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General directionality and the local behavior of argumentation semantics

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Abstract. In abstract argumentation, the directionality principle conveys the intuition that, for an unattacked set, the choice of arguments that are part of an extension should only depend on the restriction of the framework to that set. Furthermore, having made such a choice, one should be able to select arguments from the rest of the framework so as to get an extension. In this paper we show how this idea can be generalized and used for formulating SCC-recursiveness as a stronger version of directionality.

We argue that such properties characterize the information that is needed for computing the extensions of an argumentation semantics. We provide a formal approach for describing and comparing directionality-like properties. Our model provides a clear distinction between SCC-recursive semantics that use defense information and those that do not use it.

Keywords: Argumentation frameworks, argumentation semantics, SCC-recursiveness, directionality

1. Introduction

This work lies in the general setting of abstract argumentation frameworks, as proposed by Dung [6] and deals with the characterization of argumentation semantics with respect to the local computation of extensions.

Non-interference and directionality were proposed as desirable properties of argumentation semantics, conveying the idea that for some sets of arguments the selection of arguments for an extension should not depend on the rest of the framework [1,2].

SCC-recursiveness was introduced in [3] as a powerful schema for characterizing argumentation semantics with respect to the decomposition of the argumentation framework into strongly connected components (SCCs). The approach relies on the idea that the arguments selected from an SCC may only depend on arguments selected from ancestor SCCs.

In this paper we introduce strongly connected sets (SCSs) as a generalization of SCCs and we provide a stronger characterization of SCC-recursiveness using these sets. Furthermore, we propose SCS-directionality as a natural refinement of directionality.

We also introduce general directionality as a very broad formal approach for describing the behavior of argumentation semantics with respect to local computation of extensions and the information that needs to be available from the rest of the framework. This paper is an extension and refinement of ideas presented in [10].

Section 2 introduces the argumentation concepts we are going to use. We generalize SCC-recursiveness and introduce SCS-directionality, together with its properties, in Section 3. The general directionality is presented in Section 4, with an example that outlines the added value of our approach. The paper ends with conclusions and ideas for future research in Section 5.

2. Background

In this section we aim to provide a minimal argumentation background, covering three aspects that are relevant to our work: argumentation semantics, SCC-recursiveness and properties of semantics. We start with a formal definition of argumentation frameworks and the related terminology.

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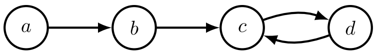

Fig. 1. Example argumentation framework.

Definition 1. An *argumentation framework* is a pair $F = (\mathcal{A}, \mathcal{R})$, where $\mathcal{A}$ is a set of arguments and $\mathcal{R} \subseteq \mathcal{A} \times \mathcal{A}$ is a binary attack relation on $\mathcal{A}$. Whenever $(a, b) \in \mathcal{R}$ we say that argument a *attacks* argument b and we write this as $a \to b$. We say that a set of arguments $S \subseteq \mathcal{A}$ *defends* an argument a iff S attacks all arguments b that attack a. We extend the notion of attack to also refer to sets of arguments, as follows:

$$\begin{aligned} a \to S &\iff \exists b(b \in S \wedge a \to b), \\ S \to a &\iff \exists b(b \in S \wedge b \to a), \\ S \to T &\iff \exists ab(a \in S \wedge b \in T \wedge a \to b). \end{aligned} \tag{1}$$

For a given argumentation framework $F = (\mathcal{A}, \mathcal{R})$ and a set of arguments $S \subseteq \mathcal{A}$, the *restriction* of F to S is the argumentation framework

$$F{\downarrow_S} = \bigl(S, \mathcal{R} \cap (S \times S)\bigr). \tag{2}$$

For example, the argumentation framework depicted in Fig. 1 is given by $\mathcal{A} = \{a, b, c, d\}$ and $\mathcal{R} = \{(a, b), (b, c), (c, d), (d, c)\}$. The restriction of F to $\{a, b, c\}$ is $F{\downarrow_{\{a,b,c\}}} = (\{a, b, c\}, \{(a, b), (b, c)\})$.

2.1. *Argumentation semantics*

In the argumentation literature, semantics refer to approaches (algorithmic, constraint-based or otherwise) for choosing sets of arguments (extensions) that can be accepted together. We first introduce the argumentation semantics defined in [6], sometimes referred to as the traditional, or classical, semantics.

Definition 2. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $E \subseteq \mathcal{A}$ be a set of arguments.

- E is *conflict-free* iff there is no attack between arguments from E. The set of all conflict-free sets of F is denoted by $\mathcal{E}_{\mathcal{CF}}(F)$.
- E is *admissible* iff E is conflict-free and E defends all the arguments it contains. The set of all admissible sets of F is denoted by $\mathcal{E}_{\mathcal{AS}}(F)$.
- E is a *complete extension* of F iff E is admissible and E contains all the arguments it defends. The set of all complete extensions of F is denoted by $\mathcal{E}_{\mathcal{CO}}(F)$.
- E is a *stable extension* of F iff E is conflict-free and E attacks all arguments that are not in E. The set of all stable extensions of F is denoted by $\mathcal{E}_{\mathcal{ST}}(F)$.
- E is a *preferred extension* of F iff E is a maximal (with respect to set inclusion) admissible set of F. The set of all preferred extensions of F is denoted by $\mathcal{E}_{\mathcal{PR}}(F)$.
- E is the *grounded extension* of F iff E is the (unique) minimal complete extension (with respect to set inclusion). The (singleton) set of grounded extensions is denoted by $\mathcal{E}_{\mathcal{GR}}(F)$.

The name of the sets (or extensions) presented in Definition 2 corresponds to the name of the respective argumentation semantics. Furthermore, whenever the extensions of a particular semantics are denoted by $\mathcal{E}_{\mathcal{S}em}(F)$, it means that $\mathcal{S}em$ is used as an abbreviation for the corresponding semantics. For example, $\mathcal{CO}$ stands for the complete semantics.

For the framework from Fig. 1, the semantics introduced in Definition 2 give the following extensions:

$$\begin{aligned} \mathcal{E}_{\mathcal{CF}}(F) &= \bigl\{\emptyset, \{a\}, \{a, c\}, \{a, d\}, \{b\}, \\ &\qquad \{b, d\}, \{c\}, \{d\}\bigr\}, \\ \mathcal{E}_{\mathcal{AS}}(F) &= \bigl\{\emptyset, \{a\}, \{a, c\}, \{a, d\}, \{d\}\bigr\}, \\ \mathcal{E}_{\mathcal{CO}}(F) &= \bigl\{\{a\}, \{a, c\}, \{a, d\}\bigr\}, \\ \mathcal{E}_{\mathcal{ST}}(F) &= \bigl\{\{a, c\}, \{a, d\}\bigr\}, \\ \mathcal{E}_{\mathcal{PR}}(F) &= \bigl\{\{a, c\}, \{a, d\}\bigr\}, \\ \mathcal{E}_{\mathcal{GR}}(F) &= \bigl\{\{a\}\bigr\}. \end{aligned} \tag{3}$$

Let us see, for example, why the set $E = \{a, c\}$ is a complete extension of F. First of all E is conflict-free, because arguments a and c do not attack one another or themselves. Furthermore, a is unattacked, so implicitly defended by E. The two attackers of c, namely b and d, are attacked respectively by a and c, which are both in E. Thus, E defends its elements, so it is an admissible set.

For completeness we need to show that E contains all the arguments it defends or, equivalently, that arguments that are not in E are not defended by E. Indeed, b is attacked by a, and d is attacked by c, while no argument from E attacks either a or c. Thus, we can conclude that E is a complete extension of F.

Several other semantics have been proposed in the literature, such as ideal [7], semi-stable [4], eager [5]. Although such semantics can also be discussed using the approach presented in this paper, we will only focus on the ones already presented, for the sake of conciseness.

2.2. SCC-recursiveness

The general idea behind SCC-recursiveness is to compute semantics taking advantage of the decomposition of the argumentation framework along its strongly connected components (SCCs). We will only provide here the definitions required for introducing the concept. For more details and the rationale behind the idea, the reader may consult [3]. We start with a formal definition for the strongly connected components.

Definition 3. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. We define the *path equivalence relation* $PE_F \subseteq \mathcal{A} \times \mathcal{A}$ as follows:

- $(a, a) \in PE_F$ for all arguments $a \in \mathcal{A}$,
- for any two arguments a and b, $(a, b) \in PE_F$ iff there is a path in $\mathcal{R}$ from a to b and a path from b to a.

PE_F so defined is an equivalence relation and its equivalence classes are called *strongly connected components* (SCCs). We will use $SCCS_F$ to refer to the set of all strongly connected components of F. We will denote the (unique) strongly connected component that contains an argument a with $SCC_F(a)$.

