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A Simple Operator Check of the Effective Fermion Mode Function during Inflation

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ABSTRACT

We present a relatively simple operator formalism which reproduces the leading infrared logarithm of the one loop quantum gravitational correction to the fermion mode function on a locally de Sitter background. This rule may serve as the basis for an eventual stochastic formulation of quantum gravity during inflation. Such a formalism would not only effect a vast simplification in obtaining the leading powers of $\ln(a)$ at fixed loop orders, it would also permit us to sum the series of leading logarithms. A potentially important point is that our rule does not seem to be consistent with any simple infrared truncation of the fields. Our analysis also highlights the importance of spin as a gravitational interaction that persists even when kinetic energy has redshifted to zero.

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1 Introduction

Gravitons and massless, minimally coupled (MMC) scalars are unique in being massless without classical conformal invariance. The combination of these properties causes the accelerated expansion of spacetime during inflation to tear long wavelength virtual quanta out of the vacuum [1, 2]. As more and more gravitons and MMC scalars emerge from the vacuum, the metric and MMC scalar field strengths experience a slow growth. The effect can be felt by any quantum field theory which involves either the undifferentiated metric or an undifferentiated MMC scalar.

An example is the one loop enhancement recently found [3, 4] for the plane wave mode functions of massless, Dirac fermions which are coupled to quantum gravity on a locally de Sitter background,

$$ds^2 = -dt^2 + a^2(t) d\vec{x} \cdot d\vec{x} \qquad \text{where} \qquad a(t) = e^{Ht} \; . \tag{1}$$

(This background solves the classical Friedmann equation, $(D-1)(\dot{a}/a)^2 = \Lambda \equiv (D-1)H^2$, where Λ is the cosmological constant. The one-loop back-reaction on $a(t)$ cannot affect the fermion mode function until two loop order.) At late times the full mode function $\Xi(x;\vec{k},s)$ behaves as if the tree order mode function $\Xi_0(x;\vec{k},s)$ was subject to a time-dependent field strength renormalization,

$$\Xi(x;\vec{k},s) \longrightarrow \frac{\Xi_0(x;\vec{k},s)}{\sqrt{Z_2(t)}} \; . \tag{2}$$

This field strength renormalization takes the form,

$$Z_2(t) = 1 - \frac{17}{4\pi} G H^2 \ln(a) + O(G^2) \; , \tag{3}$$

where G and H are the Newton and Hubble constants, respectively.

The factor of $\ln(a) = Ht$ in expression (3) is known as an *infrared logarithm.* Any quantum field theory which involves undifferentiated MMC scalars or metrics will show similar infrared logarithms in some of its Green's functions. They arise at one and two loop orders in the expectation value of the stress tensor and in the scalar self-mass-squared of a MMC scalar with a quartic self-coupling [5]. In scalar quantum electrodynamics they have been seen in the one loop vacuum polarization [6] and the two loop expectation values of scalar bilinears [7], the field strength bilinear and the stress tensor

[8]. In Yukawa theory they show up in the one loop fermion self-energy [9] and in the two loop coincident vertex function [10]. In pure quantum gravity they occur in the one loop graviton self-energy [11] and in the two loop expectation value of the metric [12]. They even contaminate loop corrections to the power spectrum of cosmological perturbations [13, 14] and other fixed-momentum correlators [15].

Infrared logarithms introduce a fascinating secular element into the usual, static results of quantum field theory. Their most intriguing property is their ability to compensate for powers of the loop counting parameter which suppress quantum loop effects. Indeed, the continued growth of $\ln(a) = Ht$ must eventually *overwhelm* the loop counting parameter, no matter how small it is. However, this does not necessarily mean that quantum loop effects become strong. The correct conclusion is rather that perturbation theory breaks down past a given point in time. One must employ a nonperturbative technique to follow what happens later.

Certain models lend themselves to resummation schemes such as the $1/N$ expansion [16], but a more general technique is suggested by the form of the expansion for $Z_2(t)$ in (3),

$$Z_2(t) = 1 + \sum_{\ell=1}^{\infty} (GH^2)^\ell \Biggl\{ c_{\ell,0} \Bigl[\ln(a)\Bigr]^\ell + c_{\ell,1} \Bigl[\ln(a)\Bigr]^{\ell-1} + \ldots + c_{\ell,\ell-1} \ln(a) \Biggr\} . \qquad (4)$$

Here the constants $c_{\ell,k}$ are pure numbers which are assumed to be of order one. The term in (4) involving $[GH^2 \ln(a)]^\ell$ is the *leading logarithm* contribution at ℓ loop order; the other terms are *subdominant logarithms.* Perturbation theory breaks down when $\ln(a) \sim 1/GH^2$, at which point the leading infrared logarithms at each loop order contribute numbers of order one. In contrast, the subleading logarithms are all suppressed by at least one factor of the small parameter $GH^2 \lesssim 10^{-12}$. So it makes sense to retain only the leading infrared logarithms,

$$Z_2(t) \longrightarrow 1 + \sum_{\ell=1}^{\infty} c_{\ell,0} \Bigl[GH^2 \ln(a)\Bigr]^\ell . \qquad (5)$$

This is known as the *leading logarithm approximation.*

Starobinskiĭ has developed a simple stochastic formalism [17] which reproduces the leading infrared logarithms at each order [18] for any scalar potential model of the form,

$$\mathcal{L} = -\frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi g^{\mu\nu} \sqrt{-g} - V(\varphi) \sqrt{-g} . \qquad (6)$$

Probabilistic representations of inflationary cosmology have been much studied in order to understand initial conditions [19] and global structure [20]. More recently they have been employed to study non-Gaussianity [21]. However, we wish here to focus on Starobinskiĭ's technique as a wonderfully simple way of recovering the most important secular effects of inflationary quantum field theory [22]. It is of particular importance for us that Starobinskiĭ and Yokoyama have shown how to take the late time limit of the series of leading infrared logarithms whenever the potential $V(\varphi)$ is bounded below [23]. This is the true analogue of what the renormalization group accomplishes in flat space quantum field theory and statistical mechanics.

The solution of Starobinskiĭ and Yokoyama is an amazing achievement, but it only gives us control over infrared logarithms which arise in scalar potential models (6). The most general theories which show infrared logarithms possess two complicating features:

- Couplings to fields other than MMC scalars and gravitons; and
- Interactions which involve differentiated MMC scalars and gravitons.[1]

An important step forward was the recent leading log solutions for MMC scalars which are either Yukawa-coupled to a massless, Dirac fermion [10] or to electrodynamics [24]. Although the second model has derivative interactions, this feature was avoided (at leading logarithm order) by working in Lorentz gauge. We still do not understand how to treat derivative interactions.

At the level of dimensionally regularized perturbation theory, the scalar leading logarithm solutions which have so far been obtained can be reduced to five simple steps [24]:

1. Expand the full scalar operator $\varphi(x)$ in powers of the free field $\varphi_0(x)$ which agrees with $\varphi(x)$ and its first derivative at the beginning of inflation as described in the recent paper by Musso [25];
2. The expectation value of any desired operator can then be expressed as vertex integrations of retarded Green's functions times products of expectation values of pairs of free fields;

[1]Of course there would be no infrared logarithms if *all* the MMC scalars and gravitons were differentiated. However, infrared logarithms must arise, in the expectation values of some operators, from interactions which involve at least one undifferentiated MMC scalar or graviton. Examples include the $h^n \partial h \partial h$ interaction of pure quantum gravity [11, 12] and scalar interactions of the form $\varphi^2 \partial\varphi\partial\varphi$ [13, 18].

3. Make the following replacement for the expectation value of two free fields:
$$\Big\langle \Omega \Big| \varphi_0(x) \varphi_0(x') \Big| \Omega \Big\rangle \longrightarrow \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!1)}{\Gamma(\frac{D}{2})} \, 2 \ln\Big[\min(a, a')\Big] \; ; \qquad (7)$$
4. Make the following replacement for the retarded Green's function:
$$G(x; x') \longrightarrow \frac{\theta(t\!-\!t') \delta^{D-1}(\vec{x}\!-\!\vec{x}')}{(D\!-\!1) H a'^{D-1}} \; ; \text{and} \qquad (8)$$
5. Evaluate the contributions from any other fields (for example, photons or fermions) exactly to the required order.

Indeed, these rules even predict the occasional null results [26] that sometimes occur at low orders.

It is straightforward to show that this old rule (7-8) does *not* suffice to recover (3). The purpose of this paper is to devise a simple rule which does work. We do not yet know if this rule applies either to other quantities or to higher loop orders. Nor do we possess a nonperturbative realization for this rule. Our rule nonetheless represents very significant progress in the struggle to solve inflationary quantum gravity in the leading logarithm approximation. Such a solution would make it simple to compute leading logarithm results at fixed order, and would also facilitate summation of the series of leading logarithms, thereby defining evolution past the breakdown of perturbation theory.

In section 2 we explain how solving the Schwinger-Keldysh effective field equations is equivalent to computing the expectation value of a suitable canonical operator. Section 3 works out the operator and its expectation value to the order we require. At this stage the result is still exact and represents no simplification of the effective field equation technique. Our simplifying rule is presented in section 4. In section 5 we demonstrate that the rule indeed reproduces the leading infrared logarithm in the one loop correction to the fermion mode function. Our conclusions comprise section 6.

2 The Effective Mode Function

We begin this section by describing the Schwinger-Keldysh formalism. This is a covariant extension of Feynman diagrams that produces true expectation values instead of in-out matrix elements[27, 28, 29, 30]. We then review the quantum-corrected Dirac equation whose solution (for spatial plane waves) gives the $\mathbb{C}$-number effective fermion mode function $\Xi_i(x;\vec{k},s)$. The section closes by giving the connection between $\Xi_i(x;\vec{k},s)$ and the fermion operator $\Psi_i(x)$.

The in-out effective field equations give a fine representation of flat space scattering problems but they are not typically suitable for cosmological settings in which particle production precludes the in vacuum from evolving to the out vacuum. Persisting with thein-out formalism on de Sitter background would result in processes being dominated by infrared divergences from the enormous spacetime volume of the infinite future [1, 31]. The better course in this case is to release the universe in a prepared state at finite time and let it evolve as it will. Problems of this sort are described by the Schwinger-Keldysh effective field equations [32].

Consider a scalar field φ whose Lagrangian (by which we mean the spatial integral of the Lagrangian density) at time t is $L[\varphi(t)]$. The fundamental relation between the canonical operator formalism and the Schwinger-Keldysh functional integral formalism is [33],

$$\Big\langle \Phi \Big| \overline{T}\Big(\mathcal{O}_2[\varphi]\Big) T\Big(\mathcal{O}_1[\varphi]\Big) \Big| \Phi \Big\rangle = \int [d\varphi_+][d\varphi_-]\, \delta\Big[\varphi_-(t_1) - \varphi_+(t_1)\Big] \Phi^*[\varphi_-(t_0)] \Phi[\varphi_+(t_0)] \\ \times \mathcal{O}_2[\varphi_-] \mathcal{O}_1[\varphi_+] \exp\Bigg[i \int_{t_0}^{t_1} \! dt \, \Big\{ L[\varphi_+(t)] - L[\varphi_-(t)] \Big\} \Bigg] . (9)$$

Here $|\Phi\rangle$ is the Heisenberg state whose wave functional in terms of the φ eigenkets at time t_0 is $\Phi[\varphi(t_0)]$. The canonical expectation value on the left hand side consists of the product of an anti-time-ordered operator $\mathcal{O}_2[\varphi]$ times a time-ordered operator $\mathcal{O}_1[\varphi]$. The value of $t_1 > t_0$ is arbitrary as long as it is in the future of the latest operator occurring in either $\mathcal{O}_1$ or $\mathcal{O}_2$.

