CARL STUMPF'S PHILOSOPHY OF MATHEMATICS

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Abstract. Like most of Franz Brentano's students, Carl Stumpf showed an interest in the philosophy of mathematics. In particular, Stumpf wrote his habilitation thesis *On the Foundations of Mathematics*, used mathematical examples in central parts of his lectures, and later returned to the topic in the posthumously published *Erkenntnislehre*. I will try to show the development and the continuity of Stumpf's position on the basis of his writings and (unpublished) lectures on logic and psychology, taking into account the Brentanist approach to the philosophy of mathematics that developed in the 1880s and 1890s in the School of Brentano.

1 Introductory remarks

In this paper I will provide an account of Carl Stumpf's approach to the foundations of mathematics from his earliest to his latest works, with a particular focus on the middle period. This discussion is situated within the historical and intellectual context of the School of Brentano and the Brentanian philosophy of mathematics developed therein.[1] With respect to the philosophy of mathematics, especially regarding the basic concepts of arithmetic, Stumpf has an important place as Franz Brentano's first student and as mentor to Edmund Husserl. In the following contribution I will of course consider Stumpf's most explicitly mathematical works, such as his 1870 habilitation thesis *Über die Grundsätze der Mathematik* (*On the Foundations of Mathematics*)[2] and his posthumous work, *Erkenntnislehre* (*Theory of Knowledge*),[3] particularly the section on the concept of number. There are some works that address Stumpf's philosophy of

[1] Regarding the idea of a "Brentanian Philosophy of Mathematics" see Ierna 2009.
[2] Recently published as: Stumpf 2008.
[3] Stumpf 1939.

mathematics,[4] and they do so naturally also by focusing on these obviously relevant texts. In order to not merely repeat or summarize such accounts, I will concentrate on Stumpf's mostly unpublished lectures from the 1880s.[5] In addition to elucidating the position he takes in his very first and in his very last works, my contribution can shed light on his position in the crucial intermediate phase. From the mid-1880s to the early 1890s numerous treatises were produced regarding the basic concepts of arithmetic and, therefore, of mathematics, both by mathematicians as well as by philosophers, specifically by Brentanists, such as, for instance, Kerry, Husserl and von Ehrenfels. Both Brentano and Stumpf clearly influenced their pupils in this regard maingly through their lectures, which therefore represent a highly significant source for establishing their respective positions and the relationships between them.

2 Background

Stumpf's philosophical development and career have already been discussed quite extensively elsewhere,[6] but I will point out a few elements that are of particular relevance for our discussion.

Stumpf had almost completed his first year of studies in Würzburg when he attended Franz Brentano's public habilitation defense in July 1866, where he advanced the famous thesis "*Vera philosophiae methodus nulla alia nisi scientiae naturalis est*" ("The true method of philosophy is none other than that of natural science").[7] His confidence and eloquence made a deep impression on Stumpf, prompting him to dedicate his life to philosophy.[8]

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[4] E.g. Ewen 2008.

[5] Ewen does not take any of Stumpf's lectures into account, not even the syllabi of his lectures on psychology and on logic published as appendices two and three in Robin Rollinger's *Husserl's Position in the School of Brentano* (Rollinger 1999). Ewen also seems to jump from the discussion of Stumpf's habilitation thesis of 1870 to Stumpf's 1907 *Einteilung der Wissenschaften* and the posthumous *Erkenntnislehre*, while his main aim is to produce a comparison between Stumpf's views and Frege's position in the 1884 *Grundlagen*.

[6] Consider, for instance, Stumpf 1919 and Stumpf 1917; Schuhmann 1996 and Rollinger 1999, pp. 83-123.

[7] Brentano 1929, 136-7, engl. transl. by Smith (1994, 28).

[8] Stumpf 1883, V.

> The way in which Brentano defended and explained his theses showed such a superiority with respect to his opponents, that I decided to attend his lectures in the winter […] We were especially pleased that he did not consider any other method for philosophy than that of the natural sciences and took this as the foundation of his hopes for a rebirth of philosophy.[9]

Stumpf became Brentano's "eldest", i.e. first, student[10] and together with Anton Marty, formed the very first generation of the School of Brentano.

As would also be the case with Husserl almost a decade later, Brentano, being merely a *Privatdozent* at the time, had to send Stumpf elsewhere to obtain his promotion. So in the summer semester of 1867 Stumpf went to Göttingen for three semesters to study with Lotze: "In the year 1867/68 I studied psychology, history of philosophy after Kant, natural philosophy and practical philosophy with him."[11] In the summer semester of 1868 Stumpf obtained his doctorate with a thesis on *Das Verhältniss des Platonischen Gottes zur Idee des Guten* (*The Relation of the Platonic God to the Idea of the Good*). The next semester he returned to Würzburg and continued to attend Brentano's lectures until the summer semester of 1870.[12] Following Brentano's example once again, Stumpf began in October 1869 to study theology at the seminary. In the wake of Brentano's religious struggles and the controversy concerning the question of papal infallibility, less than a year later Stumpf decided to quit theology and returned to Lotze for his habilitation. Stumpf obtained his habilitation in philosophy in October 1870 with the work *Über die Grundsätze der Mathematik*.

> During the vacation I completed a work on the topic of mathematical axioms and obtained my habilitation at the end of October 1870 in Göttingen. However, I did not publish this work, as the non-Euclidian approach, to which Felix Klein introduced me, in the end was a little beyond me.[13]

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[9] Stumpf 1919, 88 f. Stumpf confirms that he attended Brentano's lectures on the history of philosophy (WS 1866/67), metaphysics (SS 1867) and logic.
[10] Stumpf 1919, 145.
[11] Stumpf 1917, 7.
[12] On history and on metaphysics I (WS 1868/69); on metaphysics II (SS 1869); on history I and on deductive and inductive logic (WS 1869/70). See Schuhmann 1996, 110.
[13] Stumpf 1924, 211.