For example, the argumentation framework in Fig. 2 has four strongly connected components: $S_1 = \{a, b, c, d\}$, $S_2 = \{e\}$, $S_3 = \{f\}$, $S_4 = \{g, h\}$. Thus, we have $SCCS_F = \{S_1, S_2, S_3, S_4\}$. We can also write, for example, $SCC_F(a) = S_1$.

Given an argumentation framework F and an extension E, we can partition the elements of any strongly connected component S into three different classes, with respect to how the extension E interacts with them from outside S. These classes are introduced in Definition 4.

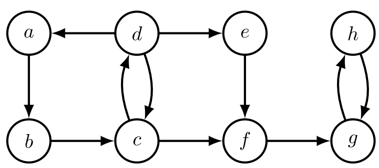

Fig. 2. Argumentation framework consisting of four SCCs.

Definition 4. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework, $E \subseteq \mathcal{A}$ a set of arguments and $S \in SCCS_F$ a strongly connected component of F. The elements of S can be partitioned into three sets with respect to E:

(1) $D_F(S, E) = \{a \in S \mid (E \setminus S) \to a\}$ – the set of arguments that are attacked by E from outside S (defeated arguments),
(2) $U_F(S, E) = \{a \in S \mid (E \setminus S) \not\to a \wedge \forall b(b \in (\mathcal{A} \setminus S) \wedge b \to a \Rightarrow (E \setminus S) \to b)\}$ – the set of arguments not attacked by E from outside S and defended by E against all attackers that are not in S (undefeated arguments),
(3) $P_F(S, E) = S \setminus (D_F(S, E) \cup U_F(S, E))$ – arguments that are neither attacked by E from outside of S, nor defended by E against attacks coming from outside S (provisionally defeated arguments).

As an example, let us look again at the argumentation framework from Fig. 2. Consider the set $E = \{b, d, f, h\}$, which is a complete extension. Let us focus on the strongly connected component $S = S_4 = \{g, h\}$. Note that the three sets introduced in Definition 4 are only concerned with $E \setminus S$. In other words, the classification of the arguments of S into either U, D or P does not depend on whether they are in E or not.

This said, let us see that g is in $D_F(S, E)$, because it is attacked by f and $f \in E \setminus S$. Argument h, on the other hand, is unattacked from outside S, so we have $h \in U_F(S, E)$. To conclude, for the chosen SCC and extension, we have the following partition: $U_F(S, E) = \{h\}$, $D_F(S, E) = \{g\}$ and $P_F(S, E) = \emptyset$.

In what follows, we will use $UP_F(S, E)$ as an abbreviation for $U_F(S, E) \cup P_F(S, E)$, which is the same as $S \setminus D_F(S, E)$. Also note that in the formulas from Definition 4 we have used $\Rightarrow$ for logical implication so that there is no confusion with the attack relation, denoted by $\to$.

Definition 5. An argumentation semantics $\mathcal{S}em$ is said to be *SCC-recursive* iff, for any argumentation framework $F = (\mathcal{A}, \mathcal{R})$, $\mathcal{E}_{\mathcal{S}em}(F) = \mathcal{GF}(F, \mathcal{A})$, where the generic recursive function $\mathcal{GF}$ is defined as follows: for any argumentation framework $F = (\mathcal{A}, \mathcal{R})$ and any two sets of arguments $E, C \subseteq \mathcal{A}$ it holds that $E \in \mathcal{GF}(F, C)$ iff

- in case $|SCCS_F| = 1$, $E \in \mathcal{BF}_{\mathcal{S}em}(F, C)$,

- otherwise, for all strongly connected components $S \in SCCS_F$, we have $(E \cap S) \in \mathcal{GF}(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E))$,

where $\mathcal{BF}_{\mathcal{S}em}$ is a base function that depends on $\mathcal{S}em$.

The idea conveyed in Definition 5 is that the arguments chosen from a strongly connected component S as elements of an extension E are selected based on what has already been selected from other SCCs, more precisely by taking into account which arguments are defended ($U_F(S, E)$) or at least not defeated ($UP_F(S, E)$) by the arguments selected for E from components that attack S.

Since this intuition is quite important for our approach, let us go a bit deeper into it. Note that, in contrast with the semantics we have presented in the previous subsection, both the generic function $\mathcal{GF}$ and the base function $\mathcal{BF}$ take two arguments. The first one, F, stands for the argumentation framework for which the extensions are computed. The second argument, C, denotes the set of arguments (from F) that are defended as far as attacks from outside F are concerned. This makes sense since the two functions can also operate on subframeworks of a given framework. The second argument is useful whenever the desired semantics should produce admissible sets as, whenever computing extensions in a certain SCC, one should be aware which arguments fail to be defended against outer attacks. This is because arguments from within an SCC cannot help defend against such attacks.

Now, in order for a given argumentation semantics $\mathcal{S}em$ to be SCC-recursive, there must exist a base function $\mathcal{BF}_{\mathcal{S}em}$ such that the extensions of $\mathcal{S}em$ can be computed by the generic function $\mathcal{GF}$ using $\mathcal{BF}_{\mathcal{S}em}$. Note that the extensions of $\mathcal{S}em$ are given by $\mathcal{GF}(F, \mathcal{A})$, i.e. at the top level all arguments are presumed defended as far as outer attacks are concerned.

Now let us look into the two properties from Definition 5, as they give hints about the SCC-recursive computation of extensions. First of all, whenever the argumentation framework F consists of a single SCC, the generic function $\mathcal{GF}$ returns the same extensions as the base function $\mathcal{BF}_{\mathcal{S}em}$. If F has several SCCs, then the intersection of any extension E with a particular SCC S can be computed via the generic function:

$$(E \cap S) \in \mathcal{GF}\big(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E)\big).$$

Note that the framework is restricted to the arguments from S that are not already defeated by $E \setminus S$. This is consistent with the fact that already defeated arguments cannot be a part of $E \cap S$, as they would violate conflict-freeness, which is required by all argumentation semantics proposed so far in the literature. The set of defended arguments is also restricted to S and, in addition, filters out arguments that are not defended against attackers from outside S.

What this tells us is that once the elements of an extension E are selected from within all but one SCC S, the elements for $E \cap S$ can be selected locally, based on what $E \setminus S$ defeats or defends in S. The crucial observation is that, in fact, $UP_F(S, E)$ and $U_F(S, E)$ do not really depend on all elements of $E \setminus S$, but only on those in $E \cap \bigcup\{T \mid T \in SCCS_F, T \rightarrow S\}$, i.e. the SCCs that attack S. With this observation, the computation of extensions can be carried out by considering a topological ordering of the SCCs (with unattacked SCCs first) and by choosing arguments from each of them locally. We are going to illustrate this approach with an example after introducing the $\mathcal{CF}2$ semantics.

Based on the previous discussion, we can conclude that SCC-recursiveness emphasizes local criteria for choosing the arguments of extensions. It is shown in [3] that all the classical semantics are SCC-recursive so this approach proves to be an alternative to their original definitions. On the other hand, the same approach can be used for defining new semantics, simply by providing a meaningful choice for the base function $\mathcal{BF}_{\mathcal{S}em}$.

Definition 6. The $\mathcal{CF}2$ semantics [3] is the SCC-recursive semantics given by the following base function:

$$\mathcal{BF}_{\mathcal{CF}2}(F, C) = \mathcal{E}_{\mathcal{MCF}}(F), \tag{4}$$

where $\mathcal{E}_{\mathcal{MCF}}(F)$ stands for the set of maximal (with respect to set inclusion) conflict-free sets, also known as *naive* extensions.

Let us consider the framework from Fig. 3, call it F. The naive extensions of F are $\mathcal{E}_{\mathcal{MCF}}(F) = \{\{a, d\}, \{a, e\}, \{b, c, e\}, \{b, d\}\}$. One way we could

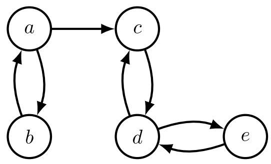

Fig. 3. Argumentation framework to illustrate the $\mathcal{CF}2$ semantics.

compute the $\mathcal{CF}2$ extensions of F is to test every possible set of arguments against the properties required by Definition 5, applied for the base function of $\mathcal{CF}2$. We might also use the fact that all $\mathcal{CF}2$ extensions are also naive extensions [8], so we can test only those. However, this is still a highly inefficient method for larger frameworks.

