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On Right and Wrong Drawings

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Abstract

Are pictorial renderings that deviate from linear perspective necessarily ‘wrong’? Are those in perfect linear perspective necessarily ‘right’? Are wrong depictions in some sense ‘impossible’? Linear perspective is the art of the peep show, making sense only from one fixed position, whereas typical art works are constructed and used more like panel presentations, that leave the vantage point free. In the latter case the viewpoint is free; moreover, a change of viewpoint has only a minor effect on pictorial experience. This phenomenologically important difference can be made explicit and formal, by considering the effects of panning eye movements when perusing scenes, and of changes of viewpoint induced by translations with respect to pictorial surfaces. We present examples from formal geometry, photography, and the visual arts.

Keywords

Perspective, peephole vision, panel vision

1. Introduction

‘Wrong’ in this context apparently means ‘impossible’. Perhaps fortunately, impossibility does not imply non-existence, though. A picture, if understood as ‘*a planar surface covered with pigments in some particular order*’, which is Maurice Denis’ famous definition (1890; Note 1), is a physical object that

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evidently exists. But the mental pictures evoked by looking 'into' such physical pictures are objects of a very different nature. They are obviously *objects*, because you are certain to 'possess' them as part of your reality. They do not exist as physical objects, but in Brentano's (1874 [1995]) terms, they have an *intentional inexistence* (Note 2). Meinong (1899) suggested the term 'subsistence' (Note 3).

Most objects, even very important ones, subsist. Science operates with them, just think of the electron's wave function. Likewise, thought operates exclusively with subsisting objects, or 'symbols'. Such objects come in different varieties. The *golden mountain* subsists, though it is merely counterfactual since it *might* exist as a physical object. But consider the *round square*, which is certainly every bit as round as it is square. It is also an object that subsists, but it is an 'impossible' object in the sense that it does not allow of a physical implementation (Note 4). You cannot comfortably draw one. Pictorial objects doubtlessly subsist, but their possibility is hard to assess: are they like the golden mountain, or like the round square? Many people will readily draw them for you, whereas others will loudly protest that they cannot possibly do so. Opinions differ.

It is sometimes held that only pictures in perfect linear perspective can be 'right'. A picture that fails to comply is 'wrong', or 'impossible', in the sense that it is at odds with some aspects of physical reality. Well known examples are Picasso's (1940s) drawings, where he draws a model as seen from all sides simultaneously (Fig. 1; Notes 5, 6). There are obvious problems with the concept of right and wrong drawings, depending upon the sense in which it is construed.

The conventional explanation takes it for granted that 'right' pictures are thus because they exactly mimic the optical input that an observer would obtain in front of some scene, a claim made mainly but not exclusively by scientists (Gibson, 1950, 1954, 1960, 1971; Gombrich, 1961; Rehkämper, 2003; Ward, 1976). That is to say, such a picture ideally implements perfect *virtual reality*. Instead of a picture on the wall, you might as well look through a window through the wall (Note 7). The purported inference is that the pictorial world evoked by the picture then 'has to be' the same as the visual world evoked by a physical scene beyond that window (Pirenne, 1970). The implicit causal connection here is ill defined. Apparently the pictorial world is supposed to somehow 'inherit' the existential status of this virtual scene. That is perhaps why so many people, especially scientists, remain unshaken in their conviction that linear perspective is the unique way to produce right pictures. Pictures in 'faulty perspective' are to be considered *wrong*, or at best pictures of impossible objects.

We mention such notions because they are what we address in this paper, but we ourselves take them as very muddled, and essentially meaningless (see

Figure 1. A drawing by Picasso from the 1940s. Notice that the model is simultaneously seen from various distinct vantage points. The 'perspective police' (Note 6) would not hesitate to call it 'wrong', even 'impossible', possibly on the basis of a conviction that such women would be singular in our world.

below). From a pragmatic point of view, artists turn out 'wrong' images in large numbers, thus one is faced with a substantial question that requires some sort of answer.

Here are some immediate objections against the above views. *Firstly*, a picture is a physical object, and thus a possible 'scene' itself, that may double as a window. Whatever is in the picture is thus seen in the most vivid virtual reality. In this interpretation any picture is at least a perfectly veridical rendering *of itself*. This is Magritte's famous '*Ceci n'est pas une pipe*' (Gombrich, 1961). *Secondly*, the optical input to the eye fails to specify any unique 'view through a window' (Koenderink *et al.,* 2001). Indeed, there exist infinitely many optically equivalent 'metameric scenes'. The Hollywood movie industry does pretty well for itself by exploiting this fact. The Ames room (Ames, 1952) is a well known instance of such a case. It implies that the term 'veridical' is meaningless in the context of pictures. Of course, this assumes that 'veridicality' is an issue that should concern one, a notion that many artists might disagree with.

The case for linear perspective as a reliable means of generating 'right' pictures would be stronger if more artists had exploited it for that purpose. In fact, historically speaking painters have hardly ever used it in its purest form, not out of ignorance or carelessness, but because in many situations its strict application leads to pictures that look 'wrong'. Initial enthusiasm among fifteenth century Italian artists for the newly published method was soon tempered by the realization that ad hoc modifications were required to avoid perceptual oddities such as the 'column paradox', known to Leonardo da Vinci and Piero della Francesca (Elkins, 1994; Gombrich, 1961). The celestial sphere and globe held by the figures widely thought to be Strabo and Ptolemy in Raphael's *School of Athens* (1509–1511) are notable cases of objects shown in incorrect linear perspective that nevertheless fit a viewer's expectations of how a sphere should look (Derksen, 1999).

Later artists, although versed in the theory and application of linear perspective, chose to refine or even disobey its rules in order to achieve their pictorial ends. When working directly from the landscape in the early nineteenth century John Constable attached a framed glass window to his easel on which, viewing with one static eye, he traced the outline of the scene before him (Kemp, 1990). Some of these traced drawings, which are effectively correct in linear perspective, still exist (Fleming-Williams, 1990). But when they are compared to the oil paintings often made from the same spot on the same day, the spatial structure is very different; the finished works deviate significantly from the sketches made in preparation (Pepperell & Haertel, 2014). Constable's contemporary, Turner, held the esteemed position of Professor of Perspective at the Royal Academy in London for thirty years. Rigorously trained as a topographical and architectural artist in his early career, for which a sound knowledge of linear perspective was a necessity, Turner became highly aware of its limitations. His lecture notes, most of which still exist, discuss in detail the problems of using it, for example, to depict wide-angle views (Davies, 1992). In mature works, such as *Rome from the Vatican* (1820) and *Petworth Park* (1828), he distinctly curves objectively straight lines in order to accommodate a wider expanse of space than would be possible using conventional geometrical methods.

Although students in European art schools continued to be trained in perspective up until the first half of the twentieth century the method was becoming increasingly discredited during the late nineteenth among more progressive artists. In his early career Vincent van Gogh had used a framing device similar to that used by Constable (Van Gogh, 1882). But comparison of his later paintings with photographs taken from the same standpoint suggests he subsequently discarded this practice (Rewald, 1942). As in the case of Constable, works such as *Bedroom in Arles (*1888/1889, see below) deviate substantially from a linear perspective structure. The same pattern can be found in the work of Paul Cézanne (Machotka, 1996), and by the early twentieth century artists were not simply deviating from linear perspective but becoming actively hostile to it. Georges Braque, for example, held it in contempt for the artificiality with which it depicted space, denouncing it in the strongest terms: ‘Scientific perspective is nothing but eye fooling illusionism; it is simply a trick — a bad trick — which makes it impossible for an artist to convey a full experience of space, since it forces the objects in a picture to disappear away from the beholder instead of bringing them within his reach…’ (Richardson, 1964). The case for the ‘rightness’ of linear perspective would not only be stronger if more artists had used it but if fewer great artists had outright rejected it.

In order to introduce the issues considered in this paper we start by discussing three mutually distinct aspects. *The first* involves the formal and conventional properties of linear perspective as a geometrical exercise. *The second* involves the way pictures are presented to viewers, or the way viewers look at pictures. *The third* involves other aspects of the phenomenology of pictorial perception, especially the influence of familiarity.

1.1. The Picture as a Window

‘Linear perspective’ is one way, for there are infinitely many alternative possibilities, to record optical structure on a surface, that is, to render a picture of a scene. We will introduce a few below. It assumes that the optical input is due to a fixed vantage point in a static scene, and that the picture is a planar surface presented such that the viewpoint coincides with the ‘center of perspective’. One way to enforce this is to use a conventional photographic camera (Eder, 1932 [1945]), and view the picture in a ‘view box’ (Zeiss and von Rohr, 1904). The result is close to a virtual reality ‘view through a window’. Indeed, many viewers become aware of a three-dimensional ‘pictorial space’ (Ames, 1925), a mysterious phenomenon known as ‘stereopsis’ (Note 8). It is an excellent method to document physical scenes, and was popular for much of the twentieth century (Note 9). Linear perspective as an artist’s technique involves the painstaking construction of a near photographic picture. It used to be taught in the classical *académies des beaux-arts* (Note 10). The result is frequently denoted ‘photographic’, and in fact it may be hard to differentiate between a monochrome photographic print and a graphite drawing. Such methods are still frequently used by illustrators and designers (Note 11). In the context of this discussion a distinction between photographs and paintings is void, we don’t pursue it.

