An Instrumentation Theory View On Students' Use Of An Applet For Algebraic Substitution

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In this paper we investigated the relationship between the use of a digital tool for algebra and students' algebraic understanding from an instrumentation theory perspective. In particular, we considered the schemes that students developed for algebraic substitution using an applet called Cover-up. The data included video registrations of three seventh-grade Indonesian students (12-13 year-olds) using the applet. The results showed that while solving equations and related word problems, the students developed schemes for algebraic substitution in which technical skills and conceptual understanding are intertwined. The schemes gradually were adapted to solve larger classes of equations. We found that crucial factors in this development called instrumental genesis are the characteristics of the applet and the task design, the role of a teacher, and the interaction among students.

1 INTRODUCTION

Proficiency in algebra is a gateway for secondary school students to pursue advanced studies at university level (Harvey, Waits and Demana, 1995; Katz, 2007; Kendal and Stacey, 2004; Morgatto, 2008). Therefore, the acquisition of algebraic expertise, including conceptual understanding and procedural skills, is an issue at an international level (e.g., Bokhove, 2011; Kendal and Stacey, 2004; Van Stiphout, 2011). Also in Indonesia much importance is attributed to students' algebra competence (Jupri, Drijvers and Van den Heuvel-Panhuizen, 2014a). However, the significance ascribed to proficiency in algebra is in sharp contrast with Indonesia's 38th position out of 42 participating countries in the domain of algebra in the 2011 Trends in International Mathematics and Science Studies (Mullis, Martin, Foy and Arora, 2012). These low results raise the question of how to improve student achievement in algebra.

Over the last decades, educational stakeholders over the world have highlighted the potential of digital technologies for mathematics education. The National Council of Teachers of Mathematics (NCTM), for instance, in its position statement, claims that "technology is an essential tool for learning mathematics in the 21st century, and all schools must ensure that all their students have access to technology" (NCTM, 2008, p.1). In Indonesia, the Ministry of National Education released a policy to introduce ICT (Information and Communication Technology) as a new subject for secondary schools, and suggested integrating the use of ICT in all school subjects, including mathematics (Depdiknas, 2007).

Whether ICT really helps and in what circumstances is not that obvious yet. On the one hand, there is research evidence that underpins the plea for technology-rich mathematics education. Review studies in mathematics education show that the use of ICT impacts positively on student mathematics achievement (Li and Ma, 2010) as well as on students' attitude towards mathematics (Barkatsas, Kasimatis and Gialamas, 2009). Specifically for algebra education, for instance, the use of ICT affects significantly on student achievement and conceptual understanding as well as procedural skills (Rakes, Valentine, McGatha and Ronau, 2010); the use of digital tools in algebra education can promote students' development of both symbol sense and procedural skills (Bokhove and Drijvers, 2010b), can be effective for improving algebraic expertise of secondary school students (Bokhove and Drijvers, 2012), and may foster the development of the function concept (Doorman, Drijvers, Gravemeijer, Boon and Reed, 2012). Moreover, the digital environment can support problem solving skills in informal algebra problems (Kolovou, Van den Heuvel-Panhuizen and Köller, 2013; Van den Heuvel-Panhuizen, Kolovou and Robitzsch, 2013). In line with this, we found that students who enrolled in a digital technology-rich intervention significantly outperformed their peers in the control condition without digital tools (Jupri, Drijvers and Van den Heuvel-Panhuizen, 2015. These results suggest that digital technology may enhance student learning of algebra.

On the other hand, however, digital technology is not a panacea for all issues in mathematics education and its integration turns out to be a non-trivial matter (Trouche and Drijvers, 2010), as is shown by the modest effect sizes found in the above studies, and even the absence of significant positive effects in others (see Drijvers, Doorman, Kirschner, Hoogveld and Boon, 2014). Because the transfer between work in a digital environment and the traditional paper-and-pencil work is not self-evident, teachers find themselves faced with the challenge of integrating new media in an appropriate way (Drijvers, Tacoma, Besamusca, Doorman and Boon, 2013). Moreover, fundamental questions about how and why digital technology works are waiting to be answered. Therefore, in the study reported in this paper, we aimed to contribute to the investigation of how the use of digital technology in algebra does foster students' algebraic thinking. In particular, we addressed the relationship between using a digital tool for algebra and the targeted algebraic understanding as well as mastery of procedural skills.

2 ALGEBRAIC SUBSTITUTION

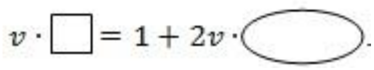
To investigate students' algebraic thinking, we focused on algebraic substitution, which is an important and sometimes indispensable method for, e.g., simplifying algebraic expressions, solving equations, and solving integration problems. From a mathematical point of view, algebraic substitution includes (1) replacing a more complex expression by one variable, and (2) replacing one variable by a more complex expression.

2.1 The first type of substitution

A well-known example of the first type of algebraic substitution, replacing a more complex expression by one variable, is provided by Wenger (1987):

Solve the equation $v \cdot \sqrt{u} = 1 + 2v \cdot \sqrt{1+u}$ for v.

Many students do not see this equation as being linear in v and thereforeare unable to solve it (Wenger, 1987; Gravemeijer, 1990). The reason is that the sub-expressions $\sqrt{u}$ and $\sqrt{1+u}$ are not considered as objects, as entities that can be covered or replaced with arbitrary variables without caring for their content. Rather, students see the square root signs as strong cues calling for algebraic manipulations. In other words, students do not have a 'global substitution principle' at their disposal that triggers them to consider sub-expressions such as $\sqrt{u}$ and $\sqrt{1+u}$ as objects (Wenger, 1987). Such a global look at sub-expressions can be stimulated by putting squared or oval tiles on the sub-expressions by which the 'object' interpretation is elicited (Freudenthal, 1962). For instance, Wenger's equation can be represented as follows:



simplify algebraic expressions to a familiar, standard form. Furthermore, Kindt (2010) stated that if the cover-up method is kept up for a time, and if sufficient variation of tasks is provided, it will encourage students to develop more formal strategies for solving problems. As an example, a relatively complex non-linear equation, such as $\frac{2015}{\sqrt{4053-2x}} = 403$, can be given to students in initial algebra to trigger them to use the cover-up method. This means that they can cover $\sqrt{4053-2x}$ by a tile and notice that its value is 5, which means that $4053 - 2x = 25$. Next, by covering $2x$ they will find that $2x = 4028$. Finally, by covering x, they will conclude that $x = 2014$ is the equation's solution.

2.2 The second type of substitution

The second type of algebraic substitution, that is, replacing one variable by a more complex expression, is important to understand composite functions and to combine different equations. For example, to find a formula for the composite function $f(2m-1)$ if $f(x) = x^2 + x$, the variable x must be replaced by the expression $2m - 1$, which after some intermediate steps leads to $f(2m-1) = 4m^2 - 2m$. Another example is shown in Figure 1 including the screen of a symbolic calculator (TI-89), on which $x = \frac{-b}{2}$ is substituted into the expression $x^2 + bx + 1$, which leads to $1 - \frac{b^2}{4}$ as its result.

