Journal of Number Theory 167 (2016) 317–335

Contents lists available at ScienceDirect

Journal of Number Theory

www.elsevier.com/locate/jnt

Fourier coefficients of Eisenstein series formed with modular symbols and their spectral decomposition

Roelof Bruggeman [a], Nikolaos Diamantis [b,*]

[a] *Universiteit Utrecht, Netherlands*
[b] *University of Nottingham, United Kingdom*

ARTICLE INFO

Article history:
Received 30 September 2015
Received in revised form 8 March 2016
Accepted 8 March 2016
Available online 28 April 2016
Communicated by David Goss

Keywords:
Shifted convolution sums
Spectral decomposition
Second order Maass forms

ABSTRACT

The Fourier coefficient of a second order Eisenstein series is described as a shifted convolution sum. This description is used to obtain the spectral decomposition of and estimates for the shifted convolution sum.

1. Introduction

We study the Fourier coefficients of Eisenstein series formed with modular symbols, and give a description in terms of the spectral decomposition of certain shifted convolution sums.

Let f be a weight 2 cuspidal eigenform

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

* Corresponding author.
E-mail addresses: r.w.bruggeman@uu.nl (R. Bruggeman), nikolaos.diamantis@nottingham.ac.uk (N. Diamantis).

http://dx.doi.org/10.1016/j.jnt.2016.03.009
0022-314X/

for $\Gamma_0(N)$, and consider the *Eisenstein series formed with modular symbols* given (for $\mathrm{Re}(s) \gg 0$) by

$$E(z, s; f, \chi) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{\chi(\gamma)} \mathrm{Im}(\gamma z)^s \int_{i\infty}^{\gamma i\infty} f(w) dw$$

where Γ_∞ is the group of $\gamma \in \Gamma_0(N)$ fixing ∞. Whereas the Fourier coefficients of the usual Eisenstein series (without the integral) have a relatively simple description in terms of divisor sums and L-functions, the n-th Fourier coefficient of $E(z, s; f, \chi)$ has a more complicated description. This is given in Theorem 2.4 and involves the analytic continuation of the following shifted convolution sum:

$$\sum_{l \geq 1, m = l - n \neq 0} \frac{a(l) \sigma^{\chi}_{2s-1}(|m|)}{l^t |m|^{2s-1}}.$$

We denote the value at $t = 1$ of this analytic continuation by $L(n, \chi; s)$. The existence of the analytic continuation along with that of a weighted version, denoted by $L(n, s, x; \chi)$ ($x \in \mathbb{R}$) is proved in Section 2. With this notation, Theorem 2.4 can be summarized as:

Theorem 1.1. *For* $\mathrm{Re}(s) > 2$ *and* $n \neq 0$, *the* n*-th coefficient in the Fourier expansion of* $E(z, s; f, \chi)$ *is*

$$\frac{iW(\bar{\chi})(\pi|n|)^{s-1}}{2\Gamma(s) N^{2s} L(\bar{\chi}, 2s)} L(n, \chi; s)$$

where $W(\chi)$ *is a Gauss sum and* $L(\bar{\chi}, 2s)$ *a Dirichlet* L*-function. The coefficient of* y^{1-s} *in the same expansion is*

$$\frac{iW(\overline{\chi})\Gamma(s - \frac{1}{2})}{2\pi^{1/2} N^{2s} \Gamma(s)} \frac{L(f \otimes \chi, 1) L(f, 2s)}{L(\overline{\chi}, 2s) L(\chi, 2s)}$$

where $L(f, s)$ *is the* L*-function of* f *and* $L(f \otimes \chi, s)$ *its twisted counterpart.*

Shifted convolution sums have been the focus of intense attention, especially because of their applications to subconvexity problems and to estimates for averages of L-functions. In many of the various approaches to such sums, a key tool is their spectral decomposition. Establishing such decompositions can be a difficult problem and various methods have been adopted to resolve it, each time introducing an additional important aspect, e.g. estimates of triple products [18], the spectral structure of $L^2(\Gamma \backslash G)$ ($\Gamma = \mathrm{PSL}_2(\mathbb{Z})$, $G = \mathrm{PSL}_2(\mathbb{R})$) [2], Sobolev norms and Kirillov models [1], multiple Dirichlet series [7–9], etc.

Here we obtain the spectral decomposition by using a combination of those approaches together with a new element. The new element is a "completion" technique which can be summarized as follows: The function $E(z, s; f, \chi)$ is a second order Maass form. It is an eigenfunction of the Laplace operator, but it is not invariant under the group $\Gamma_0(N)$. By subtracting a more simple function with the same transformation behavior we arrive at a $\Gamma_0(N)$-invariant function, which we view as the "completion" of $E(z, s; f, \chi)$. This completion turns out to be in $L^2(\Gamma_0(N)\backslash\mathbb{H}, \chi)$, and can hence be analyzed by spectral methods along the lines of [19] and [1]. From this we obtain the spectral expansion of $L(n, s, x; \chi)$.

From the point of view of the theory of Eisenstein series with modular symbols and, more generally, higher-order forms, the significance of our contribution is that, by connecting Fourier coefficients of a higher-order form with shifted convolutions, we provide an arithmetic meaning to those coefficients. In contrast to classical modular forms, Fourier coefficients of forms of higher order did not have until now an immediate arithmetic interpretation. Their number-theoretic applications originated in their expression as Poincaré series ([5,16] etc.) Here we give an arithmetic description of those coefficients.

The structure of the note is as follows: In Section 2, we compute the Fourier coefficients of $E(z, s; f, \chi)$ modified with modular symbols in terms of a shifted convolution sum $L(n, s; \chi)$. In Proposition 2.1, we discuss how this shifted convolution sum can be defined as the analytic continuation of the double Dirichlet series (8).

In Section 3 we use the constructions of the previous section to establish, in Theorem 3.2, the spectral decomposition of the weighted shifted convolution sum $L(n, s, x; \chi)$:

Theorem 1.2. *Fix $n < 0$ and $\mathrm{Re}(s) > 2$. For $x > \mathrm{Re}(s) + 5/2$ we have*

$$\begin{aligned} &|n|^{s-1} L(n, \chi, x; s) \\ &\quad = -\left(\frac{N}{2\pi}\right)^{2s} \frac{2\pi L(\bar{\chi}, 2s)}{W(\bar{\chi})\Gamma(s-1+x)\Gamma(x-s)} \\ &\qquad \times \Big(\sum_{j=1}^{\infty} \Gamma(s - \frac{1}{2} \pm ir_j)\Gamma(x - \frac{1}{2} \pm ir_j) b_j(n, \chi) L(f \otimes \eta_j, s) + \textit{cont. part}\Big). \end{aligned} \tag{1}$$

Here $W(\chi)$ is a Gauss sum, $L(\bar{\chi}, 2s)$ a Dirichlet L-function, $b_j(n, \chi)$ the n-th Fourier coefficient of the j-th element of a complete orthonormal basis of Maass cusp forms and $L(f \otimes \eta_j, s)$ a Rankin–Selberg zeta function (all will be defined in detail in Section 3). "cont. part" stands for the contribution of the continuous spectrum.

An application of this decomposition is the meromorphic continuation of the convolution sums (Theorem 3.3). As a further example of a direct implication of the spectral expansion we derive some bounds for our convolution sums. We should stress that it is not our aim to obtain the best possible bounds and it does not seem possible to compare them with previously established bounds. Indeed, to our knowledge, shifted convolution sums that are most comparable to ours have previously been considered in [7,8]. However, in contrast to our shifted convolution sums, the shift in [7,8] occurs in the Fourier coefficient of the cusp form. As J. Hoffstein pointed out to us, $L(n, s; \chi)$ could be expressed in terms of the shifted convolution sum $D'(w, v, m)$ studied in [7], but the insufficiently uniform convergence of the infinite sums involved prevents us from comparing the bounds established in [7] (or [8]) with ours. It appears that the shifted convolution sums appearing as Fourier coefficients of Eisenstein series with modular symbols are new objects.