The Zeiss Verant (Zeiss and von Rohr, 1904), which is essentially a view box honed to perfection, is an appropriate instrument to view pictures because it enforces the geometry of the window view. The optical design is intricate, it effectively removes all physiological cues that might reveal the nature of the picture as a planar surface at a certain distance from the eye (Note 12). The instrument may stand as paradigmatic for ‘right viewing’.

A poor man’s substitute for correct viewing is a large picture at a few meters distance, viewed monocularly from a carefully fixed position, in a context that minimizes cues that reveal the nature of the picture as a physical object. This fits a generic gallery setting. One might naïvely expect that such conditions would be enforced, especially in art museums. But, perhaps surprisingly, this is far from being the case. In fact, we cannot remember having visited a museum that has the correct viewpoint even *indicated*. Small wonder, because that would be highly inconvenient, since only one person at a time could then view a particular picture (Note 13). For museums it doesn't pay to have customers line up for every picture. Indeed, one frequently sees *groups* of people discuss a picture. Does any member of the group see it 'correctly'? Certainly not all, most probably none. Typically, the viewing conditions are left completely free.

When viewing pictures in a living room setting, most observers don't care much about viewing conditions either — except for proper lighting perhaps. In this case the pictures will typically be smallish, say postcard or magazine size, and seen from convenient reading distance. Few people feel committed to close an eye when viewing pictures, nor are they likely to use a standard geometry such as a viewing frame. They rather hold the picture, or magazine, with their hands, rarely bothering to get the spatial attitude or the distance 'right'. They have simultaneous binocular, and haptic information as to the location and shape of the pictures. The very notion of a 'virtual window view' would hardly occur to them.

Of course, conditions are different for the artist and the observer. The artist views the work at the original size and from an advantageous viewpoint. However, pen drawings made for reproduction are drawn at enlarged size, designs for theatrical backdrops at reduced size, and so forth. The vantage point of the artist is usually free though sometimes forced (e.g., when using a camera lucida) and is usually different when working on the canvas or stepping back and perhaps closing an eye. In this paper we concentrate on the observer side.

Thus the conditions required for the right viewing of perspective pictures are very seldom met outside of the laboratory. Yet these are an essential part of the very concept of linear perspective. Apparently users don't care about this. Nor do contemporary artists, who typically don't attempt to hide the physical nature of a work. One of the defining attributes of Modernist painting was that it gave great prominence to the surface and handling of the paint, which often became the 'subject' of the painting as much as the objects being depicted. They let the canvas structure be visible, they use very obvious 'touches', and so forth (Note 14). They often dislike frames that would isolate a work from its environment (Note 15). This goes against the grain of the 'picture as a window' concept.

1.2. The Beholder's Share

The phenomenology of pictorial perception can hardly be neglected in this discussion, nor can the phenomenology of visual perception in physical scenes. The two are very different because you can *physically* move through physical space (Gibson, 1950; Koenderink and van Doorn, 1975, 1987, 1991), whereas you can only *mentally* move in pictorial space (Koenderink *et al.*, 2001). Thus, a portrait in *en face* pose looks straight at you from wherever you are with respect to the picture, whereas you can gain both *en face* and *en profil* views of an actual person (Pirenne, 1970). Again, you see a circle, such as the rim of a soup plate, as indeed circular from almost any position in the room, and you understand an elliptical outline in a picture as 'the image of a circle' (Fig. 2). Sometimes you 'correct' for perspectival foreshortening, sometimes you don't. It all depends upon context and your current situational awareness. This is the reason why pictures in perfect perspective can be viewed from 'the wrong positions' (Pirenne, 1970; Fig. 3), and also why they are often perceived as 'distorted', even if the eye is placed exactly at the perspective center (Pont *et al.*, 2011).

In most typical cases of pictorial viewing the viewer has no sense of a preferred viewpoint, and is well aware of the picture as a physical object. The viewer has a simultaneous awareness of the space she moves in and of a

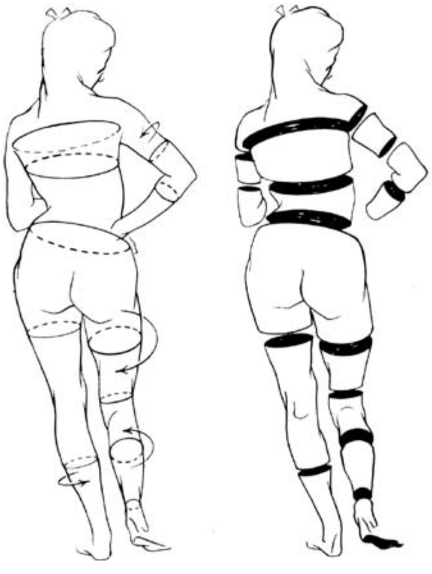

Figure 2. Left: Three dimensional shape is evoked by a variety of ways. One important pictorial device is the ellipse, which is likely to be seen as a slanted and tilted circle. Especially series of ellipses that appear as parallel cylindrical cross sections are very effective. In drawing such cross sections can be introduced in various ways. Right: In this print Felix Valloton used sections with the water surface to bring out the volumetric nature of the bodies.

Figure 3. Lord Kitchener always points right at you, even when you stand aside, and see him obliquely. All that happens is that he 'looks thinner'. The space you move in and pictorial space do not mesh. You can step aside in the space you move in, but not in pictorial space. The foreshortening happens in the space you move in, but there is no foreshortening in pictorial space. Lord Kitchener looks 'skinny' but is still seen frontally, not obliquely. This confuses many people, even smart ones. Notice that the frame, seen as a picture, is not in the same pictorial space as Lord Kitchener. One deals with a variety of mutually disjunct spaces here. The boundary of the trapezoid (the frame) looks oblique, whereas – paradoxically – its interior looks frontoparallel: it all depends upon your current intentional reference.

pictorial space. In such circumstances vision automatically 'corrects for foreshortening'. The result is that any location on the picture is experienced as essentially seen frontally. Thus the effect of eye movements is not different from a change of viewpoint, that is to say, rotations of the eyeball have the effect of translation of the picture plane. This is important and the consequences are followed up below.

The first person to remark on such issues from a scientific perspective was the mathematician Guido Hauck (1879, 1882). Hauck had a deep understanding of perspective, indeed his work was influential in the progress of computer vision at the end of the twentieth century. He also understood the importance of the 'beholder's share' (Gombrich, 1961) well. His proposal for a more appropriate drawing system is firmly based on both geometry and psychology. The classic account on the topic is Panofski's (1927) '*Perspective as symbolic form*', which is based on Kantian views, and Ernst Cassirer's philosophy (1923–1929) that later was to result in Langer's theory of the arts (1942, 1953).

1.3. Familiarity and 'Looking Right'

It has to be remembered that one's familiarity with generic scenes plays a key role in 'reading' pictures. Thus, human bodies, spheres, and cuboid objects, as boxes or buildings, are often perceived as distorted in pictures, whereas unknown shapes, or less constrained shapes such as bushes or trees are rather more immune to this. The phenomenology of pictorial perception is why many artists prefer things to 'look right', rather than be in exact perspective, and why photographers avoid extreme wide angle or tele lenses for portraits.

In cases of very wide, or very narrow angular views, it is virtually *impossible* to enforce the eye to be at the center of perspective. Here is a quantitative example: in photography one might present pictures at magazine format, say 40 cm wide, viewed from 40 cm distance, yielding a field of view of 53° of visual angle. Consider a 'full frame', that is 36 × 24 mm, sensor (Note 16). A professional set of lenses (Note 17) ranges from about 12 mm (extreme wide angle) to 600 mm (extreme tele), yielding horizontal optical fields of view of 113° and 3–4°. The wide-angle photograph 'should be' viewed from a distance of 113 mm, and the tele photograph viewed from a distance of 670 cm. Perhaps unfortunately, the former is not possible to assume for the unarmed eye, whereas the latter will not be possible in many rooms. This is evidently an inconvenience. In practice, one leaves the viewing distance at 40 cm for all pictures of the given size. But then the wide-angle photograph is viewed as *minified* by a factor of 2.1, whereas the tele photograph is viewed as *magnified* by a factor of 15.5. Phenomenologically, this results in the experience of very noticeable 'distortions' in pictures of familiar scenes. Indeed, many people will spontaneously complain. Unless one sticks to photographs taken with 'standard lenses' — here 36 mm – this cannot be avoided (Note 18). Similar considerations apply to art work. Does that mean photographers and artists should stick to the strict constraints enforced by the perspective police? But if not, then what to do in order for pictures to 'look right'? Here classical perspective offers no clue. Artists have invented various tricks, photographers carefully select the right lens for the job.

In this paper we consider the problem of how to render pictures that somehow minify or magnify the field of view so as to 'look right'. Possible solutions to the problem have implications for the generic problem of 'artistic perspective' too (Baldwin *et al.*, 2014). It is not a trivial problem, for it transcends the virtual window setting, and thus linear perspective. There exists hardly any quantitative phenomenology on the matter, with the exception of small field magnification (Note 19).