The Feynman rules follow from relation (9) in close analogy to those for in-out matrix elements. Because the same field is represented by two different dummy functional variables, $\varphi_\pm(x)$, the endpoints of lines carry a $\pm$ polarity. External lines associated with the anti-time-ordered operator $\mathcal{O}_2[\varphi]$ have the $-$ polarity whereas those associated with the time-ordered operator $\mathcal{O}_1[\varphi]$ have the $+$ polarity. Interaction vertices are either all $+$ or all $-$. Vertices with $+$ polarity are the same as in the usual Feynman rules whereas vertices with the $-$ polarity have an additional minus sign. Propagators can be $++$, $-+$, $+-$ or $--$.

From this sketch we see that the N-point one-particle-irreducible (1PI) function $\Gamma^N(x_1, \ldots, x_N)$ of the in-out formalism gives rise to 2^N different Schwinger-Keldysh 1PI functions $\Gamma^N(x_{1\pm}, \ldots, x_{N\pm})$. Now recall that the in-out effective action is the generating functional of in-out 1PI functions,

$$\Gamma[\phi] = \sum_N \frac{1}{N!} \int d^4x_1 \phi(x_1) \ldots \int d^4x_N \phi(x_N) \times \Gamma^N(x_1, \ldots, x_N) \; . \tag{10}$$

The analogous generating functional for Schwinger-Keldysh 1PI functions is,

$$\Gamma[\phi_+, \phi_-] = \sum_N \frac{1}{N!} \int d^4x_1 \phi_\pm(x_1) \ldots \int d^4x_N \phi_\pm(x_N) \times \Gamma^N(x_{1\pm}, \ldots, x_{N\pm}) \; . \tag{11}$$

The Schwinger-Keldysh effective field equations are obtained by varying this functional with respect to either ϕ_+ or ϕ_-, and then setting the two fields equal,

$$\frac{\delta \Gamma[\phi_+, \phi_-]}{\delta \phi_+(x)} \Bigg|_{\phi_\pm = \phi} = 0 \; . \tag{12}$$

It is worth being a little more explicit for the case in which the 0-point and 1-point functions vanish. If the classical action is $S[\phi]$, and the self-mass-squared is $-i M^2_{\pm\pm}(x; x')$, the Schwinger-Keldysh effective action has the following expansion,

$$\Gamma[\phi_+, \phi_-] = S[\phi_+] - S[\phi_-] - \frac{1}{2} \int d^4x \int d^4x' \times \left\{ \begin{array}{c} \phi_+(x) M^2_{++}(x; x') \phi_+(x') + \phi_+(x) M^2_{+-}(x; x') \phi_-(x') \\ + \phi_-(x) M^2_{-+}(x; x') \phi_+(x') + \phi_-(x) M^2_{--}(x; x') \phi_-(x') \end{array} \right\} + O(\phi^3) \; . \tag{13}$$

The Schwinger-Keldysh effective field equations are,

$$\frac{\delta S[\phi]}{\delta \phi(x)} - \int d^4x' \Big[M^2_{++}(x; x') + M^2_{+-}(x; x') \Big] \phi(x') + O(\phi^2) = 0 \; . \tag{14}$$

The quantum-corrected Klein-Gordon equation results from linearizing (14), and its solution for a spatial plane wave is the scalar effective mode function. The peculiar combination of $M^2_{++}(x; x') + M^2_{+-}(x; x')$ in (14) has two important properties:

- It is real, even though each self-mass-squared has a nonzero imaginary part; and

- It vanishes for any point x'^{μ} outside the past light-cone of x^{μ}.

This paper concerns our solution of the quantum-corrected Dirac equation for the effective fermion mode function [4],

$$i\partial\!\!\!/_{ij}\Xi_j(x) = \int d^4x' \left\{ \Big[{}_i\Sigma_j\Big]_{++}(x;x') + \Big[{}_i\Sigma_j\Big]_{+-}(x;x') \right\} \Xi_j(x) \; . \qquad (15)$$

Here$\partial\!\!\!/_{ij} \equiv \gamma^{\mu}_{ij}\partial_{\mu}$ and γ^{μ}_{ij} represents the usual, 4×4 gamma matrices. Note that $\Xi_i(x)$ is a 4-component $\mathbb{C}$-number field, even though the associated canonical operator $\Psi_i(x)$ is fermionic. There is no trace of the de Sitter geometry in the classical part of (15) because we work in conformal coordinates,

$$ds^2 = a^2\Big[-d\eta^2 + d\vec{x}\cdot d\vec{x}\Big] \equiv a^2\eta_{\mu\nu}dx^{\mu}dx^{\nu} \quad \text{where} \quad a = -\frac{1}{H\eta} = e^{Ht} \; . \quad (16)$$

Massless fermions are conformally invariant in any dimension so we computed the fermion self-energy using dimensional regularization for the conformally rescaled field,

$$\Psi_i(x) \equiv a^{\frac{D-1}{2}}\,\psi_i(x) \; . \qquad (17)$$

This removes any dependence upon the de Sitter scale factor from the tree order equation for $\Psi_i(x)$, and hence for $\Xi_i(x)$.

Gravity is *not* conformally invariant, so one loop quantum gravitational corrections to the fermion self-energy involve the de Sitter scale factor. In computing these corrections we fixed the local Lorentz gauge so as to allow an algebraic expression for the vierbein in terms of the metric [34]. The general coordinate gauge was fixed to make the tensor structure of the graviton propagator decouple from its spacetime dependence [35]. After absorbing the divergences with three BPHZ (Bogoliubov-Parasiuk-Hepp-Zimmermann) counterterms we took the unregulated limit of $D=4$ to obtain the following results [3]:

$$\Big[\Sigma\Big]_{++}(x;x') = \frac{i\kappa^2H^2}{2^6\pi^2}\left\{\frac{\ln(aa')}{H^2aa'}\partial\!\!\!/\partial^2 + \frac{15}{2}\ln(aa')\partial\!\!\!/ - 7\ln(aa')\overline{\partial\!\!\!/}\right\}\delta^4(x-x')$$
$$+\frac{\kappa^2}{2^8\pi^4aa'}\partial\!\!\!/\partial^4\Big[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}}\Big] + \frac{\kappa^2H^2}{2^8\pi^4}\left\{\Big(\frac{15}{2}\partial\!\!\!/\,\partial^2 - \overline{\partial\!\!\!/}\,\partial^2\Big)\Big[\frac{\ln(\mu^2\Delta x^2_{++})}{\Delta x^2_{++}}\Big]\right.$$
$$\left. + \Big(-8\overline{\partial\!\!\!/}\partial^2 + 4\partial\!\!\!/\nabla^2\Big)\Big[\frac{\ln(\frac{1}{4}H^2\Delta x^2_{++})}{\Delta x^2_{++}}\Big] + 7\partial\!\!\!/\nabla^2\Big[\frac{1}{\Delta x^2_{++}}\Big]\right\} + O(\kappa^4) \; , \quad (18)$$

$$\Big[\Sigma\Big]_{+-}(x;x') = \frac{-\kappa^2}{2^8\pi^4 a a'} \partial\!\!\!/ \partial^4 \Big[\frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}}\Big] - \frac{\kappa^2 H^2}{2^8 \pi^4} \Big\{ \Big(\frac{15}{2} \partial\!\!\!/ \partial^2 - \overline{\partial\!\!\!/} \partial^2\Big) \Big[\frac{\ln(\mu^2 \Delta x^2_{+-})}{\Delta x^2_{+-}}\Big] \\ + \Big(-8 \overline{\partial\!\!\!/} \partial^2 + 4 \partial\!\!\!/ \nabla^2\Big) \Big[\frac{\ln(\frac14 H^2 \Delta x^2_{+-})}{\Delta x^2_{+-}}\Big] + 7 \partial\!\!\!/ \nabla^2 \Big[\frac{1}{\Delta x^2_{+-}}\Big]\Big\} + O(\kappa^4) \; . \quad (19)$$

Here $\kappa^2 \equiv 16\pi G$ is the loop counting parameter of quantum gravity. The various differential and spinor-differential operators are,

$$\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \quad , \quad \nabla^2 \equiv \partial_i \partial_i \quad , \quad \partial\!\!\!/ \equiv \gamma^\mu \partial_\mu \quad \text{and} \quad \overline{\partial\!\!\!/} \equiv \gamma^i \partial_i \; . \quad (20)$$

The two conformal coordinate intervals are,

$$\Delta x^2_{++}(x;x') \equiv \|\vec{x} - \vec{x}'\|^2 - \Big(|\eta - \eta'| - i\delta\Big)^2 \; , \quad (21)$$

$$\Delta x^2_{+-}(x;x') \equiv \|\vec{x} - \vec{x}'\|^2 - \Big(\eta - \eta' + i\delta\Big)^2 \; . \quad (22)$$

Note that they agree for $\eta < \eta'$, whereas they are complex conjugates of one another for $\eta > \eta'$.

Of course we can only solve for the one loop corrections to the field because we lack the higher loop contributions to the self-energy. Suppressing spinor indices and polarities, the general perturbative expansion takes the form,

$$\Xi(x) = \sum_{\ell=0}^{\infty} \kappa^{2\ell} \Xi_\ell(x) \qquad \text{and} \quad \Big[\Sigma\Big](x;x') = \sum_{\ell=1}^{\infty} \kappa^{2\ell} \Big[\Sigma_\ell\Big](x;x') \; . \quad (23)$$

One substitutes these expansions into the effective Dirac equation (15) and then segregates powers of κ^2,

$$i \partial\!\!\!/ \Xi_0(x) = 0 \; , \quad (24)$$

$$\kappa^2 i \partial\!\!\!/ \Xi_1(x) = \kappa^2 \int d^4x' \Big\{ \Big[\Sigma_1\Big]_{++}(x;x') + \Big[\Sigma_1\Big]_{+-}(x;x') \Big\} \Xi_0(x') \; , \quad (25)$$

and so on. We considered the one loop correction $\Xi_{1i}(x;\vec{k},s)$ to a spatial plane wave of helicity s,

$$\Xi_{0i}(x;\vec{k},s) = \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k},s) e^{i\vec{k}\cdot\vec{x}} \quad \text{where} \quad k^\ell \gamma^\ell_{ij} u_j(\vec{k},s) = k \gamma^0_{ij} u_j(\vec{k},s) \; . \quad (26)$$

In the limit of late times the source term on the right hand side of (25) takes the form,

$$\kappa^2 i \partial\!\!\!/ \Xi_1(x;\vec{k},s) \longrightarrow \frac{\kappa^2 H^2}{16\pi^2} \times \frac{17}{8} i H a \gamma^0 \Xi_0(x;\vec{k},s) \; . \quad (27)$$

Hence we conclude that the late time limit of the one loop correction to the effective mode function gives a time-dependent enhancement of the tree order field strength [4],

$$\Xi_0(x;\vec{k},s) + \kappa^2 \Xi_1(x;\vec{k},s) \longrightarrow \Bigl\{ 1 + \frac{\kappa^2 H^2}{16\pi^2} \times \frac{17}{8} \ln(a) \Bigr\} \Xi_0(x;\vec{k},s) \; . \qquad (28)$$

