Equivalence of D-Brane Categories

Equivalentie van D-Braan Categorieën

(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. J. C. Stoof, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op maandag 2 maart 2009 des middags te 12:45 uur

door

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geboren op 4 juli 1977, te Manizales, Colombia

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Het onderzoek voor dit proefschrift is gefinancierd door de Koninklijke Nederlandse Akademie van Wetenschappen in het kader van het Akademiehoogleraarschap van Prof. dr. J. J. Duistermaat.

Lo que limita lo verdadero no es lo falso, sino lo insignificante.
— René Thom

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INTRODUCTION

MOTIVATION

The appearance in string theory of higher-dimensional objects known as D-branes has opened new perspectives on geometry. Of fundamental importance is the understanding that spacetime becomes a secondary concept, from the D-brane perspective, rather than a primary given. Armed with this insight, it is hoped that a deeper understanding of the nature of spacetime would be found within the study of D-branes. Various parts in the present thesis must be viewed in the light of this on-going search. To help place our analysis into context, we begin with a brief overview of how D-brane physics reflects the underlying geometry of spacetime.

In Einstein's theory of general relativity, gravity is intimately tied to geometry: general relativity, that is, classical gravity, is most concisely formulated in differential geometric terms. In that framework, one is looking for manifolds M with metrics g that solve Einstein's equations. It is precisely such pairs (M,g) that legally acquire the epithet 'spacetime'. In comparison with Newton's formulation of gravity, of which general relativity is a refinement, two features are new: firstly spacetime now becomes a dynamical entity, the evolution of which is governed by the Einstein's equations. Secondly, general relativity reveals the strong entanglement between gravity and geometry, thus connecting physical and mathematical data.

Since theoretical physicists appreciate a flavour of elegance in such physics-geometry links, Einstein's theory stimulated a geometrisation trend in physics: classical geometry was probed to 'explain' features of theoretical models. In more recent times, developments in superstring theory have pushed this process to its limits. Let us spell out more clearly what we mean by this.

In a given spacetime (M,g), a string sweeps out a two-dimensional surface W as it evolves in time; W describes the history of the string, so to speak. To separate worldsheet and spacetime has turned out to be a powerful convenience: given a two-dimensional surface Σ, the worldsheet as it is called, one studies how it is embedded in the spacetime manifold (M,g), the target space for short. Let us denote the embedding map $\phi:\Sigma\to M$.

String theory describes a profound relation between a quantum field theory on a worldsheet Σ and a corresponding quantum field theory on the target space M. In actuality, we are interested in a special case of this construction in which the theory has what is called $N=2$ supersymmetry and is in addition conformally invariant. These requirements place severe restrictions on M. Namely, M must be a complex Kähler manifold for $N=2$

supersymmetry and the metric g must be Ricci flat. It also turns out that for such superstring theories the dimension of M must be 10. Typically, we meet these constraints by taking $M = \mathbb{R}^{1,3} \times X$ where $\mathbb{R}^{1,3}$ is the familiar 4-dimensional Minkowski space and X is a compact Calabi-Yau threefold. Essentially all the nontrivial structure comes from the X part of the model. Hence, we focus our attention on X.

D-branes made their appearance in string theory via considerations of boundary conditions. Boundary conditions come naturally with open strings, and vice versa. For one thing, the latter sweep out surfaces Σ with boundaries. The requirement that the endpoints of the string be confined to a given submanifold $S \subset X$ imposes the Dirichlet boundary condition that $\phi : \partial\Sigma \to S$ (with a natural generalisation to worldsheet fermionic fields, if any). Accordingly, S becomes a geometric locus 'where open strings can end'. From the particular boundary condition type, these submanifolds borrowed their name 'D(irichlet)-branes'. In a some what loose language, D-branes get identified with (sets of) boundary conditions.

Not all submanifolds are allowed as viable D-branes. For instance, a consistent choice of boundary conditions must preserve the fundamental conformal invariance of the string theory. Determining the allowed D-branes in a given string background X is an extremely difficult problem which requires having the two-dimensional quantum field theory on the worldsheet under control. So we have to find some simplification. This leads us to the topological string.

Topological string theory is a simplified version of string theory. It may be obtained by a topological 'twisting' of the worldsheet description of ordinary string theory. As indicated by the name, topologically twisted theories are independent of the metric on the worldsheet, i.e. they are topological field theories. The topological twist can be performed in two different ways, which leads to two a priori different theories. These are usually referred to as the A-model and the B-model.

It is also possible to include D-branes in the topological models discussed above. In this setting D-branes are identified with objects in certain triangulated categories. One can think of these categories as the enriched versions of the A and B-models: while A and B-models describe closed strings, A and B-models with boundaries describe closed and open strings with all possible boundary conditions.

'New' types of D-branes show up in "non-geometric" phases of string theory, such as Landau-Ginzburg models. In such phases, spacetime is represented by some abstract topologically twisted $N = 2$ field theory. Obviously, the spacetime picture of D-branes as submanifolds then breaks down, since there is no such thing as spacetime. However, as it tuns out, the collection of boundary conditions may still be regarded as a category, and thereby used to extend the notion of D-branes to these "non-geometric" phases. In this framework, the category of D-branes arises as a natural candidate for the category of sheaves on some noncommutative "space". It is via this route that the "non-geometric" backgrounds can be interpreted as noncommutative spaces. This ties in with a general philosophy, emphasized by M. Kontsevich, that to do geometry you really don't need a space, all you need is the category of sheaves on this would-be space. In this sense, one can think of string theory as providing us with a generalisation of ordinary classical geometry. Namely, one can seek to formulate a new geometrical discipline whose basic ingredients are the D-branes.

The central theme of this thesis is the construction of equivalence of D-brane categories, in three chapters, each treating a problem where the connection between different descriptions of D-branes is visible. These equivalent ways to express the same category yield isomorphic noncommutative spaces when taken as backgrounds for string propagation.

OVERVIEW OF THE THESIS

A systematic description of the material is supplied by the introduction to individual chapters. Here we shall just give a brief overview of each chapter together with the statements of the major results.

CHAPTER 1. The original McKay correspondence starts with a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ and its natural linear action on $\mathbb{C}^2$. It was observed by McKay that the irreducible components of the exceptional divisor in the minimal resolution Y of the associated quotient singularity $\mathbb{C}^2/G$ are in one-to-one correspondence with the nontrivial irreducible representations of G. This suggests a deep relation between the geometry of Y and the G-action on $\mathbb{C}^2$. Gonzalez-Sprinberg and Verdier [38] reformulated McKay's observation as an isomorphism of the Grothendieck group of Y and the representation ring of G. In the language of derived categories, Kapranov and Vasserot [47] proved that this isomorphism lifts to an equivalence of the derived category of coherent sheaves on Y and the derived category of G-equivariant coherent sheaves on $\mathbb{C}^2$. Later on, Bridgeland, King and Reid [16] generalized the result from [47] by taking an arbitrary finite subgroup G of $\mathrm{SL}(n,\mathbb{C})$ acting on $\mathbb{C}^n$.

The appearance of the derived category in the latter version of the McKay correspondence has a very natural interpretation in terms of gauged linear sigma models with boundaries. The category of topological D-branes in a "large volume" phase is the derived category of coherent sheaves on Y, while in the "orbifold" phase it is the category of G-equivariant coherent sheaves on $\mathbb{C}^n$. The result follows at once from the observation that we must have an equivalence of D-brane categories for any pair of phases.

The idea of using the gauged linear sigma model to analyze the phase structure of D-branes and the resulting categorical equivalences leads naturally to a generalization of the McKay correspondence for Landau-Ginzburg models. We next review the basic setting. Let $M = \mathbb{C}^n$ and consider a Landau-Ginzburg superpotential $f : M \to \mathbb{C}$ with an isolated critical point at the origin and its orbifold with respect to the action of some finite subgroup G of $\mathrm{SL}(n,\mathbb{C})$. We assume that G acts on M freely outside the origin, which means that $X = M/G$ has an isolated singularity. Let Y be the irreducible component of the G-Hilbert scheme of M which contains the G-clusters of free orbits and consider the Landau-Ginzburg model (Y, g), where g is the pullback of f to Y. Our main result is as follows. (This is proved as Theorem 1.8.6.)

Theorem A. *Let the setting be as above. If* $\dim(Y \times_X Y) \leq n+1$*, then the category of singularities associated to the Landau-Ginzburg model* (Y, g) *is equivalent to the* G*-equivariant category of singularities associated to the Landau-Ginzburg orbifold* (M, f)*.*

The proof of this theorem relies heavily on the aforementioned result of Bridgeland, King and Reid, which reformulate and generalize the McKay correspondence in the language of derived categories, along with the techniques introduced in [25].

Following Orlov [70], we also establish a connection between categories of D-branes in Landau-Ginzburg orbifolds and equivariant categories of singularities. Our result is the following. (This is proved as Theorem 1.7.3.)

Theorem B. *The category of G-equivariant matrix factorizations on an affine Landau-Ginzburg orbifold (X, W) is equivalent to the G-equivariant category of singularities of the singular fiber of the map W.*

Using this result, we can now rephrase Theorem A as follows: The category of D-branes in the Landau-Ginzburg model (Y, g) is equivalent to the category of D-branes in the Landau-Ginzburg orbifold (M, f).

CHAPTER 2. Recently, Van den Bergh introduced the notion of noncommutative resolution of a singular variety and used it to reproduce and somewhat generalize Bridgeland's work on derived equivalences of flops [92, 91]. The leading principle behind this construction is that for some singularities it is possible to find a noncommutative algebra A such that the representation theory of this algebra dictates in every way the process of resolving these singularities. To be more precise, this means that all the information of the singularity can be encoded in the center of A, and that "nice" resolutions of the singularity can be constructed as moduli spaces of representations of A. What is more, the category of finitely generated modules over A is derived equivalent to the category of coherent sheaves of any resolution.

This phenomenon also appears naturally in string theory in the context of reverse geometric engineering, where one can reconstruct the singularities from quiver gauge theories. The description is in terms of a noncommutative algebra A which is constructed from the quiver diagram and the superpotential. The algebraic geometry of the center of this algebra will be identified with the target space where closed strings propagate, which might be singular, and the algebraic geometry of the noncommutative algebra A will give the resolution of those singularities. In a similar vein, one can expect that representations of A parametrize D-branes on any resolution.

Our focus is on ADE geometries. In order to state our main results, let us introduce some notation. Let us consider a particular ADE singularity X_0 and its minimal resolution $Y_0 \to X_0$. Given the semiuniversal deformation $\mathscr{X}$ of X_0 parametrized by $\mathfrak{h}/W$ as described explicitly in Sect. 2.2.1, a semiuniversal resolution $\mathscr{Y}$ exists over $\mathfrak{h}$, so that the family $\mathscr{Y}$ is the semiuniversal deformation of Y_0. We can now pullback the simultaneous resolution via a sufficiently general polynomial map $f : \mathbb{C} \to \mathfrak{h}$, obtaining a smooth threefold Y over $\mathbb{C}$ which maps to a possibly singular threefold X. For any λ with $f(\lambda) = 0$, there will be an ADE configuration of curves lying over λ.

Now let Q be an extended Dynkin quiver with vertex set I and let δ be the minimal positive imaginary root for Q. We denote by $\widehat{Q}$ the quiver which is the standard digraph associated to Q except that we add a loop u_i at each node $i \in I$. The $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$ of Q is the quotient of the path algebra of $\widehat{Q}$ by the relations described in Sect. 2.4.2. As it turns out, the fibration data is determined by the choice of τ. We write $\mathrm{Rep}(\mathfrak{A}^\tau(Q), \delta)$ for the variety of $\mathfrak{A}^\tau(Q)$-module structures on $\mathbb{C}^\delta = \bigoplus_i \mathbb{C}^{\delta_i}$. Elements of $\mathrm{Rep}(\mathfrak{A}^\tau(Q), \delta)$ are representations of $\mathfrak{A}^m(Q)$ of dimension vector δ. The group $\mathrm{G}(\delta) = \left(\prod_i \mathrm{GL}(\delta_i, \mathbb{C})\right)/\mathbb{C}^\times$ acts naturally on this variety and the orbits correspond to isomorphism classes of representations. We denote by $\mathcal{R}_Q(\tau, \delta)$ the closed subspace of

$\mathrm{Rep}(\mathfrak{A}^\tau(Q),\delta)$ corresponding to representations V for which there exists a λ such that, if $v_i \in V_i$, then $V_{u_i} v_i = \lambda v_i$. We are now ready to state the following. (This is proved as Theorem 2.4.5.)

Theorem C. *The affine quotient variety $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta)$ is isomorphic to the ADE fibered Calabi-Yau threefold X associated with Q and τ.*

Using GIT quotients it is possible to construct resolutions of singularities. If $\theta \in \mathbb{Z}^I$ satisfies $\theta \cdot \delta = 0$, then thre are notions of θ-stable and θ-semistable elements of $\mathcal{R}_Q(\tau,\delta)$, there is a GIT quotient $\mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta)$ and a projective morphism

$$\pi_\theta \colon \mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) \longrightarrow \mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta) \cong X.$$

We prove the following result. (This is proved as Theorem 2.4.8.)

Theorem D. *If θ is generic, then π_θ is a small resolution of the ADE fibered Calabi-Yau threefold X associated with Q and τ.*

Finally, it is shown that the $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$ contains exactly the same homological information as the small resolution Y. The precise result we prove is the following. (This is proved as Theorem 2.5.12.)

Theorem E. *The category of finitely generated modules over $\mathfrak{A}^\tau(Q)$ is derived equivalent to the category of coherent sheaves on the threefold Y.*

Thus the preceding theorem asserts (among other things) that D-branes on the fibered threefold Y are classified by representations of the $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$; this matches the physicists' predictions.

CHAPTER 3. In this expository chapter we review the Homological Mirror Symmetry for toric Del Pezzo surfaces. Aside perhaps with respect to the exposition, this chapter makes no claims of originality. It is meant rather to be an elementary introduction to some of the core topics of Homological Mirror Symmetry for toric Fano surfaces. Our primary goal is to show explicitly how things work in the by now familiar case of $\mathbb{P}^2$ with one point blown up. While we shall only study this one example, it should be mentioned that all of the techniques we use can be generalized to arbitrary toric Fano surfaces. Though the level of complexity grows in general the crux of Homological Mirror Symmetry is well captured by this example.

1

MCKAY CORRESPONDENCE FOR LANDAU-GINZBURG MODELS

In this chapter we prove an analogue of the McKay correspondence for Landau-Ginzburg models. Our proof is based on the theorem of Bridgeland, King and Reid [16], which gives the McKay correspondence on the derived category level. This chapter is an expanded version of the preprint [74].

1.1 INTRODUCTION

In its original form, the McKay correspondence was observed as a nice relation between the irreducible representations of a finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$ on the one hand, and the geometry of the exceptional divisor in a minimal resolution of $\mathbb{C}^2/G$ on the other hand (cf [65]). The first hint of a McKay correspondence in higher dimensions came from the work of L. Dixon, J. Harvey, C. Vafa and E. Witten. It was conjectured in [33] that for a finite subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$ acting on $\mathbb{C}^n$, the Euler characteristic of a crepant resolution Y of the quotient space $\mathbb{C}^n/G$ equals the number of conjugacy classes, or equivalently the number of equivalence classes of irreducible representations of G. If $n=2$, the equality can be viewed as a version of the McKay correspondence. As a result, this formula may be regarded as a generalization of the McKay correspondence to an arbitrary dimension n. The McKay correspondence became recently a subject of intense study in both physics and mathematics. However, the term is now primarily used to indicate a relationship between the various invariants of the actions of finite automorphism groups on quasiprojective varieties and resolutions of the corresponding quotients by such actions.

The guiding principle behind the McKay correspondence was stated by M. Reid along the following lines:

Principle 1.1.1. Let M be an algebraic variety, G a group of automorphisms of M, and Y a crepant resolution of singularities of $X = M/G$. Then the answer to any well posed question about the geometry of Y is the G-equivariant geometry of M.

Applied to the case of quotient singularities $X = \mathbb{C}^n/G$ arising from a finite subgroup $G \subset \mathrm{SL}(n,\mathbb{C})$, the content of this slogan is that the G-equivariant geometry of $M = \mathbb{C}^n$ already *knows* about the crepant resolution Y. In particular, any two crepant resolutions of X should have equivalent geometries.

Reid suggested that one manifestation of Principle 1.1.1 should be a derived equivalence $\mathbf{D}(Y) \cong \mathbf{D}^G(M)$, where $\mathbf{D}(Y)$ is the bounded derived category of coherent sheaves on Y and $\mathbf{D}^G(M)$ is the bounded derived category of G-equivariant coherent sheaves on M. This has been worked out by Kapranov and Vasserot [47] in dimension $n = 2$ and generalized to higher dimensions including all cases of finite subgroups of $\mathrm{SL}(3,\mathbb{C})$ by Bridgeland, King and Reid [16]. In the latter case the quotient singularity $X = \mathbb{C}^3/G$ always has a crepant resolution, a distinguished choice being given by the Hilbert scheme of G-orbits G-Hilb(M). This scheme is perhaps best thought of as a moduli space of representations of the skew group algebra $A = \mathbb{C}[x,y,z]\#G$ that are stable with respect to a certain choice of stability condition. Indeed, this is closely related to the physicist's understanding of D-branes as objects in the derived category.

In string theory, space-time X is represented by a two-dimensional quantum field theory with $N = 2$ supersymmetry. A quite important class of such theories are nonlinear sigma models on a Kähler manifold X. In this case, E. Witten explained how to manufacture two dimensional topological field theories. He showed that any nonlinear sigma model with a Kähler target space X admits a topologically twisted version called the A-model; if X is a Calabi-Yau manifold, there is another topologically twisted theory, the B-model. A similar construction exists in the equivariant setting. Given an action of a finite group G on a space X satisfying certain properties, one can construct a two-dimensional topological field theory which represents the G-equivariant physics of X. To be more precise, one associates a G-gauged sigma model to a presentation of the quotient stack $[X/G]$: the gauged sigma model can be interpreted as a sigma model on $[X/G]$.

Open strings are associated to extended objects, different from strings, which go under the name of D-branes. Loosely speaking, a D-brane is a 'nice' boundary condition for the two-dimensional quantum field theory. To any topologically twisted sigma model one can associate a category of D-branes. In the case of the topological B-model of a Calabi-Yau X, the category of D-branes is believed to be equivalent to the bounded derived category $\mathbf{D}(X)$ of coherent sheaves on X. In the equivariant setting this should be replaced by the bounded derived category $\mathbf{D}([X/G]) \cong \mathbf{D}^G(X)$ of G-equivariant coherent sheaves on X.

From the previous consideration we see that the McKay correspondence has a completely natural explanation in terms of nonlinear sigma models with boundaries. Indeed, arguments from topological open string theory, formalized in the 'decoupling statement' of [18], suggest that there is an equivalence $\mathbf{D}(Y) \cong \mathbf{D}([M/G])$ for any crepant resolution Y of the singularities of $X = M/G$.

In this chapter we study another class of topological field theories: topological Landau-Ginzburg models. The general definition of a Landau-Ginzburg model involves, besides a choice of a target space X, a choice of a holomorphic function $W : X \to \mathbb{C}$ called a superpotential. In particular, non-trivial Landau-Ginzburg models require a non-compact target space X. For a smooth affine variety $X = \operatorname{Spec} A$, a simple description of the category of D-branes in Landau-Ginzburg models has been proposed by M. Kontsevich and derived from physical considerations in [48]. It turns out that the category of D-branes is equivalent to the category $\mathrm{MF}(W)$ of matrix factorizations of W.

For non-affine X the following construction was proposed [70]. Suppose that we are given a Landau-Ginzburg superpotential $W : X \to \mathbb{C}$ with a single critical value at $0 \in \mathbb{C}$. Let X_0 denote the fiber of W over 0. Consider the bounded derived category of coherent sheaves on X_0. A perfect complex is an object of $\mathbf{D}(X_0)$ which is quasi-isomorphic to a bounded complex of locally free sheaves. One can define a triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(X_0)$ as the quotient of $\mathbf{D}(X_0)$ by the full subcategory of perfect complexes $\mathrm{Perf}(X_0)$. If X_0 were non-singular, the quotient would be trivial, since in that case any object in $\mathbf{D}(X_0)$ would have a finite locally free resolution. Therefore $\mathbf{D}_{\mathrm{Sg}}(X_0)$ depends only on the singular points of X_0. The main result of [70] is that the category of matrix factorizations $\mathrm{MF}(W)$ for a smooth affine $X = \operatorname{Spec} A$ is equivalent to $\mathbf{D}_{\mathrm{Sg}}(X_0)$. Thus for non-affine X the category $\mathbf{D}_{\mathrm{Sg}}(X_0)$ can be considered as a definition of the category of D-branes.

One may also consider Landau-Ginzburg models on orbifolds. Such models are particularly important because they provide an alternative description of certain Calabi-Yau sigma models. In the affine case D-branes are described by the category $\mathrm{MF}^G(W)$ of G-equivariant matrix factorizations, cf. [1, 2] and Section 1.6 of this chapter. In general, one may consider a full subcategory of perfect complexes $\mathrm{Perf}([X_0/G])$, which is formed by bounded complexes of locally free sheaves in $\mathbf{D}([X_0/G]) \cong \mathbf{D}^G(X_0)$, and also the quotient category $\mathbf{D}^G_{\mathrm{Sg}}(X_0) = \mathbf{D}^G(X_0)/\mathrm{Perf}([X_0/G])$. In Section 1.7 we show that the category of G-equivariant matrix factorizations $\mathrm{MF}^G(W)$ for a smooth affine $X = \operatorname{Spec} A$ is equivalent to $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$.

Let us assert our version of the McKay correspondence for Landau-Ginzburg models. Consider the Landau-Ginzburg model on the affine space $M = \mathbb{C}^n$ with polynomial superpotential $f : M \to \mathbb{C}$ and its orbifold with respect to the action of some finite subgroup G of $\mathrm{SL}(n, \mathbb{C})$. Let $\tau : Y \to M/G$ be a crepant resolution and consider the Landau-Ginzburg model (Y, g), where g is the pullback of f to Y. We expect the following to hold.

Assertion 1.1.2. The category of D-branes in the Landau-Ginzburg model (Y, g) is equivalent to the category of D-branes in the Landau-Ginzburg orbifold (M, f).

In this chapter we prove a special case of this assertion. The main result is the following. Consider the Landau-Ginzburg orbifold defined by (M, f), where the superpotential f is a regular G-invariant function with an isolated critical point at the origin and G is a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$ which acts on $M = \mathbb{C}^n$ freely outside the origin. In the circumstances described in Section 1.8, a crepant resolution is given by the irreducible component $Y \subset G\text{-Hilb}(M)$ dominating $X = M/G$. Then the category of singularities $\mathbf{D}_{\mathrm{Sg}}(Y_0)$ of the fiber Y_0 is equivalent to the G-equivariant category of singularities $\mathbf{D}^G_{\mathrm{Sg}}(M_0)$ of the fiber M_0. Bearing in mind that the categories of singularities are equivalent to the categories of D-branes we obtain the connection between D-branes mentioned above.

To finish this introduction we make some remarks of a more philosophical nature. Noncommutative geometry, as propagated by M. Kontsevich in [56] is based on the idea that to do geometry you really don't need a space, all you need is a category of sheaves on this would-be space. A noncommutative space X is a small triangulated $\mathbb{C}$-linear category $\mathcal{C}_X$ which is Karoubi closed and enriched over complexes of $\mathbb{C}$-vector spaces (this notion is explained in detail in [23]). If X is a smooth scheme of finite type, then X can be considered as a noncommutative space with $\mathcal{C}_X = \mathbf{D}(X)$. Any Landau-Ginzburg model (X, W) is also a noncommutative space with $\mathcal{C}_{(X,W)} = \mathbf{D}_{\mathrm{Sg}}(X_0)$. We see that the physical meaning of noncommutative space is to replace the space by the category of D-branes. If we return to the McKay correspondence, then we deduce that the noncommutative space Y is isomorphic to the noncommutative space $A = \mathbb{C}[x_1, \ldots, x_n]\#G$. This leads naturally to a generalized notion of McKay correspondence as an isomorphism of noncommutative spaces. Note that this fits well with M. Reid's Principle 1.1.1, where the word 'geometry' was left deliberately vague. We can restate assertion 1.1.2 by saying that the Landau-Ginzburg model (Y, g) and the Landau-Ginzburg orbifold (M, f) are isomorphic as noncommutative spaces.

Note. After the preprint [74] was posted on the arXiv, I have learned that similar results were obtained by S. Mehrotra in his PhD dissertation [66]. In the situation described above, he has shown that the G-equivariant category of singularities $\mathbf{D}^G_{\mathrm{Sg}}(M_0)$ embeds fully and faithfully into the category of singularities $\mathbf{D}_{\mathrm{Sg}}(Y_0)$. However, Mehrotra approach is different to ours in that it does not use the techniques of [25] in the context of the generalized McKay correspondence. Our proof uses in an essential way these techniques. It is a natural question to try and understand to what extent the result really depends on the derived McKay correspondence, but not a question we explore in this thesis.

1.2 THE PHYSICAL ARGUMENT

It is instructive to look at the physical argument involved in justifying assertion 1.1.2. The set-up is the so-called gauged linear sigma model.

The gauged linear sigma model is a very useful model which in an appropriate sense 'interpolates' between nonlinear sigma models on Calabi-Yau manifolds and Landau-Ginzburg orbifolds. Such a model is determined by a "radial" parameter r.

Here are some of the basic ideas concerning gauged linear sigma models [94]. We will just indicate enough details to see the parameter r appearing. Let us consider the U(1) gauge theory with n chiral matter superfields $X_1, \ldots, X_n$ of charge 1, and one chiral superfield P of charge $-n$. We also consider a twisted chiral superfield Σ with values in the complexification of the adjoint bundle over 2|4-superspace. Write each of these superfields in components

$$\begin{aligned} X_i &= x_i + \theta(\cdots) + \cdots \\ P &= p + \theta(\cdots) + \cdots \\ \Sigma &= \sigma + \theta(\cdots) + \cdots \end{aligned}$$

The bosonic potential is a function $V = V(x, p, \sigma)$ of the bosonic components of these superfields. It has the form

$$V = \frac{1}{2e^2} D^2 + |\sigma|^2 \Big(\sum_{i=1}^{n} |x_i|^2 + n^2 |p|^2 \Big).$$

The "D-term" is equal to

$$D = \sum_{i=1}^{n} |x_i|^2 - n|p|^2 - r.$$

This is actually a familiar function mathematically; it is the moment map generating the U(1)-action on the flat Kähler manifold $Z = \mathbb{C}^{n+1}$ with coordinates $x_1, \ldots, x_n$ and p.

The moduli space of classical vacua –that is, the special field configurations of minimal energy– for this theory is

$$\mathcal{M}_{\text{vac}} = V^{-1}(0)/\,\mathrm{U}(1).$$

The quotient by U(1) comes from the gauge symmetry. So we need to set $V = 0$ and divide by U(1). Thanks to the form of the potential, this requires that $D = 0$, and either $\sigma = 0$ or $\sum_i |x_i|^2 + n^2|p|^2 = 0$. Now, setting $D = 0$ and dividing by U(1) is the familiar mathematical operation of symplectic reduction, in which $D = 0$ defines a level set for the moment map of the U(1)-action (with the choice of r specifying the level). There is another mathematical interpretation of this process, as a quotient in the sense of GIT: we complexify the group U(1) to $\mathbb{C}^\times$ and consider the action of $\mathbb{C}^\times$ on $Z = \mathbb{C}^{n+1}$ with the same weights as before (the x_i's have weight 1 and p has weight $-n$).

It turns out that there are two possible GIT quotients depending upon the sign of r. For $r > 0$, $D = 0$ implies that not all x_i can vanish and thus σ must be zero. The variable p is free as long as the condition $D = 0$ is satisfied. Owing to these, the quotient can be interpreted as the total space $Y = \operatorname{tot}(\mathcal{O}_{\mathbb{P}^{n-1}}(-n))$ of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ (p serves as a fiber coordinate). For $r < 0$, vanishing of the D-term requires that $p \neq 0$. We can therefore use the $\mathbb{C}^\times$-action on (x_i, p) to set $p = 1$. This leaves a residual invariance under the subgroup $G = \mathbb{Z}_n$ on U(1) (because p has charge $-n$). Thus, the quotient is $\mathbb{C}^n/G$. This will therefore be what is known as an *orbifold* theory.

Let us note that r determines the "size" of the non-compact Calabi-Yau manifold Y. In this sense, the variable r can be thought of as determining the Kähler modulus of the theory. Geometrically, taking $r \to 0$ corresponds to blowing-down the $\mathbb{P}^{n-1}$ at the base of the line bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ and the geometry becomes isomorphic to $\mathbb{C}^n/G$.

The real Kähler modulus r is complexified by the θ-angle of the gauged linear sigma model (which becomes the B-field in string theory) through the combination $\frac{\theta}{2\pi} + \mathrm{i}\,r$, and the complexified Kähler moduli space has two phases. When $r \gg 0$ the infrared fixed point of the gauged linear sigma model is a nonlinear sigma model on the target space Y and this is called the Calabi-Yau phase. The phase $r \ll 0$ corresponds formally to an analytic continuation to negative Kähler class. For $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$ this means "negative size" of the $\mathbb{P}^{n-1}$, i.e., we pass to the blow-down phase where the $\mathbb{P}^{n-1}$ has been collapsed to a point, and the target is $\mathbb{C}^n/G$. The singularity at $r = 0$ can be avoided by turning on a non-zero θ-angle.

We are particularly interested in trying to understand D-branes (in particular, D-branes with B-type boundary conditions) in gauged linear sigma models with boundary. In the Calabi-Yau phase the category of D-branes is $\mathbf{D}(Y)$, the derived category of coherent sheaves on Y. In the orbifold phase, this should be replaced by the derived category $\mathbf{D}^G(\mathbb{C}^n)$ of G-equivariant sheaves on $\mathbb{C}^n$. We can try to use the boundary gauged linear sigma model as a tool to "flow" the category $\mathbf{D}^G(\mathbb{C}^n)$ to the category $\mathbf{D}(Y)$, thus realizing the equivalence of the two categories by means of a physical system. Thus D-branes give a completely natural explanation of the McKay correspondence in terms of the interpolation between small and large "volume" phase of a gauged linear sigma model with boundary.

Now it is time to supplement the gauged linear sigma model by a superpotential $W\colon Z \to \mathbb{C}$. It must be a holomorphic function on $Z = \mathbb{C}^{n+1}$. We are chiefly interested in superpotentials of the form $W = pf(x_1, \ldots, x_n)$, where f is a general homogeneus polynomial of degree d. The potential energy for this linear sigma model is

$$V = \frac{1}{2e^2}D^2 + |f|^2 + |p|^2|\mathrm{d}f|^2 + |\sigma|^2\Big(\sum_{i=1}^{n}|x_i|^2 + n^2|p|^2\Big).$$

Let us restrict attention to polynomials that are *transverse*, meaning that the equations $f = \mathrm{d}f = 0$ have no simultaneous solutions except at the origin. This implies that the hypersurface S of $\mathbb{P}^{n-1}$ defined by $f = 0$ is a smooth complex manifold. Moreover, if $d = n$ then S is a Calabi-Yau manifold. We will assume this in the sequel.

Let us analyse the spectrum of the classical theory. As before, the structure of the moduli space of classical vacua is different for $r > 0$ and $r < 0$, and we will treat these two cases separately.

First, let us take $r > 0$. In this case, $D = 0$ requires at least one x_i to be nonzero, forcing σ to vanish. If we assume $p \neq 0$, the equations $f = \mathrm{d}f = 0$ with the transversality condition imply that all x_i must vanish. However, this is inconsistent with $D = 0$. Thus p must be zero. Our equations for classical vacua become $p = 0$, $\sum_i |x_i|^2 = r$, and $f = 0$, and we must divide by the action of the gauge group U(1). This gives the hypersurface S defined by the equation $f = 0$ in $\mathbb{P}^{n-1}$, with Kähler modulus r. Thus, classically our theory can be described as a nonlinear sigma model whose target space is this hypersurface S.

Let us move to the case $r < 0$. The space of classical vacua satisfies $x_i = 0$ and $n|p|^2 = -r$. We can use a gauge transformation to fix $p = \sqrt{-r/n}$, leaving a residual gauge invariance of $G = \mathbb{Z}_n$. The local description of the theory is this: for $r \ll 0$, the field P has a large mass and can be integrated out, leaving an effective theory of n massless chiral superfields $X_1, \dots, X_n$ with an effective interaction

$$W_{\mathrm{eff}} = \mathrm{const} \cdot f(x_1, \dots, x_n).$$

Such a theory of n massless fields with a polynomial interaction is called a Landau-Ginzburg model. We should notice, however, that the Landau-Ginzburg model is not an ordinary one, but a G-gauge theory. Physical fields must be invariant under the G-action, and the configuration must be single-valued only up to the G-action. Such a gauge theory is usually called a Landau-Ginzburg orbifold.

In this way, the gauged linear sigma model interpolates between the Landau-Ginzburg orbifold and the Calabi-Yau nonlinear sigma model. These two regions can be considered as a sort of analytic continuation of each other.

In both these theories we know how to describe topological D-branes. In the Calabi-Yau phase the D-brane category is the derived category $\mathbf{D}(S)$ of coherent sheaves on S. In the Landau-Ginzburg phase, D-branes are realized as G-equivariant matrix factorizations of f. Using the gauged linear sigma model realization, the previous discussion naturally leads to the statement that there should be an equivalence of categories $\mathbf{D}(S) \cong \mathrm{MF}^G(f)$, where $\mathrm{MF}^G(f)$ is the category of G-equivariant matrix factorizations of f.

Now, we can consider $Y = \mathrm{tot}(\mathscr{O}_{\mathbb{P}^{n-1}}(-n))$ as a Landau-Ginzburg model with superpotential g given by the pullback of f to Y. As mentioned in the introduction, in this case the category of D-branes is defined as the category of singularities $\mathbf{D}_{\mathrm{Sg}}(Y_0)$, where Y_0 is the fiber of g over 0.

On the other hand, we can describe Y as a GIT quotient of an affine space $Z=\mathbb{C}^{n+1}$ by the linear action of $\mathbb{C}^{\times}$. The underlying superpotential $W=pf(x_1,\dots,x_n)$ on $Z=\mathbb{C}^{n+1}$ descends to a holomorphic function on Y that coincides with g. In the presence of a $\mathbb{C}^{\times}$-action one can also consider the category $\mathrm{MF}^{\mathrm{gr}}(W)$ of graded matrix factorizations of W. We can think of the latter as being the category of D-branes in the gauged linear sigma model.

Now we reach the crucial step. One of the main outcomes of [42], is that the categories of D-branes in the Calabi-Yau and Landau-Ginzburg phases are both quotients of $\mathrm{MF}^{\mathrm{gr}}(W)$. However, at $r>0$ and at "intermidiate energy scale" one could always choose the description as the Landau-Ginzburg model with superpotential g over Y. This superpotential gives masses to the field P and to the "transverse modes" to the hypersurface S. At "lower energies", it is more appropriate to integrate them out, and we have the nonlinear sigma model on S.

In the light of all this we can expect that the categories of D-branes $\mathbf{D}_{\mathrm{Sg}}(Y_0)$ and $\mathrm{MF}^{G}(f)$ are also equivalent. Now, our Theorem 1.7.3 gives an equivalence between the category of D-branes $\mathrm{MF}^{G}(f)$ and the G-equivariant category of singularities $\mathbf{D}^{G}_{\mathrm{Sg}}(M_0)$, where M_0 is the fiber of f over 0. So, we arrive at the statement that the category $\mathbf{D}_{\mathrm{Sg}}(Y_0)$ should be equivalent to the category $\mathbf{D}^{G}_{\mathrm{Sg}}(M_0)$. This equivalence allows us to compare the category of D-branes on the Landau-Ginzburg model (Y,g) with the category of D-branes in the Landau-Ginzburg orbifold (M,f). Given this simple observation, it is natural to think that the correspondence between D-branes in the two theories is given by a McKay correspondence.

1.3 Localization in triangulated categories

In this section we will review the definition of localization of triangulated categories. The reader is referred to [36], for example, for a more complete discussion.

Recall that a triangulated category $\mathcal{D}$ is an additive category equipped with the additional data:

(a) an additive autoequivalence $T:\mathcal{D}\to\mathcal{D}$, which is called a translation functor,

(b) a class of exact (or distinguished) triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} TX.$$

This data must satisfy a certain set of axioms (see [36], also [41]).

An additive functor $F:\mathcal{D}\to\mathcal{D}'$ between two triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ is called *exact* if it commutes with the translation functors, i.e. there is a natural isomorphism $FT\cong TF$, and it sends exact triangles to exact triangles, i.e. any exact triangle $X\to Y\to Z\to TX$ in $\mathcal{D}$ is mapped to an exact triangle

$$FX \longrightarrow FY \longrightarrow FZ \longrightarrow FTX$$

in $\mathcal{D}'$, where FTX is identified with TFX via the natural isomorphism of FT and TF.

A full additive subcategory $\mathcal{N}\subset\mathcal{D}$ is said to be a full triangulated subcategory, if the following condition holds: it is closed with respect to the translation functor in $\mathcal{D}$ and if it contains any two objects of an exact triangle in $\mathcal{D}$ then it contains the third object of this triangle as well.