2. Geometry of the Field of View

The ‘field of view’ is an optical entity, whereas the ‘visual field’ is an object of visual awareness. These are evidently categorically distinct entities. In this section we discuss optics only. For terrestrial animals like humans, important facts of life are the transparency of air, the rectilinear propagation of electromagnetic radiation, and the fact that surfaces of physical objects scatter radiation over all directions. This results in the valuable possibility to see things from a distance (Heider, 1926; Riedl, 1975). In every direction from the eye one obtains scattered radiation from the nearest opaque object. Vision may be understood as at least partly based on the labelling of directions with points of the scene in front of the eye. On abstracting the geometry, one simply deals with the structure of directions from the vantage point. It is a kind of Kolmogorov complexity theory (Kolmogorov 1963), in which the directions are the atomic objects that may be called ‘visual rays’. Euclid’s ‘Optics’ formalizes such a theory, though it is often — erroneously — construed to be a faulty theory of propagation of some physical entity (Note 20; Euclid, ca. 300 BC; Koenderink, 1982).

The visual rays are most conveniently labeled as points on a ‘viewing sphere’, Gibson’s ‘optic array’ (Gibson, 1950; Graf, 1940). This is a mere matter of convenience (Note 21), similar to the use of celestial globes that label positions of stars and stellar configurations. After all, the cosmos is not like a spherical surface, so why consider a globe at all (Note 22)? Formally speaking, it is a mere convenience. But from a psychological perspective, it allows one to obtain an ‘external view’ of the geometry that represents relations in a particularly intuïtive way. Indeed on various historical celestial globes the personifications of constellations, like Orion, are depicted as seen from behind (Fig. 4). Then the problem of perspective can be formalized as that of the mappings from the viewing *sphere* to the picture *plane*. Formally, the possibilities are infinite (Graf, 1940; Note 23). Unique ‘solutions’ can be enforced by imposing sufficient constraints. The ‘virtual reality’ condition is indeed one such a constraint, but there is much freedom in the choice. The officially ‘right’ constraint has its advantages and disadvantages, much as alternative sets of constraints have. People, when asked to come up with desirable constraints, usually specify these such that there is no solution (Graf, 1940). Apparently, *pictorial rendering is a matter of give and take*.

Consider a fixed vantage point. The observer is able to view approximately a hemisphere, albeit not with uniform clarity, the periphery being much less

Figure 4. A celestial globe represents the viewing sphere as seen from the outside. Notice that the personification of Orion is seen from behind. Moving the mental viewpoint away from the center of the sphere to its exterior involves a major switch of mind frame. It is a psychologically necessary initial step on the way to reason about the geometry of the field of view.

distinct than the area about the fixation direction (Graham, 1966). Thus, in order to obtain a good overview eye movements are required. Geometrically, such eye movements shift the viewing sphere in itself, leaving all internal metrical relations invariant. It is like rotating a geographical globe. Such rotations induce *congruencies*, or *proper movements* in the sphere. Configurations of points on the sphere are not deformed, only displaced, by an eye movement. This is perhaps the most basic property, it renders vision 'transparent' to eye movements. That is to say, you are typically not aware of involuntary eye movements, although they frequently occur.

However, if you *displace (technically 'translate')* the eye you generically induce deformations in the viewing sphere. In this case you experience the movements in terms of an apparent three-dimensionality of the visual world 'due to optic flow' (Gibson, 1950). The virtual reality condition implies that pictures in linear perspective, viewed from the perspective center, are likewise transparent with respect to eye movements. Again, if you move your eye away from the perspective center, you induce deformations in the viewing sphere. In this case the deformations are of a very different nature. You may experience them either as the flatness of the picture surface, or as a deformation of pictorial space (Pirenne, 1970). It depends upon your current mode of vision, which again depends on your current situational awareness, and so forth. The following heuristic is perhaps helpful: you may see all sides of an object by walking around it, whereas you have only a single view in the case of a picture. What has not been painted cannot be seen, no matter how you move. Thus, once you move away from the perspective center, the picture is very *unlike* a window on a scene (Note 24).

How do these insights transfer to cases of *minification* or *magnification* of the field of view? In such cases *congruences* are geometrically impossible. But one would perhaps be satisfied with *similarities*, that are isotropic scalings, as these would at least conserve 'shapes'. However, it is well known, indeed intuïtively self-evident, that the only similarity modulo movements on the sphere is the identity. Thus minifications and magnifications *necessarily* introduce distortions. They can only lead to 'wrong pictures' in the understanding of the perspective police. In the case of minifications and magnifications one has no choice but to look for different types of constraint than the virtual reality window or simply give up.

3. Constraints Reflecting the Human Condition

A bipedal agent like the human observer is well adapted to the generic terrestrial environment, which is largely constrained by the effects of gravity on solid matter. A common landscape is roughly a horizontal ground plane, from which linearly extended objects either stick out in the vertical direction, or lie flat on the ground. Such is repeated over and over again in books on 'how to draw' — and rightly so, it is a key insight. The human body is no exception, the active human stands upright. The visual system primarily enables locomotion and navigation on the ground plane. The movements are in the horizontal plane, in which the frontal direction in the plane of the body's bilateral symmetry is of primary importance (Gibson, 1950; Riedl, 1975; von Uexküll, 1909).

This polarizes the viewing sphere. The zenith and nadir define the direction of gravity. The horizon represents the ground plane, itself due to the effects of gravity. The hemisphere above the horizon is mainly occupied with sky, and usually not very informative. The hemisphere below the horizon is occupied by the terrain. It may be divided into a part of the ground plane of less than an eye-height away, an annular region that can be reached in a few steps, and a distant annular region just below the horizon. The most important region is centered upon the horizon, extending about twenty degrees above and below it. Most eye movements involve panning, shifting the horizon within itself, that is to say, a 'horizon scanning' (Note 25). Such eye movements render the effective visual field panoramic. It might well be a reason for the dominant preference for 'landscape' over 'portrait' aspect ratios in depictions of the environment. This layout is slightly different for various environments (Note 26), but it by and large captures the human condition. It evidently differs from that of birds, squirrels, and so forth (Note 27).

The consequences are numerous and important. The human observer is extremely sensitive to the direction of gravity. Thus if you hang a picture upside down you're severely handicapped in 'reading' it, whereas a left-right swap of a picture is hardly noticeable at all (Mach, 1959 [1886]). The interactions with a picture on the wall are mainly due to horizontal body movements and/or eye movements. It is rare to view pictures on the wall from the ceiling, or the floor, but quite common to view them from the left or right, necessarily so if you view the picture in company. Similarly, in viewing a scene panning (left–right) eye movements, even large ones, are the rule, whereas tilting (up–down) eye movements are comparatively rare, and typically smallish. Moreover, in any event, the vertical is special. Panning eye movements transform the family of verticals into itself, whereas the horizontals are even individually conserved, as they are shifted within themselves (Note 28).

Thus desirable objectives for a map of the viewing sphere to the picture plane are:

- to map the meridians (great circles through zenith and nadir) to vertical straight lines;
- to map the horizontals (small circles parallel to the horizon) to horizontal straight lines;
- to render the map conformal (locally a similarity) at least on the horizon, though preferably everywhere.

These are very strong constraints. In fact, they cannot generally be met. They also differ from linear perspective in various important details. [The mathematics of such projections is not entirely trivial. Most of what is needed here is conveniently described by Lambert (1772).] Linear perspective does indeed also map the meridians to vertical straight lines in case the picture plane contains the vertical. However, it fails to map the horizontals to horizontal lines, but instead it maps the horizontals to parabolic curves (Note 29). Moreover, linear perspective is only a similarity at the straight ahead direction, whereas for all other directions it fails to be conformal (Note 30). It depends critically upon the 'correct' viewpoint, and cannot–even in principle–deal with minifications or magnifications.

The crucial part of the constraints may be summarized as

> ‘*horizontal shifts of the picture should have the same effects as panning eye movements in a scene*’ (Note 31).

This is what the difference between peephole shows and panel renderings boils down to. It is very different from the virtual reality constraint. We indicate the distinction by saying that linear perspective regards an image as a *peephole show*, whereas the constraint introduced here leads to a *panel presentation* (Note 32). It is related to Guido Hauck’s ideas, who on the one side stressed the ‘principle of centricity’ (“*Prinzip der Centricität*”), but on the other side characterizes the observer as ‘the happy roamer’ (“*den fröhlichen Wanderer*”), that is to say, as making continual eye movements, either in pictorial space, or in the space they move in.

The constraint is also closely related to the phenomenology of *external local sign* (Koenderink *et al.,* 2009). Visual observers have a strong tendency to see everything in front of them, fully ignoring the divergence of visual rays (Kepler, 2000 [1604]; Koenderink *et al.,* 2010; von Helmholtz, 1866). As a consequence, *rotations* of the eye in accordance with Listing’s (1845) law of eye movements will be experienced as *translations*. Thus the above constraint merely formalizes the phenomenology of pictorial vision.

4. Extreme Magnifications

The case of extreme magnification is simple in principle, as it implies parallel projection. However, there are a number of problems. First of all, one has to pick a viewing direction. The horizontal, forward direction is natural, and indeed sometimes used, as in the Bayeux tapestry or Egyptean wall paintings (Fig. 5). However, a horizontal direction is awkward, for in order to show the floor one needs to have a ‘bird eye’s view’. This is typical for oriental scroll paintings (Fig. 5). One also needs to decide on the plane of projection, which need not be frontoparallel, although it should contain the vertical. Various, mutually very similar, possibilities result from this. They often are viewed as ‘methods of construction’, rather than ‘projections’ (Dubery and Willats, 1983). This equally applies to most artistic productions. One ‘draws in a certain manner’, rather than ‘uses a formal projection system’. In many, perhaps most, cases of interest ‘projection’ is really a misnomer. Although often not immediately apparent, it can become obvious at first blush, as in ancient Egyptian renderings.

