &= (\theta)_{(q,p)}(X_u + X_S, X_u + X_S) \\
&= \Omega_p(X_u + X_S) - \Omega_q(X_u + X_S) \\
&= 0.
\end{aligned}$$

Therefore, $\widetilde{\sigma} \in \Gamma(\mathcal{G}(I), \theta)$.

Next, we prove b). Let $\widetilde{\sigma} \in \Gamma(\mathcal{G}(I), \theta)$, i.e. $\widetilde{\sigma}$ is a local bisection of $\mathcal{G}(I)$ that satisfies $\widetilde{\sigma}^*\theta = 0$. We first show that, locally, $\widetilde{\sigma}$ descends to a local bisection σ of P_{red}. Since $\mathrm{pr} : P \to P_{\mathrm{red}}$ is a submersion, any point in $\mathrm{Dom}(\widetilde{\sigma})$ has a neighborhood $U \subset \mathrm{Dom}(\widetilde{\sigma})$

such that the fibers of $\mathrm{pr}|_U : U \to P_{\mathrm{red}}$ are connected, and we may, thus, assume that the domain of $\widetilde{\sigma}$ has this property. Let $p \in \mathrm{Dom}(\widetilde{\sigma})$, set $x := I(p)$ and set $U_x := \mathrm{Dom}(\widetilde{\sigma}) \cap \mathrm{pr}^{-1}([p])$, where $[p] = \mathrm{pr}(p) \in P_{\mathrm{red}}$. Since Σ acts freely on P, the action map $\Sigma_s \times_I P \to P$ provides a diffeomorphism $U_x \cong V_x$ between U_x and an open subset $V_x \subset s^{-1}(x) \subset \Sigma$ which maps a point $q \in U_x$ to the unique arrow $g \in \Sigma$ satisfying $g \cdot p = q$. We need to show that for any $g \in V_x$, $\widetilde{\sigma}(g \cdot p) = g \cdot \widetilde{\sigma}(p)$. Equivalently, passing to the pseudogroup point of view, writing $\widetilde{\sigma}(q) = (\phi(q), q)$ for all $q \in \mathrm{Dom}(\widetilde{\sigma})$ for some uniquely determined $\phi \in \mathrm{Diff}_{\mathrm{loc}}(P)$, we must show that $\phi(g \cdot p) = g \cdot \phi(p)$. Since V_x is connected, we can choose a path g_ϵ in V_x such that $g_0 = 1_x$ and $g_1 = g$. We show that

$$\phi(g_\epsilon \cdot p) = g_\epsilon \cdot \phi(p), \qquad \forall\, \epsilon,$$

by showing that both sides are integral curves of the same time dependent vector field, and since $\phi(g_\epsilon \cdot p) = g_\epsilon \cdot \phi(p)$ at $\epsilon = 0$, this will imply that they are equal for all ϵ. For each ϵ, $\frac{d}{d\epsilon} g_\epsilon \in T_{g_\epsilon} s^{-1}(x) \subset T_{g_\epsilon}\Sigma$, and applying $dR_{g_\epsilon^{-1}}$ gives an element of $(\mathcal{S}^{(1)})_{t(g_\epsilon)}$, where $\mathcal{S}^{(1)}$ is the Lie algebroid of Σ. Thus, we may find a time dependent section $u_\epsilon \in \Gamma(\mathcal{S}^{(1)})$ such that if $\widetilde{u}_\epsilon \in \mathfrak{X}(\Sigma)$ is the induced right invariant vector field, then the value of $\widetilde{u}_\epsilon(g_\epsilon) = \frac{d}{d\epsilon} g_\epsilon$. Now, if $X_{u_\epsilon} \in \mathfrak{X}(P)$ is the image of u_ϵ under the infinitesimal action map $\Gamma(\mathcal{S}^{(1)}) \to \mathfrak{X}(P)$, then $(X_{u_\epsilon})_{g_\epsilon \cdot p} = \frac{d}{d\epsilon} g_\epsilon \cdot p$. But by Proposition 8.3.4, $d\phi(X_{u_\epsilon}) = X_{u_\epsilon}$, and so

$$\frac{d}{d\epsilon}\phi(g_\epsilon \cdot p) = d\phi((X_{u_\epsilon})_{g_\epsilon \cdot p}) = (X_{u_\epsilon})_{\phi(g_\epsilon \cdot p)},$$

and on the other hand,

$$\frac{d}{d\epsilon} g_\epsilon \cdot \phi(p) = (X_{u_\epsilon})_{g_\epsilon \cdot \phi(p)}.$$

Thus, we may define a local section σ of $\mathcal{G}(I)_{\mathrm{red}}$ with domain $\mathrm{pr}(\mathrm{Dom}(\widetilde{\sigma}))$ by defining $\sigma([p]) := [\widetilde{\sigma}(p)]$ for some representative p of $[p]$ (and one easily checks that it is smooth). The two are related by 8.14, and choosing a section η of $\mathrm{pr} : P \to P_{\mathrm{red}}$, we see as before that $\sigma = \mathrm{pr} \circ \widetilde{\sigma} \circ \eta$. Finally, reading (8.15) from right to left, we see that $\sigma^*\theta_{\mathrm{red}} = 0$. $\square$

8.7 Examples of Reduction

Let us look at two examples of reduction.

In the first example, we illustrate the full reduction procedure through the simple but instructive case of a Lie group. Starting from the Lie pseudogroup of right translations on a Lie group, the generalized pseudogroup obtained by reduction will turn out to be the Lie group itself viewed as a generalized pseudogroup (see Example 3.6.3). In this case, the reduction procedure reveals the "true" abstract object underlying the Lie pseudogroup.

Example 8.7.1. (Lie groups) Let H be a Lie group with Lie algebra $\mathfrak{h}$. We continue from Examples 5.2.13 and 8.3.3, where we discussed the realization $(P = H, \Omega = \Omega_{\mathrm{MC}})$ of the Cartan algebroid $(\mathcal{C} = \mathfrak{h}, \mathfrak{g} = 0)$.

First, the Pfaffian groupoid induced by the realization is the pair $(\mathcal{G}(I), \theta)$, with

$$\mathcal{G}(I) = H \times H \qquad \text{and} \qquad \theta = s^*\Omega_{\text{MC}} - t^*\Omega_{\text{MC}}.$$

Furthermore,

$$E = \text{Ker}\ \rho = \mathfrak{h}$$

and thus $\theta \in \Omega^1(H \times H; \mathfrak{g})$. From (8.13), we see that $E = \mathfrak{h}$ becomes a representation of $\mathcal{S}^{(1)} = \mathfrak{h}$ by the map

$$\mathfrak{h} \mapsto \text{End}(\mathfrak{h}), \qquad X \mapsto [X, \cdot], \tag{8.17}$$

i.e. the adjoint representation.

To perform the reduction, we can use the integration of $\mathcal{S}^{(1)} = \mathfrak{h}$ to the Lie group $\Sigma = H$. The action of $\mathcal{S}^{(1)} = \mathfrak{h}$ on $P = H$ integrates to the action of $\Sigma = H$ on $P = H$ given by

$$(\Sigma = H) \times (P = H) \to (P = H), \qquad (h, h') \mapsto h \cdot h',$$

and integrating (8.17) results in the adjoint action of $\Sigma = H$ on $E = \mathfrak{h}$. With this integration, we can divide out by the action of Σ. The result, as one might expect, is that we recover the Lie group H together with its Maurer-Cartan form. More precisely, the reduced Lie-Pfaffian groupoid is

$$\begin{array}{ccc} \mathcal{G}(I)_{\text{red}} & \cong & H \\ \downarrow\downarrow & & \\ P_{\text{red}} & \cong & \{*\}, \end{array}$$

where the isomorphism $\mathcal{G}(I)_{\text{red}} \cong H$ is canonical, and, under this isomorphism, the reduced form θ_{red} corresponds to the Maurer-Cartan form on H. Let us describe the isomorphism explicitly. Elements of $\mathcal{G}(I)$ are pairs $(g, g') \in H \times H$, and, in turn, the quotient $\mathcal{G}(I)_{\text{red}}$ consists of equivalence classes of the form $[g, g']$, with the equivalence relation given by $(g, g') \sim (hg, hg')$, where $h \in H$. Finally, the isomorphism is given by $\mathcal{G}(I)_{\text{red}} \xrightarrow{\cong} H$, $[g, g'] \mapsto g^{-1}g'$. Unraveling the definitions, one readily verifies that this is indeed an isomorphism of Lie groups and that the reduced form θ_{red} corresponds to Ω_{MC} on H.

Finally, since a local solution of $(\mathcal{G}(I)_{\text{red}}, \theta_{\text{red}})$ is simply an element of H, then

$$\Gamma(\mathcal{G}(I)_{\text{red}}, \theta_{\text{red}}) = H,$$

and, unraveling the definitions, one sees that a generating element $R_{h^{-1}} \in \Gamma(H, \Omega_{\text{MC}})$ of the original pseudogroup, where $h \in H$, descends to the element $h \in \Gamma(\mathcal{G}(I)_{\text{red}}, \theta_{\text{red}})$ of the reduced generalized pseudogroup.

To summarize, the reduction procedure recovers the Lie group H from the realization (H, Ω_{MC}) and detects that $\Gamma(H, \Omega_{\text{MC}})$ comes from an action of the Lie group H on the manifold H, but viewing H as a generalized pseudogroup and not as a Lie group. $\diamondsuit$

As a second example, let us revisit Example 5.1.11 of Cartan. In this example, it is evident that the Lie pseudogroup Cartan considers comes from an action of the pseudogroup of local diffeomorphisms of $\mathbb{R}$ on $\mathbb{R}^2$, or, more precisely, that it is an isomorphic prolongation of $\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R})$. This fact is revealed by the reduction procedure:

Example 8.7.2. Let (x, y, u) be coordinates on $\mathbb{R}^3$. In Example 5.1.11, given the initial Lie pseudogroup (5.9), Cartan derives the pseudogroup in normal form Γ on $P = \mathbb{R}^3 \backslash \{y = 0\}$ generated by the local diffeomorphisms

$$(x, y, u) \mapsto (f(x), \frac{y}{f'(x)}, \frac{uf'(x) + f''(x)}{(f'(x))^2}), \qquad f \in \mathrm{Diff}_{\mathrm{loc}}(\mathbb{R}). \tag{8.18}$$

As computed in Example 5.4.1, this is the pseudogroup $\Gamma(P, \Omega)$ induced by the realization (P, Ω) of the Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over $N = \{*\}$, where: $\mathcal{C}$ is the pre-Lie algebroid over a point with basis $\{e^1, e^2\}$ and bracket

$$[e^1, e^2] = e^1;$$

$\mathfrak{g} \subset \mathrm{Hom}(\mathcal{C}, \mathcal{C})$ is the rank 1 vector bundle over a point with basis t that satisfies

$$t(e^1) = e^2 \quad \text{and} \quad t(e^2) = 0;$$

$I : P \to N$ maps all of P to the point N; $\Omega \in \Omega^1(P; \mathcal{C})$ decomposes as $\Omega = \omega_1 e^1 + \omega_2 e^2$ with

$$\omega_1 = ydx \quad \text{and} \quad \omega_2 = udx + \frac{1}{y}dy;$$

and a choice of $\Pi \in \Omega^1(P; \mathfrak{g})$ is given by $\Pi = \pi t$, with

$$\pi = \frac{1}{y}(u^2 dx - udy + du).$$

From this data we can compute the systatic action. The systatic space $\mathcal{S}$ is spanned by e^2 and the 1st systatic space $\mathcal{S}^{(1)}$ by $\{e^2, t\}$. One readily computes the bracket of $\mathcal{S}$. The action of $\mathcal{S}^{(1)}$ on P is computed by inverting (Ω, Π) and is given by

$$\mathcal{S}^{(1)} \to \mathfrak{X}(P), \qquad e^2 \mapsto y\frac{\partial}{\partial y} + uy\frac{\partial}{\partial u},\ t \mapsto y\frac{\partial}{\partial u}.$$

We see that the orbits of the action are the submanifolds of P given by $x = \mathrm{cnst}$. Thus, $P_{\mathrm{red}} = P/\mathcal{S}^{(1)} \cong \mathbb{R}$ and we can choose a coordinate x so that $\mathrm{pr} : P \to P_{\mathrm{red}}$ is simply the projection $(x, y, u) \mapsto x$.