With any pair $\mathcal{N} \subset \mathcal{D}$, where $\mathcal{N}$ is a full triangulated subcategory in a triangulated category $\mathcal{D}$, we can associate the *quotient* $\mathcal{D}/\mathcal{N}$. To construct it denote by Σ the class of morphisms s in $\mathcal{D}$ fitting into an exact triangle

$$X \xrightarrow{\;s\;} Y \longrightarrow N \longrightarrow TX$$

with $N \in \mathcal{N}$. It is not hard to see that Σ is a multiplicative system. We then define the quotient $\mathcal{D}/\mathcal{N}$ as the localization $\mathcal{D}[\Sigma^{-1}]$ and observe that it is a triangulated category. The translation functor on $\mathcal{D}/\mathcal{N}$ is induced from the translation functor in the category $\mathcal{D}$, and the exact triangles in $\mathcal{D}/\mathcal{N}$ are triangles isomorphic to the images of exact triangles.

The category $\mathcal{D}/\mathcal{N}$ has the following explicit description. The objects of $\mathcal{D}/\mathcal{N}$ are the objects of $\mathcal{D}$. The morphisms from X to Y are equivalence classes of diagrams (s, f) in $\mathcal{D}$ of the form

$$X \xleftarrow{\;s\;} Y' \xrightarrow{\;f\;} Y \qquad \text{with } s \in \Sigma,$$

where two diagrams (s, f) and (t, g) are equivalent if they fit into a commutative diagram

$$\begin{array}{ccccc} & & Y' & & \\ & \swarrow^{s} & \uparrow & \searrow^{f} & \\ X & \xleftarrow{\;r\;} & Y''' & \xrightarrow{\;h\;} & Y \\ & \nwarrow_{t} & \downarrow & \nearrow_{g} & \\ & & Y'' & & \end{array}$$

with $r \in \Sigma$.

The quotient functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ annihilates $\mathcal{N}$. Moreover, any exact functor $F : \mathcal{D} \to \mathcal{D}'$ between triangulated categories, for which $F(X) \cong 0$ when $X \in \mathcal{N}$, factors uniquely through Q. This implies the following result which will be useful later.

Lemma 1.3.1. *Let $\mathcal{N}$ and $\mathcal{N}'$ be full triangulated subcategories of triangulated categories $\mathcal{D}$ and $\mathcal{D}'$, respectively. Let $F : \mathcal{D} \to \mathcal{D}'$ and $G : \mathcal{D}' \to \mathcal{D}$ be an adjoint pair of exact functors such that $F(\mathcal{N}) \subset \mathcal{N}'$ and $G(\mathcal{N}') \subset \mathcal{N}$. Then they induce functors*

$$\overline{F} : \mathcal{D}/\mathcal{N} \longrightarrow \mathcal{D}'/\mathcal{N}' \quad \textit{and} \quad \overline{G} : \mathcal{D}'/\mathcal{N}' \longrightarrow \mathcal{D}/\mathcal{N}$$

which are adjoint as well. Moreover, if the functor $F : \mathcal{D} \to \mathcal{D}'$ is fully faithful, then the functor $\overline{F} : \mathcal{D}/\mathcal{N} \to \mathcal{D}'/\mathcal{N}'$ is also fully faithful.

1.4 TRIANGULATED CATEGORIES OF SINGULARITIES

In this section we give the definition and basic properties of triangulated categories of singularities. We refer to Orlov's papers [70] and [69] for all the proofs of the assertions below.

We are mainly interested in triangulated categories and their quotient by triangulated subcategories which are coming from algebraic geometry. Let X be a separated Noetherian scheme of finite Krull dimension over $\mathbb{C}$ such that the category of coherent sheaves $\mathrm{Coh}(X)$ has enough locally free sheaves. For future reference we denote the category of quasi-coherent sheaves on X by $\mathrm{Qcoh}(X)$.

Denote by $\mathbf{D}(X)$ the bounded derived category of coherent sheaves on X. The objects of the category $\mathbf{D}(X)$ which are isomorphic to bounded complexes of locally free sheaves on X form a full triangulated subcategory. It is called the subcategory of *perfect complexes* and is denoted by $\mathrm{Perf}(X)$.[1]

Definition 1.4.1. Define the triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(X)$ of X as the quotient category $\mathbf{D}(X)/\mathrm{Perf}(X)$.

It is known that if our scheme X is regular then the subcategory of perfect complexes $\mathrm{Perf}(X)$ coincides with the whole bounded derived category of coherent sheaves. In this case the triangulated category of singularities $\mathbf{D}_{\mathrm{Sg}}(X)$ is trivial. Thus $\mathbf{D}_{\mathrm{Sg}}(X)$ is only sensitive to singularities of X.

Let $f : X \to Y$ be a morphism of finite Tor-dimension (for example a flat morphism or a regular closed embedding). It defines the inverse image functor $\mathbf{L}f^* : \mathbf{D}(Y) \to \mathbf{D}(X)$. It is clear that the functor $\mathbf{L}f^*$ sends perfect complexes on Y to perfect complexes on X. Therefore, the functor $\mathbf{L}f^*$ induces an exact functor $\mathbf{L}\overline{f}^* : \mathbf{D}_{\mathrm{Sg}}(Y) \to \mathbf{D}_{\mathrm{Sg}}(X)$.

Suppose, in addition, that the morphism $f : X \to Y$ is proper and locally of finite type. Then the direct image functor $\mathbf{R}f_* : \mathbf{D}(X) \to \mathbf{D}(Y)$ takes perfect complexes on X to perfect complexes on Y (see [87]). Hence it determines a functor $\mathbf{R}\overline{f}_* : \mathbf{D}_{\mathrm{Sg}}(X) \to \mathbf{D}_{\mathrm{Sg}}(Y)$ which is right adjoint to $\mathbf{L}\overline{f}^*$. We should remark, however, that all the specific morphisms we consider are non-proper.

A fundamental property of triangulated categories of singularities is a property of locality. Here is a precise statement.

Proposition 1.4.2. *Let X be as above and let $j : U \to X$ be an embedding of an open subscheme such that* $\mathrm{Sing}(X) \subset U$. *Then the functor $\overline{j}^* : \mathbf{D}_{\mathrm{Sg}}(X) \to \mathbf{D}_{\mathrm{Sg}}(U)$ is an equivalence of triangulated categories.*

Triangulated categories of singularities of X have additional good properties in case the scheme is Gorenstein. Recall that a local Noetherian ring A is called Gorenstein if A as module over itself has a finite injective resolution. It can be shown that if A is Gorenstein then A has finite injective dimension and the natural map

$$M \longrightarrow \mathbf{R}\mathrm{Hom}^{\cdot}_A(\mathbf{R}\mathrm{Hom}^{\cdot}_A(M, A), A)$$

[1] Actually, a perfect complex is defined as a complex of $\mathcal{O}_X$-modules locally quasi-isomorphic to a bounded complex of locally free sheaves of finite type. But under our assumption on the scheme any such complex is quasi-isomorphic to a bounded complex of locally free sheaves of finite type (see [87]).

is an isomorphism for any finitely generated A-module M and, as a consequence, for any object from $\mathbf{D}(\operatorname{Spec} A)$. A scheme X is Gorenstein if all of its local rings are Gorenstein local rings. If X is Gorenstein and has finite dimension, then $\mathcal{O}_X$ is a dualizing complex for X, i.e. it has finite injective dimension as a quasi-coherent sheaf and the natural map

$$\mathcal{E} \longrightarrow \mathbf{R}\mathcal{H}om^{\cdot}_X(\mathbf{R}\mathcal{H}om^{\cdot}_X(\mathcal{E}, \mathcal{O}_X), \mathcal{O}_X)$$

is an isomorphism for any coherent sheaf $\mathcal{E}$. In particular, there is an integer n_0 such that $\mathcal{E}xt^i_X(\mathcal{E}, \mathcal{O}_X) = 0$ for each quasi-coherent sheaf $\mathcal{E}$ and all $i > n_0$.

The following gives a useful description of the morphism spaces in triangulated categories of singularities.

Proposition 1.4.3. *Let X be as above and Gorenstein. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves such that $\mathcal{E}xt^i_X(\mathcal{E}, \mathcal{O}_X) = 0$ for all $i > 0$. Fix n such that $\mathcal{E}xt^i_X(\mathcal{S}, \mathcal{F}) = 0$ for $i > n$ and for any locally free sheaf $\mathcal{S}$. Then*

$$\operatorname{Hom}_{\mathbf{D}_{\mathrm{Sg}}(X)}(\mathcal{E}, \mathcal{F}[n]) \cong \operatorname{Ext}^n_X(\mathcal{E}, \mathcal{F})/\mathcal{R}$$

where $\mathcal{R}$ is the subspace of elements factoring through locally free, i.e. $e \in \mathcal{R}$ if and only if $e = \alpha\beta$ with $\alpha : \mathcal{E} \to \mathcal{S}$ and $\beta \in \operatorname{Ext}^n_X(\mathcal{S}, \mathcal{F})$ where $\mathcal{S}$ is locally free.

1.5 Triangulated categories of matrix factorizations

In this section we introduce the category of matrix factorizations and give some of its basic properties. The origin of this category goes back to the work of D. Eisenbud [34] in the context of so-called maximal Cohen-Macaulay modules over local rings of hypersurface singularities.

As proposed by M. Kontsevich (see also [48]) the category of D-branes associated to a Landau-Ginzburg model can be characterized in terms of matrix factorizations. For us, a Landau-Ginzburg model is simply a pair (X, W), where X is a smooth variety (or regular scheme), and $W : X \to \mathbb{C}$ is a regular function on X called the *superpotential*. To keep things simple, we will assume throughout that W has a single critical value at the origin $0 \in \mathbb{C}$. To this data one can associate two categories: an exact category $\operatorname{Pair}(W)$ and a triangulated category $\operatorname{MF}(W)$. We give the construction of these categories under the condition that X is affine.

Let A be a commutative algebra over $\mathbb{C}$. Then one can regard A as the algebra of functions on an affine scheme $X = \operatorname{Spec} A$. Denote by Mod–$A$ the category of all right modules over A. It is a well-known fact that the global section functor

$$H^0 : \operatorname{Qcoh}(X) \longrightarrow \text{Mod–}A,$$

is an equivalence with inverse denoted by $\widetilde{(-)}$. It is also well-known that this functor restrict to an equivalence

$$H^0 : \operatorname{Coh}(X) \longrightarrow \text{mod–}A,$$

where mod–A is the category of finitely generated right modules over A. Note that under this equivalence locally free sheaves are the same as projective modules.

For a non-zero element $W \in A$, a *matrix factorization* of W is an ordered pair

$$\overline{P} = \big(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1 \big)$$

where P_0, P_1 are finitely generated projective A-modules and p_0, p_1 are A-homomorphisms such that $p_1 p_0 = W \cdot \mathrm{id}_{P_0}$ and $p_0 p_1 = W \cdot \mathrm{id}_{P_1}$. Since $p_0 p_1$ and $p_1 p_0$ are W times the identities, where W is a non-zero element of A, the rank of P_0 coincides with that of P_1. We call the rank the size of the matrix factorization.

The above construction can be reformulated in terms of $\mathbb{Z}_2$-graded A-modules as follows. A $\mathbb{Z}_2$-graded A-module $P = P_0 \oplus P_1$ can be thought of as an ordinary A-module P equipped with a $\mathbb{C}$-linear involution $\tau : P \to P$, $\tau^2 = \mathrm{id}$. The homogeneous parts P_0 and P_1 are the eigenspaces of τ corresponding to the eigenvalues 1 and -1 respectively. A pair $\overline{P}$ can be similarly thought of as a triple (P, τ, D_P) where $D_P : P \to P$ is an odd A-homomorphism satisfying $D_P^2 = W \cdot \mathrm{id}_P$. Given two matrix factorizations $\overline{P} = (P, \tau, D_P)$ and $\overline{Q} = (Q, \sigma, D_Q)$ the A-module $\mathbf{Hom}(\overline{P}, \overline{Q})$ form a $\mathbb{Z}_2$-graded complex

$$\mathbf{Hom}(\overline{P}, \overline{Q}) = \mathbf{Hom}(\overline{P}, \overline{Q})_0 \oplus \mathbf{Hom}(\overline{P}, \overline{Q})_1$$

where

$$\mathbf{Hom}(\overline{P}, \overline{Q})_0 = \mathrm{Hom}_A(P_0, Q_0) \oplus \mathrm{Hom}_A(P_1, Q_1),$$
$$\mathbf{Hom}(\overline{P}, \overline{Q})_1 = \mathrm{Hom}_A(P_0, Q_1) \oplus \mathrm{Hom}_A(P_1, Q_0),$$

and with differential D acting on homogeneous elements of degree k as

$$D\phi = D_Q \cdot \phi - (-1)^k \phi \cdot D_P.$$

The set of objects of the categories $\mathrm{Pair}(W)$ and $\mathrm{MF}(W)$ is given by the set of matrix factorizations of W. The space of morphisms $\mathrm{Hom}_{\mathrm{Pair}(W)}(\overline{P}, \overline{Q})$ in the category $\mathrm{Pair}(W)$ is the space of homogeneous morphisms of degree 0 which commute with the differential D. The space of morphisms in the category $\mathrm{MF}(W)$ is the space of morphisms in $\mathrm{Pair}(W)$ modulo null-homotopic morphisms, i.e.

$$\mathrm{Hom}_{\mathrm{Pair}(W)}(\overline{P}, \overline{Q}) = Z^0(\mathbf{Hom}(\overline{P}, \overline{Q})),$$
$$\mathrm{Hom}_{\mathrm{MF}(W)}(\overline{P}, \overline{Q}) = H^0(\mathbf{Hom}(\overline{P}, \overline{Q})).$$

Thus a morphism $\phi : \overline{P} \to \overline{Q}$ in the category $\mathrm{Pair}(W)$ is a pair of morphisms $\phi_0 : P_0 \to Q_0$ and $\phi_1 : P_1 \to Q_1$ such that $\phi_1 p_0 = q_0 \phi_0$ and $q_1 \phi_1 = \phi_0 p_1$. The morphism ϕ is null-homotopic if there are two morphisms $t_0 : P_0 \to Q_1$ and $t_1 : P_1 \to Q_0$ such that $\phi_1 = q_0 t_1 + t_0 p_1$ and $\phi_0 = t_1 p_0 + q_1 t_0$.

It is clear that the category $\mathrm{Pair}(W)$ is an exact category with respect to componentwise monomorphisms and epimorphisms (see definition in [73]).

The category $\mathrm{MF}(W)$ can be endowed with a natural structure of a triangulated category. To determine it we have to define a translation functor [1] and a class of exact triangles.

The translation functor can be defined as a functor that takes $\overline{P}$ to the object

$$\overline{P}[1] = \big(P_1 \underset{-p_0}{\overset{-p_1}{\rightleftarrows}} P_0 \big) \tag{1.5.1}$$

i.e. it changes the order of the modules and signs of the morphisms, and takes a morphism $\phi = (\phi_0, \phi_1)$ to the morphism $\phi[1] = (\phi_1, \phi_0)$. We see that the functor [2] is the identity functor.

For any morphism $\phi : \overline{P} \to \overline{Q}$ from the category Pair(W) we define a mapping cone $C(\phi)$ as an object

$$C(\phi) = \Big(Q_0 \oplus P_1 \underset{c_1}{\overset{c_0}{\rightleftarrows}} Q_1 \oplus P_0 \Big) \tag{1.5.2}$$

such that

$$c_0 = \begin{pmatrix} q_0 & \phi_1 \\ 0 & -p_1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} q_1 & \phi_0 \\ 0 & -p_0 \end{pmatrix}.$$

There are maps $\psi : \overline{Q} \to C(\phi)$, $\psi = (\mathrm{id}, 0)$ and $\xi : C(\phi) \to \overline{P}[1]$, $\xi = (0, \mathrm{id})$.

Now we define a standard triangle in the category MF(W) as a triangle of the form

$$\overline{P} \xrightarrow{\phi} \overline{Q} \xrightarrow{\psi} C(\phi) \xrightarrow{\xi} \overline{P}[1]$$

for some $\phi \in \mathrm{Hom}_{\mathrm{Pair}(W)}(\overline{P}, \overline{Q})$. A triangle $\overline{P} \to \overline{Q} \to \overline{R} \to \overline{P}[1]$ in MF(W) will be called an exact triangle if it is isomorphic to a standard one.

As a consequence we get the following.

Proposition 1.5.1. *The category* MF(W) *endowed with the translation functor* [1] *and the above class of exact triangles becomes a triangulated category.*

The proof is the same as the analogous result for a usual homotopic category (see, for example [36]).

Definition 1.5.2. The category MF(W) constructed above is called the triangulated category of matrix factorizations for the pair $(X = \operatorname{Spec} A, W)$.

Denote by X_0 the fiber of $W : X \to \mathbb{C}$ over the point 0. With any matrix factorization $\overline{P}$ we can associate a short exact sequence

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow \operatorname{coker} p_1 \longrightarrow 0.$$

We can attach to an object $\overline{P}$ the sheaf $\operatorname{coker} p_1$. This is a sheaf on X. But the multiplication by W annihilates it. Hence, we can consider $\operatorname{coker} p_1$ as a sheaf on X_0. Any morphism $\phi : \overline{P} \to \overline{Q}$ in Pair(W) gives a morphism between cokernels. This way we get a functor $\mathrm{Cok} : \mathrm{Pair}(W) \to \mathrm{Coh}(X_0)$. We have the following result, see [70, Theorem 3.9].

Theorem 1.5.3. *There is a functor F which completes the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Pair}(W) & \xrightarrow{\mathrm{Cok}} & \mathrm{Coh}(X_0) \\ \big\downarrow & & \big\downarrow \\ \mathrm{MF}(W) & \xrightarrow[F]{} & \mathbf{D}_{\mathrm{Sg}}(X_0). \end{array}$$

Moreover, the functor F is an equivalence of triangulated categories.

1.6 Orbifold categories

As is well known, for the Calabi-Yau/Landau-Ginzburg correspondence, one must consider orbifolds of D-branes in a Landau-Ginzburg theory. The definition of triangulated categories of singularities and matrix factorizations can be extended to this situation.

We start by recalling the definition and basic properties of equivariant coherent sheaves. More details can be found in [71]. Let G be a finite group acting on some scheme X. A G-equivariant coherent sheaf on X is a coherent sheaf $\mathcal{E}$ on X together with isomorphisms $\lambda_g^{\mathcal{E}} : \mathcal{E} \xrightarrow{\sim} g^*\mathcal{E}$ for all $g \in G$ subject to $\lambda_e^{\mathcal{E}} = \mathrm{id}_{\mathcal{E}}$ and $\lambda_{gh}^{\mathcal{E}} = h^*(\lambda_g^{\mathcal{E}})\lambda_h^{\mathcal{E}}$. Mumford calls this a G-linearization of $\mathcal{E}$.

If $\mathcal{E}$ and $\mathcal{F}$ are two G-equivariant coherent sheaves, then the vector space $\mathrm{Hom}_X(\mathcal{E},\mathcal{F})$ becomes a G-representation via $g \cdot \theta = (\lambda_g^{\mathcal{F}})^{-1} g^*\theta\lambda_g^{\mathcal{E}}$ for $\theta : \mathcal{E} \to \mathcal{F}$. Let $\mathrm{Coh}^G(X)$ be the category whose objects are G-equivariant coherent sheaves and whose morphisms are the G-invariant sheaf morphisms:

$$G\text{-}\mathrm{Hom}_X(\mathcal{E},\mathcal{F}) \equiv \mathrm{Hom}_X(\mathcal{E},\mathcal{F})^G.$$

This category is abelian. It is not difficult to define the usual additive functors $\otimes$, $\mathcal{H}om$ on this category. Furthermore, if $f : X \to Y$ is a G-equivariant map between G-schemes, then one defines in an obvious way the additive functors $f_* : \mathrm{Coh}^G(X) \to \mathrm{Coh}^G(Y)$, $f^* : \mathrm{Coh}^G(Y) \to \mathrm{Coh}^G(X)$. For example, if $\mathcal{E} \in \mathrm{Coh}^G(X)$, then $f_*\mathcal{E}$ is canonically a G-equivariant coherent sheaf via $f_*\lambda_g^{\mathcal{E}} : f_*\mathcal{E} \xrightarrow{\sim} f_*g^*\mathcal{E} = g^*f_*\mathcal{E}$. One now also has the usual adjunctions and relations among these functors.

We shall have to deal with the special case where G acts trivially on X. Then a G-equivariant coherent sheaf $\mathcal{E}$ is merely given by a group homomorphism $\lambda^{\mathcal{E}} : G \to \mathrm{Aut}(\mathcal{E})$. As G is finite, this representation decomposes into a direct sum over the irreducible G-representations $\rho_0, \rho_1, \ldots, \rho_n$, where we take ρ_0 to be the trivial one; i.e. $\mathcal{E} \cong \bigoplus_{i=0}^n \mathcal{E}_i \otimes_{\mathcal{O}_X} \widetilde{\rho_i}$ in $\mathrm{Coh}^G(X)$ with ordinary sheaves $\mathcal{E}_i \in \mathrm{Coh}(X)$. There exists no homomorphisms between sumands corresponding to two different representations, and hence we obtain two mutually adjoint and exact functors, the latter of which is 'taking G-invariants':

$$-\otimes \rho_0 : \mathrm{Coh}(X) \longrightarrow \mathrm{Coh}^G(X),$$
$$[-]^G : \mathrm{Coh}^G(X) \longrightarrow \mathrm{Coh}(X).$$

We come back now to the general case. Given two objects $\mathcal{E}$ and $\mathcal{F}$ in $\mathrm{Coh}^G(X)$, we consider $\mathrm{Ext}^i_X(\mathcal{E},\mathcal{F})$ as a G-representation in the usual way. Then it is easily seen that

$$G\text{-}\mathrm{Ext}^i_X(\mathcal{E},\mathcal{F}) = \mathrm{Ext}^i_X(\mathcal{E},\mathcal{F})^G.$$

Denote the bounded derived category of $\mathrm{Coh}^G(X)$ by $\mathbf{D}^G(X)$. We shall refer to $\mathbf{D}^G(X)$ as the derived category of G-equivariant coherent sheaves on X. Using induction on the length of complexes, the above relation for equivariant Ext groups translates to

$$\mathrm{Hom}^i_{\mathbf{D}^G(X)}(\mathcal{E}^\cdot,\mathcal{F}^\cdot) = \mathrm{Hom}^i_{\mathbf{D}(X)}(\mathcal{E}^\cdot,\mathcal{F}^\cdot)^G,$$

for complexes of G-equivariant coherent sheaves $\mathcal{E}^\cdot$ and $\mathcal{F}^\cdot$ in $\mathbf{D}^G(X)$. Note that all facts about G-equivariant coherent sheaves also apply to complexes of G-equivariant coherent sheaves.

It will be useful for us to look at $\mathbf{D}^G(X)$ in another way. Consider the quotient stack $[X/G]$. It is covered by one étale chart, given by the projection $X \to X/G$, or more explicitly, by the fiber diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{p} & X \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ X & \longrightarrow & X/G. \end{array}$$

Now a sheaf on the stack $[X/G]$ is just a sheaf $\mathcal{E}$ on the chart X with $p^*\mathcal{E} \cong \sigma^*\mathcal{E}$, and the descent condition translates into the linearization property. Therefore, the abelian categories $\mathrm{Coh}([X/G])$ and $\mathrm{Coh}^G(X)$ are equivalent, and consequently they give rise to equivalent derived categories.

A *perfect complex* of G-equivariant coherent sheaves is an object of $\mathbf{D}([X/G])$ which is quasi-isomorphic to a bounded complex of locally free sheaves on $[X/G]$. The perfect complexes of G-equivariant coherent sheaves form a full triangulated subcategory $\mathrm{Perf}([X/G]) \subset \mathbf{D}([X/G]) \cong \mathbf{D}^G(X)$.

Definition 1.6.1. Define the G-equivariant category of singularities $\mathbf{D}^G_{\mathrm{Sg}}(X)$ of X as the quotient category $\mathbf{D}^G(X)/\mathrm{Perf}([X/G])$.

One can show that the entire discussion we had in Section 1.4 goes through in the case of G-equivariant coherent sheaves.

It also makes sense to define G-equivariant matrix factorizations. Suppose $X = \mathrm{Spec}\, A$ is a G-scheme. It is natural to define the following abelian category $\mathrm{Mod}^G\text{–}A$. Its objects are A-modules M with the property that for every $g \in G$, there is given an A-isomorphism $\lambda^M_g : M \to g^*M$, such that for every $g, h \in G$, we have $\lambda^M_{gh} = h^*(\lambda^M_g)\lambda^M_h$ and $\lambda^M_e = \mathrm{id}_M$. Note that in this expression $g^*M = g^{-1}_*M$ is just the abelian group M with its A-module structure induced by $g^{-1} : A \to A$. A morphism $\phi : M \to N$ is just an A-homomorphism, which should satisfy the property that for all $g \in G$ and $m \in M$, we have $\phi(\lambda^M_g(m)) = \lambda^N_g(\phi(m))$. This clearly gives rise to an abelian category in a natural way. Likewise, it has an abelian subcategory determined by the full subcategory of finitely generated A-modules, which we will denote by $\mathrm{mod}^G\text{–}A$. Note that if X happens to be a trivial G-scheme, we have $\mathrm{mod}^G\text{–}A = \mathbb{C}G\text{–mod–}A$ (just a category of bimodules). We can now define in an obvious way a functor

$$H^0 : \mathrm{Qcoh}^G(X) \longrightarrow \mathrm{Mod}^G\text{–}A,$$

which is an equivalence with inverse $\widetilde{(-)}$. Moreover this functor restrict to an equivalence

$$H^0 : \mathrm{Coh}^G(X) \longrightarrow \mathrm{mod}^G\text{–}A.$$

Note that these functors are just extensions of the previous ones.

Now assume that there is an action of the group G on the Landau-Ginzburg model $(X = \mathrm{Spec}\, A, W)$ such that the superpotential W is G-equivariant. In this situation, we can consider two categories: an exact category $\mathrm{Pair}^G(W)$ and a triangulated category $\mathrm{MF}^G(W)$. Objects of these categories are ordered pairs

$$\overline{P} = \left(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1 \right)$$

where P_0, P_1 are finitely generated projective G-A-modules and p_0, p_1 are G-equivariant maps such that the compositions $p_0 p_1$ and $p_1 p_0$ are the multiplication by the element $W \in A$. A morphism $\phi : \overline{P} \to \overline{Q}$ in the category $\mathrm{Pair}^G(W)$ is a pair of G-equivariant morphisms $\phi_0 : P_0 \to Q_0$ and $\phi_1 : P_1 \to Q_1$ such that $\phi_1 p_0 = q_0 \phi_0$ and $q_1 \phi_1 = \phi_0 p_1$. Morphisms in the category $\mathrm{MF}^G(W)$ are classes of G-equivariant morphisms in $\mathrm{Pair}^G(W)$ modulo null-homotopic morphisms. The shift functor and the distinguished triangles can be constructed by imposing equivariance conditions on equations (1.5.1) and (1.5.2).

Definition 1.6.2. The category $\mathrm{MF}^G(W)$ constructed above is called the triangulated category of G-equivariant matrix factorizations for the pair $(X = \operatorname{Spec} A, W)$.

1.7 Categories of matrix factorizations and categories of singularities

Our aim now is to describe an equivalence of categories between $\mathrm{MF}^G(W)$, the category of G-equivariant matrix factorizations and $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$, the G-equivariant category of singularities. In the non-equivariant setting, we have seen in Section 1.5 that $\mathrm{MF}(W)$ is equivalent to $\mathbf{D}_{\mathrm{Sg}}(X_0)$. The generalization to the equivariant situation is straightforward. Our proofs in this section are modeled on those in [70].

With any object $\overline{P}$ in $\mathrm{Pair}^G(W)$ we associate the module $\operatorname{coker} p_1$ and its free resolution

$$0 \longrightarrow P_1 \xrightarrow{p_1} P_0 \longrightarrow \operatorname{coker} p_1 \longrightarrow 0.$$

It can be easily checked that W annihilates $\operatorname{coker} p_1$. Hence the module $\operatorname{coker} p_1$ is naturally a right G-A-module. For each object $\overline{P}$ in $\mathrm{Pair}^G(W)$ we define $\mathrm{Cok}^G(\overline{P}) = \operatorname{coker} p_1$; this is a G-equivariant coherent sheaf on X_0. If $\phi : \overline{P} \to \overline{Q}$ is a morphism in $\mathrm{Pair}^G(W)$ then ϕ induces a morphism $\mathrm{Cok}^G(\phi) : \operatorname{coker} p_1 \to \operatorname{coker} q_1$. This construction defines a functor $\mathrm{Cok}^G : \mathrm{Pair}^G(W) \to \mathrm{Coh}^G(X_0)$.

Lemma 1.7.1. *The functor* Cok^G *is full.*

Proof. This is essentially the Lemma 3.5 proved in [70]. We recall its proof for the convenience of readers. Fix two objects $\overline{P}$ and $\overline{Q}$ in $\mathrm{Pair}^G(W)$ and let $f : \operatorname{coker} p_1 \to \operatorname{coker} q_1$ be a morphism in $\mathrm{Coh}^G(X_0)$. Since P_0 and P_1 are projective f can be extended to a map of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_1 & \xrightarrow{p_1} & P_0 & \longrightarrow & \operatorname{coker} p_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle f} & & \\
0 & \longrightarrow & Q_1 & \xrightarrow{q_1} & Q_0 & \longrightarrow & \operatorname{coker} q_1 & \longrightarrow & 0.
\end{array}$$

We want to show that $\phi = (\phi_0, \phi_1)$ is a map of pairs. We have that

$$q_1(\phi_1 p_0 - q_0\phi_0) = \phi_0 p_1 p_0 - q_1 q_0 \phi_0 = \phi_0 W - W\phi_0 = 0.$$

Using that q_1 is a monomorphism, we get that $\phi_1 p_0 = q_0 \phi_0$, which shows that $\phi = (\phi_0, \phi_1)$ is a map of pairs, as required. □

Next we show that the functor Cok^G induces an exact functor between triangulated categories.

Proposition 1.7.2. *There is a functor F^G which completes the following commutative diagram*

$$\begin{array}{ccc} \mathrm{Pair}^G(W) & \xrightarrow{\mathrm{Cok}^G} & \mathrm{Coh}^G(X_0) \\ \downarrow & & \downarrow \\ \mathrm{MF}^G(W) & \xrightarrow{F^G} & \mathbf{D}^G_{\mathrm{Sg}}(X_0). \end{array}$$

Moreover, the functor F^G is an exact functor between triangulated categories.

Proof. Most of the argument is identical to the non-equivariant case proved in [70, Proposition 3.7]. We define a functor $F^G : \mathrm{Pair}^G(W) \to \mathbf{D}^G_{\mathrm{Sg}}(X_0)$ to be the composition of Cok^G and the natural functor from $\mathrm{Coh}^G(X_0)$ to $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$. To prove that F^G induces a functor from $\mathrm{MF}^G(W)$ to $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$ we need to show that any morphism $\phi = (\phi_0, \phi_1) : \overline{P} \to \overline{Q}$ in $\mathrm{Pair}^G(W)$ which is homotopic to 0 goes to 0-morphism in $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$. Fix a homotopy $t = (t_0, t_1)$ where $t_0 : P_0 \to Q_1$ and $t_1 : P_1 \to Q_0$. Consider the following decomposition of ϕ:

$$\begin{array}{ccccc} P_1 & \underset{p_0}{\overset{p_1}{\rightleftarrows}} & P_0 & \longrightarrow & \operatorname{coker} p_1 \\ \downarrow{\scriptstyle (t_1,\phi_1)} & & \downarrow{\scriptstyle (t_0,\phi_0)} & & \downarrow \\ Q_0 \oplus Q_1 & \underset{c_0}{\overset{c_1}{\rightleftarrows}} & Q_1 \oplus Q_0 & \longrightarrow & Q_0/W \\ \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow \\ Q_1 & \underset{q_0}{\overset{q_1}{\rightleftarrows}} & Q_0 & \longrightarrow & \operatorname{coker} q_1 \end{array}$$

where

$$c_0 = \begin{pmatrix} -q_0 & \mathrm{id} \\ 0 & q_1 \end{pmatrix}, \quad c_1 = \begin{pmatrix} -q_1 & \mathrm{id} \\ 0 & q_0 \end{pmatrix}.$$

This gives a decomposition of $F^G(\phi)$ through a G-equivariant locally free object Q_0/W on X_0. By Proposition 1.4.3 we have that $F^G(\phi) = 0$ in the category $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$. It is not difficult to check that F^G takes a standard triangle in $\mathrm{MF}^G(W)$ to an exact triangle in $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$. Therefore F^G is exact. □

Notice that there is a natural forgetful functor $U : \mathrm{MF}^G(W) \to \mathrm{MF}(W)$, which simply forgets the G-action. We have the natural second forgetful functor $U : \mathbf{D}^G_{\mathrm{Sg}}(X_0) \to \mathbf{D}_{\mathrm{Sg}}(X_0)$. For each $\overline{P}$ in $\mathrm{MF}^G(W)$, the two objects $UF^G\overline{P}$ and $FU\overline{P}$ coincide. More precisely, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{MF}^G(W) & \xrightarrow{F^G} & \mathbf{D}^G_{\mathrm{Sg}}(X_0) \\ \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\ \mathrm{MF}(W) & \underset{F}{\xrightarrow{\sim}} & \mathbf{D}_{\mathrm{Sg}}(X_0). \end{array}$$

We can now prove the main result of this section.

Theorem 1.7.3. *The functor* $F^G : \mathrm{MF}^G(W) \to \mathbf{D}^G_{\mathrm{Sg}}(X_0)$ *is an equivalence of triangulated categories.*

Proof. First we verify that the functor F^G is fully faithful. This follows from the arguments of [72, Lemma 5]. We repeat the proof in the current setting. Fix two objects $\overline{P}$ and $\overline{Q}$ in $\mathrm{MF}^G(W)$. By definition of morphisms in $\mathrm{MF}^G(W)$ and $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$, we have a diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{MF}(W)}(U\overline{P}, U\overline{Q}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{D}_{\mathrm{Sg}}(X_0)}(FU\overline{P}, FU\overline{Q}) \\
\uparrow & & \uparrow \\
\mathrm{Hom}_{\mathrm{MF}(W)}(U\overline{P}, U\overline{Q})^G & & \mathrm{Hom}_{\mathbf{D}_{\mathrm{Sg}}(X_0)}(UF^G\overline{P}, UF^G\overline{Q})^G \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{MF}^G(W)}(\overline{P}, \overline{Q}) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}^G_{\mathrm{Sg}}(X_0)}(F^G\overline{P}, F^G\overline{Q})
\end{array}$$

and the top morphism is a bijection. Thus the lower map of the diagram is injective, and hence F^G is faithful. To see that F^G is full as well, consider the following variation of the former diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{MF}(W)}(U\overline{P}, U\overline{Q}) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathbf{D}_{\mathrm{Sg}}(X_0)}(FU\overline{P}, FU\overline{Q}) \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\rho} \\
\mathrm{Hom}_{\mathrm{MF}(W)}(U\overline{P}, U\overline{Q})^G & & \mathrm{Hom}_{\mathbf{D}_{\mathrm{Sg}}(X_0)}(UF^G\overline{P}, UF^G\overline{Q})^G \\
\| & & \| \\
\mathrm{Hom}_{\mathrm{MF}^G(W)}(\overline{P}, \overline{Q}) & \longrightarrow & \mathrm{Hom}_{\mathbf{D}^G_{\mathrm{Sg}}(X_0)}(F^G\overline{P}, F^G\overline{Q})
\end{array}$$

using the averaging (or Reynolds) operators π and ρ. We obviously have $\pi(\phi) = \phi$ (respectively $\rho(f) = f$) if and only if ϕ (respectively f) is a G-equivariant morphism. In particular, π and ρ are surjective. The fact that the functor F is full then implies the same property for F^G.

What remains to be proved is that every object $\mathcal{A}$ in $\mathbf{D}^G_{\mathrm{Sg}}(X_0)$ is isomorphic to $F^G\overline{P}$ for some $\overline{P}$. A complete proof of this is given in [70, Theorem 3.9]; it carries over without change. □

1.8 McKay correspondence for Landau-Ginzburg models

Here we use the results from the preceding sections to prove a version of the McKay correspondence for Landau-Ginzburg models. The proof relies heavily on the main result of [16], which reformulates and generalizes the McKay correspondence in the language of derived categories.

Let $M = \mathbb{C}^n$ be the complex n-dimensional affine space, and let G be a finite subgroup of $\mathrm{SL}(n, \mathbb{C})$. Put $X = M/G$ and let $\pi : M \to X$ denote the natural projection. We assume that G acts on M freely outside the origin, which means that X has an isolated singularity[2]. Write G-Hilb(M) for the Hilbert scheme parametrising G-clusters in M, that is, the scheme parametrising G-invariant subschemes $Z \subset M$ of dimension zero with global sections $H^0(\mathcal{O}_Z)$ isomorphic as a $\mathbb{C}G$-module to the regular representation of G. Let Y be the irreducible component of G-Hilb(M) which contains the G-clusters of free orbits. There is a Hilbert-Chow morphism $\tau : G\text{-Hilb}(M) \to X$ which, on closed points, sends a G-cluster to the orbit supporting it. This morphism is always projective and the irreducible component $Y \subset G$-Hilb(M) is mapped birationally onto X. We use the same notation τ for the restriction of the map to Y. In the remainder of this section we impose on Y the condition that $\dim(Y \times_X Y) \leq n+1$.