Another, more serious, problem is that parallel projection implies exact ‘size constancy’ (Note 33). Thus you cannot simultaneously show small, a human say, and large, a mountain say, objects in a single picture. This can be solved simply and effectively, by adopting distinct, mutually hugely different

Figure 5. Left: detail from the tapestry of Bayeux. Objects are depicted as seen from infinity, with a horizontal direction of view. There is no preferred viewpoint implied, the tapestry is far to elongated for that; right: a bird eye's view in parallel projection is typical for many art works from the far east. Notice how the scale jumps discontinuously between foreground, middle ground, and background.

scales for foreground, middle ground, and background. This is very common in landscape pictures in many cultural traditions (Note 34). Landscape paintings in Western art also derive from such schemes (Clarke, 1949).

Apart from such issues, orthographic projection is in many respects the perfect solution. Different from linear perspective, it allows pictures to be of arbitrary length, say as scrolls, or wall tapestries. There is no such a thing as a natural unit size, like the ‘focal length’ in linear perspective, nor is there any preferred ‘focal point’. The method is much better suited to decorative work than linear perspective because of this. A peephole show hardly qualifies for wallpaper because all locations in a room should be treated equally (Note 35).

Because the parallel view allows one to visually compare similar objects at different locations immediately, it is very effective in many cases. A typical example is Ferdinand Hodler’s (1892) painting ‘Tired of life’ (Fig. 6), depicting five old men. The men are set in the frontoparallel plane, and can almost be superimposed by a parallel translation. The painting is huge, you can walk from man to man as you study it, of course taking your viewpoint with you. This is how Hodler drew them, each separately seen from the front. An oddity, due to Hodler’s familiarity with linear perspective, is the bench, which is the only object he rendered ‘correctly’. You can only see it ‘right’ from just one viewpoint. Had he drawn the models from that viewpoint, the outermost would be seen in half-profile (Koenderink *et al.,* 2010). As a result, the bench becomes an eyesore once you notice it (Fig. 6). The outermost men had to be rotated by almost a quarter turn relative to the bench (Koenderink *et al.,* 2010). Such awkward juxtapositions are common enough in pre-twentieth century Western art. Apparently they aren’t generally noticed.

Linear perspective degenerates to parallel projection in the limit of very narrow fields of view. But it would yield postage stamp, or smaller size paintings to be viewed from a typical reading distance, which is not very convenient.

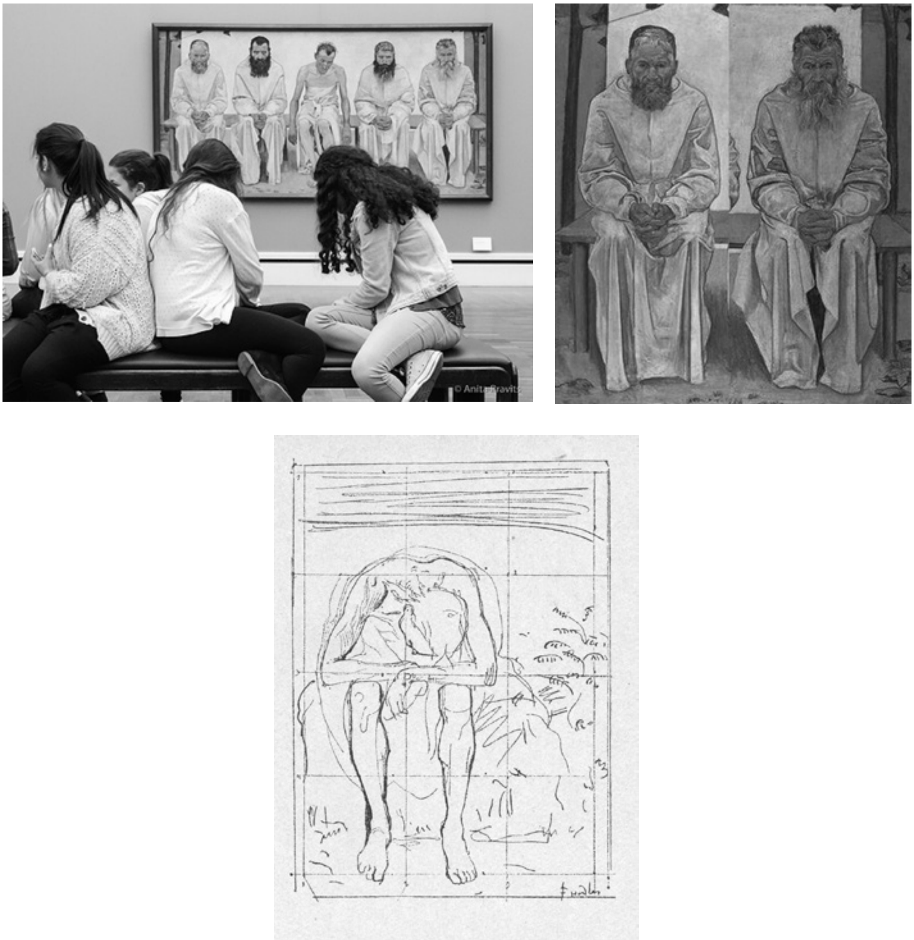

Figure 6. At left the painting by Ferdinand Hodler discussed in the text. It is a very large canvas (150 × 294 cm); spectators typically view it from various positions. At center a drawing related to the painting. Hodler drew the men separately; he probably never had models pose together in a single scene. At right we combined cut-outs of the outmost men. Notice the bench.

Blowing them up puts the correct viewpoint at infinite distance, so one will necessarily be too close, resulting in a marked flattening of pictorial space. Strong magnification simply fails in the linear perspective setting.

5. Strong Minifications

Minification of the field of view is frequently used by the visual artist. Art directors use minifying glasses to judge possible illustrations, painters simply step back from the easel to view their work. The point is that you need a small field of view to see in ‘pictorial mode’, as was forcefully argued by Adolf von Hildebrand (1893). When viewing the scene in front of you, you may use a convex mirror (Note 36), or a concave lens to obtain a minified view. Photographers often look through minifying viewfinders for the same reason. In fact, the old-fashioned (nineteenthirties to fifties) viewfinders by Leitz and Zeiss (Note 37) are pocketable, or can be hung around your neck. They make great viewers and were often used for that. In the old days the camera obscura served — among more — a similar function.

Formally, the case of strong minification is somewhat involved. Since the view sphere — a unit sphere by convention — has fixed total area, it cannot be minified as a whole. If you minify a part, other parts need to be magnified to make up for this. Alternatively, one has to omit part of the total area. Since the meridians meet at zenith and nadir, they have to be conserved as such. Apparently the minification needs to be focussed on part of the horizon.

This leads to the following heuristic. First we map the sphere on a planar strip, using Mercator's (1595) map (Note 38). This famously maps the meridians on uniformly spaced vertical lines of infinite extent, whereas the horizontals map on horizontal line segments. This is why the map was once so popular with seamen. Straight lines on the map are 'rhumb-lines', or 'loxodromes', that is to say lines of constant course, or bearing. They are easy to steer. Like the horizon, the horizontals are limited to a range of minus to plus ninety degrees in case you want to picture the half-space in front of you. This puts the nadir and zenith at minus and plus infinity, thus it is best to limit the map to some reasonable elevations, say plus or minus thirty degrees from the horizon. This map reduces to the identity at the horizon, and is neatly conformal throughout. It is perhaps unfortunate that the scale necessarily grows out of bounds as you near the zenith or nadir. But who cares? If you are mainly interested in the horizon you will not even *see* the zenith, you'd have to stretch your neck to do so. Even worse, actually looking at the zenith lets you lose contact with the horizon, a dangerous thing to do in a world full of potential earth-bound enemies. Most importantly, the effects of eye movements map on translations of the picture plane, exactly as desired. This is why the ancient mariners liked the map and it is why vision likes it too. It renders spherical geometry palatable to the generic human mind by reducing it to Euclidean, thus *turning rotations into translations*.

After this basic construction, the picture plane can be uniformly minified without any problem, retaining all these desirable properties. This basically solves the minification issue, or better, turns it into a non-problem. We refer to these depictions as 'panel renderings'.

The invariance to panning eye movements is shared by all 'cylindrical projections'. However, only the Mercator map is conformal. This is not a major advantage, since all cylindrical projections are conformal at the horizon. A well known example of another cylindrical map is the one proposed by Hauck, an 'equirectangular projection' ("*Quadraten Plattkarte*, *Plate carrée*"), introduced by Marinos von Tyros around 100 BCE.

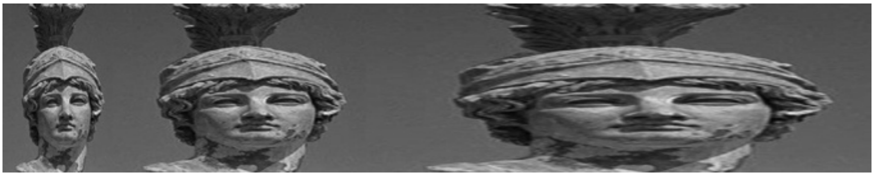

Figure 7. Deformations of familiar objects like human faces tend to be objectionable. Here are typical deformations due to linear perspective. Although technically 'right', they look 'wrong', and are typically avoided in the arts. (In this picture the faces 'all look at the lens', thus are in quite distinct spatial attitudes.)