We must now explain how the $\mathbb{C}$-number effective mode function $\Xi(x;\vec{k},s)$ relates to the canonical fermion operator $\Psi(x)$. Consider the perturbative expansions of the Heisenberg operator equations for the graviton $h_{\mu\nu}(x)$ and the (conformally rescaled) fermion $\Psi_i(x)$,

$$h_{\mu\nu}(x) = h_{0\mu\nu}(x) + \kappa h_{1\mu\nu}(x) + \kappa^2 h_{2\mu\nu}(x) + \dots \; , \qquad (29)$$

$$\Psi_i(x) = \Psi_{0i}(x) + \kappa \Psi_{1i}(x) + \kappa^2 \Psi_{2i}(x) + \dots \; . \qquad (30)$$

Long experience with such expansions permits us to anticipate how the first and second order corrections to Ψ depend upon the zeroth order fields,

$$\Psi_1 \sim h_0 \Psi_0 \qquad , \qquad \Psi_2 \sim h_0 h_0 \Psi_0 + \overline{\Psi}_0 \Psi_0 \Psi_0 \; . \qquad (31)$$

Because our state is released in free vacuum at $t = 0$ ($\eta = -1/H$), it makes sense to express the zeroth order solutions in terms of the creation and annihilation operators of this free state,

$$h_{0\mu\nu}(x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_{\lambda} \Bigl\{ \epsilon_{\mu\nu}(\eta;\vec{k},\lambda) e^{i\vec{k}\cdot\vec{x}} \alpha(\vec{k},\lambda) + \epsilon^*_{\mu\nu}(\eta;\vec{k},\lambda) e^{-i\vec{k}\cdot\vec{x}} \alpha^\dagger(\vec{k},\lambda) \Bigr\} \; , \qquad (32)$$

$$\Psi_{0i}(x) = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \sum_{s} \Bigl\{ \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k},s) e^{i\vec{k}\cdot\vec{x}} b(\vec{k},s) + \frac{e^{ik\eta}}{\sqrt{2k}} v_i(\vec{k},\lambda) e^{-i\vec{k}\cdot\vec{x}} c^\dagger(\vec{k},s) \Bigr\} \; . \qquad (33)$$

The graviton mode functions are proportional to Hankel functions whose precise specification we do not require. The Dirac wave functions $u_i(\vec{k},s)$ and $v_i(\vec{k},s)$ are precisely those of flat space by virtue of the conformal invariance of massless fermions. The canonically normalized creation and annihilation operators obey,

$$\Bigl[\alpha(\vec{k},\lambda), \alpha^\dagger(\vec{k}',\lambda') \Bigr] = \delta_{\lambda\lambda'} (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}') \; , \qquad (34)$$

$$\Bigl\{ b(\vec{k},s), b^\dagger(\vec{k}',s') \Bigr\} = \delta_{ss'} (2\pi)^{D-1} \delta^{D-1}(\vec{k} - \vec{k}') = \Bigl\{ c(\vec{k},s), c^\dagger(\vec{k}',s') \Bigr\} \; . \qquad (35)$$

We can get the $\mathbb{C}$-number mode function $\Xi_i(x;\vec{k},s)$ from the zeroth order field $\Psi_{0i}(x)$ by anti-commuting with the fermion creation operator,

$$\Xi_{0i}(x;\vec{k},s) = \Big\{\Psi_{0i}(x), b^\dagger(\vec{k},s)\Big\} = \frac{e^{-ik\eta}}{\sqrt{2k}} u_i(\vec{k},s) e^{i\vec{k}\cdot\vec{x}} \; . \qquad (36)$$

The higher order contributions to $\Psi_i(x)$ are no longer linear in the creation and annihilation operators, so anti-commuting the full solution $\Psi_i(x)$ with $b^\dagger(\vec{k},s)$ produces an operator whose general form is,

$$\Big\{\Psi, b^\dagger\Big\} \sim \Xi_0 + \kappa h_0 \Xi_0 + \kappa^2 h_0 h_0 \Xi_0 + \kappa^2 \overline{\Psi}_0 \Psi_0 \Xi_0 + O(\kappa^3) \; . \qquad (37)$$

The quantum-corrected fermion mode function we obtain by solving (15) is the expectation value of this operator in the presence of the state which is free vacuum at $t = 0$,

$$\Xi_i(x;\vec{k},s) = \Big\langle \Omega \Big| \Big\{\Psi_i(x), b^\dagger(\vec{k},s)\Big\} \Big| \Omega \Big\rangle \; . \qquad (38)$$

This is the promised relation between solving for the effective mode function and canonical operators [4].

Because we have a prediction (27) for the late time limit of $i\,\partial\!\!\!/ \Xi(x;\vec{k},s)$ it makes sense to act the free kinetic operator on (38),

$$i\partial\!\!\!/ \Xi(x;\vec{k},s) = \Big\langle \Omega \Big| \Big\{ i\partial\!\!\!/ \Psi(x), b^\dagger(\vec{k},s)\Big\} \Big| \Omega \Big\rangle \; . \qquad (39)$$

Of course this equation must hold order-by-order in the κ expansions of $\Xi(x;\vec{k},s)$ and $\Psi(x)$. The order κ^0 terms vanish identically. There is no order κ^1 correction to $\Xi(x;\vec{k},s)$, and the order κ^1 correction to $\Psi(x)$ vanishes when the expectation value is taken. The key relation for this paper comes from taking the late time limit at order κ^2,

$$\kappa^2 \Big\langle \Omega \Big| \Big\{ i\partial\!\!\!/ \Psi_2(x), b^\dagger(\vec{k},s)\Big\} \Big| \Omega \Big\rangle \longrightarrow \frac{\kappa^2 H^2}{16\pi^2} \times \frac{17}{8} i H a \gamma^0 \Xi_0(x;\vec{k},s) \; . \qquad (40)$$

3 Perturbative Operator Solution

The purpose of this section is to work out the canonical operator contributions to the left hand side of expression (40). We begin by giving the invariant action and fixing the gauge. This defines the fermion and gravtion propagators which, in turn, give the retarded Green's functions. We then perturbatively solve the Heisenberg operator equations to the required order in powers of the free fields (32-33). Our result for $\kappa^2 i\bar{\partial}\Psi_2$ is reported in Table 1. We also report the contribution of each term to $\kappa^2\langle\Omega|\{i\bar{\partial}\Psi_2(x), b^\dagger(\vec{k},s)\}|\Omega\rangle$ in Table 2. All the analysis of this section is done in D dimensions so that ultraviolet divergences are dimensionally regulated.

The invariant Lagrangian density of Dirac + Einstein is,

$$\mathcal{L} = \frac{1}{16\pi G}\Bigl(R-(D-1)(D-2)H^2\Bigr)\sqrt{-g}+\overline{\psi}e^{\mu}{}_{b}\gamma^b\Bigl(i\partial_\mu-\frac{1}{2}A_{\mu cd}J^{cd}\Bigr)\psi\sqrt{-g}\ . \quad (41)$$

Here G is Newton's constant and H is the Hubble constant. The vierbein field is $e_{\mu b}$ and $g_{\mu\nu} \equiv e_{\mu b}e_{\nu c}\eta^{bc}$ is the metric. The metric and vierbein-compatible connections are,

$$\Gamma^{\rho}{}_{\mu\nu} \equiv \frac{1}{2}g^{\rho\sigma}\Bigl(g_{\sigma\mu,\nu}+g_{\nu\sigma,\mu}-g_{\mu\nu,\sigma}\Bigr) \qquad \text{and} \qquad A_{\mu cd} \equiv e^{\nu}{}_{c}\Bigl(e_{\nu d,\mu}-\Gamma^{\rho}{}_{\mu\nu}e_{\rho d}\Bigr)\ . \quad (42)$$

The Ricci scalar is,

$$R \equiv g^{\mu\nu}\Bigl(\Gamma^{\rho}{}_{\nu\mu,\rho}-\Gamma^{\rho}{}_{\rho\mu,\nu}+\Gamma^{\rho}{}_{\rho\sigma}\Gamma^{\sigma}{}_{\nu\mu}-\Gamma^{\rho}{}_{\nu\sigma}\Gamma^{\sigma}{}_{\rho\mu}\Bigr)\ . \quad (43)$$

The gamma matrices γ^b_{ij} have spinor indices $i,j \in \{1,2,3,4\}$, obey the usual anti-commutation relations and give the usual Lorentz generators,

$$\{\gamma^b,\gamma^c\} = -2\eta^{bc}I \qquad , \qquad J^{bc} \equiv \frac{i}{4}[\gamma^b,\gamma^c]\ . \quad (44)$$

It is useful to conformally rescale the vierbein by the de Sitter scale factor $a(t)$,

$$e_{\beta b} \equiv a\,\widetilde{e}_{\beta b} \qquad \Longrightarrow \qquad e^{\beta b} = a^{-1}\,\widetilde{e}^{\beta b}\ . \quad (45)$$

Of course this implies a rescaled metric $\widetilde{g}_{\mu\nu}$,

$$g_{\mu\nu} = a^2\,\widetilde{g}_{\mu\nu} \equiv a^2\Bigl(\eta_{\mu\nu}+\kappa h_{\mu\nu}(x)\Bigr) \quad \text{where} \quad a = -\frac{1}{H\eta} = e^{Ht}\ . \quad (46)$$

The old connections can be expressed as follows in terms of the ones formed from the rescaled fields,

$$\Gamma^{\rho}{}_{\mu\nu} = a^{-1}\Bigl(\delta^{\rho}{}_{\mu}\,a_{,\nu}+\delta^{\rho}{}_{\nu}\,a_{,\mu}-\widetilde{g}^{\rho\sigma}\,a_{,\sigma}\,\widetilde{g}_{\mu\nu}\Bigr)+\widetilde{\Gamma}^{\rho}{}_{\mu\nu} \quad (47)$$

$$A_{\mu cd} = -a^{-1}\Bigl(\widetilde{e}^{\nu}{}_{c}\,\widetilde{e}_{\mu d}-\widetilde{e}^{\nu}{}_{d}\,\widetilde{e}_{\mu c}\Bigr)a_{,\nu}+\widetilde{A}_{\mu cd}\ . \quad (48)$$

We define rescaled fermion fields as,

$$\Psi \equiv a^{\frac{D-1}{2}}\,\psi \qquad \text{and} \qquad \overline{\Psi} \equiv a^{\frac{D-1}{2}}\,\overline{\psi}\;. \tag{49}$$

We employ Lorentz symmetric gauge, $e_{\mu b} = e_{b\mu}$, which permits one to perturbatively determine the vierbein in terms of the metric and their respective backgrounds [34],

$$\widetilde{e}[\widetilde{g}]_{\beta b} \equiv \Bigl(\sqrt{\widetilde{g}\eta^{-1}}\Bigr)_{\beta}^{\;\;\gamma}\eta_{\gamma b} = \eta_{\beta b} + \frac{1}{2}\kappa h_{\beta b} - \frac{1}{8}\kappa^2 h_{\beta}^{\;\;\gamma} h_{\gamma b} + \dots \tag{50}$$