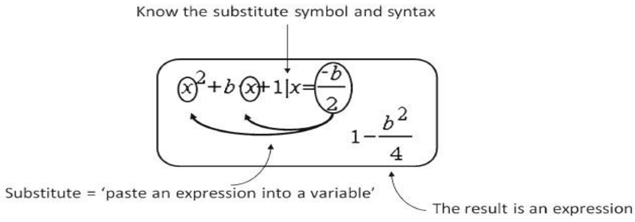


Figure 1Algebraic substitution on a TI-89 (Drijvers, Godino, Font and Trouche, 2013, p. 35)

2.3 The difficulty of substitution

Whether carried out with pencil and paper or in a digital environment, a main underlying difficulty of algebraic substitution concerns the process-object duality of an expression. In the above examples, an algebraic expression should be perceived not only operationally as a calculation process on variables, but also, and more important here, structurally as an algebraic object that can be replaced by another one. The in this case less appropriate process view on an algebraic expression often precedes an object view (Sfard, and Linchevski, 1994) and may result in the so-called lack of closure obstacle (Tall and Thomas, 1991), the discomfort in dealing with algebraic expressions that cannot be simplified any further and that do not have a numerical value. Integrating a process and an object view requires the reification of an algebraic expression as a mathematical object (Sfard, 1991). It is this reification process that is difficult to achieve for students, but that is needed for, for instance, algebraic substitution (e.g., Van Stiphout, Drijvers and Gravemeijer, 2013).

This method of putting tiles on sub-expressions, which is

3 SUBSTITUTION IN A DIGITAL ENVIRONMENT: THE COVER-UP APPLET

The Cover-up applet, developed by Peter Boon, of the Freudenthal Institute, Utrecht University, the Netherlands is an online digital environment developed to foster students' understanding of algebraic substitution and the reification of expressions. This applet allows students to solve equations of the form $f(x) = c$ by subsequently highlighting with the mouse an expression within an equation and assigning a value to it. Figure 2 shows an example of how the equation $\frac{48}{8(z+1)} = 3$ can be solved. In step 1, the equation to solve is displayed in the solution window and needs to be studied. In step2, the expression $8(z+1)$ can be highlighted; then, the applet automatically shows the expression $8(z+1) = \cdots$ in the next line. In step 3, a numerical valuemust be determined and filled in for the selected expression. In this case the correct value is 16. The applet produces feedback in the form of a yellow tick mark signifying a correct action (otherwise a red cross appears). This solution process proceeds, for instance, until step 6 and ends with $z = 1$ as the solution of the equation, leading to a green tick mark and the final feedback: "The equation is solved correctly!". In practice, a student does not necessarily follow all these six steps, but may also take shortcuts, such as jump to step 6 immediately after step 3, or make detour and may need more than six steps to solve the equation.

There are three main reasons for using the Cover-up applet for fostering students' understanding of algebraic substitution and the reification of expressions. First, the activities with the applet invite students to simultaneously develop an operational and a structural view on algebraic expressions: selecting expressions by highlighting them stresses their object character, whereas assigning numerical values relates to the outcome of a calculation process (Jupri, Drijvers and Van den Heuvel-Panhuizen, 2014b). Second, the Cover-up applet can be used to solve various kinds of equations and not just linear equations in one variable of the form $f(x) = c$. Third, the way to use the applet is close to the intended way of thinking about equations and expressions, whereas students in the meantime have the freedom to explore different pathways within this approach.

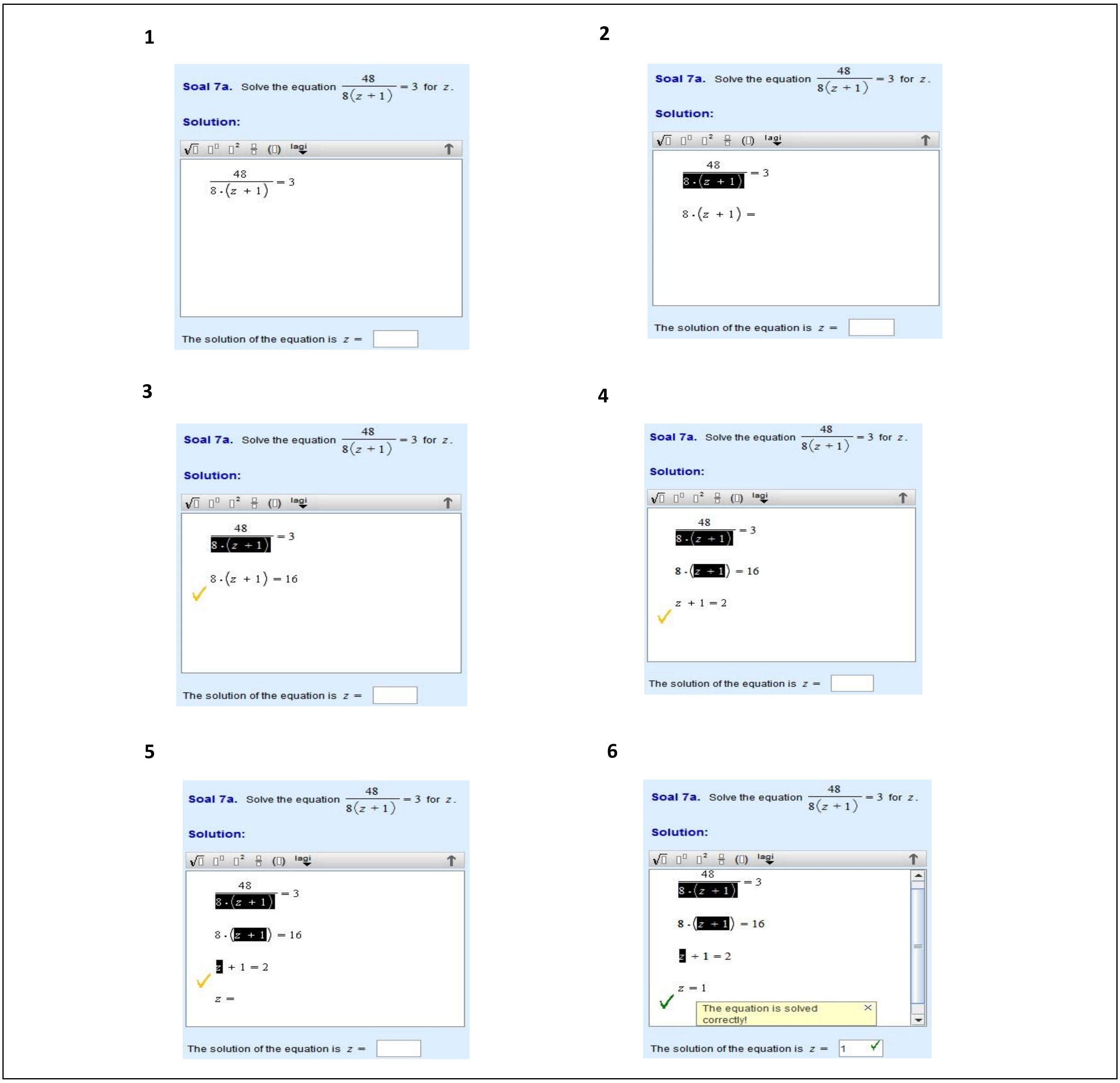


Figure 2 An equation solved by using the Cover-up applet

4 INSTRUMENTATION THEORY AS A LENS

As mentioned in the introduction, the relationship between using a digital tool for algebra and the targeted algebraic understanding and mastery of procedural skills was the focal issue in this study. We investigated this relationship for the case of carrying out algebraic substitution with the Cover-up applet. To get an in-depth understanding of this relationship, we chose the perspective of instrumentation theory. We now briefly review the core elements of this theory, also called the instrumental approach, for using digital tools in mathematics education.