2. Eisenstein series formed with modular symbols

Let

$$f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z}$$

be a cusp form of weight 2 for $\Gamma_0(N)$, with a positive integer N. The modular symbols can be described with additive twists of L-functions. For every $c, d \in \mathbb{Z}$ ($c \neq 0$) we consider for $\mathrm{Re}(t) > 3/2$ the following function:

$$L(f, t; -d/c) = \sum_{n=1}^{\infty} \frac{a(n) e^{-2\pi i n d/c}}{n^t}.$$

We then set

$$\Lambda(f, t, -d/c) := \left(\frac{c}{2\pi}\right)^t \Gamma(t) L(f, t; -d/c) = c^t \int_0^{\infty} f(-d/c + ix) x^t \frac{dx}{x}.$$

The last expression can be used to give the analytic continuation of $\Lambda(f, t, -d/c)$ to the entire t-plane for each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The function has a functional equation $\Lambda(f, t, -d/c) = -\Lambda(f, 2-t, a/c)$, where $ad \equiv 1 \bmod c$ (see, e.g. [13, A.3]) which implies the convexity bound

$$\Lambda(f, t, -d/c) \ll c^{3/2+\epsilon} \qquad \text{for } 3/2 + \epsilon > \mathrm{Re}(t) > 1 - \delta \text{ for some } \delta > 0 \tag{2}$$

with the implied constant independent of c and of t.

The modular symbol associated to the cusp form f is:

$$< f, \gamma > := \int_{i\infty}^{\gamma i\infty} f(w) dw \qquad \text{for all } \gamma \in \Gamma_0(N).$$

For each $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$ it satisfies

$$-< f, \gamma^{-1} >= -\int_{i\infty}^{-d/c} f(w)dw = i\int_0^\infty f(-d/c+iy)dy = \frac{i}{c}\Lambda(f,1,-d/c). \tag{3}$$

With this notation we introduce the Eisenstein series at the cusp ∞ with modular symbols. We first define it generally, and then specialize to characters induced by primitive Dirichlet characters mod N.

Let χ be a character on $\Gamma = \Gamma_0(N)$ which is trivial on the stabilizer of ∞. We set

$$E(z,s;f,\chi) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\chi(\gamma)} < f,\gamma > \operatorname{Im}(\gamma z)^s \tag{4}$$

where, as usual, $\Gamma_\infty = \left\{\left(\begin{smallmatrix} \pm1 & * \\ 0 & \pm1 \end{smallmatrix}\right)\right\}$ is the group of elements of Γ fixing ∞.

An explicit Fourier expansion of $E(z,s;f,\chi)$, and of more general Eisenstein series with modular symbols, is essentially given in [15, (1.1)–(1.3)]: Let for $c > 0$, $c \equiv 0 \bmod N$,

$$S(n,m,f,\chi;c) := \sum_{\gamma=\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} \overline{\chi(\gamma)} < f,\gamma > e^{2\pi i(n\frac{d}{c}+m\frac{a}{c})}$$

be the twisted Kloosterman sum. Then,

$$E(z,s;f,\chi) = \phi(s,f,\chi)y^{1-s} + \sum_{n\neq0}\phi(n,s,f,\chi)W_s(nz)$$

with

$$W_s(nz) = \sqrt{|n|y}K_{s-\frac12}(2\pi|n|y)e^{2\pi inx}$$

$$\phi(s,f,\chi) = \sqrt{\pi}\frac{\Gamma(s-\frac12)}{\Gamma(s)}\sum_{c>0,\, c\equiv0(N)} c^{-2s}S^*(0,0,f,\chi;c) \tag{5}$$

$$\phi(n,s,f,\chi) = \frac{\pi^s}{\Gamma(s)}|n|^{s-1}\sum_{c>0,\, c\equiv0(N)} c^{-2s}S^*(n,0,f,\chi;c). \tag{6}$$

Here $K_s(y)$ denotes the modified Bessel function normalized as:

$$K_s(y) = \int_0^\infty e^{-\frac{y}{2}(t+\frac1t)}t^s\frac{dt}{t}.$$

For the rest of the paper we consider a character induced by a Dirichlet character. Specifically, we take as χ a character of $\Gamma_0(N)$ induced by a primitive Dirichlet character mod N such that $\chi(-1) = 1$. (Such a character is trivial on the stabilizers of ∞ and 0). To χ we associate

$$\sigma_t^{\chi}(m) := \sum_{d|m} \chi(d) d^t \quad \text{for each } m \text{ and } W(\overline{\chi}) := \sum_{a \bmod N} \overline{\chi}(a) e^{2\pi i a/N}.$$

We also consider the Dirichlet L-function given for $\mathrm{Re}(s) > 1$ by

$$L(\overline{\chi}, s) = \sum_{n \geq 1, (n,N)=1} \frac{\overline{\chi}(n)}{n^s} = \prod_{p \nmid N;\ \text{prime}} \left(1 - \frac{\overline{\chi}(p)}{p^{2s}}\right)^{-1}. \tag{7}$$

Since the cusp form f we will be working with is fixed we omit it from the notation, and write $E^*(z, s; \chi) = E(z, s; f, \chi)$, and analogously $S^*(n, m, \chi; c)$, $\phi^*(s, \chi)$, $\phi^*(n, s, \chi)$ etc.

To express the Fourier coefficients of E^* in terms of more familiar objects we will first need the analytic continuation of the double Dirichlet series in (8) below.

Proposition 2.1. *Fix an integer n and an s with $\mathrm{Re}(s) > 2$.*

(i) *Then*

$$\sum_{l \geq 1, m \neq 0; l-m=n} \frac{a(l) \sigma_{2s-1}^{\chi}(|m|)}{l^t |m|^{2s-1}} = \sum_{l \geq 1, l \neq n} \frac{a(l) \sigma_{2s-1}^{\chi}(|l-n|)}{l^t (|l-n|)^{2s-1}} \tag{8}$$

is absolutely convergent for $\tau := \mathrm{Re}(t) > 3/2$ and has an analytic continuation to $\mathrm{Re}(t) > 1 - \delta$ for some $\delta > 0$.

(ii) *For $\mathrm{Re}(t) > \frac{3}{2}$, the quantity in (8) equals*

$$N^{2s} \frac{L(\bar{\chi}, 2s)}{W(\bar{\chi})} \frac{(2\pi)^t}{\Gamma(t)} \sum_{N|c>0} c^{-2s-t} \left(\sum_{d \bmod c, (d,c)=1} \overline{\chi(d)} e^{2\pi i n \frac{d}{c}} \Lambda(f, t, -\frac{d}{c}) \right). \tag{9}$$

Proof. From $d(|a|) = o(|a|^{\epsilon})$, $|\sigma_{2s-1}^{\chi}(|a|)| \leq d(|a|)|a|^{2\mathrm{Re}(s)-1}$ and the Ramanujan bound we have

$$\left| \frac{a(l) \sigma_{2s-1}^{\chi}(|m|)}{l^t |m|^{2s-1}} \right| \ll \left| \frac{1}{l^{t-1/2-\epsilon} m^{-\epsilon}} \right| \ll \frac{1}{l^{\tau-1/2-2\epsilon}}. \tag{10}$$

Therefore the series converges absolutely when $\tau > 3/2$.

To continue meromorphically this function, we will use a formula for the Fourier coefficients of the non-holomorphic Eisenstein series. Although this formula is not new (e.g. it can be deduced from Section 2 of [10]) we give a direct proof of the exact expression we will be using.