After a few years in Göttingen as a *Privatdozent*, Stumpf went to Würzburg in 1873, thus becoming a full professor, in 1879 to Prague and in 1884 to Halle, where he was to stay five years (1884 - 1889). Then he moved to Munich (1889) and then finally to Berlin (1894), where he remained for the rest of his life and established his school, founding a psychological institute (1900) and serving as rector (1907/1908).[14]

In addition to sharing the goal of the School of Brentano regarding the renewal of philosophy as science and to Stumpf's later specialization in the philosophy and psychology of sound and music, like so many other Brentanians, he also lectured and wrote on the subject of the philosophy of mathematics.[15]

Little is known about Stumpf's motivations for choosing this topic for his habilitation.[16] However, it is likely that Brentano's lectures on metaphysics and logic might have contributed to his decision. Brentano often discussed topics relating to the philosophy of mathematics in his lectures and it is probable that a precedent can be found in his lectures for many of the points Stumpf makes. While Stumpf had gone to Lotze to undertake both his dissertation and habilitation, thereby forging strong personal and philosophical ties with him in the process,[17] the influence of Brentano is usually regarded as more significant. Indeed, if Brentano's lectures on metaphysics in the early 1880s are indicative of the content of the lectures, which Stumpf attended before writing his habilitation essay, we must conclude that he was certainly inspired by them. On the basis of the remaining notes from his lectures on metaphysics,[18] we can see that Brentano discussed the nature of the mathematical axioms. He attempted to prove the analyticity of these axioms in the context of a discussion of Euclid and asked whether there "[a]re there any synthetic axioms?"[19] From there Brentano launches into extensive critiques of Kant and Mill, discussions of deductive and inductive, analytic and synthetic as well as a priori and

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[14] See Schuhmann 1996, 113; Sprung 2002, 91, 95.

[15] I will not go into his discussions of space and geometry here, which constitute a quite separate and extensive topic, as does the topic of the theory of probability.

[16] Ewen 2008, 32: "We can glean nothing from the collected letters and documents regarding Stumpf's motivation to write a habilitation thesis on a mathematical topic".

[17] See also Baumgartner 2002, 27.

[18] Brentano 1882/83 (Ms. Q 8: *Metaphysik*), mainly in the first part on *Transzendentalphilosophie*.

[19] Brentano, 1882/83 (Q 8), 205: "Gibt es synthetische Axiome?".

a posteriori distinctions. If this is indicative of the background of Stumpf's habilitation essay, his main question ("Is there significant scientific knowledge that is not founded either immediately or mediately on experience, and if so, what is its source?") fits very well in the context of Brentano's overall discussions.

3 The Habilitation Thesis

As Stumpf's habilitation thesis has already been extensively discussed elsewhere,[20] I will only summarize here some of its main points that are connected to the further discussion of his position in his lectures.[21]

In his habilitation thesis *Über die Grundsätze der Mathematik* (literally "*On the Fundamental Propositions of Mathematics*", usually translated as "*On the Mathematical Axioms*") Stumpf argues, against John Stuart Mill and against Immanuel Kant, that mathematics is neither inductive nor synthetic, but a deductive, analytical a priori discipline.

> The critique of Mill's theory has shown that the mathematical axioms (as well as its propositions) cannot be founded on induction, but that they must be necessary a priori. The critique of Kant's theory has shown that they cannot be synthetic a priori. The only remaining possibility is: that they must be analytic.[22]

Stumpf want to show through analysis how the basic propositions of mathematics can be decomposed into elementary concepts and how these can be reduced to tautologies.[23] The fundamental concepts of mathematics, which are found in elementary arithmetic, are "number" as well as relations such as "equality", "difference", "more" and "less".[24] A number is a "sum of unities" ("*Summe von Einheiten*"), i.e. "1+1+ …", and unity is the negation of difference.[25] Counting comes down to establishing relations of difference, i.e. discerning different unities and then "grasping these acts [of differentiation] together in

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[20] See Ewen 2008. A table of contents of Stumpf's habilitation thesis can be found in Baumgartner 2002, 28.
[21] The following is in part based on my brief discussion of Stumpf in Ierna 2009.
[22] Stumpf 2008, 18-2.
[23] Stumpf 2008, 19-1.
[24] Stumpf 2008, 19-2.
[25] Stumpf 2008, 19-3.

thought: one and one and etc."[26] The concept of number is what is given through this "comprehension" (*Zusammenfassung*). To perform such an operation, we merely need the concept of a "thing in general", i.e. "something" (*Etwas*), and then we count "a thing and an other thing, etc." Any thinkable thing can be counted in this way. Stumpf goes on to briefly present the other basic concepts in a similar style, and then states that "the whole of arithmetic and algebra" follows straightforwardly from these simple ("they might look trivial") definitions: no experience, no induction and no a priori synthesis are required.[27] The axioms of arithmetic, according to Stumpf, are so general that they do not in fact apply exclusively to arithmetic itself, but also to geometry, and in general "simply everywhere" (*schlichtweg überall*). They are "presupposed by all sciences with the same right and the same need".[28] Among the basic propositions or axioms of "arithmetic, algebra and geometry"[29] discussed by Stumpf are the Euclidean common notions such as: "things which are equal to a third are also equal to each other",[30] "equals added to equals yield equals",[31] the principle of commutativity of operations,[32] and the idea that "the whole is greater than the part".[33] Mathematics itself does not presuppose any "truths, judgments, facts", but instead requires "presentations, objects, definitions".[34] Mathematics appears to be presuppositionless in that it presupposes no truth derived from other sciences or from experience, while other sciences require, and hence presuppose, the tools of mathematics.