Within this instrumentation theory, the following terms play a key role: artefact, tool, technique, scheme, and instrument (e.g., Drijvers, Godino, Font and Trouche, 2013; Trouche and Drijvers, 2010). An *artefact* is an object, either material or not. A graphing calculator is an artefact, and mathematical language can be considered an artefact as well. In this study, the main artefact is the Cover-up applet, which can be used for solving equations, but paper and pencil form an important pair of artefacts as well. If an artefact is used for carrying out a specific task, such as solving an equation, we call it a *tool*.

An artefact is useless as a tool as long as the user has no idea for which task or how to use it. This is where the notion of *technique* comes in. In line with Artigue (2002) we define a technique as a manner of solving a task using an artefact. Techniques can be observed in the user's behavior while using the artefact. The main techniques that can be applied in the Cover-up applet are described in the previous section (see Figure 2); solving equations on paper also entails the application of techniques.

Techniques, however, do not stand on their own, but are based on cognitive foundations. It is these foundations that form the schemes. Based on the work of Piaget, Vergnaud (1996) defines a *scheme* as an invariant organization of behavior for a given class of situations. The schemes at stake in the study presented here concern solving equations and the related word problems through algebraic substitution using the cover-up strategy and will be addressed in more detail in the next section.

Schemes and techniques both share conceptual and technical elements and both involve using an artefact for solving a specific type of tasks. Nevertheless, an important difference between the two is that schemes are invisible, whereas techniques are observable. In fact, we consider techniques as the observable manifestations of the invisible schemes (Drijvers, Godino, Font and Trouche, 2013). An *instrument*, now, is a mixed entity of scheme, technique, artefact and task. As such, it is the amalgam of all the 'players' involved when a student solves a mathematical task using a digital tool (Trouche and Drijvers, 2010; Trouche, 2004).

If a type of tasks can be solved by using different artefacts, but with different, related techniques, the corresponding scheme, the different artefacts and techniques can be regarded as one single instrument (see Figure 3). In this study, the artefacts in play while solving equations through algebraic substitution were the Cover-up applet and paper and pencil, but we considered the corresponding techniques to be closely related.

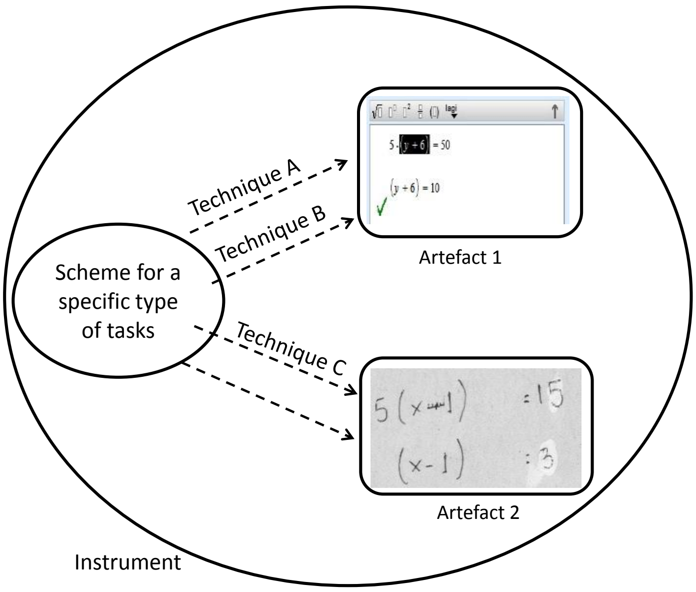


Figure 3 An instrument including different artefacts and techniques

Based on the above *instrumental genesis* can be defined as the process of the user developing instruments, consisting of cognitive schemes and observable techniques for using a specific artefact for a specific class of tasks. In principle, instrumental genesis is an individual process that usually takes place in a social context, which in this study consists of students who work in groups. For a deeper understanding of instrumentation theory we refer the reader to, for example, Artigue (2002), Lagrange (1999), Trouche (2000, 2004), and Vergnaud (2009).

Many studies have used instrumentation theory to address the relation between user and tool in the problem solving process (e.g., Artigue, 2002; Drijvers et al., 2013;

Guin and Trouche, 1999; Lagrange, 1999; Trouche, 2004; Trouche and Drijvers, 2010). However, elaborated examples of schemes are still scarce. To contribute to this, we will now provide a description of a conjectured scheme for solving equations using algebraic substitution with the Cover-up applet.

5 INSTRUMENTATION SCHEMES FOR SOLVING EQUATIONS USING ALGEBRAIC SUBSTITUTION WITH THE COVER-UP APPLET

To further investigate algebraic substitution, we first set up a conjectured instrumentation scheme for solving symbolic equations of the form $f(x) = c$ and then for related word problems. Table 1 summarizes the conjectured scheme for solving equations using algebraic substitution with the Cover-up applet. This scheme includes conceptual and technical elements which are related to each other. Even if the scheme is described for the paradigmatic task of solving the equation $\frac{48}{8(z+1)} = 3$ for z, this description has a generic character and applies to every equation of the form $f(x) = c$.

Step	Conceptual aspect	Related technical aspect
1.	By scanning the equation, recognizing the equation as being of the form $f(z) = c$, so of the form <expression> = <numerical value>, with the unknown appearing only once on the left hand side. Realizing that the task is to rewrite this equation in the form <unknown> = <value>, which provides its solution. In this case, the expression is $\frac{48}{8(z+1)}$ and the numerical value is 3. As a consequence, the cover-up strategy can be applied.	No specific techniques involved in this step.
2.	By further inspection, recognizing the structure of the expression in left hand side of the equation. In this case, for example, the division of 48 by $8(z + 1)$ should be recognized as the central operator.	No specific techniques involved in this step. The equation has already been given in the solution window, i.e., $\frac{48}{8(z+1)} = 3$.
3.	Identifying a sub-expression to be covered as to start the cover-up strategy. In this case, this could be $8(z + 1)$.	Highlighting the identified sub-expression using the mouse. The applet puts the sub-expression in a new line and adds the equal sign. In this case, the result would be $8(z + 1) = \cdots$.
4.	Assigning a numerical value to the covered sub-expression to make a new equation becomes true. In this case, this value would be 16.	Typing the value after the equal sign, and pressing enter. In this case, the result would be $8(z + 1) = 16$ with a yellow tick mark signifying a correct action.
5.	(If necessary) repeating steps 3 and 4 to the new equation obtained in step 4 until the equation is simplified to $z = \ldots$ In the example, this would lead to $z = 1$.	Highlighting a sub-expression from the new equation, typing a numerical value, and pressing enter. In this case, the sub-expressions and corresponding numerical values would be: $z + 1 = 2$, and $z = 1$, respectively. Finally, the solution is indicated by the feedback provided by the applet: a green tick mark and the text "The equation is solved correctly!"

Table 1 Conjectured scheme for solving equations using algebraic substitution with the Cover-up applet

In terms of the operational-structural duality, steps 1, 2, and 3 in Table 1 mainly appeal for a structural view on equation as the equivalence of an expression and a number, and on expression as an algebraic object. However, to assign a numerical value to the selected sub-expression (step 4) appeals for an operational view on the equation: if the output of the central operation is known, the value of the operand can be found. It is this integration of operational and structural views that makes this scheme, and the corresponding use of this applet, relevant for reification. Note that the screen captures provided in Figure 2 matches with the above scheme description. For solving word problems, which can be translated into equations of the form $f(x) = c$, the previous scheme has to be extended. The conjectured scheme for solving word problems with the Cover-up applet is summarized in Table 2.