Thus, already at this point, we see that the reduction will yield a Lie-Pfaffian groupoid over $\mathbb{R}$. As in the previous example, one now proceeds to construct the Pfaffian groupoid from the realization, integrate the 1st systatic algebroid and the action, and compute the reduced Lie-Pfaffian groupoid and its induced generalized pseudogroup of local solutions. We will leave it to the reader to fill in the details and show that the étale groupoid induced by the reduced generalized pseudogroup and the étale groupoid induced by $\mathrm{Diff}_{\mathrm{loc}}(\mathbb{R})$ are isomorphic. $\diamondsuit$

8.8 Appendix: Basic Forms

In this appendix to the chapter, we recall the notion of a basic form in the setting of Lie groupoid and Lie algebroid actions.

Let us begin with the more familiar case of Lie groups. Let $\pi : P \to B$ be a principal bundle with structure group G, hence P is equipped with a free and proper left action of the Lie group G, and let V be a representation of the Lie group G. A V-valued differential form $\theta \in \Omega^*(P; V)$ is said to be basic if it is horizontal and G-equivariant. Horizontal means that θ vanishes if applied to at least one vertical tangent vector, while G-equivariance means that

$$L_g^*\theta = g \cdot \theta, \qquad \forall\, g \in G,$$

where $L_g : P \to P,\ p \mapsto g \cdot p$. Basic V-valued forms on P, which we denote by $\Omega^*_{\text{bas}}(P; V)$, are precisely those V-valued forms on P that come from forms on the base B. More precisely, recalling that the associated vector bundle $E = E(P, V)$ on B is the vector bundle obtained as the quotient of the trivial vector bundle $P \times V \to P$ by the induced action of G on $P \times V$ given by $g \cdot (p, v) = (g \cdot p, g \cdot v)$, the pull-back by π gives a linear isomorphism

$$\pi^* : \Omega^*(B; E) \xrightarrow{\simeq} \Omega^*_{\text{bas}}(P; V).$$

Note that on the right hand side we are implicitly using the canonical isomorphism $P \times V \xrightarrow{\simeq} \pi^*E$ which at a point p maps $v \mapsto [p, v]$. See [32] (Section II.5) for more details.

The notion of a basic form generalizes naturally to the realm of Lie groupoids. For simplicity, we restrict to the case of 1-forms, which is of relevance to us. For brevity, given a surjective submersion $\pi : P \to M$ and a vector bundle $E \to M$, we write $\Omega^1(P; E)$ for the space of π^*E-valued 1-forms on P.

Definition 8.8.1. *Let $\pi : P \to M$ be a surjective submersion equipped with an action of a Lie groupoid $\mathcal{G} \rightrightarrows M$ and let $E \to M$ be a representation of $\mathcal{G}$. A 1-form $\theta \in \Omega^1(P; E)$ is **horizontal (with respect to $\mathcal{G}$)** if it vanishes on all vectors that are tangent to the orbits of the action of $\mathcal{G}$. A 1-form $\theta \in \Omega^1(P; E)$ is **basic (with respect to $\mathcal{G}$)** if it is both horizontal and satisfies*

$$\theta(g \cdot X) = g \cdot \theta(X), \tag{8.19}$$

for all $g \in \mathcal{G}$ and $X \in T_pP$ for which $s(g) = \pi(p)$. We denote the space of basic 1-forms by $\Omega^1_{bas}(P; E)$.

Let us clarify the left hand side of (8.19). This is best understood in terms of the action groupoid $\mathcal{G} \ltimes P \rightrightarrows P$ associated with the action of $\mathcal{G}$ on P. We regard θ as a form on the base of the action groupoid. The fact that θ is horizontal is equivalent to the condition that the restriction of θ to any orbit $\mathcal{O} \subset P$ of $\mathcal{G} \ltimes P$ must vanish. If θ is horizontal, then it descends to a map $\theta : N\mathcal{O} \to E|_{\mathcal{O}}$ on the normal bundle to an orbit. With this in mind, (8.19) should be read as $\theta((g, p) \cdot [X]) = g \cdot \theta([X])$, for all $(g, p) \in \mathcal{G} \ltimes P$ and $X \in T_pP$, where (g, p) acts on $[X]$ via the normal representation (see Section 2.1).

The notion of a basic form can also be defined at the infinitesimal level.

Definition 8.8.2. *Let $\pi : P \to M$ be a surjective submersion equipped with an action $a : \pi^*A \to TP$ of a Lie algebroid $A \to M$ and let $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ be a representation of A. A 1-form $\theta \in \Omega^1(P; E)$ is* ***horizontal (with respect to** A**)*** *if $\theta(a(\alpha)) = 0$ for all $\alpha \in \Gamma(A)$. A 1-form $\theta \in \Omega^1(P; E)$ is* ***basic (with respect to** A**)*** *if it is both horizontal and satisfies*

$$\theta([a(\alpha), X]) = (\pi^*\nabla)_{\pi^*\alpha}\theta(X), \tag{8.20}$$

for all $\alpha \in \Gamma(A)$ and $X \in \mathfrak{X}(P)$.

Let clarify the right-hand side of (8.20). The action of A on P induces the action algebroid $\pi^*A \to P$, and the representation $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ of A induces a representation $\pi^*\nabla : \Gamma(\pi^*A) \times \Gamma(\pi^*E) \to \Gamma(\pi^*E)$ of π^*A. The construction is analogous to the construction of the pull-back of a usual connection. Namely, on pull-back sections one defines $(\pi^*\nabla)_{\pi^*\alpha}(\pi^*\sigma)|_p := \nabla_\alpha(\sigma)|_{\pi(p)}$, and then one extends the definition by the Leibniz identity. The connection is easily seen to be flat. Note that if the action of A on P and the representation ∇ come from an action of $\mathcal{G}$ on P and a representation E, then the representation $\pi^*\nabla$ is induced by a representation π^*E of $\mathcal{G} \ltimes P$. Namely, the one in which an arrow $(g, p) \in \mathcal{G} \ltimes P$ acts on a vector $v \in (\pi^*E)_p$ by $(g, p) \cdot v = g \cdot v \in (\pi^*E)_{g \cdot p}$.

In the following proposition, we give two alternative characterizations of basic forms and relate the global and infinitesimal descriptions.

Proposition 8.8.3. *Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid acting on a surjective submersion $\pi : P \to M$ and let $E \to M$ be a representation of $\mathcal{G}$. Given any $\theta \in \Omega^1(P; E)$, the following are equivalent:*

1. *$s^*\theta - t^*\theta = 0$, where s, t is the source and target maps of $\mathcal{G} \ltimes P$.*
2. *$\theta - (t \circ \sigma)^*\theta = 0$ for all $\sigma \in \mathrm{Bis}_{\mathrm{loc}}(\mathcal{G} \ltimes P)$.*
3. *θ is basic with respect to $\mathcal{G}$.*

If A is the Lie algebroid of $\mathcal{G}$ (together with the induced representation of $\mathcal{G}$ and action on P), then conditions 1-3 imply that:

4. *θ is basic with respect to A.*

If $\mathcal{G}$ is s-connected, then conditions 1-4 are equivalent.

Remark 8.8.4. A few words of explanation are in order. In condition 2, we view both $s^*\theta$ and $t^*\theta$ as elements of $\Omega^1(\mathcal{G} \ltimes P; t^*E)$. The latter is clear, while for the former one makes use of the representation, namely $(s^*\theta)_g(X) = g \cdot \theta(ds(X)) \in E_{t(g)}$. In condition 3, we view both θ and $(t \circ \sigma)^*\theta$ as locally defined elements of $\Omega^1(P; E)$ whose domains of definitions are the domain of σ. For the former one simply restricts θ to $\mathrm{Dom}(\phi)$, while in the latter one uses again the representation. Namely, $((t \circ \sigma)^*\theta)_p = \sigma(p)^{-1} \cdot ((t \circ \sigma)^*\theta)_p$ for all $p \in \mathrm{Dom}(\sigma)$. $\diamondsuit$

Proof. We first prove the equivalence of 1-3. To go from 1 to 2, simply pull back by σ. Next, assume 2. First, let $X \in T_pP$ such that X is tangent to an orbit. One can always find an $\widetilde{X} \in T_{1_p}(\mathcal{G} \ltimes P)$ such that $ds(\widetilde{X}) = -dt(\widetilde{X}) = X$ (since X is tangent to an orbit, there exist vectors in $T_{1_p}(s^{-1}(p))$ and $T_{1_p}(t^{-1}(p))$ projecting to X, so simply take their difference) and a local bisection σ of $\mathcal{G} \ltimes P$ such that $\sigma(p) = 1_p$ and $d\sigma(X) = \widetilde{X}$. Applying $\theta - (t \circ \sigma)^*\theta$ on X, we see that $2\theta(X) = 0$, and hence θ is horizontal. To prove (8.19), for every $g \in \mathcal{G}$ and $X \in T_pP$ such that $s(g) = \pi(p)$, simply choose a bisection σ such that $\sigma(p) = g$ and note that $\theta(g \cdot X) = \theta(d(t \circ \sigma)(X))$. Finally, assume 3. Let $(g, p) \in \mathcal{G} \ltimes P$ and $X \in T_{(g,p)}(\mathcal{G} \ltimes P)$, then $(s^*\theta - t^*\theta)(X) = g \cdot \theta(ds(X)) - \theta(dt(X))$, which vanishes by (8.19), since, by the definition of the normal representation, $[dt(X)] = [g \cdot ds(X)]$.

The most direct approach for proving that 1-3 imply 4 is to go from 3 to 4 by differentiating (8.19) and discovering (8.20). More conceptually, we will go from 1 to 4 by appealing to the notion of multiplicative forms and their infinitesimal counterparts, the Spencer operators. In [13], the notion of a representation-valued multiplicative form on a Lie groupoid is studied (see also Section 2.4), and in Theorem 1 of that paper it is shown that such a form linearizes to a so called Spencer operator on the associated Lie algebroid (see Section 2.5) and that the map sending the multiplicative form to the Spencer operator is injective if the Lie groupoid is source-connected (note that while the authors actually assume source-simply connectedness in the statement of the theorem, source-connectedness is sufficient for the injectivity assertion). To apply the theorem to our problem, one first notes that the form $\omega := s^*\theta - t^*\theta \in \Omega^1(\mathcal{G} \ltimes P; t^*E)$ is multiplicative (this simple fact is proven in Proposition 8.4.2 and also mentioned Example 2.4 in [13]). Next, one computes the induced Spencer operator $D_\omega : \mathfrak{X}(P) \times \Gamma(\pi^*A) \to \Gamma(\pi^*E)$ on the Lie algebroid π^*A of $\mathcal{G} \ltimes P$, obtaining the formula

$$(D_\omega)_X(\pi^*\alpha) := \theta([a(\alpha), X]) - (\pi^*\nabla)_{\pi^*\alpha}\theta(X), \qquad \forall\, \alpha \in \Gamma(A),\ X \in \mathfrak{X}(P).$$

This formula is given in Example 2.10 in [13] (and it can be computed by the same method used to prove Lemma 2.5.2). Now, since the vanishing of ω implies that D_ω vanishes, and hence that (8.20) is satisfied (and we already saw that 1 implies that θ is horizontal), then 1 implies 4. Conversely, if $\mathcal{G}$ is s-connected, and hence $\mathcal{G} \ltimes P$ is s-connected, then, by the injectivity of the map described above, the vanishing of D_ω implies the vanishing of ω and hence 4 implies 1. □

Now, as for Lie groups, if the action of $\mathcal{G}$ on P is free and proper, then the basic forms are precisely those that descend to the orbit space. To be more precise, recall that the orbit space, which we denote by $P_{\text{red}} = P/\mathcal{G}$, has a unique smooth structure with which the projection $pr : P \to P_{\text{red}}$ is a surjective submersion. The vector bundle $\pi^*E \to P$, in turn, descends to a vector bundle $E_{\text{red}} = E_{\text{red}}(P, \mathcal{G})$ over P_{red}, the **associated vector bundle**. The construction is as in the case of Lie groups. Namely, E_{red} is obtained as the quotient of π^*E by the action of $\mathcal{G}$ given by $g \cdot (p, v) = (g \cdot p, g \cdot v)$, for all $p \in P$, $v \in E_{\pi(p)}$ and $g \in s^{-1}(\pi(p))$. Thus, $[g \cdot p, v] = [p, g^{-1} \cdot v] \in (E_{\text{red}})_{[p]}$.