Here and throughout this paper graviton indices are raised and lowered with the Lorentz metric, e.g., $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. The same convention applies as well to derivatives ($\partial^\mu = \eta^{\mu\nu}\partial_\nu$) and gamma matrices ($\gamma_\mu = \eta_{\mu\nu}\gamma^\nu$). The general coordinate freedom is fixed by adding the gauge fixing term,

$$\Delta\mathcal{L} = -\frac{1}{2}a^{D-2}\eta^{\mu\nu}F_\mu F_\nu \qquad F_\mu \equiv \eta^{\rho\sigma}\Bigl(h_{\mu\rho,\sigma} - \frac{1}{2}h_{\rho\sigma,\mu} + (D-2)Hah_{\mu\rho}\delta^0_\sigma\Bigr). \tag{51}$$

After some judicious partial integrations the gauge fixed Lagrangian density has the following expansion,

$$\begin{aligned}\mathcal{L}_{\rm GF} = {} & \overline{\Psi}i\partial\!\!\!/\Psi + \frac{\kappa}{2}\Bigl[h\overline{\Psi}i\partial\!\!\!/\Psi - h^{\mu\nu}\overline{\Psi}\gamma_\mu i\partial_\nu\Psi - h_{\mu\rho,\sigma}\overline{\Psi}\gamma^\mu J^{\rho\sigma}\Psi\Bigr] \\ & + \kappa^2\Bigl[\frac{1}{8}h^2 - \frac{1}{4}h^{\rho\sigma}h_{\rho\sigma}\Bigr]\overline{\Psi}i\partial\!\!\!/\Psi + \kappa^2\Bigl[-\frac{1}{4}hh^{\mu\nu} + \frac{3}{8}h^{\mu\rho}h_\rho^{\;\;\nu}\Bigr]\overline{\Psi}\gamma_\mu i\partial_\nu\Psi \\ & + \kappa^2\Bigl[-\frac{1}{4}hh_{\mu\rho,\sigma} + \frac{1}{8}h^\nu_{\;\;\rho}h_{\nu\sigma,\mu} + \frac{1}{4}(h^\nu_{\;\;\mu}h_{\nu\rho})_{,\sigma} + \frac{1}{4}h^\nu_{\;\;\sigma}h_{\mu\rho,\nu}\Bigr]\overline{\Psi}\gamma^\mu J^{\rho\sigma}\Psi + O(\kappa^3) \\ & + \frac{1}{2}h^{\mu\nu}D_{\mu\nu}^{\;\;\;\;\rho\sigma}h_{\rho\sigma} + \Bigl(\text{Pure Gravity Interactions}\Bigr)\;. \end{aligned} \tag{52}$$

The explicit form of the graviton kinetic operator $D_{\mu\nu}^{\;\;\;\;\rho\sigma}$ is not needed here; it can be found in ref. [3].

The $++$ and $+-$ fermion propagators are related to the conformal scalar propagator in the usual way,

$$i[S]_{+\pm}(x-x') = i\partial\!\!\!/\, i\Delta^{\rm cf}_{+\pm}(x-x') \equiv i\partial\!\!\!/ \times \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}}\frac{1}{\Delta x^{D-2}_{+\pm}}\;. \tag{53}$$

The two conformal coordinate intervals $\Delta x^2_{+\pm}$ were defined in (21-22).

The graviton propagator takes the form of a sum of three scalar propagators times constant tensor factors [35],

$$i\Big[{}_{\mu\nu}\Delta_{\rho\sigma}\Big]_{+\pm}(x;x') = \sum_{I=A,B,C} \Big[{}_{\mu\nu}T^I_{\rho\sigma}\Big]\, i\Delta^I_{+\pm}(x;x') \; . \qquad (54)$$

Because our gauge (51) treats time and space differently it is useful to have expressions to the purely spatial parts of the Lorentz metric and the Kronecker delta,

$$\overline{\eta}_{\mu\nu} \equiv \eta_{\mu\nu} + \delta^0_\mu \delta^0_\nu \qquad \text{and} \qquad \overline{\delta}^\mu_\nu \equiv \delta^\mu_\nu - \delta^\mu_0 \delta^0_\nu \; . \qquad (55)$$

With this convention, the three tensor factors in (54) are,

$$\Big[{}_{\mu\nu}T^A_{\rho\sigma}\Big] = 2\,\overline{\eta}_{\mu(\rho}\overline{\eta}_{\sigma)\nu} - \frac{2}{D\!-\!3}\overline{\eta}_{\mu\nu}\overline{\eta}_{\rho\sigma} \; , \qquad (56)$$

$$\Big[{}_{\mu\nu}T^B_{\rho\sigma}\Big] = -4\delta^0_{(\mu}\overline{\eta}_{\nu)(\rho}\delta^0_{\sigma)} \; , \qquad (57)$$

$$\Big[{}_{\mu\nu}T^C_{\rho\sigma}\Big] = \frac{2}{(D\!-\!2)(D\!-\!3)}\Big[(D\!-\!3)\delta^0_\mu\delta^0_\nu + \overline{\eta}_{\mu\nu}\Big]\Big[(D\!-\!3)\delta^0_\rho\delta^0_\sigma + \overline{\eta}_{\rho\sigma}\Big] \; . \qquad (58)$$

We follow the usual convention that parenthesized indices are symmetrized.

The three scalar propagators in (54) can be expressed in terms of the appropriate de Sitter invariant length function $y_{+\pm}(x;x')$,

$$y_{+\pm}(x;x') \equiv a(t)a(t')H^2 \Delta x^2_{+\pm}(x;x') \; . \qquad (59)$$

The B-type and C type propagators are hypergeometric functions,

$$i\Delta^B_{+\pm}(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!2)\Gamma(1)}{\Gamma(\frac{D}{2})}\, {}_2F_1\Big(D\!-\!2,1;\frac{D}{2};1\!-\!\frac{y_{+\pm}}{4}\Big) \; , \qquad (60)$$

$$i\Delta^C_{+\pm}(x;x') = \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!3)\Gamma(2)}{\Gamma(\frac{D}{2})}\, {}_2F_1\Big(D\!-\!3,2;\frac{D}{2};1\!-\!\frac{y_{+\pm}}{4}\Big) \; . \qquad (61)$$

The A-type propagator has the intimidating expansion,

$$\begin{aligned} i\Delta^A_{+\pm}(x;x') &= i\Delta^{\rm cf}_{+\pm}(x;x') \\ &+ \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!1)}{\Gamma(\frac{D}{2})} \Bigg\{ \frac{D}{D\!-\!4} \frac{\Gamma^2(\frac{D}{2})}{\Gamma(D\!-\!1)} \Big(\frac{4}{y_{+\pm}}\Big)^{\frac{D}{2}-2} - \pi \cot\Big(\frac{\pi}{2}D\Big) + \ln(aa')\Bigg\} \\ &+ \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \sum_{n=1}^{\infty} \Bigg\{ \frac{1}{n} \frac{\Gamma(n\!+\!D\!-\!1)}{\Gamma(n\!+\!\frac{D}{2})} \Big(\frac{y_{+\pm}}{4}\Big)^n - \frac{1}{n\!-\!\frac{D}{2}\!+\!2} \frac{\Gamma(n\!+\!\frac{D}{2}\!+\!1)}{\Gamma(n\!+\!2)} \Big(\frac{y_{+\pm}}{4}\Big)^{n-\frac{D}{2}+2} \Bigg\} . \end{aligned} \qquad (62)$$

We need retarded Green's functions in order to develop an expansion for the full fields in terms of the free fields of the initial time. There is a very simple relation between the retarded Green's function of any field and the corresponding $++$ and $+-$ propagators. If the field's kinetic operator is $\mathcal{D}$ then the two propagators obey,

$$\mathcal{D}\, i\Delta_{++}(x;x') = i\delta^D(x-x') \qquad \text{and} \qquad \mathcal{D}\, i\Delta_{+-}(x;x') = 0 \; . \tag{63}$$

The associated retarded Green's function is,

$$G(x;x') = i\Big[i\Delta_{++}(x;x') - i\Delta_{+-}(x;x')\Big] \; . \tag{64}$$

From (63) one easily sees that it obeys the required equation,

$$\mathcal{D}\, G(x;x') = -\delta^D(x-x') \; . \tag{65}$$

It also obeys the retarded condition of vanishing for $\eta < \eta'$ because the conformal coordinate intervals (21) and (22) are equal in that case.

The Heisenberg operator equation for the fermion is,

$$\begin{aligned} i\partial\!\!\!/\Psi = & \frac{\kappa}{2}\Big\{-hi\partial\!\!\!/ + h^{\mu\nu}\gamma_\mu i\partial_\nu + h_{\mu\rho,\sigma}\gamma^\mu J^{\rho\sigma}\Big\}\Psi \\ & -\kappa^2\Big\{\frac{1}{8}h^2 - \frac{1}{4}h^{\rho\sigma}h_{\rho\sigma}\Big] i\partial\!\!\!/\Psi + \kappa^2\Big\{\frac{1}{4}hh^{\mu\nu} - \frac{3}{8}h^{\mu\rho}h_\rho^{\ \nu}\Big]\gamma_\mu i\partial_\nu\Psi \\ & +\kappa^2\Bigg\{\frac{1}{4}hh_{\mu\rho,\sigma} - \frac{1}{8}h^\nu_{\ \rho}h_{\nu\sigma,\mu} - \frac{1}{4}(h^\nu_{\ \mu}h_{\nu\rho})_{,\sigma} - \frac{1}{4}h^\nu_{\ \sigma}h_{\mu\rho,\nu}\Bigg\}\gamma^\mu J^{\rho\sigma}\Psi + O(\kappa^3) \; . \end{aligned} \tag{66}$$

We only require the analogous equation for the graviton to first order, and we only need the terms that involve fermions,

$$\begin{aligned} D_{\mu\nu}^{\ \ \rho\sigma}h_{\rho\sigma} = & \frac{\kappa}{2}\Bigg\{-\eta_{\mu\nu}\overline{\Psi}i\partial\!\!\!/\Psi + \overline{\Psi}\gamma_\mu i\partial_\nu\Psi - \partial^\sigma\Big[\overline{\Psi}\gamma_\mu J_{\nu\sigma}\Psi\Big]\Bigg\} + O(\kappa^2) \\ & + \Big(\text{Pure Gravity Interactions}\Big) \; . \end{aligned} \tag{67}$$

The next step is to expand Heisenberg operators in powers of κ,

$$\Psi = \Psi_0 + \kappa\Psi_1 + \kappa^2\Psi_2 + \dots \quad , \quad h_{\mu\nu} = h_{0\mu\nu} + \kappa h_{1\mu\nu} + \dots \; . \tag{68}$$

Of course the zeroth order equations ($D_{\mu\nu}^{\ \ \rho\sigma}h_{0\rho\sigma} = 0$ and $i\partial\!\!\!/\Psi_0 = 0$) just give the zeroth order solutions $h_{0\mu\nu}$ and Ψ_0 we already encountered in expressions (32) and (33), respectively. The order κ fermion equation implies,

$$i\partial\!\!\!/\Psi_1 = \frac{1}{2}\Big[-h_0\, i\partial\!\!\!/ + h_0^{\mu\nu}\gamma_\mu i\partial_\nu + h_0^{\mu\rho,\sigma}\gamma_\mu J_{\rho\sigma}\Big]\Psi_0 = \frac{1}{2}\Big[h_0^{\mu\nu}\gamma_\mu i\partial_\nu + h_0^{\mu\rho,\sigma}\gamma_\mu J_{\rho\sigma}\Big]\Psi_0 \; . \tag{69}$$