Lemma 2.2. *Let N be a positive integer and χ be a primitive character* mod N*. Let $m \in \mathbb{Z} \smallsetminus \{0\}$ and s with* $\mathrm{Re}(s) > 1$*. Then, the m-th coefficient $\phi(m, s; \chi)$ of $W_s(mz)$ in the Fourier expansion of*

$$E(z, s; \chi) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \overline{\chi(\gamma)} \mathrm{Im}(\gamma z)^s$$

is

$$\phi(m, s; \chi) = \left(\frac{\pi}{N^2}\right)^s \frac{W(\bar{\chi})}{\Gamma(s)} \frac{1}{L(\bar{\chi}, 2s)} \frac{\sigma^{\chi}_{2s-1}(|m|)}{|m|^s}. \tag{11}$$

Proof of lemma. It is well-known that the m-th Fourier coefficient of $E(z, s; \chi)$ is

$$\frac{\pi^s |m|^{s-1}}{\Gamma(s) N^{2s}} \sum_{c>0} c^{-2s} \sum_{0 \le d < Nc,\ (d,Nc)=1} \overline{\chi(d)}\, e^{2\pi i |m| d/Nc}. \tag{12}$$

To compute the double sum we follow the method of [3] (Section 3):

$$\sum_{c>0} \sum_{0 \le d < Nc} \sum_{\delta | (d,Nc)} \mu(\delta)\, c^{-2s}\, \overline{\chi(d)}\, e^{2\pi i |m| d/Nc}$$
$$= \sum_{d \ge 0} \sum_{\delta | d} \mu(\delta) \sum_{c > \frac{d}{N},\ \delta | Nc} c^{-2s}\, \overline{\chi(d)}\, e^{2\pi i |m| d/Nc}.$$

Since $\delta | Nc$ if and only if $(\delta/(N,\delta)) | c$ the last sum equals

$$\sum_{d \ge 0} \sum_{\delta | d} \mu(\delta) \sum_{n:\ n\delta/(N,\delta) > d/N} \big(n\delta/(N,\delta)\big)^{-2s}\, \overline{\chi(d)}\, e^{2\pi i |m| d (N,\delta)/\delta n N}$$
$$= \sum_{\delta \ge 1} \mu(\delta) \left(\frac{(N,\delta)}{\delta}\right)^{2s} \sum_{l \ge 0} \sum_{n > l(N,\delta)/N} n^{-2s}\, \overline{\chi(\delta l)}\, e^{2\pi i |m| l \frac{(N,\delta)}{Nn}}.$$

Since $\chi(\delta l) = 0$ if $(\delta, N) \neq 1$, this becomes

$$\sum_{\delta \ge 1, (N,\delta)=1} \frac{\mu(\delta)}{\delta^{2s}} \sum_{l \ge 0} \sum_{n > l/N} n^{-2s}\, \overline{\chi(\delta l)}\, e^{\frac{2\pi i |m| l}{Nn}} = L(\bar{\chi}, 2s)^{-1} \sum_{n \ge 1} n^{-2s} \sum_{l=0}^{Nn-1} \overline{\chi(l)}\, e^{2\pi i |m| l/Nn}.$$

Lemma 3.1.3(1) of [14] implies that

$$\sum_{n \ge 1} n^{-2s} \sum_{l=0}^{Nn-1} \overline{\chi(l)}\, e^{\frac{2\pi i |m| l}{Nn}} = \sum_{n | m} n^{-2s} \sum_{l=0}^{Nn-1} \overline{\chi(l)}\, e^{\frac{2\pi i |m| l}{Nn}} = \sum_{n | m} n^{-2s+1} \sum_{l=0}^{N-1} \overline{\chi(l)}\, e^{\frac{2\pi i l \left(\frac{|m|}{n}\right)}{N}}.$$

Recall that χ is primitive mod N and that we have tacitly used the same notation for the character mod Nn induced by χ. Lemma 3.1.1(1) in [14] implies that the last expression is

$$\sum_{n|m} n^{-2s+1} W(\overline{\chi})\chi(|m|/n) = W(\overline{\chi})|m|^{1-2s}\sigma^{\chi}_{2s-1}(|m|). \qquad \square$$

We proceed with the proof of the proposition. Lemma 2.2 with $m = l - n$ combined with (12) imply that, for $\mathrm{Re}(s) > 1$ and $\tau > 3/2$ the series (8) equals

$$\frac{L(\overline{\chi}, 2s)}{W(\overline{\chi})} \sum_{l\ge 1, l\neq n} \frac{a(l)}{l^t} \sum_{c>0} c^{-2s} \sum_{d \bmod Nc,\ (d,Nc)=1} \overline{\chi(d)} e^{2\pi i |n-l| \frac{d}{Nc}}.$$

Since $\chi(-1) = 1$, $|n - l|$ can be replaced by $n - l$ in the sum. Further, the primitive character χ modulo $N > 1$ is not equal to 1 and hence we can omit the condition $l \neq n$. Since for $\mathrm{Re}(s) > 1$ and $\tau > 3/2$,

$$\sum_{l\ge 1} \frac{|a(l)|}{l^\tau} \sum_{c>0} c^{-2\mathrm{Re}(s)} \sum_{d \bmod Nc,\ (d,Nc)=1} \left| \overline{\chi(d)} e^{2\pi i (n-l) \frac{d}{Nc}} \right| \ll \sum_{l\ge 1} l^{1/2+\epsilon-\tau} \sum_{c>0} c^{-2\mathrm{Re}(s)+1}$$

converges uniformly, we can interchange the order of summation to get

$$\sum_{c>0} c^{-2s} \sum_{d \bmod Nc,\ (d,Nc)=1} \overline{\chi(d)} e^{2\pi i n \frac{d}{Nc}} \sum_{l\ge 1} \frac{a(l)}{l^t} e^{-2\pi i l \frac{d}{Nc}}.$$

With the definition of $\Lambda(f, t, -d/c)$ this implies (9).

To prove the analytic continuation of (8) we note that each $\Lambda(f, t, -d/c)$ has an analytic continuation to the entire plane. Since it further satisfies (2), the double sum of (9) is uniformly convergent for $3/2 + \epsilon > \mathrm{Re}(t) > 1 - \delta$ (and our fixed s with $\mathrm{Re}(s) > 2$) giving an analytic function there. In the region $\mathrm{Re}(t) > 3/2 + \epsilon$, our series is already analytic because it is absolutely convergent there by the first part of the assertion. $\square$

With (10) we notice that, for $x > \mathrm{Re}(s) + 1/2$, the series

$$\sum_{l\ge 1, m=l-n\neq 0} \frac{a(l)\sigma^{\chi}_{2s-1}(|m|)}{l|m|^{s+x-1}}$$

is absolutely convergent. In view of this remark and Proposition 2.1 we can define the two shifted convolution sums we will be using in this paper. The first one has a weight which allows for the spectral expansion to have a more symmetric form.

Definition 2.3. Let $n \neq 0$ and consider a cusp form of weight 2 for $\Gamma_0(N)$

$$f(z) = \sum_{l=1}^{\infty} a(l) e^{2\pi i l z}.$$

For each s with $\mathrm{Re}(s) > 2$ and $x > \mathrm{Re}(s) + 1/2$ we define $L(n, \chi, x; s)$ to be the value at $t = 1$ of the analytic continuation of

$$\sum_{l\geq 1, m=l-n\neq 0} \frac{a(l)}{l^t} \frac{\sigma^{\chi}_{2s-1}(|m|)}{|m|^{2s-1}} \left(1 - \left|\frac{m}{n}\right|^{s-x}\right).$$

We also denote by $L(n, \chi; s)$ the value at $t = 1$ of the analytic continuation of

$$\sum_{l\geq 1, m=l-n\neq 0} \frac{a(l)\sigma^{\chi}_{2s-1}(|m|)}{l^t|m|^{2s-1}}.$$

For $n < 0$, $L(n, \chi; s)$ can be thought of as the "value at $x = \infty$" of $L(n, \chi, x; s)$. Although the functions $L(n, \chi, x; s)$ and $L(n, \chi; s)$ depend on the function f, we do not include it in the notation to avoid burdening it further.