The more efficient method is the recursive computation of extensions, as suggested in the discussion following Definition 5. First, let us see that F consists of two strongly connected components $S_1 = \{a, b\}$ and $S_2 = \{c, d, e\}$. To compute the $\mathcal{CF}2$ extensions of F, we first have $\mathcal{E}_{\mathcal{CF}2}(F) = \mathcal{CF}2(F, \mathcal{A})$, where $\mathcal{CF}2$ stands for the generic function $\mathcal{GF}$ from Definition 5. Since there are two SCCs, the following must hold for any $\mathcal{CF}2$ extension E:

$$\begin{aligned} &E \cap S_1 \\ &\quad \in \mathcal{CF}2\big(F{\downarrow}_{UP_F(S_1,E)}, \mathcal{A} \cap U_F(S_1, E)\big), \\ &E \cap S_2 \\ &\quad \in \mathcal{CF}2\big(F{\downarrow}_{UP_F(S_2,E)}, \mathcal{A} \cap U_F(S_2, E)\big). \end{aligned} \tag{5}$$

Note that, since S_1 is unattacked, the partition of S_1 with respect to E does not depend on arguments selected from outside S_1. Indeed, we have $UP_F(S_1, E) = S_1$ and $U_F(S_1, E) = S_1$ and, thus, we can write $E \cap S_1 \in \mathcal{CF}2(F{\downarrow}_{S_1}, S_1)$. Since $F{\downarrow}_{S_1}$ consists of a single SCC, we can write further $E \cap S_1 \in \mathcal{E}_{\mathcal{MCF}}(F{\downarrow}_{S_1}) = \{\{a\}, \{b\}\}$. We now need to consider both cases for $E \cap S_1$ and extend them to $\mathcal{CF}2$ extensions of F.

First, $E \cap S_1 = \{a\}$ leads to $UP_F(S_2, E) = U_F(S_2, E) = \{d, e\}$ and, thus, we have that $E \cap S_2 \in \mathcal{CF}2(F{\downarrow}_{\{d,e\}}, \{d, e\}) = \mathcal{E}_{\mathcal{MCF}}(F{\downarrow}_{\{d,e\}}) = \{\{d\}, \{e\}\}$. Hence, we have found the $\mathcal{CF}2$ extensions $\{a, d\}$ and $\{a, e\}$.

In the second case we have $E \cap S_1 = \{b\}$, which leads to $UP_F(S_2, E) = U_F(S_2, E) = S_2$ and, thus, $E \cap S_2 \in \mathcal{CF}2(F{\downarrow}_{S_2}, S_2) = \mathcal{E}_{\mathcal{MCF}}(F{\downarrow}_{S_2}) = \{\{c, e\}, \{d\}\}$. This observation generates another two $\mathcal{CF}2$ extensions, namely $\{b, c, e\}$ and $\{b, d\}$. While for this example all naive extensions turned out to be also $\mathcal{CF}2$ extensions, it is not generally the case.

Note that the base function of the $\mathcal{CF}2$ semantics makes no use of the second argument (the set of arguments defended against attacks from outside). As such, its extensions are generally not admissible sets.

2.3. Properties of argumentation semantics

A principle-based evaluation of argumentation semantics was proposed in [2] and extended in [1]. We only provide formal definitions here for the properties that are relevant for this work.

Definition 7. An argumentation semantics $\mathcal{S}em$ is *universally defined* iff, for any argumentation framework F, $\mathcal{E}_{\mathcal{S}em}(F) \neq \emptyset$.

In words, universally defined argumentation semantics provide at least one extension for any argumentation framework. Of the semantics we have presented here, only $\mathcal{ST}$ is not universally defined [2].

Definition 8. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. A set of arguments $S \subseteq \mathcal{A}$ is *isolated* iff there are no attacks between S and the rest of the framework:

$$\mathcal{R} \cap \big(\big(S \times (\mathcal{A} \setminus S)\big) \cup \big((\mathcal{A} \setminus S) \times S\big)\big) = \emptyset. \tag{6}$$

The set of all isolated sets of F is denoted by $\mathcal{IS}(F)$.

An argumentation semantics $\mathcal{S}em$ is said to satisfy the *non-interference* principle iff, for any argumentation framework $F = (\mathcal{A}, \mathcal{R})$ and every isolated set $S \in \mathcal{IS}(F)$ it holds that:

$$\mathcal{AE}_{\mathcal{S}em}(F, S) = \mathcal{E}_{\mathcal{S}em}(F{\downarrow}_S), \tag{7}$$

where $\mathcal{AE}_{\mathcal{S}em}(F, S) = \{E \cap S \mid E \in \mathcal{E}_{\mathcal{S}em}(F)\}$.

The intuition behind non-interference is that the elements chosen for an extension E from an isolated set S can be computed locally in the restricted framework $F{\downarrow}_S$ by using the same argumentation semantics. Furthermore, for any such locally computed extension, it should be possible to select additional arguments from the rest of the framework so as to get an extension of F. Again, of the semantics we have formally introduced, only $\mathcal{ST}$ fails to satisfy non-interference.

Definition 9. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. A set of arguments S is *unattacked* iff S is not attacked by any argument that is not in S:

$$(\mathcal{A} \setminus S) \not\rightarrow S. \tag{8}$$

The set of all unattacked sets of F is denoted by $\mathcal{US}(F)$.

An argumentation semantics $\mathcal{S}em$ satisfies the *directionality* principle iff, for any argumentation framework $F = (\mathcal{A}, \mathcal{R})$ and any unattacked set $S \in \mathcal{US}(F)$, it holds that:

$$\mathcal{AE}_{\mathcal{S}em}(F, S) = \mathcal{E}_{\mathcal{S}em}(F{\downarrow_S}). \tag{9}$$

Note that directionality and non-interference impose the same constraint, but applied to different kinds of sets. Furthermore, directionality implies non-interference [1]. The stable semantics does not satisfy directionality [2], nor does the naive semantics [8]. The other semantics discussed in this paper do satisfy the property.

If we think of SCC-recursiveness as a property as well, we have already mentioned that all classical semantics satisfy it. Furthermore, $\mathcal{CF}2$ satisfies it by definition. On the other hand, it is shown in [2] that all SCC-recursive semantics that are universally defined satisfy directionality. Since $\mathcal{MCF}$ does not satisfy directionality, it follows that it is not SCC-recursive either. On the other hand, ideal semantics satisfies directionality [2], but is not SCC-recursive [9], so the two properties are indeed distinct. Table 1 summarizes the results we have mentioned in this subsection.

In our work, we were inspired by the similarities that exist between non-interference, directionality and SCC-recursiveness with respect to what they describe as reasonable for the intersection of an extension with a set of arguments.

3. SCS-directionality

We have seen that non-interference and directionality require that the intersection of an extension with an isolated (or unattacked) set S can be computed as

Table 1
Satisfaction of the evaluation principles

Semantics	1	2	3	4
$\mathcal{CF}$	Yes	Yes	Yes	Yes
$\mathcal{AS}$	Yes	Yes	Yes	Yes
$\mathcal{CO}$	Yes	Yes	Yes	Yes
$\mathcal{ST}$	No	No	No	Yes
$\mathcal{PR}$	Yes	Yes	Yes	Yes
$\mathcal{GR}$	Yes	Yes	Yes	Yes
$\mathcal{MCF}$	Yes	Yes	No	No
$\mathcal{CF}2$	Yes	Yes	Yes	Yes

Notes: 1 – universally defined, 2 – non-interference, 3 – directionality, 4 – SCC-recursiveness.

an extension of the restricted framework $F{\downarrow_S}$. In this section we wish to define a new type of sets, one that includes unattacked and isolated sets but also generalizes strongly connected components, then formulate a principle about the intersection of extensions with such sets.

Definition 10. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $S \subseteq \mathcal{A}$ be a set of arguments. S is a *strongly connected set* iff, for all arguments $a \in \mathcal{A}$, if there is an attack path from a to an argument in S and an attack path from a (possibly different) argument of S to a, then a is in S. We will use $\mathcal{SCS}(F)$ to refer to all strongly connected sets of F. More formally, we can write:

$$\begin{aligned}&\mathcal{SCS}(F)\\&\quad = \big\{S \subseteq \mathcal{A} \mid \forall a(a \rightarrow^* S \wedge S \rightarrow^* a\\&\qquad\qquad \Longrightarrow a \in S)\big\},\end{aligned} \tag{10}$$

where $\rightarrow^*$ stands for the reflexive and transitive closure of the attack relation, given by:

- $a \rightarrow^* a$, for all $a \in \mathcal{A}$,
- $a \rightarrow^* b$ iff there exists a path of attacks from a to b

and is extended to sets of arguments as in Definition 1.

Note that strongly connected sets can also be seen as convex sets with respect to attack path accessibility. Furthermore, the path equivalence relation used in Definition 3 can be written as: $(a, b) \in PE_F \Leftrightarrow a \rightarrow^* b \wedge b \rightarrow^* a$. This observation leads to the following result.