Now let $\mathscr{Z} \subset Y \times M$ denote the universal closed subscheme, and consider its structure sheaf $\mathcal{O}_{\mathscr{Z}}$. We remark that $\mathcal{O}_{\mathscr{Z}}$ has finite homological dimension, because $\mathcal{O}_{\mathscr{Z}}$ is flat over Y and M is nonsingular. Let $\mathbf{D}(Y)$ and $\mathbf{D}^G(M)$ denote the bounded derived categories of coherent sheaves on Y and G-equivariant coherent sheaves on M, respectively. If π_Y and π_M are the projections from $Y \times M$ to Y and M, we define a functor $\Phi : \mathbf{D}(Y) \to \mathbf{D}^G(M)$ by the formula

$$\Phi(-) = \mathbf{R}\pi_{M*}(\mathcal{O}_{\mathscr{Z}}^{\vee}[n] \otimes^{\mathbf{L}} \pi_Y^*(- \otimes \rho_0))$$

where $\mathcal{O}_{\mathscr{Z}}^{\vee}$ denotes the derived dual $\mathbf{R}\mathscr{H}om^{\cdot}_{Y\times M}(\mathcal{O}_{\mathscr{Z}}, \mathcal{O}_{Y\times M})$. In this situation, we can apply the theorem of Bridgeland, King, and Reid [16] to prove the following result.

Theorem 1.8.1. *Let the assumptions and notations be as above. Then $\tau : Y \to X$ is a crepant resolution and Φ is an equivalence of triangulated categories.*

The quasi-inverse $\Psi : \mathbf{D}^G(M) \to \mathbf{D}(Y)$ is given by

$$\Psi(-) = [\mathbf{R}\pi_{Y*}(\mathcal{O}_{\mathscr{Z}} \otimes^{\mathbf{L}} \pi_M^*(-))]^G.$$

Before we proceed, we need to make a remark. In [16], the definitions of Φ and Ψ differ slightly from the ones we took. Bridgeland, King and Reid define

$$\Phi(-) = \mathbf{R}\pi_{M*}(\mathcal{O}_{\mathscr{Z}} \otimes^{\mathbf{L}} \pi_Y^*(- \otimes \rho_0)),$$
$$\Psi(-) = [\mathbf{R}\pi_{Y*}(\mathcal{O}_{\mathscr{Z}}^{\vee}[n] \otimes^{\mathbf{L}} \pi_M^*(-))]^G.$$

It is clear that this difference does not really change the proof of Theorem 1.8.1. The only difference is that everywhere $\mathcal{O}_{\mathscr{Z}}$ and $\mathcal{O}_{\mathscr{Z}}^{\vee}$ become interchanged.

Assume now that $f : M \to \mathbb{C}$ is a regular function with an isolated critical point at the origin which is invariant with respect to the action of G on M. We can regard M as a

[2] This is for the purpose of simplicity –the method would seem to be applicable to the general case with some modifications.

Landau-Ginzburg orbifold with superpotential f. We denote by M_0 the fiber of the map f over the point $0 \in \mathbb{C}$. Next, let $\overline{f} : X \to \mathbb{C}$ be the unique morphism such that $f = \overline{f}\pi$. Another Landau-Ginzburg model consists of the variety Y and superpotential $g : Y \to \mathbb{C}$ obtained by pullback of $\overline{f}$ to Y. We let Y_0 be the fiber of g over the point 0. Note that Y_0 contains the exceptional set $\tau^{-1}(\pi(0))$ of the resolution. Note also that the function g will, generally speaking, have non-isolated critical points. For future use, we let $i_0 : Y_0 \to Y$ and $j_0 : M_0 \to M$ denote the corresponding closed immersions of fibers.

We now head towards proving the main result of this section, which asserts that there is an equivalence between the category of singularities of Y_0 and the G-equivariant category of singularities of M_0. First, however, we must provide preliminary results. Let us denote by p_Y and p_M the projections of the fiber product $Y \times_{\mathbb{C}} M$ onto its factors so that we have the following cartesian diagram:

$$\begin{array}{ccccc} & & Y \times_{\mathbb{C}} M & & \\ & \swarrow^{p_Y} & & \searrow^{p_M} & \\ Y & & & & M \\ & \searrow_{g} & & \swarrow_{f} & \\ & & \mathbb{C} & & \end{array}$$

The universal sheaf $\mathcal{O}_{\mathcal{Z}}$ on $Y \times M$ is actually supported on the closed subscheme $j : Y \times_{\mathbb{C}} M \hookrightarrow Y \times M$. Thus there is a sheaf $\mathcal{P}$ on $Y \times_{\mathbb{C}} M$, flat over Y, such that $\mathcal{O}_{\mathcal{Z}} = j_* \mathcal{P}$.

Let $\mathbf{D}(Y_0)$ denote the bounded derived category of coherent sheaves on Y_0 and $\mathbf{D}^G(M_0)$ the bounded derived category of G-equivariant coherent sheaves on M_0. Write k_0 for the natural immersion $Y_0 \times M_0 \hookrightarrow Y \times_{\mathbb{C}} M$. Then $\mathcal{P}_0^{\cdot} = \mathbf{L}k_0^* \mathcal{P}$ has finite homological dimension and we may define a functor $\Psi_0 : \mathbf{D}^G(M_0) \to \mathbf{D}(Y_0)$ by the formula

$$\Psi_0(-) = [\mathbf{R}\pi_{Y_0 *}(\mathcal{P}_0^{\cdot} \otimes^{\mathbf{L}} \pi_{M_0}^*(-))]^G,$$

where π_{Y_0} and π_{M_0} are the projections of $Y_0 \times M_0$ to Y_0 and M_0. That the functor $\mathbf{R}\pi_{Y_0 *}(\mathcal{P}_0^{\cdot} \otimes^{\mathbf{L}} -)$ takes $\mathbf{D}^G(Y_0 \times M_0)$ to $\mathbf{D}^G(Y_0)$ can easily be seen from the argument of [25, Lemma 2.1] since the support of $\mathcal{P}_0^{\cdot}$ is proper over Y_0.

We obtain a useful and probably well-known result, a version of which can be found in [25, Lemma 6.1].

Lemma 1.8.2. *There is a natural isomorphism of functors:*

$$i_{0*}\Psi_0(-) \cong \Psi j_{0*}(-).$$

Proof. We first note that there exist a natural isomorphism between the functors

$$\mathbf{D}^G(Y_0) \xrightarrow{[-]^G} \mathbf{D}(Y_0) \xrightarrow{i_{0*}} \mathbf{D}(Y)$$

and

$$\mathbf{D}^G(Y_0) \xrightarrow{i_{0*}} \mathbf{D}^G(Y) \xrightarrow{[-]^G} \mathbf{D}(Y)\ .$$

The cartesian diagram

$$\begin{array}{ccc} Y_0 \times M_0 & \xrightarrow{k_0} & Y \times_{\mathbb{C}} M \\ {\scriptstyle \pi_{M_0}}\downarrow & & \downarrow{\scriptstyle p_M} \\ M_0 & \xrightarrow[j_0]{} & M \end{array}$$

shows that

$$k_{0*}\pi_{M_0}^*(-) \cong p_M^* j_{0*}(-) \cong \mathbf{L}j^*\pi_M^* j_{0*}(-).$$

By the projection formula, we can then write

$$\begin{aligned} j_*k_{0*}(\mathbf{L}k_0^*\mathscr{P} \otimes^{\mathbf{L}} \pi_{M_0}^*(-)) &\cong j_*(\mathscr{P} \otimes^{\mathbf{L}} k_{0*}\pi_{M_0}^*(-)) \\ &\cong j_*(\mathscr{P} \otimes^{\mathbf{L}} \mathbf{L}j^*\pi_M^* j_{0*}(-)) \\ &\cong j_*\mathscr{P} \otimes^{\mathbf{L}} \pi_M^* j_{0*}(-) \\ &\cong \mathscr{O}_{\mathscr{Z}} \otimes^{\mathbf{L}} \pi_M^* j_{0*}(-). \end{aligned}$$

Putting these observations together, we obtain the desired isomorphism:

$$\begin{aligned} i_{0*}\Psi_0(-) &= i_{0*}[\mathbf{R}\pi_{Y_0*}(\mathscr{P}_0^{\cdot} \otimes^{\mathbf{L}} \pi_{M_0}^*(-))]^G \\ &\cong [i_{0*}\mathbf{R}\pi_{Y_0*}(\mathscr{P}_0^{\cdot} \otimes^{\mathbf{L}} \pi_{M_0}^*(-))]^G \\ &\cong [\mathbf{R}\pi_{Y*}(i_0 \times j_0)_*(\mathscr{P}_0^{\cdot} \otimes^{\mathbf{L}} \pi_{M_0}^*(-))]^G \\ &\cong [\mathbf{R}\pi_{Y*}j_*k_{0*}(\mathbf{L}k_0^*\mathscr{P} \otimes^{\mathbf{L}} \pi_{M_0}^*(-))]^G \\ &\cong [\mathbf{R}\pi_{Y*}(\mathscr{O}_{\mathscr{Z}} \otimes^{\mathbf{L}} \pi_M^* j_{0*}(-))]^G \\ &= \Psi j_{0*}(-). \end{aligned}$$

□

We want now to consider a correspondence in the opposite direction. The main problem is the right adjoint to $\mathbf{R}\pi_{Y_0*}$ as π_{Y_0} is manifestly non-proper. However, using Deligne's construction of $\pi_{Y_0}^!$ in the context of general Grothendieck duality theory (cf. [32, 78, 62, 63]) we can still obtain a right adjoint to $\mathbf{R}\pi_{Y_0*}$ for the full subcategory of $\mathbf{D}^G(Y_0 \times M_0)$ consisting of objects whose support is proper over Y_0. Let us see how this comes about.

Let $\overline{M}_0$ be the closure of M_0 in the projective space $\mathbb{P}^n$. Then the map π_{Y_0} factorizes as $\pi_{Y_0} = \overline{\pi}_{Y_0}\iota$ where $\iota : Y_0 \times M_0 \hookrightarrow Y_0 \times \overline{M}_0$ is an open immersion and $\overline{\pi}_{Y_0} : Y_0 \times \overline{M}_0 \to Y_0$ is the projection. In this way we get an extension of π_{Y_0} which is a proper map. Now define the functor $\pi_{Y_0}^! : \mathbf{D}(Y_0) \to \mathbf{D}(Y_0 \times M_0)$ to be $\iota^*\overline{\pi}_{Y_0}^!$. A reasoning as in [63, Lemma 4] shows that there is a functorial isomorphism

$$\mathrm{Hom}_{\mathbf{D}(Y_0)}(\mathbf{R}\pi_{Y_0*}\mathscr{E}^{\cdot}, \mathscr{F}^{\cdot}) \cong \mathrm{Hom}_{\mathbf{D}(Y_0 \times M_0)}(\mathscr{E}^{\cdot}, \pi_{Y_0}^!\mathscr{F}^{\cdot}),$$

for every object $\mathscr{E}^{\cdot}$ in $\mathbf{D}(Y_0 \times M_0)$ whose support is proper over Y_0 and any $\mathscr{F}^{\cdot}$ in $\mathbf{D}(Y_0)$. Furthermore, since the map π_{Y_0} is of finite Tor-dimension and of finite type, it follows from [62, Theorem 4.9.4] that there is a functorial isomorphism

$$\pi_{Y_0}^!\mathscr{F}^{\cdot} \cong \pi_{Y_0}^!\mathscr{O}_{Y_0} \otimes^{\mathbf{L}} \pi_{Y_0}^*\mathscr{F}^{\cdot},$$

for any $\mathcal{F}^\cdot \in \mathbf{D}(Y_0)$. Let us remark that the above extends straightforwardly to the corresponding G-equivariant categories.

Let $\Phi_0 : \mathbf{D}(Y_0) \to \mathbf{D}^G(M_0)$ denote the functor in the other direction defined as

$$\Phi_0(-) = \mathbf{R}\pi_{M_0*}\mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathcal{P}_0^\cdot, \pi_{Y_0}^!(-\otimes\rho_0)).$$

Observe that the fact that τ is proper implies that the support of $\mathcal{P}_0^\cdot$ is proper over M_0. Arguing as before one can check that $\mathbf{R}\pi_{M_0*}\mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathcal{P}_0^\cdot, -)$ sends $\mathbf{D}^G(Y_0\times M_0)$ to $\mathbf{D}^G(M_0)$, so Φ_0 is well-defined.

The following is an immediate consequence of the definition.

Lemma 1.8.3. *Φ_0 is right adjoint to Ψ_0.*

Proof. Indeed, for any $\mathcal{E}^\cdot \in \mathbf{D}(Y_0)$ and $\mathcal{F}^\cdot \in \mathbf{D}^G(M_0)$ one has a sequence of isomorphisms:

$$\begin{aligned}
&\mathrm{Hom}_{\mathbf{D}^G(M_0)}(\mathcal{F}^\cdot, \Phi_0\mathcal{E}^\cdot)\\
&\quad= \mathrm{Hom}_{\mathbf{D}^G(M_0)}(\mathcal{F}^\cdot, \mathbf{R}\pi_{M_0*}\mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathcal{P}_0^\cdot, \pi_{Y_0}^!(\mathcal{E}^\cdot\otimes\rho_0)))\\
&\quad\cong \mathrm{Hom}_{\mathbf{D}^G(Y_0\times M_0)}(\pi_{M_0}^*\mathcal{F}^\cdot, \mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathcal{P}_0^\cdot, \pi_{Y_0}^!(\mathcal{E}^\cdot\otimes\rho_0)))\\
&\quad\cong \mathrm{Hom}_{\mathbf{D}^G(Y_0\times M_0)}(\mathcal{P}_0^\cdot \otimes^{\mathbf{L}} \pi_{M_0}^*\mathcal{F}^\cdot, \pi_{Y_0}^!(\mathcal{E}^\cdot\otimes\rho_0))\\
&\quad\cong \mathrm{Hom}_{\mathbf{D}^G(Y_0)}(\mathbf{R}\pi_{Y_0*}(\mathcal{P}_0^\cdot \otimes^{\mathbf{L}} \pi_{M_0}^*\mathcal{F}^\cdot), \mathcal{E}^\cdot\otimes\rho_0)\\
&\quad\cong \mathrm{Hom}_{\mathbf{D}(Y_0)}([\mathbf{R}\pi_{Y_0*}(\mathcal{P}_0^\cdot \otimes^{\mathbf{L}} \pi_{M_0}^*\mathcal{F}^\cdot)]^G, \mathcal{E}^\cdot)\\
&\quad\cong \mathrm{Hom}_{\mathbf{D}(Y_0)}(\Psi_0\mathcal{F}^\cdot, \mathcal{E}^\cdot).
\end{aligned}$$

Here, the third isomorphism is the aforementioned duality for π_{Y_0}, which can be applied since $\mathcal{P}_0^\cdot$ has proper support over Y_0. □

We now make an observation to be applied in the subsequent argument.

Lemma 1.8.4. *There is an isomorphism:*

$$\mathbf{L}k_0^*\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^!\mathcal{O}_Y) \cong \mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathcal{P}_0^\cdot, \pi_{Y_0}^!\mathcal{O}_{Y_0}).$$

Proof. We have to prove that the natural morphism

$$\mathbf{L}k_0^*\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^!\mathcal{O}_Y) \longrightarrow \mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathbf{L}k_0^*\mathcal{P}, \pi_{Y_0}^!\mathcal{O}_{Y_0})$$

is an isomorphism. Since k_0 is a closed immersion, it is enough to prove that the induced morphism

$$k_{0*}\mathbf{L}k_0^*\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^!\mathcal{O}_Y) \longrightarrow k_{0*}\mathbf{R}\mathcal{H}om^\cdot_{Y_0\times M_0}(\mathbf{L}k_0^*\mathcal{P}, \pi_{Y_0}^!\mathcal{O}_{Y_0})$$

is an isomorphism. Consider the cartesian diagram

$$\begin{array}{ccc}
Y_0\times M_0 & \xrightarrow{k_0} & Y\times_{\mathbb{C}} M\\
{\scriptstyle \pi_{Y_0}}\downarrow & & \downarrow{\scriptstyle p_Y}\\
Y_0 & \xrightarrow[i_0]{} & Y.
\end{array}$$

We have $k_{0*}\pi^*_{Y_0}(-) \cong p_Y^* i_{0*}(-)$. By the projection formula, we deduce that the first member is isomorphic to

$$\begin{aligned}
&\mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} k_{0*}\mathcal{O}_{Y_0\times M_0}\\
&\quad\cong \mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} k_{0*}\pi^*_{Y_0}\mathcal{O}_{Y_0}\\
&\quad\cong \mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} p_Y^* i_{0*}\mathcal{O}_{Y_0}\\
&\quad\cong \mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! i_{0*}\mathcal{O}_{Y_0})
\end{aligned}$$

where the last step follows from the observation that $\mathcal{P}$ has finite homological dimension. The second member is isomorphic to

$$\mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, k_{0*}\pi^!_{Y_0}\mathcal{O}_{Y_0})$$

by the adjoint property of $\mathbf{L}k_0^*$ and k_{0*}. Thus, we have to prove that the natural morphism

$$\mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! i_{0*}\mathcal{O}_{Y_0}) \longrightarrow \mathbf{R}\mathcal{H}om^{\cdot}_{Y\times_{\mathbb{C}}M}(\mathcal{P}, k_{0*}\pi^!_{Y_0}\mathcal{O}_{Y_0})$$

is an isomorphism. Then, it is enough to see that $p_Y^! i_{0*}\mathcal{O}_{Y_0} \cong k_{0*}\pi^!_{Y_0}\mathcal{O}_{Y_0}$. This follows from the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}(Y\times_{\mathbb{C}}M)}(\mathcal{E}^{\cdot}, p_Y^! i_{0*}\mathcal{O}_{Y_0}) &\cong \mathrm{Hom}_{\mathbf{D}(Y)}(\mathbf{R}p_{Y*}\mathcal{E}^{\cdot}, i_{0*}\mathcal{O}_{Y_0})\\
&\cong \mathrm{Hom}_{\mathbf{D}(Y_0)}(\mathbf{L}i_0^*\mathbf{R}p_{Y*}\mathcal{E}^{\cdot}, \mathcal{O}_{Y_0})\\
&\cong \mathrm{Hom}_{\mathbf{D}(Y_0)}(\mathbf{R}\pi_{Y_0*}\mathbf{L}k_0^*\mathcal{E}^{\cdot}, \mathcal{O}_{Y_0})\\
&\cong \mathrm{Hom}_{\mathbf{D}(Y_0\times M_0)}(\mathbf{L}k_0^*\mathcal{E}^{\cdot}, \pi^!_{Y_0}\mathcal{O}_{Y_0})\\
&\cong \mathrm{Hom}_{\mathbf{D}(Y\times_{\mathbb{C}}M)}(\mathcal{E}^{\cdot}, k_{0*}\pi^!_{Y_0}\mathcal{O}_{Y_0})
\end{aligned}$$

which hold for any object $\mathcal{E}^{\cdot}$ in $\mathbf{D}(Y\times_{\mathbb{C}} M)$ whose support is proper over Y (here we used the base change theorem for the above cartesian diagram; see [43, Sect. 1]). □

Before stating our next result, it will be convenient to provide the following piece of information. As we pointed out earlier, there exist a functorial isomorphism $\pi^!_{Y_0}(-) \cong \pi^!_{Y_0}\mathcal{O}_{Y_0} \otimes^{\mathbf{L}} \pi^*_{Y_0}(-)$. Using the fact that $\mathcal{P}_0^{\cdot}$ has finite homological dimension we obtain an isomorphism

$$\begin{aligned}
\mathbf{R}\mathcal{H}om^{\cdot}_{Y_0\times M_0}(\mathcal{P}_0^{\cdot}, \pi^!_{Y_0}(-)) &\cong \mathbf{R}\mathcal{H}om^{\cdot}_{Y_0\times M_0}(\mathcal{P}_0^{\cdot}, \pi^!_{Y_0}\mathcal{O}_{Y_0} \otimes^{\mathbf{L}} \pi^*_{Y_0}(-))\\
&\cong \mathbf{R}\mathcal{H}om^{\cdot}_{Y_0\times M_0}(\mathcal{P}_0^{\cdot}, \pi^!_{Y_0}\mathcal{O}_{Y_0}) \otimes^{\mathbf{L}} \pi^*_{Y_0}(-).
\end{aligned}$$

Thus, denoting $\mathcal{K}_0^{\cdot} = \mathbf{R}\mathcal{H}om^{\cdot}_{Y_0\times M_0}(\mathcal{P}_0^{\cdot}, \pi^!_{Y_0}\mathcal{O}_{Y_0})$, we can rewrite Φ_0 as

$$\Phi_0(-) \cong \mathbf{R}\pi_{M_0*}(\mathcal{K}_0^{\cdot} \otimes^{\mathbf{L}} \pi^*_{Y_0}(-\otimes\rho_0)).$$

Combining these remarks with Lemma 1.8.4 we have the following.

Lemma 1.8.5. *There is a natural isomorphism of functors:*

$$j_{0*}\Phi_0(-) \cong \Phi i_{0*}(-).$$

Proof. The argument is very similar to that used in the proof of Lemma 1.8.2. We give it for the sake of completeness. To begin with, we observe that there is a natural isomorphism between the functors

$$\mathbf{D}(Y_0) \xrightarrow{-\otimes\rho_0} \mathbf{D}^G(Y_0) \xrightarrow{i_{0*}} \mathbf{D}^G(Y)$$

and

$$\mathbf{D}(Y_0) \xrightarrow{i_{0*}} \mathbf{D}(Y) \xrightarrow{-\otimes\rho_0} \mathbf{D}^G(Y).$$

Invoking Lemma 1.8.4 and the projection formula, we obtain that

$$\begin{aligned} & j_* k_{0*}(\mathcal{K}_0^\cdot \otimes^{\mathbf{L}} \pi_{Y_0}^*(-\otimes\rho_0)) \\ &\quad \cong j_* k_{0*}(\mathbf{L}k_0^* \mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} \pi_{Y_0}^*(-\otimes\rho_0)) \\ &\quad \cong j_*(\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} k_{0*}\pi_{Y_0}^*(-\otimes\rho_0)) \\ &\quad \cong j_*(\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} p_Y^* i_{0*}(-\otimes\rho_0)) \\ &\quad \cong j_*(\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} \mathbf{L}j^*\pi_Y^* i_{0*}(-\otimes\rho_0)) \\ &\quad \cong j_*\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) \otimes^{\mathbf{L}} \pi_Y^* i_{0*}(-\otimes\rho_0). \end{aligned}$$

On the other hand, by relative Grothendieck duality, we get

$$\begin{aligned} j_*\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, p_Y^! \mathcal{O}_Y) &\cong j_*\mathbf{R}\mathcal{H}om^\cdot_{Y\times_{\mathbb{C}}M}(\mathcal{P}, j^!\pi_Y^! \mathcal{O}_Y) \\ &\cong \mathbf{R}\mathcal{H}om^\cdot_{Y\times M}(j_*\mathcal{P}, \pi_Y^! \mathcal{O}_Y) \\ &\cong \mathbf{R}\mathcal{H}om^\cdot_{Y\times M}(\mathcal{O}_{\mathscr{Z}}, \pi_Y^! \mathcal{O}_Y) \\ &\cong \mathcal{O}_{\mathscr{Z}}^\vee[n], \end{aligned}$$

where we used the isomorphism $\pi_Y^! \mathcal{O}_Y \cong \mathcal{O}_{Y\times M}[n]$ which follows from the triviality of the canonical bundle ω_M. Hence

$$j_* k_{0*}(\mathcal{K}_0^\cdot \otimes^{\mathbf{L}} \pi_{Y_0}^*(-\otimes\rho_0)) \cong \mathcal{O}_{\mathscr{Z}}^\vee[n] \otimes^{\mathbf{L}} \pi_Y^* i_{0*}(-\otimes\rho_0).$$

Wrapping things up, we conclude that

$$\begin{aligned} j_{0*}\Phi_0(-) &= j_{0*}\mathbf{R}\pi_{M_0 *}(\mathcal{K}_0^\cdot \otimes^{\mathbf{L}} \pi_{Y_0}^*(-\otimes\rho_0)) \\ &\cong \mathbf{R}\pi_{M*}(i_0\times j_0)_*(\mathcal{K}_0^\cdot \otimes^{\mathbf{L}} \pi_{Y_0}^*(-\otimes\rho_0)) \\ &\cong \mathbf{R}\pi_{M*} j_* k_{0*}(\mathcal{K}_0^\cdot \otimes^{\mathbf{L}} \pi_{Y_0}^*(-\otimes\rho_0)) \\ &\cong \mathbf{R}\pi_{M*}(\mathcal{O}_{\mathscr{Z}}^\vee[n] \otimes^{\mathbf{L}} \pi_Y^*(i_{0*}(-)\otimes\rho_0)) \\ &= \Phi i_{0*}(-), \end{aligned}$$

as asserted. □

The following result is the goal we have been striving for throughout this whole section.

Theorem 1.8.6. *The functors Φ_0 and Ψ_0 define inverse equivalences between $\mathbf{D}(Y_0)$ and $\mathbf{D}^G(M_0)$. These equivalences induce equivalences $\overline{\Phi}_0$ and $\overline{\Psi}_0$ between $\mathbf{D}_{\mathrm{Sg}}(Y_0)$ and $\mathbf{D}^G_{\mathrm{Sg}}(M_0)$.*

Proof. Let us prove that the composition $\Phi_0\Psi_0$ is isomorphic to the identity functor on $\mathbf{D}^G(M_0)$. The composition in the different order is computed similarly. Consider an object $\mathscr{E}^\cdot \in \mathbf{D}^G(M_0)$ and denote the cone of the adjunction morphism $\mathscr{E}^\cdot \to \Phi_0\Psi_0\mathscr{E}^\cdot$ by $\mathscr{F}^\cdot$. Applying j_{0*} yields an exact triangle

$$j_{0*}\mathscr{E}^\cdot \longrightarrow j_{0*}\Phi_0\Psi_0\mathscr{E}^\cdot \longrightarrow j_{0*}\mathscr{F}^\cdot \longrightarrow j_{0*}\mathscr{E}^\cdot[1]\ .$$

Combining Lemma 1.8.5 and Lemma 1.8.2 we get

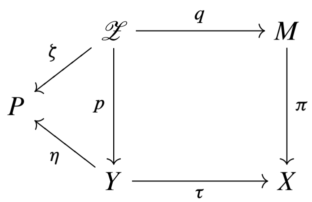
$$j_{0*}\Phi_0\Psi_0(-) \cong \Phi i_{0*}\Psi_0(-) \cong \Phi\Psi j_{0*}(-).$$

Hence, $j_{0*}\mathscr{F}^\cdot$ is isomorphic to the cone of the morphism $j_{0*}\mathscr{E}^\cdot \to \Phi\Psi j_{0*}\mathscr{E}^\cdot$. Since Φ is an equivalence and j_0 is a closed immersion, one obtains $\mathscr{F}^\cdot \cong 0$. The conclusion is that the adjunction morphism $\mathscr{E}^\cdot \to \Phi_0\Psi_0\mathscr{E}^\cdot$ is an isomorphism.

We next show that the functors Φ_0 and Ψ_0 induce equivalences between $\mathbf{D}_{\mathrm{Sg}}(Y_0)$ and $\mathbf{D}^G_{\mathrm{Sg}}(M_0)$. Let us first make an observation. Let $\mathscr{E}^\cdot$ be a perfect complex on $Y_0 \times M_0$ and let us consider the object $\mathbf{R}\pi_{M_0*}(\mathscr{K}_0^\cdot \otimes^{\mathbf{L}} \mathscr{E}^\cdot)$ in the derived category of coherent sheaves on M_0. We claim that $\mathbf{R}\pi_{M_0*}(\mathscr{K}_0^\cdot \otimes^{\mathbf{L}} \mathscr{E}^\cdot)$ is a perfect complex on M_0. To substantiate this claim, it suffices to verify that $\mathbf{R}\pi_{M_0*}(\mathscr{K}_0^\cdot \otimes^{\mathbf{L}} \mathscr{E}^\cdot) \otimes^{\mathbf{L}} \mathscr{F}^\cdot$ is an object of $\mathbf{D}(M_0)$ for every $\mathscr{F}^\cdot$ in $\mathbf{D}(M_0)$ (see, e.g. [44, Lemma 1.2]). But this follows at once from the projection formula for the morphism π_{M_0}. Similarly, we check that $\mathbf{R}\pi_{Y_0*}(\mathscr{P}_0^\cdot \otimes^{\mathbf{L}} \mathscr{E}^\cdot)$ is a perfect complex on Y_0. The same situation prevails in the equivariant setting.

Now, the functors $\pi^*_{Y_0}(-\otimes\rho_0)$ and $\pi^*_{M_0}$ are exact and take perfect complexes to perfect complexes. By what we have just seen, the functors $\mathbf{R}\pi_{M_0*}(\mathscr{K}_0^\cdot \otimes^{\mathbf{L}} -)$ and $[\mathbf{R}\pi_{Y_0*}(\mathscr{P}_0^\cdot \otimes^{\mathbf{L}} -)]^G$ also preserve perfect complexes. Hence, owing to Lemma 1.3.1, we obtain a functor $\overline{\Phi}_0 : \mathbf{D}_{\mathrm{Sg}}(Y_0) \to \mathbf{D}^G_{\mathrm{Sg}}(M_0)$ and this functor has the left adjoint $\overline{\Psi}_0 : \mathbf{D}^G_{\mathrm{Sg}}(M_0) \to \mathbf{D}_{\mathrm{Sg}}(Y_0)$. As the composition $\Phi_0\Psi_0$ is isomorphic to the identity functor, the composition $\overline{\Phi}_0\overline{\Psi}_0$ is also isomorphic to the identity functor on $\mathbf{D}^G_{\mathrm{Sg}}(M_0)$. A similar argument shows that the composition $\overline{\Psi}_0\overline{\Phi}_0$ is isomorphic to the identity functor on $\mathbf{D}_{\mathrm{Sg}}(Y_0)$. The result then follows immediately. $\square$

It seems appropriate to conclude by examining the implications of this result in the specific context of Section 1.2. Let $G = \mathbb{Z}_n$ be a cyclic group in $\mathrm{SL}(n,\mathbb{C})$ acting on $M = \mathbb{C}^n$ and let Y be the canonical crepant resolution of the quotient $X = M/G$. Explicitly we choose coordinates $x_1,\ldots,x_n$ on M in terms of which the action of the generator in G is given by $(x_1,\ldots,x_n) \mapsto (\varepsilon x_1,\ldots,\varepsilon x_n)$ where $\varepsilon = \exp(2\pi i/n)$ is a fixed nth root of unity. The space Y is the blow up of the unique singular point of X. It can be described explicitly as follows. Write $P = \mathbb{P}^{n-1}$ for the projective space with homogeneous coordinates $x_1,\ldots,x_n$. Then $Y = \mathrm{tot}(\mathscr{O}_P(-n))$ is the total space of the line bundle $\mathscr{O}_P(-n)$ and the natural map $\tau\colon Y \to X$ is simply contracting the zero section. Let $\mathscr{Z} \subset Y \times M$ denote the fiber product of Y and M over X. Then $\mathscr{Z}$ can be identified with the total space $\mathscr{Z} = \mathrm{tot}(\mathscr{O}_P(-1))$ and the map $q\colon \mathscr{Z} \to M$ is again the contraction of the zero section, this time to a smooth point –the origin $0 \in M$. All this data can be conveniently organized in the commutative diagram



where $\eta\colon Y \to P$ and $\zeta\colon \mathscr{Z} \to P$ denote the natural projections and $p\colon \mathscr{Z} \to Y$ is the map of taking a quotient by G. Note that the group G acts on $\mathscr{Z}$ by simply multiplying by ε along the fibers of $\mathscr{O}_P(-1) \to P$ and so the map $p\colon \mathscr{Z} \to Y$ can also be viewed as the map raising into nth power along the fibers of the line bundle $\mathscr{O}_P(-1)$. Conversely, we can view $\mathscr{Z}$ as the canonical nth root cover of Y which is branched along the zero section $Q \subset Y$ of η.

Now let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree n. Then f can be viewed as a regular function on M with a critical point at the origin which is invariant with respect to the action of G on M. This way, we get a singular Landau-Ginzburg model (M, f) with an action of G. Let S be the hypersurface of degree n in $P = \mathbb{P}^{n-1}$ given by the homogeneous equation $f = 0$. Consider the associated affine cone over S, namely, the hypersurface M_0 given in $M = \mathbb{C}^n$ by exactly the same equation $f = 0$. It is evident that the singular fiber of the map $f\colon M \to \mathbb{C}$ over the point $0 \in \mathbb{C}$ is precisely M_0. Let $g\colon Y \to \mathbb{C}$ be defined as before, and let Y_0 denote the fiber of g over the point 0. Then Y_0 is a normal crossing variety with irreducible components Y_0' and Y_0''. One component Y_0' is isomorphic to the total space of the line bundle $\mathscr{O}_P(-n)|_S$ over S. The second component Y_0'' is isomorphic to Q.

It is proved in [10, Proposition 2.40] that G-Hilb(M) is isomorphic to Y. Moreover, the tautological bundles on G-Hilb(M) (see [76] for the definition) are $\eta^*\mathscr{O}_P$, $\eta^*\mathscr{O}_P(1), \dots, \eta^*\mathscr{O}_P(n-1)$. It then follows from [15, Example 4.3] that Φ is an equivalence of categories and we can apply Theorem 1.8.6 to obtain $\mathbf{D}_{\mathrm{Sg}}(Y_0) \cong \mathbf{D}^G_{\mathrm{Sg}}(M_0)$. We are now set to establish the claim made at the end of Section 1.2.

Corollary 1.8.7. *Let the context be as above. Then the category of D-branes in the Landau-Ginzburg model (Y, g) is equivalent to the category of D-branes in the Landau-Ginzburg orbifold (M, f).*

Addendum. In a recent preprint P. Seidel [80] gave another example illustrating the use of Theorem 1.8.6 in the context of Homological Mirror Symmetry.

2

NONCOMMUTATIVE RESOLUTIONS OF ADE FIBERED CALABI-YAU THREEFOLDS

In this chapter we construct noncommutative resolutions of ADE fibered Calabi-Yau threefolds. The construction is in terms of a noncommutative algebra introduced by V. Ginzburg in [37], which we call the "$N = 1$ ADE quiver algebra". This chapter is based on joint work with Alex Boer [75].

2.1 INTRODUCTION

In recent years, there has been a great deal of interest in noncommutative algebra in connection with algebraic geometry, in particular to the study of singularities and their resolutions. The underlying idea in this context is that the resolutions of a singularity are closely linked to the structure of a noncommutative algebra.

The case of Kleinian singularities $X = \mathbb{C}^2/G$, for G a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$, was the first non-trivial example of this phenomenom, studied in [24]. It was shown that the minimal resolution of X can be described as a moduli space of representations of the preprojective algebra associated to the action of G. This preprojective algebra is known to be Morita equivalent to the skew group algebra $\mathbb{C}[x,y]\#G$, so we could alternatively use this algebra to construct the minimal resolution of X. Later, M. Kapranov and E. Vasserot [47] showed that there is a derived equivalence between $\mathbb{C}[x,y]\#G$ and the minimal resolution of X. A similar statement was established by T. Bridgeland, A. King and M. Reid [16] for crepant resolutions of quotient singularities $X = \mathbb{C}^3/G$ arising from a finite subgroup $G \subset \mathrm{SL}(3,\mathbb{C})$. In this case the crepant resolution of X is realized as a moduli space of representations of the McKay quiver associated to the action of G, subject to a certain natural commutation relations (see [28] and §4.4 of [37]).

Various steps in the direction mentioned above have been taken in a series of papers [61, 60, 39, 85, 27, 93], where several concrete examples have been discussed. More abstract approaches have also been put forward in [59, 11]. The lesson to be drawn from these works is that for some singularities it is possible to find a noncommutative algebra A such that the representation theory of this algebra dictates in every way the process of resolving these singularities. More precisely, it is shown that:

- the centre of A corresponds to the coordinate ring of the singularity;
- the algebra A is finitely generated over its centre;
- "nice" resolutions of the singularity are obtained via the moduli space of representations of A;
- the category of finitely generated modules over A is derived equivalent to the category of coherent sheaves on an appropriate resolution.

Following the terminology of M. Van den Bergh (cf. [91, 92]) we may think of A as a "noncommutative resolution".

This phenomenom also appears naturally in string theory in the context of coincident D-branes. There the singularity X should be a Calabi-Yau threefold and one studies Type IIB string theory compactified on X. It turns out that a collection of D-branes located at the singularity gives rise to a noncommutative algebra A, which can be described as the path algebra of a quiver with relations. For a fixed quiver Q, this construction only depends on a 'noncommutative function' called the superpotential. As a consequence, the aforementioned derived equivalence establishes a correspondence between two different ways of describing a D-brane: as an object of the derived category of coherent sheaves on a resolution of X and as a representation of the quiver Q.