Theorem 2.4. *For* $\mathrm{Re}(s) > 2$ *and* $n \neq 0$*, we have*

$$L(n, \chi; s) = \frac{2\Gamma(s)N^{2s}L(\bar{\chi}, 2s)}{iW(\bar{\chi})(\pi|n|)^{s-1}} \phi^*(n, s, \chi).$$

The coefficient of y^{1-s} *in the Fourier expansion of* $E^*(z, s; \chi)$ *is*

$$\phi^*(s, \chi) = \frac{iW(\overline{\chi})\Gamma(s-\frac{1}{2})}{2\pi^{1/2}N^{2s}\Gamma(s)} \frac{L(f\otimes\chi, 1)L(f, 2s)}{L(\overline{\chi}, 2s)L(\chi, 2s)}. \tag{13}$$

Proof. We first observe that, because of (3),

$$S^*(n, m, \chi; c) = \lim_{t\to 1} \frac{i}{c^t} \sum_{ad\equiv 1 \pmod{c}} \overline{\chi(d)}\Lambda(f, t, -d/c) \cdot e^{2\pi i(n\frac{d}{c}+m\frac{a}{c})}. \tag{14}$$

From (6), (ii) of Proposition 2.1 and the definition of $L(s, \chi, s)$ we get

$$\begin{aligned}\phi^*(n, s, \chi) &= \frac{\pi^s}{\Gamma(s)}|n|^{s-1} \lim_{t\to 1} \sum_{c>0, N|c} ic^{-2s-t} \sum_{d \bmod c, (d,c)=1} \overline{\chi(d)} e^{2\pi i n\frac{d}{c}} \Lambda(f, t, -\frac{d}{c}) \\ &= \frac{i\pi^s|n|^{s-1}}{\Gamma(s)} \lim_{t\to 1} \frac{W(\bar{\chi})\Gamma(t)}{L(\bar{\chi}, 2s)(2\pi)^t N^{2s}} \sum_{l\geq 1, m=l-n\neq 0} \frac{a(l)\sigma^{\chi}_{2s-1}(|m|)}{l^t|m|^{2s-1}}.\end{aligned}$$

This implies the first part of the result.

Now, (ii) of Proposition 2.1 can be applied to yield

$$\phi^*(s, \chi) = \frac{iW(\overline{\chi})N^{-2s}\Gamma(s-\frac{1}{2})}{2\pi^{1/2}\Gamma(s)L(\overline{\chi}, 2s)} \lim_{t\to 1} \Big(\sum_{l\geq 1} \frac{a(l)\sigma^{\chi}_{2s-1}(l)}{l^{t+2s-1}}\Big). \tag{15}$$

Since f is an eigenform of the Hecke operators we have, for each $m, n \geq 1$ the identity

$$a(mn) = \sum_{d|(m,n)} \mu(d)da(m/d)a(n/d)$$

and this implies that the sum in (15)

$$\sum_{l,d,m\geq 1} \frac{\mu(l)l\chi(dl)a(d)a(m)}{(dl)^t(ml)^{t+2s-1}} = \frac{L(f\otimes\chi,t)L(f,t+2s-1)}{L(\chi,2t+2s-2)}.$$

Upon passing to the limit at $t = 1$ we obtain the result. □

Remark. In some applications of convolution sums, e.g. to second moment problems (cf. [7]), it is necessary to consider sums ranging over m, l such that $l_2 l - l_1 m = n$ for fixed integers n, l_1, l_2. These more general sums can be also parametrized by Eisenstein series with modular symbols as above and they have a spectral expansion. Here, we only discuss the simpler case because the notation for general l_1, l_2 becomes more complicated and that would obscure the main point of our construction.

3. Spectral expansions

We will first derive a general decomposition which will then be used to derive spectral expansions of shifted convolution sums.

As before, we fix $N \in \mathbb{N}^*$ and an even primitive character χ modulo N and a cusp form $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi inz}$ of weight 2 for $\Gamma = \Gamma_0(N)$. Set

$$F(z) = \int_{i\infty}^{z} f(w)dw$$

and, for $\mathrm{Re}(s) \gg 0$

$$G(z,s;\chi) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma)F(\gamma z)\mathrm{Im}(\gamma z)^s.$$

We note that, for each $\gamma \in \Gamma_0(N)$ the modular symbol satisfies

$$< f, \gamma^{-1} >= - < f, \gamma > \qquad \text{and} \;\; < f, \gamma >= F(\gamma z) - F(z). \tag{16}$$

Therefore,

$$E^*(z,s;\chi) = G(z,s;\chi) - F(z)E(z,s;\chi). \tag{17}$$

The function G has been studied in [5] and shown to be absolutely convergent for $\mathrm{Re}(s) > 2$ and to belong to $L^2(\Gamma\backslash\mathfrak{H};\chi)$. As such, it is amenable to a spectral expansion. Relation (17) represents the basis of the "completion" mentioned in the introduction:

Adding the more explicit term $F(z)E(z,s;\chi)$ to $E^*(z,s;\chi)$, we obtain an element of $L^2(\Gamma\backslash\mathfrak{H};\chi)$.

We next consider a generalized Poincaré series which will allow us to retrieve weighted shifted convolutions. Let h be an element of $C^\infty(0,\infty)$ which is $\ll y^{1/2-\epsilon}$, as $y \to \infty$ and $\ll y^{1+\epsilon}$ as $y \to 0$ for some $\epsilon > 0$. For $n \neq 0$, set

$$P(n,h,\chi;z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \overline{\chi}(\gamma)e^{2\pi i n\mathrm{Re}(\gamma z)}h(2\pi|n|\mathrm{Im}(\gamma z)).$$

A comparison with $E(z,s;\chi)$ shows that this is absolutely convergent and that it belongs to $L^2(\Gamma\backslash\mathfrak{H};\chi)$.

Let $k \in L^2(\Gamma\backslash\mathfrak{H},\chi)$ have a Fourier expansion

$$k(z) = \sum_{m\in\mathbb{Z}} c_m(y)e^{2\pi i m x} \quad \text{with } c_m(y) = \underset{y\to 0}{O}(y^A), = \underset{y\to\infty}{O}(y^B) \qquad (-A, B < \epsilon).$$

Then, by unfolding the integral in the Petersson scalar product we obtain:

$$\langle k, P(n,h,\chi)\rangle = \int_0^\infty c_n(y)\overline{h(2\pi|n|y)}\frac{dy}{y^2}. \tag{18}$$

Parseval's formula Since both series G and P defined above are in $L^2(\Gamma\backslash\mathfrak{H},\chi)$, we have

$$\begin{aligned}&\langle G, P(n,h,\chi)\rangle \\ &= \sum_{j=1}^\infty \langle G,\eta_j\rangle\langle\eta_j, P(n,h,\chi)\rangle \\ &\quad + \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^\infty \langle G, E_{\mathfrak{a}}(\cdot, 1/2+ir,\chi)\rangle\langle E_{\mathfrak{a}}(z,1/2+ir,\chi), P(n,h,\chi)\rangle dr \end{aligned} \tag{19}$$

where

$$\eta_j(z) = \sum_{n\neq 0} b_j(n,\chi)W_{s_j}(nz) \tag{20}$$

form a complete orthonormal basis of Maass cusp forms with character χ, with corresponding positive eigenvalues $s_j(1-s_j) \to \infty$. We write $s_j = 1/2 + ir_j$. For almost all j, $r_j \geq 0$. For at most finitely many of them we have $ir_j \in (0,1)$. The last sum ranges over a set of singular inequivalent cusps and

$$E_{\mathfrak{a}}(z,s,\chi) = \delta_{\mathfrak{a}\infty}y^s + \phi_{\mathfrak{a}}(s,\chi)y^{1-s} + \sum_{n\neq 0}\phi_{\mathfrak{a}}(n,s;\chi)W_s(nz) \tag{21}$$

denotes the weight 0 non-holomorphic Eisenstein series at the cusp $\mathfrak{a}$.