Step	Conceptual aspect	Related technical aspect
a.	Recognizing the possibility of solving the word problem through re-phrasing it in terms of a mathematical equation.	Reading the word problem aloud (if necessary).
b.	Setting up the equation, i.e., transforming each phrase into an algebraic expression, and altogether the word problem into an equation. After setting up an equation from the word problem, the next steps are the same as described in Table 1.	Typing the equation using the Cover-up applet's equation editor (if necessary), and pressing enter to check whether the equation is correct or not.

Table 2 Conjectured scheme extension for solving word problems with the Cover-up applet

6 RESEARCH QUESTION

Based on the lens of instrumentation theory we specified our initial questions phrased in the Introduction and decided to examine the relationship between using a digital tool for algebra and the targeted algebraic understanding as well as mastery of procedural skills through the following theory-guided research question:

Which schemes do students develop for solving equations using algebraic substitution with the Cover-up applet and which relationships between techniques and understanding are developed?

In this question, the 'schemes' should be understood in the perspective of instrumentation theory. In particular, these include a scheme for solving equations using algebraic substitution with the Cover-up applet, and an extended one for solving related word problems. The problems on which the focus in this research question are (mainly linear) equations of the form $f(x) = c$ with the unknown, in this case x, appearing only once on the left hand side, as well as related word problems.

7 METHOD

To answer the research question we carried out a case study. This case study was part of a larger experimental study (Jupri, Drijvers and Van den Heuvel-Panhuizen, 2015) in which a learning arrangement was designed for learning to solve (mainly) linear equations in one variable which in Indonesia is part of the grade VII curriculum. The designed learning arrangement included activities with the Cover-up applet (see section 3)which is embedded within the Digital Mathematics Environment (DME).

The DME is a web-based learning environment providing (i) interactive digital tools for algebra, geometry, and other mathematical domains; (ii) a design of open online tasks and immediate feedback; (iii) access to the environment at any time and place, as long as technological infrastructure and conditions are met, and (iv) a storage for student work (Boon, 2006; Drijvers, Boon, Doorman, Bokhove and Tacoma, 2013). In a Delphi study (Bokhove and Drijvers, 2010a) where four groups of criteria (algebra didactics, theories on tool use, assessments, and general characteristics of digital tools) were used to evaluate digital environments for mathematics education it was shown that the DME compared to other digital tools was recognized as a suitable environment for research in algebra education addressing the co-emergence of procedural skills and conceptual understanding.

The case study was based on one lesson carried out in one seventh-grade classroom. The lesson was given by the classroom teacher who was informed on how to implement the learning arrangement activities through the teacher guide. The lesson lasted for 80 minutes and consisted of three respective parts. First, a paper-and-pencil activity was done and included posing problems and whole-class discussion. In this activity, the teacher introduced the concept of equation through posing problems and guided class discussion, while the students paid attention, did the problems with paper and pencil, and were actively involved in the discussion. This was followed by a whole-class demonstration of how to work with the Cover-up applet for solving equations and a group-based digital activity done by students. During the group work, students were asked to solve a series of tasks embedded in the DME using the Cover-up applet under the teacher's guidance. For example, through explaining the meaning of an equation, the teacher guided students to choose an appropriate expression within the equation for algebraic substitution. Finally, the students were requested to do individually paper-and-pencil tasks and the teacher was guiding the students to reflect upon the lesson. The tasks used in the lesson, in both digital and paper-and-pencil activities, consisted of two types: bare tasks and word problems.

To analyse the relationship between the use of the Cover-up applet and students' conceptual understanding and skills, we focused on the data of one group of three male Indonesian students (12-13 year-old). In the paper these students are named Ali, Quni and Widan. The group we chose was based on the teacher's recommendation with the criteria: students in the group would have heterogeneous mathematical abilities, and they would feel free to do mathematical activities in front of camera. During their work the three students were observed and video-recorded by the first author. Moreover, during the lesson, the first author also helped these students. In this way, he acted as a substitute teacher, while the teacher took care of the other groups of students in the class. The analysed data included the video recordings of the three students during their group work in the digital environment, the corresponding student digital work stored in the DME, the students' written work on individual tasks, and observation notes made by the first author. An integrative qualitative analysis on these data, with the help of Atlas.ti software, was carried out to investigate the students' scheme development and the relationship between the use of the applet and the targeted algebraic understanding. This analysis included transcribing the video data, studying the students' written and digital work, and analysing the relationship between the use of the Cover-up applet and student conceptual understanding and skills, respectively, with an instrumentation theory lens, that is, in terms of the conjectured schemes in Tables 1 and 2.

8 RESULTS

In this section we present the results of the observations of the group work during the Cover-up activity. For both bare problems and word problems, we first provide an analysis of one paradigmatic task with the conjectured scheme described in section 5 as a frame of reference. Next, we describe one scheme that the students used for solving the task. Finally, to follow students' development throughout the activity, we present our observation on schemes and related techniques for all tasks treated in this one lesson, discuss the observation and relate to student written work.

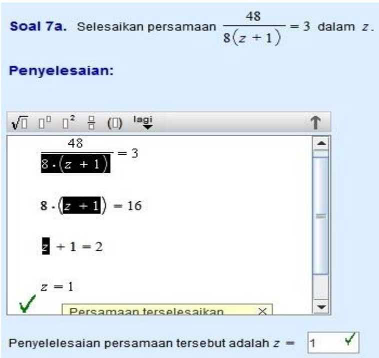
Observation	Commentary
Task 7a. Solve the equation $\frac{48}{8(z+1)} = 3$for z. 	Task 7a is a bare problem in the Cover-up activity. The figure that is below the task shows the corresponding student digital work stored in the DME.
Widan reads out the task aloud.	Reading the equation aloud might help the students to recognize the equation as being of the form $f(z) = c$ and to realize that $z =<$ value $>$ would provide the solution of the equation (step 1); and to see the structure of the algebraic expression in the left-hand side of the equation. In particular, it is important to perceive the division of 48 and $8(z+1)$ as the central operator that produces 3, the numerical value in the right side (step 2). This enables the student to determine the sub-expression to cover and to assign a numerical value.
Quni: Please cover that part [$8(z+1)$]. [Ali highlights $8(z+1)$ with the mouse, the applet yields $8(z+1) = \cdots$ in the next line]. Good! Quni: The value of [$8(z+1)$] is... Ali: This is 48 [divided by $8(z+1)$ equals 3]. So, [the value of $8(z+1)$ is] 16. Quni: Yes, yes, you are right! It is 16. [He types 16 and presses enter. It is correct.].	Quni and Ali correctly apply the cover-up strategy to solve the equation. That is, they recognize the first sub-expression to cover, i.e., $8(z+1)$, and they are able to assign a correct numerical value for that, i.e., 16 (steps 3 and 4).
Widan: How did you get 16? Quni: What is the value of 3×16? Widan: Yes, it is 48. [$3 \times 16 = 48$].	The question posed by Widan indicates that he initially does not understand why his friends assign 16 to $8(z+1)$, which means he probably does not recognize the division as the central operation of the expression in the left-hand side. Therefore, Quni explains to Widan by asking the value of 3×16.
Quni: $8(z + 1) = 16$. So, $z + 1 = 2$. Ali: Yes it is 2 [he highlights $z + 1$, types 2, and presses enter. It is correct.] Quni: Now, z, z, z [to be covered]. And its value is 1. [Ali highlights z, types 1 and presses enter. It is correct. Also, he inputs the solution in the answer box, and presses enter.]	Quni and Ali carry out step 5. Quni is able to identify a sub-expression to cover from the new equation $8(z + 1) = 16$, i.e., $z+1$, and assigns 2 to it. Ali agrees, and carries out the technique in the applet. Finally, Quni identifies z to cover from $z + 1 = 2$ and assigns 1 to it. Ali carries out the technique. Both of them finally get $z = 1$ as the solution of the equation.
The students immediately proceed to a next task without checking their solution mentally or orally.	Students do not check the solution because the applet has already provided feedback in each step, thus confirming a correct action and solution.