Term	Contribution to $\kappa^2 i\partial\!\!\!/ \Psi_2(x)$
1a	$-\frac{1}{4}\kappa^2 {h_0}^{\mu\nu}(x)\gamma_\mu i\partial_\nu i\partial\!\!\!/ \int d^Dx'\, G_{\rm cf}(x{-}x')h_0^{\rho\sigma}(x')\gamma_\rho i\partial'_\sigma\Psi_0(x')$
1b	$-\frac{1}{4}\kappa^2 {h_0}^{\mu\nu}(x)\gamma_\mu i\partial_\nu i\partial\!\!\!/ \int d^Dx'\, G_{\rm cf}(x{-}x')h_0^{\rho\sigma,\beta}(x')\gamma_\rho J_{\sigma\beta}\Psi_0(x')$
2a	$-\frac{1}{4}\kappa^2 {h_0}^{\mu\nu,\alpha}(x)\gamma_\mu J_{\nu\alpha} i\partial\!\!\!/ \int d^Dx'\, G_{\rm cf}(x{-}x')h_0^{\rho\sigma}(x')\gamma_\rho i\partial'_\sigma\Psi_0(x')$
2b	$-\frac{1}{4}\kappa^2 {h_0}^{\mu\nu,\alpha}(x)\gamma_\mu J_{\nu\alpha} i\partial\!\!\!/ \int d^Dx'\, G_{\rm cf}(x{-}x')h_0^{\rho\sigma,\beta}(x')\gamma_\rho J_{\sigma\beta}\Psi_0(x')$
3a	$-\frac{1}{4}\kappa^2 \int d^Dx'\, [{}^{\mu\nu}G^{\rho\sigma}](x;x')\overline{\Psi}_0(x')\gamma_\rho i\partial'_\sigma\Psi_0(x') \times \gamma_\mu i\partial_\nu\Psi_0(x)$
3b	$\frac{1}{4}\kappa^2 \int d^Dx'\, [{}^{\mu\nu}G^{\rho\sigma}](x;x')[\overline{\Psi}_0(x')\gamma_\rho J_{\sigma\beta}\Psi_0(x')]^{,\beta} \times \gamma_\mu i\partial_\nu\Psi_0(x)$
4a	$-\frac{1}{4}\kappa^2 \partial^\alpha \int d^Dx'\, [{}^{\mu\nu}G^{\rho\sigma}](x;x')\overline{\Psi}_0(x')\gamma_\rho i\partial'_\sigma\Psi_0(x') \times \gamma_\mu J_{\nu\alpha}\Psi_0(x)$
4b	$\frac{1}{4}\kappa^2 \partial^\alpha \int d^Dx'\, [{}^{\mu\nu}G^{\rho\sigma}](x;x')[\overline{\Psi}_0(x')\gamma_\rho J_{\sigma\beta}\Psi_0(x')]^{,\beta} \times \gamma_\mu J_{\nu\alpha}\Psi_0(x)$
5	$-\frac{3}{8}\kappa^2 {h_0}^{\mu\nu}(x)\, h_0^{\rho\sigma}(x)\, \eta_{\mu\rho}\gamma_\nu i\partial_\sigma\Psi_0(x)$
6	$-\frac{1}{8}\kappa^2 h_0^{\mu\nu}(x)\, h_0^{\rho\sigma,\alpha}(x)\, \eta_{\mu\rho}\gamma_\alpha J_{\nu\sigma}\Psi_0(x)$
7	$-\frac{1}{4}\kappa^2 [h_0^{\mu\nu}(x)\, h_0^{\rho\sigma}(x)]^{,\alpha}\, \eta_{\mu\rho}\gamma_\nu J_{\sigma\alpha}\Psi_0(x)$
8	$-\frac{1}{4}\kappa^2 h_0^{\mu\nu}(x)\, h_0^{\rho\sigma,\alpha}(x)\, \eta_{\mu\alpha}\gamma_\rho J_{\sigma\nu}\Psi_0(x)$

Table 1: Free Field Expansion of $\kappa^2 i\partial\!\!\!/ \Psi_2(x)$

Hence the order κ correction to the fermion operator is,

$$\Psi_1(x) = -\frac{1}{2} i\partial\!\!\!/ \int d^D x' G_{\rm cf}(x-x') \Big[h_0^{\mu\nu}(x')\gamma_\mu i\partial'_\nu + h_0^{\mu\rho,\sigma}(x')\gamma_\mu J_{\rho\sigma}\Big] \Psi_0(x'). \quad (70)$$

In the same way we obtain the first order correction to the graviton,

$$h_1^{\mu\nu}(x) = -\frac{1}{2}\int d^D x' [{}^{\mu\nu}G^{\rho\sigma}](x;x') \Big\{ \overline{\Psi}_0(x')\gamma_\rho i\partial'_\sigma \Psi_0(x') - \partial'^\alpha \Big[\overline{\Psi}_0(x')\gamma_\rho J_{\sigma\alpha}\Psi_0(x')\Big]\Big\} + \Big(\text{Pure Gravity Terms}\Big) \; . \quad (71)$$

This brings us to the order κ^2 correction to the fermion. We can of course drop any factors of $i\partial\!\!\!/\Psi_0 = 0$. With some further simplifications based on the first order equations we reach the form,

$$i\partial\!\!\!/\Psi_2 = \frac{1}{2}\Big[h_0^{\mu\nu}\gamma_\mu i\partial_\nu + h_0^{\mu\rho,\sigma}\gamma_\mu J_{\rho\sigma}\Big]\Psi_1 + \frac{1}{2}\Big[h_1^{\mu\nu}\gamma_\mu i\partial_\nu + h_1^{\mu\rho,\sigma}\gamma_\mu J_{\rho\sigma}\Big]\Psi_0 - \frac{3}{8} h_0^{\mu\rho} h_{0\;\,\rho}^{\;\nu}\gamma_\mu i\partial_\nu\Psi_0 - \Big[\frac{1}{8} h_{0\;\,\nu}^{\;\rho} h^{\nu\sigma,\mu} + \frac{1}{4}\Big(h_{0\;\,\nu}^{\;\mu} h_0^{\nu\rho}\Big)^{,\sigma} + \frac{1}{4} h_{0\;\,\nu}^{\;\sigma} h^{\mu\rho,\nu}\Big]\gamma_\mu J_{\rho\sigma}\Psi_0 \; . \quad (72)$$

Table 1 gives the free field expansion of $i\partial\!\!\!/\Psi_2$, excepting only the contributions from the pure gravity corrections to $h_{1\mu\nu}$ which vanish when the expectation value in (40) is taken.

Each contribution to Table 1 contains three free fields. It remains to evaluate the source term (40),

$$\kappa^2 \Big\langle \Omega \Big| \Big\{ i\partial\!\!\!/\Psi_2(x), b^+(\vec{k},s)\Big\}\Big| \Omega\Big\rangle + O(\kappa^4) \; . \quad (73)$$

This is done by using the anti-commutator to absorb a Ψ_0 and then exploiting the fundamental Schwinger-Keldysh relation (9) to express the expectation value of the two remaining free fields in terms of the propagator of appropriate polarity. To be definite, suppose the two remaining free fields are scalars $\varphi_0(x)$ and $\varphi_0(x')$. Here is where the factor ordering matters. From relation (9) we see that the $+-$ propagator emerges from the order $\varphi_0(x') \times \varphi(x)$,

$$\Big\langle \Omega \Big| \varphi_0(x')\varphi_0(x)\Big|\Omega\Big\rangle = i\Delta_{+-}(x;x') \; . \quad (74)$$

The order $\varphi_0(x) \times \varphi_0(x')$ gives the $-+$ propagator, however, this is equivalent to the $++$ propagator when account is taken of the factor of $\theta(\eta-\eta')$ in the retarded Green's function that is always present,

$$G(x;x') \times \Big\langle \Omega\Big|\varphi_0(x)\varphi_0(x')\Big|\Omega\Big\rangle = G(x;x') \times i\Delta_{-+}(x;x') \; , \quad (75)$$

$$= G(x;x') \times i\Delta_{++}(x;x') \; . \quad (76)$$

Term	Contribution to $\kappa^2\Big\langle\Omega\Big\vert\Big\{i\not\partial\Psi_2(x), b^\dagger(\vec{k},s)\Big\}\Big\vert\Omega\Big\rangle$
1a	$\frac{i\kappa^2}{4}\int d^Dx'\, i[^{\mu\nu}\Delta^{\rho\sigma}]_{++}(x;x')\,\gamma_\mu\partial_\nu\not\partial\, G_{\rm cf}(x-x')\gamma_\rho\partial'_\sigma\Xi_0(x')$
1b	$\frac{\kappa^2}{4}\int d^Dx'\,\partial'^\beta i[^{\mu\nu}\Delta^{\rho\sigma}]_{++}(x;x')\,\gamma_\mu\partial_\nu\not\partial\, G_{\rm cf}(x-x')\gamma_\rho J_{\sigma\beta}\Xi_0(x')$
2a	$\frac{\kappa^2}{4}\int d^Dx'\,\partial^\alpha i[^{\mu\nu}\Delta^{\rho\sigma}]_{++}(x;x')\,\gamma_\mu J_{\nu\alpha}\not\partial\, G_{\rm cf}(x-x')\gamma_\rho\partial'_\sigma\Xi_0(x')$
2b	$-\frac{i\kappa^2}{4}\int d^Dx'\,\partial^\alpha\partial'^\beta i[^{\mu\nu}\Delta^{\rho\sigma}]_{++}(x;x')\,\gamma_\mu J_{\nu\alpha}\not\partial\, G_{\rm cf}(x-x')\gamma_\rho J_{\sigma\beta}\Xi_0(x')$
3a	$\frac{i\kappa^2}{4}\int d^Dx'\,[^{\mu\nu}G^{\rho\sigma}](x;x')\,\gamma_\mu\partial_\nu\not\partial\, i\Delta^{\rm cf}_{+-}(x-x')\gamma_\rho\partial'_\sigma\Xi_0(x')$
3b	$\frac{\kappa^2}{4}\int d^Dx'\,\partial'^\beta[^{\mu\nu}G^{\rho\sigma}](x;x')\,\gamma_\mu\partial_\nu\not\partial\, i\Delta^{\rm cf}_{+-}(x-x')\gamma_\rho J_{\sigma\beta}\Xi_0(x')$
4a	$\frac{\kappa^2}{4}\int d^Dx'\,\partial^\alpha[^{\mu\nu}G^{\rho\sigma}](x;x')\,\gamma_\mu J_{\nu\alpha}\not\partial\, i\Delta^{\rm cf}_{+-}(x-x')\gamma_\rho\partial'_\sigma\Xi_0(x')$
4b	$-\frac{i\kappa^2}{4}\int d^Dx'\,\partial^\alpha\partial'^\beta[^{\mu\nu}G^{\rho\sigma}](x;x')\gamma_\mu J_{\nu\alpha}\not\partial\, i\Delta^{\rm cf}_{+-}(x-x')\gamma_\rho J_{\sigma\beta}\Xi_0(x')$
5	$-\frac{3i\kappa^2}{8} i[^{\mu\nu}\Delta^{\rho\sigma}](x;x)\,\eta_{\mu\rho}\gamma_\nu\partial_\sigma\Xi_0(x)$
6	$-\frac{\kappa^2}{8}\lim_{x'\to x}\partial'^\alpha i[^{\mu\nu}\Delta^{\rho\sigma}](x;x')\,\eta_{\mu\rho}\gamma_\alpha J_{\nu\sigma}\Xi_0(x)$
7	$-\frac{\kappa^2}{4}\partial^\alpha i[^{\mu\nu}\Delta^{\rho\sigma}](x;x)\,\eta_{\mu\rho}\gamma_\nu J_{\sigma\alpha}\Xi_0(x)$
8	$-\frac{\kappa^2}{4}\lim_{x'\to x}\partial'^\alpha i[^{\mu\nu}\Delta^{\rho\sigma}](x;x')\,\eta_{\mu\alpha}\gamma_\rho J_{\sigma\nu}\Xi_0(x)$

Table 2: Contribution to $\kappa^2\Big\langle\Omega\Big\vert\Big\{i\not\partial\Psi_2(x), b^\dagger(\vec{k},s)\Big\}\Big\vert\Omega\Big\rangle$ from each term in the free field expansion.