For simplicity, we sometimes write $\phi_{\mathfrak{a}}(s), \phi_{\mathfrak{a}}(n,s)$ and $b_j(n)$ instead of $\phi_{\mathfrak{a}}(s,\chi)$, $\phi_{\mathfrak{a}}(n,s;\chi)$ and $b_j(n;\chi)$.

The first inner products in the series (resp. integral) are computed essentially in [5]. Let $L(f \otimes \eta_j, s)$ and $L(f \otimes E_{\mathfrak{a}}(\cdot, \frac{1}{2}+ir,\chi), s)$ denote the Rankin–Selberg zeta function defined, for $\mathrm{Re}(s) \gg 0$ by

$$2\sum_{n=1}^{\infty} \frac{a(n)\overline{b_j(n)}}{n^s} \quad \text{and} \quad 2\sum_{n=1}^{\infty} \frac{a(n)\overline{\phi_{\mathfrak{a}}(n,\frac{1}{2}+ir)}}{n^s}$$

respectively. With this notation, we have

$$\langle G(\cdot,s;\chi), \eta_j\rangle = \frac{\Gamma(s-\frac{1}{2}+ir_j)\Gamma(s-\frac{1}{2}-ir_j)}{(4\pi)^s i\Gamma(s)} L(f\otimes \eta_j, s) \tag{22}$$

and

$$\langle G(\cdot,s;\chi), E_{\mathfrak{a}}(\cdot, \frac{1}{2}+ir,\chi)\rangle = \frac{\Gamma(s-\frac{1}{2}+ir)\Gamma(s-\frac{1}{2}-ir)}{(4\pi)^s i\Gamma(s)} L(f\otimes E_{\mathfrak{a}}(\cdot,\frac{1}{2}+ir,\chi), s). \tag{23}$$

The remaining inner products in the series are computed with (18):

$$\langle \eta_j, P(n,h,\chi)\rangle = \sqrt{2\pi}|n| b_j(n)(\mathcal{K}(\bar{h}))(ir_j)$$

where

$$\mathcal{K}(h)(s) := \int_0^\infty K_s(y)h(y)\frac{dy}{y^{3/2}}.$$

Fourier coefficients By (17) and Lemma 2.2 we see that the Fourier coefficient of $e^{2\pi inx}$ in the expansion of $G(z,s;\chi)$ equals

$$\phi^*(n,s;\chi)\sqrt{|n|y}K_{s-\frac{1}{2}}(2\pi|n|y) + \delta_{n>0}\frac{a(n)(y^s+\phi(s,\chi)y^{1-s})}{2\pi i n e^{2\pi n y}}$$
$$+ \frac{W(\bar{\chi})\pi^{s-1}\sqrt{y}}{2iN^{2s}L(\bar{\chi},2s)\Gamma(s)} \sum_{1\le l\ne n} \frac{a(l)}{l}\frac{K_{s-1/2}(2\pi|n-l|y)}{e^{2\pi l y}}\frac{\sigma^{\chi}_{2s-1}(|n-l|)}{|n-l|^{s-1/2}}. \tag{24}$$

Theorem 2.4 implies that this Fourier coefficient equals

$$\frac{iW(\bar{\chi})(\pi|n|)^{s-1}}{2\sqrt{2\pi}\Gamma(s)N^{2s}L(\bar{\chi},2s)}\Big(L(n,\chi;s)\sqrt{2\pi|n|y}K_{s-\frac{1}{2}}(2\pi|n|y)$$
$$-\delta_{n>0}\frac{2\Gamma(s)N^{2s}L(\bar{\chi},2s)}{\sqrt{2\pi}W(\bar{\chi})(\pi|n|)^{s-1}}\frac{a(n)}{n}\frac{(y^s+\phi(s,\chi)y^{1-s})}{e^{2\pi n y}}$$
$$-\frac{\sqrt{2\pi|n|y}}{|n|^{s-1/2}}\sum_{1\le l\ne n}\frac{a(l)}{l}\frac{K_{s-1/2}(2\pi|n-l|y)}{e^{2\pi l y}}\frac{\sigma^{\chi}_{2s-1}(|n-l|)}{|n-l|^{s-1/2}}\Big). \tag{25}$$

Thus, integrating against $\bar{h}(2\pi|n|y)/y^2$, we obtain with (18),

$$\langle G, P(n,h,\chi)\rangle = \frac{iW(\bar{\chi})\pi^{s-1/2}}{\sqrt{2}\Gamma(s)N^{2s}L(\bar{\chi},2s)}\Bigg[|n|^s L(n,s;\chi)\mathcal{K}(\bar{h})(s-1/2)$$
$$- \delta_{n>0}\frac{\sqrt{2}\Gamma(s)L(\bar{\chi},2s)}{W(\bar{\chi})\pi^{s-\frac{1}{2}}N^{-2s}}a(n)$$
$$\times\left((2\pi|n|)^{-s}\int_0^\infty y^{s-1}e^{-y}\bar{h}(y)\frac{dy}{y} + (2\pi|n|)^{s-1}\phi(s)\int_0^\infty y^{-s}e^{-y}\bar{h}(y)\frac{dy}{y}\right)$$
$$- \sum_{l\geq 1, \neq n}\frac{a(l)}{l}\frac{\sigma^{\chi}_{2s-1}(|n-l|)}{|n-l|^{s-1}}\mathcal{K}\left(\overline{h\left(\left|\frac{n}{n-l}\right|\cdot\right)e^{\frac{-l}{|n-l|}\cdot}}\right)(s-1/2)\Bigg]. \quad (26)$$

The interchange of integration and infinite summation in the last term is justified by the asymptotics of $K_{s-1/2}(y)$: $\ll y^{1/2-s}$ as $y \to 0^+$ and $\ll e^{-y}y^{-1/2}$ as $y \to \infty$. Together with the growth conditions of $h(y)$ at 0 and ∞ and the bound $e^{-x} \ll x^{-1-\epsilon}$, they imply that the l-th term of the series is $\ll e^{-4\pi l y}y^A l^B \ll e^{-4\pi l}l^C$ for some A, B, C as $y \to \infty$ and $\ll l^{-1-\epsilon}y^{\epsilon}$ as $y \to 0$. Therefore, the series converges uniformly in $(1,\infty)$ and in $(0,1)$.