Table 3A commented observation of the group's work on Task 7a

8.1 Students' scheme for solving equations using algebraic substitution with the Cover-up applet

An observation of a group working on one of the tasks

Table 3 presents a two-minute observation of the group's work on Task 7a. In the right column we provide corresponding commentaries, which are based on the conjectured scheme presented in Table 1.

In the light of the conjectured scheme, this observation shows that the group's scheme is in line with the conjectured scheme described in Table 1, even if one of the students did not fully understand the solution process. To summarize this observation, Figure 4visualises the main conceptual elements of the students' scheme: recognizing the equation as suitable for the cover-up strategy, and wanting to rewrite in the form$<$ unknown $>=<$ value $>$, identifying a sub-expression to cover, assigning a numerical value to the covered sub-expression, and repeating these steps as long as needed.

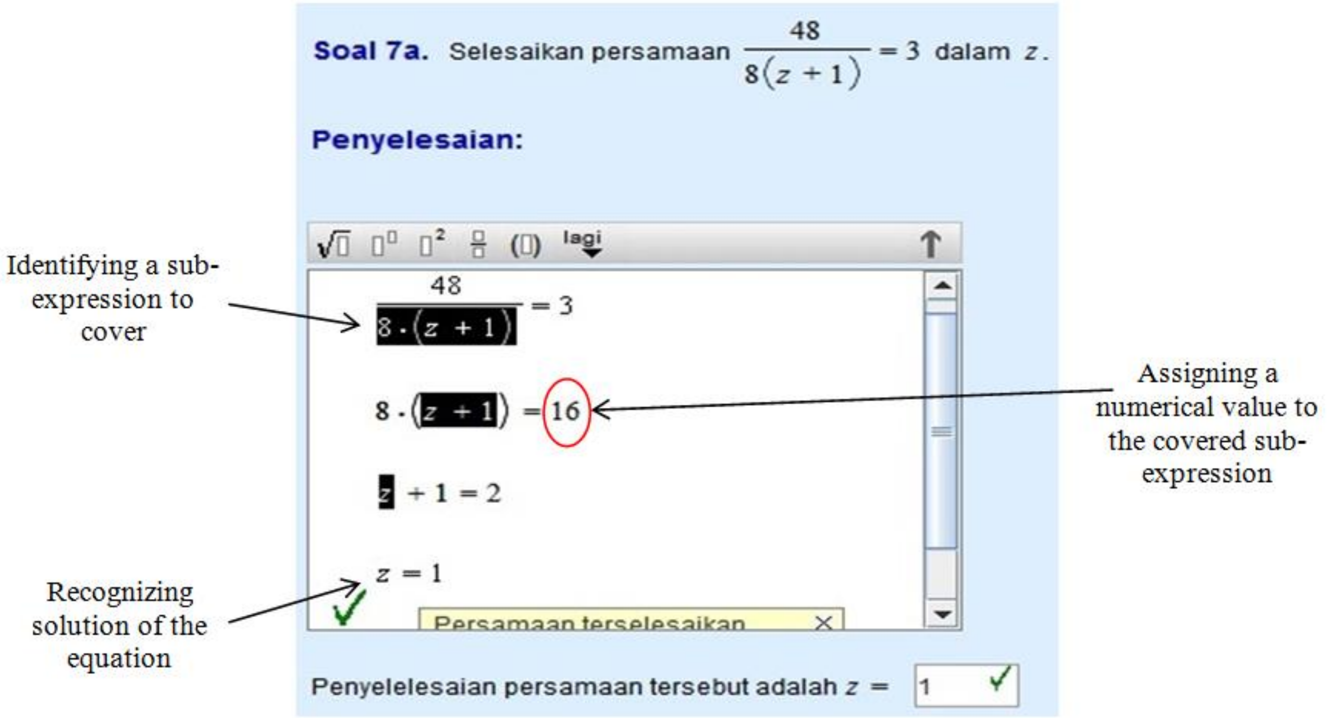


Figure 4 The main elements of the substitution scheme observed in the group of three students

An analysis of the group's work over the different tasks within one lesson

Table 4summarisesstudents' scheme and related techniques of the observed group for bare algebra problems treated during the one-lesson Cover-up activity.

From this observation we noted that even if the three students were finally able to solve the given equations by applying the cover-up strategy using the applet, still they encountered difficulties while doing so.

The main difficulties encountered by students concerned arithmetical calculations errors, as shown for the case of tasks 3a, 8, and 9a. For instance, when solving the task 3a, the students assigned 10 as a numerical value to $2p+5$ instead of 15. Also, the observer in some cases gave too much guidance as shown in the observations of tasks 2a, 3a, 4a, 8, and 9a.

Three points in this observation deserve further attention. First, Widan seemed to experience difficulties – in the sense that he often could not follow his peers' thinking – while solving equations during the activity. In our view, these difficulties were caused by Widan's limited understanding of an equation as a structural equivalence between two objects (an algebraic expression and a number).

This lack of understanding was manifest when working on the tasks 6a, 7a, 9a and 9b. For example, Widan did not understand why his peers assigned 16 to $8(z+1)$ while solving the equation $\frac{48}{8(z+1)} = 3$.