Now, let us explain the situation on which we will focus. We shall study a kind of singular Calabi-Yau threefolds with isolated singularities, known as ADE fibered Calabi-Yau threefolds. They have been defined and studied in the work of F. Cachazo, S. Katz and C. Vafa [21] from the point of view of $N = 1$ quiver gauge theories. The quiver diagrams of interest here are the extended Dynkin quivers of type A, D or E. Following [37], for such a quiver Q, we associate a noncommutative algebra $\mathfrak{A}^\tau(Q)$ which we call the "$N = 1$ ADE quiver algebra". The choice of τ is encoded in the fibration data. The goal of this chapter is to show that the $N = 1$ ADE quiver algebra realizes a noncommutative resolution of the ADE fibered Calabi-Yau threefold associated with Q and τ. The proof of this result depends on two ingredients. On the one hand, we use the results in [24], on the construction of deformations of Kleinian singularities and their simultaneous resolutions in terms of Q. On the other hand, we use the results in [31] to contruct a Morita equivalence between $\mathfrak{A}^\tau(Q)$ and a noncommutative crepant resolution A^τ in the sense of Van den Bergh; this allows us to use the techniques developed in [91] to show that the derived category of finitely generated modules over $\mathfrak{A}^\tau(Q)$ is equivalent to the derived category of coherent sheaves on the small resolution of the ADE fibered Calabi-Yau threefold.

Some related results using different methods were obtained by B. Szendrői in [86]. He considers threefolds X fibered over a general curve C by ADE singularities and shows that D-branes on a small resolution of X are classified by representations with relations of a Kronheimer-Nakajima-type quiver in the category $\mathrm{Coh}(C)$ of coherent sheaves on C. The correspondence is given by a derived equivalence between the small resolution of X

and a sheaf of noncommutative algebras on C. In particular, there is a substantial overlap between Section 2.5 of this chapter and the results of Ref. [86].

An effort has been made to make this chapter as self-contained as possible. The chapter is structured as follows. In Section 2.2 we will review the theory of deformations of Kleinian singularities and their simultaneous resolutions and we define ADE fibered Calabi-Yau threefolds. In Section 2.3 we outline the results of Cassens and Slodowy on the construction of deformations of Kleinian singularities and their simultaneous resolutions in terms of Q. In Section 2.4 we define the $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$, describe some of its basic structure and prove that small resolutions of ADE fibered Calabi-Yau threefolds are obtained via the moduli space of representations of $\mathfrak{A}^\tau(Q)$. We conclude with a discussion of the derived equivalence between $\mathfrak{A}^\tau(Q)$ and the small resolution of an ADE fibered Calabi-Yau threefold in Section 2.5.

2.2 Preliminaries

In this section we will briefly review the definition of ADE fibered Calabi-Yau threefolds, as developed in [21]. We begin by outlining the theory of deformations of Kleinian singularities and their simultaneous resolutions. For more details on these topics, we refer the reader to [50], [55] and [40].

2.2.1 Deformations of Kleinian singularities.

We remind briefly the basic setting. Recall that Kleinian singularities are constructed as the quotient of $\mathbb{C}^2$ by a finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$. Such finite G are known to fall into an ADE classification: up to conjugacy, they are in one-to-one correspondence with the Dynkin diagrams of type A, D or E. The quotient $\mathbb{C}^2/G$ can be realized as a hypersurface $X_0 \subset \mathbb{C}^3$ with an isolated singularity at the origin. The defining equation for the singularity is determined by the Dynkin diagram associated to the group G:

$$\begin{aligned}
A_n : &\qquad x^2 + y^2 + z^{n+1} = 0 \\
D_n : &\qquad x^2 + y^2 z + z^{n-1} = 0 \\
E_6 : &\qquad x^2 + y^3 + z^4 = 0 \\
E_7 : &\qquad x^2 + y^3 + yz^3 = 0 \\
E_6 : &\qquad x^2 + y^3 + z^5 = 0
\end{aligned}$$

Write $\pi : Y_0 \to X_0$ for the minimal resolution. The exceptional divisor of π consists of (-2)-curves C_i intersecting transversally. This configuration is best explained via the dual graph Γ of X_0: each curve C_i is a vertex, and two vertices are joined by an edge if and only if the corresponding curves intersect in Y_0. For Kleinian singularities, the resolution graph is just a Dynkin diagram of type A, D or E. For this reason, these singularities are also called ADE singularities.

The Dynkin diagram appears in the classification of simple Lie algebras. Thus we have a complex simple Lie algebra $\mathfrak{g}$ corresponding to Γ. We let $\mathfrak{h}$ be the complex Cartan subalgebra of $\mathfrak{g}$ and W the corresponding Weyl group. On $\mathfrak{h}$, we have a natural action of W.

For later use we briefly discuss the notion of semiuniversal deformation of a singularity. Let S be a complex space with a distinguished point s_0. A *deformation* of a complex space X_0 consists of a flat morphism $\varphi : \mathscr{X} \to S$ together with an isomorphism $X_0 \cong \mathscr{X}_{s_0} = \varphi^{-1}(s_0)$. The space S is called the *base* of the deformation φ.

An isomorphism of two deformations $\varphi : \mathscr{X} \to S$ and $\varphi' : \mathscr{X}' \to S$ of X_0 over S is an isomorphism $\theta : \mathscr{X} \to \mathscr{X}'$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{X} & \xrightarrow{\ \theta\ } & \mathscr{X}' \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi'} & \\ & S & \end{array}$$

If $\varphi : \mathscr{X} \to S$ is a deformation of X_0 and $u : T \to S$ is any morphism, then the pull-back $\psi : \mathscr{X} \times_S T \to T$ of φ by u is flat again, hence a deformation of X_0 over T, which will be called the deformation induced by u from φ.

A deformation $\varphi : \mathscr{X} \to S$ of X_0 is called *semiuniversal* if any other deformation $\varphi' : \mathscr{X}' \to S'$ of X_0 is isomorphic to a deformation induced from φ by a base change $u : S' \to S$ whose differential at $s_0' \in S'$ is uniquely determined by φ'. It follows immediately that semiuniversal deformations are unique up to isomorphism.

In general it is very hard to determine semiuniversal deformations. Fortunately, semiuniversal deformations are easy to write down explicitly for any isolated hypersurface singularity. They can be constructed in the following way. Let $X_0 = \{f = 0\} \subset \mathbb{C}^k$ be an isolated hypersurface singularity at the origin. Choose polynomials $g_1, \ldots, g_n$ which descend to a basis of the vector space $\mathbb{C}[x_1, \ldots, x_k]/(f, \partial f/\partial x_1, \ldots, \partial f/\partial x_k)$. Define the spaces

$$\mathscr{X} = \left\{ f(x_1, \ldots, x_k) + \sum_{i=1}^{n} t_i g_i(x_1, \ldots, x_k) = 0 \right\} \subset \mathbb{C}^k \times \mathbb{C}^n$$

and $S = \mathbb{C}^n$, and let $\varphi : \mathscr{X} \to S$ be the composition of the embeding $\mathscr{X} \to \mathbb{C}^k \times \mathbb{C}^n$ and the second projection. Then the map φ realizes a semiuniversal deformation of X_0.

Now we want to construct semiuniversal deformations of Kleinian singularities. From what we have discussed above, each Kleinian singularity has an associated Dynkin diagram Γ whose Weyl group W acts on the complex Cartan subalgebra $\mathfrak{h}$ of the associated Lie algebra $\mathfrak{g}$. A model for the base of the deformation is given by $S = \mathfrak{h}/W$. The defining polynomial of $\mathscr{X}$ is simple to write in the A_n and D_n cases, namely

$$A_n : \qquad x^2 + y^2 + z^{n+1} + \sum_{i=2}^{n+1} \alpha_i z^{n+1-i}$$

$$D_n : \qquad x^2 + y^2 z + z^{n-1} - \sum_{i=1}^{n-1} \delta_{2i} z^{n-1-i} + 2\gamma_n y$$

where α_i is the ith elementary symmetric function of $t_1, \ldots, t_{n+1}$, δ_{2i} is the ith elementary symmetric function of $t_1^2, \ldots, t_n^2$ and $\gamma_n = t_1 \cdots t_n$. The corresponding equations for E_6, E_7 and E_8 in terms of the t's are more complicated and we refer the reader to [50].

2.2.2 **Simultaneous resolutions for Kleinian singularities.** Kleinian singularities are exceptional among the other surface singularities because they admit simultaneous resolutions.

We start by recalling the definition of a simultaneous resolution. Let $\varphi : X \to S$ be a flat morphism of complex spaces. A *simultaneous resolution* of φ is a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\pi} & X \\ {\scriptstyle\varphi'}\downarrow & & \downarrow{\scriptstyle\varphi} \\ T & \xrightarrow[u]{} & S \end{array}$$

such that $\varphi' : X' \to T$ is flat, u is finite and surjective, π is proper, and for all $t \in T$, the morphism $\pi|_{X'_t} : X'_t \to X_{u(t)}$ is a resolution of singularities. If S is a point, then a simultaneous resolution of $\varphi : X \to S$ is the same as a resolution of X.

Brieskorn and Tyurina [17, 88] showed that the semiuniversal deformation $\varphi : \mathscr{X} \to \mathfrak{h}/W$, for any Kleinian singularity X_0, admits a simultaneous resolution after making the base change $\mathfrak{h} \to \mathfrak{h}/W$. More precisely, the family $\mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$ may be resolved explicitly and one obtains a simultaneous resolution $\mathscr{Y} \to \mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$ of φ inducing the minimal resolution $Y_0 \to X_0$. The situation can be conveniently summarized by the diagram

$$\begin{array}{ccccc} \mathscr{Y} & \longrightarrow & \mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h} & \longrightarrow & \mathscr{X} \\ & \searrow & \downarrow & & \downarrow \\ & & \mathfrak{h} & \longrightarrow & \mathfrak{h}/W \end{array}$$

We should mention that there is a more general construction obtained from simultaneous *partial* resolutions of X_0; if we let $\Gamma_0 \subset \Gamma$ be the subdiagram for the part of the singularity that is not being resolved and denote by $W_0 \subset W$ the subgroup generated by reflections of the simple roots corresponding to $\Gamma \setminus \Gamma_0$, then we can define a deformation of X_0, which has a model for the base given by $\mathfrak{h}/W_0$, and there is a simultaneous partial resolution $\mathscr{Z}$ of the family $\mathscr{X} \times_{\mathfrak{h}/W_0} \mathfrak{h}$.

We will illustrate this with an example.

Example 2.2.1. We consider a singularity of type A_n. In the space $\mathbb{C}^{n+1}$ with coordinates $t_1, \dots, t_{n+1}$, there is a natural identification of $\mathfrak{h}$ with the hyperplane given by the equation $\sum_{i=1}^{n+1} t_i = 0$. The Weyl group W is isomorphic to the symmetric group $\mathfrak{S}_{n+1}$. The quotient map $\mathfrak{h} \to \mathfrak{h}/W$ can be realized by the formula $\alpha_i = \sigma_i(t_1, \dots, t_{n+1})$, where σ_i is the ith elementary symmetric function of $t_1, \dots, t_{n+1}$. Making a finite base change $\mathfrak{h} \to \mathfrak{h}/W$, one gets a family $\mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$ given in $\mathbb{C}^3 \times \mathfrak{h}$ by the equation

$$x^2 + y^2 + \prod_{i=1}^{n+1} (z + t_i) = 0$$

with discriminant locus $\prod_{i<j} (t_i - t_j)^2 = 0$. This can be rewritten after an analytic coordinate change as

$$xy = \prod_{i=1}^{n+1} (z + t_i).$$

The resolution $\mathscr{Y}$ can be constructed explicitly as the closure of the graph of the rational map $\mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h} \to (\mathbb{P}^1)^n$ defined by

$$(x,y,z,t_i) \longmapsto \left(\prod_{i=1}^{j} (z+t_i) : y \right).$$

Let $(u_j : v_j)$ be homogeneous coordinates on the jth $\mathbb{P}^1$ arising from the resolution described above. Then

$$\begin{aligned}
xy &= \prod_{i=1}^{n+1} (z+t_i) \\
xv_j &= u_j \prod_{i=j+1}^{n+1} (z+t_i), \quad 1 \le j \le n \\
yu_j &= v_j \prod_{i=1}^{j} (z+t_i), \quad 1 \le j \le n \\
u_j v_k &= u_k v_j \prod_{i=k+1}^{j} (z+t_i), \quad 1 \le k < j \le n
\end{aligned}$$

are the equations for $\mathscr{Y}$. The projection $\mathbb{C}^3 \times \mathfrak{h} \times (\mathbb{P}^1)^n \to \mathfrak{h}$ induces a map $\psi : \mathscr{Y} \to \mathfrak{h}$, and the composition $\mathbb{C}^3 \times \mathfrak{h} \times (\mathbb{P}^1)^n \to \mathbb{C}^3 \times \mathfrak{h} \to \mathbb{C}^3 \times \mathfrak{h}/W$ induces a map $\pi : \mathscr{Y} \to \mathscr{X}$. By construction, the following diagram is commutative:

$$\begin{array}{ccc}
\mathscr{Y} & \xrightarrow{\ \pi\ } & \mathscr{X} \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\
\mathfrak{h} & \longrightarrow & \mathfrak{h}/W
\end{array}$$

We claim that this diagram is a simultaneous resolution of φ. Let $U_0, U_1, \dots, U_n$ be the open subsets of $\mathbb{C}^3 \times \mathfrak{h} \times (\mathbb{P}^1)^n$ defined by

$$\begin{aligned}
U_0 &= \{v_1 \neq 0\}, \\
U_k &= \{u_k \neq 0, v_{k+1} \neq 0\}, \quad 1 \le k \le n-1 \\
U_n &= \{u_n \neq 0\},
\end{aligned}$$

and on U_k, let

$$\xi_k = v_k/u_k, \qquad \eta_k = u_{k+1}/v_{k+1}.$$

In $\mathscr{Y} \cap U_k$ we can solve for the $(u_j : v_j)$, $j \neq k, k+1$, by

$$(u_j : v_j) = \left(1 : \xi_k \prod_{i=j+1}^{k} (z+t_i) \right), \quad \text{for } j < k$$

$$(u_j : v_j) = \left(\eta_k \prod_{i=k+2}^{j} (z+t_i) : 1 \right), \quad \text{for } j > k+1.$$

An elementary computation shows that $\mathscr{Y} \cap U_k$ may be defined by the above equations together with the equations

$$x = \eta_k \prod_{i=k+2}^{n+1} (z + t_i),$$
$$y = \xi_k \prod_{i=1}^{k} (z + t_i),$$
$$z = \xi_k \eta_k - t_{k+1}.$$

Consequently, it may be inferred that $\mathscr{Y}$ is non-singular, and ψ has maximal rank, whence ψ is flat. It is now routine to verify that the diagram above is a simultaneous resolution of φ.

2.2.3 **ADE fibered Calabi-Yau threefolds.** We now describe a certain class of Calabi-Yau threefolds constructed in [21] by F. Cachazo, S. Katz and C. Vafa (see also [20, 49, 84, 96]).

We begin with some preliminaries. By a Gorenstein Calabi-Yau threefold X we always mean a quasiprojective threefold X with only terminal Gorentein singularities such that $K_X = 0$. Fix a Gorenstein Calabi-Yau threefold. A *small resolution* $\pi : Z \to X$ is a proper birational morphism from a normal variety Z to X such that the expectional locus is a curve. We are mainly interested in the case where $P \in X$ is an isolated Gorenstein threefold singularity. In this case $C = \pi^{-1}(P)$ is a transverse union of smooth rational $(-1,-1)$-curves C_i.

We point out the following important result, originally due to M. Reid (cf. [77]).

Proposition 2.2.2. *Let $P \in X$ be an isolated Gorenstein threefold singularity and $\pi : Z \to X$ a small resolution. Then the generic hyperplane section X_0 through P is a Kleinian singularity, and its proper transform Z_0 on Z has isolated singularities, and hence is normal, and is dominated by the minimal resolution of X_0.*

In light of this result, we can view X as the total space of a one parameter family of deformations of its generic hyperplane X_0, and the small resolution Z as the total space of a one parameter family of deformations of a partial resolution Z_0 of X_0. We are thus in the presence of the phenomenon of simultaneous partial resolutions of Kleinian singularities.

Using these observations we can define a broader class of Calabi-Yau threefolds as follows. We want to obtain a Gorenstein Calabi-Yau threefold X by fibering the total space of the semiuniversal deformation of a Kleinian singularity over a complex plane whose coordinate we denote by λ. This fibration is implemented by allowing the t_i's to be polynomials in λ. In more concrete terms, let $f : \mathbb{C} \to \mathfrak{h}$ be a polynomial map[1]. Via the defining equation for the family $\mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$, we can view X as the total space of a one parameter family defined by f. Similarly, the simultaneous resolution $\mathscr{Y} \to \mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$

[1]More general and more local constructions can be obtained from polynomial maps $f : \Delta \to \mathfrak{h}$ with Δ a complex disk, but we choose $\mathbb{C}$ as the domain of f for application to physics.

can be used to construct a Calabi-Yau threefold Y. That is, we get a cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathscr{Y} \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & \mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C} & \xrightarrow{f} & \mathfrak{h} \end{array}$$

where Y is the pullback of $\mathscr{Y}$ by f and X is the pullback of $\mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$ by f. One can show that if f is sufficiently general, then Y is smooth, X is Gorenstein with an isolated singular point, and $\pi : Y \to X$ is a small resolution. The genericity condition is that f is transverse to the hyperplanes $\Pi_\rho \subset \mathfrak{h}$ orthogonal to each positive root ρ of $\mathfrak{h}$. This class of Calabi-Yau threefolds we call ADE fibered Calabi-Yau threefolds.

Note that if $f = 0$, then the singular threefold X is isomorphic to a direct product of the form $X_0 \times \mathbb{C}$. In particular X has a line of Kleinian singularities. The resolution Y is isomorphic to the direct product $Y_0 \times \mathbb{C}$, where Y_0 is the minimal resolution of X_0. This of course is *not* a small resolution, as it has an exceptional divisor over a curve. The main point for us here is that ADE fibered Calabi-Yau threefolds are related to $X_0 \times \mathbb{C}$ by a complex structure deformation. For a full discussion of these matters consult [84, 86].

We illustrate with the following example.

Example 2.2.3. Let $P \in X$ be an isolated A_n singularity. We keep the notation of Example 2.2.1. A map $f = (t_1, \dots, t_{n+1}) : \mathbb{C} \to \mathfrak{h}$ would give the t_i's as polynomial functions of λ. The singular threefold X can be induced from $\mathscr{X} \times_{\mathfrak{h}/W} \mathfrak{h}$ by the morphism f. In other words, X is given by

$$\left\{ (x, y, z, \lambda) \,\middle|\, xy = \prod_{i=1}^{n+1} (z + t_i(\lambda)) \right\} \subset \mathbb{C}^4.$$

Y is the resolution of the above given by taking the closure of the graph of the mapping $X \to (\mathbb{P}^1)^n$ given by

$$(u_j : v_j) = \left(\prod_{i=1}^{j} (z + t_i(\lambda)) : y \right), \quad 1 \le j \le n.$$

As with Example 2.2.1 we define $n+1$ open subsets on $\mathbb{C}^4 \times (\mathbb{P}^1)^n$ as follows:

$$\begin{aligned} U_0 &= \{v_1 \neq 0\}, \\ U_k &= \{u_k \neq 0, v_{k+1} \neq 0\}, \quad 1 \le k \le n-1 \\ U_n &= \{u_n \neq 0\}. \end{aligned}$$

Setting $\xi_k = v_k/u_k$, $\eta_k = u_{k+1}/v_{k+1}$ for $1 \le k \le n-1$ and observing that $Y \subset \mathbb{C}^4 \times (\mathbb{P}^1)^n$

satisfies

$$x v_j = u_j \prod_{i=j+1}^{n+1} (z + t_i(\lambda)), \quad 1 \le j \le n$$

$$y u_j = v_j \prod_{i=1}^{j} (z + t_i(\lambda)), \quad 1 \le j \le n$$

$$u_j v_k = u_k v_j \prod_{i=k+1}^{j} (z + t_i(\lambda)), \quad 1 \le k < j \le n$$

one can solve for the $(u_j : v_j)$, $j \neq k, k+1$, by

$$(u_j : v_j) = \left(1 : \xi_k \prod_{i=j+1}^{k} (z + t_i(\lambda))\right), \quad \text{for } j < k$$

$$(u_j : v_j) = \left(\eta_k \prod_{i=k+2}^{j} (z + t_i(\lambda)) : 1\right), \quad \text{for } j > k+1.$$

It follows that Y is obtained by glueing $n+1$ copies of $\mathbb{C}^3$ with coordinates (ξ_k, η_k, λ) and transition maps

$$\xi_{k+1} = \xi_k^{-1}, \qquad \eta_{k+1} = \eta_k(\xi_k \eta_k - t_{k+1}(\lambda)), \qquad \lambda = \lambda.$$

The contraction map $\pi : Y \to X$ is defined on U_k by

$$x = \eta_k \prod_{i=k+2}^{n+1} (z + t_i(\lambda)),$$

$$y = \xi_k \prod_{i=1}^{k} (z + t_i(\lambda)),$$

$$z = \xi_k \eta_k - t_{k+1}(\lambda),$$

which extends over the whole of Y.

2.3 DEFORMATIONS AND SIMULTANOEUS RESOLUTIONS OF KLEINIAN SINGULARITIES REVISITED

A new approach to the deformation and resolution theory of Kleinian singularities was given by H. Cassens and P. Slodowy [24]. Their construction starts directly from the McKay graph of the finite subgroup G of $\mathrm{SL}(2, \mathbb{C})$ and uses geometric invariant theory. In this section, we describe some aspects of this construction we shall need in the sequel.

2.3.1 **Quivers.** First of all, let us briefly remind the definitions and the basic properties of quivers and their representations.

Recall that a *quiver* $Q = (Q_0, Q_1, h, t)$ consists of a finite set Q_0 of vertices, a finite set Q_1 of arrows and two maps $h, t : Q_1 \to Q_0$ which assign to each arrow $a \in Q_1$ its head $h(a)$ and tail $t(a)$, respectively. We shall write $a : i \to j$ for an arrow starting in i and ending in j.

A *path* in Q of length $m \geq 0$ is a sequence $p = a_1 a_2 \cdots a_m$ of arrows such that $t(a_{i+1}) = h(a_i)$ for $1 \leq i \leq m-1$. We let $h(p) = h(a_m)$ and $t(p) = t(a_1)$ denote the initial and final vertices of the path p. For each vertex $i \in Q_0$, we let e_i denote the trivial path of length zero which starts and ends at the vertex i. The *path algebra* $\mathbb{C}Q$ associated to a quiver Q is the $\mathbb{C}$-algebra whose underlying vector space has basis the set of paths in Q, and with the product of paths given by concatenation. Thus, if $p = a_1 \cdots a_m$ and $q = b_1 \cdots b_n$ are two paths, then $pq = a_1 \cdots a_m b_1 \cdots b_n$ if $t(q) = h(p)$ and $pq = 0$ otherwise. We also have

$$e_i e_j = \begin{cases} e_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

$$e_i p = \begin{cases} p & \text{if } t(p) = i \\ 0 & \text{if } t(p) \neq i \end{cases},$$

$$p e_i = \begin{cases} p & \text{if } h(p) = i \\ 0 & \text{if } h(p) \neq i \end{cases},$$

for $p \in \mathbb{C}Q$. This multiplication is associative. Note that $\sum_{i \in Q_0} e_i$ is the identity element of $\mathbb{C}Q$. We denote by mod–$\mathbb{C}Q$ the category of finitely generated right modules over $\mathbb{C}Q$.

The path algebra $\mathbb{C}Q$ is sometimes too big to be of interest and so often instead we wish to consider the path algebra modulo an ideal. This ideal is often defined by using relations on the quiver. Formally, a *relation* σ on a quiver Q is a $\mathbb{C}$-linear combination of paths $\sigma = c_1 p_1 + \cdots + c_n p_n$ with $c_i \in \mathbb{C}$ and $h(p_1) = \cdots = h(p_n)$ and $t(p_1) = \cdots = t(p_n)$. If $\rho = \{\sigma_r\}$ is a set of relations on Q, the pair (Q, ρ) is called a *quiver with relations*. Associated with (Q, ρ) is the $\mathbb{C}$-algebra $A = \mathbb{C}Q/(\rho)$, where (ρ) denotes the ideal in $\mathbb{C}Q$ generated by the set of relations ρ.

A *representation* V of Q is a collection $\{V_i \mid i \in Q_0\}$ of finite dimensional vector spaces over $\mathbb{C}$ together with a collection $\{V_a : V_{t(a)} \to V_{h(a)} \mid a \in Q_1\}$ of $\mathbb{C}$-linear maps. The *dimension vector* of the representation V is the vector $\underline{\dim} V \in \mathbb{Z}^{Q_0}$ whose ith component is $\dim V_i$. If V and W are two representations of Q, then a *morphism* $\phi : V \to W$ is a collection of $\mathbb{C}$-linear maps $\{\phi_i : V_i \to W_i \mid i \in Q_0\}$ such that for all arrows $a \in Q_1$ we have that $W_a \phi_{t(a)} = \phi_{h(a)} V_a$. A morphism ϕ is an isomorphism if all the components ϕ_i are isomorphisms.

Suppose that V and V' are representations of Q. We say that V' is a *subrepresentation* of V if $V'_i \subset V_i$ for all $i \in Q_0$ and $V'_a = V_a|_{V_i}$ for each arrow $a : i \to j$. A representation V is *trivial* if $V_i = 0$ for all $i \in Q_0$ and *simple* if its only subrepresentations are the trivial representation and V itself.

For a quiver Q we can form a category rep(Q) whose objects are representations of Q with the morphisms as defined above. With each $\mathbb{C}Q$-module M we associate a representation of Q given by $V_i = e_i M$ for each vertex $i \in Q_0$ and $V_a : V_i \to V_j, m \mapsto am$ for each arrow $a : i \to j$ in Q. This construction is functorial. In fact it is an equivalence between rep(Q) and mod–$\mathbb{C}Q$.

For a given quiver with relations (Q,ρ) we define the category $\operatorname{rep}(Q,\rho)$ of representations to be the full subcategory of $\operatorname{rep}(Q)$ whose objects are the V with $V_\sigma = 0$ for each relation σ in ρ. As before, the categories $\operatorname{rep}(Q,\rho)$ and mod–A are equivalent and we view the equivalence as an identification.

For a dimension vector $\alpha \in \mathbb{Z}^{Q_0}$ we define the representation space

$$\operatorname{Rep}(Q,\alpha) = \bigoplus_{a\in Q_1} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{t(a)}}, \mathbb{C}^{\alpha_{h(a)}})$$

which is an irreducible affine variety of dimension $\sum_{a\in Q_1} \alpha_{t(a)}\alpha_{h(a)}$. Given a point $x \in \operatorname{Rep}(Q,\alpha)$, we define a representation $V(x)$ of Q of dimension vector α by declaring that $V(x)_i = \mathbb{C}^{\alpha_i}$ and that $V(x)_a$ is the $\mathbb{C}$-linear map x_a. There is a canonical action of the linear reductive group $\operatorname{GL}(\alpha) = \prod_{i\in Q_0} \operatorname{GL}(\alpha_i,\mathbb{C})$ on the representation space $\operatorname{Rep}(Q,\alpha)$ determined for all elements $x \in \operatorname{Rep}(Q,\alpha)$ and all group elements $g = (g_i)_{i\in Q_0}$ of $\operatorname{GL}(\alpha)$ by the rule

$$(g\cdot x)_a = g_{h(a)} x_a g_{t(a)}^{-1}.$$

It is clear that the orbits of $\operatorname{GL}(\alpha)$ on $\operatorname{Rep}(Q,\alpha)$ are precisely the isomorphism classes of representations. If ρ is a set of relations for Q, we denote by $\operatorname{Rep}(A,\alpha)$ the closed subspace of $\operatorname{Rep}(Q,\alpha)$ corresponding to representations for $A = \mathbb{C}Q/(\rho)$. As before, we can see that the orbits of $\operatorname{GL}(\alpha)$ on $\operatorname{Rep}(A,\alpha)$ correspond to isomorphism classes of representations of A of dimension α.

2.3.2 **Deformation of Kleinian singularities revisited.** This subsection is devoted to constructing the semiuniversal deformation of a Kleinian singularity in terms of the associated quiver. We follow the general approach described in Ref. [31] (see also [29] and [30]).

We begin by defining the deformed preprojective algebra. Let Q be a quiver with vertex set I. We denote by $\overline{Q}$ the *double* of Q, obtained by adding a reverse arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. If $\tau \in \mathbb{C}^I$ then the *deformed preprojective algebra of weight* τ is defined by

$$\Pi^\tau(Q) = \mathbb{C}\overline{Q}\Big/\left(\sum_{a\in Q}[a,a^*] - \sum_{i\in I}\tau_i e_i\right).$$

where $[a,a^*]$ is the commutator $aa^* - a^*a$. The algebra $\Pi(Q) = \Pi^0(Q)$ is known as the *preprojective algebra*.

The representations of $\overline{Q}$ of dimension vector α are parametrized by

$$\operatorname{Rep}(\overline{Q},\alpha) = \operatorname{Rep}(Q,\alpha) \oplus \operatorname{Rep}(Q^{\mathrm{op}},\alpha),$$

where Q^{op} is the *opposite* quiver to Q with an arrow $a^* : j \to i$ for each arrow $a : i \to j$ in Q. As earlier, we have a natural action of the reductive group $\operatorname{GL}(\alpha) = \prod_{i\in I} \operatorname{GL}(\alpha_i,\mathbb{C})$ on $\operatorname{Rep}(\overline{Q},\alpha)$ defined by conjugation, and its orbits correspond to isomorphism classes of representations. The diagonal subgroup $\mathbb{C}^\times \subset \operatorname{GL}(\alpha)$ acts trivially, leaving a faithful action of $\operatorname{G}(\alpha) = \operatorname{GL}(\alpha)/\mathbb{C}^\times$. The Lie algebra of $\operatorname{GL}(\alpha)$ is given by $\operatorname{End}(\alpha) = \prod_{i\in I} \operatorname{End}_{\mathbb{C}}(\mathbb{C}^{\alpha_i})$.

Using the trace pairing we may identify $\mathrm{End}(\alpha) \cong \mathrm{End}(\alpha)^*$, and, exchanging the components corresponding to arrows a and a^*, also $\mathrm{Rep}(Q^{\mathrm{op}},\alpha) \cong \mathrm{Rep}(Q,\alpha)^*$. Thus, there is an identification of the cotangent bundle $T^*\mathrm{Rep}(Q,\alpha)$ of $\mathrm{Rep}(Q,\alpha)$ with $\mathrm{Rep}(\overline{Q},\alpha)$:

$$\mathrm{Rep}(\overline{Q},\alpha) \cong \mathrm{Rep}(Q,\alpha) \times \mathrm{Rep}(Q,\alpha)^* = T^*\mathrm{Rep}(Q,\alpha).$$

The natural symplectic form on $T^*\mathrm{Rep}(Q,\alpha)$ correspond to the form

$$\omega(x,y) = \sum_{a\in Q} \mathrm{tr}(x_{a^*}y_a) - \mathrm{tr}(x_a y_{a^*})$$

on $\mathrm{Rep}(\overline{Q},\alpha)$. Associated to the action of $\mathrm{G}(\alpha)$ there is a *moment map* $\mu_\alpha : \mathrm{Rep}(\overline{Q},\alpha) \to \mathrm{End}(\alpha)_0$ given by

$$\mu_\alpha(x)_i = \sum_{\substack{a\in Q\\ h(a)=i}} x_a x_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} x_{a^*} x_a,$$

where

$$\mathrm{End}(\alpha)_0 = \left\{ \phi \in \mathrm{End}(\alpha) \,\middle|\, \sum_{i\in I} \mathrm{tr}(\phi_i) = 0 \right\}.$$

If one uses the trace pairing to identify $\mathrm{End}(\alpha)_0$ with the dual of the Lie algebra of $\mathrm{G}(\alpha)$, then this is a moment map in the usual sense.

Now, suppose that $\tau \in \mathbb{C}^I$. The category of $\Pi^\tau(Q)$-modules is equivalent to the category of representations V of $\overline{Q}$ which satisfy

$$\sum_{\substack{a\in Q\\ h(a)=i}} V_a V_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} V_{a^*} V_a = \tau_i \,\mathrm{id}_{V_i}$$

for each vertex i. If $\alpha\cdot\tau = 0$, then τ can be identified with the element $(\tau_i \,\mathrm{id}_{\mathbb{C}^{\alpha_i}})$ in the centre of $\mathrm{End}(\alpha)_0$, and

$$\mathrm{Rep}(\Pi^\tau(Q),\alpha) = \mu_\alpha^{-1}(\tau)$$

is the closed subset of $\mathrm{Rep}(\overline{Q},\alpha)$ that classifies representations of $\Pi^\tau(Q)$ of dimension vector α.

We are now ready to start our study of semiuniversal deformations of Kleinian singularities. Let G be a finite group of $\mathrm{SL}(2,\mathbb{C})$ and let $\mathbb{C}^2/G$ be the corresponding Kleinian singularity. Let $\rho_0,\dots,\rho_n$ be the irreducible representations of G with ρ_0 trivial, and let V be the natural 2-dimensional representation of G. The *McKay graph* of G is the graph with vertex set $I = \{0,1,\dots,n\}$ and with the number of edges between i and j being the multiplicity of ρ_i in $V\otimes\rho_j$. (Since V is self-dual, this is the same as the multiplicity of ρ_j in $V\otimes\rho_i$.) McKay observed that this graph is an extended Dynkin diagram of type A, D or E. Let Q be the quiver obtained from the McKay graph by choosing any orientation of the edges, and let $\delta \in \mathbb{N}^I$ be the vector with $\delta_i = \dim\rho_i$. We consider the moment map $\mu_\delta : \mathrm{Rep}(\overline{Q},\delta) \to \mathrm{End}(\delta)_0$ for the action of $\mathrm{G}(\delta)$ on $\mathrm{Rep}(\overline{Q},\delta)$. Let $\mathfrak{h}$ be the hyperplane defined by $\{\tau\in\mathbb{C}^I \mid \delta\cdot\tau = 0\}$. Recall that if $\tau\in\mathfrak{h}$, then $\mu_\delta^{-1}(\tau)$ is identified with the space of representations of $\Pi^\tau(Q)$ of dimension vector δ. Hence, we see that $\mathrm{G}(\delta)$ acts naturally on $\mu_\delta^{-1}(\mathfrak{h})$. We denote by $\mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta)$ the affine quotient, and by $\varphi : \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta) \to \mathfrak{h}$ the map which is obtained from the universal property of the quotient. Then we have the following result, see [31, 24].

Theorem 2.3.1. *The map* $\varphi : \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta) \to \mathfrak{h}$ *is isomorphic in the category of complex spaces to a lift through the Weyl group of the semiuniversal deformation of* $\mathbb{C}^2/G$*.*

Taking fibers, it follows that the quotients $\mathrm{Rep}(\Pi^\tau(Q),\delta) /\!\!/ \mathrm{G}(\delta)$ with $\delta \cdot \tau = 0$ are isomorphic as complex spaces to the fibers of the semiuniversal deformation. In particular, one has that $\mathbb{C}^2/G \cong \mathrm{Rep}(\Pi(Q),\delta) /\!\!/ \mathrm{G}(\delta)$.

2.3.3 **Simultaneous resolutions of Kleinian singularities revisited.** We now turn our attention to studying the resolution of Kleinian singularities and the simultaneous resolution of their deformations in terms of the corresponding quiver. Our treatment follows Cassens and Slodowy [24].

Let us start by giving the basic definitions. Let $A = \mathbb{C}Q/(\rho)$ be a finite-dimensional $\mathbb{C}$-algebra where ρ is a set of relations for the quiver Q. Let I be the set of vertices of Q. As before, we denote by $\mathrm{Rep}(A,\alpha)$ the space of representations of A of dimension vector α. The group $\mathrm{G}(\alpha) = \mathrm{GL}(\alpha)/\mathbb{C}^\times$ acts by conjugation on $\mathrm{Rep}(A,\alpha)$ and its orbits correspond to isomorphism classes of representations of A of dimension vector α. Now, every character of $\mathrm{GL}(\alpha)$ is of the form

$$\chi_\theta(g) = \prod_{i\in I} \det(g_i)^{\theta_i}$$

for some $\theta \in \mathbb{Z}^I$. As the diagonal subgroup $\mathbb{C}^\times \subset \mathrm{GL}(\alpha)$ acts trivially on $\mathrm{Rep}(A,\alpha)$ we are only interested in the characters χ_θ such that $\theta\cdot\alpha = 0$. Such a vector θ can also be interpreted as a homomorphism $\mathbb{Z}^I \to \mathbb{Z}$ by putting $\theta(\beta) = \sum_{i\in I}\theta_i\beta_i$. We say that θ is *generic* with respect to α if $\theta(\beta) \neq 0$ for all $0 < \beta < \alpha$. Note that such a θ exists if and only if α is indivisible, meaning that the α_i have no common divisors.

To proceed further we need the notion of θ-stability introduced by A. King [54]. Let θ be a homomorphism $\mathbb{Z}^I \to \mathbb{Z}$. A representation V of A is said to be *θ-stable* if $\theta(\underline{\dim} V) = 0$, but $\theta(\underline{\dim} V') > 0$ for every proper subrepresentation $V' \subset V$. The notion of *θ-semistable* is the same with "$\geq$" replacing "$>$". We may note in passing that if θ is generic for $\underline{\dim} V$ the notions of θ-semistability and θ-stability coincide.