Proposition 1. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. Every strongly connected set $S \in \mathcal{SCS}(F)$ can be written as the union of zero or more strongly connected components of F.*

Proof. Let S be an arbitrary strongly connected set of F. Given an argument $a \in S$, for all arguments $b \in SCC_F(a)$ we have that $a \rightarrow^* b$ and $b \rightarrow^* a$, which leads to $b \in S$ and, thus, $SCC_F(a) \subseteq S$. It follows that $S = \bigcup_{a \in S} SCC_F(a)$. $\square$

Note that the converse of Proposition 1 is not always true, i.e. the union of several SCCs is not necessarily an SCS. Indeed, consider the argumentation framework F form Fig. 4. The set $S = \{a, b, d, e\}$, which is the

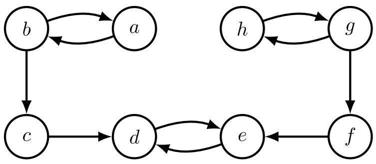

Fig. 4. Argumentation framework for illustrating look-back and look-ahead.

union of the strongly connected components $\{a, b\}$ and $\{d, e\}$, is not a strongly connected set, because $S \to^* c$ and $c \to^* S$, but $c \notin S$.

Next, we introduce several notations and concepts that will help provide another characterization for strongly connected sets, one that gives a better intuition about the actual property of SCCs that is preserved for strongly connected sets.

Definition 11. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $S \subseteq \mathcal{A}$ be a set of arguments. We define the *look-ahead function* $\mathcal{LA}_F$ and the *look back function* $\mathcal{LB}_F$ as follows:

$$\begin{aligned} \mathcal{LA}_F(S) &= S \cup \bigcup_{a\in S, SCC_F(a)\to b} SCC_F(b), \\ \mathcal{LB}_F(S) &= S \cup \bigcup_{a\in S, b\to SCC_F(a)} SCC_F(b). \end{aligned} \tag{11}$$

Intuitively, the look-back function returns the arguments that have an impact on the acceptability of arguments from S with respect to some SCC-recursive semantics. Indeed, note that $\mathcal{LB}_F(S)$ contains all arguments from S, together with all arguments that are part of strongly connected components that attack S. The look-ahead function captures a similar intuition, but looking at SCCs that are attacked by S. In order to obtain a fine grained description of the possible local behavior of argumentation semantics, we will partition the arguments of an argumentation framework with respect to a set S into layers that can reasonably impact the acceptability of arguments from S. We use $\mathcal{LA}_F$ and $\mathcal{LB}_F$ as building blocks for this partitioning.

Definition 12. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $S \subseteq \mathcal{A}$ be a set of arguments. For any integer $n \geqslant 0$, the *level n look-ahead* $\mathcal{LA}_n$ and the *level n look-back* $\mathcal{LB}_n$ are defined as:

$$\begin{aligned} \mathcal{LA}_n(F, S) &= \mathcal{LA}_F^{(n)}(S) \setminus \mathcal{LA}_F^{(n-1)}(S), \\ \mathcal{LB}_n(F, S) &= \mathcal{LB}_F^{(n)}(S) \setminus \mathcal{LB}_F^{(n-1)}(S), \end{aligned} \tag{12}$$

where we have used $f^{(n)}$ to stand for the repeated application of f n times and by convention we have taken the following:

$$\begin{aligned} \mathcal{LA}_F^{(0)}(S) &= \mathcal{LB}_F^{(0)}(S) = \bigcup_{a\in S} SCC_F(a), \\ \mathcal{LA}_F^{(-1)}(S) &= \mathcal{LB}_F^{(-1)}(S) = \emptyset. \end{aligned} \tag{13}$$

Furthermore, we define the *total look-ahead* $\mathcal{LA}^*$ and the *total look-back* $\mathcal{LB}^*$ as follows:

$$\begin{aligned} \mathcal{LA}^*(F, S) &= \bigcup_{n\geqslant 1} \mathcal{LA}_n(F, S), \\ \mathcal{LB}^*(F, S) &= \bigcup_{n\geqslant 1} \mathcal{LB}_n(F, S). \end{aligned} \tag{14}$$

In other words, the level n look-ahead (look-back) consists of the union of all SCCs whose minimum distance from an SCC of an argument from S is equal to n, using forward (backward) attack paths. Let us illustrate these concepts with an example. Let F be the argumentation framework from Fig. 4 and consider the set $S = \{a, b, d\}$. We can write:

$$\begin{aligned} &\mathcal{LA}_F^{(0)}(S) = \mathcal{LB}_F^{(0)} = \{a, b, d, e\}, \\ &\mathcal{LA}_F^{(n)}(S) = \{a, b, c, d, e\} \quad \text{for } n \geqslant 1, \\ &\mathcal{LB}_F^{(1)}(F, S) = \{a, b, c, d, e, f\}, \\ &\mathcal{LB}_F^{(n)}(F, S) = \{a, b, c, d, e, f, g, h\} \quad \text{for } n \geqslant 2. \end{aligned} \tag{15}$$

Note that we have $\mathcal{LA}_F^{(0)} \subseteq \mathcal{LA}_F^{(1)} \subseteq \cdots$, and similarly for $\mathcal{LB}_F$. This property is true in general and, among other things, guarantees the existence of $\mathcal{LA}^*$ and $\mathcal{LB}^*$, which can be alternatively written as:

$$\begin{aligned} \mathcal{LA}^*(F, S) &= \bigcup_{n\geqslant 1} \mathcal{LA}_F^{(n)}(S) \setminus \mathcal{LA}_F^{(0)}, \\ \mathcal{LB}^*(F, S) &= \bigcup_{n\geqslant 1} \mathcal{LB}_F^{(n)}(S) \setminus \mathcal{LB}_F^{(0)}. \end{aligned} \tag{16}$$

Proposition 2. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and $S \subseteq \mathcal{A}$ a set of arguments. For any integer $n \geqslant 0$ it holds that*:

$$\begin{aligned} \mathcal{LA}_F^{(n-1)}(S) &\subseteq \mathcal{LA}_F^{(n)}(S), \\ \mathcal{LB}_F^{(n-1)}(S) &\subseteq \mathcal{LB}_F^{(n)}(S). \end{aligned} \tag{17}$$

Proof. For $n = 0$ the property follows from the fact that the empty set is included in any other set. Furthermore, for $n \geqslant 2$, the property follows from Definition 11, as we have $S \subseteq \mathcal{LA}_F(S)$ and similarly for $\mathcal{LB}_F$. The interesting case is $n = 1$. We consider only the case of $\mathcal{LA}_F$, the proof for $\mathcal{LB}_F$ is very similar. We need to show that:

$$\bigcup_{a \in S} SCC_F(a) \subseteq S \cup \bigcup_{a \in S, SCC_F(a) \to b} SCC_F(b). \tag{18}$$

Let c be an argument such that $c \in SCC_F(a)$ for some $a \in S$. If $c \in S$, there is nothing to prove. If $c \notin S$, then there exists an attack path from c to a containing only arguments from $SCC_F(a)$ and, thus, $SCC_F(a) \to a$, which leads to $c \in \bigcup_{a \in S, SCC_F(a) \to b} SCC_F(b)$. This concludes the proof. $\square$

For our working example, the results from (15) lead to:

$$\begin{aligned}
&\mathcal{LA}_0(F, S) = \mathcal{LB}_0(F, S) = \{a, b, d, e\},\\
&\mathcal{LA}_1(F, S) = \{c\},\\
&\mathcal{LA}_n(F, S) = \emptyset \quad \text{for } n \geqslant 2,\\
&\mathcal{LA}^*(F, S) = \{c\},\\
&\mathcal{LB}_1(F, S) = \{c, f\},\\
&\mathcal{LB}_2(F, S) = \{g, h\},\\
&\mathcal{LB}_n(F, S) = \emptyset \quad \text{for } n \geqslant 3,\\
&\mathcal{LB}^*(F, S) = \{c, f, g, h\}.
\end{aligned} \tag{19}$$

Note that having argument e in the 0th layer is more intuitive than not having it in any of the layers, thus justifying our convention for $n = 0$. We have seen that the inclusion property from Proposition 2 holds for $n = 0$ as well. Let us now see that our choice for $\mathcal{LA}_F^{(0)}$ and $\mathcal{LB}_F^{(0)}$ is also consistent from a computational point of view, i.e. it does not violate the identity $f^{(n+1)} = f \circ f^{(n)}$.