Finally, the proof of the following proposition, which we omit, is analogous to the case of Lie groups:

Proposition 8.8.5. *Let $\pi : P \to M$ be a surjective submersion equipped with a free and proper action of a Lie groupoid $\mathcal{G} \rightrightarrows M$, and let $E \to M$ be a representation of $\mathcal{G}$. The pull-back by the projection $pr : P \to P_{red}$ gives a 1-1 correspondence*

$$pr^* : \Omega^1(P_{red}; E_{red}) \xrightarrow{\simeq} \Omega^1_{bas}(P; E).$$

Chapter 9

Prolongations

In this chapter, we study Cartan's construction of a *prolongation* of a pseudogroup in normal form. Given a normal form data (the data encoding a set of structure equations, see Definition 5.1.1), Cartan constructs a new normal form data on a larger space that, in a sense, extends the original one. This construction plays an important role in various classification problems of Lie pseudogroups that Cartan addresses in [8] (e.g., see pp. 26-28, and, in particular, the fundamental theorem on p. 27). In terms of the Lie pseudogroup induced by the normal form data, the connection between a normal form data and its prolongation is very much related to the connection between the k-th order defining equations of a Lie pseudogroup and their differential consequences, the induced $k+1$-th order equations.

In this chapter, we study Cartan's construction in the context of our modern framework of Cartan algebroids and realizations. We define the *prolongation of a realization* and, at the infinitesimal level, the *prolongation of Cartan algebroid*. We identify the obstructions to performing a single prolongation step in both cases and discuss the notion of *formal integrability*, i.e. infinite prolongation steps. We will see that this notion of prolongation is very much related to the notion of prolongation of PDEs that was discussed in Chapter 1. In a sense, Cartan's structure theory is an abstract framework that encodes the properties that are essential for the study of the notion of prolongation of PDEs that define Lie pseudogroups. This way of viewing the structure theory was influential in the development of the notions of Pfaffian bundles, Pfaffian groupoids and Pfaffian algebroids, the abstract approach to PDEs and prolongation that was studied in [62] and discussed in Sections 1.9 and 2.7.

9.1 Cartan's Formulation

We begin by explaining Cartan's construction of the prolongation of a pseudogroup in normal form (see pp. 229-234 in [6] and p. 26 in [8]). In contrast to our presentation in the previous sections titled "Cartan's Formulation", we found it easier and clearer in this case to use the modern language to convey Cartan's ideas. We encourage the reader to compare this short discussion with Cartan's own writings.

Let (P,Ω) be a realization of a Cartan algebroid $(\mathcal{C},\mathfrak{g})$ and let us assume that $\mathfrak{g}^{(1)}$ is of constant rank. Consider the pseudogroup $\Gamma(P,\Omega)$ that is defined by the invariance property (5.14). Given an element $\psi\in\Gamma(P,\Omega)$, the fact that ψ preserves Ω implies that ψ also preserves the Maurer-Cartan expression $d\Omega+\frac{1}{2}[\Omega,\Omega]$ (easy to check). This fact, together with the structure equation (5.12), implies that any $\Pi\in\Omega^1(P;I^*\mathfrak{g})$ as in

Definition 5.2.11 of a realization satisfies

$$(\psi^*\Pi - \Pi) \wedge \Omega = 0.$$

This last observation led Cartan to the following construction: consider the space $P^{(1)}$ consisting of all elements $\Pi \in \mathrm{Hom}(TP, I^*\mathfrak{g})$ that satisfy (5.12) and (5.13) pointwise. As a consequence of the structure equation, $P^{(1)}$ has the structure of an affine bundle over P modeled on the vector bundle $I^*\mathfrak{g}^{(1)}$ (see discussion preceding Proposition 5.2.20). Now, an element $\psi \in \Gamma$ induces a local diffeomorphism $\widehat{\psi}$ of $P^{(1)}$ by pull-back, i.e. $\hat{\psi} : \Pi \mapsto \psi^*\Pi$. The set of all local diffeomorphisms that arise in this fashion induces a pseudogroup on $P^{(1)}$ which is an isomorphic prolongation of $\Gamma(P, \Omega)$. This pseudogroup, denoted by $\Gamma^{(1)}(P, \Omega)$, is called the *1st prolongation* of $\Gamma(P, \Omega)$.

Cartan then proceeds to show that $\Gamma^{(1)}(P, \Omega)$ is again a pseudogroup in normal form, i.e. that there exists a normal form data for which $\Gamma^{(1)}(P, \Omega)$ is the induced pseudogroup (see Equation (2) on the second page of [8]). In the next section, inspired by Cartan's construction, we define the 1st prolongation of a realization and identify the conditions under which it itself is a realization.

9.2 Prolongation of a Realization

In this section, guided by Cartan's notion of a prolongation of a pseudogroup in normal form, we construct the prolongation of a realization. Cartan's prolongation is intimately related to the notion of a prolongation in the setting of PDEs (see Chapter 1). This relationship has influenced us in both directions. On the one hand, our understanding of Cartan's prolongation was largely facilitated by the modern understanding of prolongations and formal integrability of PDEs. On the other hand, the ideas in Cartan's approach led us to the simplified proof of formal integrability of PDEs that was presented in Section 1.6.

Let (P, Ω) be a realization of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ and assume that the 1st prolongation $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ (Definition 1.2.5) is of constant rank. Out of this data we construct a new pair $(P^{(1)}, \Omega^{(1)})$ as follows. First,

$$P^{(1)} := \{ \Pi \in \mathrm{Hom}(TP, I^*\mathfrak{g}) \mid d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega \text{ and } (\Omega, \Pi) : TP \xrightarrow{\simeq} I^*(\mathcal{C}, \mathfrak{g}) \}.$$

Here we are being slightly sloppy with the notation. If $\Pi \in \mathrm{Hom}(TP; I^*\mathfrak{g})$ belongs to the fiber over $p \in P$, then the conditions should be read as: $(d\Omega + \frac{1}{2}[\Omega, \Omega])_p = \Pi \wedge \Omega_p$ and $(\Omega_p, \Pi) : T_pP \to (\mathcal{C}, \mathfrak{g})_{I(p)}$ is a linear isomorphism. The space $P^{(1)}$ is equipped with two projections:

$$\pi : P^{(1)} \to P \qquad \text{and} \qquad I : P^{(1)} \to N,$$

where π maps Π to its base point and I is the composition of $\pi : P^{(1)} \to P$ with $I : P \to N$. The projection $\pi : P^{(1)} \to P$ is an affine bundle modeled on the vector bundle $I^*\mathfrak{g}^{(1)}$ over P, since a choice of $\Pi_0 \in \Omega^1(P; I^*\mathfrak{g})$ as in Definition 5.2.11 induces the trivialization

$$P^{(1)} \to I^*\mathfrak{g}^{(1)}, \qquad P^{(1)}_p \ni \Pi \mapsto (\Pi - (\Pi_0)_p) \in \mathfrak{g}^{(1)}_{I(p)},$$

where $\Pi - (\Pi_0)_p$ is viewed as an element of $\mathfrak{g}^{(1)}$ by the isomorphism given by (5.20).

Secondly, setting

$$\mathcal{C}^{(1)} := \mathcal{C} \oplus \mathfrak{g},$$

we define the "tautological form" on $P^{(1)}$ by

$$\Omega^{(1)} \in \Omega^1(P^{(1)}; I^*\mathcal{C}^{(1)}), \qquad \big(\Omega^{(1)}\big)_\Pi := (\Omega, \Pi) \circ d\pi.$$

Definition 9.2.1. *Let (P, Ω) be a realization of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$. The pair $(P^{(1)}, \Omega^{(1)})$ is called the **1st prolongation** of (P, Ω).*

The main question we address in this section is whether $(P^{(1)}, \Omega^{(1)})$ is itself a realization. To start off, we describe its underlying pre-Cartan algebroid. For this purpose, we only need a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ over N. From this data, we endow the pair $(\mathcal{C}^{(1)}, \mathfrak{g}^{(1)})$ with the structure of a pre-Cartan algebroid.

First, the pre-Lie algebroid structure on $\mathcal{C}^{(1)}$ is defined precisely as in Section 6.6, where we constructed a Cartan pair out of a given Cartan algebroid. Recall that the bracket on $\mathcal{C}^{(1)}$ depends on a choice of $t \in \Gamma(\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}))$ and $\nabla : \Gamma(\mathcal{C}) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$, as in Definition 6.2.1. Fixing such a pair (t, ∇), the bracket

$$[\cdot, \cdot] : \Gamma(\mathcal{C}^{(1)}) \times \Gamma(\mathcal{C}^{(1)}) \to \Gamma(\mathcal{C}^{(1)}) \tag{9.1}$$

is defined by

$$[(\alpha, S), (\alpha', S')] := ([\alpha, \alpha'] + S'(\alpha) - S(\alpha'),\ t_{\alpha,\alpha'} + \nabla_\alpha S' - \nabla_{\alpha'} S - [S, S']),$$

for all $\alpha, \alpha' \in \Gamma(\mathcal{C}), S, S' \in \Gamma(\mathfrak{g})$, and the anchor is an extension of the anchor of $\mathcal{C}$, namely

$$\rho : \mathcal{C}^{(1)} \to TN, \qquad (\alpha, S) \mapsto \rho(\alpha). \tag{9.2}$$

Next, $\mathfrak{g}^{(1)} \subset \mathrm{Hom}(\mathcal{C}, \mathfrak{g})$ is viewed as a vector subbundle of $\mathrm{Hom}(\mathcal{C}^{(1)}, \mathcal{C}^{(1)})$ via the inclusion

$$\mathfrak{g}^{(1)} \hookrightarrow \mathrm{Hom}(\mathcal{C}^{(1)}, \mathcal{C}^{(1)}), \qquad \xi \mapsto \Big((\alpha, S) \mapsto (0, \xi(\alpha)\Big). \tag{9.3}$$

Note that the commutator of any two sections of $\mathfrak{g}^{(1)}$ vanishes and, hence, $\mathfrak{g}^{(1)}$ is a bundle of abelian Lie algebras.