As an example we work out the $(4b)$ term. It is useful to begin by partially integrating the ∂'^β without retaining the temporal surface term,

$$(4b) \longrightarrow -\frac{\kappa^2}{4} \int d^D x' \, \partial^\alpha \partial'^\beta \Bigl[{}^{\mu\nu} G^{\rho\sigma} \Bigr](x;x') \, \overline{\Psi}_0(x') \gamma_\rho J_{\sigma\beta} \Psi_0(x') \times \gamma_\mu J_{\nu\alpha} \Psi_0(x) \; . \tag{77}$$

The term that contributes to the effective field equations is the expectation value of the anti-commutator of $(4b)$ with $b^\dagger(\vec{k},s)$,

$$\Bigl\langle \Omega \Bigl| \Bigl\{ (4b), b^\dagger(\vec{k},s) \Bigr\} \Bigr| \Omega \Bigr\rangle = \frac{\kappa^2}{4} \int d^D x' \, \partial^\alpha \partial'^\beta \Bigl[{}^{\mu\nu} G^{\rho\sigma} \Bigr](x;x') \\ \times \Bigl\langle \Omega \Bigl| \overline{\Psi}_0(x') \gamma_\rho J_{\sigma\beta} \Bigl\{ \Psi_0(x'), b^\dagger(\vec{k},s) \Bigr\} \times \gamma_\mu J_{\nu\alpha} \Psi_0(x) \Bigr| \Omega \Bigr\rangle \; , \tag{78}$$

$$= \frac{\kappa^2}{4} \int d^D x' \, \partial^\alpha \partial'^\beta \Bigl[{}^{\mu\nu} G^{\rho\sigma} \Bigr](x;x') \\ \times \Bigl\langle \Omega \Bigl| \overline{\Psi}_0(x') \gamma_\rho J_{\sigma\beta} \, \Xi_0(x') \times \gamma_\mu J_{\nu\alpha} \Psi_0(x) \Bigr| \Omega \Bigr\rangle \; . \tag{79}$$

At this stage the spinor indices become confusing so we write them out explicitly, and also remove all $\mathbb{C}$-numbers from the expectation value,

$$\Bigl\langle \Omega \Bigl| \Bigl\{ (4b)_i, b^\dagger(\vec{k},s) \Bigr\} \Bigr| \Omega \Bigr\rangle = \frac{\kappa^2}{4} \int d^D x' \, \partial^\alpha \partial'^\beta \Bigl[{}^{\mu\nu} G^{\rho\sigma} \Bigr](x;x') \\ \times \Bigl\langle \Omega \Bigl| \overline{\Psi}_{0k}(x') \Bigl(\gamma_\rho J_{\sigma\beta} \Bigr)_{k\ell} \Xi_{0\ell}(x') \times \Bigl(\gamma_\mu J_{\nu\alpha} \Bigr)_{ij} \Psi_{0j}(x) \Bigr| \Omega \Bigr\rangle \; , \tag{80}$$

$$= \frac{\kappa^2}{4} \int d^D x' \, \partial^\alpha \partial'^\beta \Bigl[{}^{\mu\nu} G^{\rho\sigma} \Bigr](x;x') \times \Bigl(\gamma_\mu J_{\nu\alpha} \Bigr)_{ij} \times \Bigl(\gamma_\rho J_{\sigma\beta} \Bigr)_{k\ell} \Xi_{0\ell}(x') \\ \times \Bigl\langle \Omega \Bigl| \overline{\Psi}_{0k}(x') \Psi_{0j}(x) \Bigr| \Omega \Bigr\rangle \; . \tag{81}$$

The expectation value on the final line of (81) is minus the $+-$ fermion propagator,

$$\Bigl\langle \Omega \Bigl| \overline{\Psi}_{0k}(x') \Psi_{0j}(x) \Bigr| \Omega \Bigr\rangle = -i \Bigl[{}_j S_k \Bigr]_{+-}(x;x') = -i \partial\!\!\!/_{jk} i \Delta^{\rm cf}_{+-}(x-x') \; . \tag{82}$$

The minus sign derives from the fact that the preferred order for the fermion propagator is $\Psi \overline{\Psi}$. Substituting (82) into (81) gives an expression we can write without resort to explicit spinor indices,

$$\Bigl\langle \Omega \Bigl| \Bigl\{ (4b), b^\dagger(\vec{k},s) \Bigr\} \Bigr| \Omega \Bigr\rangle \\ = -\frac{i\kappa^2}{4} \int d^D x' \, \partial^\alpha \partial'^\beta \Bigl[{}^{\mu\nu} G^{\rho\sigma} \Bigr](x;x') \gamma_\mu J_{\nu\alpha} \partial\!\!\!/ i \Delta^{\rm cf}_{+-}(x-x') \gamma_\rho J_{\sigma\beta} \Xi_0(x') \; . \tag{83}$$

Table 2 gives our results for each entry in Table 1.

4 Our Rule

The first eight entries of Table 2 provide a somewhat cumbersome re-expression of the nonlocal contributions to the order κ^2 source term of expression (25). The original source has the generic form of a difference of $++$ and $+-$ terms, with each polarity being a product of contributions from the graviton and contributions from the fermion. Table 2 effects the following re-grouping,

$$\int d^Dx' \Big\{ (_{++})_h \times (_{++})_\psi - (_{+-})_h \times (_{+-})_\psi \Big\} \, \Xi_0(x') = \int d^Dx' \, (_{++})_h \\ \times \Big\{ (_{++})_\psi - (_{+-})_\psi \Big\} \, \Xi_0(x') + \int d^Dx' \Big\{ (_{++})_h - (_{+-})_h \Big\} \, (_{+-})_\psi \Xi_0(x') \; . \quad (84)$$

From expression (64) we see that the difference of $++$ and $+-$ propagators for any field gives $-i$ times the retarded Green's function of that same field. The first eight entries come in pairs of this form: $(1a)$-$(3a)$, $(1b)$-$(3b)$, $(2a)$-$(4a)$ and $(2b)$-$(4b)$. This is an illuminating insight but it represents no simplification of the original calculation.

We cannot simplify the propagators and retarded Green's functions associated with the fermion. In contradistiction to the graviton, the fermion is a "passive" field which cannot produce infrared logarithms [10, 24]. Passive fields contribute factors of order one that derive from both the infrared and the ultraviolet. To correctly recover these factors the passive field must be treated exactly.

Our simplification concerns the propagators and retarded Green's functions of the graviton. The A-type graviton polarizations are the "active" fields which cause infrared logarithms, whereas the B-type and C-type polarizations are passive. Because this particular calculation involves only one graviton propagator or Green's function there is no chance of getting an infrared logarithm unless the A-type part of the graviton propagator is involved.

Even within the A-type polarization, only the following tiny portion of the infinite series expansion (62) of $i\Delta^A_{+\pm}$ plays any role in generating infrared logarithms,

$$i\delta\Delta^A_{+\pm}(x;x') \equiv \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D\!-\!1)}{\Gamma(\frac{D}{2})} \Big\{ -\pi \cot\Big(\frac{D}{2}\pi\Big) + \ln(aa') \Big\} \\ + \frac{H^2}{8\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}\!+\!1)}{D\!-\!4} \frac{(aa')^{2-\frac{D}{2}}}{\Delta x^{D-4}_{+\pm}} \; . \quad (85)$$

Our rule is accordingly to make the following simplifications on the graviton propagators and Green's functions,

$$i\Big[{}^{\mu\nu}\Delta^{\rho\sigma}\Big](x;x') \;\longrightarrow\; \Big[{}^{\mu\nu}T_A^{\rho\sigma}\Big] \times i\delta\Delta^A_{+\pm}(x;x') \; , \qquad (86)$$

$$\Big[{}^{\mu\nu}G^{\rho\sigma}\Big](x;x') \;\longrightarrow\; \Big[{}^{\mu\nu}T_A^{\rho\sigma}\Big] \times \delta G_A(x;x') \; , \qquad (87)$$

where the A-type tensor factor is (56) and we define $\delta G_A(x;x')$ to be,

$$\delta G_A(x;x') \;\equiv\; i\Big[i\delta\Delta^A_{++}(x;x') - i\delta\Delta^A_{+-}(x;x')\Big] \; , \qquad (88)$$

$$= \frac{iH^2}{8\pi^{\frac{D}{2}}} \frac{\Gamma(\frac{D}{2}+1)}{D-4} \Big[\frac{(aa')^{2-\frac{D}{2}}}{\Delta x_{++}^{D-4}} - \frac{(aa')^{2-\frac{D}{2}}}{\Delta x_{+-}^{D-4}}\Big] \; . \qquad (89)$$

In the next section we demonstrate that applying our replacements (86-87) to the various terms in Table 2 reproduces the source term (40) whose integration gives the infrared logarithm (3).

We close this section by commenting on the relation between our rule and the replacements (7-8) that have been shown to reproduce the leading infrared logarithms to all orders in scalar models without derivative couplings [10, 23, 24]. The rules are certainly not identical but they do seem to agree, at leading logarithm order, for $D = 4$ and for certain treatments of the spatial coordinate separation. To see this, first take the $D = 4$ limits of (85) and (89),

$$\lim_{D\to 4} i\delta\Delta^A_{+\pm}(x;x') \;=\; -\frac{H^2}{8\pi^2}\Big\{\ln\Big[\frac{1}{4}H^2\Delta x^2_{+\pm}\Big] + \frac{1}{2}\Big\} \; , \qquad (90)$$

$$\lim_{D\to 4} \delta G_A(x;x') \;=\; \frac{H^2}{4\pi}\,\theta\Big(\eta-\eta'-\|\vec{x}-\vec{x}'\|\Big) \; . \qquad (91)$$

If we set $\vec{x}' = \vec{x}$ in (90) the result is,

$$-\frac{H^2}{8\pi^2}\Big\{\ln\Big[\frac{1}{4}\Big(\frac{1}{a'}-\frac{1}{a}\Big)^2\Big] + \frac{1}{2}\Big\} = \frac{H^2}{4\pi^2}\Big\{\ln\Big[\min(a,a')\Big] + O(1)\Big\} \; . \qquad (92)$$

At leading logarithm order this indeed agrees with the $D = 4$ limit of our previous rule (7). Similarly, the spatial integral of (91) is,

$$\int d^3x\, \frac{H^2}{4\pi}\,\theta\Big(\eta-\eta'-\|\vec{x}-\vec{x}'\|\Big) = \frac{1}{3H}\Big(\frac{1}{a'}-\frac{1}{a}\Big)^3 = \frac{1}{3Ha'^3}\Big[1 + O\Big(\frac{a'}{a}\Big)\Big] \; . \qquad (93)$$

At leading logarithm order this agrees with the $D = 4$ limit of the spatial integral of (8).