Proposition 3. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and $S \subseteq \mathcal{A}$ a set of arguments. Then the following hold*:

$$\begin{aligned}
\mathcal{LA}_F(S) &= \mathcal{LA}_F\Big(\bigcup_{a \in S} SCC_F(a)\Big),\\
\mathcal{LB}_F(S) &= \mathcal{LB}_F\Big(\bigcup_{a \in S} SCC_F(a)\Big).
\end{aligned} \tag{20}$$

Proof. We will only prove the result for $\mathcal{LA}_F$, the proof for $\mathcal{LB}_F$ is very similar. From Definition 11 it is easy to see that $\mathcal{LA}_F$ is monotonic with respect to set inclusion. Since $S \subseteq \bigcup_{a \in S} SCC_F(S)$, we can conclude that $\mathcal{LA}_F(S) \subseteq \mathcal{LA}_F(\bigcup_{a \in S} SCC_F(a))$. We now need to show that we also have

$$\mathcal{LA}_F\Big(\bigcup_{a \in S} SCC_F(a)\Big) \subseteq \mathcal{LA}_F(S).$$

Let c be an argument in $\mathcal{LA}_F(\bigcup_{a \in S} SCC_F(a))$. If $c \in \bigcup_{a \in S} SCC_F(a)$, then we can use Proposition 2 to conclude that $c \in \mathcal{LA}_F(S)$. Otherwise, it must be that $c \in SCC_F(b)$, for an argument b such that there exists $d \in \bigcup_{a \in S} SCC_F(a)$ with $SCC_F(d) \to b$. Let e be the particular argument in S for which $d \in SCC_F(e)$. Then we have $SCC_F(d) = SCC_F(e)$, so $SCC_F(e) \to b$, which leads to $c \in \bigcup_{a \in S, SCC_F(a) \to b} SCC_F(b)$ and, thus, $c \in \mathcal{LA}_F(S)$. This concludes our proof. $\square$

The purpose of look-back and look-ahead is to provide us with formal means for talking about the (union of the) parent and ancestor, respectively child and descendant strongly connected components of a given SCS. Furthermore, they allow us to define isolated, unattacked and strongly connected sets in a common framework based on SCCs as building blocks.

Proposition 4. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $S \subseteq \mathcal{A}$ be a set of arguments. The following relations hold*:

(a) $S \in \mathcal{IS}(F)$

$$\iff S = \bigcup_{a \in S} SCC_F(a) \wedge \mathcal{LA}^*(F, S) = \mathcal{LB}^*(F, S) = \emptyset,$$

(b) $S \in \mathcal{US}(F)$

$$\iff S = \bigcup_{a \in S} SCC_F(a) \wedge \mathcal{LB}^*(F, S) = \emptyset,$$

(c) $S \in \mathcal{SCS}(F)$

$$\iff \quad S = \bigcup_{a \in S} SCC_F(a) \wedge \mathcal{LA}^*(F, S) \cap \mathcal{LB}^*(F, S) = \emptyset.$$

Proof. (a) For the direct implication, note that the set S is isolated, hence any attacker a of S is in S. Furthermore, the strongly connected component containing a is included in S, leading to $\mathcal{LA}_1(F, S) = \mathcal{LB}_1(F, S) = \emptyset$, which leads to both $S = \bigcup_{a \in S} SCC_F(a)$ and $\mathcal{LA}^*(F, S) = \mathcal{LB}^*(F, S) = \emptyset$. For the converse, suppose that the right hand side holds, but S is not isolated. Then there must exist some argument a not in S such that either $S \to a$ or $a \to S$, so either $a \in \mathcal{LA}_F(S)$ or $a \in \mathcal{LB}_F(S)$. Furthermore, since $a \notin S = \bigcup_{a \in S} SCC_F(a) = \mathcal{LA}_F^{(0)}(S) = \mathcal{LB}_F^{(0)}(S)$, this would lead to either $\emptyset \neq SCC_F(a) \subseteq \mathcal{LA}_1(F, S) \subseteq \mathcal{LA}^*(F, S)$ or $\emptyset \neq SCC_F(a) \subseteq \mathcal{LB}_1(F, S) \subseteq \mathcal{LB}^*(F, S)$, in both cases contradicting the assumption we made.

(b) Similar to (a).

(c) For the forward direction, we can use Proposition 1 to get $S = \bigcup_{a \in S} SCC_F(a)$. Furthermore, assume that there exists an argument $a \in \mathcal{LA}^*(F, S) \cap \mathcal{LB}^*(F, S)$. Then $S \to^* a$ and $a \to^* S$ and, since S is strongly connected, $a \in S$. But S is disjoint from the total look-back and look-ahead.

For the converse, note that an argument a that satisfies $S \to^* a$ but is not in S must be in $\mathcal{LA}^*(F, S)$, since we already know that $\bigcup_{a \in S} SCC_F(a) \setminus S = \emptyset$. Similarly, if a also indirectly attacks S, then a is in $\mathcal{LB}^*(F, S)$. Given the right hand side of the relation, such an argument a cannot exist, so S is a strongly connected set. □

The advantage of using look-back and look-ahead consists in clustering the attackers of S and the arguments attacked by S into strongly connected components and grouping them according to their distance from S.

Note that only one-way attacks are possible between a strongly connected set S and a strongly connected component T that is not included in S. Indeed, suppose that $S \to T$ and consider $a \in T$ such that $S \to a$. Suppose that also $T \to S$. Then it must be that $a \to^* S$. Coupled with $S \to a$, this leads to $a \in S$ and $T \cap S \neq \emptyset$, which in turn gives $T \subseteq S$, which is a contradiction. On the other hand, the property does not necessarily hold for two SCSs. Indeed, consider again the argumentation framework from Fig. 4 and the sets $S_1 = \{a, b, f\}$ and $S_2 = \{c, g, h\}$. It is easy to see that S_1 and S_2 are strongly connected sets that satisfy both $S_1 \to S_2$ and $S_2 \to S_1$.

Corollary 1. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. All isolated sets of F are also unattacked and all unattacked sets of F are also strongly connected sets in F:*

$$\mathcal{IS}(F) \subseteq \mathcal{US}(F) \subseteq \mathcal{SCS}(F). \tag{21}$$

Note that the minimal set that contains a given set of arguments S and is the union of zero or more SCCs is $\mathcal{LA}_F^{(0)}(S) = \mathcal{LB}_F^{(0)}(S)$. Furthermore, the minimal unattacked set that contains S is $\mathcal{LB}_F^{(0)}(S) \cup \mathcal{LB}^*(F, S) = \bigcup_{n \geqslant 0} \mathcal{LB}_F^{(n)}(S)$. Similarly, the minimal SCS that contains S is given by $\mathcal{LB}_F^{(0)}(S) \cup \mathcal{LB}^*(F, S) \cup \mathcal{LA}^*(F, S)$.

Proposition 5. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $S \in \mathcal{SCS}(F)$ be a strongly connected set in F. Then, for any set of arguments $T \subseteq \mathcal{A}$, it holds that S intersects T at a strongly connected set of the restricted framework $F{\downarrow_T}$:*

$$S \cap T \in \mathcal{SCS}(F{\downarrow_T}). \tag{22}$$

Proof. Let a be an argument from $F{\downarrow_T}$ (i.e. $a \in T$) such that $a \to^* (S \cap T)$ and $(S \cap T) \to^* a$ in $F{\downarrow_T}$. Then the same relations must also hold in the unrestricted framework F. From this we also get that $S \to^* a$ and $a \to^* S$ in F. But S is an SCS, so we must have $a \in S$. Coupled with $a \in T$, this leads to $a \in S \cap T$, which proves that $S \cap T \in \mathcal{SCS}(F{\downarrow_T})$. □

We are now ready to prove one of the main results of this paper, the generalization of SCC-recursiveness.

Theorem 1. *Let $\mathcal{GF}$ be the generic function of an SCC-recursive argumentation semantics. Then, for any argumentation framework $F = (\mathcal{A}, \mathcal{R})$ and for any partition of $\mathcal{A}$ into disjoint strongly connected sets $\mathcal{P}art = \{S_1, S_2, \ldots, S_n\}$, the following holds:*

$$E \in \mathcal{GF}(F, C) \iff \forall S\big(S \in \mathcal{P}art \implies (E \cap S) \in \mathcal{GF}\big(F{\downarrow_{UP_F(S,E)}}, C \cap U_F(S, E)\big)\big). \tag{23}$$

Proof. We know that the result holds if the partition consists of all the SCCs of F. What we need to show is that it holds for other partitions as well. We proceed by induction on the size of F. For frameworks with a single argument there will be a single SCC and a single partition into strongly connected sets, so the result holds trivially.

For the induction step, we assume that the claim is true for all frameworks that have fewer arguments than F and we prove it for F. We start with the converse. Let us see that if $\mathcal{P}art = \{\mathcal{A}\}$ the claim trivially holds. In what follows we focus on partitions containing at least two distinct strongly connected sets. We must prove that if $E \cap S \in \mathcal{GF}(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E))$ for all $S \in \mathcal{P}art$, then $E \in \mathcal{GF}(F, C)$.

