How to Find the Logarithm of Any Number Using Nothing But a Piece of String

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The shape of a freely hanging chain suspended from two points is called the catenary, from the Latin word for chain. In principle, any piece of string would do, but one speaks of a chain since a chain with fine links embodies in beautifully concrete form the ideal physical assumptions that the string is nonstretchable and that its elements have complete flexibility independent of each other.

In modern terms, the catenary can be expressed by the equation $y = (e^x + e^{-x})/2$. As we shall see, Leibniz did not state this formula explicitly, but he understood well the relation it expresses, calling it a "wonderful and elegant harmony of the curve of the chain with logarithms" [**6**, p. 436]. (English translations of [**5**] and [**6**] are given in [**10**].) Indeed, he continued, the close link between the catenary and the exponential function means that logarithms can be determined by simple measurements on an actual catenary. "This may be helpful since during long journeys one may lose one's table of logarithms ... In case of need the catenary can then serve in its place" [**7**, p. 152]. Leibniz's recipe for determining logarithms in this way is delightfully simple and can easily be carried out in practice using, for example, a cheap necklace pinned to a cardboard box with sewing needles.

Leibniz's recipe

Refer to Figure 1 and the following description.

(a) Suspend a chain from two horizontally aligned nails. Draw the horizontal through the endpoints, and the vertical axis through the lowest point.

(b) Put a third nail through the lowest point and extend one half of the catenary horizontally.

(c) Connect the endpoint to the midpoint of the drawn horizontal, and bisect the line segment. Drop the perpendicular through this point, draw the horizontal axis through the point where the perpendicular intersects the vertical axis, and take the distance from the origin of the coordinate system to the lowest point of the catenary to be the unit length. We will show below that the catenary now has the equation $y = (e^x + e^{-x})/2$ in this coordinate system.

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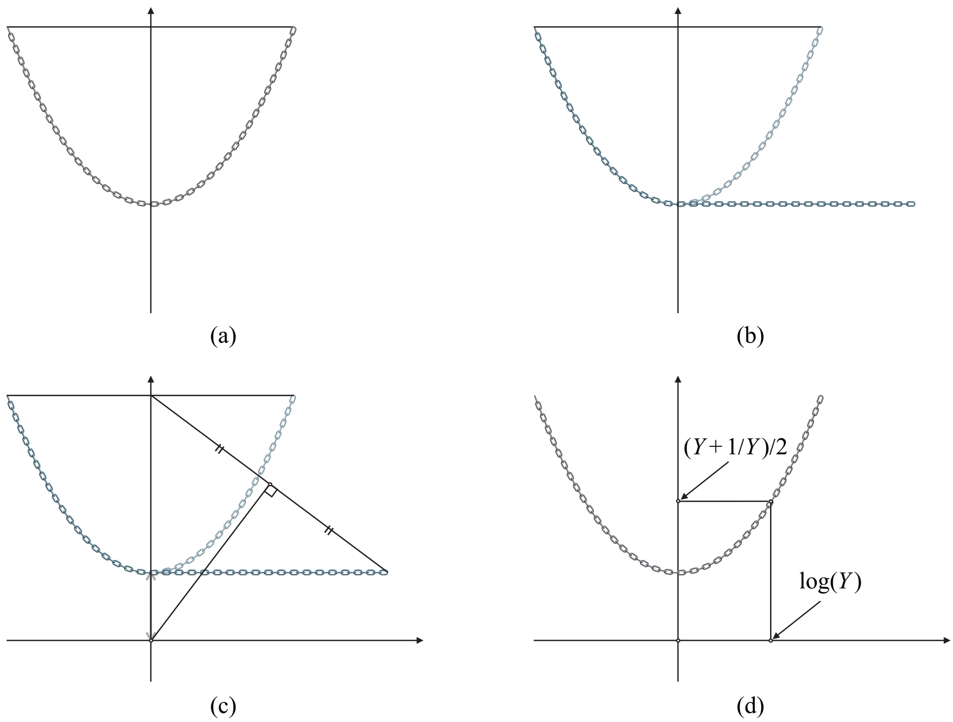


Figure 1. Leibniz's recipe for determining logarithms from the catenary.

(d) To find $\log(Y)$, find $(Y + 1/Y)/2$ on the y-axis and measure the corresponding x-value (on the catenary returned to its original form). This assumes that $Y > 1$. To find logarithms of negative values, use the fact that $\log(1/Y) = -\log(Y)$. If you seek the logarithm of a very large value, then you may end up too high on the y-axis; in such cases you can either try hanging the endpoints closer together or using logarithm laws to express the desired logarithm in terms of those of lower values.

The last step in this construction is given in Leibniz's catenary papers [**5, 7, 8**] where, however, the preceding steps are implicit at best; Leibniz later spelled these steps out in [**11**, No. 199].