These correspondences seem to mean that our new replacements (86-87) would reproduce the leading infrared logarithms of the simple scalar models previously studied. However, it is straightforward to check that the old replacements (7-8) do *not* reproduce the result (40) we get from quantum gravity, whereas our new replacements (86-87) do. It therefore seems that our new rule represents a successful generalization of the old rule to the more singular environment that arises when derivative couplings are present. What is not yet clear is whether or not the rule can be simplified.

5 Analysis

The purpose of this section is to show that applying our rule (86-87) to Table 2 reproduces the result (40) of our explicit computation. We begin by observing that any terms involving derivatives of Ξ_0 cannot contribute at leading order. That reduces the problem to considering the nonlocal contributions $1b$, $2b$, $3b$ and $4b$, and the local contributions 6, 7 and 8. The local contributions were evaluated in an earlier effort to understand our result (40) on a qualitative level by making the Hartree approximation [4], so we concentrate on the nonlocal contributions. We first introduce a systematic classification for the myriads of distinct terms they give when the A-type tensor factor and the factors of $\gamma_\mu J_{\nu\alpha}$ and $\gamma_\rho J_{\sigma\beta}$ are broken up. Then we explicitly evaluate four of the contributions from ($2b$) as an example. Final results for all nonlocal and local contributions are reported in tables.

It is important to understand that we only seek the leading late time behaviors of the various source terms in Table 2. By considering the form of quantum gravity interactions we see that the one loop mode function can be enhanced by at most a single infrared logarithm [24],

$$\kappa^2 \Big\langle \Omega \Big| \Big\{ \Psi_2(x), b^\dagger(\vec{k}, s) \Big\} \Big| \Omega \Big\rangle \sim \kappa^2 H^2 \times \ln(a) \times \Xi_0(x) \; . \tag{94}$$

The source terms of Table 2 should be $i\not{\partial}$ times this, which gives the loop counting parameter $\kappa^2 H^2$ times $iaH\gamma^0 \Xi_0(x)$.

Now consider how the derivatives of Table 2 act. Any which act on the tree order mode function $\Xi_0(x')$ will bring down factors of the wave number k. This factor of k must persist, even after the integration over x'^μ, because the integral remains finite for $\vec{k} = 0$. Further, this factor of k will always be accompanied by a factor of $1/a$ to make the wave number physical. It follows that the fastest growth possible for any $\partial' \Xi_0(x')$ term is $\ln(a) k \gamma^0 \Xi_0(x)$. We can therefore forget about nonlocal contributions from ($1a$), ($2a$), ($3a$) or ($4a$), and also the local contribution from (5). For the same reason we can make the following simplification in the nonlocal contributions from ($1b$), ($2b$), ($3b$) and ($4b$),

$$\Xi_0(x';\vec{k},s) = \Xi_0(x;\vec{k},s) \times e^{-ik_\mu (x-x')^\mu} \longrightarrow \Xi_0(x;\vec{k},s) \times 1 \; . \tag{95}$$

We turn now to the problem of classifying the many distinct contributions that derive from ($1b$), ($2b$), ($3b$) and ($4b$). These four terms all involve a single factor of the A-type tensor and either one or two factors of the Lorentz generators. The A-type tensor indices are purely spatial, for example,

$$\Bigl[{}^{\mu\nu} T^{\rho\sigma}_{A}\Bigr] \times \Bigl(\gamma_\mu J_{\nu\alpha}\Bigr) \times \Bigl(\gamma_\rho J_{\sigma\beta}\Bigr) = \Bigl[{}^{ij} T^{k\ell}_{A}\Bigr] \times \Bigl(\gamma_i J_{j\alpha}\Bigr) \times \Bigl(\gamma_k J_{\ell\beta}\Bigr) \; . \tag{96}$$

Our classification system is based upon decomposing the $\gamma \cdot J$ factors as follows,

$$\gamma_i J_{j\beta} = \frac{i}{2} \gamma_{[i} \gamma_j \gamma_{\alpha]} + \frac{i}{2} \delta_{i\alpha} \gamma_j - \frac{i}{2} \delta_{ij} \gamma_\alpha \; . \tag{97}$$

The totally anti-symmetrized term drops out because the A-type tensor factor is symmetric in i and j. We label the other two terms by Roman numerals "I" and "II" as follows,

$$\mathrm{I} \quad \Longleftrightarrow \quad \frac{i}{2} \delta_{i\alpha} \gamma_j \qquad \text{and} \qquad \frac{i}{2} \delta_{k\beta} \gamma_\ell \; , \tag{98}$$

$$\mathrm{II} \quad \Longleftrightarrow \quad \frac{i}{2} \delta_{ij} \gamma_\alpha \qquad \text{and} \qquad \frac{i}{2} \delta_{k\ell} \gamma_\beta \; . \tag{99}$$

The indices α — which appears only in ($2b$) and ($4b$) — and β, contract into derivative operators ∂^α and ∂'^β that act upon the graviton propagator or retarded Green's function. The indices on I-type terms must be spatial but those on II-type terms can be either spatial — denoted by just II — or temporal — denoted by II$'$. The type-II term always produces a contraction of the A-type tensor factor, for example,

$$\Bigl[{}^{ij} T^{k\ell}_{A}\Bigr] \times \delta_{k\ell} = -\frac{4}{D-3} \, \delta^{ij} \; . \tag{100}$$

However, the type-I term can receive distinct contributions from each of the three terms in $[{}^{ij}T^{k\ell}_A]$. Where the results are distinct we label these "A", "B" and "C" as follows,

$$\mathrm{A} \Longleftrightarrow \delta^{ik}\delta^{j\ell} \qquad , \qquad \mathrm{B} \Longleftrightarrow \delta^{i\ell}\delta^{jk} \qquad , \qquad \mathrm{C} \Longleftrightarrow -\frac{2}{D\!-\!3}\delta^{ij}\delta^{k\ell} \; . \tag{101}$$

The various classifications are arranged in a prescribed order, and are separated by periods. First comes the term designation — 1b, 2b, 3b or 4b. Next comes the leftmost of the $\gamma \cdot J$ factor designations — I, II or II′. If there is a second $\gamma \cdot J$ factor, its designator comes next. The final designator is the A, B, or C from the tensor factor, with no designator denoting the presence of all three terms. As an example, consider the full $(1b)$ term,

$$\frac{\kappa^2}{4}\int d^Dx' \, \partial'^{\beta} i\delta\Delta^A_{++}(x;x') \left[{}^{ij}T^{k\ell}_A\right] \gamma_i \partial_j \not{\partial} \, G_{\rm cf}(x\!-\!x')\gamma_k J_{\ell\beta}\Xi_0(x') \; . \tag{102}$$

The 1b.II′ contribution is,

$$\begin{aligned} \text{1b.II}' &= \frac{\kappa^2}{4}\int d^Dx' \times \Bigl(\partial'^0\Bigr) \times i\delta\Delta^A_{++}(x;x') \times \Bigl(-\frac{4}{D\!-\!3}\,\delta^{ij}\Bigr) \\ &\qquad \times \gamma_i\partial_j \not{\partial} \, G_{\rm cf}(x\!-\!x') \times \Bigl(-\frac{i}{2}\gamma_0\Bigr) \times \Xi_0(x') \; , &(103)\\ &= \frac{i\kappa^2}{2(D-3)}\int d^Dx' \, \partial'_0 i\delta\Delta^A_{++}(x;x') \overline{\not{\partial}} \not{\partial} \, G_{\rm cf}(x\!-\!x')\gamma^0 \Xi_0(x') \; . &(104) \end{aligned}$$

In contrast, the contribution from 1b.I.B is,

$$\begin{aligned} \text{1b.I.B} &= \frac{\kappa^2}{4}\int d^Dx' \times \Bigl(\partial'_k\Bigr) \times i\delta\Delta^A_{++}(x;x') \times \Bigl(\delta^{i\ell}\delta^{jk}\Bigr) \\ &\qquad \times \gamma_i\partial_j \not{\partial} \, G_{\rm cf}(x\!-\!x') \times \Bigl(\frac{i}{2}\gamma_\ell\Bigr) \times \Xi_0(x') \; , &(105)\\ &= -\frac{i\kappa^2}{8}\int d^Dx' \, \partial_k i\delta\Delta^A_{++}(x;x')\gamma^\ell \partial_k \not{\partial} \, G_{\rm cf}(x\!-\!x')\gamma^\ell \Xi_0(x') \; . &(106) \end{aligned}$$

Note the minus sign from converting ∂'_k to $-\partial_k$.

To describe the evaluation technique we have chosen four of the contributions from $(2b)$: 2b.II′.I, 2b.II.I, 2b.I.II′ and 2b.I.II. Each of these involves a single derivative of the conformal Green's function,

$$\not{\partial} G_{\rm cf}(x\!-\!x') = \frac{i\Gamma(\frac{D}{2})}{2\pi^{\frac{D}{2}}}\Bigl[-\frac{1}{\Delta x^D_{++}} + \frac{1}{\Delta x^D_{+-}}\Bigr]\gamma^\mu \Delta x_\mu \; . \tag{107}$$

Term	Contribution from $\Big\langle\Omega\Big\vert\Big\{(1b),b^\dagger(\vec{k},s)\Big\}\Big\vert\Omega\Big\rangle$	Coef.
1b.II′	$\frac{i\kappa^2}{2(D-3)}\int d^Dx'\partial_0' i\delta\Delta^A_{++}(x;x')\overline{\partial}\!\!\!/\partial\!\!\!/\, G_{\rm cf}(x-x')\gamma^0\Xi_0(x')$	$+\frac{1}{4}$
1b.II	$-\frac{i\kappa^2}{2(D-3)}\int d^Dx'\partial_k i\delta\Delta^A_{++}(x;x')\overline{\partial}\!\!\!/\partial\!\!\!/\, G_{\rm cf}(x-x')\gamma^k\Xi_0(x')$	$-\frac{1}{4}$
1b.I.A	$-\frac{i\kappa^2}{8}\int d^Dx'\partial_k i\delta\Delta^A_{++}(x;x')\gamma^k\partial_\ell\partial\!\!\!/\, G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$-\frac{1}{16}$
1b.I.B	$-\frac{i\kappa^2}{8}\int d^Dx'\partial_k i\delta\Delta^A_{++}(x;x')\gamma^\ell\partial_k\partial\!\!\!/\, G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$-\frac{3}{16}$
1b.I.C	$\frac{i\kappa^2}{4(D-3)}\int d^Dx'\partial_k i\delta\Delta^A_{++}(x;x')\overline{\partial}\!\!\!/\partial\!\!\!/\, G_{\rm cf}(x-x')\gamma^k\Xi_0(x')$	$+\frac{1}{8}$
Total	$\frac{\kappa^2}{4}\int d^Dx'\partial'^\beta i\delta\Delta^A_{++}(x;x')[^{ij}T^{k\ell}_A]\gamma_i\partial_j\partial\!\!\!/ G_{\rm cf}(x-x')\gamma_k J_{\ell\beta}\Xi_0(x')$	$-\frac{1}{8}$

Table 3: The full contribution from each $(1b)$ term consists of its numerical coefficient times $\frac{i\kappa^2H^2}{16\pi^2}Ha\gamma^0\Xi_0(x)$.