Lemma 9.2.2. *Let $(\mathcal{C}, \mathfrak{g})$ be a Cartan algebroid and fix a choice of (t, ∇) as in Definition 6.2.1. The induced pair $(\mathcal{C}^{(1)}, \mathfrak{g}^{(1)})$, with the structure define above, is a pre-Cartan algebroid.*

Proof. The anchor ρ of $\mathcal{C}^{(1)}$ is surjective because the anchor ρ of $\mathcal{C}$ is surjective. From the formula (9.2), we readily see that the bracket of $\mathcal{C}^{(1)}$ satisfies the Leibniz identity because the bracket of $\mathcal{C}$ and ∇ do. Let us verify the compatibility of the anchor with the bracket:

$$\begin{aligned}\rho([(\alpha,S),(\alpha',S')]) &= \rho([\alpha,\alpha']+S'(\alpha)-S(\alpha'))\\ &= \rho[\alpha,\alpha']\\ &= [\rho(\alpha),\rho(\alpha')]\\ &= [\rho(\alpha,S),\rho(\alpha',S')].\end{aligned}$$

In the second equality, we used the fact that $\mathfrak{g}\subset \mathrm{Hom}(\mathcal{C},\mathrm{Ker}\ \rho)$ and hence $\rho\circ S = \rho\circ S' = 0$. $\square$

Definition 9.2.3. *Let $(\mathcal{C},\mathfrak{g})$ be a Cartan algebroid and fix a choice of (t,∇) as in Definition 6.2.1. The induced pre-Cartan algebroid $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$ is called the* ***1st prolongation*** *of $(\mathcal{C},\mathfrak{g})$.*

The pre-Cartan algebroid $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$ models the space $P^{(1)}$ in the sense that we have a short exact sequence of vector bundles

$$0\to I^*\mathfrak{g}^{(1)}\to TP^{(1)}\xrightarrow{\Omega^{(1)}} I^*\mathcal{C}^{(1)}\to 0, \tag{9.4}$$

where the left map is the inclusion together with the canonical isomorphism between $T^\pi P^{(1)}$, the vertical bundle of $\pi : P^{(1)}\to P$, and $I^*\mathfrak{g}^{(1)}$, the vector bundle modeling $\pi : P^{(1)}\to P$. Explicitly, as for any affine bundle, a vector in $T^\pi P^{(1)}$ over the point $\Pi\in P^{(1)}_p$, with $p\in P$, is represented by a difference $\Pi'-\Pi$, where $\Pi'\in P^{(1)}_p$, and in turn $\Pi'-\Pi$ is identified with an element of $\mathfrak{g}^{(1)}_{I(p)}$ via (5.20). Any left splitting $\Pi^{(1)}$ of (9.4), thus a 1-form $\Pi^{(1)}\in\Omega^1(P^{(1)};I^*\mathfrak{g}^{(1)})$ such that $\pi^{(1)}$ restricted to $T^\pi P^{(1)}$ is the canonical identification with $I^*\mathfrak{g}^{(1)}$, induces an isomorphism

$$(\Omega^{(1)},\Pi^{(1)}) : TP^{(1)}\xrightarrow{\simeq} I^*(\mathcal{C}^{(1)}\oplus\mathfrak{g}^{(1)}). \tag{9.5}$$

The inverse of this isomorphism induces a map of sections

$$\begin{aligned}\alpha\in\Gamma(\mathcal{C}) &\ \mapsto\ X_\alpha\in\mathfrak{X}(P^{(1)}),\\ S\in\Gamma(\mathfrak{g}) &\ \mapsto\ X_S\in\mathfrak{X}(P^{(1)}),\\ \xi\in\Gamma(\mathfrak{g}^{(1)}) &\ \mapsto\ X_\xi\in\mathfrak{X}(P^{(1)}).\end{aligned} \tag{9.6}$$

Although this is left out of the notation, one must keep in mind that this map depends on the choice of $\Pi^{(1)}$.

Having described the pre-Cartan algebroid underlying the pair $(P^{(1)},\Omega^{(1)})$, we turn to the main theorem. Recall first that as part of the Spencer complex (1.47) associated with the tableau bundle $\mathfrak{g}$, we have the sequence

$$\mathrm{Hom}(\mathcal{C},\mathfrak{g}^{(1)})\xrightarrow{\delta}\mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g})\xrightarrow{\delta}\mathrm{Hom}(\Lambda^3\mathcal{C},\mathcal{C}), \tag{9.7}$$

and that we denoted the cocycles at $\mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g})$ by $Z^{0,2}(\mathfrak{g})$ and the cohomology group at $\mathrm{Hom}(\Lambda^2\mathcal{C},\mathfrak{g})$ by $H^{0,2}(\mathfrak{g})$.

Theorem 9.2.4. *Let $(\mathcal{C},\mathfrak{g})$ be a Cartan algebroid and and fix a choice of (t,∇) as in Definition 6.2.1, inducing the pre-Cartan algebroid $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$, the 1st prolongation of $(\mathcal{C},\mathfrak{g})$. Let (P,Ω) be a realization of $(\mathcal{C},\mathfrak{g})$. If $H^{0,2}(\mathfrak{g})=0$ and $\mathfrak{g}^{(1)}$ is of constant rank, then $(P^{(1)},\Omega^{(1)})$ is a realization of $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$.*

To prove this theorem, we need to construct a $\Pi^{(1)}$ that satisfies

$$d\Omega^{(1)}+\frac{1}{2}[\Omega^{(1)},\Omega^{(1)}]=\Pi^{(1)}\wedge\Omega^{(1)}.$$

As usual, we write

$$\mathrm{MC}_{\Omega^{(1)}}:=d\Omega^{(1)}+\frac{1}{2}[\Omega^{(1)},\Omega^{(1)}].$$

We already have natural candidates for such a $\Pi^{(1)}$, namely left splittings of (9.4). This initial guess, as we will see, will be close but not exactly right. Its failure to satisfy the structure equation is controlled by the Spencer cohomology of $\mathfrak{g}$, which is the content of the following lemma. In the proof of the theorem, that will follow the lemma, we will use our assumption on the Spencer cohomology to perturb the initial $\Pi^{(1)}$ to one that does satisfy the structure equation.

Lemma 9.2.5. *Let (P,Ω) be a realization of a Cartan algebroid $(\mathcal{C},\mathfrak{g})$. Around each point $p\in P$ there exists an open neighborhood $U\subset P$ and $\Pi^{(1)}\in\Omega^1(P^{(1)}|_U;I^*\mathfrak{g}^{(1)}|_U)$ that satisfies the restriction of* (9.5) *to U (inducing the maps* (9.6)*) and such that*

$$\begin{aligned}I^*(\alpha,\alpha') &\mapsto MC_{\Omega^{(1)}}(X_\alpha,X_{\alpha'}) &\in\Gamma(I^*Z^{0,2}(\mathfrak{g})),\\ I^*\alpha &\mapsto MC_{\Omega^{(1)}}(X_\alpha,X_S) &\in\Gamma(I^*\mathfrak{g}^{(1)}),\end{aligned}\tag{9.8}$$

for all $S\in\Gamma(\mathfrak{g})$, and

$$\begin{aligned}&\Big(MC_{\Omega^{(1)}}-\Pi^{(1)}\wedge\Omega^{(1)}\Big)(X_S,X_{S'})=0,\\ &\Big(MC_{\Omega^{(1)}}-\Pi^{(1)}\wedge\Omega^{(1)}\Big)(X_\alpha,X_\xi)=0,\\ &\Big(MC_{\Omega^{(1)}}-\Pi^{(1)}\wedge\Omega^{(1)}\Big)(X_S,X_\xi)=0,\\ &\Big(MC_{\Omega^{(1)}}-\Pi^{(1)}\wedge\Omega^{(1)}\Big)(X_\xi,X_{\xi'})=0,\end{aligned}$$

for all $\alpha\in\Gamma(\mathcal{C}),S,S'\in\Gamma(\mathfrak{g}),\xi,\xi'\in\Gamma(\mathfrak{g}^{(1)})$.

Proof. Around any point $p\in P$ there exists a neighborhood $U\subset P$ such that the projection $\pi|_U:P^{(1)}|_U\to U$ admits a flat Ehresmann connection ($\pi:P^{(1)}\to P$ is a affine bundle and hence any choice of a local frame of the underlying vector bundle induces the desired connection). Such a connection induces a $\Pi^{(1)}\in\Omega^1(P^{(1)}|_U;I^*\mathfrak{g}^{(1)}|_U)$ by the condition that $\Pi^{(1)}$ vanish when applied on horizontal vectors and be the canonical isomorphism $T^\pi P^{(1)}|_U\cong I^*\mathfrak{g}^{(1)}|_U$ when restricted to $T^\pi P^{(1)}|_U$. This choice of $\Pi^{(1)}$ will do the job, as we now show. To simplify notation, we assume that $P=U$.

Having fixed such a $\Pi^{(1)}$, we have the induced isomorphism (9.6). Note that because $\Pi^{(1)}$ comes from a flat connection, the vector fields of the type X_α and X_S (which are killed by $\Pi^{(1)}$) are tangent to the leaves of the induced foliation. Recall from Lemma 5.2.6 that for a pair $X_\alpha, X_{\alpha'} \in \mathfrak{X}(P^{(1)})$ coming from $\alpha, \alpha' \in \Gamma(\mathcal{C})$, we have

$$\mathrm{MC}_{\Omega^{(1)}}(X_\alpha, X_{\alpha'}) = -\Omega^{(1)}([X_\alpha, X_{\alpha'}]) + I^*[(\alpha, 0), (\alpha', 0)],$$

and similarly for all other combinations. Let us now compute the six different properties from the statement of the Lemma.

The first three properties are direct consequences of Lemma 5.2.16 and Corollary 6.3.3. Let us look for example at the first property. Let $\alpha, \alpha' \in \Gamma(\mathcal{C})$. Computing $[X_\alpha, X'_\alpha]$ at a point $\Pi \in P^{(1)}$ amounts to computing it in the leaf that passes through Π. Such a leaf is an extension of Π to a $\widetilde{\Pi} \in \Omega^1(P; I^*\mathfrak{g})$ that satisfies (5.12) and (5.13). Now, given this $\widetilde{\Pi}$, we have induced vector fields $\hat{X}_\alpha, \hat{X}_{\alpha'} \in \mathfrak{X}(P)$ on the base and the restrictions of $X_\alpha, X_{\alpha'}$ to the leaf $\widetilde{\Pi}$ are π-related to $\hat{X}_\alpha, \hat{X}_{\alpha'} \in \mathfrak{X}(P)$. Hence,

$$-\Omega^{(1)}([X_\alpha, X_{\alpha'}]\big|_\Pi) = -(\Omega, \widetilde{\Pi}) \circ d\pi([X_\alpha, X_{\alpha'}]\big|_\Pi) = -(\Omega, \widetilde{\Pi})([\hat{X}_\alpha, \hat{X}_{\alpha'}]\big|_{\pi(\Pi)}).$$

Next, from Lemma 5.2.16 and Corollary 6.3.3, we know that

$$-(\Omega, \widetilde{\Pi})([\hat{X}_\alpha, \hat{X}_{\alpha'}] + I^*([\alpha, \alpha'], t_{\alpha,\alpha'}) \quad \in \Gamma(Z^{0,2}(\mathfrak{g})).$$

This proves the first property. The second and third property follow in a similar fashion.

We turn to the last three properties. The last property follows from the fact that $X_\xi, X_{\xi'}$ are π-related to 0 and hence $\Omega^{(1)}([X_\xi, X_{\xi'}]\big|_\Pi) = (\Omega, \Pi) \circ d\pi([X_\xi, X_{\xi'}]\big|_\Pi) = 0$. For the fifth property, note that Lemma 5.2.17 implies that the vector field $X_S \in \mathfrak{X}(P^{(1)})$ is projectable to a vector field on P, and since X_ξ is π-related to 0, it follows that $\Omega^{(1)}([X_\xi, X_S]\big|_\Pi) = 0$ (the fact that X_S is projectable could have been used to compute the third property as well).