The validity of this construction may be confirmed as follows. Figure 2 shows the forces acting on a segment of a catenary starting from its lowest point: the tension forces at the endpoints, which act tangentially, and the gravitational force, which is proportional to the arc s from T_0 to T. Since the catenary is in equilibrium, it is evident that the horizontal and vertical components of T balance with T_0 and the weight as, respectively, so $T_x = -T_0$ and $T_y = -as$. But since T acts in the direction of the tangent, we also know $T_y/T_x = dy/dx$. Thus we obtain $dy/dx = as/T_0$. On the left half of the catenary, where s is negative, we get instead $T_y = as$ and $-T_y/T_x = dy/dx$, which gives the same result. For convenience we choose the units of force and mass so that $a/T_0 = 1$, which gives $dy/dx = s$ as the differential equation for the catenary. Squaring both sides of this equation and using the Pythagorean identity $(dx)^2 + (dy)^2 = (ds)^2$ to eliminate dx leads to $(dy)^2 = s^2(ds^2 - dy^2)$ or $(1 + s^2)(dy)^2 = s^2(ds)^2$ and, by separating the variables and taking square roots,

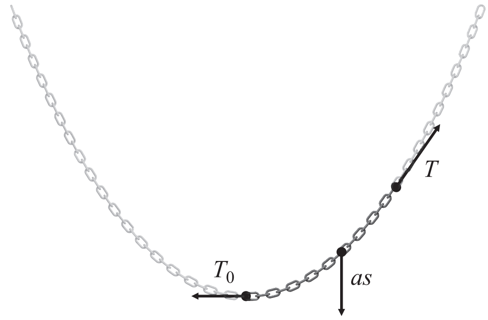


Figure 2. The forces acting on a segment of a catenary.

$$dy = \frac{s\,ds}{\sqrt{1+s^2}},$$

which integrates to $y = \sqrt{1+s^2}$. Thus $s = \sqrt{y^2-1}$, which we can substitute into the original differential equation for the catenary to obtain $dy/dx = \sqrt{y^2-1}$. It is now straightforward to check that $y = (e^x + e^{-x})/2$ is the solution to this differential equation that passes through (0, 1).

It remains to verify that the coordinate system assumed in this solution is the same as that defined by the construction of Figure 1. The key to Leibniz's verification turns out to be the intermediate step $y = \sqrt{1+s^2}$ above. To see this, consider Figure 3, which is Figure 1(c) with additional notation. We know from above that the catenary FAL is given by $y = (e^x + e^{-x})/2$ in a certain coordinate system whose origin O is at a vertical distance $OA = 1$ below the lowest point of the catenary. Consider the particular y-value $OH = y$ and the associated arc $AL = s$, then construct the horizontal segment AM with the same length s. It follows by the Pythagorean theorem that $OM = \sqrt{1+s^2}$. But above we saw that $y = \sqrt{1+s^2}$, which means in terms of this figure that $OH = OM$. Thus OHM is an isosceles triangle and so the perpendicular bisector of its base HM passes through the vertex O. This shows that the construction of Figure 1 does indeed give a way of recovering the coordinate system associated with the solution $y = (e^x + e^{-x})/2$, as we needed to show. From here it is a simple matter of algebra to check the final step of Figure 1.

In a 17th-century context

Finding logarithms from a catenary may seem like an oddball application of mathematics today, but to Leibniz it was a very serious matter—not because he thought this method so useful in practice, but because it pertained to the very question of what it means to solve a mathematical problem. Today we are used to thinking of a formula such as $y = (e^x + e^{-x})/2$ as the answer to the question of the shape of the catenary, but this would have been considered a naïve view in the 17th century. Leibniz and his contemporaries discovered this relation between the catenary and the exponential function in the 1690s, but they never wrote this equation in any form, even though they understood perfectly well the relation it expresses. Nor was this for lack of familiarity with exponential expressions, at least in Leibniz's case, as he had earlier used such expressions to describe curves with considerable facility [**11**, No. 6].

Why, indeed, should one express the solution as a formula? What kind of solution to the catenary problem is $y = (e^x + e^{-x})/2$, anyway? The 17th-century philosopher