The cylindrical maps ensure a uniform distribution of all directions in the vicinity of the horizon. It is very different from the unequal distribution due to linear perspective. For instance, in the latter case, for a hundred-and-twenty degrees visual field (a *very* wide angle) the horizontal spacings at the picture edges are *four times* that at the center, whereas the vertical spacings are *two times* that at the center. This shows up as an extreme distortion when you strongly minify the field of view. It does things to the image of a human head that makes most observers strongly complain (Vangorp *et al.*, 2013; Fig. 7).

6. The Helmholtz Alternative

Von Helmholtz (1866) based a different map on the basis of his analysis of human eye movements. Eye movements satisfy Listing's (1845) law, making them an Abelian group (Note 39), in contradistinction to the full rotation group which fails to be commutative (Note 40). This is no doubt a great convenience in terms of eyeball control. Von Helmholtz shows that the curves shifted into themselves by Listing's constrained eye movements are small circles in the viewing sphere that pass through the anterior pole — that is the antipode of the forward direction. Helmholtz noticed that they map to straight lines if you project the viewing sphere on a frontoparallel plane from the anterior pole as center. It yields a 'stereographic projection' (Note 41) of the viewing sphere.

This is a very intriguing idea. It has often been quoted to account for the subjective curvatures of objectively straight lines (Oomes *et al.,* 2009; Panofski, 1925; Pirenne, 1970; Rogers, 2008; Rogers and Brecher, 2007). However, the issue of whether this 'explains' the phenomenology of 'subjective curvatures' remains essentially undecided.

As a way of rendering the scene in front of you, stereographic projection has much going for it. It is conformal, and the horizon is rendered as a straight horizontal line, the main meridian as a vertical one. However, both the generic verticals and horizontals map on circular arcs.

Flocon and Barre (1968) proposed a map similar to this for general documentary use (Note 42). It is not quite stereographic, which is a pity, because

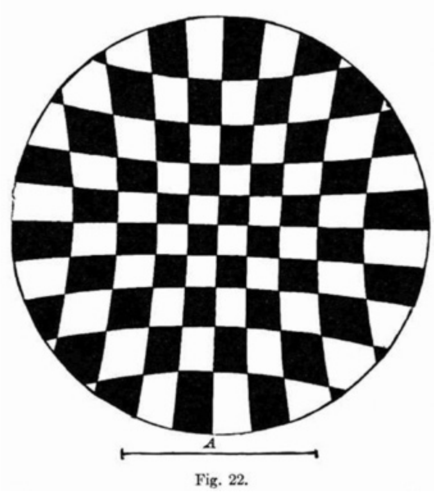

Figure 8. At left a famous demonstration by Helmholtz. If you view the figure from the indicated distance you are supposed to see a regular chequers pattern. At right a rendering in the Postel map, proposed by Flocon and Barre, showing a full hemisphere. This is the map most commonly used in the design of fisheye lenses (Note 45).

the stereographic projection has unique geometrical properties (Note 43) that have been appreciated since antiquity (Note 44). On the other hand, the Flocon and Barre map minimizes peripheral scaling as compared to the stereographic case. This reflects the phenomenology of the visual field somewhat better (Pepperell and Haertel, 2014).

The Helmholtz geometry maps the half-space in front of you into a circular disk (Fig. 8, Note 45). Von Helmholtz suggests that this disk should be looked at centrally, orthogonal to its plane, at a distance equal to its radius (Note 46). Indeed, we have found — as Helmholtz suggests — that most human observers experience the half-space in front of them as subtending a cone with square top angle (Koenderink *et al.,* 2009). Flocon and Barre (1968) do not put any constraint on size, suggesting that their pictures are meant to be viewed any way you like. This makes pragmatic sense, but it severs the link with Helmholtz's theory. We refer to these renderings collectively as 'fisheye views'.

7. Examples

7.1. Formal Coordinate Grids

In Fig. 9 we show the viewing sphere 'as seen from the outside'. Such a projection has also been recommended to artists (Bonbon, 1985). Here we show it because it is familiar — it looks like a geographical globe — and it allows us to introduce the 'window' used for the examples in this section. The window

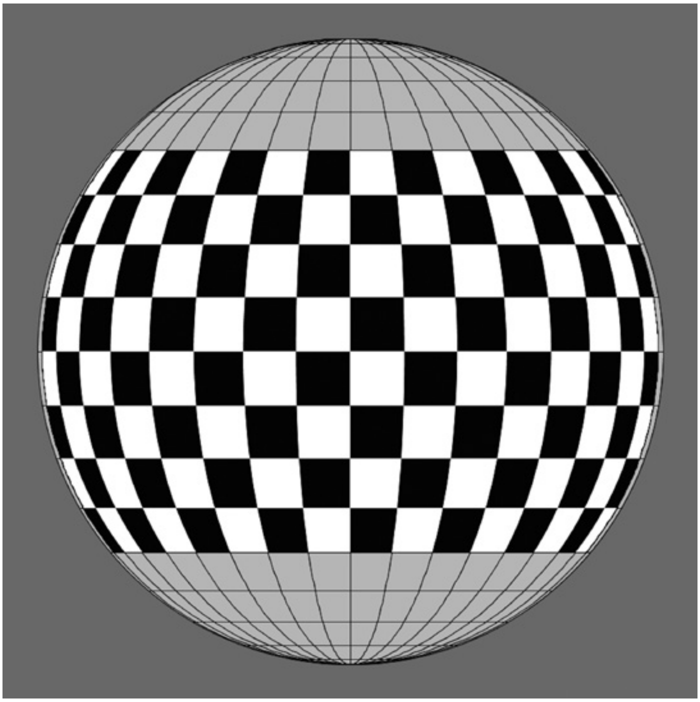

Figure 9. The viewing sphere seen in parallel perspective. The primary visual direction is at the center, zenith at top, nadir at bottom. The coordinate lines, verticals ('meridians'), and horizontals ('latitude circles') are drawn at 10° intervals. The 'viewing window' used in this section for demonstration purposes is 160° wide, and ± 40° high. In photography a fisheye lens (Hill, 1926) would cover this, in the arts one occasionally encounters landscapes or city views up to such extents (Cornish, 1935). Notice that a landscape format rectangular picture of, for example, golden ratio (1.61803…) aspect ratio, covering 180° over its diagonal would subtend 153° horizontally, and ± 48° vertically, practically defining a limit for pictorial rendering. It applies to conventional (Postel map) fisheye lenses covering the frame.

used for the images in this section measures — of course rather arbitrarily — a subtense of ± 80° in the horizontal ('azimuth'), and ± 40° in the vertical ('elevation'). The primary viewing direction is at the center (zero azimuth, zero elevation). With a horizontal subtense of 160°, this is an extreme wide angle view, only a little narrower than the full field of view of a human observer. The full field is so large that — as an observer — you will typically see parts of your own body, spectacle rims, and so forth (Pepperell, in press).

In Figs 10, and 11 (top), we compare the renderings of the window in peep-hole shows and in panel presentations. Notice that the panel renderings are invariant with respect to horizontal shifts — and, indeed, approximately invariant for shifts in *any* direction — thus a painting in this system can be viewed from any vantage point. In contradistinction, the peep show may perhaps look

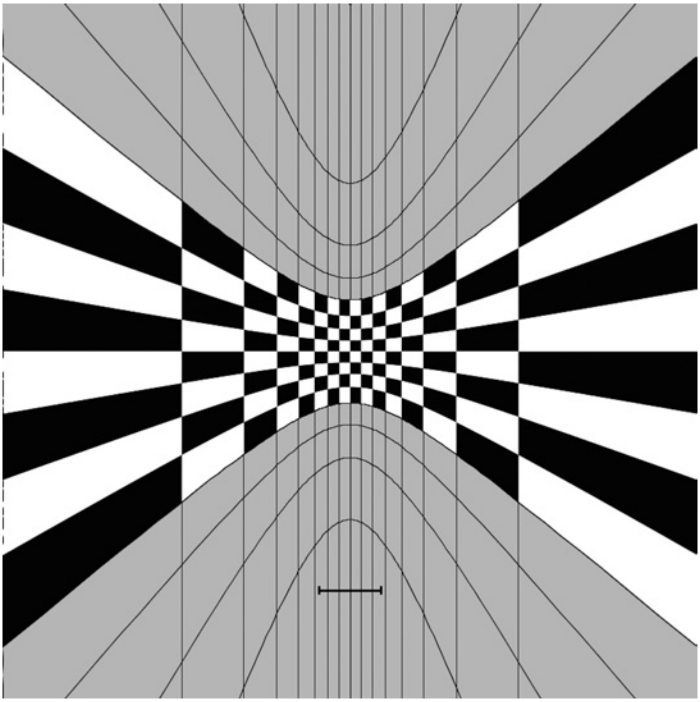

Figure 10. The window introduced in Fig. 9 in peepshow presentation. The peepshow rendering — in applications one would only show a rectangular cutout — should be viewed from the distance indicated by the thick line segment. Any change immediately reveals the large deformations that vary from place to place. Notice that the frontal hemisphere maps on the full (infinite!) plane in linear perspective, thus cannot be represented.

great from the perfect vantage point, but looks extremely deformed from any other. The deformations away from the horizon that occur in the panel renderings are hardly noticeable with this window aperture.