Task	Observation: scheme and techniques
Task 2a. Solve the equation $5(y+6)=50$ for y. Hint: Cover $y+6$ at the first step and assign a value to it.	As suggested by the observer, students follow the hint. They assign the values to$y+6$ and y correctly. An observed technical obstacle concerns covering parts of the equation, i.e., they initially highlight $5(y+6)$ rather than $y+6$.
Task 3a. Solve the equation $4(2p+5)=60$ for p. Hint: Cover $2p+5$ at the first step and assign a value to it.	Students follow the hint, but they assign 10 (i.e., students conclude that 60 divided by 4 is 10) rather than 15 to $2p+5$. After the observer explains that the equation means"4 times something equals 60", students assign 15 to $2p+5$. Once the equation is reduced to $2p=10$, the technical obstacle of covering $2p$ rather than p appears again. They are finally able to find $p=5$ as the solution.
Task 4a. Solve the equation for w. $$\frac{8(3w+2)}{5}-2=6$$ Hint: Cover $\frac{8(3w+2)}{5}$ at the first step and assign a value to it.	After getting the observer's explanation, students follow the hint and assign 8 to $\frac{8(3w+2)}{5}$. Next, they identify correct values for $8(3w+2)$, $3w+2$, $3w$ and w, respectively, by themselves.
Task 5a. Solve for a: $$\frac{18}{5a-2}=6$$ Hint: Choose one of three sub-expressions from the answer box to cover at the first step.	Even if Ali assigns 3 to $5a-2$ correctly, Quni misunderstands it as $5-2=3$. Next, Quni suggests tocover a directly and assigns 3 to it, rather than assigning $5a$ as suggested by Ali. Overall, the students finally solve the equation correctly. The technical obstacle of covering $5a-2$ appears at the initial step of solving the equation.
Task 6a. Solve for x: $$\frac{30}{2x+3}+4=6$$ Hint: Choose one of three sub-expressions from the answer box to cover at the first step.	After reading the task, students are able to identify parts of the equation to cover and to assign proper numerical values to those parts by themselves. However, Widan seems to not fully understand when Ali says that he will cover $\frac{30}{2x+3}$ at first.
Task 7a. Solve for z: $$\frac{48}{8(z+1)}=3.$$	Overall, students are able to identify parts of the equation to cover and to assign proper numerical values to those parts. However, Widan seems to not fully understand why his colleagues assign 16 to $8(z+1)$ (see Table 3 for a detailed description).
Task 8. Solve for positive q: $$(q+3)^2=49.$$ Hint: Choose one of three sub-expressions from the answer box to cover at the first step.	After reading a worked example and getting an explanation from the observer, the students work on Task 8. Even if Quni identifies $(q+3)$ to cover in the first step, he assigns 9 rather than 7 to it. Overall, the students are able to solve the equation.
Task 9a. Solve for positive x: $$(x-1)^2+3=12.$$	With the observer guidance, students solve the equation without a serious difficulty. A calculation mistake appears when Widan assigns 4 to $(x-1)$ rather than 3 as suggested by Quni.
Task 9b. Solve for positive r: $$(2r+1)^2+1=26.$$	After reading the equation aloud, Quni identifies 5 for the value of $2r+1$. When this group concludes 2 for r from $2r=4$,Widan does not know why his colleagues assigned it – indicating that he does not understand the solution process.

Table 4 Students' schemes for the different tasks within one lesson

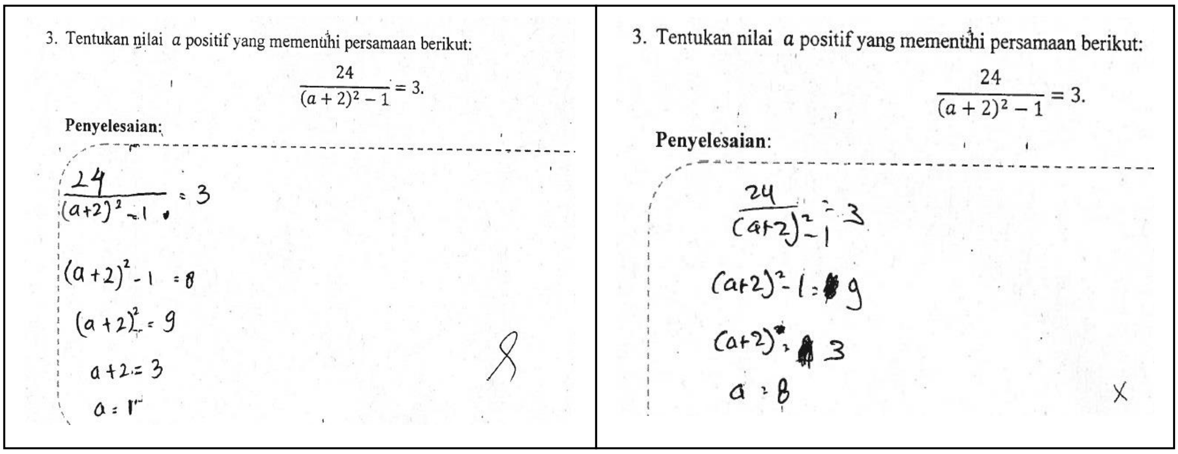


Figure 5 Exemplary written work by Ali (left) and Widan (right) after engaging in the Cover-up activity

This result shows that the three students in the group acquired different conceptual understanding and skills. This also is manifest in their individual written work working on a paper-and-pencil task shown in Figure 5.

The left part shows Ali's work. It is a correct solution and Ali is able to select appropriate expressions as well as able to assign correct numerical values to them, and to successfully apply the cover-up strategy. The right part shows Widan's work, which is to a certain degree similar to Quni's work. Widan was able to select appropriate expressions and seemed to have a correct view of the solution process. Although his writing was not correct in the in-between step – maybe because of doing some steps 'in his head'– he arrived correctly at $(a + 2) = 3$, but unfortunately then he did not manage to derive the correct value of a. This final step suggests that he mixes up the variable$a$ and the expression $(a + 2)^2 - 1$.

.

The second point concerns the applet's technical limitations. Whereas covering an expression within an equation with the mouse was expected to foster reification, the applet proved to be too sensitive to mouse movements. As a consequence, students often highlighted an expression that they did not intend to cover, such as covering $\frac{18}{5a-2}$ instead of $5a - 2$ in $\frac{18}{5a-2} = 6$. However, once students got used to working with the applet, they became more skilful in using the mouse for covering an expression.

The third point concerns the order of tasks. While designing the instructional sequence, we ordered the tasks according to their conjectured difficulty, as reflected in many curricula: linear equations first (Tasks 2a-4a), next rational equations (Tasks 5a-7a) of the form $\frac{k}{g(x)} = c$, where k and c are constants, and $g(x)$ is a linear expression, and, finally, quadratic equations (Tasks 8-9b). Indeed, the data show that linear equations were easier than rational equations, but the rational equations, which in addition contain divisions, seemed to be more difficult than the quadratic ones. However, the students were able to apply the scheme and techniques to new types of equations, which can be seen as a modest form of instrumental genesis.

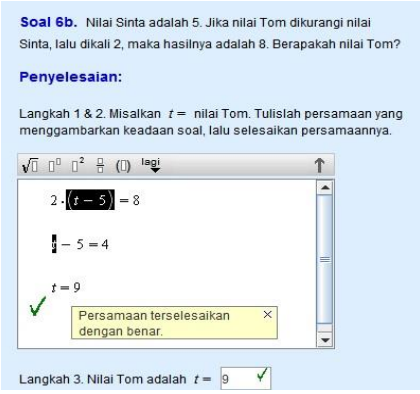
To summarise, the observed scheme was in line with the conjectured scheme described in Table 1 and Figure 4. As presented in Table 4, the main obstacles encountered by students included arithmetical calculation errors. This scheme and the related techniques were applied to increasingly complex equations, which suggest the development of a structural view on equations and expressions, and instrumental genesis.

8.2 Students' scheme for solving word problems using algebraic substitution with the Cover-up applet

An observation of a group working on one of the tasks

The word problems addressed in the activity after the proper transformation of problems into mathematical models all concern linear equations of the form $f(x) = c$. Table 5 presents a three-minute observation of the group's work on Task 6b. In the right column we provide commentaries based on the conjectured scheme presented in Table 2 (steps a-b) and Table 1 (steps 1-5).

<table>
<tr><th>Observation</th><th>Commentary</th></tr>
<tr><td>
Task 6b. Sinta's grade is 5. If Tom's grade is subtracted by Sinta's grade, next multiplied by 2, the result is 8. Find Tom's grade. Hint: Let t be Tom's grade.*



</td><td>
Task 6b is a word problem in the Cover-up activity. The figure below is the student digital work stored in the DME.
</td></tr>
<tr><td>
Students read out the task aloud together.
</td><td>
Reading the task aloud seems to help the students to recognize that the word problem can be transformed into a mathematical equation, to identify the given information and an unknown, and to prepare possible strategies for finding the unknown (step a).
</td></tr>
<tr><td>
Quni and Widan: t is subtracted by 5, next divided by 2.