For the fourth property, we must compute $\Omega^{(1)}([X_\alpha, X_\xi]\big|_\Pi)$. Fix $\Pi \in P^{(1)}_p$, with which we fix an isomorphism $(\Omega, \Pi) : T_pP \to (\mathcal{C} \oplus \mathfrak{g})_{I(p)}$ identifying $\alpha \in \mathcal{C}_{I(p)}$ with $\hat{X}_\alpha \in T_pP$. Since $P^{(1)}$ is an affine bundle, a vertical vector $X_\xi \in T^\pi P^{(1)}$ at Π associated with $\xi \in \mathfrak{g}^{(1)}_{I(p)}$ is the same thing the difference $\Pi' - \Pi$, with $\Pi' \in P^{(1)}_p$, where the identification is given by the condition $(\Pi' - \Pi)(\hat{X}_\alpha) = \xi(\alpha)$ for all $\alpha \in \mathcal{C}_{I(p)}$. The vector $X_\xi = \Pi' - \Pi$ is represented by the straight path in $P^{(1)}_p$,

$$\Pi + \lambda(\Pi' - \Pi), \qquad \lambda \in \mathbb{R}.$$

Next, if we set $\hat{X}_\alpha := d\pi(X_\alpha\big|_\Pi)$ and $\hat{X}'_\alpha := d\pi(X_\alpha\big|_{\Pi'})$, the projections down of X_α at the points Π and Π', then $d\pi(X_\alpha\big|_{\Pi+\lambda(\Pi'-\Pi)}) = \hat{X}_\alpha + \lambda(\hat{X}'_\alpha - \hat{X}_\alpha)$, since

$$\begin{aligned}\big(\Omega, \Pi + \lambda(\Pi' - \Pi)\big)\big(\hat{X}_\alpha + \lambda(\hat{X}'_\alpha - \hat{X}_\alpha)\big) &= \big(\alpha_{I(p)}, (1-\lambda)\big(\Pi(\hat{X}'_\alpha) + \Pi'(\hat{X}_\alpha)\big)\big) \\ &= (\alpha, 0)_{I(p)}.\end{aligned}$$

In the last equality we used the fact that $\Pi(\hat{X}'_\alpha) = -\Pi'(\hat{X}_\alpha)$, which follows from

$$0 = ((\Pi' - \Pi) \wedge \Omega)(\hat{X}_\alpha - \hat{X}'_\alpha, \hat{X}_\beta) = \Pi'(\hat{X}_\alpha)(\beta) + \Pi(\hat{X}'_\alpha)(\beta) \qquad \forall \beta \in \Gamma(\mathcal{C}).$$

It is now not difficult to see (e.g. by means of local coordinates) that

$$d\pi([X_\xi, X_\alpha]\big|_\Pi) = \frac{\partial}{\partial \lambda}(\hat{X}_\alpha + \lambda(\hat{X}'_\alpha - \hat{X}_\alpha)) = (\hat{X}'_\alpha - \hat{X}_\alpha),$$

and hence $\Omega^{(1)}([X_\alpha, X_\xi]\big|_\Pi) = (0, \Pi(\hat{X}_\alpha - \hat{X}'_\alpha)) = (0, (\Pi' - \Pi)(\hat{X}_\alpha)) = (0, \xi(\alpha))_{I(p)}$, so that

$$\Big(\mathrm{MC}_{\Omega^{(1)}} - \Pi^{(1)} \wedge \Omega^{(1)}\Big)(X_\alpha, X_\xi) = -\Omega^{(1)}([X_\alpha, X_\xi]) + I^*(0, \xi(\alpha)) = 0.$$

$\square$

The above lemma proves the existence of local $\Pi^{(1)}$'s that are close to satisfying the structure equation. Their failure to do so is concentrated in the two expressions (9.8). For any choice of a (local) splitting $\Pi^{(1)}$ of (9.4), we call

$$c_{\Pi^{(1)}} := \Big(\ I^*(\alpha, \alpha') \ \mapsto\ \mathrm{MC}_{\Omega^{(1)}}(X_\alpha, X_{\alpha'})\ \Big) \ \in \Gamma(I^* Z^{0,2}(\mathfrak{g}))$$

the **weak curvature** of $\Pi^{(1)}$.

Proof of Theorem 9.2.4. Our assumption that the Spencer cohomology vanishes at $\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g})$, i.e. that the sequence (9.7) is exact, implies that $Z^{0,2}$ is a vector bundle (since it is both the image and the kernel of a vector bundle map and hence of constant rank) and that there exists a splitting $\eta : Z^{0,2}(\mathfrak{g}) \to \mathrm{Hom}(\mathcal{C}, \mathfrak{g}^{(1)})$ of the surjective vector bundle map $\delta : \mathrm{Hom}(\mathcal{C}, \mathfrak{g}^{(1)}) \to Z^{0,2}(\mathfrak{g})$.

Let $\Pi^{(1)}$ be as in Lemma 9.2.5 (in particular, it is only locally defined) and let (9.6) be the induced map of sections. Our strategy will be to perturb $\Pi^{(1)}$ to a $\Pi'^{(1)} = \Pi^{(1)} + \widetilde{\Pi}$, with some suitable choice of $\widetilde{\Pi} \in \Omega^1(P^{(1)}; I^*\mathfrak{g}^{(1)})$, so that $\Pi'^{(1)}$ satisfy the structure equation. Let us denote the section $\alpha \mapsto \mathrm{MC}_{(\Omega^{(1)})}(X_\alpha, X_S)$ by $\mathrm{MC}_{\Omega^{(1)}}(\cdot, X_S)$. We set

$$\begin{aligned}
\widetilde{\Pi}(X_\alpha) &:= -(I^*\eta)\,(c_{\Pi^{(1)}})(I^*\alpha) && \forall\ \alpha \in \Gamma(\mathcal{C}),\\
\widetilde{\Pi}(X_S) &:= -\big(\mathrm{MC}_{(\Omega^{(1)})}(\cdot, X_S)\big) && \forall\ S \in \Gamma(\mathfrak{g}),\\
\widetilde{\Pi}(X_\xi) &:= 0 && \forall\ \xi \in \Gamma(\mathfrak{g}^{(1)}).
\end{aligned}$$

Lemma 9.2.5 tells us that $\widetilde{\Pi}$ is indeed a section of $\mathfrak{g}^{(1)}$. Also by Lemma 9.2.5,

$$\begin{aligned}
(\Pi'^{(1)} \wedge \Omega^{(1)})(X_\alpha, X_{\alpha'}) &= (\widetilde{\Pi} \wedge \Omega^{(1)})(X_\alpha, X_{\alpha'})\\
&= -(I^*\eta)(c_{\Pi^{(1)}})(I^*\alpha)(I^*\alpha') + (I^*\eta)(c_{\Pi^{(1)}})(I^*\alpha')(I^*\alpha)\\
&= (I^*(\delta \circ \eta))(c_{\Pi^{(1)}})(I^*(\alpha, \alpha'))\\
&= \mathrm{MC}_{\Omega^{(1)}}(X_\alpha, X_{\alpha'}),\\
(\Pi'^{(1)} \wedge \Omega^{(1)})(X_\alpha, X_S) &= (\widetilde{\Pi} \wedge \Omega^{(1)})(X_\alpha, X_S)\\
&= \big(\mathrm{MC}_{\Omega^{(1)}}(\cdot, X_S)\big)(I^*\alpha)\\
&= \mathrm{MC}_{\Omega^{(1)}}(X_\alpha, X_S),
\end{aligned}$$

for all $\alpha, \alpha' \in \Gamma(\mathcal{C})$ and $S \in \Gamma(\mathfrak{g})$. The four other cases one must check follow directly from (9.2.5) in Lemma 9.2.5. Thus, $\Pi^{(1)}$ satisfies the structure equation. Finally, since the affine combination of solutions $\Pi^{(1)}$ to the structure equation are again solutions, then a standard partition of unity argument produces a global solution and it is simple to see that the resulting affine combination still satisfies the coframe condition (9.5). $\square$

Example 9.2.6. The notion of prolongation appears in Cartan's work in the context of pseudogroups in normal form, i.e. the pseudogroups of local symmetries of realizations, that come from Lie pseudogroups via the first and second fundamental theorem. Let us explain the abstract notion of a prolongation of a realization in terms of the Lie pseudogroups that realizations come form.

Let Γ be a Lie pseudogroup of order k on M. Recall that Γ is the set of solutions of the PDE $J^k\Gamma$. Assuming that $(\mathfrak{g}^k)^{(1)}$, the 1st prolongation of the k-th symbol space of Γ, is of constant rank, we can construct the realization $(J^k\Gamma, (dt, \omega))$ of the Cartan algebroid $(TM \oplus A^{k-1}, \mathfrak{g}^k)$. This was the content of Proposition 5.3.3. Given this realization, we can construct its 1st prolongation $((J^k\Gamma)^{(1)}, (dt, \omega)^{(1)})$, as we described in the current chapter, and if $H^{0,2}(\mathfrak{g}^k) = 0$, then $((J^k\Gamma)^{(1)}, (dt, \omega)^{(1)})$ is a realization of the Cartan algebroid $((TM \oplus A^{k-1})^{(1)}, (\mathfrak{g}^k)^{(1)})$.

Alternatively, we can arrive at this 1st prolongation by prolonging the defining PDE $J^k\Gamma$ of Γ. In Chapter 1, we discussed the notion of a prolongation of a PDE, and we proved the formal integrability theorem, Theorem 1.5.8. From our proof of the theorem in Section 1.6, we read off that the condition for the PDE $J^k\Gamma$ to be integrable up to order $k+1$ (Definition 1.5.1) is that $(\mathfrak{g}^k)^{(1)}$ be of constant rank and the cohomological condition $H^{0,2}(\mathfrak{g}^k) = 0$. Precisely the condition we had above. In this case, the 1st prolongation $(J^k\Gamma)^{(1)}$ is smooth and the projection $\pi : (J^k\Gamma)^{(1)} \to J^k\Gamma$ is surjective. This implies that Γ is also a Lie pseudogroup of order $k+1$, and that $J^{k+1}\Gamma = (J^k\Gamma)^{(1)}$. Thus, using Proposition 5.3.3 again, we can construct the realization $(J^{k+1}\Gamma, (dt, \omega))$ of the Cartan algebroid $(TM \oplus A^k, \mathfrak{g}^{k+1})$. This, one can show, coincides with the 1st prolongation of the realization $(J^k\Gamma, (dt, \omega))$ of the Cartan algebroid $(TM \oplus A^{k-1}, \mathfrak{g}^k)$ (e.g., we saw that $J^{k+1}\Gamma = (J^k\Gamma)^{(1)}$), which was obtained by the abstract mechanism of prolongations of realizations.

Thus, our modern formulation shows that Cartan's abstract notion of prolongation of structure equations, as well as the obstructions to prolongation, coincide in this example with the more concrete notion of prolongation of a PDE. $\diamondsuit$

Remark 9.2.7. Referring to the last example, not only do the notions of prolongation of a realization and prolongation of a PDE coincide for realizations coming from Lie pseudogroups, as well as the obstructions to prolongation, but also the proof of Theorem 9.2.4 should remind the reader of our proof of the formal integrability theorem of PDEs, Theorem 1.5.8. The initial choice of $\Pi^{(1)}$ as a splitting of (9.4) is analogous to a choice of a Cartan-Ehresmann connection, and correcting $\Pi^{(1)}$ by a choice of a splitting in the Spencer cohomology so that it satisfy the structure equation is analogous to correcting the Cartan-Ehresmann connection so that it be integral. In fact, this is beyond an analogy.

If one unravels the proof of Theorem 9.2.4 for the case in which a realization comes from a Lie pseudogroup, then one will discover the proof of formal integrability for the PDE defining the Lie pseudogroup. This was our inspiration for the alternative proof of Theorem 1.5.8. This connection between the two settings also explains our choice to attribute the notion of a Cartan-Ehresmann connection in the setting of PDEs to Cartan (as well as to Ehresmann, for obvious reasons). $\diamondsuit$

9.3 Prolongation a Cartan Algebroid

If a Cartan algebroid $(\mathcal{C},\mathfrak{g})$ admits a realization (P,Ω) whose 1st prolongation $(P^{(1)},\Omega^{(1)})$ is again a realization (e.g. if the conditions of Theorem 9.2.4 are satisfied), then this implies that the 1st prolongation $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$, which is a priori a pre-Cartan algebroid, is again a Cartan algebroid by Theorem 6.3.1. The situation is depicted in the following diagram:

$$\begin{tikzcd}
(P^{(1)},\Omega^{(1)}) \arrow[r, rightsquigarrow] \arrow[d, "\pi"] & (\mathcal{C}^{(1)},\mathfrak{g}^{(1)}) \arrow[d, "\pi"] \\
(P,\Omega) \arrow[r, rightsquigarrow] & (\mathcal{C},\mathfrak{g}).
\end{tikzcd}$$

However, we can also phrase the following purely infinitesimal question: given a Cartan algebroid $(\mathcal{C},\mathfrak{g})$, under which conditions is the pre-Cartan algebroid $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$ a Cartan algebroid. An answer is given in the following theorem:

Theorem 9.3.1. *Let $(\mathcal{C},\mathfrak{g})$ be a Cartan algebroid and choose a pair (t,∇) as in Definition 6.2.1, inducing the pre-Cartan algebroid $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$, i.e. the 1st prolongation of $(\mathcal{C},\mathfrak{g})$. If $H^{0,2}(\mathfrak{g})=0$, $H^{0,3}(\mathfrak{g})=0$ and $\mathfrak{g}^{(1)}$ is of constant rank, then the 1st prolongation $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$ is a Cartan algebroid.*

Proof. We must show that there exist

$$t'\in\Gamma(\mathrm{Hom}(\Lambda^2\mathcal{C}^{(1)},\mathfrak{g}^{(1)})) \qquad \text{and} \qquad \nabla':\Gamma(\mathcal{C}^{(1)})\times\Gamma(\mathfrak{g}^{(1)})\to\Gamma(\mathfrak{g}^{(1)})$$

for $(\mathcal{C}^{(1)},\mathfrak{g}^{(1)})$ such that the three conditions in Definition 6.2.1 are satisfied. Condition 1 is automatically satisfied since $\mathfrak{g}^{(1)}$ is a bundle of abelian Lie algebras.