Notice that there is little difference between the Mercator and Hauck renderings. Although the Mercator is conformal but the Hauck map not, this hardly shows. For even larger elevation ranges the Hauck map may well be preferred, because the conformal property of the Mercator is bought at the cost of a steep scaling gradient of solid angle (spherical area) at high elevations. In practice, there is little to choose here.

In Fig. 11 (bottom) we compare the fisheye renderings. They share many of the advantages with the panel rendering, and are generally to be preferred over the 'right' drawing. However, most contemporary Westerners are likely to complain loudly about the curvature of the verticals (Note 47). It is of some interest to compare these two renderings, and also compare them with the rendering of Fig. 10. *Firstly*, there are differences in conformity. Stereographic

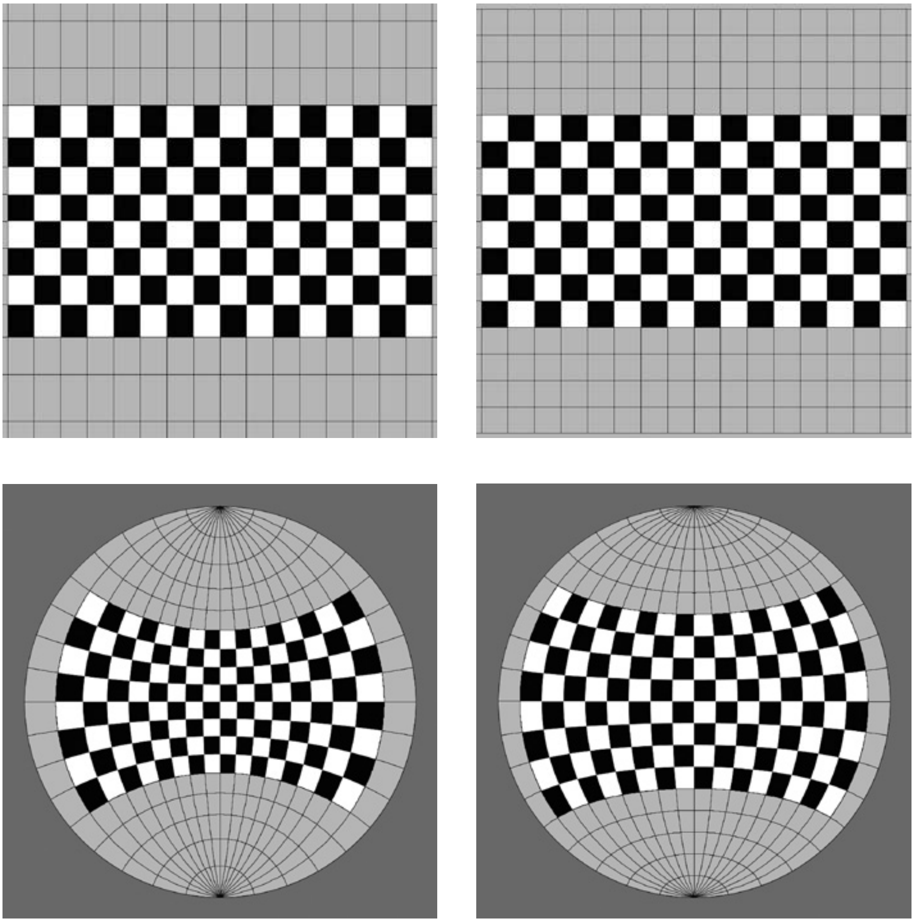

Figure 11. At top left the Mercator, and at top right the Hauck map. Both are ‘panel presentations’. Thus panning eye movements appear as horizontal translations, the presentation is invariant to them. Both ‘projections’ map the full frontal half-space on a strip of finite width of 180°. In the Hauck map the full hemisphere maps on a square, in the Mercator map on a strip of infinite height. The Mercator map is conformal throughout, the Hauck map only at the horizon. Here we show only the window defined in Fig. 9 — in this window the differences are hardly apparent. At bottom left the stereographic projection (Von Helmholtz, 1866), at bottom right the Postel map (Flocon and Barre, 1968). They are similar, but different. Both represent the frontal hemisphere in a circular disk. In the stereographic projection all coordinate lines are circular arcs, in the Postel map they are more complicated curves. The stereographic projection is conformal, whereas the Postel map has deformations outside the center. For artistic purposes it is perhaps more attractive. However, for the given window the differences are slight.

is conformal throughout, whereas Postel (1581) shows (mild) deformations near the boundary. *Secondly*, there are differences in the magnification ratio between centre and periphery. *Thirdly*, the curvature of the horizontals differs appreciably. It is hard to assess the relative advantages and disadvantages (Note 48), but it seems likely that many artists would favor the stereographic rendering least because it stresses the periphery rather too heavily. Perhaps oddly, the rendering of Fig. 11 (bottom right) might be preferred by some because of the slightly greater straightness of the horizontals, and the somewhat larger relative importance of the center as compared to Fig. 11 (bottom left). In practice there is little to choose here.

Something not often remarked upon are differences *other than geometrical*. If you use a rectangular frame for a very wide angle view in linear perspective, this has various consequences of a *non-geometrical* nature, that we have not seen discussed. The elevation at the top of the frame is much less at the left and right sides than at the center. But this implies, for instance, that the color of the sky near the top of the frame varies — perhaps rather strongly — with position (Note 49). No artist seems to suggest that in painting. At least, we never encountered an instance. This is natural from an artistic point of view, since it would destroy the unity. We are not aware of any references to such effects, the perspective police being mainly interested in geometry. This suggests another desirable constraint, namely *to have the elevation range invariant with azimuth for a rectangular frame*. The constraint is automatically met by any cylindrical projection, like the Mercator, or the Hauck maps.

7.2. Photographs

In Fig. 12 we show a wide angle photograph, converted to peep show, scaled panel (Hauck), and fisheye formats (Postel). They are presented at similar size. In practice one would cut a rectangular picture out of these renderings. In the past we have studied pictures from a large range of viewing distances (Pont *et al*., 2011), and a huge range of angular sizes. This is a major hassle, but the simple result was that people apparently apply a *template*. Here we decide on a fixed size because it is the realistic choice. People view pictures at the size of their screens, or as printed post cards or magazine pictures, typically at normal reading distance. It all makes sense.

Things work out differently for pictures with various types of subject matter. Presence of 'gauge objects' and in-depth configuration are important. For instance, a frontoparallel flat object will yield the *same picture* for any minification or magnification, thus the depth range is vital. Familiarity is important in a different way. For instance, the curvature of a line expected to be straight, such as the horizon, or the edge of a building, is a dead giveaway that the rendering is not 'in perspective' — the perspective police need look no further.

So are the aspect ratios of objects expected to be roughly spherical, such as human heads. Moreover, pictures of objects contained in a compact spatial volume, like a group of people, and pictures that show extended surfaces in depth, such as the ground, usually lead to different visual experiences. These effects are of a psychological, rather than geometrical nature.

Figure 12. At top left a photograph in linear perspective, taken with a 14mm lens on a 36x24mm sensor. This yields a field of 104° horizontally, and $\pm$ 41° vertically (114° over the diagonal). At top right a transformation to a panel rendering (Hauck). In the black areas data is lacking, because the lens does not cover these parts. In the latter case one has a feeling of being 'immersed' in the scene, in the former the pictorial space looks like a 'tunnel'. Of course, this depends upon the viewing geometry, but here one could impossibly place the (single!) eye at the 'correct' position–about a quarter of the width–anyway. For the panel rendering there is no such a correct position. It looks good from any reasonable viewpoint, and you may keep both eyes open. Although the panel rendering is conformal, it shows deformations on larger scales. Notice the circular table, which is not rendered elliptically. At bottom a rendering according to Postel. Here the curvature of verticals is noticeable in the outline of the house seen at the top right corners of the pictures. Notice that a sense of 'presence' (Lombard & Ditton, 1997) is least in the case of true perspective rendering.

7.3. Visual Arts

Artists often use a 'free' form of rendering that only approximately resembles one of the familiar formal systems. When they apparently use the 'right' methods, a more in depth study is likely to reveal various 'corrections', necessary to not only *be right*, but also *look right*. These tend to be loudly pointed out as 'mistakes' by the perspective police, who take pride in this exercise. In a number of instances, some well known, one notices a systematic trend towards systems distinct from the norm. We mention a few examples.

Jean Fouquet (1450s) has a remarkable image (Fig. 13) that seems to be constructed on the same principles as the panel rendering described above. Notice the straight verticals, and curved horizontals. Fouquet knew about perspective and indeed has 'right' images that predate this one. Thus there is no doubt that his curved horizontals are intentional.

Vincent van Gogh's (1888/1889) '*Bedroom in Arles*' (Fig. 14) shows much the same features as the Fouquet example. Again, there is no doubt that the apparent curvatures are intentional, for they are not present in one of his drawings, which is in 'correct' perspective.