Interestingly, as we will also see later on, Stumpf bases the possibility of conceptualizing higher, i.e. bigger, numbers on the development of the system of numerals. Without such a system of regular sign construction we probably could not conceive numbers beyond three.[35] Higher numbers are conceived, within such a system, with the help of

[26] Stumpf 2008, 19-3.
[27] Stumpf 2008, 20-1.
[28] Stumpf 2008, 20-2.
[29] Stumpf 2008, 20-4.
[30] Stumpf 2008, 21-1, which would seem a rough formulation of the principle of transitivity.
[31] Stumpf 2008, 22-1.
[32] Stumpf 2008, 22-4.
[33] Stumpf 2008, 23-1.
[34] Stumpf 2008, 35-4.
[35] Stumpf 2008, 24-1.

relations to lower numbers. For instance, we cannot conceive 100 except as 10 times 10. While Stumpf does not explicitly talk about proper and symbolic presentations in the Brentanian sense, we have effectively a very early sketch of this distinction. Improper or symbolic presentations of number would simply be presentations through relations. Higher numbers are presented "mediately" through their relations to lower numbers and, in progressive analysis, may be reduced to numbers that can be "immediately" conceived as merely consisting of sums of unities. Stumpf, similarly to Husserl, argues that "by constructing a numerical system, we have constructed the higher numbers themselves",[36] i.e. it is only by constructing a system of signs for the concepts we can conceive them at all, albeit improperly, symbolically. Stumpf will return to the topic of symbolic presentations and signs in mathematics in his lectures delivered in the 1880s.

4 The Lectures on Logic and Psychology

As Schuhmann argues,[37] due to the lack of other materials, the remaining notes (conserved at the Husserl-Archives Leuven) detailing the lectures that Stumpf held in Halle in the 1880s are among the most revealing sources for understanding Stumpf's development and his place in the Brentano School. This is particularly true with respect to topics concerning the elementary concepts in the philosophy of mathematics. The lectures we will take into consideration here are those on *Psychology* and those on *Logic* respectively delivered in the winter semester of 1886/87[38] and in the summer semester of 1887.[39] While they do not discuss philosophy of mathematics *per se*, they often contain examples or digressions relating to and involving mathematical topics. This is especially relevant in cases where mathematical examples are used to illustrate central concepts of Brentanism. Moreover, more than 15 years after his habilitation Stumpf also addresses again part of his habilitation thesis in these lectures, specifically his arguments in favor of establishing mathematics as a deductive, analytical, and a priori science and his refutation of Mill and Kant. This indicates a high degree of continuity and suggests that we can probably assume the pres-

[36] Stumpf 2008, 24-2.
[37] Schuhmann 1996, 113-115.
[38] Stumpf 1886/87 (Ms. Q 11/II: *Vorlesungen über Psychologie*).
[39] Stumpf 1887 (Ms. Q 14: *Logik und Enzyklopädie der Philosophie*).

ence of a Brentanian influence throughout, from the early period up to and including the 1880s, on Stumpf's thought.

In order to introduce the notion of symbolic presentations, in his lectures on logic Brentano provided the following example:

> We improperly present that of which we have no precisely corresponding presentation and often can have none. […] We name objects, the single features of which we could presumably grasp, but which are for us no longer presentable due to their complication. A million, a billion, we cannot properly present any longer and we name them without understanding the name precisely.[40]

> Language aids our thought in the same way as the signs of the mathematician aid his calculation when he uses them instead of more complex expressions. From now on he considers the reference only as the object to which this sign refers; similarly as he already does with most ordinary number signs, where the sum of numbers passed a certain limit. Who could conceive of a million in any other way but as of a great number, as a 1 with six zeroes? Thus we have an example here, where language helps out thought in such a way, that it overcomes difficulties of the highest degree, even overcoming impossibilities.[41]

If we then consider Stumpf's lectures on psychology, we see close similarities with the definition of symbolic presentations to the point that it even would seem that Stumpf perused Brentano's lectures in the preparation of his own, as he repeats the same examples almost *verbatim:*

> § 29. Symbolic Presentations. By this we mean those presentations which occur only as signs for others by replacing them for the use of judgement. Seldom is the case where a name completely expresses a

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[40] Brentano Ms. EL 80/13060, quoted from Rollinger 2009, 81 f.

[41] Brentano 1884/85 (Ms. Y 2: *Die elementare Logik und die in ihr nötigen Reformen I*), 29 f. "Endlich fördert die Sprache auch noch in der Weise das Denken wie die Zeichen des Mathematikers seine Rechnung fördert, wenn er sie statt eines komplizierten Ausdrucks setzt. Er denkt von nun an an das Bezeichnete nur in dem Sinne eines von diesem Zeichen Bezeichneten; ähnlich macht er <es> schon bei den meisten gewöhnlichen Zahlzeichen, wo die Summe der Zahl über ein gewisses Maß hinausgewachsen ist. Wer kann eine Million anders denken als eine große Zahl, etwa 1 mit sechs Nullen. Hier haben wir also ein Beispiel, wo die Sprache dem Denken in der Art zu Hilfe kommt, daß sie über Schwierigkeiten des höchsten Grades, ja über Unmöglichkeiten hinweghilft."

> thought; but only usually a certain part of it. Sometimes, as in the case of larger numbers, the adequate presentation [*adäquate Vorstellung*] is altogether impossible for us and we think, instead of it, the indeterminate concept of a large number together with certain relations of the number we mean (we intend) [*der gemeinten (intendierten) Zahl*] to other numbers. E.g. 1000 = 10x100, 100 = 10x10.[42]

> The importance of names is still greater in cases in which a presentation is altogether completely unthinkable. In these cases the linguistic sign performs a similar function as the numerical sign does for the mathematician. A million: one can easily perform operations in thought with these contents, which are in themselves enormous and cannot be fully thought out.[43]

Stumpf, like Brentano, uses a mathematical example to elucidate the concept, thereby reinforcing the idea that mathematics is to be understood as a science entirely based on symbolic presentations, i.e. signs. While Husserl extensively thanks Brentano for the "deeper understanding" of the significance of the distinction between proper and improper (symbolic) presentations, he also refers to Meinong's *Hume Studien* II, where "indirect presentations" are defined through relative determinations and attributes. Indeed, Meinong regarded Husserl's claim that Brentano had made this distinction "all along" in his lectures as a veiled accusation of plagiarism and claimed himself authorship of it.[44] However, Husserl's most direct sources are probably the lectures delivered by Stumpf, which he attended while writing his habilitation thesis, which would form the basis for the *Philosophy of Arithmetic*.