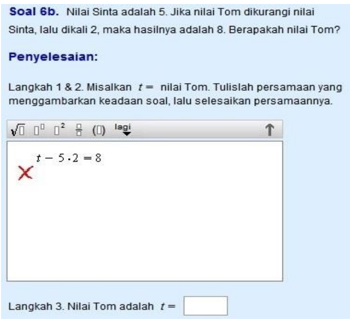
Ali: No, it is multiplied by 2.

Quni and Widan: Yes, it is incorrect. It must be multiplied by 2.

Quni: t is subtracted by 5, and then multiplied by 2.
</td><td>
Students are trying to set up an equation (step b). However, Quni and Widan mistranslated the second phrase of the problem. Ali corrects them.
</td></tr>
<tr><td>
Quni and Widan: t minus 5, multiplied by 2, equals 8. [Quni types $t - 5 \times 2 = 8$.]

Ali: Enter!

Quni, Ali and Widan: [After Quni presses enter] Incorrect! Why is it still wrong? [A red cross appears in the solution window signifying that the formulated equation is incorrect.]



Observer: Maybe you typed the equation incorrectly! Please you type it again! Please you type 2 [first]. [Quni erases the previous incorrect equation, and types 2 firstly.]

Observer: So, 2 times...
</td><td>
Even if the three students understand the word problem, they are not able to express it in a proper equation. In other words, they have difficulties in setting up an equation (step b).

Observing this situation, the observer gives guidance.
</td></tr>
<tr><td>
Quni, Ali and Widan: [2 times] t subtracted by 5, equals 8. [One of students types $2(t - 5) = 8$.]

Observer: Enter!

Quni, Ali and Widan: [One of students presses enter! It is correct!] Ooo...

Observer: Do you understand why I suggested you to type the 2 [in front of] $(t - 5)$? [No reply, maybe they are thinking].

Observer: Because, as you said, first you do t minus 5, and then
</td><td>
The observer suggests to type 2 in front of $t - 5$, and, as an indirect consequence, to use a bracket for $t - 5$, so that they get a correct equation. With the observer's guidance, students are finally able to set up a correct equation (step b). However, the observer seems to give too much guidance as if he is part of the group wanting to solve the task correctly.
</td></tr>
</table>

multiplied by 2. The bracket means something that should be carried out firstly.	
Ali: Cover$t-5$. [Quni highlights $t-5$, the applet provides $t-5=\cdots$ in the next line.] Quni, Ali and Widan: [$t-5=$] 4. [Quni types 4 and presses enter. It is correct.] Quni: [The value of t is] 6. Quni: Eh, no, [it is] 4. Widan: [No! It is] 5. Ali: [Hi, the value of *t* must be] 9. Quni and Widan: Yes, 9, you are right! [Quni types 9, so it becomes $t=9$.] Ali: Enter! Quni: [Presses enter] Correct! [So, the solution is $t=9$.]	Quni and Ali recognize the equation of the form $f(t)=c$ and its structure (steps 1 and 2). Even if they make mistakes during the solution process, they are finally able to solve the equation using the cover-up strategy (steps 3-5), i.e., they are able to identify the sub-expression to cover (step 3), assign a numerical value to the covered sub-expression (step 4), repeat steps 3 and 4 while applying the cover-up strategy (step 5) and recognize $t=9$ as the solution.
The students proceed directly to the next task without checking the solution.	They do not check the solution because the applet has already given feedback in each step that their actions are correct.

Table 5 A commented observation of a group's work on Task 6b
This task is adapted from one of tasks in the Indonesian textbooks. Even if the task is considered to be less appropriate from a didactical point of view – one can argue if grades can be multiplied – we decided to use it to connect to Indonesian textbook.

Concerning students' mistakes when assigning a numerical value to$t$ in the equation $t-5=4$, we conjecture that the students referred to the equation $2(t-5)=8$ when determining the value of t for the equation $t-5=4$. Quni seemed to see the addition as the central operation in the expression $2(t-5)$ rather than the multiplication of 2 and $t-5$. As a result, Quni assigned 6 to the value of $(t-5)$, but he mistakenly considered this as the value for t. Next, when he saw $2(t-5)$ as a multiplication of 2 and $(t-5)$, he assigned$(t-5)=4$ (again he mistakenly considered $t-5$ as t). Widan, who assigned $t=5$, might have guessed an arbitrary value. The mistakeof recognising the central operation for the expression $2(t-5)$ as an addition of 2 and $(t-5)$ suggests that Quni lacks a structural view on the algebraic expression. As an aside, we notice that he also interchanged $t-5$ for t, which concerns the difficulty in understanding a variable.

From the observations such as the one described in Table 5, we conclude that the students' scheme is in line with the conjectured scheme outlined in Tables 2 and 1. To summarize, the main conceptual elements in this scheme – which is an extension of the scheme visualized in Figure 4, include: setting up an equation, and working towards the form$<$ unknown $>=<$ value $>$, which provides the solution of the equation. The step of setting up an equation is the difference between this scheme and the scheme in Figure 4. For this step, the role of the applet concerns providing feedback stating whether a formulated equation typed in the solution window is correct or not: when it is correct, a yellow tick mark appears, otherwise a red cross mark emerges. In this way, students can improve their ideas while formulating an equation.

An analysis of the group's work over the different word problems within one lesson

Table 6 summarises the students' scheme and related techniques of the observed group for the case of word problems treated during the one-lesson Cover-up activity. Similar to the observation for the bare problems as shown in Table 4, the students' main obstacles observed included arithmetical calculation mistakes (such as the ones in tasks 3b and 4b) and mistakes in transforming word problems properly into equations. Technical obstacles included the use of the equation editor, such as, typing fractional expressions in the solution window, which was not needed in the bare equation solving tasks. The already noted difficulties of highlighting expressions with the mouse re-appeared. Also, the observer gave too much guidance while the students were solving the tasks, such as for the case of task 6b.

Tasks	Observation: scheme and techniques
Task 2b. Budin's height is 130 cm. If Adin's height is divided by 3, next added to Budin's height, the final result is 175 cm. Find Adin's height. Hints: Given the equation $\frac{a}{3}+130=175$ representing the word problem, students are required to choose one out of three options from the answer box for the meaning of a.	Through questions, the observer guides students. In this way, students are able to apply the cover-up strategy step-by-step: identify the respective sub-expressions $\frac{a}{3}$ and a, and assign numerical values. An observed obstacle includes typing the fractional expression $\frac{a}{3}$ with the equation editor.
Task 3b. Two times a number is added to 3, then divided by 5, and finally added by 1. If the final result is 4, find the number. Hint: Let m be the number to find, students are required to select one out of three equations representing the problem from the answer box.	Students choose $\frac{2m+3}{5}+1=4$ as the equation expressing the word problem correctly. Next, they type it correctly through the equation editor. Even if the observer guides them, the students improperly identify the first sub-expression to cover: $2m, 2m+3$ and m respectively. As a result, they cannot assign numerical values to these expressions. Next, after correctly choosing $\frac{2m+3}{5}$ as the sub-expression to cover, students make calculation mistakes. This may indicate that students do not understand yet how to apply the cover-up strategy. By the observer's guidance, students are finally able to solve the equation.
Task 4b. Udin is 4 years older than Tom. If Tom's and Udin's ages are 30, find Tom's age. Hint: Let t be Tom's age, students are required to choose one out of three equations representing the problem.	After typing $t+t+4=30$, students do not simplify the equation into $2t+4=30$. Rather, they directly cover $t+t$, and finally assign $t=13$. Calculation mistakes appear during the process of assigning t from the equation $t+t=26$. The technical obstacle of covering a sub-expression also appears when covering $t+t$.
Task 5b. The price of a glass of ice is Rp 1000. Doni has Rp 2000. If the price of a bowl of meatballs is subtracted by the price of a glass of ice, next divided by 3, then the results is equal to Doni's money. How much is a bowl of meatballs? Hint: Let b be the price of a bowl of meatballs.	Even if students are able to translate the word problem in a proper equation, they type it with a minor incorrect notation: rather than typing for example 1000 for the price of a glass of ice, students typed it as Rp 1000. Next, with the help from the observer, students are able to type the equation correctly. Then, they properly apply the cover-up strategy (in spite of some calculation mistakes).
Task 6b. Sinta's grade is 5. If Tom's grade is subtracted by Sinta's grade, next multiplied by 2, then the result is 8. Find Tom's grade. Hint: Let t be Tom's grade.	Students find it is difficult to set-up a correct equation from the word problem. With the observer's guidance, students are finally able to do this. However, calculation errors emerge while applying the cover-up strategy in solving the equation (see Table 5 for a detailed description).