Let us start with t' and condition 2. We compute the Jacobiator

$$\mathrm{J}\in\Gamma(\mathrm{Hom}(\Lambda^3\mathcal{C}^{(1)},\mathcal{C}^{(1)})),\ \mathrm{J}_{(\alpha,S),(\alpha',S'),(\alpha'',S'')}=[[(\alpha,S),(\alpha',S')],(\alpha'',S'')]+\text{c.p.},$$

i.e. the left hand side of condition 2. Here, c.p. stands for cyclic permutations. For brevity, let us write $\mathrm{J}_{\alpha,\alpha',\alpha''}$ instead of $\mathrm{J}_{(\alpha,0),(\alpha',0),(0,\alpha'')}$, and similarly for all other combinations. A direct computation shows that

1. $\mathrm{J}_{\alpha,\alpha',\alpha''}=(0,t_{[\alpha,\alpha'],\alpha''}-\nabla_{\alpha''}(t_{\alpha,\alpha'})+\text{c.p.})$,

2. $\mathrm{J}_{\alpha,\alpha',S} = (0, \nabla_{[\alpha,\alpha']}S - \nabla_\alpha\nabla_{\alpha'}S + \nabla_{\alpha'}\nabla_\alpha S - [t_{\alpha,\alpha'}, S] - t_{S(\alpha),\alpha'} - t_{\alpha,S(\alpha')})$,
3. $\mathrm{J}_{\alpha,S,S'} = (0, \nabla_\alpha([S,S']) - [\nabla_\alpha(S), S'] - [S, \nabla_\alpha(S')] + \nabla_{S(\alpha)}(S') - \nabla_{S'(\alpha)}(S))$,
4. $\mathrm{J}_{S,S',S''} = (0,0)$.

In the computation, one notes that the first component in expressions 1-3 vanishes precisely due to axioms 1-3 in Definition 6.2.1 of a Cartan algebroid, respectively (note that this fact was already used in Lemma 6.6.2 in the construction of a Cartan pair out of a Cartan algebroid). Thus, J takes values in $\mathfrak{g}$. We split J into the following three pieces:

1. $\mathrm{J}^1 \in \Gamma(\mathrm{Hom}(\Lambda^3\mathcal{C}, \mathfrak{g}))$ defined by $\mathrm{J}^1(\alpha,\alpha',\alpha'') = \mathrm{J}_{\alpha,\alpha',\alpha''}$,
2. $\mathrm{J}^2 : \Gamma(\mathfrak{g}) \to \Gamma(\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}))$ defined by $\mathrm{J}^2(S)(\alpha,\alpha') = \mathrm{J}_{\alpha,\alpha',S}$,
3. $\mathrm{J}^3 : \Gamma(\Lambda^2\mathfrak{g}) \to \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}))$ defined by $\mathrm{J}^3(S,S')(\alpha) = \mathrm{J}_{\alpha,S,S'}$.

One easily checks that all three maps come from vector bundle maps (which amounts to verifying that J is indeed a tensor). A rather long but not difficult computation shows that

1. $\mathrm{J}^1 \in \Gamma(Z^{0,3}(\mathfrak{g}))$, where
$$Z^{0,3}(\mathfrak{g}) \subset \mathrm{Hom}(\Lambda^3\mathcal{C}, \mathfrak{g}),$$
are the cocycles at $\mathrm{Hom}(\Lambda^3\mathcal{C}, \mathfrak{g})$ in the Spencer complex of $\mathfrak{g}$,
2. $\mathrm{J}^2(S) \in \Gamma(Z^{0,2}(\mathfrak{g}))$ for all $S \in \Gamma(\mathfrak{g})$, where
$$Z^{0,2}(\mathfrak{g}) \subset \mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g})$$
are the cocycles at $\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g})$ in the Spencer complex of $\mathfrak{g}$,
3. $\mathrm{J}^3(S,S') \in \Gamma(\mathfrak{g}^{(1)})$ for all $S, S' \in \Gamma(\mathfrak{g})$.

For instance, in the second case, the computation is as follows: fix an $S \in \Gamma(\mathfrak{g})$, then

$$\begin{aligned}
\delta(\mathrm{J}^2(S))(\alpha,\alpha',\alpha'') &= \mathrm{J}_{\alpha,\alpha',S}(\alpha'') + (\text{c.p. of } \alpha,\alpha',\alpha'') \\
&= -\Big((\nabla_\alpha(\nabla_{\alpha'}S))(\alpha'') - (\nabla_{\alpha''}(\nabla_{\alpha'}S))(\alpha)\Big) \\
&\quad - \Big(t_{S(\alpha),\alpha'}(\alpha'') + t_{\alpha'',S(\alpha)}(\alpha') + t_{\alpha',\alpha''}(S(\alpha))\Big) \\
&\quad + (\nabla_{[\alpha,\alpha']}S)(\alpha'') + S(t_{\alpha,\alpha'}(\alpha'')) + (\text{c.p. of } \alpha,\alpha',\alpha''),
\end{aligned}$$

and now use the three axioms of the Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ (in this case only two are needed) to show that the expression vanishes. Now, by our assumption on the Spencer cohomology, both $Z^{0,2}(\mathfrak{g})$ and $Z^{0,3}(\mathfrak{g})$ are vector bundles and the vector bundle maps

$$\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}^{(1)}) \xrightarrow{\delta} Z^{0,3}(\mathfrak{g}) \qquad \text{and} \qquad \mathrm{Hom}(\mathcal{C}, \mathfrak{g}^{(1)}) \xrightarrow{\delta} Z^{0,2}(\mathfrak{g})$$

are surjective. Thus, we may choose splittings

$$\eta^1 : Z^{0,3}(\mathfrak{g}) \to \mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}^{(1)}) \qquad \text{and} \qquad \eta^2 : Z^{0,2}(\mathfrak{g}) \to \mathrm{Hom}(\mathcal{C}, \mathfrak{g}^{(1)})$$

of both maps, and we define t' by

$$\begin{aligned} t'_{(\alpha,0),\left(\alpha',0\right)}(\alpha'', S'') &:= \left(0, \eta^1(\mathrm{J}^1)(\alpha,\alpha')(\alpha'')\right), \\ t'_{(\alpha,0),(0,S')}(\alpha'', S'') &:= \left(0, \eta^2(\mathrm{J}^2(S))(\alpha)(\alpha'')\right), \\ t'_{(0,S),(0,S')}(\alpha'', S'') &:= \left(0, \mathrm{J}^3(S,S')(\alpha'')\right). \end{aligned}$$

With this choice of t', condition 2 is satisfied:

$$\begin{aligned} &t'_{(\alpha,0),(\alpha',0)}(\alpha'',0) + \text{c.p.} = (\delta\circ\eta^1)(\mathrm{J}^1)(\alpha,\alpha',\alpha'') = \mathrm{J}^1(\alpha,\alpha',\alpha'') = \mathrm{J}_{(\alpha,0),(\alpha',0),(\alpha'',0)}, \\ &t'_{(\alpha,0),(\alpha',0)}(0,S) + \text{c.p.} = (\delta\circ\eta^2)(\mathrm{J}^2(S))(\alpha,\alpha') = \mathrm{J}^2(S)(\alpha,\alpha') = \mathrm{J}_{(\alpha,0),(0,S),(0,S')}, \\ &t'_{(\alpha,0),(0,S)}(0,S') + \text{c.p.} = \mathrm{J}^3(S,S')(\alpha) = \mathrm{J}_{(\alpha,0),(0,S),(0,S')}. \end{aligned}$$

We move on to ∇'. The strategy is the same as for t'. Consider the map

$$\begin{aligned} &\mathrm{Ad} : \Gamma(\mathfrak{g}^{(1)}) \times \Gamma(\Lambda^2\mathcal{C}^{(1)}) \to \Gamma(\mathcal{C}^{(1)}), \\ &\quad \mathrm{Ad}_{(\alpha,S),(\alpha',S')}\xi := \xi([(\alpha,S),(\alpha',S')]) - [\xi(\alpha,S),(\alpha',S')] - [(\alpha,S),\xi(\alpha',S')], \end{aligned}$$

i.e. the left hand side of condition 3. A direct computation shows that

1. $\mathrm{Ad}_{\alpha,\alpha'}\xi = (0, \xi([\alpha,\alpha']) - \nabla_\alpha(\xi(\alpha')) + \nabla_{\alpha'}(\xi(\alpha)))$
2. $\mathrm{Ad}_{\alpha,S}\xi = (0, \xi(S(\alpha)) + [\xi(\alpha), S])$
3. $\mathrm{Ad}_{S,S'}\xi = (0,0)$.

In the first case, the first component vanishes because $\xi(\alpha)(\alpha') = \xi(\alpha')(\alpha)$, and in the other two, it is identically zero. Thus, Ad takes values in $\mathfrak{g}$. We split Ad into the following two pieces:

1. $\mathrm{Ad}^1 : \Gamma(\mathfrak{g}^{(1)}) \to \Gamma(\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}))$ defined by $\mathrm{Ad}^1(\xi)(\alpha,\alpha') = \mathrm{Ad}_{\alpha,\alpha'}\xi$,
2. $\mathrm{Ad}^2 : \Gamma(\mathfrak{g}^{(1)} \otimes \mathfrak{g}) \to \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}))$ defined by $\mathrm{Ad}^2(\xi, S)(\alpha) = \mathrm{Ad}_{\alpha,S}\xi$.

A simple computations, using both the axioms of a Cartan algebroid (only axiom 2 is needed) and the fact that $\xi(\alpha)(\alpha') = \xi(\alpha')(\alpha)$, shows that

1. $\mathrm{Ad}^1(\xi) \in \Gamma(Z^{0,2}(\mathfrak{g}))$ for all $\xi \in \Gamma(\mathfrak{g}^{(1)})$,
2. $\mathrm{Ad}^1(\xi, S) \in \Gamma(\mathfrak{g}^{(1)})$ for all $\xi \in \mathfrak{g}^{(1)}, S \in \Gamma(\mathfrak{g})$.