Using the Hilbert-Mumford criterion the notion of χ_θ-(semi)stability can be translated into the language of representations of the algebra, see [54, Proposition 3.1].

Proposition 2.3.2. *A point in* $\mathrm{Rep}(A,\alpha)$ *corresponding to a representation V of A is χ_θ-stable (respectively χ_θ-semistable) if and only if V is θ-stable (respectively θ-semistable).*

Denote by $\mathrm{Rep}(A,\alpha)_\theta^{ss}$ the subset of $\mathrm{Rep}(A,\alpha)$ consisting of the points x such that the corresponding representation V is θ-semistable. Then $\mathrm{Rep}(A,\alpha)_\theta^{ss}$ is open (maybe empty) in the Zariski topology of $\mathrm{Rep}(A,\alpha)$.

We next recall the notion of semi-invariant polynomial. A polynomial function $f \in \mathbb{C}[\mathrm{Rep}(A,\alpha)]$ is called a *θ-semi-invariant of weight k* if

$$f(g\cdot x) = \chi_\theta(g)^k f(x) \quad \text{for all } g \in \mathrm{G}(\alpha)$$

where χ_θ is the character of $\mathrm{G}(\alpha)$ corresponding to θ. The set of θ-semi-invariants of a given weight k is a vector subspace of $\mathbb{C}[\mathrm{Rep}(A,\alpha)]$, and we denote this space by

$\mathbb{C}[\mathrm{Rep}(A,\alpha)]^{\mathrm{G}(\alpha)}_{\chi_\theta^k}$. Obviously, θ-semi-invariants of weight zero are just polynomial invariants and a product of θ-semi-invariants of weights k, l is again a θ-semi-invariant of weight $k+l$. In particular, this means that the ring of all θ-semi-invariants

$$\mathbb{C}[\mathrm{Rep}(A,\alpha)]^{\mathrm{G}(\alpha)}_{\chi_\theta} = \bigoplus_{k\geq 0} \mathbb{C}[\mathrm{Rep}(A,\alpha)]^{\mathrm{G}(\alpha)}_{\chi_\theta^k}$$

is a graded algebra with part of degree zero $\mathbb{C}[\mathrm{Rep}(A,\alpha)]^{\mathrm{G}(\alpha)}$. Hence, the corresponding invariant quotient of $\mathrm{Rep}(A,\alpha)$ by $\mathrm{G}(\alpha)$, which we shall denote by $\mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha)$ can be described as

$$\mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha) = \mathrm{Proj}\left(\mathbb{C}[\mathrm{Rep}(A,\alpha)]^{\mathrm{G}(\alpha)}_{\chi_\theta}\right)$$

which is projective over the affine quotient $\mathrm{Rep}(A,\alpha) /\!\!/ \mathrm{G}(\alpha)$. The variety $\mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha)$ gives a 'good quotient' of $\mathrm{Rep}(A,\alpha)^{ss}_\theta$; namely, there is a surjective affine $\mathrm{G}(\alpha)$-invariant morphism $\pi : \mathrm{Rep}(A,\alpha)^{ss}_\theta \to \mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha)$ such that the induced map $\pi^* : \mathbb{C}[U] \to \mathbb{C}[\pi^{-1}(U)]^{\mathrm{G}(\alpha)}$ is an isomorphism for any open subset $U \subset \mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha)$. From this it follows that $\mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha)$ can be identified with the affine quotient variety $\mathrm{Rep}(A,\alpha)^{ss}_\theta /\!\!/ \mathrm{G}(\alpha)$.

There is a projective morphism

$$\pi_\theta : \mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha) \longrightarrow \mathrm{Rep}(A,\alpha) /\!\!/ \mathrm{G}(\alpha)$$

such that all fibers of π_θ are projective varieties. We are now ready to quote the following fundamental result, compare [54, Proposition 3.2].

Proposition 2.3.3. *The closed orbits of* $\mathrm{G}(\alpha)$ *on* $\mathrm{Rep}(A,\alpha)^{\mathrm{ss}}_\theta$ *correspond to isomorphism classes of direct sums of* θ*-stable representations of* A *of dimension* α.

We shall refer to the variety $\mathrm{Rep}(A,\alpha) /\!\!/_{\chi_\theta} \mathrm{G}(\alpha)$ as the 'moduli space of θ-semistable representations of A of dimension α' and we will denote it by $\mathcal{M}_\theta(A,\alpha)$. (The justification for using the term 'moduli space' is given in [54, §5]; see also Sect. 2.5.2 of this chapter.) If θ is generic then all θ-semistable representations of dimension α are θ-stable, so the points of the moduli space correspond to isomorphism classes of θ-stable representations.

We now apply this to Kleinian singularities. Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Let Q be an orientation of the McKay graph. We consider the preprojective algebra $\Pi(Q)$. Recall that the flat family $\varphi : \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta) \to \mathfrak{h}$ realizes the semiuniversal deformation of $\mathbb{C}^2/G$, or rather its lift through a Weyl group action. Choose $\theta : \mathbb{Z}^I \to \mathbb{Z}$ with $\theta(\delta) = 0$. Now, there is a projective morphism

$$\pi_\theta : \mu_\delta^{-1}(\mathfrak{h}) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) \longrightarrow \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta).$$

We denote by φ_θ the composition of π_θ and φ so that we have the following commutative diagram:

$$\begin{array}{ccc} \mu_\delta^{-1}(\mathfrak{h}) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) & \xrightarrow{\quad \pi_\theta \quad} & \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta) \\ & {\scriptstyle \varphi_\theta} \searrow \quad \swarrow {\scriptstyle \varphi} & \\ & \mathfrak{h} & \end{array}$$

We obtain the following statement, see [24].

Theorem 2.3.4. *If θ is generic, then the above diagram is a simultaneous resolution of φ.*

In the special case $\tau = 0$, then $\mu_\delta^{-1}(0) /\!\!/_{\chi_\theta} \mathrm{G}(\delta)$ is identified with the moduli space $\mathcal{M}_\theta(\Pi(Q), \delta)$ of θ-semistable representations of $\Pi(Q)$ of dimension δ. It follows immediately that the projective morphism

$$\mathcal{M}_\theta(\Pi(Q), \delta) \longrightarrow \mathrm{Rep}(\Pi(Q), \delta) /\!\!/ \mathrm{G}(\delta) \cong \mathbb{C}^2/G$$

is the minimal resolution of the Kleinian singularity.

2.4 ADE fibered Calabi-Yau threefolds and their small resolutions revisited

This section studies the reconstruction of ADE fibered Calabi-Yau threefolds and their small resolutions in terms of a noncommutative algebra, which we call the "$N = 1$ ADE quiver algebra". We begin with some motivating ideas from physics.

2.4.1 **Physical and mathematical context.** We would like to explain the motivation and background that have led to the results that are described in this chapter. The work arose as a result of an attempt to understand the reverse geometric engineering of ADE fibered Calabi-Yau threefolds and their small resolutions.

We begin the discussion by describing the geometric engineering of gauge theories. To this end we need to make a digression and discuss D-branes. Recall that ordinary superstrings, known as Type II strings, are described by maps from a Riemann surface Σ, the "worldsheet" as it is called, to a 10-dimensional "spacetime" manifold M through which the string propagates. M is also referred to as the "target space". The physical definition of a D-brane is " a submanifold of M on which open strings can end". This means that if a D-brane is present, then one needs to consider maps from a Riemann surface with boundaries to M such that the boundaries are mapped to a certain submanifold $S \subset M$, usually referred to as the "worldvolume". In this case one says that there is a D-brane wrapped on S. If S is connected and has dimension $p+1$, then one says that one is dealing with a Dp-brane. In general, S can have several components with different dimensions, and then each component corresponds to a D-brane.

In the context of conformal field theory, a D-brane may be defined to be a boundary condition preserving conformal invariance. Conformal field theories on Riemann surfaces with boundaries can be described axiomatically as a functor from a geometric category to an algebraic category. Simple considerations of gluing show that the boundary conditions should be regarded as objects in an additive (in fact $\mathbb{C}$-linear) category.

Another important property of D-branes is the fact that the dynamics on its worldvolume is a gauge theory. More concretely, the data specifying the boundary conditions for the conformal field theory on Σ include a choice of a unitary vector bundle E on S and a connection on it. In the simplest case this bundle has rank 1, but one can also have "multiple" D-branes, described by bundles of rank $r > 1$. Such bundles describe r coincident D-branes wrapped on the same submanifold S.

The idea behind geometric engineering is to look at the gauge theories that arise on D-branes at singularities. To be more concrete, let us consider Type II string theory on the 10-dimensional space $M = \mathbb{R}^{1,3} \times X$ where $\mathbb{R}^{1,3}$ is 4-dimensional Minkowski space and

X is a Calabi-Yau threefold. We will assume that X has an isolated singularity at $P \in X$ and that there are r coincident D3-branes wrapped on $\mathbb{R}^{1,3} \times \{P\}$. We want to determine the gauge theory on such set of D3-branes. Much of this problem can be analyzed in the context of purely topological B-branes, where the field theory on a set of D3-branes is described in terms of a noncommutative algebra A. The idea is that the effective theory on a set of D3-branes is an $N = 1$ supersymmetric gauge theory on $\mathbb{R}^{1,3}$ whose matter content can be conveniently encoded in a quiver with relations coming from a 'superpotential'. This will therefore be what is known as an $N = 1$ quiver gauge theory.

Let us be more precise about this. If Q is a quiver and $\mathbb{C}Q$ is the corresponding path algebra, then a *superpotential* is a formal sum of oriented cycles on the quiver, i.e. an element of the vector space $\mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$. On this space we can define for every arrow a a 'derivation' ∂_a that takes any occurrences of the arrow in an oriented cycle and removes them leading to a path from the head of a to its tail. An $N = 1$ *quiver gauge theory* consists of a quiver Q together with a choice of a superpotential W and a dimension vector $\alpha \in \mathbb{N}^{Q_0}$. Given an $N = 1$ quiver gauge theory, one can construct the $N = 1$ *quiver algebra*

$$A = \mathbb{C}Q/(\partial_a W \mid a \in Q_1).$$

The affine variety $\operatorname{Rep}(A, \alpha)$ is said to be the space of 'vacua' and the affine quotient $\operatorname{Rep}(A, \alpha) /\!\!/ \mathrm{G}(\alpha)$ is called the 'moduli space of classical vacua'. This algebraic construction also has a noncommutative geometric interpretation in which A is viewed as the noncommutative coordinate ring of the 'critical locus' of W.

Back in the general situation, one can expect that all the information of the Calabi-Yau singularity X can be encoded in the $N = 1$ quiver algebra A and that the moduli space of classical vacua will recover the geometry of X. From this perspective, the field theory on the D-brane is the primary concept, whereas the spacetime itself is a secondary, derived concept.

The above discussion suggest that it would be possible to reconstruct singular Calabi-Yau threefolds from $N = 1$ quiver gauge theories. Following the terminology in [8], we call this process reverse geometric engineering of singularities. The basic idea of this construction may be summarized as follows: under some particular conditions (namely that the $N = 1$ quiver algebra A is finitely generated as a module over its centre $Z(A)$), the centre $Z(A)$ is the coordinate ring of a three-dimensional algebraic variety, to be identified as the singularity X. One then needs to show that the moduli space of classical vacua can be identified with $\operatorname{Spec} Z(A)$. The physical interpretation now is that X will correspond to the "spacetime" in which closed strings propagate, while A is associated to a noncommutative geometry that D-branes see. This is in the spirit of [79] where there are two geometries at play, one for closed strings and another for open strings.

A fundamental observation made in [9] is that the $N = 1$ quiver algebra A also encodes the process of resolving the singular Calabi-Yau threefold X. For us, this means that string theory can be used to resolve singularities in algebraic geometry. Indeed, a resolution of the singularity X is obtained via the moduli space of certain stable representations of the associated $N = 1$ quiver algebra A. The physical content of this result is as follows. We mentioned earlier that it is convenient to think of a D-brane as an object of an additive category. In the case of IIB theory, which lives on a *crepant* resolution Y of the singularity X, the D-branes are objects of the bounded derived category $\mathbf{D}^b(\mathrm{Coh}(Y))$ of coherent sheaves on Y. What is happening here is that topological B-branes on the resolution Y are described by representations of the $N=1$ quiver algebra A. More precisely, there is an equivalence of triangulated categories

$$\mathbf{D}^b(\mathrm{Coh}(Y)) \cong \mathbf{D}^b(\mathrm{mod}\text{–}A),$$

where $\mathbf{D}^b(\mathrm{mod}\text{–}A)$ is the bounded derived category of finitely generated right modules over A. In M. Van den Berg's terminology, A is a "noncommutative crepant resolution".

The situation that we actually wish to apply this to is the case of ADE fibered Calabi-Yau threefolds and their small resolutions. The relevant $N=1$ quiver gauge theory was written down in [21] (see also [20, 49, 96]). We will explicitly carry out the previous construction in the subsequent subsections.

2.4.2 **ADE fibered Calabi-Yau threefolds revisited.** In this paragraph we show how to construct the ADE fibered Calabi-Yau threefold in terms of the associated quiver. We start by summarising some of the neccessary definitions.

Let Q be an extended Dynkin quiver with vertex set I and let $\overline{Q}$ be the double of Q. Let $\widehat{Q}$ be the quiver obtained from $\overline{Q}$ by attaching an additional edge-loop u_i for each vertex $i \in I$. We write $\mathbb{C}\overline{Q}$ and $\mathbb{C}\widehat{Q}$ for the path algebras of $\overline{Q}$ and $\widehat{Q}$. Let $B = \bigoplus_{i\in I} \mathbb{C}e_i$ be the semisimple commutative subalgebra of $\mathbb{C}\overline{Q}$ spanned by the trivial paths and consider the algebra $B[u]$ of polynomials of a central variable u with coefficients in B. For an element $\tau \in B[u]$, we will write $\tau(u) = \sum_i \tau_i(u)e_i$ where $\tau_i \in \mathbb{C}[u]$. Let $\mathbb{C}\overline{Q} *_B B[u]$ denote the free product of $\mathbb{C}\overline{Q}$ with $B[u]$ over B. We have an isomorphism

$$\mathbb{C}\overline{Q} *_B B[u] \xrightarrow{\sim} \mathbb{C}\widehat{Q} : u \longmapsto \textstyle\sum_i u_i .$$

This isomorphism sends the element $e_i u e_i$ to u_i, the additional edge-loop at the vertex i. We also have an isomorphism

$$B[u] = \Big(\textstyle\bigoplus_{i\in I} \mathbb{C}e_i\Big) \otimes \mathbb{C}[u] \xrightarrow{\sim} \textstyle\bigoplus_{i\in I} \mathbb{C}[u_i] : e_i \otimes u \longmapsto u_i .$$

Therefore, choosing an element $\tau \in B[u]$ amounts to choosing a collection of polynomials $\{\tau_i \in \mathbb{C}[u_i] \mid i \in I\}$.

If $\tau \in B[u]$ then the $N=1$ *ADE quiver algebra* determined by Q is defined by

$$\mathfrak{A}^\tau(Q) = \mathbb{C}\widehat{Q} \Big/ \left(\sum_{a\in Q} [a, a^*] - \sum_{i\in I} \tau_i(u) e_i \right).$$

Compare this with the definition of [37, §4.3]. We point out that the defining relations for $\mathfrak{A}^\tau(Q)$ are generated by the superpotential

$$W = u \sum_{a\in Q} [a, a^*] - \sum_{i\in I} w_i(u) e_i ,$$

where each $w_i \in \mathbb{C}[u]$ satisfies $w_i'(u) = \tau_i(u)$. We also note that if $\tau(u)$ is identified with the element $\sum_i \tau_i(u_i)$ then $\mathfrak{A}^\tau(Q)$ is the same as the quotient of $\mathbb{C}\widehat{Q}$ by the relations

$$\sum_{\substack{a\in Q\\ h(a)=i}} aa^* - \sum_{\substack{a\in Q\\ t(a)=i}} a^*a - \tau_i(u_i) = 0, \quad au_i = u_j a,$$

for each vertex i, and for each arrow $a : i \to j$ in $\overline{Q}$. This is helpful when considering representations of $\mathfrak{A}^\tau(Q)$, as they can be identified with representations V of $\widehat{Q}$ which satisfy

$$\sum_{\substack{a\in Q\\ h(a)=i}} V_a V_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} V_{a^*} V_a - \tau_i(V_{u_i}) = 0, \quad V_a V_{u_i} = V_{u_j} V_a, \tag{$*$}$$

for each vertex i, and for each arrow $a : i \to j$ in $\overline{Q}$.

The following is immediate from what we have just seen.

Lemma 2.4.1. *Let V be a representation of $\mathfrak{A}^\tau(Q)$, and let v_i be an eigenvector of V_{u_i} with eigenvalue λ. If $a : i \to j$ is any arrow in $\overline{Q}$, then $V_a v_i$ is either an eigenvector of V_{u_j} with eigenvalue λ or the zero vector.*

Proof. If v_i is an eigenvector of V_{u_i} corresponding to the eigenvalue λ, then we have

$$V_{u_j} V_a v_i = V_a V_{u_i} v_i = \lambda V_a v_i,$$

by virtue of $(*)$. The assertion follows. □

We also want to make the following observation.

Lemma 2.4.2. *Let V be a simple representation of $\mathfrak{A}^\tau(Q)$. Then there exists a λ with the property that $V_{u_i} v_i = \lambda v_i$ whenever $0 \neq v_i \in V_i$.*

Proof. Set $J = \{i \in I \mid V_i \neq 0\}$. Then J is connected, since otherwise V is not simple. Let $l = \min J$, and let v_l be an eigenvector of V_{u_l} with eigenvalue λ. For every $i \in J$ let us denote by U_i the λ-eigenspace of V_{u_i}. Applying Lemma 2.4.1, we conclude that $U = \{U_i \mid i \in J\}$ is a subrepresentation of V. Since V is simple it follows that $U = V$, which proves the result. □

If $\alpha \in \mathbb{N}^I$, then representations of $\widehat{Q}$ of dimension vector α are given by elements of the variety

$$\operatorname{Rep}(\widehat{Q},\alpha) = \operatorname{Rep}(\overline{Q},\alpha) \oplus \left(\bigoplus_{i\in I} \operatorname{End}_{\mathbb{C}}(\mathbb{C}^{\alpha_i})\right).$$

We denote by $\operatorname{Rep}(\mathfrak{A}^\tau(Q),\alpha)$ the closed subspace of $\operatorname{Rep}(\widehat{Q},\alpha)$ corresponding to representations for $\mathfrak{A}^\tau(Q)$. The group $\mathrm{G}(\alpha) = \mathrm{GL}(\alpha)/\mathbb{C}^\times$ acts on both these spaces, and the orbits correspond to isomorphism classes.

We have the following easily verified result.

Lemma 2.4.3. *If $x \in \operatorname{Rep}(\mathfrak{A}^\tau(Q),\alpha)$ and V is the corresponding representation, then* $\sum_i \operatorname{tr} \tau_i(V_{u_i}) = 0$.

Proof. Given $a \in \overline{Q}$, we have $\operatorname{tr}(V_a V_{a^*}) = \operatorname{tr}(V_a V_{a^*})$. Taking traces to relations $(*)$ and summing over all vertices $i \in I$, one obtains $\sum_i \operatorname{tr} \tau_i(V_{u_i}) = 0$, as required. □

We define $\mathcal{R}_Q(\tau,\alpha)$ to be the subset of $\operatorname{Rep}(\mathfrak{A}^\tau(Q),\alpha)$ consisting of the representation for which there exists a λ with the property that $V_{u_i} v_i = \lambda v_i$ whenever $v_i \in V_i$. It is clear that this is a locally closed subset of $\operatorname{Rep}(\mathfrak{A}^\tau(Q),\alpha)$, so a variety. In view of Lemma 2.4.2, we have that $\mathcal{R}_Q(\tau,\alpha)$ contains the open subset $\operatorname{Rep}(\mathfrak{A}^\tau(Q),\alpha)_s$ consisting of simple representations of $\mathfrak{A}^\tau(Q)$.

The next result is an easy consequence of Lemma 2.4.3.

Corollary 2.4.4. *If* $x \in \mathcal{R}_Q(\tau,\alpha)$ *then* $\sum_i \alpha_i \tau_i(\lambda) = 0$.

We now begin our project of describing ADE fibered Calabi-Yau threefolds in terms of the associated quiver. As we have already remarked, the $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$ is supposed to encode all information about such singular Calabi-Yau threefolds. Our aim is to show that this is indeed the case. First, however, it will be convenient to provide the following piece of information. Every extended Dynkin quiver Q arises by orienting the McKay graph of some finite subgroup G of $\mathrm{SL}(2,\mathbb{C})$. We denote by $\rho_0,\dots,\rho_n$ the irreducible representations of G with ρ_0 trivial, and set $I = \{0,\dots,n\}$. Let $\delta \in \mathbb{N}^I$ be the vector with $\delta_i = \dim \rho_i$. We keep the notation employed in Sect. 2.3.2, so μ_δ is the map $\mathrm{Rep}(\overline{Q},\delta) \to \mathrm{End}(\delta)_0$ and $\mathfrak{h}$ is the hyperplane $\{\tau \in \mathbb{C}^I \mid \delta\cdot\tau = 0\}$. By Corollary 2.4.4, the dimension vector δ satisfies $\sum_i \delta_i \tau_i(\lambda) = 0$. Furthermore, for any vertex i we have

$$\mu_\delta(x)_i = \sum_{\substack{a\in Q\\ h(a)=i}} x_a x_{a^*} - \sum_{\substack{a\in Q\\ t(a)=i}} x_{a^*} x_a = \tau_i(\lambda).$$

Therefore one can identify $\mathcal{R}_Q(\tau,\delta)$ with the fiber product

$$\begin{array}{ccc} \mathcal{R}_Q(\tau,\delta) & \longrightarrow & \mu_\delta^{-1}(\mathfrak{h}) \\ \big\downarrow & & \big\downarrow{\scriptstyle \mu_\delta} \\ \mathbb{C} & \xrightarrow{\;\;\widehat{\tau}\;\;} & \mathfrak{h} \end{array}$$

where $\widehat{\tau} : \mathbb{C} \to \mathfrak{h}$ is the map corresponding to τ. Observe that $\mathrm{G}(\delta)$ acts naturally on $\mathcal{R}_Q(\tau,\delta)$ in such a way that all maps in the fiber product are equivariant (where the action on $\mathbb{C}$ is trivial). Now $\mathcal{R}_Q(\tau,\delta) \to \mathbb{C}$ is flat since it is the pullback of μ_δ, which is flat by [31, Lemma 8.3]. From this it follows that the map $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta) \to \mathbb{C}$ is also flat and surjective.

Incidentally, if $\tau \in \mathfrak{h}$, then $\mu_\delta^{-1}(\tau)$ is irreducible by [30, Lemma 6.3], which implies that every fiber of the map $\mathcal{R}_Q(\tau,\delta) \to \mathbb{C}$ is irreducible. It follows from [30, Lemma 6.1] that $\mathcal{R}_Q(\tau,\delta)$ is irreducible, so that the set $\mathrm{Rep}(\mathfrak{A}^\tau(Q),\delta)_s$ of simple representations is either empty or dense.

We have now accumulated all the information neccessary to prove the following result.

Theorem 2.4.5. *Assume that* $\widehat{\tau}$ *is sufficiently general. The affine quotient* $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta)$ *is isomorphic to the ADE fibered Calabi-Yau threefold associated with* Q *and* $\widehat{\tau}$.

Proof. Let $\mathbb{C}[\mu_\delta^{-1}(\mathfrak{h})]$ and $\mathbb{C}[\mathfrak{h}]$ be the coordinate rings of $\mu_\delta^{-1}(\mathfrak{h})$ and $\mathfrak{h}$, respectively. Then the coordinate ring of $\mathcal{R}_Q(\tau,\delta)$ is given by

$$\mathbb{C}[\mathcal{R}_Q(\tau,\delta)] = \mathbb{C}[\mu_\delta^{-1}(\mathfrak{h})] \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[u].$$

Since $\mathrm{G}(\delta)$ is linear reductive and acts trivially on $\mathbb{C}[u]$, we see that

$$\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)} \cong \mathbb{C}[\mu_\delta^{-1}(\mathfrak{h})]^{\mathrm{G}(\delta)} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[u].$$

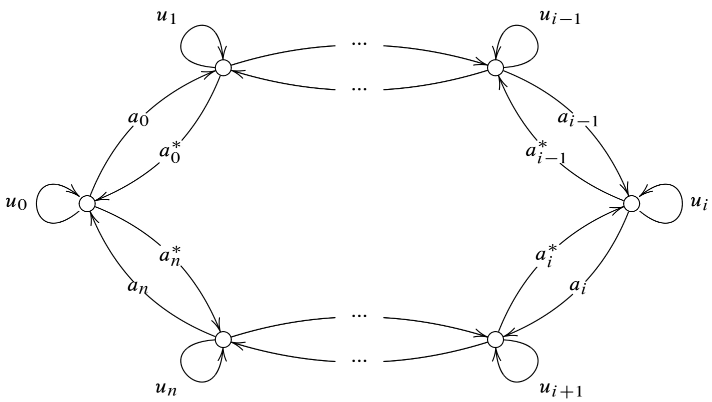
Accordingly, we have $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta) \cong \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta) \times_{\mathfrak{h}} \mathbb{C}$. Hence we obtain the affine quotient $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta)$ as the fiber product

$$\begin{array}{ccc} \mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta) & \longrightarrow & \mu_\delta^{-1}(\mathfrak{h}) /\!\!/ \mathrm{G}(\delta) \\ \downarrow & & \downarrow \varphi \\ \mathbb{C} & \xrightarrow{\quad \widehat{\tau} \quad} & \mathfrak{h} \end{array}$$

The desired assertion is now a consequence of Theorem 2.3.1. □

We illustrate with the following concrete example.

Example 2.4.6. Suppose that Q is of type $\widetilde{A}_n$, so that $\delta_i = 1$ for all vertices i. The arrows a_i in Q connect vertices i and $i+1$ (identifying $n+1$ with zero). Thus $\widehat{Q}$ has shape



As above let $\tau \in B[u]$; recall that it is specified by a set of polynomials $\{\tau_i \in \mathbb{C}[u_i] \mid 0 \le i \le n\}$. Because $\delta = (1,\dots,1)$, a representation of $\widehat{Q}$ of dimension δ involves placing a one-dimensional vector space at each vertex i and assigning a complex number to each arrow a_i, a_i^*, u_i. Hence, we may identify $\mathrm{Rep}(\widehat{Q},\delta)$ with the space $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}$ so that

$$\begin{aligned} \mathrm{Rep}(\mathfrak{A}^\tau(Q),\delta) &\cong \{(x_i, y_i, \lambda_i) \mid -x_i y_i + x_{i+1} y_{i+1} = \tau_i(\lambda_i), 0 \le i \le n\} \\ &\subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1}. \end{aligned}$$

Also one can identify

$$\begin{aligned} \mathcal{R}_Q(\tau,\delta) &\cong \{(x_i, y_i, \lambda) \mid -x_i y_i + x_{i+1} y_{i+1} = \tau_i(\lambda), 0 \le i \le n\} \\ &\subset \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}. \end{aligned}$$

The relations for $\mathcal{R}_Q(\tau,\delta)$ lead to the condition that $\sum_{i=0}^n \tau_i(\lambda) = 0$. From this it follows that the map $\widehat{\tau} = (\tau_0,\dots,\tau_n) : \mathbb{C} \to \mathbb{C}^{n+1}$ corresponding to τ has its image in $\mathfrak{h}$. Without loss of generality it is possible to suppose that $\tau_i = t_i - t_{i+1}$ for some polynomial map $f = (t_0,\dots,t_n) : \mathbb{C} \to \mathfrak{h}$.

Now the action of $\mathrm{G}(\delta) = (\mathbb{C}^\times)^{n+1}/\mathbb{C}^\times$ on $\mathcal{R}_Q(\tau,\delta)$ is by

$$(x_i, y_i, \lambda) \longmapsto (g_{i+1} g_i^{-1} x_i, g_i g_{i+1}^{-1} y_i, \lambda)$$

for $(g_i) \in \mathrm{G}(\delta)$ and $(x_i, y_i, \lambda) \in \mathcal{R}_Q(\tau,\delta)$. It is easily seen that the ring of invariants $\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}$ is generated by

$$\begin{aligned} x &= x_0 \cdots x_n, \\ y &= y_0 \cdots y_n, \\ z_i &= x_i y_i, \quad 0 \le i \le n. \end{aligned}$$

These invariants satisfy the relation

$$xy = z_0 \cdots z_n.$$

On the other hand, the relations for $\mathcal{R}_Q(\tau,\delta)$ imply that

$$z_i = z_n - \sum_{j=0}^{i} \tau_j(\lambda), \quad 0 \le i \le n.$$

Bearing in mind that $\sum_{j=0}^{i} \tau_j(\lambda) = t_0(\lambda) - t_{i+1}(\lambda)$, we derive

$$\sum_{i=0}^{n} z_i = (n+1)(z_n - t_0(\lambda)).$$

Setting $z = \frac{1}{n+1}\sum_{i=0}^{n} z_i$, we therefore deduce that

$$z_i = z + t_{i+1}(\lambda), \quad 0 \le i \le n.$$

The conclusion is that the affine quotient variety $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta)$ is given as the hypersurface

$$\left\{ (x,y,z,\lambda) \,\middle|\, xy = \prod_{i=0}^{n} (z + t_{i+1}(\lambda)) \right\} \subset \mathbb{C}^4$$

which is the total space of the family describing the A_n fibration over the λ-plane.

2.4.3 **Small resolutions of ADE fibered Calabi-Yau threefolds revisited.** We continue with the same hypothesis and notation as in the previous subsection. Our main aim in this paragraph is to show how the small resolution of an ADE fibered Calabi-Yau threefold can be derived from a moduli space of representations of the $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$. We begin with some generalities.

Let S and T be two graded A-algebras. We suppose that S is generated by S_1 as an S_0-algebra and that T is generated by T_1 as an T_0-algebra. We define the *Cartesian product* $S \times_A T$ to be the graded ring $\bigoplus_{d\ge 0} S_d \otimes_A T_d$. If we denote $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$ then $\operatorname{Proj}(S \times_A T) \cong X \times_{\operatorname{Spec} A} Y$. To see this note that for any homogeneous decomposable element $s \otimes t$ in $S \times_A T$ there is an isomorphism $(S \times_A T)_{(s\otimes t)} \cong S_{(s)} \otimes_A T_{(t)}$. With this observation in mind, we prove the following.

Lemma 2.4.7. *Let S be a graded A-algebra which is generated by S_1 as an S_0-algebra, and let R be an A-algebra. Then*

$$\operatorname{Proj} S \times_{\operatorname{Spec} A} \operatorname{Spec} R \cong \operatorname{Proj}(S \otimes_A R).$$

Proof. The polynomial ring in one variable $R[t]$ is a graded ring by assigning $\deg R = 0$, $\deg t = 1$. In this case the structure morphism

$$\operatorname{Proj} R[t] \xrightarrow{\sim} \operatorname{Spec} R$$

is an isomorphism. From our previous remark it then follows that

$$\operatorname{Proj} S \times_{\operatorname{Spec} A} \operatorname{Spec} R \cong \operatorname{Proj}(S \times_A R[t]).$$

Now it is easily checked that $S \times_A R[t] \cong S \otimes_A R$. This finishes the proof. □

Let us now proceed with the construction of the small resolution. We keep the notations introduced in Sect. 2.3.3. Let $\theta : \mathbb{Z}^I \to \mathbb{Z}$ satisfy $\theta(\delta) = 0$. Recall that the moduli space $\mathcal{M}_\theta(\mathfrak{A}^\tau(Q), \delta)$ corresponds to the graded ring of semi-invariant functions with character χ_θ. As a closed subset of this, there is a moduli space $\mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta)$ of θ-semistable representations of $\mathfrak{A}^\tau(Q)$ for which there exists a λ with the property that $V_{u_i} v_i = \lambda v_i$ whenever $v_i \in V_i$. Also, as before, there is a projective morphism

$$\pi_\theta : \mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) \longrightarrow \mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta).$$

We have seen earlier that $\mathcal{R}_Q(\tau,\delta)$ is irreducible, and the general element is a simple representation of $\mathfrak{A}^\tau(Q)$, hence θ-stable. Then the morphism π_θ is a birational map of irreducible varieties.

We are at last in a position to attain our main objective, which is to prove the following result.

Theorem 2.4.8. *Assume that $\widehat{\tau}$ is sufficiently general. If θ is generic, then π_θ is a small resolution of the ADE fibered Calabi-Yau threefold associated with Q and $\widehat{\tau}$.*

Proof. We keep the notation employed in the proof of Theorem 2.4.5. To begin with, we observe that there is an isomorphism

$$\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}_{\chi_\theta^n} \cong \mathbb{C}[\mu_\delta^{-1}(\mathfrak{h})]^{\mathrm{G}(\delta)}_{\chi_\theta^n} \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[u].$$

Invoking Lemma 2.4.7, it follows that

$$\operatorname{Proj}\left(\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}_{\chi_\theta}\right) \cong \operatorname{Proj}\left(\mathbb{C}[\mu_\delta^{-1}(\mathfrak{h})]^{\mathrm{G}(\delta)}_{\chi_\theta}\right) \times_{\mathfrak{h}} \mathbb{C},$$

which entails $\mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) \cong \mu_\delta^{-1}(\mathfrak{h}) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) \times_{\mathfrak{h}} \mathbb{C}$. Therefore we obtain the moduli space $\mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta)$ as the fiber product

$$\begin{array}{ccc} \mathcal{R}_Q(\tau,\delta) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) & \longrightarrow & \mu_\delta^{-1}(\mathfrak{h}) /\!\!/_{\chi_\theta} \mathrm{G}(\delta) \\ \big\downarrow & & \big\downarrow {\scriptstyle \varphi_\theta} \\ \mathbb{C} & \xrightarrow{\quad \widehat{\tau} \quad} & \mathfrak{h} \end{array}$$

The required result now follows by virtue of Theorem 2.3.4. □

We end this section with the following illustration of Theorem 2.4.8.

Example 2.4.9. Assume that Q is a quiver of Dynkin type $\widetilde{A}_n$. We use the notations introduced in Example 2.4.6. Consider the generic stability parameter $\theta = (-n, 1, \dots, 1)$. By definition, the ring of θ-semi-invariants is spanned by the monomials $\prod_{i=0}^{n} x_i^{\alpha_i} y_i^{\beta_i}$ satisfying $-\alpha_0 + \alpha_n + \beta_0 - \beta_n = -n$ and $-\alpha_i + \alpha_{i-1} + \beta_i - \beta_{i-1} = 1$ for $1 \le i \le n$. Given $j = 0, \dots, n-1$, put

$$u_j = x_0 \cdots x_j,$$
$$v_j = y_{j+1} \cdots y_n.$$

Then we have the following relations

$$xv_j = u_j z_{j+1} \cdots z_n, \quad 0 \le j \le n-1$$
$$yu_j = v_j z_0 \cdots z_j, \quad 0 \le j \le n-1$$
$$u_j v_k = u_k v_j z_{k+1} \cdots z_j, \quad 0 \le k < j \le n-1$$

or, using the fact that $z_i = z + t_{i+1}(\lambda)$ for $0 \le i \le n$,

$$xv_j = u_j \prod_{i=j+1}^{n} (z + t_{i+1}(\lambda)), \quad 0 \le j \le n-1$$
$$yu_j = v_j \prod_{i=0}^{j} (z + t_{i+1}(\lambda)), \quad 0 \le j \le n-1$$
$$u_j v_k = u_k v_j \prod_{i=k+1}^{j} (z + t_{i+1}(\lambda)), \quad 0 \le k < j \le n-1$$

Analyzing possibilities for α_i, β_i $(0 \le i \le n)$ it is easily seen that the ring of θ-semi-invariants is generated as a polynomial ring by

$$u_I v_{I'} = u_{i_1} \dots u_{i_p} v_{i'_1} \dots v_{i'_q},$$

where $I = (i_1, \dots, i_p)$ is a multi-index of $\{0, \dots, n-1\}$ and $I' = (i'_1, \dots, i'_q)$ denotes the complementary index. It is also not difficult to see that this space is the module over $\mathbb{C}[\mathcal{R}_Q(\tau, \delta)]^{\mathrm{G}(\delta)}$ generated by

$$f_0 = v_0 \cdots v_{n-2} v_{n-1},$$
$$f_1 = v_0 \cdots v_{n-2} u_{n-1},$$
$$\cdots$$
$$f_n = u_0 \cdots u_{n-2} u_{n-1}.$$

Further we have

$$\mathbb{C}[u_I v_{I'} \mid I = (i_1, \dots, i_p), I' = (i'_1, \dots, i'_q)]$$
$$\cong \mathbb{C}[u_0, v_0] \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} \mathbb{C}[u_{n-1}, v_{n-1}],$$

so that

$$\begin{aligned}&\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}[f_0,\dots,f_n]\\&\quad\cong \mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}[u_0,v_0]\times_{\mathbb{C}}\cdots\times_{\mathbb{C}}\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}[u_{n-1},v_{n-1}].\end{aligned}$$

Hence the Proj quotient $\mathcal{R}_Q(\tau,\delta)/\!\!/_{\chi_\theta}\mathrm{G}(\delta)$ can be identified with a closed subvariety of $\mathbb{C}^4\times(\mathbb{P}^1)^n$ with $(u_j:v_j)$ the homogeneous coordinates on the jth $\mathbb{P}^1$.