Table 6 Students' schemes over the different tasks within one lesson

From the above observations, we conclude that the students' scheme development for solving word problems is in line with the conjectured scheme described in Tables 2 and 1. Also, similar to the case of bare equation problems, this scheme and techniques were applied to increasingly complex equations, which again suggest the development of a structural view on equations and expressions and instrumental genesis.

9 CONCLUSIONS AND REFLECTION

To investigate the relationship between the use of a digital tool and student conceptual understanding we examined which schemes students develop for solving equations using algebraic substitution with the Cover-up applet. In particular, we focused on the relationship between the technique of highlighting expressions with the mouse, and the ability to identify and select appropriate expressions, which requires both an object and a process view.

A first conclusion is that the scheme which students developed for solving an equation using algebraic substitution with the Cover-up applet is in line with the conjectured scheme formulated in Table 1. It includes recognizing the equation as suitable for the cover-up strategy and the task to rewrite it in the form $<$ unknown $>=<$ value $>$, identifying a sub-expression within the equation to cover as well as assigning a numerical value toit (in each cover-up strategy step), and repeating this until the desired form is found. Within this scheme we noticed the interplay between on the one hand the techniques of highlighting an expression, typing a numerical value for the highlighted part, and pressing enter to check; and on the other hand the ability to see expressions as objects and to identify an appropriate expression to advance towards the desired form.

Second, the scheme that students developed for solving corresponding word problems is in line with the conjectured scheme described in Tables 2 and 1 as well. This scheme includes setting up an equation from the word problem, entering it, and put into action the scheme described above. As such it is an extension of the previous scheme, which includes some additional interplay between technique and understanding. For instance, entering an algebraic expression corresponding to a phrase in the word problem using the equation editor is a technique that reflects the mental activity of recognising the algebraic structure within that phrase.

Third, as the equations in the digital activity gradually get more complex, we observed that the students' schemes develop in the sense that their application is extended to a wider category of problems. The fact that the schemes and techniques 'survived' when facing an increasing complexity is considered as a form of instrumental genesis. Further instrumental genesis would be expected in a more extended period of use.

Reflecting on these conclusions, we feel that three factors play an important role in fostering the co-emergence of techniques and understanding, and as such the instrumental genesis: the characteristics of the applet and the corresponding task design, the role of the observer who acted as a teacher for the students, and the interaction among students. Concerning the first factor, central in the characteristics of the applet is that the techniques that the applet makes available – in this case a quite limited set of possible techniques – correspond to mathematical notions and operations. We might call this the applet's mathematical fidelity. The applet's feedback helped students to overcome algebraic errors and to improve their method. For word problems, the applet also provided feedback on the syntactical correctness of the equations the student enters. This feedback allows students to improve their work in the digital environment, as they would not have been able to do in a paper-and-pencil environment. This, in our view, has also contributed to the development of students' schemes and techniques. The Cover-up applet can be criticized because of the limited construction room it provides to the students. Indeed, the repertoire of possible techniques that the applet makes available is small. From a didactical point of view, this is a draw-back; for a case study on instrumental genesis, however, this limitation allowed us to focus on the instrumental genesis.

Concerning the task design, ordering the tasks from a relatively simple to more complex problems helped the students to gradually develop their thinking and as such contributed to the development of schemes and techniques. As an aside, the relatively simple tasks in this study are more complex than the tasks in the regular Indonesian mathematics curriculum. In fact, one of the interesting features of the cover-up strategy is that it can easily be applied to any equation of the form $f(x) = c$ with the variable appearing only once in the left hand side, and is not restricted to linear equations. The hints provided indirect guidance to students on initial actions in the solution process. In the design of the word problems, we observed that student difficulties in setting up equations were not caused by their inability to understand each word or phrase in the problem, but by their inability to represent them in an appropriate expression or equation (Jupri and Drijvers, in press), which reflects a limited understanding of the problems (Verschaffel, De Corte and Lasure,1994).This concerns the process of transforming the problem situation into the world of mathematics, also called horizontal mathematization (Treffers, 1987; Van den Heuvel-Panhuizen, 2003).

The second crucial factor concerns the role of the observer, who acted as a teacher for the observed group. On the one hand, this is part of an orchestration that guides students' instrumental genesis, as the regular teacher did for the other groups of students. On the other hand, we acknowledge that the observer in some cases gave strong guidance, which may have affected the instrumental genesis. In line with Swan (2008), for future teaching design, we recommend the observer to take a more distant stance to avoid this effect.

The third and final factor concerns the interaction among students during the group activity. Even if the three students helped each other during the problem solving, they seem to have gained different conceptual understanding and skills. In particular, the higher ability student, Ali acquired a better understanding than his peers and he played an important role in the group's success in solving the problems. The different conceptual understanding acquired by the students is manifest in their written work, and in the conversations during the group work. For example, when solving $\frac{48}{8(z+1)} = 3$, Ali concluded the value of $8(z+1) = 16$, and Quni agreed; yet Widan did not understand how to get 16.

As a final reflection, we wonder how the specific conclusions on scheme development for algebraic substitution can be extrapolated to the general issue of the relationship between using a digital tool and the targeted mathematical understanding, of the "why and how" of using digital technology in mathematics education. Even if this paper reports on a small case study, we feel that the main conclusions go beyond this case. The correspondence between techniques that the digital environment invites and the targeted mathematical understanding, that is so well phrased in the vocabulary of the instrumentation theory, is an indispensable condition for a fruitful use of digital tools for mathematical learning. Using digital technology in mathematics education will really work, is our strong conviction, because of this intertwinement of technique and mathematical concept, according to which the techniques encapsulate mathematical thinking. This is a start to answering the "why"-question: digital tools work because they allow us to represent mathematical ideas in an efficient and challenging way. As a consequence, answering the "how"-question might start with designing tasks and orchestrating the learning process in a way that exploits the affordances of the so crucial connection between technique and mathematical understanding. This being said, we are of course aware that these reflections are but a start in the engaging enterprise of fruitfully integrating digital technology in mathematics education.

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BIOGRAPHICAL NOTES

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