Choosing again a splitting $\eta : \Gamma(Z^{0,2}(\mathfrak{g})) \to \Gamma(\mathrm{Hom}(\mathcal{C}, \mathfrak{g}^{(1)}))$ as before, we define ∇' by

$$\begin{aligned} \nabla'_{(\alpha,0)}(\xi) &:= \eta(\mathrm{Ad}^1(\xi))(\alpha), \\ \nabla'_{(0,S)}(\xi) &:= \mathrm{Ad}^2(\xi, S). \end{aligned} \tag{9.9}$$

There is, however, a subtlety with this definition. Namely, ∇' may fail to be a connection. More precisely, $\nabla'_{(\alpha,0)}(f\xi) - f\nabla'_{(\alpha,0)}(\xi) - \rho(\alpha, 0)(f)\xi$ may fail to vanish for all $f \in C^\infty(N)$. We can overcome this problem as follows: define a local connection ∇' by imposing (9.9) on a local frame of $\mathfrak{g}^{(1)}$ and extending by the Leibniz identity. For such a local connection, one has that on any element ξ of the local frame of $\mathfrak{g}^{(1)}$,

$$\begin{aligned} (\nabla'_{(\alpha',0)}\xi)(\alpha, 0) - (\nabla'_{(\alpha,0)}\xi)(\alpha', 0) &= (\delta \circ \eta)(\mathrm{Ad}^1(\xi))(\alpha, \alpha') \\ &= \mathrm{Ad}^1(\xi)(\alpha, \alpha') \\ &= \mathrm{Ad}_{(\alpha,0),(\alpha',0)}\xi, \\ (\nabla'_{(0,S)}\xi)(\alpha, 0) - (\nabla'_{(\alpha',0)}\xi)(0, S) &= \mathrm{Ad}^2(\xi, S)(\alpha) \\ &= \mathrm{Ad}_{(\alpha,0),(0,S)}\xi, \end{aligned}$$

which is precisely condition 3. Furthermore, the desired equalities hold for any local section of $\mathfrak{g}^{(1)}$ since $(\nabla'_{(\alpha',S')}\xi)(\alpha, S) - (\nabla'_{(\alpha,S)}\xi)(\alpha', S') - \mathrm{Ad}_{(\alpha,S),(\alpha',S')}\xi$ is $C^\infty(N)$-linear in all slots, as one readily verifies. Finally, we obtain the desired connection ∇' by gluing the local connections by means of a partition of unity. □

Example 9.3.2. In Example 6.5.7, we mentioned that a Cartan algebroid over a point is the same thing (modulo gauge equivalence) as the truncated Lie algebras of Singer and Sternberg from [64]. In that paper, the authors also define the notion of a prolongation of a truncated Lie algebra and find conditions under which the prolongation is again a truncated Lie algebra (see Lemma 4.3 in the paper). Theorem 9.3.1 is a generalization of their result to the setting of Cartan algebroids. ♢

9.4 Higher Prolongations and Formal Integrability

Starting with a realization (P, Ω) of a Cartan algebroid $(\mathcal{C}, \mathfrak{g})$, if its 1st prolongation $(P^{(1)}, \Omega^{(1)})$ is again a realization of the Cartan algebroid $(\mathcal{C}^{(1)}, \mathfrak{g}^{(1)})$, then we can proceed to construct its 1st prolongation $((P^{(1)})^{(1)}, (\Omega^{(1)})^{(1)})$, as in Section 9.2, and ask whether it is a realization of the pre-Cartan algebroid $((\mathcal{C}^{(1)})^{(1)}, (\mathfrak{g}^{(1)})^{(1)})$. Setting $(\mathcal{C}^{(0)}, \mathfrak{g}^{(0)}) := (\mathcal{C}, \mathfrak{g})$ and $(P^{(0)}, \Omega^{(0)}) := (P, \Omega)$, we can proceed inductively by defining

$$(\mathcal{C}^{(k)}, \mathfrak{g}^{(k)}) := (\mathcal{C}^{(k-1)}, \mathfrak{g}^{(k-1)}) \qquad \text{and} \qquad (P^{(k)}, \Omega^{(k)}) := (P^{(k-1)}, \Omega^{(k-1)}).$$

At every stage, assuming that $(P^{(k-1)}, \Omega^{(k-1)})$ is a realization of the Cartan algebroid $(\mathcal{C}^{(k-1)}, \mathfrak{g}^{(k-1)})$, we can ask whether $(P^{(k)}, \Omega^{(k)})$ is a realization of the pre-Cartan algebroid $(\mathcal{C}^{(k)}, \mathfrak{g}^{(k)})$. This process is depicted in the following diagram:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
(P^{(2)},\Omega^{(2)}) & \rightsquigarrow & (\mathcal{C}^{(2)},\mathfrak{g}^{(2)}) \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
(P^{(1)},\Omega^{(1)}) & \rightsquigarrow & (\mathcal{C}^{(1)},\mathfrak{g}^{(1)}) \\
\Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\
(P,\Omega) & \rightsquigarrow & (\mathcal{C},\mathfrak{g}).
\end{array}$$

Definition 9.4.1. *Let $(\mathcal{C},\mathfrak{g})$ be a Cartan algebroid. The pair $(\mathcal{C}^{(k)},\mathfrak{g}^{(k)})$ is called the **k-th prolongation** of $(\mathcal{C},\mathfrak{g})$.*

Definition 9.4.2. *Let (P,Ω) be a realization of a Cartan algebroid $(\mathcal{C},\mathfrak{g})$. The pair $(P^{(k)},\Omega^{(k)})$ is called the **k-th prolongation** of the realization (P,Ω).*

Two question come to mind:

1. When is a realization **formally integrable**, i.e. when are all of its prolongations also realizations?

2. When is a Cartan algebroid **formally integrable**, i.e. when are all of its prolongations also Cartan algebroids?

The first question is the analog of the formal integrability problem for PDEs in the realm of Lie pseudogroups, as we already pointed out in Remark 9.2.7. The second question is the analog of the formal integrability problem for linear PDEs in the realm of sheaves of Lie algebras of vector fields. In fact, a solution to the second question was given in [64] (mainly Lemma 4.3) in the so called "transitive case", i.e. for the case of Cartan algebroids over a point which are known as truncated Lie algebras (see Example 6.5.7 and Example 9.3.2).

In this section, we address the first of the two questions. In fact, to answer either one of these questions, we can simply apply Theorem 9.2.4 or Theorem 9.3.1 inductively. There is however a problem with this approach, namely the Spencer cohomological condition at each stage lives in a different cohomology. In the case of realizations, for example, the condition in the first step lives in $H^{0,2}(\mathfrak{g})$, the condition in the second step lives in $H^{0,2}(\mathfrak{g}^{(1)})$, and so forth. This is somewhat unsatisfactory. One way around this is to understand the Spencer complexes of the prolongations of $\mathfrak{g}$ in terms of the Spencer complex of $\mathfrak{g}$. For example, Proposition 3.3 in [64] says that if $\mathfrak{g}$ is involutive then $\mathfrak{g}^{(1)}$ is involutive. From this we immediately conclude that involutivity of $\mathfrak{g}$ implies formal integrability of both the realization (P,Ω) of $(\mathcal{C},\mathfrak{g})$ as well as of the Cartan algebroid $(\mathcal{C},\mathfrak{g})$.

It would be more satisfactory, however, to identify the precise conditions to formal integrability at each step. Let us treat the case of formal integrability of realizations and consider the second prolongation step. In Lemma 9.2.5, the obstruction to performing the first prolongation step was identified in the form of the weak curvature associated with $\Pi^{(1)}$, which induced a class in $H^{0,2}(\mathfrak{g})$. We would like to obtain the analog of Lemma 9.2.5 for the second step. Starting with the realization $(P^{(1)}, \Omega^{(1)})$ of the Cartan algebroid $(\mathcal{C}^{(1)}, \mathfrak{g}^{(1)})$, we construct $(P^{(2)}, \Omega^{(2)})$. At a point $\Pi^{(1)} \in P^{(2)}$ that projects to $\Pi \in P^{(1)}$,

$$\Omega^{(2)}_{\Pi^{(1)}} = (\Omega, \Pi, \Pi^{(1)}) \circ d\pi.$$

Proceeding as in the lemma, we begin with a (local) $\Pi^{(2)} \in \Omega^1(P^{(2)}; I^*\mathfrak{g}^{(2)})$ that comes from a flat connection so that we have induced maps of sections. As in the Lemma, the failure of $\Pi^{(2)}$ to satisfy the structure equation originates from Corollary 6.3.3 applied to the realization $(P^{(1)}, \Omega^{(1)})$. Thus, choosing $\Pi^{(1)}$ for this realization, we obtain the maps of sections

$$\begin{aligned} &\alpha \in \Gamma(\mathcal{C}) \mapsto X_\alpha \in \mathfrak{X}(P^{(1)}), \\ &S \in \Gamma(\mathfrak{g}) \mapsto X_S \in \mathfrak{X}(P^{(1)}), \\ &\xi \in \Gamma(\mathfrak{g}^{(1)}) \mapsto X_\xi \in \mathfrak{X}(P^{(1)}), \end{aligned}$$

and we would like to compute terms of the type $\Pi^{(1)}([X_\alpha, X_S])$ for all possible combinations. Proceeding as in the proof of Corollary 6.3.3, such terms are computed by computing the "differential consequence" of the structure equation $d\Omega^{(1)} + \frac{1}{2}[\Omega^{(1)}, \Omega^{(1)}] = \Pi^{(1)} \wedge \Omega^{(1)}$. The result is that the only new obstruction is the weak curvature of $\Pi^{(2)}$,

$$c_{\Pi^{(2)}} := (\alpha, \alpha') \mapsto \Pi^{(1)}([X_\alpha, X_{\alpha'}]) - I^* t^{(1)}_{\alpha,\alpha'} \quad \in \Gamma(Z^{1,2}(\mathfrak{g})),$$

where $Z^{1,2}(\mathfrak{g})$ are the cocycles in the Spencer complex at the term $\mathrm{Hom}(\Lambda^2\mathcal{C}, \mathfrak{g}^{(1)})$, and $t^{(1)} \in \Gamma(\mathrm{Hom}(\Lambda^2\mathcal{C}^{(1)}, \mathfrak{g}^{(1)}))$ is a choice associated with the Cartan algebroid $(\mathcal{C}^{(1)}, \mathfrak{g}^{(1)})$.

Proceeding in this fashion, we identify the obstruction at step k as the weak curvature of $\Pi^{(k)}$,

$$c_{\Pi^{(k)}} := (\alpha, \alpha') \mapsto \Pi^{(k)}([X_\alpha, X_{\alpha'}]) - I^* t^{(k-1)}_{\alpha,\alpha'} \quad \in \Gamma(Z^{k-1,2}(\mathfrak{g})).$$

Note that, by Lemma 1.5.6, the condition that $\mathfrak{g}$ be 2-acyclic, together with the assumption that $\mathfrak{g}^{(1)}$ is of constant rank, ensure that all higher prolongations $\mathfrak{g}^{(k)}$ are of constant rank, which ensures smoothness of all the objects.

We conclude that:

Theorem 9.4.3. *Let (P, Ω) be a realization of the Cartan algebroid $(\mathcal{C}, \mathfrak{g})$ and assume that $\mathfrak{g}^{(1)}$ is of constant rank. If $\mathfrak{g}$ is 2-acyclic, then (P, Ω) is formally integrable, i.e. its k-th prolongation is a realization for all $k > 0$.*

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A Summary for the Non-Mathematician

In 1904-05, the French mathematician Élie Cartan published two pioneering papers [5, 6] in which he introduced a structure theory for *Lie pseudogroups*. Loosely speaking, a Lie pseudogroup is a set of local transformations of a manifold that is defined as the set of solutions of a system of partial differential equations. Lie pseudogroups arise in mathematics as sets of local symmetries of differential equations or geometric structures such as Riemannian manifolds (for the laymen, manifolds are "curved" spaces such as the surface of a ball or a doughnut, and a geometric structure is a manifold equipped with extra auxiliary structure such as a "ruler" to measure distances).

Lie pseudogroups, as the name suggests, were first studied by Cartan's predecessor Sophus Lie. Lie's writings were mainly dedicated to a special class of these objects known as Lie pseudogroups of *finite type*. Lie's work on this special case gave rise to the modern theory of Lie groups, a theory which has become a mainstream subject in mathematics and which plays a central role in many areas of mathematics and physics. Cartan, building on Lie's work, moved on to study the general case of Lie pseudogroups. He realized that the attempts that were made to extend Lie's ideas to the general case were leading to dead ends and devised a new approach for the study of Lie pseudogroups: he studied them by means of their defining system of partial differential equations.

Probably the most remarkable aspect of Cartan's work on this subject is the vast amount of basic mathematical concepts and tools that he developed in the process. It is quite surprising, then, that while these basic concepts and tools have become so fundamental in mathematics, the theory for which they were created failed to reach the same level of maturity as its "older sibling", the theory of Lie groups. In this thesis, driven by the certainty that there are still many hidden treasures waiting to be discovered, and equipped with the mathematical innovations of the past few decades (some of which were inspired by precisely this same reason), we revisit Cartan's original writings and formulate his structure theory for Lie pseudogroups in modern mathematical language, placing a special emphasis on remaining as close as possible to Cartan's ideas. Of course, this thesis builds on the work of many great mathematicians who have taken this same path. We refer the reader to the introduction chapter for more details on their contributions.