Now let $U_0,U_1,\dots,U_n$ be the open subsets of $\mathbb{C}^4\times(\mathbb{P}^1)^n$ defined by

$$\begin{aligned}U_0&=\{v_0\neq 0\},\\U_k&=\{u_{k-1}\neq 0, v_k\neq 0\},\quad 1\le k\le n-1\\U_n&=\{u_{n-1}\neq 0\},\end{aligned}$$

and on U_k, let

$$\xi_k=v_{k-1}/u_{k-1},\qquad \eta_k=u_k/v_k.$$

Direct computations show that $\mathcal{R}_Q(\tau,\delta)/\!\!/_{\chi_\theta}\mathrm{G}(\delta)\cap U_k$ is defined by equations

$$\begin{aligned}(u_j:v_j)&=\left(1:\xi_k\prod_{i=j+1}^{k-1}(z+t_{i+1}(\lambda))\right),\quad \text{for } j<k-1\\(u_j:v_j)&=\left(\eta_k\prod_{i=k+1}^{j}(z+t_{i+1}(\lambda)):1\right),\quad \text{for } j>k.\end{aligned}$$

and

$$\begin{aligned}x&=\eta_k\prod_{i=k+1}^{n}(z+t_{i+1}(\lambda)),\\y&=\xi_k\prod_{i=0}^{k-1}(z+t_{i+1}(\lambda)),\\z&=\xi_k\eta_k-t_{k+1}(\lambda).\end{aligned}$$

It then follows from Example 2.2.3 that the moduli space $\mathcal{R}_Q(\tau,\delta)/\!\!/_{\chi_\theta}\mathrm{G}(\delta)$ is isomorphic to the small resolution of the threefold for a A_n fibration over the λ-plane.

2.5 Derived equivalence

In this section, it is shown how to describe the derived category of the small resolution of an ADE fibered Calabi-Yau threefold in terms of the associated $N=1$ ADE quiver algebra, in the spirit of noncommutative crepant resolutions of M. Van den Bergh. Assertions of this sort have already been considered in [86]. Our work is mostly based on the ideas and constructions of [31] and [91].

2.5.1 **The algebra A^τ.** As explained in the previous section, the $N = 1$ ADE quiver algebra encodes the process of resolving ADE fibered Calabi-Yau threefolds. This algebra should be treated as an algebra that can be naturally associated to the dual graph of the small resolution. Thus the $N = 1$ ADE quiver algebra is defined with prior knowledge of the small resolution. Here we introduce a noncommutative algebra which also dictates the process of resolving ADE fibered Calabi-Yau threefolds and can be defined without prior knowledge of the small resolution.

We begin by fixing some notation. Let G be a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. Let V be the natural 2-dimensional representation of G equipped with a nondegenerate symplectic form ω, and TV its tensor algebra. For our convenience, we denote by $R = \mathbb{C}[u]$ the ring of polynomials in a dummy variable u. We write $T_R V$ for the induced R-algebra $TV \otimes R$. Let $T_R V \# G$ denote the skew group algebra, with G acting naturally on TV and trivially on R. Denote by $Z(RG)$ the centre of the group algebra RG.

For $\tau \in Z(RG)$ we define the algebra A^τ as the quotient of $T_R V \# G$ by the relations

$$v_1 \otimes v_2 - v_2 \otimes v_1 = \omega(v_1, v_2) \cdot \tau,$$

for all $v_1, v_2 \in V$. The algebra A^τ was introduced and studied by W. Crawley-Boevey and M. Holland in [31] (compare also to [86]).

It is convenient to choose and fix a symplectic basis x, y for V, such that $\omega(x, y) = 1$, and to identify V with $\mathbb{C}^2$. Then $T_R V$ gets identified with the free algebra on two generators $R\langle x, y\rangle$. We may therefore conclude that

$$A^\tau = (R\langle x, y\rangle \# G)/(xy - yx - \tau).$$

This relation allows us to put all elements of A^τ into a normal form and we find that A^τ is the free R-module with a basis consisting of all the words of the form $x^i y^j g$ with $i, j \geq 0$ and $g \in G$. Observe that if $\tau = 0$ then we recover the skew group algebra $R[x, y]\# G$. In other words, A^τ is a deformation of $R[x, y]\# G$ for every choice of τ.

Before we proceed, some comments of a general nature may be helpful. A classical theorem of Hilbert (see, e.g. [7, Theorem 1.3.1]) asserts that $R[x, y]$ is a finitely generated $R[x, y]^G$-module. From this it follows that $R[x, y]\# G$ is also a finitely generated $R[x, y]^G$-module. Being finitely generated over its centre, the properties of $R[x, y]\# G$ are closely connected to those of $R[x, y]^G$. One of the chief motives for introducing deformations of $R[x, y]\# G$ is to improve our understanding of deformations of the singular Calabi-Yau threefold $\mathbb{C}^2/G \times \mathbb{C}$, whose coordinate ring is of course $R[x, y]^G$. As we indicated earlier, ADE fibered Calabi-Yau threefolds can be obtained from $\mathbb{C}^2/G \times \mathbb{C}$ by such deformations.

We now return to the general discussion. The algebra A^τ carries a natural filtration, given by $\deg x = \deg y = 1$, $\deg u = 0$ and $\deg g = 0$ for any $g \in G$. Let $\mathrm{gr}\, A^\tau$ denote the associated graded algebra. There is a natural surjective homomorphism $R\langle x, y\rangle \# G \to \mathrm{gr}\, A^\tau$. Letting $\bar{x}, \bar{y}$ denote the images of x, y, it is clear that $\bar{x}\bar{y} - \bar{y}\bar{x} = 0$. Thus we obtain a surjective morphism $R[x, y]\# G \to \mathrm{gr}\, A^\tau$ which is easily seen to be an isomorphism. As a consequence of these observations we obtain the following result, see [31, Lemma 1.1].

Lemma 2.5.1. *We have* $\mathrm{gr}\, A^\tau \cong R[x, y]\# G$.

Before stating our next result, it is convenient to make some definitions. Recall that a $\mathbb{C}$-algebra R is said to be *Auslander-Gorenstein* if it has finite injective dimension and if, for every finitely generated R-module M and every submodule $N \subset \mathrm{Ext}^j_R(M,R)$, one has $\mathrm{Ext}^i_R(N,R) = 0$ for $i < j$. An Auslander-Gorenstein ring of finite global dimension is called *Auslander-regular*. Finally, and Auslander-Gorenstein ring R is said to be *Cohen-Macaulay* if $j(M) + \mathrm{GK}\, M = \mathrm{GK}\, R$ for all nonzero, finitely generated R-modules M. Here $j(M)$ is the *grade* of M, that is, $\inf\{i \mid \mathrm{Ext}^i_R(M,R) \neq 0\}$, and $\mathrm{GK}\, M$ is the GK dimension of M. Standard filtered-graded techniques can be used to deduce the following result, see [31, Sect. 1].

Lemma 2.5.2. *A^τ is a prime noetherian maximal order which is Auslander-regular and Cohen-Macaulay of GK dimension* 3.

Write $e = |G|^{-1} \sum_{g \in G} g$ for the averaging idempotent, viewed as an element in A^τ. Define a subalgebra C^τ of A^τ to be $eA^\tau e$. The increasing filtration on A^τ induces a filtration on C^τ. It is well known that $C^0 = e(R[x,y]\#G)e \cong R[x,y]^G$. Further e lies in the degree zero part of the filtration of A^τ and therefore

$$\mathrm{gr}\, C^\tau \cong e\, \mathrm{gr}\, A^\tau e \cong R[x,y]^G.$$

Since $R[x,y]^G$ is a finitely generated, integrally closed noetherian domain of GK dimension 3, these properties pass up to C^τ. This shows that C^τ is a deformation of the coordinate ring of the singular Calabi-Yau threefold $\mathbb{C}^2/G \times \mathbb{C}$, as required.

In order to make further progress we need to bring in the notion of noncommutative crepant resolution introduced by M. Van den Bergh [91, 92]. Let R be an integrally closed Gorenstein domain. If A is an R-algebra that is finite as an R-module, then A is said to be homologically homogeneous if A is a maximal Cohen-Macaulay R-module and $\mathrm{gldim}\, A_\mathfrak{p} = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$. A *noncommutative crepant resolution* of R is a homologically homogeneous R-algebra of the form $A = \mathrm{End}_R(M)$, where M is a finitely generated reflexive R-module. We remind the reader that an R-module M is said to be reflexive if the natural morphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ is an isomorphism.

Our aim now is to show that A^τ is a noncommutative crepant resolution of C^τ. The following preliminary result will clear our path.

Lemma 2.5.3. *$A^\tau e$ is a finitely generated reflexive C^τ-module. In addition, we have $A^\tau \cong \mathrm{End}_{C^\tau}(A^\tau e)$.*

Proof. One can adapt the techniques of [31, Lemma 1.4] to the present situation. Note first that $A^\tau e A^\tau$ is a finitely generated ideal of A^τ. We write $A^\tau e A^\tau = \sum_{i=1}^m x_i A^\tau$ and $x_i = \sum_j r_{ij} e s_{ij}$ for some $r_{ij}, s_{ij} \in A^\tau$. Choose $a \in A^\tau$. Then $ae \in A^\tau e = (A^\tau e A^\tau)e$, and so $ae = \big(\sum_i x_i a_i\big)e = \sum_{i,j} r_{ij} e s_{ij} a_i e$. This proves that the elements r_{ij} generate $A^\tau e$ as a C^τ-module.

For the condition on the endomorphism ring, there are natural inclusions

$$A^\tau \subset \mathrm{End}_{C^\tau}(A^\tau e) \subset \mathrm{End}_{A^\tau}(A^\tau e A^\tau).$$

Let Q denote the simple artinian quotient ring of A^τ. The fact that $Qe \cong Q \otimes_{A^\tau} A^\tau e$ implies that $\mathrm{End}_{C^\tau}(A^\tau e) \subset \mathrm{End}_{C^\tau}(Qe)$. But C^τ is a maximal order in eQe so that $\mathrm{End}_{C^\tau}(Qe) = \mathrm{End}_{eQe}(Qe)$. Because Q is simple, we also have $Q \cong \mathrm{End}_{eQe}(Qe)$. Thus the endomorphism ring $\mathrm{End}_{C^\tau}(A^\tau e)$ can be identified with a subring of Q. From this it follows that

$$\mathrm{End}_{C^\tau}(A^\tau e) \cong \{q \in Q \mid qA^\tau e \subset A^\tau e\}.$$

Similarly, it can be shown that

$$\mathrm{End}_{A^\tau}(A^\tau e A^\tau) \cong \{q \in Q \mid qA^\tau e A^\tau \subset A^\tau e A^\tau\} = A^\tau,$$

the latter equality being an immediate consequence of the definition of a maximal order. The conclusion is that $A^\tau \cong \mathrm{End}_{C^\tau}(A^\tau e)$, as asserted.

It remains to check that $A^\tau e$ is reflexive. A similar argument to the one above can be applied to show that $A^\tau \cong \mathrm{End}_{C^\tau}(eA^\tau)$. Hence

$$A^\tau e \cong \mathrm{Hom}_{C^\tau}(eA^\tau, eA^\tau)e \cong \mathrm{Hom}_{C^\tau}(eA^\tau, C^\tau),$$

and

$$eA^\tau \cong e\,\mathrm{Hom}_{C^\tau}(A^\tau e, A^\tau e) \cong \mathrm{Hom}_{C^\tau}(A^\tau e, C^\tau),$$

proving that $A^\tau e \cong \mathrm{Hom}_{C^\tau}(\mathrm{Hom}_{C^\tau}(A^\tau e, C^\tau), C^\tau)$. This completes the proof of the lemma. □

We are now ready to prove our promised result.

Proposition 2.5.4. *The algebra A^τ is a noncommutative crepant resolution of C^τ.*

Proof. By Lemma 2.5.3, it suffices to show that A^τ is homologically homogeneous. We already know that A^τ is Cohen-Macaulay. Further, by Lemma 2.5.3 and [64, Corollary 6.18], A^τ has finite global dimension. The desired assertion is now a consequence of [91, Lemma 4.2]. □

We now study the relationship between the algebra A^τ and the $N = 1$ ADE quiver algebra. As before, the irreducible representations of G are $\rho_0, \dots, \rho_n$, with ρ_0 trivial, and $I = \{0, 1, \dots, n\}$. Let Q be the quiver with vertex set I obtained by choosing any orientation of the McKay graph, and let $\delta \in \mathbb{N}^I$ be the vector with $\delta_i = \dim \rho_i$. Fix an isomorphism $\mathbb{C}G \cong \bigoplus_{i \in I} M_{\delta_i}(\mathbb{C})$ and for every ordered pair (p,q), $1 \le p,q \le \delta_i$, take e_{ipq} to be the matrix with p,q entry 1 and zero elsewhere. Given $i \in I$, put $f_i = e_{i11}$. Then $\{f_0, \dots, f_n\}$ is a set of nonzero orthogonal idempotents with the property $\mathbb{C}Gf_i \cong \rho_i$ for all $i \in I$. Hence we get that $f = f_0 + \cdots + f_n$ is idempotent. Furthermore $f_0 = e$, so $e = ef = fe$. Observe also that the map $R^I \to Z(RG)$ given by $\tau \mapsto \sum_{i \in I}(\tau_i/\delta_i) f_i$ is a bijection, and we use this to identify R^I and $Z(RG)$.

Before going on to give the connection between the algebra A^τ and the $N = 1$ ADE quiver algebra $\mathfrak{A}^\tau(Q)$, it is convenient to point out the following description of $\mathfrak{A}^\tau(Q)$. We keep the notation of Sect. 2.4.2. Following Crawley-Boevey and Holland [31], given an element $\tau \in R^I$ we define $\Pi^{R,\tau}(Q)$ to be

$$R\overline{Q} \Big/ \left(\sum_{a \in Q} [a, a^*] - \sum_{i \in I} \tau_i e_i \right).$$

Because u is central, we must have $R\overline{Q} = \mathbb{C}\overline{Q} \otimes R \cong \mathbb{C}\overline{Q} *_B B[u]$. Hence it follows that $\mathfrak{A}^\tau(Q) \cong \Pi^{R,\tau}(Q)$. With this understood, we get the following.

Proposition 2.5.5. *A^τ is Morita equivalent to $\mathfrak{A}^\tau(Q)$ and $C^\tau \cong e_0 \mathfrak{A}^\tau(Q) e_0$.*

Proof. The first part of the proposition follows from the the fact that $fA^\tau f \cong \mathfrak{A}^\tau(Q)$ established in [31, Theorem 3.4]. Under this isomorphism, e corresponds to the trivial path e_0. Using that $e = ef = fe$, we have

$$C^\tau = eA^\tau e = efA^\tau fe \cong e_0\mathfrak{A}^\tau(Q)e_0,$$

as desired. □

For simplicity of notation we fix an isomorphism between C^τ and $e_0\mathfrak{A}^\tau(Q)e_0$ and henceforth identify $C^\tau = e_0\mathfrak{A}^\tau(Q)e_0$. We recall from Sect. 2.4.2 that one can identify $\mathcal{R}_Q(\tau,\delta)$ with the fiber product $\mu_\delta^{-1}(\mathfrak{h}) \times_{\mathfrak{h}} \operatorname{Spec} R$. Now, the coordinate ring of $\operatorname{Rep}(\overline{Q},\delta)$ is the polynomial ring $\mathbb{C}[s_{apq} \mid a \in \overline{Q}, 1 \le p \le \delta_{h(a)}, 1 \le q \le \delta_{t(a)}]$ where the indeterminate s_{apq} picks out the p,q entry of the matrix x_a, corresponding to $x \in \operatorname{Rep}(\overline{Q},\delta)$. It is fairly straightforward to see that $\mathcal{R}_Q(\tau,\delta)$ has coordinate ring $R[s_{apq}]/J_\tau$, where J_τ is generated by the elements

$$\sum_{\substack{a\in Q\\ h(a)=i}} \sum_{r=1}^{\delta_{t(a)}} s_{apr}s_{a^*rq} - \sum_{\substack{a\in Q\\ t(a)=i}} \sum_{r=1}^{\delta_{h(a)}} s_{a^*pr}s_{arq} - \delta_{pq}\tau_i$$

for each vertex i and for $1 \le p,q \le \delta_i$. Letting $\ell = \sum_i \delta_i$ there is a natural ring homomorphism $R\overline{Q} \to M_\ell(R[s_{apq}])$ sending an arrow a to the matrix whose entries are the relevant s_{apq}. By our previous remark this homomorphism descends to a map $\mathfrak{A}^\tau(Q) \to M_\ell(\mathbb{C}[\mathcal{R}_Q(\tau,\delta)])$. Now $\delta_0 = 1$, so this restricts to a homomorphism $e_0\mathfrak{A}^\tau(Q)e_0 \to \mathbb{C}[\mathcal{R}_Q(\tau,\delta)]$. It is apparent that the elements in the image of this map are invariant under the action of $\mathrm{G}(\delta)$. In this way we get a map $\phi_\tau : C^\tau \to \mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}$. It follows from [31, Corollary 8.12] that if $\tau \in R^I$ satisfies $\sum_i \delta_i\tau_i = 0$, then the map ϕ_τ is an isomorphism. Thus, we arrive to the following result.

Proposition 2.5.6. *If $\sum_i \delta_i\tau_i = 0$, then $C^\tau \cong \mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}$.*

One immediate consequence of this is that $\mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}$ is an integrally closed domain, so the quotient scheme $\mathcal{R}_Q(\tau,\delta) /\!\!/ \mathrm{G}(\delta)$ is normal.

Another application of Proposition 2.5.6 is given by the following.

Corollary 2.5.7. *If $\sum_i \delta_i\tau_i = 0$, then the rings A^τ and C^τ have Krull dimension* 3.

Proof. Using Lemma 2.5.3 and [64, Corollary 13.4.9] one sees immediately that the rings A^τ and C^τ are PI rings, and so their Krull dimension coincides with their GK dimension. The assertion follows. □

We finish this subsection with an observation which will be central to our main result. Here we denote the centres of A^τ and C^τ by $Z(A^\tau)$ and $Z(C^\tau)$ respectively.

Proposition 2.5.8. *The map $\phi : A^\tau \to C^\tau$ given by $\phi(a) = eae$ for all a in A^τ restricts to an algebra isomorphism from $Z(A^\tau)$ to $Z(C^\tau)$.*

Proof. It is a straightforward calculation to show that $\phi|_{Z(A^\tau)}$ is an algebra homomorphism with image in $Z(C^\tau)$. To see that it is an algebra isomorphism we construct the inverse map. First we note that an element ξ in $Z(C^\tau)$ implements a C^τ-endomorphism of $A^\tau e$ via right multiplication by ξ. Thanks to Lemma 2.5.3, this endomorphism can be regarded as an element a_ξ of A^τ. Then the algebra homomorphism $\psi : Z(C^\tau) \to A^\tau$ given by $\psi(\xi) = a_\xi$ for all ξ in $Z(C^\tau)$ has its image in $Z(A^\tau)$ because the right multiplication by ξ on $A^\tau e$ commutes with left multiplication by A^τ. It is readily verified that this homomorphism is inverse to $\phi|_{Z(A^\tau)}$. This completes the proof of the proposition. □

This result shows a second vital feature of C^τ: its structure determines the centre of A^τ. Now, if $\sum_i \delta_i \tau_i = 0$, then we know from Proposition 2.5.6 that C^τ is commutative. According to Proposition 2.5.8, in this case $C^\tau \cong Z(A^\tau)$.

2.5.2 **Brief account of Van den Bergh's construction.** In this subsection we describe some of Van den Bergh's results concerning noncommutative crepant resolutions.

We first consider the following more general situation. Let R be a commutative noetherian algebra over $\mathbb{C}$ and let A be an R-algebra which is finitely generated as an R-module. Let $\{e_0,\dots,e_n\}$ be a complete set of primitive orthogonal idempotents in A and set $I = \{0,1,\dots,n\}$. We wish to construct a moduli space of A-modules. To do this we introduce a stability condition.

Let us fix a field K and a ring homomorphism $R \to K$. If M is a finite dimensional $A \otimes_R K$-module, its dimension vector $\underline{\dim}\, M$ is the element of $\mathbb{N}^I$ whose ith component is $\dim_K(e_i M)$. Let θ be a homomorphism $\mathbb{Z}^I \to \mathbb{Z}$. As before, a finite dimensional $A \otimes_R K$-module M is said to be *θ-stable* (or *θ-semistable*) if $\theta(\underline{\dim}\, M) = 0$, but $\theta(\underline{\dim}\, M') > 0$ (or $\theta(\underline{\dim}\, M') \geq 0$) for every proper submodule $M' \subset M$. We say that θ is *generic* for α if every θ-semistable $A \otimes_R K$-module of dimension α is θ-stable. Just as in Sect. 2.3.3, there is a generic θ if and only if α is indivisible. As a matter of fact, the condition $\theta(\beta) \neq 0$ for all $0 < \beta < \alpha$ ensures θ is generic.

Next we recall the notion of family from [39]. Fix a dimension vector $\alpha \in \mathbb{N}^I$. A family of A-modules of dimension α over an R-scheme S is a locally free sheaf $\mathcal{F}$ over S together with an R-algebra homomorphism $A \to \mathrm{End}_S(\mathcal{F})$ such that $e_i\mathcal{F}$ has constant rank α_i for all $i \in I$. Two such families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there is a line bundle $\mathcal{L}$ on S and an isomorphism $\mathcal{F} \cong \mathcal{F}' \otimes_{\mathcal{O}_S} \mathcal{L}$. Finally we say that a family $\mathcal{F}$ is *θ-stable* (or *θ-semistable*) if for every field K and every morphism $\phi : \mathrm{Spec}\, K \to S$ we have that $\phi^*\mathcal{F}$ is θ-stable (or θ-semistable) as $A \otimes_R K$-module.

We have the following result, see [91, Proposition 6.2.1].

Proposition 2.5.9. *If θ is generic, then the functor which assigns to a scheme S the set of equivalence classes of families of θ-stable A-modules of dimension α over S is representable by a projective scheme $\mathcal{M}_\theta(A,\alpha)$ over $X = \mathrm{Spec}\, R$.*

We now illustrate how to use this result to construct a crepant resolution starting from a noncommutative one. Let R be an integrally closed Gorenstein domain admitting a noncommutative crepant resolution $A = \mathrm{End}_R(M)$ and set $X = \mathrm{Spec}\, R$. Let $M = \bigoplus_{i \in I} M_i$ be any decomposition of M corresponding to idempotents $e_0,\dots,e_n \in A = \mathrm{End}_R(M)$, and let $\alpha \in \mathbb{N}^I$ be the vector with $\alpha_i = \mathrm{rank}\, M_i$. By Proposition 2.5.9 we know that, for generic θ, there is a fine moduli space $\mathcal{M}_\theta(A,\alpha)$ of θ-stable A-modules of dimension α. Let us denote by $\phi : \mathcal{M}_\theta(A,\alpha) \to X$ the structure morphism. If we let $U \subset X$ be the open subset over which M is locally free then it follows from [91, Lemma 6.2.3] that $\phi^{-1}(U) \to U$ is an isomorphism. Each point $y \in \phi^{-1}(U)$ is a θ-stable A-module of dimension α so there is an embedding $U \hookrightarrow \mathcal{M}_\theta(A,\alpha)$. Let $W \subset \mathcal{M}_\theta(A,\alpha)$ be the irreducible component of $\mathcal{M}_\theta(A,\alpha)$ containing the image of this morphism. Then W is fine, in that W is projective and there is a universal sheaf $\mathcal{U}$ on $W \times X$. We denote by $\mathcal{P}$ the restriction of $\mathcal{U}$ to W. Notice that $\mathcal{P}$ is a sheaf of A-modules on W.

Now let $\mathbf{D}^b(\mathrm{Coh}(W))$ denote the bounded derived category of coherent shea-ves on W and $\mathbf{D}^b(\text{mod–}A)$ the bounded derived category of finitely generated right modules over A. The method of Bridgeland, King and Reid generalises to prove the following result, see [91, Theorem 6.3.1].

Theorem 2.5.10. *Let the setting be as above. If* $\dim(W \times_X W) \times_X \operatorname{Spec} \mathcal{O}_{X,x} \leq n+1$ *for every point* $x \in X$ *of codimension* n, *then* $\phi : W \to X$ *is a crepant resolution and the functors* $\mathbf{R}\Gamma(- \otimes^{\mathbf{L}}_{\mathcal{O}_W} \mathcal{P})$ *and* $- \otimes^{\mathbf{L}}_A \mathbf{R}\mathcal{H}om^{\cdot}_W(\mathcal{P}, \mathcal{O}_W)$ *define inverse equivalences between* $\mathbf{D}^b(\mathrm{Coh}(W))$ *and* $\mathbf{D}^b(\text{mod–}A)$.

2.5.3 **Application to our situation.** We now return to the concrete situation of Sect. 2.5.1. Our main aim is to show how the ideas developed in the previous subsection can be used to prove that the small resolution of an ADE fibered Calabi-Yau threefold is derived equivalent to the corresponding $N = 1$ ADE quiver algebra. We start with some preliminary observations.

We have seen in Proposition 2.5.4 that the algebra A^τ is a noncommutative crepant resolution of C^τ. Hereafter we assume that $\tau \in R^I$ satisfies $\sum_i \delta_i \tau_i = 0$. As we pointed out earlier, this implies that $C^\tau \cong Z(A^\tau) \cong \mathbb{C}[\mathcal{R}_Q(\tau,\delta)]^{\mathrm{G}(\delta)}$. Setting $X = \operatorname{Spec} C^\tau$, it follows, from Theorem 2.4.5, that X is isomorphic to an ADE fibered Calabi-Yau threefold.

As usual, let $\rho_0, \ldots, \rho_n$ denote the irreducible representations of G with ρ_0 trivial, and set $I = \{0, \ldots, n\}$. For each $i \in I$, let f_i be the idempotent in $\mathbb{C}G$ with $\mathbb{C}G f_i \cong \rho_i$. In view of Lemma 2.5.1, we may regard the f_i's as elements of A^τ. Then $f_i A^\tau e$ is a submodule of $A^\tau e$ for all $i \in I$ and $A^\tau e = \bigoplus_{i \in I} f_i A^\tau e$. Bearing in mind that $f_i A^\tau e \cong \operatorname{Hom}_{\mathbb{C}G}(\rho_i, A^\tau e)$ we have $\delta_i = \dim \rho_i = \operatorname{rank}(f_i A^\tau e)$. Let $\mathcal{M}_\theta(A^\tau,\delta)$ be the moduli space, as constructed in the previous subsection, of θ-stable A^τ-modules of dimension δ (equivalently, isomorphic to $\mathbb{C}G$), and let W be the irreducible component of $\mathcal{M}_\theta(A^\tau,\delta)$ that maps birationally to X.

With the aid of Theorem 2.5.10, we easily derive the following.

Proposition 2.5.11. *With the notation above,* W *is a crepant resolution of* X *and there is an equivalence of categories between* $\mathbf{D}^b(\mathrm{Coh}(W))$ *and* $\mathbf{D}^b(\text{mod–}A^\tau)$.

Proof. Define Δ to be the diagonal of $W \times W$. As in the previous subsection, we write $\phi : W \to X$ for the structure morphism. This is a birational projective mapping, so it is closed. Let us take non-empty open subsets $V \subset W$ and $U \subset X$, such that ϕ restricts to an isomorphism $\phi : V \to U$. Denote by Z the complement of V. We may assume without loss of generality that $\phi(Z) \cap U = \varnothing$. It therefore follows that $W \times_X W \subset \Delta \cap (Z \times Z)$. Since $\dim X = 3$ we have $\dim Z \leq 2$, which ensures that $\dim(W \times_X W) \leq 4$. Now we are in the situation of Theorem 2.5.10 and the assertion follows. □

We now apply Proposition 2.5.11 to prove the result promised in the beginning of this subsection.

Theorem 2.5.12. *Let the context be as above. If $\pi : Y \to X$ is a small resolution of the ADE fibered Calabi-Yau threefold X, then there is an equivalence of categories*

$$\mathbf{D}^b(\mathrm{Coh}(Y)) \cong \mathbf{D}^b(\mathrm{mod}\text{–}\mathfrak{A}^\tau(Q)),$$

where $\mathfrak{A}^\tau(Q)$ is the associated $N = 1$ ADE quiver algebra.

Proof. It is well-known (see, e.g. [26, Proposition 16.4]) that π is a crepant resolution. Owing to Proposition 2.5.11, there exists another crepant resolution $\phi : W \to X$ associated to A^τ. Let $f : Y \to W$ be the birational map over X such that f is isomorphic in codimension 1. Then, by [55, Theorem 6.38], f is a composition of finitely many flops. A result of Bridgeland [14, Theorem 1.1] provides an equivalence of categories $\mathbf{D}^b(\mathrm{Coh}(Y)) \cong \mathbf{D}^b(\mathrm{Coh}(W))$. Invoking Propositions 2.5.11 and 2.5.5, we therefore deduce that

$$\mathbf{D}^b(\mathrm{Coh}(Y)) \cong \mathbf{D}^b(\mathrm{mod}\text{–}A^\tau) \cong \mathbf{D}^b(\mathrm{mod}\text{–}\mathfrak{A}^\tau(Q)),$$

as we wished to show. □

3

HOMOLOGICAL MIRROR SYMMETRY FOR TORIC DEL PEZZO SURFACES

In this chapter we provide a survey of basic ideas relating to Homological Mirror Symmetry for toric Del Pezzo surfaces and their mirror Landau-Ginzburg models. Our treatment follows the outstanding articles by Auroux, Katzarkov and Orlov [4, 3], for the most part.

3.1 INTRODUCTION

Mirror symmetry is a phenomenon in string theory. It really attracted the attention of the mathematicians when it was used to obtain predictions for the number of rational curves per degree on a Calabi-Yau manifold. Since then a whole theory has been developed around this phenomenon. In 1994, M. Kontsevich presented the idea of Homological Mirror Symmetry which states that there should be an equivalence of categories behind mirror duality, one category being the derived category of coherent sheaves on a Calabi-Yau manifold X and the other one being the Fukaya category of the mirror manifold X'. Thus, in some sense, mirror symmetry relates the complex structure of a Calabi-Yau manifold with the symplectic structure of its mirror.

In a different direction, the framework of Homological Mirror Symmetry has been extended to the non Calabi-Yau setting, and in particular to Fano varieties. If X is a Fano variety, then its mirror is conjectured to be a Landau-Ginzburg model (M, W). The derived category of coherent sheaves of X is then expected to be equivalent to the derived category of Lagrangian vanishing cycles associated to the singularities of W. This version of Homological Mirror Symmetry has been confirmed for certain toric Del Pezzo surfaces [89, 90] as well as weighted projective planes and Hirzebruch surfaces [4]. For all these examples, the toric structure plays a crucial role in determining the geometry of the mirror Landau-Ginzburg model.

Our aim in this chapter is to provide an essentially self contained introduction to Homological Mirror Symmetry in the particular case of toric Del Pezzo surfaces. A special emphasis will be on the interplay of quiver representations and A_∞-algebras. Sect. 3.4.3 will do Homological Mirror Symmetry by example: the correspondence will be computed in the case of $\mathbb{P}^2$ blown up at one point. This captures the essential idea behind the more general case.

Before we come to Sect. 3.4.3 we need quite a few preparations. Hopefully a brief outline of the strategy will help the reader not to get lost among all the details. The basic idea is to obtain a convenient description of the higher order products for both the derived category and the category of Lagrangian vanishing cycles. After calculating them in both cases we note that the expressions match if we make the right identifications.

The goal of Sect. 3.3.2 is to give a detailed description of the derived category of coherent sheaves on a toric Del Pezzo surface in terms of an A_∞-algebra. In Sect. 3.3.3 we use this description to calculate the composition of morphisms and higher order products in the case of $\mathbb{P}^2$ with one point blown up.

In Sect. 3.4.3, we turn our attention to the category of Lagrangian vanishing cycles on the mirror Landau-Ginzburg models. Again we first discuss the class of objects that we consider and give an explicit description of the morphisms between them. Subsequently we use this description to calculate higher order products. Finally, after a discussion of Maslov index and grading, we establish an explicit equivalence between the derived category of coherent sheaves and the derived category of Lagrangian vanishing cycles.

3.2 PRELIMINARIES

In this section we consider some primarily unrelated basic topics that we shall need in varying degree throughout the chapter. We include the material here for the reader's convenience, and to fix notation.

3.2.1 **Homological algebra of quiver representations.**

Our aim here is to discuss some basic homological properties of quiver representations. We refer to [5] for a complete exposition (see also [19]).

Let $Q = (Q_0, Q_1)$ be a quiver with vertices Q_0 and arrows Q_1 and denote by $\mathbb{C}Q$ the corresponding path algebra. We are mainly interested in the algebras $A = \mathbb{C}Q/(\rho)$ where ρ is a set of admissible relations for the quiver Q. Here, a set of relations ρ is called admissible if $J^m \subset (\rho) \subset J^2$ for some $m \in \mathbb{N}$ where J denotes the ideal generated by the arrows of Q. Note that A is finite dimensional over $\mathbb{C}$. The abelian category of finitely generated right modules over A will be denoted by mod–A. We interpret mod–A as the category of finite dimensional representations of (Q, ρ).

It is useful to describe the projective and simple objects in mod–A directly. With every vertex $i \in Q_0$ corresponds canonically an idempotent e_i in A given by the trivial path. We have that $1 = \sum_{i \in Q_0} e_i$ is a decomposition of 1 into a sum of orthogonal idempotents, and $A = \bigoplus_{i \in Q_0} e_i A$. This shows that $P_i = e_i A$ is a projective object in mod–A. We note also that P_i is spanned by all paths starting at i. What is more, any indecomposable projective A-module is isomorphic to P_i for some vertex i. We obtain in this way a complete set of representatives from the isomorphism classes of indecomposable projective A-modules.

Furthermore, for any right A-module M we have $\mathrm{Hom}(P_i, M) \cong Me_i$ and in particular $\mathrm{Hom}(P_i, P_j) \cong e_j A e_i$ is the vector space spanned by all paths from vertex j to vertex i.

Similarly, for each vertex $i \in Q_0$ we have a simple object S_i in mod–A; this is the representation which assigns the field $\mathbb{C}$ to the vertex i and 0 to any other vertex and where each arrow gives the zero map. Again, we obtain in this way a complete set of representatives from the isomorphism classes of simple A-modules. Another basic fact about these modules which is easily proven is that $\mathrm{Hom}(P_i, S_j) \cong \delta_{ij}\mathbb{C}$.

Our aim now is to describe the projective resolutions of the simple A-modules. First we introduce a little terminology.

For each $i \in Q_0$ let $P_i^* = \mathrm{Hom}(P_i, A) \cong Ae_i$, the left A-module spanned by all paths ending in i. Given an arrow arrow $a \in Q_1$, we denote by $\delta_a : \mathbb{C}Q \to P_{h(a)}^* \otimes P_{t(a)}$ the linear map defined in the following manner. Any element in $\mathbb{C}Q$ is a sum of paths in the quiver. For each element in the sum, locate all occurrences of the arrow a. Then, the path can be written as paq with $t(a) = h(q)$ and $h(a) = t(p)$. This defines an element $p \otimes q \in P_{h(a)}^* \otimes P_{t(a)}$. If a occurs multiple times in the path, then do this for each occurrence. The value of δ_a in $P_{h(a)}^* \otimes P_{t(a)}$ is the sum of all elements so obtained. Observe that this linear map is defined on $\mathbb{C}Q$, not A.

With this notation, one can show that A has a projective resolution which starts

$$\cdots \to \bigoplus_{r\in\rho} P_{h(r)}^* \otimes P_{t(r)} \xrightarrow{f} \bigoplus_{a\in Q_1} P_{h(a)}^* \otimes P_{t(a)} \xrightarrow{g} \bigoplus_{i\in Q_0} P_i^* \otimes P_i \xrightarrow{m} A \to 0,$$

where f is defined by $f(e_{h(r)} \otimes e_{t(r)}) = \sum_{a\in Q_1} \delta_a r$, g is defined by $g(e_{h(a)} \otimes e_{t(a)}) = a \otimes e_{t(a)} - e_{h(a)} \otimes a$, and m is multiplication. Notice that $P_j^* \otimes P_i$ is isomorphic as right A-module to a direct sum of copies of P_i, indexed by a basis of P_j^*. Thus the terms are indeed projective A-modules.

Now to obtain a projective resolution of an arbitrary simple A-module S_i, we simply tensor the above resolution with S_i over A, getting

$$\cdots \longrightarrow \bigoplus_{j\in Q_0} P_j^{\oplus r_{ij}} \longrightarrow \bigoplus_{j\in Q_0} P_j^{\oplus n_{ij}} \longrightarrow P_i \longrightarrow S_i \longrightarrow 0;$$

here n_{ij} is equal to the number of arrows in the quiver from vertex i to vertex j and r_{ij} represents the number of independent relations on paths from i to j.