Besides in the definition of symbolic presentations, there are (unsurprisingly) other parallels with Brentano's lectures with respect to topics relating to the philosophy of mathematics. For instance, in his logic lectures from 1887, Stumpf briefly discusses the "English logicians" De Morgan, Boole, and Jevons:

[42] Stumpf 1886/87, 504. Translation from Rollinger 1999, 301.
[43] Stumpf 1886/87, 510: "Noch größer wird die Bedeutung der Namen da, da wo eine Vorstellung vollständig überhaupt nicht denkbar ist. Da übt das sprachlichen Zeichen eine ähnliche Funktion wie das Zahlzeichen für den Mathematiker. Million: Mit Leichtigkeit operiert man im Denken mit diesen an sich ungeheuren und nicht auszudenkenden Inhalten".
[44] Meinong to Husserl, 19-6-1891, see Husserl 1994, 129. With regard to this discussion, also see Ierna 2009, 7-36.

> Other English logicians, who were also mathematicians, wanted to apply the formal language of mathematics: De Morgan, Boole, Jevons, and in Germany Wundt. According to these logicians deduction, for example, is nothing other than the inference of a third equation from two given equations on the basis of substitution.[45]

Brentano had made the same point in his lectures on logic in 1884/5, while discussing Boole and Jevons[46] and specifically quoting and commenting Jevons' opinion that all judgements are equations (*Gleichungen*).[47]

Like Brentano, Stumpf places mathematics at the top of the scientific hierarchy: "the most exact science: mathematics".[48] Its concepts and laws find application everywhere, since everywhere there is "something" that can be counted. To what exactly does mathematics owe this status as being the most rigorous and exact of sciences? What is its foundation? For Kant, mathematics was the prime example of a science built on synthetic a priori judgements, which are capable of providing new knowledge.

> We ask: are there really a priori synthetic judgements, resp. are the examples adduced by Kant really of this kind? Let us take mathematical propositions [*Sätze*]. Already pure arithmetic contains a priori synthetic judgements. 5 + 7 = 12: in 5 lies nothing of 12, in 7 there is likewise nothing, and also nothing in 5 + 7. It is a different concept from the concept of 12 after all. Such is the case of all arithmetical propositions. They are not after all purely tautological. They do not repeat the subject, they add something new.[49]

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[45] Stumpf 1887 (Ms. Q 14: *Logik und Enzyklopädie der Philosophie*), 53a-b: "Andere englische Logiker, die zugleich auch Mathematiker waren, wollten geradezu die mathematische Formelsprache in Anwendung bringen: De Morgan, Boole, Jevons, in Deutschland Wundt. Der Schluß z.B. ist nach diesen Logikern nichts als die Herleitung einer dritten Gleichung aus zwei gegebenen Gleichungen aufgrund der Substitution."

[46] Brentano 1884/85 (Y 2), 36-37

[47] Brentano 1884/85 (Y 2), 38-39.

[48] Stumpf, Q 14, 4b. Compare Franz Brentano, *Psychology from an Empirical Standpoint,* London, 1995, 17, 21; Franz Brentano, *Psychologie vom empirischen Standpunkte*, Leipzig, 1874, 28 f., 34.

[49] Stumpf 1887 (Q 14), 86b: "Wir fragen: Gibt es wirklich synthetische Urteile a priori, bzw. sind die Beispiele, die Kant angeführt hat, von solcher Natur, wie sie ihnen Kant zuschreibt? Zunächst die mathematischen Sätze. Schon die reine Arithmetik enthält synthetische Urteile a priori. 5 + 7 = 12: In 5 liegt nichts von 12, in 7 auch nichts,

After providing an outline of Kant's arguments and examples, Stumpf strongly opposes such a view and argues instead that mathematics is essentially analytic:

> The propositions are analytic. The subject really contains the predicate.
> Let us take 1 + 2 = 3, then 2 means: 1 + 1, and 3 means: 1 + 1 + 1. If we substitute the concepts for the signs, then the proposition is identical.
> The difference with respect to the previous version lies only in the fact that the units [*Einheiten*] in the second version are grouped differently, that is, every unit is placed on a par with every other, while in a certain sense we place them into brackets: (1 + 1) + 1 = 1 + 1 + 1 = 1 + (1 + 1).
> This is a different grouping of the units. Only this lies in the concepts of sum and of number, that is, the grouping of the units does not make any difference. This is established right from the start in the concept of sum. This is why the proposition is evident a priori: a + b = b + a, a proposition that many have regarded likewise as a synthetic a priori proposition.[50]

When we properly understand the concept of number and elementary arithmetical operations, we see that different numbers such as 5, 7, and 12, instead of being *toto genere* different and thus involving a form of synthesis, turn out to consist of the same basic building blocks: unity and addition. This was also Brentano's view:

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auch in 5 + 7 nichts. Es ist doch ein anderer Begriff als der Begriff der 12. So in allen arithmetischen Sätzen. Sie sind doch nicht rein tautologisch. Sie wiederholen nicht das Subjekt, sie fügen etwas Neues hinzu".

[50] Stumpf 1887 (Q 14), 87a: "Die Sätze sind analytisch. Das Subjekt enthält wirklich das Prädikat.
Nehmen wir 1 + 2 = 3, dann bedeutet 2: 1 + 1, 3: 1 + 1 + 1. Setzen wir die Begriffe für die Zeichen, so ist der Satz identisch.
Der Unterschied gegenüber der früheren Fassung liegt nur darin, daß die Einheiten in der zweiten Fassung verschieden gruppiert sind, jede Einheit gleichwertig neben der anderen steht, während wir gewissermaßen Klammern setzen: (1 + 1) + 1 = 1 + 1 + 1 = 1 + (1 + 1).
Es ist eine verschiedene Gruppierung der Einheiten. Allein das liegt im Begriff der Summe und der Zahl überhaupt, daß die Gruppierung der Einheiten keine Unterschiede macht. Das bedingen wir uns von vornherein im Begriff der Summe aus. Deshalb leuchtet ja auch der Satz apriori ein: a + b = b + a, ein Satz, welchen ebenfalls manche als einen synthetischen Satz bezeichnet haben apriori."