Multiplying both sides with $i\Gamma(s)2^{2s-1/2}\pi^{s-\frac{1}{2}}/|n|$ and taking into account the definition of $L(n,\chi;s)$ we deduce the following proposition. (We use the following notational simplification:

$$\Gamma(a \pm b) := \Gamma(a+b)\Gamma(a-b).)$$

Proposition 3.1. *Consider s with $\mathrm{Re}(s) > 2$ and an $h \in C^\infty(0,\infty)$ which is $\ll y^{1/2-\epsilon}$, as $y \to \infty$ and $\ll y^{\mathrm{Re}(s)+5/2+\epsilon}$ as $y \to 0$ for some $\epsilon > 0$. If $n < 0$, we have*

$$\frac{-W(\bar{\chi})|n|^{s-1}}{N^{2s}L(\bar{\chi},2s)(2\pi)^{1-2s}}$$
$$\times\lim_{t\to 1}\Big(\sum_{l>0}\frac{a(l)}{l^t}\frac{\sigma^{\chi}_{2s-1}(|n-l|)}{|n-l|^{2s-1}}\mathcal{K}\left(\bar{h} - \left|\frac{n-l}{n}\right|^s\bar{h}\left(\frac{|n|\cdot}{|n-l|}\right)e^{-\frac{l\cdot}{|n-l|}}\right)(s-\frac{1}{2})\Big)$$
$$= \sum_{j=1}^{\infty}\Gamma(s-\frac{1}{2}\pm ir_j)\mathcal{K}(\bar{h})(ir_j)b_j(n,\chi)L(f\otimes\eta_j,s)$$
$$+ \frac{1}{4\pi}\sum_{\mathfrak{a}}\int_{-\infty}^{\infty}\Gamma(s-\frac{1}{2}\pm ir)\mathcal{K}(\bar{h})(ir)\phi_{\mathfrak{a}}(n,\frac{1}{2}+ir;\chi)L(f\otimes E_{\mathfrak{a}}(\cdot,\frac{1}{2}+ir;\chi),s)dr. \quad (27)$$

If $n > 0$ then the same identity holds with the term

$$\frac{\sqrt{2}\Gamma(s)a(n)(2\pi)^{2s-1}}{\pi^{s-1/2}n}\left((2\pi|n|)^{-s}\int_0^\infty e^{-y}y^{s-1}\bar{h}(y)\frac{dy}{y}+(2\pi|n|)^{s-1}\phi(s)\int_0^\infty y^{-s}e^{-y}\bar{h}(y)\frac{dy}{y}\right)$$

subtracted from the left hand side.

This proposition allows us to derive the spectral decomposition for the twisted shifted convolution sum we have been studying.

Theorem 3.2. *Fix* $n<0$ *and* $\mathrm{Re}(s)>2$. *For* $x>\mathrm{Re}(s)+5/2$ *we have*

$$\begin{aligned}&|n|^{s-1}L(n,\chi,x;s)\\&=-\left(\frac{N}{2\pi}\right)^{2s}\frac{2\pi L(\bar{\chi},2s)}{W(\bar{\chi})\Gamma(s-1+x)\Gamma(x-s)}\\&\quad\times\Big(\sum_{j=1}^{\infty}\Gamma(s-\frac{1}{2}\pm ir_j)\Gamma(x-\frac{1}{2}\pm ir_j)b_j(n,\chi)L(f\otimes\eta_j,s)+\textit{cont. part}\Big).\end{aligned}\tag{28}$$

Proof. We apply Proposition 3.1 with the test function $h(y)=h_x(y):=e^{-y}y^x$. It has the nice property that for $\tilde{h}(y)=h_x\big(|n|y/|n-l|\big)\,e^{-ly/n-l|}$ we have

$$(\mathcal{K}\tilde{h})(y)\ =\ \left|\frac{n}{n-l}\right|^x\mathcal{K}h_x(y)\,.$$

So the transform $\mathcal{K}$ occurring in the left hand side of (27) is equal to $\big(1-\left|\frac{n-l}{n}\right|^{s-x}\big)\mathcal{K}h_x$. The explicit form of the K-Bessel transforms appearing in the formula are given by [6, 6.621.3]. □

An implication of this is the meromorphic continuation and bounds of $L(n,\chi,x;s)$.

Theorem 3.3. *For every integer* $n<0$, *the function* $L(n,\chi,z;s)$ *can be meromorphically continued to* $(s,z)\in\mathbb{C}^2$. *For each* $\epsilon>0$ *and* $z=x+iy$, $s=\sigma+it$ *such that* $x,\sigma\in(1/2,\theta+3)$ *and such that* $L(n,\chi,z;s)$ *does not have a pole there, we have*

$$\begin{aligned}&\Gamma(s-1+z)\Gamma(z-s)L(n,\chi,z;s)\\&\quad\ll|n|^{1-\sigma+\epsilon+\theta}e^{-\pi\max(|y|,|t|)}(1+\max(|y|,|t|))^{4\theta+2\sigma+2x+6\epsilon+7}\end{aligned}\tag{29}$$

where θ *is the best exponent towards the Ramanujan conjecture for Maass cusp forms and the implied constant depends on* N, f, θ *and* ϵ.

Remark. The presently best known value $\theta=\frac{7}{64}$ is due to Kim–Sarnak [12]. In Corollary 3.4 we will work with that value.

Proof. Each of the terms in the RHS of (28) is meromorphic in s and z. To show that the convergence of each of the series/integrals is uniform on compacta we recall (A16) of [19]

$$b_j(n,\chi) \ll e^{\pi r_j/2}|n|^{\epsilon+\theta} \tag{30}$$

where θ is the best exponent towards the Ramanujan conjecture.

To bound the Rankin–Selberg zeta functions appearing in (28) we first observe that, with (30) and the Ramanujan bound we have, for $\mathrm{Re}(s) > \theta + 3$

$$L(f \otimes \eta_j, s) \ll \sum_{n=1}^{\infty} \frac{n^{1/2+\epsilon} n^{\epsilon+\theta} e^{\pi r_j/2}}{n^{\mathrm{Re}(s)}} \ll e^{\pi r_j/2}. \tag{31}$$

Similarly, $L(f \otimes E_{\mathfrak{a}}(\cdot, 1/2+ir), s) \ll (1+|r|)^{\epsilon} e^{\pi|r|/2}$ and thus, for a $M > \theta + 3$ and $\eta = \eta_j$ or $E_{\mathfrak{a}}(\cdot, 1/2+ir)$, we have

$$L(\bar{\chi}, 2s) L(f \otimes \eta, s) \ll (1+|r|)^{\epsilon} e^{\pi|r|/2} \qquad \text{for } \mathrm{Re}(s) = M.$$

To obtain the growth at $\mathrm{Re}(s) = 1 - M$ we recall the functional equation of $L(f \otimes \eta_j, s)$ (e.g. [4, Lemma 1]) which, in our case can be written as:

$$\frac{\Gamma(s-1/2 \pm ir_j)}{(4\pi)^s \Gamma(s)} \vec{L}(f \otimes \eta_j, s) = \frac{\Gamma(1/2 - s \pm ir_j)}{(4\pi)^{1-s}\Gamma(1-s)} \Phi(s,\chi) \vec{L}(f \otimes \eta_j, 1-s) \tag{32}$$

where $\Phi(s,\chi)$ is the scattering matrix of the Eisenstein series and $\vec{L}(s)$ is the column vector of Rankin–Selberg zeta functions $L_{\mathfrak{a}_{\mathfrak{i}}}$ of f, η_j at the cusp $\mathfrak{a}_{\mathfrak{i}}$, (ranging over a set of inequivalent cusps of $\Gamma_0(N)$ in whose stabilizers χ is trivial). For our purposes, the only information about $L_{\mathfrak{a}_{\mathfrak{i}}}$ ($\mathfrak{a}_{\mathfrak{i}} \neq \infty$) we need is that they are bounded when $\mathrm{Re}(s) > \theta + 3$. We further use the formula for the $\infty\mathfrak{b}$-entry of the scattering matrix for $\Gamma_0(N)$: It is 0, unless $\mathfrak{b} = 0$ in which case it equals

$$W(\bar{\chi}) N^{1-3s} \pi^{2s-1} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{L(\chi, 2-2s)}{L(\bar{\chi}, 2s)}$$