Umberto Boccioni's (Tisdall and Bozzolla, 1977) '*The street invades the house*' (Fig. 15) is a wide angle view in remarkable perspective. It has affinity

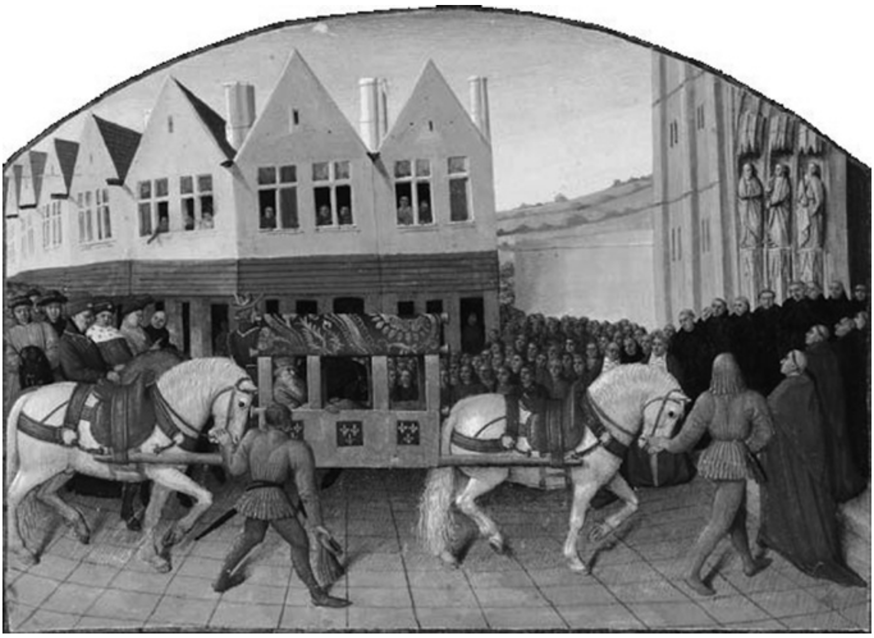

Figure 13. Jean Fouquet '*Entrée de l'empereur Charles IV à Saint-Denis*' (1455–1460). Notice the verticals and horizontals.

Figure 14. Vincent van Gogh, '*Bedroom in Arles*' (1888/1889). The drawing is from a letter to Gauguin, the painting is the third version. Compare the horizontals in the drawing and the painting. Van Gogh drew 'right', but composed the painting 'wrongly'.

Figure 15. Umberto Boccioni, '*La strada entra nella casa*' (1911). In a catalogue Boccioni wrote 'The principles of Röntgen rays is applied to the work, allowing the personages to be studied from all sides, objects both at the front and the back are in the painter's memory' (Tisdall and Bozzolla, 1977).

Figure 16. David Hockney, '*Pearblossom highway*' (1986). A mosaic of 650 mounted photographs of the Mohave desert landscape. Hockney took the photographs from a variety of vantage points. The resulting quilt looks like a coherent cylindrical projection.

with fisheye renderings, and is certainly not 'right' in the eye of the perspective police. This evidently contributes much to its impact. The circular composition, which explains the unusual square shape of the frame, emphasizes the centre, and relates it to the periphery.

David Hockney's '*Pearblossom highway*' (Fig. 16) is remarkable in many respects. Notice the straight horizon and straight verticals. It is evidently akin to a cylindrical projection, although perhaps not the Mercator one. Hockney used many vantage points (as Picasso did in Fig. 1) as he took the polaroid photographs that make up the work.

Such examples could be multiplied almost ad infinitum. Art cannot be pigeonholed into categories like 'perspective right or wrong'.

8. Discussion

Are pictures that are not in correct linear perspective 'wrong', or at least objectionable? A pragmatic answer comes from a cursory view of the art market. Clients don't buy or reject pictures because of the 'correctness' or 'incorrectness' of the perspective, whereas painters do whatever is necessary to achieve their pictorial aims. So much for the brute facts. In spite of this, there remains a strong feeling, especially among scientists, that linear perspective is the *uniquely correct* way to represent physical reality on the picture plane.

The issue has much to do with the ontology of pictures regarded as *representations*. A common kind of reasoning has that — ultimately — the only veridical representation of an object is the object itself. What's wrong with that? From a visual point of view a perfectly colored wax replica has to come next (Note 50), for it looks the same. Only when the representation *has* to be flat, like in a book illustration, a photograph is the best bet. At least it is an *objective copy*. If photography is inconvenient, as in the representation of a hatching dinosaur, some artist may be hired to emulate the camera as a last resort (Gurney, 2010). Unfortunately, this lets in some subjective — thus potentially *wrong* — elements. Anything beyond that is either merely childish, or just plainly wrong. This is the kind of reasoning one might well apply in documentary work, say for a museum of natural history. But the artist may prefer to think of pictures as *presentations*, rather than *representations*. Presentations are designed to evoke certain experiences in prospective viewers. No holds are barred in making this come true. The resulting visual experiences have no 'right or wrong' status at all.

Here we pursued the topic of which type of formal 'projection' looks best, if any. Such a question is of some interest to illustrators and designers. It is clearly different from the issue of 'correctness', or even 'possibility'. It depends upon the intention of the author. It is perfectly possible to illustrate the notion of a 'round square', but there is evidently no unique 'correct' way to do so, certainly no formal drawing method will apply. If the work is intended as a plan to help in constructing something — a building say, or explain the structure of a complicated object — a windmill say, a formal drawing method is evidently preferable. For typical applications the window will be limited, implying that differences between the various maps become less pronounced. Major choices that remain are between linear perspective (the 'peep show'), panel representation — with hardly any difference between Mercator and Hauck, and fish-eye representation — with hardly any difference between stereographic and Postel. Of course, infinitely many different renderings are possible, but only a few seem useful. The three generic types suggested here by and large exhaust the topic: *planar*, *cylindrical*, and *spherical* renderings. Although these terms are formally awkward, the visual artist will grasp the meaning.

In informal observations, naïve observers, when confronted with smallish renderings of wide angle scenes, prefer the Hauck rendering in more than half of the cases. Only one out of five favors true perspective rendering. We present a formal study elsewhere (Koenderink *et al.*, in press).

Due to size constraints, the virtual reality cachet of *veridicality* usually doesn't apply, which implies that observers will have to come to terms with the 'distortions'. Indeed, one has to *learn* to 'read' pictures in linear perspective, something that must appear incomprehensible to those who, like Gibson (1950), believe that seeing a picture is like looking out of a window. The initial resistance, and final acceptance, of linear perspective has been documented in non-Western cultures like the Japanese (Sasaki, 2013). Panofsky's (1925) notion of perspective as a 'symbolic form' appears quite apt.

Notes

1. Maurice Denis (1870–1943) famously wrote « Se rappeler qu'un tableau, avant d'être un cheval de bataille, une femme nue ou une quelconque anecdote, est essentiellement une surface plane recouverte de couleurs en un certain ordre assemblées » (Denis, 1890).
2. Franz Brentano (1838–1917) wrote 'Diese intentionale Inexistenz ist den psychischen Phänomenen ausschließlich eigentümlich. Kein physisches Phänomen zeigt etwas Ähnliches.' (Brentano, 1874, p. 124).
3. Alexius Meinong (1853–1920) discusses the nature of objecthood in Meinong (1899).
4. The examples of the golden mountain and the round square are taken from Meinong (1899).
5. Picasso was first inspired to experiment with depicting objects from multiple viewpoints after his decisive encounter with African sculpture around 1907 (Richardson, 1996). As in most cultures in the world, there is no historical tradition of linear perspective in African art. To judge by his wide popularity, the apparent distortions in the multi-view Picassos are evidently no longer a source of pictorial distress for the art-loving public but meaningful descriptions of other truths about the way we visually experience objects in the world.
6. We will use the term 'perspective police' to denote the virtual community that is ready to denounce any deviation from strict linear perspective rendering as obviously wrong. This perspective police does not officially exist. It is perhaps more like the secret police as it is supposed to exist in various non-democratic societies. Yet it is a power with remarkable influence, and thus very real.
7. Leon Battista Alberti (1404–1472) wrote *Della Pittura* in 1435. He famously describes perspective as projection on a window, using a wire frame (rete), thus providing a Cartesian (avant la lettre) coordinate frame in the picture plane.
8. Dictionaries define 'stereopsis' as due to binocular disparity. This is wrong. 'Stereopsis' is a general term that does not imply binocularity. The dictionaries should have said 'binocular stereopsis'. The monocular

variant appears mysterious to the mainstream, because one has no causal account. Apparently mainstream observers experience photographs as planar distributions of pigments on a paper substrate. Thus one speaks of ‘paradoxical stereopsis’ in this case (Claparède, 1904; Enright, 1991; Koenderink *et al.,* 1994, 2011; Schlosberg, 1941).