> A multiplicity, a number, is composed from smaller numbers, 12 from 6 + 6. But when you go further back, you reach the unity. Is that still a number? It is not a number at all. Thus multiplicity is ultimately composed of non-multiplicity.[51]

Also in his lectures on metaphysics Brentano argued much in the same vein against Kant and Mill, presenting his own solution in terms of a decomposition of the sum 7 + 5 = 12 into a summation of units distinguished only by their grouping.[52]

Unsurprisingly, Husserl, as pupil of both Brentano and Stumpf, defended broadly the same position.[53] The main difference with Kant consists in their psychological analysis of the concept of number which allows the reduction to simplest elements, such as unity, and which can then explain the "synthesis", or rather, the construction of new numbers in such a way that it consists simply of tautologies on a deeper level. One of the necessary conditions of this analysis, however, is the fundamental distinction between proper and symbolic concepts and the role of signs therein:

> In propositions that involve higher numbers, already with 7 + 5 = 12, there is a further difference in that we do not present larger numbers in a complete and proper way [*in vollständiger und eigentlicher Weise vorstellen*], by considering every unit for itself. We use signs, such as "12" [or] "7" for the concepts that we do not completely and exactly think of. These propositions then are only mediately evident. They can be reduced to propositions of a simpler kind. We think e.g. 7 = 5 + 2 (where 5 and 2 are presented properly), then 9 = 7 + 2, etc. In this way we can reduce [*zurückführen*] the propositions of higher numbers to

[51] Brentano 1884/85a (Ms. Y 3: *Die elementare Logik und die in ihr nötigen Reformen II*), 47: "Eine Vielheit, eine Zahl setzt sich zusammen aus kleineren Zahlen, 12 aus 6 + 6. Aber wenn Sie weiter zurückgehen, kommen Sie auf die Einheit. Ist das noch eine Zahl? Es ist gar keine Zahl. So setzt sich im letzten Grunde die Vielheit aus Nichtvielheit zusammen."

[52] Brentano 1882/83 (Q 8), on Kant 165-176, Mill is briefly discussed in 177 ff. and again Mill and the nature of mathematics, 242 ff.

[53] See Husserl 1970, 132 and Husserl 2003, 139. Husserl develops his discussion in dialogue with Frege's argument in the *Grundlagen* that number is the answer to the question "How many?". Husserl then points out that one and zero cannot be positive answers to the question, since they are "not-many". Nevertheless zero and one can be introduced as numbers on computational grounds.

> those of lower numbers. If the first propositions are analytic, then the later ones are also analytic.[54]

The requirements for a proper conception of numbers beyond a certain limit, whether 5, 10, or 12, are simply too demanding for the limited cognitive capacities of the "narrow mind of man".[55] Beyond such a limit we have to operate with signs, with symbolic numbers, which explains why we do not immediately see that nothing new is synthesized in our acts and we must analyze and reduce such "higher" numbers first to properly conceivable ones in order to grasp the analytic nature of the equations involved. Every number conceived symbolically can in the end be reduced to a normal form consisting merely of units and addition. Hence Stumpf's conclusion that: "the whole of mathematics is nothing but an abbreviated counting."[56] This conception of course explains the focus of the entire School of Brentano (and of the present article), with respect to the philosophy of mathematics, on elementary arithmetic as foundation for the whole of mathematics.

Stumpf was also concerned with such fundamental mathematical concepts in his lectures on Psychology from 1886/87, where he introduces mathematical concepts as a special class:

> Finally a special class of concepts, the mathematical concepts: the concept of number, of counting and the geometrical concepts. Here again we find special and very significant difficulties. It is certain that regarding the concept of number the perception of relations plays a role. When I say, here are two things a b, then it is clear that I must have distinguished each one from the other and that thus the perception of a difference plays a role. When I say, there are three things, a b c, then one could at first answer that this is a double perception. I perceive the difference of a and b and the difference of b and c. But this

[54] Stumpf 1887 (Q 14), 87a f.: "In Sätzen, bei welchen höhere Zahlen vorkommen, schon bei 7 + 5 = 12, ist noch ein Unterschied, daß wir größere Zahlen nicht mehr in vollständiger und eigentlicher Weise vorstellen, indem wir jede Einheit für sich denken. Da gebrauchen wir Zeichen wie das Zeichen 12, 7 für die Begriffe, die wir nicht vollständig und genau denken. Allein diese Sätze leuchten dann mittelbar ein. Sie lassen sich auf Sätze einfacherer Art zurückführen. Wir denken etwa 7 = 5 + 2 (wo 5 und 2 eigentlich vorgestellt werden), dann 9 = 7 + 2 usf. So können wir die Sätze für höhere Zahlen auf die für niedere Zahlen zurückführen. Sind die ersten Sätze analytisch, so sind die späteren auch analytisch."

[55] Also see Stumpf 1886/87 (Q 11/II), 611 f. on "*Enge des Bewußtseins*".

[56] Stumpf 1887 (Q 14), 114a: "die ganze Mathematik [ist] nichts anderes als ein abgekürztes Zählen".

> would be circular. When I say, double perception, then I reintroduce the concept of number again with the "double." Thus, there must also be here relations of a more complex kind, relations of a higher order, on which the concept of number is based.[57]

It is noteworthy, especially in connection with Husserl's early works, that Stumpf here mentions the concept of number and the act of counting as fundamental notions in mathematics. These were also the main focus of Husserl's work during these years and were clearly due for a large part to the influence of Stumpf's lectures. Additionally, the suggestion that relations would form the basis of the concept of number is developed more extensively in Husserl's habilitation work and subsequently in the *Philosophy of Arithmetic.* Indeed, Husserl takes over quite literally Stumpf's account of "relations of relations", including its pictorial representation.[58] However, the main issue remains the role of symbolic presentations in mathematics. Since there are presentations that we cannot ever properly and completely conceive of (such as "Europe", "the earth", "the solar system"), these will always remain symbolic, characterized by merely contingent (pictorial) associations. This is of course also the case of numbers beyond a certain threshold:

> 1000, indeed maybe already 20, 10. How many items can one present exactly sensuously and at the same time be aware of their number as such? One will not get much beyond 5 this way. Through sensuous intuition [*sinnliche Anschauung*] we will not be able to distinguish 20 from 21, except through counting, i.e. mediating presentation. Thus a concrete presentation of 1000 [is] impossible. Yet the child can calculate with it easily. This happens because it counts with number signs.