We would like to compute some Ext groups which are central to our analysis. This is very easy in the current context. Noting that Ext^n is the nth derived functor of Hom and that $\mathrm{Hom}(P_i, S_j) \cong \delta_{ij}\mathbb{C}$, we can compute $\mathrm{Ext}^n_A(M, N)$ by taking a projective resolution

$$\cdots \longrightarrow \mathcal{P}_2 \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow M \longrightarrow 0,$$

where the $\mathcal{P}_i$ are direct sums of the P_i and from it constructing the complex

$$0 \longrightarrow \mathrm{Hom}(\mathcal{P}_0, N) \longrightarrow \mathrm{Hom}(\mathcal{P}_1, N) \longrightarrow \mathrm{Hom}(\mathcal{P}_2, N) \longrightarrow \cdots$$

The cohomology of this complex in the nth position is then $\mathrm{Ext}^n_A(M, N)$. Using this method one can show that

$$\dim \mathrm{Ext}^1_A(S_i, S_j) = n_{ij},$$
$$\dim \mathrm{Ext}^2_A(S_i, S_j) = r_{ij}.$$

One should therefore think of the arrows in the quiver as representing Ext^1's between the simples S_i and Ext^2's arising because of the relations in the quiver.

3.2.2 **A_∞-algebras and A_∞-categories.** This paragraph is a short discussion about A_∞-algebras and A_∞-categories. For more details concerning most of the following definitions the reader can consult [53].

An A_∞-algebra is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{p\in\mathbb{Z}} A^p$ equipped with linear maps $m_k : A^{\otimes k} \to A$ for $k \geq 1$ of degree $2-k$ satisfying

$$\sum_{i+j+k=n} (-1)^{i+jk} m_l(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = 0,$$

for every $n \geq 1$, where $l = i+1+k$. Here, id denotes the identity map of A. Note that when these formulas are applied to elements, additional signs appear due to the Koszul sign rule

$$(f \otimes g)(a \otimes b) = (-1)^{\deg(g)\deg(a)} f(a) \otimes g(b)$$

where f and g are graded maps, a and b are homogeneous elements.

The above condition may appear somewhat mysterious, and replaces the associativity condition in an ordinary algebra. Let us try to understand this by examining it in particular cases. For $n = 1$, it states

$$m_1 m_1 = 0.$$

Notice that this, along with $\deg(m_1) = 1$ shows that m_1 is a differential on the graded vector space A, with respect to which one may take the cohomology. To wit, let us examine the condition when $n = 2$. It then states m_2 is of degree zero and satisfies

$$m_1 m_2 = m_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1).$$

Thus m_2 is a chain map, and induces a product on cohomology. Finally, $n = 3$ yields

$$\begin{aligned} &m_2(\mathrm{id} \otimes m_2 - m_2 \otimes \mathrm{id}) \\ &\qquad = m_1 m_3 + m_3(m_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes m_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes m_1). \end{aligned}$$

The left-hand side of this equation is the associator of m_2 and the right-hand side may be viewed as the boundary of m_3 in the morphism complex $\mathrm{Hom}^\cdot(A^{\otimes 3}, A)$. Thus m_2 remains associative *up to homotopy*, but more is true. The homotopy is provided by m_3, which is built into the definition of A.

One should note that grading is essential to nontrivial A_∞-algebras. An A_∞-algebra concentrated in degree 0 is necessarily an associative algebra (all m_k vanish for $k \neq 0$).

For a pair of A_∞-algebras A and B there is a natural notion of an A_∞-morphism from A to B. Namely, such a morphism consists of the data $(f_n, n \geq 1)$ where $f_n : A^{\otimes n} \to B$ is a linear map of degree $1-n$ such that

$$\sum_{i+j+k=n} (-1)^{i+jk} f_l(\mathrm{id}^{\otimes i} \otimes m_j \otimes \mathrm{id}^{\otimes k}) = \sum_{\substack{1\leq r\leq n \\ i_1+\cdots+i_r=n}} (-1)^q m_r(f_{i_1} \otimes \cdots \otimes f_{i_r}),$$

for any $n \geq 1$ and $l = i + 1 + k$ again. The sign on the right is given by $q = (r-1)(i_1 - 1) + (r-2)(i_2 - 1) + \cdots + (i_{r-1} - 1)$. For $n = 1$ this yields the following condition

$$f_1 m_1 = m_1 f_1.$$

For $n = 2$ the condition is already more complicated

$$f_1 m_2 = m_2(f_1 \otimes f_1) + m_1 f_2 + f_2(m_1 \otimes \mathrm{id} + \mathrm{id} \otimes m_1).$$

The first equation implies that f_1 defines a morphism of complexes. The second equation implies that f_1 preserves the product given by m_2 up to a homotopy given by f_2. More generally one might say that $f = (f_n, n \geq 1)$ preserves the m_n up to homotopy.

An A_∞-morphism f is called a *quasi-isomorphism* if f_1 is a quasi-isomorphism. Two A_∞-algebras A and B are said to be quasi-isomorphic as A_∞-algebras if there is an A_∞-morphism $f : A \to B$ that is a quasi-isomorphism.

One can compose A_∞-morphisms in the natural way. The identity A_∞-morphism consists of $f_1 = \mathrm{id}$, $f_n = 0$ for $n \geq 2$. If $f : A \to B$ is an A_∞-morphism such that f_1 is an isomorphism of underlying abelian groups, then f_1^{-1} extends to an A_∞-morphism $B \to A$ which is inverse to f. In the case A and B have the same underlying spaces and $f_1 = \mathrm{id}$ we will call the data $(f_n, n \geq 2)$ a *strict A_∞-isomorphism*.

We now come to a result that allows us to pass to cohomology of an A_∞-algebra without losing too much information. Let A be an A_∞-algebra. As noted before, m_1 gives A the structure of a graded differential complex, and we may take cohomology to yield $H^\cdot(A)$. By choosing representatives of each cohomology class we may define an embedding $i : H^\cdot(A) \hookrightarrow A$. Thanks to a theorem of Kadeishvili [46], we may define an A_∞-structure on $H^\cdot(A)$ such that $m_1 = 0$, and there is a quasi-isomorphism f from $H^\cdot(A)$ to A with f_1 equal to the embedding i. Here, m_1 refers to the A_∞-structure on $H^\cdot(A)$. This A_∞-structure is not unique, but it is unique up to a strict A_∞-isomorphism. An A_∞-algebra with $m_1 = 0$ is called a *minimal* A_∞-algebra; thus the above may be interpreted as saying that each A_∞-algebra has an essentially unique minimal model.

It is quite easy to construct the minimal model in practice. A rather simple example of an A_∞-algebra is given by $m_k = 0$ for $k \geq 3$. Such an algebra is called a *differential graded algebra*, or DG algebra. In this chapter, we will need to put an A_∞-structure on the cohomology of a DG algebra, which may be done explicitly as follows [58, 67]. Let m_1 on A be denoted d, and let m_2 be denoted μ. We set $Z^n = \ker(d^n : A^n \to A^{n+1})$ and $B^n = \mathrm{im}(d^{n-1} : A^{n-1} \to A^n)$. For $n \geq 1$, there are subspaces L^n and H^n of A^n such that $Z^n = B^n \oplus H^n$ and $A^n = B^n \oplus H^n \oplus L^n$. Of course, there are many different choices of H^n and L^n. In what follows, we identify $H^n(A)$ with H^n. Let $P : A \to H^\cdot(A)$ be the projection to $H^\cdot(A)$. Now we define a linear map $G : A \to A$ of degree -1 with the following properties. For $n \geq 1$, $G^n : A^n \to A^{n-1}$ is defined as $G^n = 0$ when restricted to $H^n(A) \oplus L^n$, and $G^n = (d^{n-1}|_{L^{n-1}})^{-1}$ when restricted to B^n. The higher order products m_k for $k \geq 2$ are defined by the formula

$$m_k = \sum_T m_{k,T},$$

where the sum is over all planar rooted trees T with k leaves. Here, $m_{k,T}$ for a planar rooted tree T is defined as the composition of the product μ for each internal vertex, the map G for each internal edge and the projection P for the root edge. Applying this recipe one easily finds the following expressions for the first few higher products

$$\begin{aligned} m_1(a) &= 0, \\ m_2(a_1,a_2) &= P\mu(a_1,a_2), \\ m_3(a_1,a_2,a_3) &= P\mu(G\mu(a_1,a_2),a_3) + P\mu(a_1,G\mu(a_2,a_3)). \end{aligned}$$

Finally, we briefly address the important topic of A_∞-categories. The definition of an A_∞-category is similar to that of an A_∞-algebra. Namely, an A_∞-category $\mathcal{A}$ consists of a class of objects $\operatorname{Ob}\mathcal{A}$, for every pair of objects E and E', a graded space of morphisms $\operatorname{Hom}^\cdot_{\mathcal{A}}(E,E')$, and a collection of linear maps

$$m_n : \operatorname{Hom}^\cdot_{\mathcal{A}}(E_0,E_1)\otimes\cdots\otimes\operatorname{Hom}^\cdot_{\mathcal{A}}(E_{n-1},E_n) \longrightarrow \operatorname{Hom}^\cdot_{\mathcal{A}}(E_0,E_n)$$

of degree $2-n$ for all $n \geq 1$. The associativity constraint is that these compositions define a structure of A_∞-algebra on $\bigoplus_{i,j}\operatorname{Hom}^\cdot_{\mathcal{A}}(E_i,E_j)$ for every collection $E_0,\dots,E_n \in \operatorname{Ob}\mathcal{A}$.

An A_∞-functor $\phi : \mathcal{A} \to \mathcal{B}$ between A_∞-categories consists of a map $\phi : \operatorname{Ob}\mathcal{A} \to \operatorname{Ob}\mathcal{B}$ and of a collection of linear maps

$$f_n : \operatorname{Hom}^\cdot_{\mathcal{A}}(E_0,E_1)\otimes\cdots\operatorname{Hom}^\cdot_{\mathcal{A}}(E_{n-1},E_n) \longrightarrow \operatorname{Hom}^\cdot_{\mathcal{B}}(\phi(E_0),\phi(E_n))$$

of degree $1-n$ for $n \geq 1$, that define A_∞-morphisms $\bigoplus_{i,j}\operatorname{Hom}^\cdot_{\mathcal{A}}(E_i,E_j) \to \bigoplus_{i,j}\operatorname{Hom}^\cdot_{\mathcal{B}}(\phi(E_i),\phi(E_j))$.

One can check that m_1 defines a differential on $\operatorname{Hom}^\cdot_{\mathcal{A}}(E,E')$ for all pairs $E,E' \in \operatorname{Ob}\mathcal{A}$. Using this differential we can construct an ordinary category out of an A_∞-category by keeping the same set of objects and replacing the Hom-spaces by their cohomology. The composition is then defined by m_2, which is associative on the cohomology. We denote the category that we obtain in this way by $H^\cdot(\mathcal{A})$.

3.3 DERIVED CATEGORIES OF COHERENT SHEAVES ON TORIC DEL PEZZO SURFACES

One reason, that makes possible the proof of the Homological Mirror Symmetry conjecture for toric Del Pezzo surfaces, is the fact that the bounded derived category of coherent sheaves on those surfaces has a well understood structure. The principal goal of this section is to give a convenient description of this category.

3.3.1 **Exceptional collections of toric Del Pezzo surfaces.** The purpose of this part is to describe the bounded derived category of coherent sheaves on a toric Del Pezzo surface in terms of exceptional collections.

We start by recalling the classification of toric Del Pezzo surfaces. In general, a smooth projective surface X is called a Del Pezzo surface if the anticanonical sheaf $\mathcal{O}_X(-K_X)$ is ample (i.e., a Del Pezzo surface is a Fano variety of dimension 2). The toric Del Pezzo surfaces, i.e., those which are equipped with an algebraic action of a 2-dimensional torus $T = (\mathbb{C}^\times)^2$, and contain an open dense orbit, constitute a special subclass within the entire class of Del Pezzo surfaces. Any smooth complete toric surface can be described by a complete regular fan Σ in $\mathbb{R}^2$. (The reader is referred to [68, 35] for the precise definition.)

The convex hull of the primitive vertices of the one-dimensional cones of Σ is a convex polytope Δ, containing the origin as an interior point. As shown in [6], there is a simple condition on Δ to ensure that the resulting toric suface X id Del Pezzo. Quite simply, X is Del Pezzo if the vertices of Δ are primitive elements of $\mathbb{Z}^2$; in this case, we say that Δ is a *reflexive* polytope. Thus, in order to study toric Del Pezzo surfaces one should understand reflexive polytopes. It is a classical result that there exist exactly five different toric Del Pezzo surfaces up to isomorphism. They are the plane $\mathbb{P}^2$, the quadric $\mathbb{P}^1 \times \mathbb{P}^1$, and the three surfaces obtained by blowing up $\mathbb{P}^2$ at a set of $k \leq 3$ points. Up to equivalence, the corresponding polytopes are as in Figure 3.1.

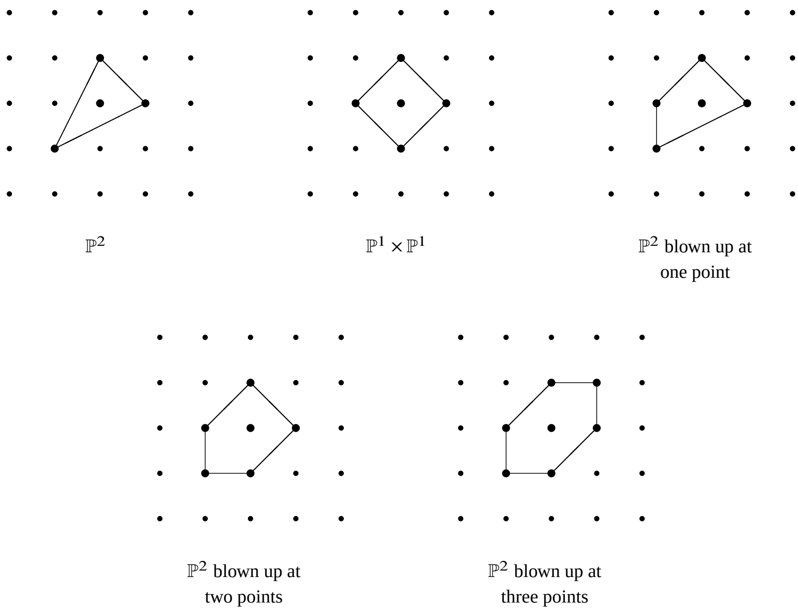


Figure 3.1: Polytopes for the five toric Del Pezzo surfaces

Next, let us recall the notion of exceptional collection. An object E of a $\mathbb{C}$-linear triangulated category $\mathcal{D}$ is said to be *exceptional* if $\mathrm{Hom}(E, E[k]) = 0$ for all $k \neq 0$, and $\mathrm{Hom}(E, E) = \mathbb{C}$. An ordered set of exceptional objects $\sigma = (E_0, \ldots, E_n)$ is called an *exceptional collection* if $\mathrm{Hom}(E_j, E_i[k]) = 0$ for $j > i$ and all k. The exceptional collection σ is said to be *strong* if it satisfies the additional condition $\mathrm{Hom}(E_j, E_i[k]) = 0$ for all i, j and for $k \neq 0$. Finally, it is called *full* if it generates the category $\mathcal{D}$, i.e. the minimal triangulated subcategory of $\mathcal{D}$ containing all objects E_i coincides with $\mathcal{D}$.

We now specialize to $\mathcal{D}$ being the bounded derived category $\mathbf{D}^b(\mathrm{Coh}(X))$ of coherent sheaves on a smooth projective variety X. Assume that this category has an exceptional collection $(E_0, \ldots, E_n)$ which is strong and full. In this case we will say that X possesses a full strong exceptional collection. In what follows we denote by B the algebra of endomorphisms of the object $\mathcal{E} = \bigoplus_{i=0}^{n} E_i$, i.e. $B = \mathrm{End}(\mathcal{E})$.

Our first observation is that the algebra B is finite dimensional over $\mathbb{C}$. Denote by mod–B the category of finitely generated right modules over B. For any coherent sheaf $F \in \mathrm{Coh}(X)$ the space $\mathrm{Hom}(\mathcal{E}, F)$ has the structure of a right B-module. Let us denote by P_i the modules $\mathrm{Hom}(\mathcal{E}, E_i)$ for $i = 0, \dots, n$. All these are projective B-modules and we have a decomposition $B = \bigoplus_{i=0}^{n} P_i$. We learn, at the same time, that the algebra B has $n+1$ primitive idempotents e_i, $i = 0, \dots, n$ such that $1 = e_0 + \cdots + e_n$ and $e_i e_j = 0$ if $i \neq j$. The right projective modules P_i coincide with $e_i B$. The morphisms between them can be easily described since

$$\mathrm{Hom}(P_i, P_j) = \mathrm{Hom}(e_i B, e_j B) \cong e_j B e_i \cong \mathrm{Hom}(E_i, E_j).$$

It follows from this discussion that the algebra B has finite homological dimension. (See [13] or [22] for details.)

Observe also that the algebra B is basic. This means that the quotient of B by the radical $\mathrm{rad}(B)$ is isomorphic to the direct sum of $k+3$ copies of the field $\mathbb{C}$. The category mod–B has $k+3$ simple modules which will be denoted S_i, $i = 0, \dots, k+2$, and $B/\mathrm{rad}(B) = \bigoplus_{i=0}^{k+2} S_i$. The modules S_i are chosen so that $\mathrm{Hom}(P_i, S_j) \cong \delta_{ij}\mathbb{C}$.

Finally, a fundamental result of Bondal [12] asserts that $\mathbf{D}^b(\mathrm{Coh}(X))$ is equivalent to the bounded derived category $\mathbf{D}^b(\text{mod–}B)$. This equivalence is given by the functor $\mathbf{R}\mathrm{Hom}^{\cdot}(\mathcal{E}, -)$.

It is known that there exist full strong exceptional collections of sheaves on all toric Del Pezzo surfaces. For example, $(\mathcal{O}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$ is a full strong exceptional collection on $\mathbb{P}^2$. If X is obtained by blowing up $\mathbb{P}^2$ at $k \leq 3$ points, and $l_1, \dots, l_k$ are the exceptional curves, then the sequence

$$(\mathcal{O}, \mathcal{O}_X(h-l_1), \mathcal{O}_X(h-l_2), \dots, \mathcal{O}_X(h-l_k), \mathcal{O}_X(h), \mathcal{O}_X(2h-l_1-\cdots-l_k)),$$

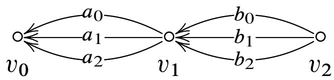
where h is the hyperplane divisor, is a full strong exceptional collection on X. In particular, there is an equivalence

$$\mathbf{D}^b(\mathrm{Coh}(X)) \cong \mathbf{D}^b(\text{mod–}B),$$

where B is the algebra of endomorphisms of the object $\mathcal{E} = \mathcal{O} \oplus \bigoplus_{i=1}^{k} \mathcal{O}_X(h-l_i) \oplus \mathcal{O}_X(h) \oplus \mathcal{O}_X(2h-l_1-\cdots-l_k)$.

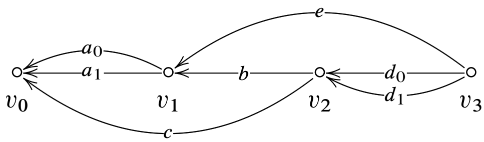
For later purposes, it will be useful to represent the algebra B as the path algebra of a quiver with relations. The quiver contains $k+3$ vertices corresponding to the idempotents e_i, and an arrow from vertex j to vertex i for each irreducible map involved in $\mathrm{Hom}(P_i, P_j)$ (cf. [22]).

As an example, consider the full strong exceptional collection $(\mathcal{O}, \mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$ on $\mathbb{P}^2$. We have $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{P}^2}(1)) \cong \mathbb{C}^3$ and $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2)) \cong \mathbb{C}^3$. Denote these maps which are just multiplication by the homogeneous coordinates on $\mathbb{P}^2$, by x_i and y_i respectively, $i = 0, 1, 2$. We also have $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{P}^2}(2)) \cong \mathbb{C}^6$; these maps are multiplication by homogeneous degree two polynomials in the homogeneous coordinates. Any element of $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{P}^2}(2))$ is given by an element of $\mathrm{Hom}(\mathcal{O}, \mathcal{O}_{\mathbb{P}^2}(1))$ composed with an element of $\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(1), \mathcal{O}_{\mathbb{P}^2}(2))$, and thus no extra arrows are needed between vertex 0 and vertex 2. In addition, we have an obvious relation $x_i y_j = x_j y_i$ for the composition of such maps. Thus, after reversing the arrows in accord with the above description, we see that the quiver corresponding to $\mathbb{P}^2$ takes the form



with relations $a_i b_j = a_j b_i$.

Another example which will be thoroughly dealt with below is X given by $\mathbb{P}^2$ with a single point blown up. Using the full strong exceptional collection $(\mathcal{O}, \mathcal{O}_X(h-l_1), \mathcal{O}_X(h), \mathcal{O}_X(2h-l_1))$, the corresponding quiver is given by



subject to the relations $a_1 b d_0 - a_0 b d_1 = 0$, $a_0 e - c d_0 = 0$, and $c d_1 - a_1 e = 0$.

3.3.2 **Twisted complexes and Koszul duality.** The aim of this subsection is to give another description of the derived category $\mathbf{D}^b(\mathrm{Coh}(X))$. It was shown above that this category is equivalent to the derived category $\mathbf{D}^b(\text{mod–}B)$. We introduce an A_∞-algebra $C^\cdot$ and prove that the category $\mathbf{D}^b(\mathrm{Coh}(X))$ is equivalent to the derived category of $C^\cdot$. First, however, we must develop our vocabulary.

Let $\mathcal{A}$ be an A_∞-category. Denote by $\mathbb{Z}\mathcal{A}$ the A_∞-category obtained from $\mathcal{A}$ by adjoining formal shifts of objects. Its objects are the symbols $E[n]$, where E is an object of $\mathcal{A}$ and $n \in \mathbb{Z}$. The space of morphisms between two objects $E[n]$, $E'[n']$ is the shifted complex

$$\mathrm{Hom}^\cdot_{\mathbb{Z}\mathcal{A}}(E[n], E'[n']) = \mathrm{Hom}^\cdot_{\mathcal{A}}(E, E')[n'-n].$$

The higher order products $m_k^{\mathbb{Z}\mathcal{A}}$ of $\mathbb{Z}\mathcal{A}$ are defined using those of $\mathcal{A}$ as follows: for $a_1 \in \mathrm{Hom}^\cdot_{\mathbb{Z}\mathcal{A}}(E_0[n_0], E_1[n_1]), \dots, a_k \in \mathrm{Hom}^\cdot_{\mathbb{Z}\mathcal{A}}(E_{k-1}[n_{k-1}], E_k[n_k])$,

$$m_k^{\mathbb{Z}\mathcal{A}}(a_1, \dots, a_k) = (-1)^{n_0} m_k^{\mathcal{A}}(a_1, \dots, a_k).$$

It is customary to identify $\mathcal{A}$ with the full A_∞-subcategory of $\mathbb{Z}\mathcal{A}$ on the set of objects $E[0]$.

Next we construct a new A_∞-category $\Sigma\mathcal{A}$ by taking the additive completion of $\mathbb{Z}\mathcal{A}$. The objects of $\Sigma\mathcal{A}$ are formal direct sums $C = \bigoplus_i E_i[n_i]$ with finitely many $E_i \neq 0$. The space of morphisms between two objects $C = \bigoplus_i E_i[n_i]$ and $C' = \bigoplus_j E'_j[n'_j]$ is given by the complex

$$\mathrm{Hom}^\cdot_{\Sigma\mathcal{A}}(C, C') = \bigoplus_{i,j} \mathrm{Hom}^\cdot_{\mathbb{Z}\mathcal{A}}(E_i[n_i], E'_j[n'_j]).$$

The higher products $m_k^{\mathbb{Z}\mathcal{A}}$ extend to higher products $m_k^{\Sigma\mathcal{A}}$ on $\Sigma\mathcal{A}$ in the obvious manner, making $\Sigma\mathcal{A}$ into an A_∞-category.

We now come to a very important notion. A *twisted complex* over $\mathcal{A}$ is an ordered pair (C, Φ) where C is an object in $\Sigma\mathcal{A}$ and Φ is an element of $\mathrm{Hom}^1_{\Sigma\mathcal{A}}(C, C)$ satisfying the condition

$$\sum_{k \geq 1} m_k^{\Sigma\mathcal{A}}(\Phi, \dots, \Phi) = 0. \tag{3.3.1}$$

If we express C as $\bigoplus_i E_i[n_i]$ with $E_i \in \mathcal{A}$ and $n_i \in \mathbb{Z}$, then this gives a decomposition of Φ as $\sum_{i,j} \Phi_{ij}$ where the Φ_{ij} are elements of $\mathrm{Hom}^1_{\mathbb{Z}\mathcal{A}}(E_i[n_i], E_j[n_j])$. In this notation equation (3.3.1) becomes

$$m_1^{\mathbb{Z}\mathcal{A}}(\Phi_{ij}) + \sum_{k\geq 2} \sum_{j_1,\dots,j_{k-1}} m_k^{\mathbb{Z}\mathcal{A}}(\Phi_{ij_1},\dots,\Phi_{j_{k-1}j}) = 0.$$

Twisted complexes over $\mathcal{A}$ form an A_∞-category denoted by $\mathrm{Tw}(\mathcal{A})$ in [81, 51]. The morphisms between two such objects (C,Φ) and (C',Φ') are

$$\mathrm{Hom}^{\cdot}((C,\Phi),(C',\Phi')) = \mathrm{Hom}^{\cdot}_{\Sigma\mathcal{A}}(C,C').$$

The action of the higher products m_k on morphisms $a_1 \in \mathrm{Hom}^{\cdot}((C_0,\Phi_0),(C_1,\Phi_1)),\dots,$ $a_k \in \mathrm{Hom}^{\cdot}((C_{k-1},\Phi_{k-1}),(C_k,\Phi_k))$ is defined by

$$\begin{aligned} &m_k(a_1,\dots,a_k) \\ &= \sum_{j_0,\dots,j_k\geq 0} m^{\Sigma\mathcal{A}}_{k+j_0+\dots+j_k}(\overbrace{\Phi_0,\dots,\Phi_0}^{j_0}, a_1, \overbrace{\Phi_1,\dots,\Phi_1}^{j_1},\dots,a_k,\overbrace{\Phi_k,\dots,\Phi_k}^{j_k}). \end{aligned}$$

The m_1-closed morphisms of degree zero between twisted complexes will be called *twisted morphisms*.

Let us now recall the construction of the derived category of an A_∞-category following [81]. First some terminology is required. A twisted complex (C,Φ) over an A_∞-category $\mathcal{A}$ is called *one-sided* if $\Phi_{ij} = 0$ for $i \geq j$. We denote by $\mathrm{Tw}^+(\mathcal{A})$ the full A_∞-subcategory of $\mathrm{Tw}(\mathcal{A})$ whose objects are the one-sided twisted complexes. Notice that $\mathrm{Tw}^+(\mathcal{A})$ is closed under formal shifts $(C,\Phi)[1] = (C[1],-\Phi)$.

We define the derived category $\mathbf{D}^b(\mathcal{A})$ to be the degree zero cohomology of the A_∞-category $\mathrm{Tw}^+(\mathcal{A})$. It has a natural structure of a triangulated category. We briefly indicate the construction of the exact triangles. Let (C,Φ) and (C',Φ') be two objects in $\mathrm{Tw}^+(\mathcal{A})$ and $a : C \to C'$ a twisted morphism from (C,Φ) to (C',Φ'). By the mapping cone of this morphism we mean the object $\mathrm{Cone}(a) = (C'',\Phi'')$ for which

$$C'' = C \oplus C'[1], \quad \Phi'' = \begin{pmatrix} \Phi & a \\ 0 & \Phi' \end{pmatrix}.$$

We have in $\mathrm{Tw}^+(\mathcal{A})$ the obvious triangle

$$(C,\Phi) \xrightarrow{a} (C',\Phi') \longrightarrow \mathrm{Cone}(a) \longrightarrow (C,\Phi)[1],$$

determining also a triangle in $\mathbf{D}^b(\mathcal{A})$. By the exact triangles in $\mathbf{D}^b(\mathcal{A})$ we mean those triangles isomorphic to triangles of this form.

We now restrict our attention to the special case where the A_∞-category $\mathcal{A}$ has only finitely many objects $E_0,\dots,E_n$. As is clear from the construction above, in this case $\mathbf{D}^b(\mathcal{A})$ always admits a full exceptional collection. Indeed, the objects E_i of $\mathcal{A}$, seen as one-sided twisted complexes with zero differential, form a full exceptional collection of the derived category.

It is also useful to make the following remark. One can encode the categorical data of $\mathcal{A}$ in an equivalent, but more amenable form. To be more specific, one can form the total morphism algebra $A = \bigoplus_{0\le i,j\le n} \operatorname{Hom}^{\cdot}_{\mathcal{A}}(E_i, E_j)$ which is a bimodule over the semisimple algebra $\mathbb{C}^{n+1}$. As pointed out in [81], the A_∞-category $\mathcal{A}$ can be described as an A_∞-algebra on the bimodule A. This algebraic formulation allows one to avoid the notational morass of the category-theoretic description. From the outset, we will therefore replace the A_∞-category $\mathcal{A}$ by the A_∞-algebra A, so we write $\mathbf{D}^b(A)$ in place of $\mathbf{D}^b(\mathcal{A})$.

Now let us go back to the specific context of the previous subsection. Denote by S the algebra $B/\operatorname{rad}(B)$ and consider it as a right B-module, isomorphic to the sum $\bigoplus_{i=0}^{k+2} S_i$ of all simple modules. The chain complex of endomorphisms $\mathbf{R}\operatorname{Hom}^{\cdot}_B(S,S)$ has a natural structure of a DG algebra. Multiplication is given by composition of endomorphisms. As explained in Sect. 3.2.2, there is an A_∞-structure on the full Ext algebra $\operatorname{Ext}^{\cdot}_B(S,S)$ with $m_1 = 0$ and m_2 is induced by the multiplication of $\mathbf{R}\operatorname{Hom}^{\cdot}_B(S,S)$. Of course, the induced A_∞-algebra structure on $\operatorname{Ext}^{\cdot}_B(S,S)$ is unique up to a strict A_∞-isomorphism.

Define an A_∞-category $\mathcal{C}$ as an A_∞-category with $k+3$ objects, say $v_0,\dots,v_{k+2}$, and the spaces of morphisms between which are the complexes

$$\operatorname{Hom}^{\cdot}(v_j, v_i) \cong \operatorname{Ext}^{\cdot}_B(S_i, S_j)$$

with the natural A_∞-structure induced by that of the A_∞-algebra $\operatorname{Ext}^{\cdot}_B(S,S)$. It follows from the definition of the A_∞-algebra $\operatorname{Ext}^{\cdot}_B(S,S)$ that

$$\operatorname{Hom}^{\cdot}(v_j, v_i) = 0 \quad \text{when } j < i.$$

We now come to the main point. Define the A_∞-algebra $C^{\cdot}$ as the total morphism algebra of the A_∞-category $\mathcal{C}$, i.e.

$$C^{\cdot} = \bigoplus_{0\le i,j\le k+2} \operatorname{Hom}^{\cdot}(v_j, v_i) \cong \operatorname{Ext}^{\cdot}_B(S,S).$$

It follows from a general result of Keller [53, Theorem 3.1], that the triangulated subcategory of $\mathbf{D}^b(\text{mod–}B)$ generated by the S_i is equivalent to the derived category $\mathbf{D}^b(C^{\cdot})$. Since S_i, $i = 0,\dots,k+2$ generate the derived category $\mathbf{D}^b(\text{mod–}B)$, we obtain an equivalence between $\mathbf{D}^b(\text{mod–}B)$ and $\mathbf{D}^b(C^{\cdot})$. Furthermore, by [52, Proposition 1], $C^{\cdot}$ is generated, as an A_∞-algebra, by C^0 and C^1. This fact allows us to say (somewhat improperly) that the A_∞-algebra $C^{\cdot}$ is the Koszul dual to the algebra B.

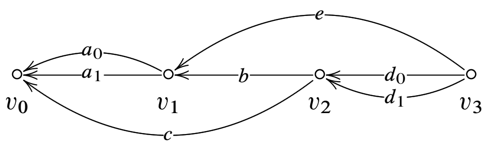
Combining our previous remarks in this section we have the following.

Proposition 3.3.1. *The derived category of coherent sheaves* $\mathbf{D}^b(\operatorname{Coh}(X))$ *on a toric Del Pezzo surface* X *is equivalent to the derived category* $\mathbf{D}^b(C^{\cdot})$.

This illustrates a basic point: the structure of the derived category $\mathbf{D}^b(\operatorname{Coh}(X))$ is controlled by the relatively simple A_∞-algebra $C^{\cdot}$. It may be noted at this point that the knowledge of $C^{\cdot}$ is sufficient for all the calculations we will carry out.

3.3.3 **An example: $\mathbb{P}^2$ blown up at one point.** Let us pause here, and illustrate how the general formalism of the previous subsection applies in the concrete case of $\mathbb{P}^2$ with one point blown up. We presume the notation of Section 3.2.

We have already noted that a complete strongly exceptional collection is $(\mathcal{O}, \mathcal{O}_X(h - l_1), \mathcal{O}_X(h), \mathcal{O}_X(2h - l_1))$ where h is the hyperplane divisor. The quiver then takes the form



with relations $r_0 = a_1 b d_0 - a_0 b d_1 = 0$, $r_1 = a_0 e - c d_0 = 0$, and $r_2 = c d_1 - a_1 e = 0$. Denote as usual by S_i the simple module corresponding to the vertex i and by P_i the corresponding projective module, for $i = 0, 1, 2, 3$. We have projective resolutions

$$0 \longrightarrow P_0 \longrightarrow S_0 \longrightarrow 0,$$

$$0 \longrightarrow P_0^{\oplus 2} \xrightarrow{(a_0\ a_1)} P_1 \longrightarrow S_1 \longrightarrow 0,$$

$$0 \longrightarrow P_0 \oplus P_1 \xrightarrow{(c\ b)} P_1 \longrightarrow S_1 \longrightarrow 0,$$

$$0 \longrightarrow P_0^{\oplus 3} \xrightarrow{\begin{pmatrix} 0 & a_0 & -a_1 \\ -a_0 b & 0 & c \\ a_1 b & -c & 0 \end{pmatrix}} P_1 \oplus P_2^{\oplus 2} \xrightarrow{(e\ d_0\ d_1)} P_3 \longrightarrow S_3 \longrightarrow 0.$$

We start by choosing specific generators of the Ext^i. Recall that the $\mathrm{Ext}^\cdot$ can be represented as morphisms between resolutions of the S_i. Define $\underline{a}_0$ and $\underline{a}_1$ to be the following generators of $\mathrm{Ext}^1_B(S_1, S_0)$:

$$\underline{a}_0 = \begin{array}{ccc} P_0^{\oplus 2} & \longrightarrow & P_1, \\ \big\downarrow {\scriptstyle (0\ 1)} & & \\ P_0 & & \end{array}$$

$$\underline{a}_1 = \begin{array}{ccc} P_0^{\oplus 2} & \longrightarrow & P_1. \\ \big\downarrow {\scriptstyle (1\ 0)} & & \\ P_0 & & \end{array}$$

Next, the three generators $\underline{b}$, $\underline{c}$ and $\underline{e}$ of $\mathrm{Ext}^1_B(S_2, S_1)$, $\mathrm{Ext}^1_B(S_2, S_0)$ and $\mathrm{Ext}^1_B(S_3, S_1)$, respectively, can be represented by

$$\underline{b} = \begin{array}{ccccc} & & P_0 \oplus P_1 & \longrightarrow & P_2, \\ & & \big\downarrow {\scriptstyle (0\ 1)} & & \\ P_0^{\oplus 2} & \longrightarrow & P_1 & & \end{array}$$

$$\underline{c} = \begin{array}{ccc} P_0 \oplus P_1 & \longrightarrow & P_2. \\ \big\downarrow {\scriptstyle (1\ 0)} & & \\ P_0 & & \end{array}$$

$$\underline{e} = \begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \big\downarrow {\scriptstyle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}} & & \big\downarrow {\scriptstyle (1\ 0\ 0)} & & \\ P_0^{\oplus 2} & \longrightarrow & P_1 & & \end{array}$$

As far as the generators $\underline{d}_0$ and $\underline{d}_1$ of $\mathrm{Ext}^1_B(S_3, S_2)$ we take

$$\underline{d}_0 = \begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \big\downarrow {\scriptstyle \begin{pmatrix} 0 & 0 & 1 \\ -a_0 & 0 & 0 \end{pmatrix}} & & \big\downarrow {\scriptstyle (0\ 1\ 0)} & & \\ P_0 \oplus P_1 & \longrightarrow & P_2 & & \end{array}$$

$$\underline{d}_1 = \begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \big\downarrow {\scriptstyle \begin{pmatrix} 0 & -1 & 0 \\ a_1 & 0 & 0 \end{pmatrix}} & & \big\downarrow {\scriptstyle (0\ 0\ 1)} & & \\ P_0 \oplus P_1 & \longrightarrow & P_2 & & \end{array}$$

Finally, we have the relations $\underline{r}_0$, $\underline{r}_1$ and $\underline{r}_2$ in $\mathrm{Ext}^2_B(S_3, S_0)$ represented by

$$\underline{r}_0 = \begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \big\downarrow {\scriptstyle (1\ 0\ 0)} & & & & \\ P_0 & & & & \end{array}$$

$$\underline{r}_1 = \begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \big\downarrow {\scriptstyle (0\ 1\ 0)} & & & & \\ P_0 & & & & \end{array}$$

$$\underline{r}_2 = \begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \big\downarrow {\scriptstyle (0\ 0\ 1)} & & & & \\ P_0 & & & & \end{array}$$

We want to compute all higher products m_k of the various Ext^1's. By the definitions of $\underline{a}_i$, $\underline{b}$, $\underline{c}$, $\underline{d}_i$ and $\underline{e}$, we infer that the possible nonzero products of $C^\cdot$ are $m_2(\underline{a}_i, \underline{e})$,

$m_2(\underline{b},\underline{d}_i)$ and $m_2(\underline{c},\underline{d}_i)$. We find, by direct computation

$$\begin{array}{ll} m_2(\underline{a}_1,\underline{e}) = \underline{r}_1, & m_2(\underline{c},\underline{d}_0) = \underline{r}_2, \\ m_2(\underline{c},\underline{d}_1) = -\underline{r}_1, & m_2(\underline{a}_0,\underline{e}) = -\underline{r}_2. \end{array}$$

On the other hand consider $m_2(\underline{b},\underline{d}_0)$. The composition $\mu(\underline{b},\underline{d}_0)$ gives a map

$$\begin{array}{ccccccc} & & P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3. \\ & & \Big\downarrow{\scriptstyle (-a_0\ 0\ 0)} & & & & \\ P_0^{\oplus 2} & \longrightarrow & P_1 & & & & \end{array}$$

We observe that $\mu(\underline{b},\underline{d}_0)$ is exact, given by d applied to

$$\begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3. \\ \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle 0} & & \\ P_0^{\oplus 2} & \longrightarrow & P_1 & & \end{array} \tag{3.3.2}$$

It therefore follows that $m_2(\underline{b},\underline{d}_0) = 0$ and $G\mu(\underline{b},\underline{d}_0)$ is given by minus (3.3.2).