Let $S \in \mathcal{P}art$ be an arbitrary SCS. Then $E \cap S \in \mathcal{GF}(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E))$. We know from Proposition 1 that S is the union of one or more SCCs of F. For any such component T, we have that $T' = T \cap UP_F(S, E)$ is a strongly connected set in $F{\downarrow}_{UP_F(S,E)}$ (from Proposition 5), so we can apply the induction hypothesis for $F' = F{\downarrow}_{UP_F(S,E)}$ and the extension $E' = E \cap S$, since F' has strictly fewer arguments than F. We get that $E' \cap T' \in \mathcal{GF}(F'{\downarrow}_{UP_{F'}(T',E')}, C' \cap U_{F'}(T', E'))$, where $C' = C \cap U_F(S, E)$. Our goal is to deduce that $E \cap T \in \mathcal{GF}(F{\downarrow}_{UP_F(T,E)}, C \cap U_F(T, E))$.

Since $E \cap S \subseteq UP_F(S, E)$ and $T \subseteq S$, we can write $(E \cap S) \cap (T \cap UP_F(S, E)) = E \cap T$, which means that $E' \cap T' = E \cap T$.

Furthermore, let us see that

$$UP_{F'}\big(T', E'\big) = UP_F(T, E).$$

Indeed, for any argument $a \in \mathcal{A}$, we have $a \in UP_{F'}(T', E') \Leftrightarrow a \in T' \wedge (E' \setminus T') \not\to a \Leftrightarrow a \in T \wedge a \in UP_F(S, E) \wedge ((E \cap S) \setminus (T \cap UP_F(S, E))) \not\to a \Leftrightarrow a \in T \wedge (E \setminus S) \not\to a \wedge ((E \cap S) \setminus T) \not\to a \Leftrightarrow a \in T \wedge (E \setminus T) \not\to a \Leftrightarrow a \in UP_F(T, E)$. We have relied on $(E \cap S) \setminus (T \cap UP_F(S, E)) = (E \cap S) \setminus T$, which follows from $UP_F(S, E) \subseteq S$.

Similarly, we show that $C' \cap U_{F'}(T', E') = C \cap U_F(T, E)$. We do this by proving that $U_F(S, E) \cap U_{F'}(T \cap UP_F(S, E), E \cap S) = U_F(T, E)$ by double inclusion. We start with $\subseteq$. Let a be an argument from $U_F(S, E) \cap U_{F'}(T \cap UP_F(S, E), E \cap S)$ and let b be an attacker of a, with $b \in \mathcal{A} \setminus T$. If $b \notin S$, then $(E \setminus S) \to b$, because $a \in U_F(S, E)$. If $b \in S$, we have either $b \in D_F(S, E)$, in which case $(E \setminus S) \to b$, or $b \in UP_F(S, E)$ and then $(E \cap S) \setminus (T \cap UP_F(S, E)) \to b$ because $a \in U_{F'}(T \cap UP_F(S, E), E \cap S)$. In all three cases we can infer $(E \setminus T) \to b$. Thus, we conclude that $a \in U_F(T, E)$.

For the other inclusion, let us consider $a \in U_F(T, E)$ and b an attacker of a. It follows that $(E \setminus T) \to b$. If $b \in UP_F(S, E) \setminus T$ (b is an attacker of a in F' as well), then $b \notin D_F(S, E)$, which means that $(E \setminus S) \not\to b$. But since $(E \setminus T) \to b$, it must be that $((E \cap S) \setminus T) \to b$, so we can conclude that $a \in U_{F'}(T \cap UP_F(S, E), E \cap S)$.

If $b \in \mathcal{A} \setminus S$ we still have $(E \setminus T) \to b$, but what we need for $a \in U_F(S, E)$ is $(E \setminus S) \to b$. Suppose that this is not the case, i.e. there exists $c \in (E \cap S) \setminus T$ such that $c \to b$. Since both c and a are in S, we have $S \to^* b$ and $b \to^* S$ so we must have $b \in S$ because S is strongly connected. However, this contradicts the assumption that $b \in \mathcal{A} \setminus S$. Thus, we have shown that $a \in UP_F(S, E)$. Note that this is the point in our proof where it is crucial that S is a strongly connected set.

Putting it all together, we got

$$E \cap T \in \mathcal{GF}\big(F{\downarrow}_{UP_F(T,E)}, C \cap U_F(T, E)\big).$$

And this holds for every strongly connected set $S \in \mathcal{P}art$ and for every strongly connected component $T \subseteq S$. Since $\mathcal{P}art$ is a partition, this covers all possible SCCs of F so, based on the SCC-recursiveness of $\mathcal{S}em$, we can conclude that $E \in \mathcal{GF}(F, C)$, as desired.

We now discuss the direct implication. So we have $E \in \mathcal{GF}(F, C)$. Just as before, we consider $S \in \mathcal{P}art$ and $T \in SCCS_F$ such that $T \subseteq S$. From SCC-recursiveness we have $E \cap T \in \mathcal{GF}(F{\downarrow}_{UP_F(T,E)}, C \cap U_F(T, E))$. We now use the same equalities and notations as before in order to obtain $E' \cap T' \in \mathcal{GF}(F'{\downarrow}_{UP_{F'}}(T', E'), C' \cap U_{F'}(T', E'))$. Since F' has fewer arguments than F, we use the induction hypothesis and we get $E' = E \cap S \in \mathcal{GF}(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E))$. This completes our proof. □

The very strong result covered by Theorem 1 reveals that we do not need to split a framework along its SCCs, it is enough to choose any partition into strongly connected sets in order to get the same property as that required by SCC-recursiveness. On a more practical note, this also means that whenever we put two frameworks together and only add attacks from one of them towards the other, we can compute the extensions of the union framework using the SCC-recursive approach. The theorem also has two immediate corollaries.

Corollary 2. *Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $\mathcal{GF}$ be the generic function of* *an SCC-recursive argumentation semantics. For every* $E, C \subseteq \mathcal{A}$, *the following relation holds*:

$$E \in \mathcal{GF}(F, C) \iff \forall S\big(S \in \mathcal{SCS}(F) \implies (E \cap S) \in \mathcal{GF}\big(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E)\big)\big). \tag{24}$$

Proof. This result follows from the fact that for every strongly connected set S we can devise an SCS partition $\mathcal{P}art$ that contains S. For example, we can partition $\mathcal{A} \setminus S$ into strongly connected components (which are also strongly connected sets). □

Corollary 3. *Let* $F = (\mathcal{A}, \mathcal{R})$ *be an argumentation framework and let* $\mathcal{S}em$ *be an SCC-recursive argumentation semantics whose generic function does not use its second argument. Then, for every set of arguments* E, *we have*:

$$E \in \mathcal{E}_{\mathcal{S}em}(F) \iff \forall S\big(S \in \mathcal{SCS}(F) \implies (E \cap S) \in \mathcal{E}_{\mathcal{S}em}(F{\downarrow}_{UP_F(S,E)})\big). \tag{25}$$

Let us see that the result from Corollary 3 is somewhat similar to the condition that characterizes both non-interference and directionality, in the sense that the same semantics is applied for a restricted framework. Based on this similarity, we formalize the idea of SCS-directionality.

On the other hand note that, while both non-interference and directionality state that the local computation of an extension (its intersection with an isolated or unattacked set) should be computable independently of the rest of the framework, the core intuition behind SCC-recursiveness is quite different. What we need to say for strongly connected sets is that the local computation does depend on the computation performed for another, well defined, set of arguments and, furthermore, the dependence is reasonable.

Definition 13. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework. An argumentation semantics $\mathcal{S}em$ satisfies the *SCS-directionality* principle iff, for any strongly connected set $S \in \mathcal{SCS}(F)$ and any $T \in \mathcal{AE}_{\mathcal{S}em}(F, \mathcal{LB}^*(F, S))$, we have

$$\mathcal{BE}_{\mathcal{S}em}\big(F, S, \mathcal{LB}^*(F, S), T\big) = \mathcal{E}_{\mathcal{S}em}(F{\downarrow}_{UP_F(S,T)}), \tag{26}$$

where, for any sets of arguments S, D, T, we define

$$\mathcal{BE}_{\mathcal{S}em}(F, S, D, T) = \big\{E \cap S \mid E \in \mathcal{E}_{\mathcal{S}em}(F) \wedge E \cap D = T\big\}. \tag{27}$$

Recall that $\mathcal{AE}_{\mathcal{S}em}(F, S)$ denotes the possible results of the intersection of S with a $\mathcal{S}em$ extension of F. On the other hand, $\mathcal{BE}_{\mathcal{S}em}(F, S, D, T)$ denotes the possible results of the intersection of S with only a subset of the $\mathcal{S}em$ extensions of F, namely those that intersect a given set D at T. The intuition here is that the extensions taken into account have a fixed behavior with respect to a chosen set D that contains arguments which can meaningfully determine the acceptability of arguments from S. In the definition, this set D is taken to be the total look-back of S in F. Note that the set T is chosen from $\mathcal{AE}_{\mathcal{S}em}(F, \mathcal{LB}^*(F, S))$, so as to ensure that only acceptability values that are compatible with $\mathcal{S}em$ are given to arguments from $D = \mathcal{LB}^*(F, S)$.