[57] Stumpf 1886/87 (Q 11/II), 494: "Endlich eine besondere Klasse von Begriffen, die mathematischen Begriffe: der Begriff der Zahl, des Zählens und die geometrische Begriffe. Hier finden sich wieder besondere und sehr bedeutende Schwierigkeiten. Das ist gewiß, daß beim Begriff der Zahl die Wahrnehmung von Verhältnissen eine Rolle spielt. Wenn ich sage, hier sind zwei Dinge a b, so ist klar, daß ich beide voneinander unterschieden haben muß, daß also die Wahrnehmung eines Unterschiedes eine Rolle spielt. Wenn ich sage, es sind drei Dinge a b c, so könnte man zunächst antworten, es ist eine doppelte Wahrnehmung. Ich nehme den Unterschied von a und b wahr und den Unterschied b und c wahr. Aber das wäre ein Zirkel. Wenn ich sage, doppelte Wahrnehmung, so habe ich mit dem "doppelt" den Zahlbegriff wieder hineingenommen. Es müssen also hier auch Verhältnisse komplizierter Art, Verhältnisse höherer Ordnung sein, die dem Zahlbegriff zugrunde liegen."

[58] See Stumpf 1886/87 (Q 11/II), 493 and Husserl 1970 (*Über den Begriff der Zahl*, 321, and *Philosophie der Arithmetik*, 53), transl. in Husserliana 2003, 338, 54.

> These are sensuous presentations, also vocal presentations [*Lautvorstellungen*]. These signs are linked to a certain content of presentation: the concept of a more or less greater number. Further, the presentation of certain relations of this number to other numbers. 1000 = a large number that stands in relation to the large number 100 by being its tenfold, this 100 again, etc. Thus develops the remarkable art of economizing with surrogates. At the same time, the relations between numbers are precisely the important factor for mathematical thought. Numbers are only important to us because of the relations that hold between them.[59]

As mentioned above, whether at 5, 10, or 7±2, our minds do arrive at a limit and require signs in order to go beyond it. It is only through the mediation of signs that we can obtain a presentation, even if only an improper one, of "large" numbers. Indeed, if we take seriously the claims that zero and one are not numbers (not multiplicities of units)[60] and that "we can hardly count beyond three in the proper sense",[61] this leads to the quite extreme (finitist) view that properly speaking the only numbers are two and three. Numbers beyond the threshold are obtained through relations to other numbers, i.e. symbolic presentations of large numbers are intelligible due to their ultimate reducibility to proper presentations of smaller numbers. Apart from this foundation, mathematics is symbol manipulation: operating with signs as surrogates for the unattainable properly presented numbers.

[59] Stumpf 1886/87 (Q 11/II), 506: "1000, ja vielleicht schon 20, 10. Wieviel Exemplare kann man sich genau sinnlich vorstellen und dabei der Zahl als solcher bewußt sein? So wird man nicht viel über 5 bekommen. Durch sinnliche Anschauung wird man nicht 20 von 21 unterscheiden können, es sei denn durch Zählen, also vermittelnde Vorstellung. Also eine konkrete Vorstellung von 1000 unmöglich. Doch rechnet das Kind damit in Leichtigkeit. Es geschieht, indem es mit Zahlzeichen rechnet. Das sind sinnliche Vorstellungen, auch Lautvorstellungen. An diese Zeichen knüpft sich ein gewisser Vorstellungsinhalt: Begriff einer mehr oder minder großen Zahl. Ferner die Vorstellung gewisser Verhältnisse dieser Zahl zu anderen Zahlen. 1000 = eine große Zahl, die im Verhältnis zu der großen Zahl 100 steht, daß sie das Zehnfache davon ist, dieses 100 wieder etc. So entsteht die merkwürdige Art, mit Surrogaten zu wirtschaften. Indessen sind gerade für das mathematische Denken die Verhältnisse der Zahlen das Wichtige darin. Die Zahlen sind uns nur wichtig wegen der Verhältnisse zwischen ihnen."

[60] See Brentano 1884/85a (Y 3), 47.

[61] Husserl's habilitation thesis (Husserl 1970, 339; Husserliana 2003, 357).

> The algebraic and arithmetical sign systems are the grandest and subtlest that we have. From the signs themselves certain relations become immediately apparent. In the case of arithmetical signs: certain relations are already indicated by the position of the sign.[62]

Arithmetic and algebra provide systems of signs that allow precisely the construction of "large" numbers through the regular application of relations. For instance, in the decimal system each step left in the position of a numeral indicates a tenfold increase in magnitude. While we can only conceive higher numbers with the help of this system, which allows for the construction of increasingly complex and therefore increasingly "improper" numbers through signs, this does not, according to Stumpf, challenge the a priori nature of mathematics. While we do need the crutch of sense-perceptible signs to calculate with higher numbers, mathematics remains a priori through and through.

> Not all immediate pieces of knowledge are called a priori truths, but only the class of axioms (a). Whenever perception is involved we already have a posteriori knowledge. However, we do not call a priori truths only the immediate insights from the concepts (axioms), but also all judgements that can be inferred through deduction from them. If we assume that all mathematical knowledge is deducible from the axioms, then every mathematical theorem should be considered as a priori knowledge.[63]

In general we can see that Stumpf still holds broadly the same views as he did at the time of his habilitation and we can notice that his position shows many parallels with Brentano's. This continuity confirms the existence of a significant Brentanian influence on Stumpf's habilitation work.

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[62] Stumpf 1886/87 (Q 11/II), 507: "Die algebraischen und arithmetischen Zeichensysteme sind die großartigsten und feinsten, die wir besitzen. Aus den Zeichen selbst sind gewisse Relationen unmittelbar ersichtlich. Bei den arithmetischen Zeichen: daß durch die Stellung des Zeichens schon gewisse Relationen angedeutet werden."