([11], Prop. 13.7). With this formula, (32) implies that

$$\begin{aligned} &L(\bar{\chi}, 2(1-s)) L(f \otimes \eta, 1-s) \\ &\quad = W(\bar{\chi}) N^{3s-2} (2\pi)^{-2(2s-1)} \frac{\Gamma(s-1/2 \pm ir)}{\Gamma(1/2 - s \pm ir)} L(\chi, 2s) L_0(f \otimes \eta, s) \end{aligned} \tag{33}$$

where, as before, $\eta = \eta_j$ or $E_{\mathfrak{a}}(\cdot, 1/2+ir)$. Choose M so that $M - 1/2 \pm ir$ has a distance at least δ from $\mathbb{Z}$. Then Stirling's estimate implies that, for $\mathrm{Re}(s) = M$ and $r \in \mathbb{R}$

$$W(\bar{\chi}) N^{3s-2} (2\pi)^{-2(2s-1)} \frac{\Gamma(s-1/2 \pm ir)}{\Gamma(1/2 - s \pm ir)} \ll_{M,\delta} (1+|t+r|)^{2M-1} (1+|t-r|)^{2M-1}.$$

If $ir \in (0,1)$, the factor is $\ll_{M,\delta} (1+|t|)^{4M-2}$.

Then, Phragmén–Lindelöf [17, Th. 12.9] applied to the function $L(\chi, 2s)L(f \otimes \eta, s)$ (which is entire by [4, Corollary to Lemma 1]) implies that for $r \in \mathbb{R}$ and s with $\mathrm{Re}(s) \in [1-M, M]$

$$L(\chi, 2s)L(f \otimes \eta, s) \ll_{M,\delta} (1+|r|)^{\epsilon} e^{\pi|r|/2}(1+|t+r|)^{2M-1}(1+|t-r|)^{2M-1}$$

whereas, for $ir \in (0,1)$, it is bounded by $(1+|t|)^{4M-2}$.

Since Stirling's estimate implies:

$$\Gamma(s-1/2 \pm ir) \ll \begin{cases} e^{-\pi|t|}((1+|t|)^2-|r|^2)^{\sigma-1}, & |t|>|r| \\ e^{-\pi|r|}((1+|r|)^2-|t|)^{\sigma-1}, & |t|<|r| \end{cases} \tag{34}$$

we deduce that the normalized function

$$\Lambda(f \otimes \eta, s) := (2\pi)^{-2s}\Gamma(s-1/2 \pm ir)L(\bar{\chi}, 2s)L(f \otimes \eta, s)$$

is bounded by an constant depending on M, δ, times

$$(1+|r|)^{\epsilon}e^{\pi|r|/2} \begin{cases} e^{-\pi|t|}((1+|t|)^2-r^2)^{2M+\sigma-2}, & |t|>|r| \\ e^{-\pi|r|}((1+|r|)^2-|t|^2)^{2M+\sigma-2}, & |t|<|r| \end{cases} \tag{35}$$

for $1-M \le \mathrm{Re}(s) < M$.

We can now rewrite (28) in terms of Λ in order to use the bounds we just established.

$$L(n, \chi, z; s) = \frac{-N^{2s}|n|^{1-s}2\pi}{W(\bar{\chi})\Gamma(s-1+z)\Gamma(z-s)} \times \Big(\sum_{j=1}^{\infty}\Gamma(z-\frac{1}{2} \pm ir_j)b_j(n,\chi)\Lambda(f \otimes \eta_j, s) + \frac{1}{4\pi}\sum_{\mathfrak{a}} I_{\mathfrak{a}}\Big)$$

with

$$I_{\mathfrak{a}}(z,s) = \int_{\mathbb{R}} \Gamma(z-\frac{1}{2} \pm ir)\phi_{\mathfrak{a}}(n, \frac{1}{2}+ir;\chi)\Lambda(f \otimes E_{\mathfrak{a}}(\cdot, \frac{1}{2}+ir;\chi), s)dr. \tag{36}$$

We first show that the sum and the integrals on the right hand side converges uniformly in compacta in (s,z) to yield a meromorphic function in $\mathbb{C}^2$. Indeed, for (s,z) in a compact set S not containing any poles, the bound (35) can be simplified as:

$$\Lambda(f \otimes \eta, s) \ll \begin{cases} e^{-\pi|t|+\pi|r|/2}(1+|t|)^B, & |t|>|r| \\ e^{-\pi|r|/2}(1+|r|)^B, & |t|<|r| \end{cases} \tag{37}$$

with $B > 0$ depending on S only. Weyl's law gives for the number of j such that $r_j \le T$ the asymptotic formula $c_N T^2 + O_N(T \log T)$. So for $l \in \mathbb{N}^*$ the number of j with $l-1 \le |t_j| \le l$ is bounded by $l^{1+\epsilon}$ for each $\epsilon > 0$. Hence, for $(z,s) = (x+iy, \sigma+it) \in S$:

$$\begin{aligned}
&\sum_{j=1}^{\infty} \Gamma(z - \frac{1}{2} \pm ir_j) b_j(n,\chi) \Lambda(f \otimes \eta_j, s) \\
&\ll \sum_{l<\max(|t|,|y|)} \Gamma(z - \frac{1}{2} \pm il) b_j(n,\chi) \Lambda(f \otimes \eta_j, s) \\
&\quad + \sum_{l>\max(|t|,|y|)} e^{-\pi l}(1+l)^{2x-2} |n|^{\theta+\epsilon} e^{\frac{\pi l}{2}} l^2 e^{-\pi l/2} (1+l)^B \\
&\ll \sum_{l<\max(|t|,|y|)} \Gamma(z - \frac{1}{2} \pm il) b_j(n,\chi) \Lambda(f \otimes \eta_j, s) \\
&\quad + |n|^{\theta+\epsilon} \sum_{l=1}^{\infty} e^{-\pi(l+\max(|t|,|y|))} (1 + l + \max(|t|,|y|))^C
\end{aligned} \tag{38}$$

where $C > 1$ is a constant depending only on S. Since $\max(|t|,|y|)$ is bounded by a constant depending on S only and $\sum_{l=1}^{\infty} e^{-\pi l}(A+l)^C$ is convergent for $A > 0$, we deduce the required uniform convergence.

For $\mathrm{Re}(z), \mathrm{Re}(s) > 1/2$, the uniform convergence of the integrals in the continuous part of the spectrum is similarly yielding a meromorphic function there. The integral is not changed if we deform the path of integration in a compact set. Let s be near the line $\mathrm{Re}(s) = \frac{1}{2}$. We deform the path of integration so that s is to the left of the path of $\frac{1}{2} + ir$ and $1-s$ to the right of it. This involves moving the path over two singularities, and we pick up residues that are in general multiples of Eisenstein series. This gives a meromorphic continuation of $I_{\mathfrak{a}}(z,s)$ to a larger region. When $\mathrm{Re}(s) < \frac{1}{2}$ in this region we can move back the line of integration and we get an expression given by the integral in (36) plus terms coming from the residues. For z we get in a similar way meromorphic continuation across the line $\mathrm{Re}(z) = \frac{1}{2}$ and across the lines $\mathrm{Re}(s)$ or $\mathrm{Re}(z) = \frac{1}{2} - m$ with $m \ge 1$