Thus, the intuition behind SCS-directionality is that, given a strongly connected set S, the arguments that we can choose from S for an extension E are determined by what we have chosen in E from the total look-back of S. Furthermore, for any meaningful choice T that we can make in $\mathcal{LB}^*(F, S)$, the arguments from S can be selected using the same semantics, but applied to a restricted framework that accounts for T.

Note that the novelty of SCS-directionality lies more in the use of strongly connected sets instead of strongly connected components and in the actual formalization using $\mathcal{BE}$. The total look-back, on the other hand, is in fact equivalent to the union of all ancestor SCCs that are mentioned in [3]. To be even more precise, we can replace total look-back with level 1 look-back in the definition (leading to the union of parent SCCs).

Proposition 6. *Every universally defined SCC-recursive semantics* $\mathcal{S}em$ *whose generic function does not use its second argument satisfies SCC-directionality.*

Proof. Let $F = (\mathcal{A}, \mathcal{R})$ be an argumentation framework and let $S \in \mathcal{SCS}(F)$ be a strongly connected set. Also, let $T \in \mathcal{AE}_{\mathcal{S}em}(F, \mathcal{LB}^*(F, S))$. Given the set $U \in \mathcal{BE}_{\mathcal{S}em}(F, S, \mathcal{LB}^*(F, S), T)$, we have that there

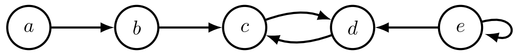

Fig. 5. Argumentation framework showing that several semantics do not satisfy SCC-directionality.

exists $E \in \mathcal{E}_{\mathcal{S}em}(F)$ such that $E \cap \mathcal{LB}^*(F, S) = T$ and $E \cap S = U$. Furthermore, from Corollary 3 we have that $E \cap S \in \mathcal{E}_{\mathcal{S}em}(F\downarrow_{UP_F(S,E)})$. However, let us see that, in fact, $UP_F(S, E) = UP_F(S, E \cap \mathcal{LB}^*(F, S))$, which leads to $UP_F(S, E) = UP_F(S, T)$. Thus, we have proved that $\mathcal{BE}_{\mathcal{S}em}(F, S, \mathcal{LB}^*(F, S), T) \subseteq \mathcal{E}_{\mathcal{S}em}(F\downarrow_{UP_F(S,T)})$.

For the other inclusion, let us consider

$$U \in \mathcal{E}_{\mathcal{S}em}(F\downarrow_{UP_F(S,T)}).$$

We denote $S_1 = S \cup \mathcal{LB}^*(F, S)$ and $S_2 = \mathcal{A} \setminus S_1$ and take the partition $\mathcal{P}art = \{S_1, S_2\}$. Both elements are strongly connected sets and, since $\mathcal{S}em$ is universally defined, we have $\mathcal{E}_{\mathcal{S}em}(F\downarrow_{UP_F(S_2,T\cup U)}) \neq \emptyset$ so we can choose an extension V from this set. but then $T \cup U \cup V$ satisfies the conditions of Theorem 1, which leads to $T \cup U \cup V \in \mathcal{E}_{\mathcal{S}em}(F)$. □

The previous result covers the $\mathcal{CF}$ and $\mathcal{CF}2$ semantics. For the other semantics, let us consider the argumentation framework from Fig. 5 and see that SCC-directionality is violated for $\mathcal{AS}$. We consider $S = \{c, d\}$. Then $\mathcal{LB}^*(F, S) = \{a, b, e\}$. Since $\{a, c\}$ is admissible, we have that $\{a\} \in \mathcal{AE}_{\mathcal{AS}}(F, \mathcal{LB}^*(F, S))$, so we can choose $T = \{a\}$. But then $\mathcal{E}_{\mathcal{AS}}(F\downarrow_{UP_F(S,T)}) = \{\emptyset, \{c\}, \{d\}\}$. However, there is no admissible set that contains d. The proof for the other semantics based on admissibility ($\mathcal{CO}$, $\mathcal{ST}$, $\mathcal{GR}$ and $\mathcal{PR}$) is similar.

Note that this result is not really surprising. It just means that the proposed refinement of directionality is too strong. Among other things, SCS-directionality goes against admissibility. Indeed, SCC-recursive semantics cannot achieve admissibility without the use of the second argument of their respective generic function. With this, we see that too strong decomposition-based properties of argumentation semantics, although intuitive and desirable themselves, may go against more fundamental properties such as admissibility. For a generalization of directionality that can also accommodate admissibility, see the next section.

Furthermore, let us see that, for universally defined semantics, the non-interference and directionality principles are just special cases of the newly introduced SCS-directionality, where the relation is required to hold only for isolated (respectively unattacked) sets. The proof relies on the fact that, for both isolated and unattacked sets, $\mathcal{BE}_{\mathcal{S}em}(F, S, \mathcal{LB}^*(F, S)) = \mathcal{AE}(F, S)$ and $UP_F(S, T) = S$.

4. General directionality

In this section we introduce general directionality, based on the intuition that, for certain sets of arguments S, the intersection of an extension E with S depends on some set of arguments D in the sense that this intersection can be computed locally, using a generic function that works with information compiled from the intersection of E with D.

First, let us consider again non-interference, directionality and SCS-directionality and discuss their building blocks. We have seen in the previous section that the three properties can be seen as instances of the same property but applied to different types of sets of arguments (isolated, unattacked, respectively strongly connected). Thus, in a general framework for describing directionality properties, we should be able to formulate the property with respect to various kinds of sets. In what follows, we will use $\mathcal{S}ets(F)$ to stand for a general function that returns particular sets for a given argumentation framework F.

We have seen that SCS-directionality attempts to characterize the intersection of an extension E with a strongly connected set S in terms of the intersection of the same extension with a particular set of arguments (which depends on S) that can meaningfully determine the acceptability of arguments from S. More formally, we have used $\mathcal{BE}_{\mathcal{S}em}(F, S, \mathcal{LB}^*(F, S), T)$ to stand for the intersection with S of all $\mathcal{S}em$ extensions of F that intersect $\mathcal{LB}^*(F, S)$ at T. We have already seen that we can use $\mathcal{LB}_1$ instead of $\mathcal{LB}^*$ and get an equivalent definition of SCS-directionality. In a more general setting, the determining set need not be based only on look-back. We will use $\mathcal{D}et(F, S)$ for a generic function that gives the proper set of arguments that determines acceptability of arguments from S. It is implied that $\mathcal{D}et(F, S)$ is disjoint from S.

The property conveyed by SCS-directionality is the fact that the intersection of $\mathcal{S}em$ extensions with a strongly connected set S, as discussed above, can be computed locally using the same semantics on some restricted argumentation framework. As we have seen, enforcing such a strong constraint can go against fundamental properties of argumentation semantics, such as admissibility. This happens because more information is needed for computing an admissible extension locally. To accommodate for this, we rely on the general idea behind SCC-recursiveness, where instead of using the same semantics, a generic function is used and, moreover, the generic function can use additional information (the set of arguments defended against attacks from outside).

The expression $\mathcal{GF}(F{\downarrow}_{UP_F(S,E)}, C \cap U_F(S, E))$ that we have seen for SCC-recursiveness can be read in several ways. We can assume that $\mathcal{GF}$ operates on the reduced framework $F{\downarrow}_{UP_F(S,E)}$ and uses as additional information the set of defended arguments $U_F(S, E)$. This is the most intuitive and also the most specific interpretation. A more general interpretation is to consider that $\mathcal{GF}$ operates on $F{\downarrow}_S$ and uses as additional information both the set of defended arguments $U_F(S, E)$ and the set of defeated arguments $D_F(S, E)$. Based on this intuition, we will consider the general case where $\mathcal{GF}$ operates on some restriction of F, given by $\mathcal{L}ocal(F, S)$ and uses additional information given by some function $\mathcal{I}nfo$. The most general reasonable constraint for this additional information is that it should be somehow computable locally using just arguments from $\mathcal{L}ocal(F, S)$ and possibly from $\mathcal{D}et(F, S)$ as well as the actual intersection T of E with $\mathcal{D}et(F, S)$.