[63] Stumpf 1886/87 (Q 11/II), 547 f.: "Apriorische Wahrheiten nennt man nicht alle unmittelbaren Erkenntnisse, sondern nur die Klasse a) der Axiome. Überall, wo eine Wahrnehmung beteiligt ist, haben wir schon eine aposteriorische Erkenntnis. Wir nennen aber apriorische Wahrheiten nicht bloß die unmittelbaren Erkenntnisse aus den Begriffen (Axiome), sondern auch alle Urteile, die daraus durch Folgerungen hergeleitet sind. Nehmen wir an, daß die gesamte mathematische Erkenntnis aus Axiomen herleitbar ist, so wäre jeder mathematische Lehrsatz als eine apriorische Erkenntnis zu bezeichnen."

5 The *Erkenntnislehre*

In his posthumous work on *Erkenntnislehre*, Stumpf dedicates a central chapter to the concept of number. Moreover, we find throughout the work many observations concerning not only the basic concepts of arithmetic, but also more generally the epistemology and ontology of mathematics:

> The relations of numbers and magnitudes form the object of mathematics. Here we do not merely recognize [*erkennen*] that something is thus and so, but that it must necessarily be thus and so and *cannot* be otherwise. 2 x 2 is not just = 4 in fact, but with absolute necessity, and indeed this is not merely a case of *psychological* necessity, to judge so as false judgements can also be psychologically necessary, as any superstition that has become second nature, but it is a matter of objective necessity [*eine sachliche Notwendigkeit*].[64]

Stumpf underscores once again that relations of numbers and magnitudes form the core of mathematics, but additionally he clearly rejects psychologism in mathematics. Mathematics as an analytical a priori discipline consists of necessarily true propositions or judgements, whose necessary truth lies in the subject matter itself. Relations of numbers are relations of ideas, in Hume's parlance:

> If this concerns only relations among concepts, as is the case with the propositions of pure mathematics, then only logical possibility comes into questions (which the mathematician regards as existence of the conceptual composition [*Begriffszusammensetzung*]).[65]

Contrary to applied mathematics or to the natural sciences, pure mathematics does not deal with physical possibilities, but only with logical possibilities. On this level, non-contradiction already involves what may be indicated as "existence". In this sense, mathematical objects do not exist either in the physical or in the psychical domain:

> There is knowledge [*Erkenntnisse*] that does not regard reality at all, whether physical or psychical, whether outer or inner, since it only concerns, strictly and seriously, what follows if certain presuppositions are made. For instance the proposition […]: "If A=C and B=C, then A=B." Even though there would be no exact identicals anywhere

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[64] Stumpf 1939, 48-49.

[65] Stumpf 1939, 54.

> in the world, still this logical connection would be evident and true. It is essentially a mere hypothetical judgement. However, all propositions of pure geometry are also of this kind, since nowhere in the world there is a mathematically straight line, or even any line in the mathematical sense at all, nowhere a triangle with a precisely straight angle, yet the theorems [*Lehrsätze*] regarding these are valid in all strictness, and they can be proven precisely for all of these constructions [*Gebilde*] that are never and nowhere exactly realized. Indeed, it all depends on what is *not* realized [*verwirklicht*] […]. What the mathematician calls the existence of his objects, is not real existence, but only non-contradiction [*Widerspruchslosigkeit*].[66]

In the chapter on the concept of number Stumpf underscores the conceptual nature of numbers: "Numbers are concepts" ("*Zahlen sind Begriffe*").[67] Every number expresses a general concept and is not a label for any perceptual content. When we perceive pluralities or quantities, we do not merely perceive the elements of the quantity, but also the quantity itself and as such.[68] Plurality itself cannot be considered as a relation between sensory contents, since every relation already presupposes the presence of a plurality of members. This of course leads us to the following question: Do we need to first perceive each of the elements of a plurality by itself before we can achieve an impression of plurality? According to Stumpf this is not the case as experience would deny this. The general impression of a plurality does not necessarily involve that of a temporal ordering.[69] Nevertheless, there are other requirements: "A certain equality in kind [*Gleichartigkeit*] of the elements is required for a perception of quantity [*Mengenwahrnehmung*]".

> There must always be an impetus [*Antrieb*] for the collection [*Zusammenfassen*] given by an apparent similarity [*Gleichartigkeit*] that crosses a certain threshold.[70]

While this is not to say that whoever perceives the plurality has to group its elements under a certain concept, it does suggest that, like for Husserl at the time of the *Philosophy of Arithmetic*, a certain level

[66] Stumpf 1939, 59
[67] Stumpf 1939, 97.
[68] Stumpf 1939, 98.
[69] Stumpf 1939, 99-100.
[70] Stumpf 1939, 100.

of "*Einheitlichkeit*" and *Gestalt* does play a role in the perception of quantities as quantities. Where Ewen points out that Stumpf excludes that *Gestalt* would be of relevance to the concept of number, he sees this as a veiled critique of Husserl's position[71], which I do not think is the case. Husserl does not introduce the concept of *Gestalt* in the (symbolic) concept of number, but only in the (symbolic) concept of quantity (*Menge*). Husserl's and Stumpf's basic definition of (properly conceived) numbers as the result of counting, i.e. addition of units, is the same.[72] Units or unities (*Einheiten*) and zero are not numbers properly speaking, but are introduced into the field of numbers as possible results of operations.[73] This is a position that, as we already saw, was expressed by Husserl in the *Philosophy Arithmetic*. Stumpf argues that such an "extended" concept of number, involving "improper" or "quasi" numbers, must remain the province of the mathematician.