Let us now turn to describe some of our main results. Cartan introduces his structure theory for Lie pseudogroups by means of what he calls the *three fundamental theorems*. His main idea is to associate with any given Lie pseudogroup the following set of equations known as the *structure equations* (see also the back cover of the thesis):

$$d\omega_i + \frac{1}{2}c_i^{jk}\omega_j \wedge \omega_k = a_i^{\lambda j}\pi_\lambda \wedge \omega_j.$$

The coefficients that appear in these equations, the *structure functions* c_i^{jk} and $a_i^{\lambda j}$, play a very important role. They are the *infinitesimal data* that is associated with the Lie pseudogroup, and, in a certain sense, they encode the Lie pseudogroup itself.

In Chapters 4 – 6, the core of the thesis, we present a modern formulation of Cartan's fundamental theorems. We begin by introducing two new geometric structures: 1) a *Cartan algebroid*, an object that encodes Cartan's notion of structure functions, and 2) a *realization* of a Cartan algebroid, an object that encodes Cartan's notion of structure equations and whose main property is the following modern incarnation of the structure equations (see also the front cover of the thesis):

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega.$$

We then proceed to study the properties of these structures and to present new and more conceptual proofs of Cartan's first two fundamental theorems.

A recurring and fundamental problem in differential geometry, one which appears in a wide variety of guises throughout the field, is the problem of integration of infinitesimal structures. In the theory of Lie pseudogroups, this problem appears as the problem of finding a realization of a given Cartan algebroid and it is known as the *realization problem*. In his third fundamental theorem, Cartan proves the existence of *local* solutions to this problem in the so called *analytic* setting. Although a significant step, it is, nevertheless, only a partial solution to the problem. Nowadays, we are also very much interested in what is called the *smooth* setting and in the problem of existence of *global* solutions. Both of these problems are notoriously difficult and remain open till this very day. In Chapter 7 of the thesis, we introduce a new method which can be used in tackling the realization problem and which sheds light on the role of a certain obstruction to the problem known as the *Jacobi identity*. We apply the method to a few related (but simpler) integrability problems, namely to the integration of Lie algebras and Lie algebroids and to the problem of existence of local symplectic realizations of Poisson manifolds.

In Chapter 8 of the thesis, we move on to study another aspect of Cartan's theory, the notion of *reduction*. Cartan shows that by examining the structure equations, one can isolate a system of equations which he calls the *systatic system*, and he then proceeds to use this system in order to simplify a given Lie pseudogroup to one which is "smaller" but "equivalent". Such a procedure is useful, for instance, when one would like to classify Lie pseudogroups, i.e. understand the space of all Lie pseudogroups. Using our modern formulation, we present a reduction procedure which is highly inspired by but goes beyond Cartan's procedure. This, of course, comes at a cost, namely the need to extend the classical notion of a pseudogroup to the more general notion of a *generalized pseudogroup*. While in this chapter we break our self-imposed rule of remaining as close as possible to Cartan, we do believe that our reduction procedure is natural and is "what Cartan would have discovered" had he the necessary tools.

To conclude, let us briefly touch upon the chapters that were not mentioned in this summary. Chapters 1 – 3 are introductory in nature, giving a survey of essential background material, such as the geometric theory of partial differential equations, and providing some new and simplified proofs of known results. The final chapter, Chapter 9, covers yet another important aspect in Cartan's theory which goes under the name of *prolongations of structure equations*. Prolongations play a role in the classification of Lie pseudogroups and in various integrability problems that arise in the theory.

Een samenvatting voor de niet-wiskundige

In 1904–1905 publiceerde de wiskundige Élie Cartan twee baanbrekende artikelen [5, 6] waarin hij de structuurtheorie van *Lie-pseudogroepen* introduceert. Grofweg, een Lie-pseudogroep is een verzameling lokale transformaties van een variëteit gegeven door de oplossingen van een stelsel partiële differentiaalvergelijkingen. Lie-pseudogroepen komen natuurlijk voor als de locale symmetriën van differentiaalvergelijkingen of meetkundige structuren zoals Riemann-variëteiten (voor de leek: variëteiten zijn 'gekromde' ruimtes zoals bijvoorbeeld het oppervlak van een bal of donut, en een meetkundige structuur is een variëteit met extra structuur zoals een 'lineaal' om afstanden te kunnen meten).

Lie-pseudogroepen zijn, zoals de naam suggereert, geïntroduceert door Cartan's voorganger Sophus Lie, wiens werk voornamelijk was toegewijd aan een speciale klasse van zulke objecten, zogeheten Lie-pseudogroepen van *eindig type*. Lie's werk bracht de moderne theorie van Lie-groepen voort, die een centrale rol speelt in veel gebieden van de wiskunde en natuurkunde. Cartan, voortbouwend op Lie's werk, bestudeerde vervolgens de algemene theorie van Lie-pseudogroepen. Hij realizeerde zich dat voorgaande pogingen om Lie's ideeën uit te breiden tot een dood spoor leidden en ontwikkelde een volledig nieuwe aanpak van de studie van Lie-pseudogroepen: hij concentreerde zich op hun bepalende stelsel partiële differentiaalvergelijkingen.

Wellicht het meest uitzonderlijke aspect aan dit onderdeel van Cartan's werk is de enorme hoeveelheid aan fundamentele wiskundige concepten en gereedschappen die hij daarbij ontwikkeld heeft. Het is dan ook verbazingwekkend dat deze concepten zo fundamenteel zijn geworden, terwijl de theorie waar zij vandaan komen nooit dezelfde mate van volwassenheid heeft bereikt als zijn 'oudere broer', de theorie van Lie-groepen. Gedreven door de verwachting dat er nog veel verborgen schatten op ons liggen te wachten, en bewapend met de wiskundige innovaties van de afgelopen decennia (waarvan sommigen uit dezelfde motivatie voortkomen), bezoeken we in dit proefschrift nogmaals Cartan's originele geschriften en formuleren we zijn structuurtheorie van Lie-pseudogroepen in moderne wiskundige taal. Hierbij leggen we er in het bijzonder nadruk op om zo dicht mogelijk bij de oorspronkelijke ideeën te blijven. Dit proefschrift bouwt natuurlijk voort op het werk van menig groot wiskundige; we verwijzen de lezer naar de introductie voor meer details over hun bijdragen.

Laten we ons nu gaandeweg op onze resultaten richten: Cartan introduceerde zijn theorie van Lie-pseudogroepen aan de hand van wat hij de *drie fundamentele stellingen* noemde. Zijn idee was om aan elke Lie-pseudogroep de volgende zogeheten *structuurvergelijkingen* toe te kennen (zie ook de achterkant van dit proefschrift):

$$d\omega_i + \frac{1}{2}c_i^{jk}\omega_j \wedge \omega_k = a_i^{\lambda j}\pi_\lambda \wedge \omega_j.$$

De coëfficienten die in deze vergelijkingen voorkomen, de *structuurfuncties* c_i^{jk} en $a_i^{\lambda j}$, spelen een centrale rol. Zij vormen de *infinitesimale gegevens* geassocieerd met de Lie-pseudogroep en coderen de Lie-pseudogroep tot op een zekere hoogte.

In hoofdstukken 4 – 6, de kern van het proefschrift, presenteren we een moderne formulering van Cartan's fundamentele stellingen. We introduceren eerst twee volledig nieuwe wiskundige structuren: 1) een *Cartan algebroïde*, een object dat Cartan's structuurfuncties codeert; en 2) een *realisatie* van een Cartan algebroïde, een object dat Cartan's begrip van structuurvergelijkingen codeert en wiens voornaamste eigenschap de volgende moderne belichaming van de structuurvergelijking is:

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = \Pi \wedge \Omega.$$

We bestuderen vervolgens de eigenschappen van deze structuren en geven nieuwe, en vooral ook inzichtelijke, bewijzen van Cartan's eerste twee fundamentele stellingen.

Het integreren van infinitesimale structuren is een terugkerend en essentieel probleem in de differentieerbare meetkunde dat allerlei gedaantes aanneemt. Dit probleem toont zich in de theorie van Lie-pseudogroepen als het vinden van een realisatie van een gegeven Cartan algebroïde, en staat bekend als het *realisatieprobleem*. Cartan bewijst in zijn derde fundamentele stelling het bestaan van *locale* oplossingen voor dit probleem in de zogeheten *analytische* categorie. Het is een significante stap, maar desondanks slechts een deel van de oplossing: tegenwoordig is er veel belangstelling voor de zogeheten *gladde* categorie en het bestaan van *globale* oplossingen. Beide problemen zijn berucht moeilijk en blijven open tot op deze dag. In hoofdstuk 7 van het proefschrift introduceren we een nieuwe methode om het realisatieprobleem aan te pakken, en werpen we wat nieuw licht op de rol van de *Jacobi-identiteit* als obstakel tot realisatie. We passen deze methode toe op wat gerelateerde (maar eenvoudigere) integratieproblemen, namelijk de integratie van Lie algebras en Lie algebroïdes, en het bestaan van locale symplectische realisaties van Poisson variëteiten.

In hoofdstuk 8 van het proefschrift bestuderen we *reductie*, een ander aspect van Cartan's theorie. Cartan heeft aangetoond dat, door de structuurvergelijkingen nauwkeurig te bestuderen, men een speciaal systeem van vergelijkingen kan afleiden dat hij het *systatische systeem* noemt. Hij gebruikte dit systeem om een 'kleinere' maar 'equivalente' Lie-pseudogroep af te leiden. Zo'n reductieprocedure is nuttig om bijvoorbeeld Lie-pseudogroepen te klassificeren. Met onze moderne formulering presenteren we een reductieprocedure die sterk gebaseerd is op Cartan's procedure, maar veel verder reikt. Dit komt echter met een prijs, namelijk dat het begrip pseudogroep uitgebreid moet woorden naar zogeheten *gegeneraliseerde pseudogroepen*. Ondanks dat we hierbij de ons zelf opgelegde regel om zo dicht mogelijk bij het bronmateriaal te blijven hebben gebroken, zijn we ervan overtuigd dat onze reductieprocedure natuurlijk is, en dat het is "wat Cartan zelf ontdekt zou hebben" als hij het benodigde gereedschap had.

Laten we nog kort de overgebleven hoofdstukken bespreken: Hoofdstukken 1 – 3 zijn ter introductie, geven een overzicht van het essentiele achtergrondmateriaal, zoals de meetkundige theorie van partiële differentiaalvergelijkingen, en geven enkele nieuwe en vereenvoudigde bewijzen van bekende resultaten. Het laatste hoofdstuk, Hoofdstuk 9, behandelt nog een ander belangrijk aspect van Cartan's theorie: *prolongatie van structuurvergelijkingen*. Prolongaties spelen een rol in de klassificatie van Lie-pseudogroepen en in verschillende integratieproblemen die in de theorie voorkomen.

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Curriculum Vitae

Ori Yudilevich was born in the city of Haifa, Israel, on July 12, 1980, son to Eitan and Bruria. He has two siblings, an older sister Gali and a younger brother Dan. He lived during the majority of his childhood in the city of Karmiel, Israel. In the years 1993-1997, due to his father's work, he lived in the USA where he graduated from Walter Johnson High School in Bethesda, Maryland.

After completing his mandatory military service, he attended the Hebrew University of Jerusalem in the years 2003-2007. He followed a bachelor's program in physics and chemistry and obtained his degree with magna cum laude distinction.

In 2007-2009 he attended a master's program in theoretical physics at Utrecht University, obtaining his degree with cum laude distinction. His thesis, titled “Methods of Calculating Loop Corrections in QCD”, was written under the supervision of Prof. Eric Laenen.

In 2009-2010, making a switch from physics to mathematics, he attended a master class program focused on the subject of non-commutative geometry at Utrecht University. His thesis, titled “Zeta Functions on Riemannian Manifolds and Noncommutative Spaces”, was written under the supervision of Prof. Gunther Cornelissen.

In the second half of 2010 he spent one semester in Guatemala City where he taught bachelor level courses in physics at the Universidad del Valle de Guatemala.

In the years 2011-2016 he was a Ph.D. student at Utrecht University under the guidance of Prof. Marius Crainic. He defended his thesis, titled “Lie Pseudogroups à la Cartan from a Modern Perspective”, on September 14, 2016.

He is currently working as a postdoctoral researcher at KU Leuven in Belgium. Ori has a 5 year old daughter, Erikah.