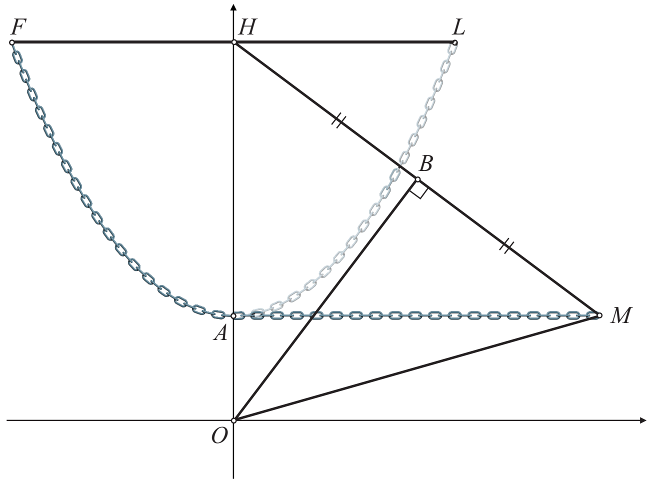
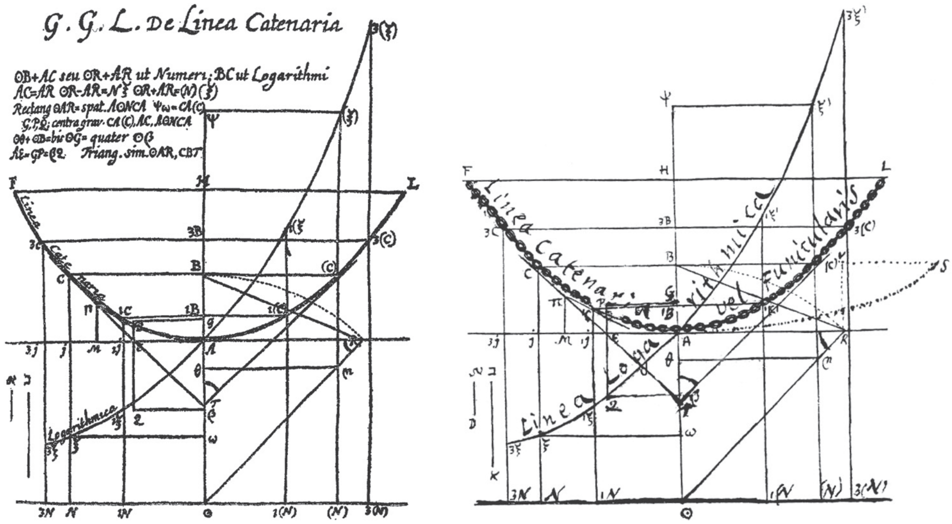


Figure 3. Figure used by Leibniz [**11**, No. 199] to justify the construction shown in Figure 1.

Thomas Hobbes once quipped that the pages of the increasingly algebraical mathematics of the day looked "as if a hen had been scraping there" [**4**, p. 330] and what indeed is an expression such as $y = (e^x + e^{-x})/2$ but some chicken-scratches on a piece of paper? It accomplishes nothing unless e^x is known already, i.e., unless e^x is more basic than the catenary itself. But is it? The fact that it is a simple formula of course proves nothing; we could just as well make up a symbolic notation for the catenary and then express the exponential function in terms of it. And, however one thinks of the graph of e^x, it can hardly be easier to draw than hanging a chain from two nails. So why not reverse the matter and let the catenary be the basic function and e^x the application? Modern tastes may have it that pure mathematics is primary and its applications to physics secondary, but what is the justification for this hierarchy? Certainly none that would be very convincing to a 17th-century mind.

The 17th-century point of view also had the authority of tradition on its side. Euclid's *Elements* had been the embodiment of the mathematical method for two millennia and one of its most conspicuous aspects is its insistence on constructions. Euclid never proves anything about a geometrical configuration that he has not first shown how to construct by ruler and compass. These constructions are what gave meaning to mathematics and defined its ontology. This paradigm remained as strong as ever in the 17th century. When Descartes introduced analytic geometry in his *Géométrie* of 1637, nothing was further from his mind than a scheme to replace the construction-based conception of mathematics by one centered on formulas. On the contrary, his starting point was a new curve-tracing method, which he presented as a generalization of the ruler and compass of classical geometry, and he accepted algebraic curves only once he had established that they could be generated in this manner [**3**].

It is in this context that we must understand Leibniz's construction: He sees the catenary not as an applied problem to be reduced to mathematical formulas, but as a fundamental construction device analogous to the ruler and the compass of Euclidean geometry. (See Figure 4 for two of his original figures.) Extending the constructional toolbox with new curve-tracing devices along these lines was a major research program in the late 17th century. Beside the catenary, other physical curves were also called upon for this purpose, such as the elastica [**1**] and the tractrix [**2**].



(a) (b)

Figure 4. Leibniz's figures for the catenary, showing its relation to the exponential function. (From [**5**] and [**8**], respectively.)

Thus 17th-century mathematicians had reason to reject the "chicken-scratch mathematics" that we take for granted today. They published not formulas but the concrete, constructional meaning that underlies them. If you want mathematics to be about something, then this is the only way that makes sense. It is prima facie absurd to define mathematics as a game of formulas and at the same time to assume naïvely a direct correspondence between its abstraction and the real world, such as $y = (e^x + e^{-x})/2$ with the catenary. It makes more sense to turn the tables: to define the abstract in terms of the concrete, the construct in terms of the construction, the exponential function in terms of the catenary. It was against this philosophical backdrop that Leibniz published his recipe for determining logarithms using the catenary. We see, therefore, that it was by no means a one-off quirk, rather it was a natural part of a concerted effort to safeguard meaning in mathematics.

Summary. We present Leibniz's 1691 recipe for determining logarithms using the catenary and discuss why this odd-looking application in fact made good sense in its historical context.

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