Stumpf often mentions the issue of synthetic vs. analytic a priori judgments and provides once again an extensive critique of Kant's theory of the synthetic a priori.[74] Stumpf's conclusion is that the distinctions between analytic/synthetic and a priori / a posteriori are not orthogonal, but congruent: a priori truths are those that can be proven through the analysis of the concepts involved. All analytic judgements are a priori and all synthetic judgements are a posteriori.[75] However, Stumpf makes a quite fundamental distinction between the deductive and inductive sciences, which he points out is not identical with the a priori and a posteriori distinction. Mathematics is analytic (and hence a priori) and additionally a wholly deductive discipline. Theoretical physics and astronomy may be deductive, but are certainly not a priori.[76] As Stumpf had already pointed out in 1907 in his "Zur Einteilung der Wissenschaften":

> The difference in method shows itself as the foremost characteristic that serves to distinguish mathematics from all other sciences: the a priori as opposed to the a posteriori.[77]

[71] Ewen 2008, 164.
[72] Stumpf 1939, 109.
[73] Stumpf 1939, 119.
[74] Stumpf 1939, 201 ff.
[75] Stumpf 1939, 205.
[76] Stumpf 1939, II, 374.
[77] Stumpf 1907, 65.

6 Concluding Remarks

In conclusion we see how Stumpf's approach to the fundamental concepts of arithmetic and the philosophy of mathematics in general is a part of the School of Brentano and develops out of core concerns within it. Apart from the development of his own position, Stumpf's interest in and engagement with mathematical topics from a Brentanist point of view is also significant due to the ties that bind him to Brentano and Husserl as pupil and master respectively. From his first to his last works we have seen that there is great continuity in the basic position that Stumpf takes towards mathematics as an analytical, deductive and a priori science. He endorses at all stages of his thought central tenets from the Brentanist approach: number is a multiplicity of units, given by the mental operation of counting and "*Zusammenfassung*" ("grasping together", i.e. collecting) operated on "somethings", in which case one and zero are not, properly speaking, numbers. Also higher numbers, beyond our presentational capacity are to be understood as presented improperly, i.e. symbolically, through signs provided in a systematic way by a (positional) numeral system. Numbers are neither empirical facts, nor platonic entities, but essentially conceptual in nature, without being purely subjective creations, but have "existence" in the sense of being logically possible, i.e. non-contradictory. Mathematics, then, is objective, presuppositionless and the most exact science.

However, here we have broached but part of the mathematical issues in the School of Brentano. We have discussed above the fundamental concepts and operations that constitute the foundations of basic arithmetic and mathematics. A further problem (besides space and geometry) is posed by the application of mathematics: first, in the justification of the use of quantitative mathematical methods in psychology in general (an issue famously raised by Kant) and second, more specifically, the justification of the application of the calculus of probability as a tool in psychological experiments. I hope however that the sources and theories presented here can serve as a first step towards further work on such topics and more generally towards a more comprehensive and detailed account of the philosophy of mathematics in the School of Brentano.

References

Manuscripts

Brentano, Franz. 1882/83. Ms. Q 8: Metaphysik.

— 1884/85. Ms. Y 2: Die elementare Logik und die in ihr nötigen Reformen I.

— 1884/85a. Ms. Y 3: Die elementare Logik und die in ihr nötigen Reformen II.

Stumpf, Carl. 1886/87. Ms. Q 11/II: Vorlesungen über Psychologie.

— 1887. Ms. Q 14: Logik und Enzyklopädie der Philosophie.

Primary

Brentano, Franz. 1874. *Psychologie vom empirischen Standpunkte*. Leipzig: Duncker & Humblot.

— 1929. *Über die Zukunft der Philosophie*. Leipzig: Meiner.

— 1995. *Psychology from an Empirical Standpoint*. London: Routledge.

Husserl, Edmund. 1970. *Philosophie der Arithmetik. Mit ergänzenden Texten (1890-1901)* (ed. L. Eley) (Husserliana XII). Dordrecht: Nijhoff.

— 1994. *Briefwechsel.* Vol. I: *Die Brentanoschule.* Dordrecht: Kluwer.

— 2003. *Philosophy of Arithmetic*. Psychological and Logical Investigations, with Supplementary Texts from 1887-1901 (tr. D. Willard) (Husserliana Collected Works X). Dordrecht: Kluwer.

Stumpf, Carl. 1883. *Tonpsychologie*. Bd. 1, Leipzig: Hirzel.

— 1907. "Zur Einteilung der Wissenschaften", Abhandlungen der Königl. Preuss. Akademie der Wissenschaften. Berlin.

— 1917. "Zum Gedächtnis Lotzes", in *Kant - Studien*, XXII.

— 1919. "Erinnerungen an Franz Brentano", in Kraus, Oscar (ed.) *Franz Brentano: zur Kenntnis seines Lebensund seiner Lehre*. München: Beck.

— 1924. "Selbstdarstellung", in Raymund Schmidt (ed.) *Philosophie der Gegenwart in Selbstdarstellungen*. Leipzig: Meiner.

— 1939. *Erkenntnislehre*. Bd. 1, Leipzig: Barth.

— 2008. *Über die Grundsätze der Mathematik* (ed. W. Ewen). Würzburg: Königshausen & Neumann.

Secondary

Baumgartner, Elisabeth, and Wilhelm Baumgartner. 2002. "Der junge Carl Stumpf" in *Brentano Studien (2000/2001),* IX: 23-50.

Ewen, Wolfgang. 2008. *Carl Stumpf und Gottlob Frege*. Würzburg: Königshausen & Neumann.

Ierna, Carlo. 2009. "Relations in the Early Works of Meinong and Husserl" in *Meinong Studien* III: 7-35.

— 2011. "Brentano and Mathematics" in *Revue Roumaine de Philosophie*, 55/1: 149-167.

Rollinger, Robin. 1999. *Husserl's Position in the School of Brentano* (Phaenomenologica 150). Dordrecht: Kluwer.

— 2009. "Brentano's Logic and Marty's Early Philosophy of Language" in *Brentano Studien* XII: 77-98.

Schuhmann, Karl. 1996. "Carl Stumpf (1848-1936)", in Albertazzi, Liliana, Massimo Libardi, and Roberto Poli (eds.) *The School of Franz Brentano*, Dordrecht, Kluwer.

Smith, Barry. 1994. *Austrian Philosophy: The Legacy of Franz Brentano*. Chicago: Open Court.

Sprung, Helga, and Lothar Sprung. 2002. "Carl Stumpf in Berlin (1894-1936)" in *Brentano Studien (2000/2001)*, IX: 89-116.