Now compose this with $\underline{a}_1$ to form $\mu(\underline{a}_1, G\mu(\underline{b},\underline{d}_0))$ given by

$$\begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3, \\ \Big\downarrow{\scriptstyle (1\ 0\ 0)} & & & & \\ P_0 & & & & \end{array}$$

Bearing in mind that $\mu(\underline{a}_1,\underline{b}) = 0$, and hence $\mu(G\mu(\underline{a}_1,\underline{b}),\underline{d}_0) = 0$, we compute

$$m_3(\underline{a}_1,\underline{b},\underline{d}_0) = P\mu(G\mu(\underline{a}_1,\underline{b}),\underline{d}_0) + P\mu(\underline{a}_1, G\mu(\underline{b},\underline{d}_0)) = \underline{r}_0.$$

Now do the same with $m_2(\underline{b},\underline{d}_1)$. We compute $\mu(\underline{b},\underline{d}_1)$ to be

$$\begin{array}{ccccccc} & & P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3. \\ & & \Big\downarrow{\scriptstyle (a_1\ 0\ 0)} & & & & \\ P_0^{\oplus 2} & \longrightarrow & P_1 & & & & \end{array}$$

We point out again that $\mu(\underline{b},\underline{d}_1)$ is exact, given by d applied to

$$\begin{array}{ccccc} P_0^{\oplus 3} & \longrightarrow & P_1 \oplus P_2^{\oplus 2} & \longrightarrow & P_3. \\ \Big\downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)} & & \Big\downarrow{\scriptstyle 0} & & \\ P_0^{\oplus 2} & \longrightarrow & P_1 & & \end{array} \tag{3.3.3}$$

The conclusion is that $m_2(\underline{b},\underline{d}_1) = 0$ and $G\mu(\underline{b},\underline{d}_1)$ is given by minus (3.3.3). We find as before

$$m_3(\underline{a}_0,\underline{b},\underline{d}_1) = -\underline{r}_0.$$

These are all the nonzero higher order products in this example.

In summary we have for the higher products:

$$\begin{array}{lll} m_3(\underline{a}_1,\underline{b},\underline{d}_0) = \underline{r}_0, & m_2(\underline{a}_1,\underline{e}) = \underline{r}_1, & m_2(\underline{c},\underline{d}_0) = \underline{r}_2, \\ m_3(\underline{a}_0,\underline{b},\underline{d}_1) = -\underline{r}_0, & m_2(\underline{c},\underline{d}_1) = -\underline{r}_1, & m_2(\underline{a}_0,\underline{e}) = -\underline{r}_2. \end{array}$$

Thus essentially one knows these products as soon as one knows all the relations in the quiver.

3.4 The mirror Landau-Ginzburg models

In this section study the Landau-Ginzburg models mirror to toric Del Pezzo surfaces. We start by elucidating the statement of Homological Mirror Symmetry for Fano varieties and making it precise. Next we introduce the category of Lagrangian vanishing cycles associated to a symplectic Lefschetz fibration, and outline the main steps involved in its determination. Finally we perform a check of the Homological Mirror Symmetry conjecture in the case of $\mathbb{P}^2$ with one point blown up. It is hoped that this will give the reader a feel for how Homological Mirror Symmetry for toric surfaces actually works. The entire section follows [4] rather closely.

3.4.1 **Categories behind mirror symmetry.** Mirror symmetry is a many-facet correspondence between symplectic and complex geometry developed to understand formulas for the number of rational curves on some 3-dimensional Calabi-Yau manifolds discovered by physicists. These formulas involve Hodge theory on *mirror dual* Calabi-Yau manifolds. At present we do not have a precise definition of the notion of a mirror dual pair that would encompass all known examples of such pairs.

In an insightful paper [57], M. Kontsevich formulated a conjecture which relates the properties of a Calabi-Yau with those of its mirror and suggested that it captures the essence of mirror symmetry. He observed that to any Calabi-Yau manifold X, one can associate two triangulated categories: the bounded derived category of coherent sheaves $\mathbf{D}^b(\mathrm{Coh}(X))$ and the so-called derived Fukaya category of X, denoted by $\mathbf{D}^b(\mathrm{Fuk}(X))$. The Homological Mirror Symmetry conjecture asserts that two Calabi-Yau manifolds X and X' are mirror if and only if $\mathbf{D}^b(\mathrm{Coh}(X))$ is equivalent to $\mathbf{D}^b(\mathrm{Fuk}(X'))$, and vice-versa.

The Homological Mirror Symmetry conjecture can be reinterpreted in physical terms. Recall that topological strings on a Calabi-Yau manifold X come in two flavours. The A-model and the corresponding A-branes depending only on the symplectic structure on X, while the B-model and B-branes depend only on the complex structure. According to Witten [95], holomorphic vector bundles are examples of B-branes, and spaces of morphisms between vector bundles are global Ext groups. Further, Witten showed that examples of A-branes are provided by Lagrangian submanifolds equipped with vector bundles with flat connections, and spaces of morphisms between them are Floer homology groups. Now, Ext groups are spaces of morphisms in the derived category of coherent sheaves, therefore it is natural to conjecture that arbitrary complexes of coherent sheaves are also examples of

B-branes, and morphisms between them are morphisms in the derived category. Similarly it is reasonable to assume that arbitrary "complexes" of Lagrangian submanifolds with flat vector bundles are examples of A-branes. The Homological Mirror Symmetry conjecture is basically the statement that *all* topological B-branes and A-branes arise in this way. In other words, the Homological Mirror Symmetry conjecture would follow if we could prove that the category of A-branes (respectively B-branes) is equivalent to $\mathbf{D}^b(\mathrm{Fuk}(X))$ (respectively $\mathbf{D}^b(\mathrm{Coh}(X))$).

One can also consider more general topologically twisted $N = 2$ field theories and the corresponding D-branes. One class of such theories is given by sigma models whose target is a Fano variety. Another set of examples is provided by $N = 2$ Landau-Ginzburg models. In many cases these two classes of $N = 2$ theories are related by mirror symmetry [45]. In particular, the Homological Mirror Symmetry conjecture remains meaningful and non-trivial in this case. That is to say, B-branes on a Fano variety are described by the derived category of coherent sheaves, and under mirror symmetry they correspond to the A-branes of a mirror Landau-Ginzburg model. These A-branes are described by a suitable analogue of the Fukaya category, namely the derived category of Lagrangian vanishing cycles. A rigorous definition of this category has been proposed by P. Seidel [82] in the case where the critical points of the superpotential are isolated and non-degenerate.

In more concrete terms, for a Fano variety X and a mirror Landau-Ginzburg model $W : M \to \mathbb{C}$, the Homological Mirror Symmetry conjecture can be formulated as follows (see Sect. 3.4.2 below for details).

Conjecture 3.4.1. The derived category of Lagrangian vanishing cycles $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$ is equivalent to the derived category of coherent sheaves $\mathbf{D}^b(\mathrm{Coh}(X))$.

It should be noted that Homological Mirror Symmetry also predicts another equivalence of categories. Namely, viewing now X as a symplectic manifold and M as a complex manifold, the derived category of B-branes of the Landau-Ginzburg model $W : M \to \mathbb{C}$, which is defined as the product $\prod_{\lambda\in\mathbb{C}} \mathbf{D}_{\mathrm{Sg}}(W^{-1}(\lambda))$, should be equivalent to the derived Fukaya category of X. We will not dwell on this aspect of mirror symmetry here.

We close with a remark about the mirror construction in the context of smooth complete toric varieties. Recall that a smooth complete toric variety X is given by a complete regular fan Σ in $\mathbb{R}^n$. The mirror of such a variety is expected to be a Laurent polynomial W on $(\mathbb{C}^\times)^n$ which can be explicitly obtained from the fan Σ as follows. Consider the one-dimensional cones of Σ. These are rays starting at the origin which have rational slope, so they intersect the lattice at a unique primitive vector. Let A be the set which consists of the primitive vertices of these one-dimensional cones. The mirror of X is the family of Laurent polynomials

$$W = \sum_{\alpha\in A} c_\alpha z^\alpha,$$

where $z^\alpha = z_1^{a_1} \cdots z_n^{a_n}$ whenever $\alpha = (a_1, \ldots, a_n)$. From our point of view the specific coefficients c_α are irrelevant as long as they are chosen generically.

3.4.2 **The category of vanishing cycles.** This preliminary subsection will outline the definition of a Fukaya-type A_∞-category associated to a symplectic Lefschetz fibration. For a more extensive discussion we refer to [83].

Let (M,ω) be an open symplectic manifold, and let $W : (M,\omega) \to \mathbb{C}$ be a symplectic Lefschetz fibration, i.e. an smooth complex-valued function with isolated non-degenerate critical points $p_0,\dots,p_r$ near which W is given in local complex coordinates by $W(z_1,\dots,z_n) = W(p_i) + z_1^2 + \cdots + z_n^2$, where the fibers of W are symplectic submanifolds of M. Assume for simplicity that the critical values $\lambda_0,\dots,\lambda_r$ of W are distinct. Fix a regular value λ_* of W, and consider a collection of arcs $\gamma_0,\dots,\gamma_r \subset \mathbb{C}$ joining λ_* to the critical values λ_i of W, intersecting each other only at λ_*, and ordered in the clockwise direction around λ_*. Using the horizontal distribution defined by the symplectic form, we can define a parallel transport along the arc γ_i to obtain a Lagrangian thimble D_i and a vanishing cycle $L_i = \partial D_i$ in the fiber $\Sigma_* = W^{-1}(\lambda_*)$. After a small perturbation we can always assume that the vanishing cycles L_i intersect each other transversely inside Σ_*.

The directed category of vanishing cycles $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ is an A_∞-category with objects $L_0,\dots,L_r$ corresponding to the vanishing cycles; the space of morphisms between two such objects is defined as follows

$$\mathrm{Hom}^\cdot(L_i,L_j) = \begin{cases} CF^\cdot(L_i,L_j) = \mathbb{C}^{|L_i \cap L_j|} & \text{if } i < j \\ \mathbb{C}\cdot \mathrm{id} & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases}$$

This space is $\mathbb{Z}$-graded by the Maslov index of the intersections. The differential m_1, composition m_2 and higher order products m_k are defined in terms of Lagrangian Floer homology inside Σ_*.

For completeness we give the general expression for m_k, though we emphasize that for most purposes the details are not necessary. For $i_0 < \cdots < i_k$, we choose points $p_{i_\ell} \in L_{i_\ell} \cap L_{i_{\ell+1}}$ ($\ell = 0,\dots,k$, where $i_{k+1} = i_0$) and define $\mathcal{M}(p_{i_0},\dots,p_{i_k})$ to be the moduli space of all pseudo-holomorphic maps u from a unit disk D^2 with $k+1$ cyclic marked points $z_\ell \in \partial D^2$ to Σ_* (equipped with a generic ω-compatible almost complex structure), such that $u(z_\ell) = p_{i_\ell}$ and the part of the boundary between z_ℓ and $z_{\ell+1}$ is sent to $L_{i_{\ell+1}}$. In the present context, this moduli space has a natural compactification which is a manifold with corners. We denote $\mathcal{M}(p_{i_0},\dots,p_{i_k})$ by $\mathcal{M}_0(p_{i_0},\dots,p_{i_k})$ if it has dimension zero, otherwise $\mathcal{M}_0(p_{i_0},\dots,p_{i_k})$ will be the empty set. Using the above notation one defines

$$m_k : \mathrm{Hom}^\cdot(L_{i_0},L_{i_1}) \otimes \cdots \otimes \mathrm{Hom}^\cdot(L_{i_{k-1}},L_{i_k}) \longrightarrow \mathrm{Hom}^\cdot(L_{i_0},L_{i_k})[2-k]$$

via

$$m_k(p_{i_0},\dots,p_{i_{k-1}}) = \sum_{r \in L_{i_0} \cap L_{i_k}} \left(\sum_{u \in \mathcal{M}_0(p_{i_0},\dots,p_{i_{k-1}},r)} \pm \exp\left(-\int_{D^2} u^*\omega \right) \right) r.$$

The reason for the sign ambiguity is that one has to make a choice concerning the orientation of the moduli spaces $\mathcal{M}_0(p_{i_0},\dots,p_{i_{k-1}},r)$.

One should note that a priori the category $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ depends on the chosen ordered collection of arcs $\{\gamma_i\}$. However, Seidel has obtained the following result, see [82].

Theorem 3.4.2. *If the ordered collection* $\{\gamma_i\}$ *is replaced by another one* $\{\gamma_i'\}$, *then the categories* $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i\})$ *and* $\mathrm{Lag}_{\mathrm{vc}}(W,\{\gamma_i'\})$ *differ by a sequence of mutations.*

Consequently, the category naturally associated to the fibration W is not the finite A_∞-category defined above, but rather its derived category as defined in Sect. 3.3.2. If two categories differ by mutations, then their derived categories are equivalent; hence the derived category $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$ depends only on the symplectic topology of W and not on the choice of an ordered system of arcs [82].

We will restrict ourselves to the case where Σ_* is an affine elliptic curve and the vanishing cycles are homotopically non-trivial closed loops. In this context, the pseudo-holomorphic disks in Σ_* that we have to consider are nothing but immersed polygonal regions bounded by the vanishing cycles, satisfying a local convexity condition at each corner point.

Also, the Maslov class vanishes identically, so we have a well-defined $\mathbb{Z}$-grading by Maslov index on the Floer complexes $CF^{\cdot}(L_i,L_j)$ once we choose graded Lagrangian lifts of the vanishing cycles. We can do this by fixing a holomorphic volume form Ω on Σ_* and choosing a real lift of the phase function $\phi_i = \arg(\Omega|_{L_i}) : L_i \to S^1$ for each vanishing cycle. The degree of a given intersection point $p \in L_i \cap L_j$ is then determined by the difference between the phases of L_i and L_j at p. In Sect. 3.4.3 below we will make concrete computations on the Landau-Ginzburg model mirror to $\mathbb{P}^2$ blown up at one point.

Finally, the orientation on the moduli spaces is determined by a choice of a spin structure for each vanishing cycle L_i; in our case this spin structure must extend to the thimble, so it is necessarily the non-trivial one. Again we will see this in detail in the concrete calculations of Sect. 3.4.3.

3.4.3 **Homological Mirror Symmetry by example: $\mathbb{P}^2$ blown up at one point.** Throughout this subsection X denotes a Del Pezzo surface obtained by blowing up $\mathbb{P}^2$ at one point. Our aim is to test the Homological Mirror Symmetry conjecture by comparing the derived category of coherent sheaves on X with the derived category of Lagrangian vanishing cycles on the mirror Landau-Ginzburg model.

Let X be given as a toric variety by the fan with one-dimensional cones generated by $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (-1,0)$ and $v_4 = (-1,-1)$. See Sect. 3.3.1, Figure 3.1. As discussed above, the mirror Landau-Ginzburg model consists of $M = (\mathbb{C}^\times)^2$ equipped with a superpotential of the form

$$W = x + y + \frac{a}{x} + \frac{b}{xy}$$

for some non-zero constants a,b. In addition, we endow $M = (\mathbb{C}^\times)^2$ with the symplectic form

$$\omega = \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}\overline{x}}{\overline{x}} + \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}\overline{y}}{\overline{y}}.$$

Since different values of the constants a,b lead to mutually isotopic symplectic Lefschetz fibrations, the actual choices do not matter. To fix ideas, we choose $a=-1, b=1$.

Let $(\lambda_i)_{0\le i\le 3}$ be the four critical values of W, ordered clockwise around the origin so that $\mathrm{Im}(\lambda_0) > 0$, $\mathrm{Im}(\lambda_1) > 0$, $\mathrm{Im}(\lambda_2) < 0$, and $\mathrm{Im}(\lambda_3) < 0$. We choose $\Sigma_0 = W^{-1}(0)$ as

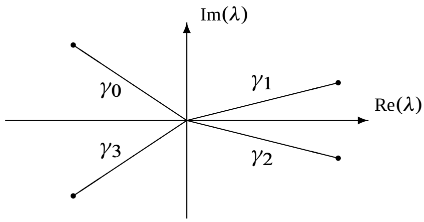


Figure 3.2: Ordered collection of arcs $\{\gamma_i\}$

our reference fiber, and consider the ordered system of arcs $(\gamma_i)_{0\leq i\leq 3}$, as drawn in Figure 3.2, i.e. each $\gamma_i \subset \mathbb{C}$ is a straight line segment joining the origin to λ_i. In order to determine the vanishing cycles of W, we need to understand how the fiber $\Sigma_\lambda = W^{-1}(\lambda)$ degenerates as λ approaches a critical value of W. This is done by considering the projection π_x to the x variable, which realizes Σ_λ as a double cover of $\mathbb{C}^\times$ branched at four points. The fiber Σ_λ becomes singular when two branch points of $\pi_x : \Sigma_\lambda \to \mathbb{C}^\times$ merge with each other, giving rise to a nodal point. The manner in which two of the branch points approach each other as one moves from $\lambda = 0$ to $\lambda = \lambda_j$ along the arc γ_j defines an arc $\delta_j \subset \mathbb{C}^\times$ as pictured in Figure 3.3.

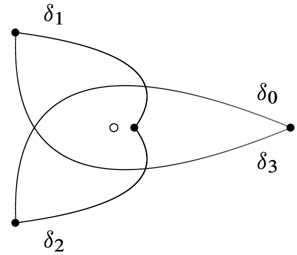


Figure 3.3: The projection of the vanishing cycles of W

The preceding discussion give us a topological description of the vanishing cycles, up to homotopy. Namely, the symplectic vanishing cycle L_i, obtained by parallel transport using the symplectic connection, is homotopic to a loop $L_i' \subset \Sigma_0$, obtained as a double lift via $\pi_x : \Sigma_0 \to \mathbb{C}^\times$ of the arc $\delta_i \subset \mathbb{C}^\times$. This loop is a topological vanishing cycle; i.e., it shrinks to a point in Σ_λ when the value of λ tends to the critical value λ_i.

The next thing to notice is that the vanishing cycle L_i is invariant by complex conjugation; i.e. complex conjugation maps L_i to itself in an orientation-preserving manner, and the same is true of L_i'. This comes about because the symplectic form ω is anti-invariant by complex conjugation. Since L_i and L_i' are homotopic to each other in Σ_0, their invariance under complex conjugation is sufficient to imply that they are Hamiltonian isotopic, which means that for the purpose of determining categories of vanishing cycles, L_i and L_i' are interchangeable. In the sequel, we implicitly identify L_i with L_i'.

We make one further general observation. It is possible to compactify Σ_0 into a smooth elliptic curve $\overline{\Sigma}_0$ by adding four points. Hence, equipping $\overline{\Sigma}_0$ with a compatible flat metric, we can identify $\overline{\Sigma}_0$ with the quotient of $\mathbb{C}$ by a lattice, and represent the vanishing cycles L_i by closed geodesics parallel to those represented in Figure 3.4. (The open circles denote the points of $\overline{\Sigma}_0$ which are missing.) Here we use the fact that all homotopically nontrivial simple closed curves on the flat torus are Hamiltonian isotopic to geodesics.

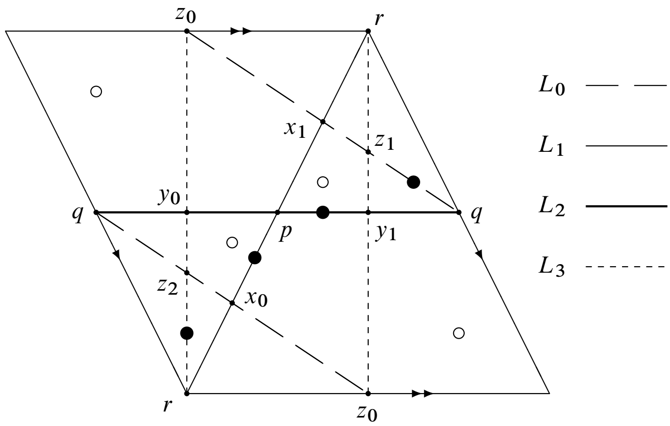


Figure 3.4: The vanishing cycles of W

The next aim is to determine the Maslov indices of the various intersection points, by choosing graded Lagrangian lifts of the vanishing cycles. We denote by x_0, x_1 (respectively y_0, y_1 and z_0, z_1, z_2) the generators of $\mathrm{Hom}^\cdot(L_2, L_3)$ (respectively $\mathrm{Hom}^\cdot(L_0, L_1)$ and $\mathrm{Hom}^\cdot(L_0, L_3)$) corresponding to the intersection points represented in Figure 3.4. Moreover, we denote by p (respectively q and r) the generators of $\mathrm{Hom}^\cdot(L_1, L_2)$ (respectively $\mathrm{Hom}^\cdot(L_1, L_3)$ and $\mathrm{Hom}^\cdot(L_0, L_2)$) corresponding to the intersection points between these vanishing cycles. We have the following.

Lemma 3.4.3. *There exists a natural choice of gradings for which* $\deg(x_i) = \deg(y_i) = \deg(p) = \deg(q) = \deg(r) = 1$ *and* $\deg(z_i) = 2$.

Proof. Following our remarks above, equip $\overline{\Sigma}_0$ with a constant holomorphic 1-form $\Omega = c\,dz$, where z is a global complex coordinate and $c > 0$ is a positive real number. Then the phase functions $\phi_i = \arg(\Omega|_{L_i}) : L_i \to \mathbb{R}/2\pi\mathbb{Z}$ are constant equal to the angle giving the direction of the tangent vectors to the L_i defining the orientation of L_i. Thus, to choose graded Lagrangian lifts for our vanishing cycles L_i we simply need to choose real lifts $\widetilde{\phi}_i \in \mathbb{R}$ of the angles. In Figure 3.4 we see that it is possible to choose $\widetilde{\phi}_i \in \mathbb{R}$ in such a way that $0 < \widetilde{\phi}_0 < \pi/2 < \widetilde{\phi}_1 < \widetilde{\phi}_2 = \pi$, $\widetilde{\phi}_3 = 3\pi/2$. Now, in the 1-dimensional case, the relationship between Maslov index and phase is very simple, as follows from [81, Lemma 6.8]: given a transverse intersection point p between two graded Lagrangians $L, L' \subset \Sigma_0$, the Maslov index of $p \in CF^\cdot(L, L')$ is equal to the smallest integer greater that $\frac{1}{\pi}(\widetilde{\phi}_{L'}(p) - \widetilde{\phi}_L(p))$. Using this remark, the result follows immediately. □

The next step is to study the Floer differentials and products in $\mathrm{Lag}(W, \{\gamma_i\})$ by counting pseudo-holomorphic maps from $(D^2, \partial D^2)$ to $(\Sigma_0, \cup_i L_i)$. As explained in Sect. 3.4.2, each pseudo-holomorphic map $u : (D^2, \partial D^2) \to (\Sigma_0, \cup_i L_i)$ is counted with a coefficient of the form $\pm \exp(-\int_{D^2} u^*\omega)$. In the present case, the symplectic form ω is exact ($\omega = d\theta$ for some 1-form θ) and the L_i are exact Lagrangian submanifolds in Σ_0 (i.e. $\theta|_{L_i} = dg_i$ is also exact). Thus, the symplectic area can be expressed in terms of the primitives g_i

of θ over L_i, and can be eliminated from the description simply by rescaling the chosen bases of the Floer complexes.

In order to identify the signs, one needs to orient the relevant moduli spaces of pseudo-holomorphic discs in some consistent way, which requires the choice of a spin structure over each Lagrangian L_i. As observed at the end of Sect. 3.4.2, we need to endow each L_i with the spin structure which extends to the corresponding thimble, i.e. the non-trivial one.

We now describe a convenient recipe for determining the correct signs in the one-dimensional case, due to Seidel [81]. We start with the case of trivial spin structures. Then to each intersection point $p \in L_i \cap L_j$ $(i < j)$ one can associate an orientation line $\sigma(p)$. This orientation line is canonically trivial when $\deg(p)$ is even, whereas in the odd degree case, a choice of trivialization of $\sigma(p)$ is equivalent to a choice of orientation of the line T_pL_j. If one considers a pseudo-holomorphic map $u : D^2 \to \Sigma_0$ contributing to m_k, whose image is a polygonal region with $k+1$ vertices $p_0, \dots, p_k$, then the corresponding sign factor is actually an element of the tensor product $\Lambda = \sigma(p_0) \otimes \cdots \otimes \sigma(p_k)$. We can define a preferred trivialization of Λ by choosing, at each vertex of odd degree, the orientation of the vanishing cycle which agrees with the positive orientation on the boundary of the image of u. The sign factor associated to u is then equal to $+1$ with respect to this trivialization of Λ (or -1 with respect to the other trivialization). In the presence of non-trivial spin structures, this rule needs to be modified as follows: fix a marked point on each L_i carrying a non-trivial spin structure (distinct from its intersection points with the other vanishing cycles); then the sign associated to u is affected by a factor of -1 for each marked point that the boundary of u passes through.

We have now at our disposal all the information necessary to prove the following.

Lemma 3.4.4. *Except for those involving identity morphisms, the possible nonzero compositions and higher products of* $\mathrm{Lag}(W, \{\gamma_i\})$ *are*

$$m_3(x_0, p, y_1) = z_0, \quad m_2(x_1, r) = z_1, \quad m_2(q, y_0) = z_2,$$
$$m_3(x_1, p, y_0) = -z_0, \quad m_2(q, y_1) = -z_1, \quad m_2(x_0, r) = -z_2.$$

Moreover, the Floer differential is trivial, i.e. $m_1 = 0$.

Proof. To start with, it is immediate from an observation of Figure 3.4 that the only contribution to the product $m_3(x_0, p, y_1)$ comes from the immersed polygonal region T with vertices x_0, p, y_1, z_0. We choose trivializations of the orientation lines as follows: for every point $x \in L_i \cap L_j$ of degree 1 (i.e., one of x_0, p, y_1), we orient T_xL_j consistently with the boundary orientation of T. If we consider trivial spin structures, then with this convention the sign factor associated to this polygonal region is by definition equal to $+1$. Next we consider the product $m_3(x_1, p, y_0)$. Once again the only contribution comes from the immersed polygonal region T' with vertices x_1, p, y_0, z_0. In this case, at each of the three vertices x_1, p, y_0 of degree 1 the chosen trivialization of T_xL_j disagrees with the boundary orientation of T', so that for trivial spin structures we get a sign factor of $(-1)^3 = -1$. Since we need to consider nontrivial spin structures, we must introduce a marked point on each L_i; we choose this marked points as depicted in Figure 3.4. With this choice, the boundary of T passes through precisely two marked points while the boundary of T' does not meet any marked point. Therefore, with this conventions, we have $m_3(x_0, p, y_1) = z_0$ and $m_3(x_1, p, y_0) = -z_0$.

By a similar argument, we can study the compositions $m_2(x_i, r)$ and $m_2(q, y_i)$ by looking at the triangular regions delimited by the vanishing cycles in Σ_0. With the above choice, we can easily show that $m_2(x_1, r) = z_1$, $m_2(q, y_1) = -r_1$, $m_2(q, y_0) = z_2$, and $m_2(x_0, r) = -z_2$.

Finally, we observe that there cannot be any contribution to the Floer differential m_1, since Σ_0 contains no nonconstant immersed disc with boundary in $L_i \cup L_j$. □

We can now draw the conclusion at which we have been aiming.

Theorem 3.4.5. *The derived category of Lagrangian vanishing cycles* $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$ *is equivalent to the derived category of coherent sheaves of the Del Pezzo surface* X.

Proof. We first recall, from the discussion preceding Proposition 3.3.1, that the derived category of Lagrangian vanishing cycles $\mathbf{D}^b(\mathrm{Lag}_{\mathrm{vc}}(W))$ admits a full strong exceptional collection (L_0, L_1, L_2, L_3). Thanks to Proposition 3.3.1, to see that we have an equivalence of categories as stated, we need only verify that $\mathrm{Lag}_{\mathrm{vc}}(W, \{\gamma_i\})$ is equivalent to the A_∞-category $\mathcal{C}$ considered in Sect. 3.3.2. This is a direct consequence of Lemmas 3.4.3 and 3.4.4. □

SAMENVATTING

In het nabije verleden heeft snaartheorie nieuwe inzichten in meetkunde opgeleverd. Oorspronkelijk is snaartheorie ontworpen als poging om de sterke kernkracht te beschrijven, maar geleidelijk is ze veranderd in een onderzoeksdiscipline die een sterke wisselwerking vertoont met verschillende andere takken van de fysica en wiskunde. Hedentendage omvat ze een enorm gebied variërend van 'quantum zwaartekracht' tot 'niet-commutatieve meetkunde'. De aanhalingstekens moeten de lezer eraan herinneren dat deze begrippen, ondanks de enorme moeite die gedaan is in de afgelopen decennia, nog steeds in een wat premature fase verkeren. Onder de verschillende motivaties om onderzoek te doen naar snaartheorie is er één die erg bekoorlijk is: snaartheorie zou de ingrediënten kunnen bevatten voor het antwoord op de vraag

"Wat is de aard van ruimtetijd?"

Het feit dat snaartheorie klassieke meetkundige begrippen kan generaliseren heeft zijn oorsprong in de definitie van perturbatieve snaartheorie: zij wordt bestudeerd als (een verzameling van) tweedimensionale veldentheorieën die, om consistentie te garanderen, conform zijn. Men zegt dat conforme veldentheorieën snaarvacua definiëren. In dit raamwerk leveren ruimtetijden die voldoen aan de Einstein vergelijkingen specifieke voorbeelden van conforme veldentheorieën, maar naast deze bestaat er nog een enorme verzameling van conforme veldentheorieën. Derhalve kan de verzameling klassieke ruimtetijden worden uitgebreid met snaarvacua.

Open snaren worden in verband gebracht met objecten, die verschillen van snaren en bekend staan onder de naam 'D-branen'. Een D-braan is ruwweg een 'mooie' randvoorwaarde voor een conforme veldentheorie. We plaatsen onszelf in de context van topologische snaartheorie, waar D-branen kunnen worden geïnterpreteerd als objecten in een getrianguleerde categorie. Door dit categorietheoretische raamwerk is het mogelijk het begrip D-branen uit te breiden van klassieke meetkundes naar snaarvacua, waar de betekenis van D-branen a priori niet duidelijk was. Wij denken over deze objecten als complexen van schoven op een 'niet-commutatieve ruimte'.

Het terugkerende thema in dit proefschrift is de constructie van equivalenties van D-braan categorieën. De equivalente manieren om een categorie te beschrijven corresponderen met equivalente (duale) maar verschillende wijzen om te kijken naar de fysica van deze D-branen. Ze geven ook een concrete realisatie van wat het voor een snaar zou betekenen om te propageren in een niet-commutatieve ruimte.

ACKNOWLEDGEMENTS

The particular choice of topic for this PhD thesis proved to be very rewarding as it allowed me to explore many unrelated areas of mathematics and physics that are of great interest to me. Thus, I would like first to express my profound gratitude to my supervisor, Jan Stienstra, for encouraging me to pursue this topic and for providing me with very friendly and thoughtful guidance when it was needed. I also wish to express my appreciation to Professor Hans Duistermaat for keeping a keen eye on the logistics and planning of the thesis as well as for some essential conversations.

Next, I would like to thank the members of the reading committee, Alastair Craw, Bernhard Keller, Alastair King, Eduard Looijenga and Ieke Moerdijk for the time they have invested in reading the first draft of this thesis. In particular, I wish to thank Alastair Craw and Alastair King for many useful comments that helped me to correct the content and improve the exposition of the manuscript.

Conversations with many mathematicians and physicists have provided many insights and new motivations during my years as a graduate student. It is a great pleasure to mention here Alastair Craw, Manfred Herbst, Kentaro Hori, Paul Horja, Alastair King, Calin Lazaroiu, Eduard Looijenga, Eric Sharpe, Dirk Siersma and Dmitri Orlov. I am especially grateful to Tom Bridgeland, Michel Van den Bergh and Balázs Szendrői for the the valuable feedback on the first draft of Chapter 2.

The intellectual atmosphere at the institute has been very stimulating throughout the whole project and I have fond memories of all the discussions I had with my colleagues during this time. I would like to make special mention of Camilo Arias Abad, David Carchedi, Gil Cavalcanti, Benoit Dherin, Bart van den Dries, Jaap Eldering, Luuk Hoevenaars, Bas Janssens, Charlene Kalle, Wilberd van der Kallen, Job Kuit, Oliver Lorscheid, Vincent van der Noort, Frans Oort, Hessel Posthuma, Dmitry Roytenberg, Jeroen Sijsling, Rogier Swierstra and Ittay Weiss. It is furthermore a pleasure to acknowledge the fruitful discussions and the joint work with Alex Boer.

I enjoyed sharing an office with Job Kuit and Albert-Jan Yzelman. I am also indebted to Arthur van Dam and Albert-Jan Yzelman for their generous computer assistance and help with LaTeX. Thanks are also due to Job Kuit, Sebastiaan Janssens and Luuk Hoevenaars for correcting the English summary and translating it into Dutch.

A special thank you to Calin Lazaroiu, Nils Carqueville and Daniel McNamee. Our many enjoyable discussions and conversations, on all kinds of topics, were a constant source of motivation.

In Amsterdam, I enjoyed socializing with colleagues at the Institute for Theoretical Physics. I retain good memories of the always interesting discussions in an early stage of my research with Sheer El-Showk and Lotte Hollands and regret not having had enough time for these in later stages of my research.

I would like to send my best thanks to Dimos Ntanis for his friendship, and for the useful and interesting discussions, both academic and otherwise. My Utrecht housemate, Paul Reska, also deserve thanks for providing a pleasant and sociable home environment.

I thank Ainhoa Mariezcurrena for the good times we had together.

A very special thanks goes out to Juan Diego Vélez, without whose motivation and encouragement I would not have considered a graduate career in mathematics. Juan Diego is one professor/teacher who truly made a difference in my life. He provided me with direction, technical support and became more of a mentor and friend, than a professor. I doubt that I will ever be able to convey my appreciation fully, but owe him my eternal gratitude.

This thesis would not have been accomplished without the support of my family. My most heartfelt thanks go to my parents for allowing me the freedom to choose my own direction while at the same time providing just the right amount of guidance, and for instilling in me a love of learning. You spared no effort to provide me with the best education possible, and for that I am eternally grateful. My thanks equally go to my brothers and my sister for their understanding. I hope I can express my love to them.

Last but not least, I would like to express my gratitude to my wife for her support and interest in my undertakings –even when following my passion leads me to move abroad.

CURRICULUM VITAE

The author of this thesis was born on July 4th 1977 in Manizales, Colombia. He grew up in Medellín and attended primary and secondary education there.

In 1996 he began to study civil engineering at the Universidad Nacional de Colombia sede Medellín, where he graduated in 2001. During his studies he became more and more interested in theoretical physics, especially in the mathematical end of theoretical physics. In 2001 he decided to enroll for the MSc programme of mathematics at the Universidad Nacional under the supervision of Prof. Juan Diego Vélez. He received his MSc degree in 2003. He spent the academic year 2003-2004 at Utrecht University, taking part in the MRI Masterclass on Noncommutative Geometry.

With the aid of the Academy Professorships Programme he started working as an “Assistent in Opleiding” (AIO) at Utrecht University in 2005, under the supervision of Dr. Jan Stienstra and Prof. Dr. Hans Duistermaat. The results of the research carried out during the last four years are presented in this thesis.

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