All the intuition presented so far is formalized in Definition 14.

Definition 14. An argumentation semantics $\mathcal{S}em$ is said to satisfy *general directionality with signature* $\Delta(\mathcal{S}ets, \mathcal{D}et, \mathcal{L}ocal, \mathcal{I}nfo)$ iff, for any argumentation framework $F = (\mathcal{A}, \mathcal{R})$, any set $S \in \mathcal{S}ets(F)$ and any set $T \in \mathcal{AE}_{\mathcal{S}em}(F, \mathcal{D}et(F, S))$, the following holds:

$$\mathcal{BE}_{\mathcal{S}em}\big(F, S, \mathcal{D}et(F, S), T\big) = \mathcal{GF}_{\mathcal{S}em}\big(F{\downarrow}_{\mathcal{L}ocal(F,S)}, \mathcal{I}nfo(F{\downarrow}_{\mathcal{D}et(F,S)\cup\mathcal{L}ocal(F,S)}, S, T)\big), \tag{28}$$

where:

- $\mathcal{GF}_{\mathcal{S}em}$ is a function that depends on $\mathcal{S}em$ (and on the signature of the general directionality).
- $\mathcal{S}ets(F)$ returns the sets of arguments S for which the relation holds. For example, $\mathcal{S}ets \in \{\mathcal{IS}, \mathcal{US}, \mathcal{SCS}\}$.
- $\mathcal{D}et(F, S)$ returns the set of arguments that reasonably influences the intersection of extensions with S; it must satisfy $\mathcal{D}et(F, S) \cap S = \emptyset$. For example, $\mathcal{D}et \in \{\mathcal{LB}_n, \mathcal{LB}^*, \mathcal{LA}_n, \mathcal{LA}^*, \mathcal{LB}_1 \cup \mathcal{LA}_1\}$.
- $\mathcal{L}ocal(F, S)$ gives the set that can be used for restricting the framework that is available to $\mathcal{GF}$; by convention, we use $\mathcal{LS}(F, S) = S$ and $\mathcal{A}ll(F, S) = \mathcal{A}$. For example $\mathcal{L}ocal \in \{\mathcal{LS}, \mathcal{LS} \cup \mathcal{LB}_1\}$.
- $\mathcal{I}nfo(F, S, T)$ encodes the knowledge of T into information that is included in $\mathcal{L}ocal(F, S)$; based on the way we used it in the definition, this information should be computable in the restriction of F to $\mathcal{L}ocal(F, S) \cup \mathcal{D}et(F, S)$. Furthermore, we define the following possible values for $\mathcal{I}nfo$: $\mathcal{D}(F, S, T) = D_F(S, T)$, $\mathcal{U}(F, S, T) = U_F(S, T)$, $\mathcal{S}(F, S, T) = S$ and $\mathcal{T}(F, S, T) = T$.

Proposition 7. *Any argumentation semantics satisfies* $\Delta(\mathcal{S}ets, \mathcal{D}et, \mathcal{A}ll, (\mathcal{S}, \mathcal{T}, \mathcal{D}et))$*-directionality, for any* $\mathcal{S}ets$ *and* $\mathcal{D}et$.

Proof. We can take $\mathcal{GF}_{\mathcal{S}em}(F, (S, T, \mathcal{D}et(F, S))) = \mathcal{BE}_{\mathcal{S}em}(F, S, \mathcal{D}et(F, S), T)$. □

Proposition 8. *For universally defined semantics, SCS-directionality is equivalent to* $\Delta(\mathcal{SCS}, \mathcal{LB}^*, \mathcal{LS}, (\mathcal{S}, \mathcal{D}))$*-directionality.*

Proof. The result follows from $UP_F(S, T) = S \setminus D_F(S, T) = S \setminus D_{F{\downarrow}_{S\cup\mathcal{LB}^*(F,S)}}(S, T)$. □

Furthermore, it is easy to see that we can get the equivalent characterization of non-interference and directionality in the general directionality setting by replacing $\mathcal{SCS}$ with $\mathcal{IS}$, respectively $\mathcal{US}$ in the signature.

We also provide the following result, without proof:

Proposition 9. *Any SCC-recursive argumentation semantics* $\mathcal{S}em$ *that is universally defined satisfies* $\Delta(\mathcal{SCS}, \mathcal{LB}^*, \mathcal{LS}, (\mathcal{D}, \mathcal{U}))$*-directionality.*

If we note that the actual computation of D_F and U_F only depends on the parent SCCs of S, we can refine the results in Proposition 8 and Proposition 9 by replacing $\mathcal{LB}^*$ with $\mathcal{LB}_1$.

If we compare SCC-recursiveness with respect to the use of the second argument (the defense information), we can see that our model for describing directionality properties clearly shows the distinction between the two classes of SCC-recursiveness, namely the defense information $\mathcal{U}$. Furthermore, the defeat information, which is somewhat hidden in the original definition of SCC-recursiveness, is present in our model.

All the results presented so far were based on the look-back information. A simple example that uses forward information can be given for conflict-free sets:

Proposition 10. *The* $\mathcal{CF}$ *semantics satisfies directionality with signature* $\Delta(\mathcal{SCS}, \mathcal{LA}_1, \mathcal{LS}, (\mathcal{D}'))$, *where* $\mathcal{D}'(F, S, T) = \{a \in S \mid a \to (T \setminus S)\}$.

Proof. Conflict-free sets are independent of the direction of the attacks. If all attacks are reversed, the meaning of forward and backward information interchanges, while the defeat information $\mathcal{D}$ becomes $\mathcal{D}'$. $\square$

In future work we will attempt to provide similar look-ahead characterizations for other semantics as well.

5. Conclusions and future work

The first contribution of this paper is the generalization of strongly connected components as strongly connected sets and their use for a stronger version of SCC-recursiveness. We have also turned the simple SCC-recursiveness (without defense information) into a directionality-like property that refines plain directionality.

Based on the similarities and differences between non-interference, directionality and SCC-recursiveness, we have provided in this paper a unifying model that can capture these properties and at the same time provide the means for comparing such properties with respect to various aspects.

The fact that all argumentation semantics satisfy some (trivial) form of general directionality suggests the possibility of searching for "minimal" kinds of directionality that are satisfied by each semantics. For SCC-recursiveness we have seen already that ancestor information can be replaced with just parent information.

Future work will explore the relations between various kinds of general directionality. We will also give more attention to the look-ahead information, since it might be relevant for argumentation semantics such as stage or semi-stable.

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References

[1] P. Baroni, M. Caminada and M. Giacomin, An introduction to argumentation semantics, *The Knowledge Engineering Review* **26**(4) (2011), 365–410.

[2] P. Baroni and M. Giacomin, On principle-based evaluation of extension-based argumentation semantics, *Artificial Intelligence* **171**(10–15) (2007), 675–700.

[3] P. Baroni, M. Giacomin and G. Guida, SCC-recursiveness: A general schema for argumentation semantics, *Artificial Intelligence* **168**(1,2) (2005), 162–210.

[4] M. Caminada, Semi-stable semantics, in: *Computational Models of Argument: Proceedings of COMMA 2006*, P.E. Dunne and T.J.M. Bench-Capon, eds, Frontiers in Artificial Intelligence and Applications, Vol. 144, IOS Press, 2006, pp. 121–130.

[5] M. Caminada, Comparing two unique extension semantics for formal argumentation: Ideal and eager, in: *Proceedings of the 19th Benelux Conference on Artificial Intelligence (BNAIC 2007)*, 2007, pp. 81–87.

[6] P.M. Dung, On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and *n*-person games, *Artificial Intelligence* **77**(2) (1995), 321–357.

[7] P.M. Dung, P. Mancarella and F. Toni, Computing ideal sceptical argumentation, *Artificial Intelligence* **171**(10–15) (2007), 642–674.

[8] W. Dvorak and S.A. Gaggl, Incorporating stage semantics in the SCC-recursive schema for argumentation semantics, in: *Proceedings of the 14th International Workshop of Non-Monotonic Reasoning (NMR 2012)*, Rome, Italy, June 2012, 2012.

[9] C. Gratie and A.M. Florea, SCC-recursiveness revisited, in: *9th International Workshop on Argumentation in Multi-Agent Systems (ArgMAS 2012)*, 2012.

[10] C. Gratie, A.M. Florea and J.-J.C. Meyer, General directionality and the local behavior of argumentation semantics, in: *Proceedings of the First International Conference on Agreement Technologies (AT 2012)*, S. Ossowski, F. Toni and G.A. Vouros, eds, Dubrovnik, Croatia, 15–16 October 2012, CEUR Workshop Proceedings, Vol. 918, 2012, pp. 113–127.