We now turn to the proof of the estimate (29). Assume first that $|t| \le |y|$. We employ (30), (34) and (35) to get for $1 - M < \mathrm{Re}(s) < M$:

$$\begin{aligned}
&\sum_{j=1}^{\infty} \Gamma(z - \frac{1}{2} \pm ir_j) b_j(n,\chi) \Lambda(f \otimes \eta_j, s) \\
&\ll |n|^{\theta+\epsilon} \Big(\sum_{l \le |t| \le |y|} l^{1+\epsilon} e^{-\pi|y|} (1+l)^{\epsilon} (1+|y|)^{2x-2} e^{\pi l/2} e^{-\pi|t| + \pi l/2} (1+|t|)^{2(2M+\sigma-2)} \\
&\quad + \sum_{|t|<l\le|y|} l^{1+\epsilon} e^{-\pi|y|} (1+l)^{\epsilon} (1+|y|)^{2x-2} e^{\pi l/2} e^{-\pi l/2} (1+l)^{2(2M+\sigma-2)}
\end{aligned}$$

$$+ \sum_{|t|\le|y|<l} l^{1+\epsilon} e^{-\pi l}(1+l)^{\epsilon}(1+l)^{2x-2} e^{\pi l/2} e^{-\pi l/2}(1+l)^{2(2M+\sigma-2)}\Big)$$

$$\ll |n|^{\theta+\epsilon}\Big(e^{-\pi|y|}\left(1+|y|\right)^{2x-2}\left(1+|t|\right)^{2(2M+\sigma-2)} e^{-\pi|t|}\int_{u=0}^{|t|} e^{\pi u}\, u^{1+2\epsilon}\, du$$

$$+ e^{-\pi|y|}\left(1+|y|\right)^{2x-2}\int_{u=|t|}^{|y|}(1+u)^{4M+2\sigma-3+2\epsilon} du$$

$$+ \int_{u=|y|}^{\infty} e^{-\pi u}\,(1+u)^{4M+2\sigma+2x+2\epsilon-5}\, du\Big)$$

$$\ll |n|^{\theta+\epsilon}\Big(2\times e^{-\pi|y|}\left(1+|y|\right)^{2x-2}\left(1+|t|\right)^{4M+2\sigma+2\epsilon-3}$$

$$+ 2\times e^{-\pi|y|}\left(1+|y|\right)^{4M+2x+2\sigma+2\epsilon-5}\Big)$$

$$\ll e^{-\pi|y|}(1+|y|)^{2x-2}(1+|t|)^{4M+2\sigma+2\epsilon-3}.$$

Setting $M=\theta+3+\epsilon$ we get the estimate. Similarly we obtain the same bound for the continuous spectrum term, and the bounds for the case $|y|\le|t|$. □

Corollary 3.4. *For integer $n<0$, s in the strip $\sigma=\mathrm{Re}(s)\in(1/2,103/64)$ outside a neighborhood of the exceptional values s_j, and $\epsilon<103/64-\sigma$ we have*

$$L(n,\chi;s) \ll |n|^{71/64-\sigma+\epsilon}(1+|t|)^{151/16+2\sigma+6\epsilon} + |n|^{3/2+\epsilon}. \tag{39}$$

The implied constant depends on N, f and ϵ.

Proof. Set $x:=\sigma+3/2+\epsilon\in(1/2,199/64)$. We can then apply the theorem in this range with $\theta=7/64$, the Kim–Sarnak bound [12]. We first observe that by definition,

$$L(n,\chi;s) = L(n,\chi,x;s) + |n|^{x-s}\sum_{l\ge1}\frac{a(l)}{l}\frac{\sigma^{\chi}_{2s-1}(l-n)}{(l-n)^{s-1+x}}.$$

To estimate the first term note Stirling's estimate

$$(\Gamma(s-1+z)\Gamma(z-s))^{-1} \ll e^{\pi\max(|t|,|y|)}(1+|t+y|)^{3/2-\sigma-x}(1+|t-y|)^{\sigma+1/2-x}.$$

Then, with $y=0$, the bound (29) in the theorem simplifies as

$$L(n,\chi,x;s) \ll |n|^{71/64-\sigma+\epsilon}(1+|t|)^{151/16+2\sigma+6\epsilon}.$$

On the other hand, with $d(l)$ denoting the number of divisors of l,

$$\sum_{l\geq 1}\left|\frac{a(l)}{l}\frac{\sigma_{2s-1}^{\chi}(l-n)}{(l-n)^{s-1+x}}\right| \leq \sum_{l\geq 1}\frac{l^{1/2+\epsilon}}{l}\frac{d(l-n)(l-n)^{2\sigma-1}}{(l-n)^{\sigma-1+x}}$$
$$\ll \sum_{l\geq 1}\frac{1}{l^{1/2-\epsilon}}\frac{1}{(l-n)^{3/2+\epsilon}} = O(1)\,.$$

This implies the result. □

Acknowledgments

We thank G. Harcos for suggesting to us the interest of establishing a spectral decomposition for our shifted convolution series. We also thank J. Hoffstein for showing an alternative method of spectral decomposition to us, and J. Louko, D. Milicevic, F. Strömberg and M. Raum for many stimulating conversations on this topic. Finally, we are indebted to the referees for remarks that helped make the objectives and the presentation of the paper clearer.

References

[1] V. Blomer, G. Harcos, The spectral decomposition of shifted convolution sums, Duke Math. J. 114 (2008) 321–339.
[2] R. Bruggeman, Y. Motohashi, A new approach to the spectral theory of the fourth moment of the Riemann zeta-function, J. Reine Angew. Math. 579 (2005) 75–114.
[3] J.B. Conrey, H. Iwaniec, The cubic moment of central values of automorphic L-functions, Ann. of Math. 151 (2000) 1175–1216.
[4] J. Deshouillers, H. Iwaniec, The nonvanishing of Rankin–Selberg zeta-functions at special points, in: The Selberg Trace Formula and Related Topics, Brunswick, Maine, 1984, in: Contemp. Math., vol. 53, Amer. Math. Soc., Providence, RI, 1986, pp. 51–95.
[5] D. Goldfeld, The distribution of modular symbols, in: Number Theory in Progress, 1999, pp. 849–865.
[6] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series and Products, fifth edition, Academic Press Inc.
[7] J. Hoffstein, Multiple Dirichlet series and shifted convolutions, arXiv:1110.4868v1.
[8] J. Hoffstein, M. Lee, Shifted multiple Dirichlet series, arXiv:1412.5917.
[9] J. Hoffstein, T.A. Hulse, A. Reznikov, Multiple Dirichlet series and shifted convolutions, J. Number Theory (2016), in press, arXiv:1110.4868v2.
[10] M. Huxley, Scattering matrices for congruence subgroups, in: Modular Forms, Durham, 1983, Ellis Horwood Ser. Math. Appl., Chichester, 1984, pp. 141–156.
[11] H. Iwaniec, Topics in Classical Automorphic Forms, Graduate Studies in Mathematics, vol. 17, AMS, 1991.
[12] H. Kim, Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2, with appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak, J. Amer. Math. Soc. 16 (1) (2003) 139–183.
[13] E. Kowalski, P. Michel, J. VanderKam, Rankin–Selberg L-functions in the level aspect, Duke Math. J. 114 (1) (2002) 123–191.
[14] T. Miyake, Modular Forms, Springer, 2003.
[15] C. O'Sullivan, Properties of Eisenstein series formed with modular symbols, J. Reine Angew. Math. 518 (2000) 163–186.
[16] Y. Petridis, M. Risager, Modular symbols have a normal distribution, Geom. Funct. Anal. 14 (5) (2004) 1013–1043.
[17] W. Rudin, Real and Complex Analysis, third edition, McGraw-Hill, 1987.
[18] P. Sarnak, Integrals of products of eigenfunctions, Int. Math. Res. Not. IMRN 6 (1994) 251–260.
[19] P. Sarnak, Estimates for Rankin–Selberg L-functions and quantum unique ergodicity, J. Funct. Anal. 184 (2001) 419–453.