9. One often viewed color slides through a slide viewer, which is a (somewhat primitive) version of the Zeiss Verant (Zeiss and von Rohr, 1904).
10. Cosimo I de Medici founded the Accademia e Compagnia della Arti dei Disegno at Florence in 1563. It taught geometry and anatomy. The Accademia de San Luca, founded 1577 in Rome, was especially concerned with art theory. The latter served as the model for the French Académie Royale de Peinture et de Sculpture (founded 1648), which became the Académie des Beaux-Arts. The French model was followed all over Europe, e.g., the well known British Royal Academy. ‘Academic Art’ ruled till the late nineteenth century. One thinks of artists like Bougereau, Cabanel, or Gérôme to take France as an example.
11. Try ‘graphite drawing’ in Google Images to find numerous examples.
12. On physiological cues see Graham (1966). The designer of the Zeiss Verant, Moritz von Rohr, was a physicist who closely cooperated with the Danish ophthalmologist Alvar Gullstrand. They managed to remove virtually all monocular depth cues listed in the textbooks.
13. Pozzo’s famous ceiling on the cupola of San Ignazio, Rome has the correct position indicated on the floor, and indeed looks terribly wrong if you don’t exactly stand there. The case is discussed in detail in Pirenne (1970).
14. The academies put much value on smooth finish. The brush strokes should be invisible, and the picture surface should not force itself on the observer. Thus impressionist paintings could not be seen as ‘finished’ works. By the end of the 19th century this had changed, although the Denis quote suggests that the idea had not been fully accepted. In the early twentieth century virtually all artists displayed the picture surface as a physical surface, using a variety of means. Similar stories apply to framing. The early gilded, elaborate frames served to isolate the work from its surroundings. By the turn of the 19th into the 20th century the avant garde had dropped the idea of the picture as a hole in the wall, and made conscious efforts to show it as a physical object.
15. The classical frame was ornate and gilded, looking very different from both the picture and the wall. Thus it served to strengthen the isolation of the picture plane from the environment. At the end of the nineteenth

century painters started to paint the frame, later the frame was often omitted altogether.

16. The ‘full frame’ derives from Oskar Barnack’s development at the Leitz factory at Wetzlar of what became the Leitz ‘Leica’ camera in the 1920s. Barnack used 35 mm perforated cine film, and combining two cine images arrived at the 36 × 24 mm format that is still in use.

17. Here ‘professional’ means so much as ‘not specialized’. For special purposes one might use fisheye lenses, or telescopes. This range of fields of view (or focal lengths) is what a well equipped general reporter might carry. The range is defined by technical possibilities on the wide angle size, and pragmatic considerations of weight and size on the tele side of the range.

18. Indeed, if you view the 40 cm wide print from 40 cm viewing distance, the focal length for the ‘normal’ lens should equal the width of the full frame sensor, that is 36 mm. In practice the industry sells ‘normal lenses’ in the range 35–55 mm. Most photographers have a personal preference. If they travel light, they depend on their trusty normal lens. Cameras with fixed lenses, such as the iPhone, use a similar definition.

19. When scanning the horizon with binoculars many people complain of the ‘globe effect’ (Koenderink, 2014; Merlitz, 2010). This is due to distortions that the optical industry has not been able to correct, because the cause is not physical, but psychical.

20. Euclid’s (ca. 300 BC) *Optics* is often understood as a treatise on the propagation of radiation. If you do so, it is obviously wrong, one speaks of the ‘extramission theory of vision’. If you understand the *Optics* as a treatise on the possibilities of vision, that is as a theory of structural information, it makes far more sense. In fact, there is nothing wrong with this treatise (Koenderink, 1982). Another way to misunderstand the *Optics* is to consider it to be a work on linear perspective. If so interpreted, the work is obviously faulty. But again, it is not. When interpreted in a more general way, there is again nothing wrong with the work.

21. This is a constant source of confusion that pervades the literature. The sphere is merely a means to conveniently label visual rays. You might as well use a cube, or any boundary of a convex volume. Thus it has nothing to do with the sphericity of the eyeball, nor is the fact that the pupil is not at the center of the eyeball of any relevance.

22. Again (Note 21) this is a matter of convenience. However, in the ancient cosmographies the various concentric ‘spheres’ played a key role – think of the ‘music of the spheres’.

23. Graf's paper is interesting for illustrating a large range of 'projections', and discussing their mutual advantages/disadvantages. For the illustrations he used the field of view of the Leitz Hektor 2.8 cm (considered wide-angle at the time) on the 36×24 mm Leica image, yielding a two times 37°40' scope.

24. The well known demonstration is the famous poster of Lord Kitchener pointing right at you. This happens from any viewing position (Hagen, 1976; Koenderink *et al.,* 2004). People are often surprised by this, because they confuse visual and pictorial space.

25. The horizontal and vertical saccades are controlled separately (although there is obvious cross-talk). They are different in their amplitudes and speeds (Bahill and Stark, 1975; Sparks, 2002).

26. Of course, exceptions occur, think of interior spaces, or forests. In this paper we concentrate on open spaces, but it is of obvious interest to consider a number of different generic environments. They might lead to alternative constraints.

27. Von Uexküll's (1909) notions of Umwelt, Merkwelt and Wirkwelt capture such differences.

28. In order to see this think of the geographical globe. As the globe rotates the latitude circles shift within themselves, whereas the meridians are displaced from one to the other.

29. The horizontals are small circles on the viewing sphere with the vertical as axis. Thus a central projection on a plane that contains the vertical will yield a parabola.

30. A central projection on a plane will by necessity introduce deformations. These deformations are like an 'inverse foreshortening' in the sense that they necessarily disappear if you view the projection from its center.

31. Horizontal shifts of the picture would be induced as you assume different viewpoints in the typical wall painting setup.

32. That is why an artist like Adolf Hildebrand (1893) considers it unartistic, and essentially in the same category as panoramas (this was the late 19th c.) or Madame Tussaud's.

33. 'Size constancy' implies that you experience size changes of smallish objects in the optic array as changes of distance from the vantage point.

34. Parallel projection involves a double limit: both the angular size of the field of view, and the relative distance range of visible objects should be small. For instance, in conjunctions of Jupiter and the moon, the former subtends a much

smaller angle (about forty seconds of arc) than the latter (about thirty minutes of arc), despite the fact that Jupiter is immensely larger than our moon.

35. It is not uncommon to see huge wall paintings in perspective in state rooms of castles. Then the perspective center is often situated in the main entrance of the room. So one has a 'right' view on entering when the wall appears in pictorial mode, and once inside shifts to visual object mode, and treats the painting as mere wall paper.
36. The 'Claude glass' was a popular viewing device. It served various functions; one of these was the minification of the apparent scene.
37. Since the early rangefinder cameras (Leica and Contax) had a fixed finder magnification, additional finders were commonly used before 1950.
38. The Mercator map is a cylindrical 'projection' presented by the Flemish geographer and cartographer Gerardus Mercator in 1569. Mercator published it as a huge planisphere measuring 202×124 cm printed in eighteen sheets. Latitude and longitude circular arcs are straight and mutually perpendicular. The map does not fully show the polar areas, because the magnification becomes arbitrarily high at the poles. The map is conformal, that is to say angles are preserved around all locations, whereas magnification varies with latitude. Formally, the horizontals are just a Cartesian representation of longitude, where the verticals are scaled according to the inverse Gudermannian function of the latitude. In terms of standard functions the inverse Gudermannian of variable x can be written as, for instance, $\log(\sec x + \tan x)$, although various equivalent expressions are in common use.
39. For an Abelean group the concatenation of group operations is commutative, that is to say, the result is independent of the order. The group of rotations in space fails to be Abelian, but the translation group is.
40. Although the group of rotations in space fails to be Abelian, the subgroup of rotations satisfying Listing's law is. As a result, a sequence of rotations leaves the eyeball in a well-defined spatial attitude.
41. Stereographic projection was first described by Ptolemy (2nd c. AD). It allows one to make a flat model of the sphere. Thus the astrolabe is a flat, conveniently pocketable, armillary sphere. Its simple geometrical properties have been known since antiquity. They allow one to do spherical constructions in a convenient manner, not much more complicated than planar Euclidean geometry. One uses circular arcs instead of straight lines.
42. The Flocon and Barre rendering is due to Guillaume Postel (1510–1581). Formally, it is an application of 'Riemann (1854) normal coordinates', but, of course, Postel was ignorant of that. Riemann normal coordinates

have various formal advantages that are irrelevant to our investigation, except for the fact that they minimize metrical deformations.

43. Flocon and Barre proposed Postel's map, which is different from stereographic projection. But, perhaps surprisingly, their proposed constructions are essentially those of stereographic geometry. Thus their drawing instructions are only approximate. This inconsistency doesn't seem to have bothered the authors, nor to have been noticed.
44. Ptolemy's '*Planisphere*'.
45. The industry has somehow agreed on this 'linear angle' representation (Kumler and Bauer, 2000). This is perhaps surprising, as a — technically equally feasible — design on the basis of the stereographic projection would have allowed them to claim their fisheye lenses to be 'distortion free'.
46. The figure in Helmholtz's book cannot be viewed as intended unless you copy it, and print a much enlarged copy of it.
47. In Guido Hauck's view most of us are collinearly infected, with the possible exception of natural man, especially women. Apart from being politically incorrect, this seems untrue today. Any Western intellectual is likely to expect straight lines in architectural scenes.
48. One classically studied deformations locally, by way of the Tissot (1881) indicatrices. However, for artistic purposes one might prefer more global measures (Goldberg and Gott III, 2007).
49. Near the horizon the sky tends to become a less saturated blue, in fact often turns whitish. This is an immediate consequence of Koschmieder's (1924) theory of the air light.
50. With the increasing availability of 3D scanners and printers this has become a viable option.

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