They also involve two derivatives of the A-type propagator,

$$\partial_k\partial_0' i\delta\Delta^A_{+\pm}(x;x') \;=\; \frac{H^2\Gamma(\frac{D}{2}+1)}{8\pi^{\frac{D}{2}}(aa')^{\frac{D}{2}-2}}\,\Delta x^k\Bigg\{\frac{(D-2)\Delta\eta}{\Delta x^D_{+\pm}}+\frac{(D-4)Ha'}{2\Delta x^{D-2}_{+\pm}}\Bigg\}\,, \tag{108}$$

$$\partial_k\partial_0 i\delta\Delta^A_{+\pm}(x;x') \;=\; \frac{H^2\Gamma(\frac{D}{2}+1)}{8\pi^{\frac{D}{2}}(aa')^{\frac{D}{2}-2}}\,\Delta x^k\Bigg\{-\frac{(D-2)\Delta\eta}{\Delta x^D_{+\pm}}+\frac{(D-4)Ha}{2\Delta x^{D-2}_{+\pm}}\Bigg\}\,, \tag{109}$$

$$\partial_k\partial_\ell i\delta\Delta^A_{+\pm}(x;x') \;=\; \frac{H^2\Gamma(\frac{D}{2}+1)}{8\pi^{\frac{D}{2}}(aa')^{\frac{D}{2}-2}}\Bigg\{-\frac{\delta^{k\ell}}{\Delta x^{D-2}_{+\pm}}+\frac{(D-2)\Delta x^k\Delta x^\ell}{\Delta x^D_{+\pm}}\Bigg\}\,. \tag{110}$$

In these and all subsequent expressions we define the coordinate differences,

$$\Delta x^\mu \equiv x^\mu - x'^\mu \qquad \text{and} \qquad \Delta\eta \equiv \eta-\eta'\,. \tag{111}$$

Each of the four terms we are considering takes the form of a common integral operator acting upon a different integrand. The integral operator is,

$$\frac{\kappa^2H^2}{64\pi^D}\frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}{D-3}\int d^Dx'\,(aa')^{2-\frac{D}{2}}\Bigg[-\frac{1}{\Delta x^D_{++}}+\frac{1}{\Delta x^D_{+-}}\Bigg]\times\,. \tag{112}$$

The four different integrands are,

$$\text{2b.II}'\text{.I} \;\Longrightarrow\; \gamma^0\gamma^\mu\gamma^k\Xi_0(x')\times\Delta x_\mu\Delta x^k\Bigg\{-\frac{(D-2)\Delta\eta}{\Delta x^D_{++}}+\frac{(D-4)Ha}{2\Delta x^{D-2}_{++}}\Bigg\}\,, \tag{113}$$

Term	Contribution from $\left\langle\Omega\middle|\left\{(3b), b^{\dagger}(\vec{k},s)\right\}\middle|\Omega\right\rangle$	Coef.
3b.II′	$\frac{i\kappa^2}{2(D-3)}\int d^Dx' \partial_0' \delta G_A(x;x') \overline{\partial}\partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x')\gamma^0\Xi_0(x')$	$+\frac{1}{4}$
3b.II	$-\frac{i\kappa^2}{2(D-3)}\int d^Dx' \partial_k \delta G_A(x;x') \overline{\partial}\partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x')\gamma^k\Xi_0(x')$	$-\frac{1}{4}$
3b.I.A	$-\frac{i\kappa^2}{8}\int d^Dx' \partial_k \delta G_A(x;x')\gamma^k\partial_\ell \partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x')\gamma^\ell\Xi_0(x')$	$-\frac{1}{16}$
3b.I.B	$-\frac{i\kappa^2}{8}\int d^Dx' \partial_k \delta G_A(x;x')\gamma^\ell\partial_k \partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x')\gamma^\ell\Xi_0(x')$	$-\frac{3}{16}$
3b.I.C	$\frac{i\kappa^2}{4(D-3)}\int d^Dx' \partial_k \delta G_A(x;x') \overline{\partial}\partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x')\gamma^k\Xi_0(x')$	$+\frac{1}{8}$
Total	$\frac{\kappa^2}{4}\int d^Dx' \partial'^\beta \delta G_A(x;x')[^{ij}T^{k\ell}_A]\gamma_i\partial_j\partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x')\gamma_k J_{\ell\beta}\Xi_0(x')$	$-\frac{1}{8}$

Table 4: The full contribution from each $(3b)$ term consists of its numerical coefficient times $\frac{i\kappa^2 H^2}{16\pi^2} H a\gamma^0\Xi_0(x)$.

$$\text{2b.II.I} \implies \gamma^k\gamma^\mu\gamma^\ell\Xi_0(x') \times \Delta x_\mu \left\{-\frac{\delta^{k\ell}}{\Delta x^{D-2}_{++}} + \frac{(D-2)\Delta x^k \Delta x^\ell}{\Delta x^D_{++}}\right\}, \qquad (114)$$

$$\text{2b.I.II}' \implies \gamma^k\gamma^\mu\gamma^0\Xi_0(x') \times \Delta x_\mu \Delta x^k \left\{-\frac{(D-2)\Delta\eta}{\Delta x^D_{++}} - \frac{(D-4)Ha'}{2\Delta x^{D-2}_{++}}\right\}, \qquad (115)$$

$$\text{2b.I.II} \implies \gamma^k\gamma^\mu\gamma^\ell\Xi_0(x') \times \Delta x_\mu \left\{-\frac{\delta^{k\ell}}{\Delta x^{D-2}_{++}} + \frac{(D-2)\Delta x^k \Delta x^\ell}{\Delta x^D_{++}}\right\}. \qquad (116)$$

If we ignore the difference between x'^μ and x^μ in the wavefunction $\Xi_0(x')$ and perform the angular averages, the various integrands take the form,

$$\text{2b.II}'\text{.I} \implies \gamma^0\Xi_0(x) \times \left\{\frac{(D-2)\Delta\eta\|\Delta\vec{x}\|^2}{\Delta x^D_{++}} - \frac{(D-4)Ha\|\Delta\vec{x}\|^2}{2\Delta x^{D-2}_{++}}\right\}, \qquad (117)$$

$$\text{2b.II.I} \implies \gamma^0\Xi_0(x) \times \left\{\frac{(D-1)\Delta\eta}{\Delta x^{D-2}_{++}} - \frac{(D-2)\Delta\eta\|\Delta\vec{x}\|^2}{\Delta x^D_{++}}\right\}, \qquad (118)$$

$$\text{2b.I.II}' \implies \gamma^0\Xi_0(x) \times \left\{\frac{(D-2)\Delta\eta\|\Delta\vec{x}\|^2}{\Delta x^D_{++}} + \frac{(D-4)Ha'\|\Delta\vec{x}\|^2}{2\Delta x^{D-2}_{++}}\right\}, \qquad (119)$$

$$\text{2b.I, II} \implies \gamma^0\Xi_0(x) \times \left\{\frac{(D-1)\Delta\eta}{\Delta x^{D-2}_{++}} - \frac{(D-2)\Delta\eta\|\Delta\vec{x}\|^2}{\Delta x^D_{++}}\right\}. \qquad (120)$$

Each of these four terms can be written as a common factor times a sum of integrals. The common factor is,

$$\frac{\kappa^2 H^2}{64\pi^{\frac{D}{2}+2}} \frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}{D-3} \, \gamma^0 \Xi_0(x) \; . \tag{121}$$

The four fundamental integrals are,

$$I_1 \equiv (D-2) \int d^D x' \, (aa')^{2-\frac{D}{2}} \Bigg[-\frac{1}{\Delta x^D_{++}} + \frac{1}{\Delta x^D_{+-}} \Bigg] \frac{\Delta\eta \|\Delta\vec{x}\|^2}{\Delta x^D_{++}} \; , \tag{122}$$

$$I_2 \equiv \frac{1}{2}(D-4) \int d^D x' \, (aa')^{2-\frac{D}{2}} \Bigg[-\frac{1}{\Delta x^D_{++}} + \frac{1}{\Delta x^D_{+-}} \Bigg] \frac{Ha \|\Delta\vec{x}\|^2}{\Delta x^{D-2}_{++}} \; , \tag{123}$$

$$I_3 \equiv (D-1) \int d^D x' \, (aa')^{2-\frac{D}{2}} \Bigg[-\frac{1}{\Delta x^D_{++}} + \frac{1}{\Delta x^D_{+-}} \Bigg] \frac{\Delta\eta}{\Delta x^{D-2}_{++}} \; , \tag{124}$$

$$I_4 \equiv \frac{1}{2}(D-4) \int d^D x' \, (aa')^{2-\frac{D}{2}} \Bigg[-\frac{1}{\Delta x^D_{++}} + \frac{1}{\Delta x^D_{+-}} \Bigg] \frac{Ha' \|\Delta\vec{x}\|^2}{\Delta x^{D-2}_{++}} \; . \tag{125}$$

And the four terms under consideration are,

$$\text{2b.II}'\text{.I} \Longrightarrow \frac{\kappa^2 H^2}{64\pi^D} \frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}{D-3} \, \gamma^0 \Xi_0(x) \times \Big\{ I_1 - I_2 \Big\} \; , \tag{126}$$

$$\text{2b.II.I} \Longrightarrow \frac{\kappa^2 H^2}{64\pi^D} \frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}{D-3} \, \gamma^0 \Xi_0(x) \times \Big\{ I_3 - I_1 \Big\} \; , \tag{127}$$

$$\text{2b.I.II}' \Longrightarrow \frac{\kappa^2 H^2}{64\pi^D} \frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}{D-3} \, \gamma^0 \Xi_0(x) \times \Big\{ I_1 + I_4 \Big\} \; , \tag{128}$$

$$\text{2b.I.II} \Longrightarrow \frac{\kappa^2 H^2}{64\pi^D} \frac{\Gamma(\frac{D}{2})\Gamma(\frac{D}{2}+1)}{D-3} \, \gamma^0 \Xi_0(x) \times \Big\{ I_3 - I_1 \Big\} \; . \tag{129}$$

The procedure for evaluating I_{1-4} is,

1. Perform the angular integrations;
2. Note that (for $\delta \to 0$) the radial integrand vanishes for $r > \Delta\eta$;
3. Make the change of variable $r = \Delta\eta\sqrt{x}$, which reduces the radial integrals to beta functions; and
4. Make the change of variable $\eta' = -1/(Hat)$.

As an example, consider I_1. The first step brings it to,

$$I_1 = (D-2) \times \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \times \int_{-\frac{1}{H}}^{\eta} d\eta' \, (aa')^{2-\frac{D}{2}} \int_0^{\infty} dr \, r^{D-2} \times \Biggl\{ -\frac{1}{[r^2 - (\Delta\eta - i\delta)^2]^{\frac{D}{2}}} + \frac{1}{[r^2 - (\Delta\eta + i\delta)^2]^{\frac{D}{2}}} \Biggr\} \frac{\Delta\eta \, r^2}{[r^2 - (\Delta\eta - i\delta)^2]^{\frac{D}{2}}} \; . \qquad (130)$$

Step 2 is accomplished by noting that the $\mp i\delta$ factors serve to fix the phase of the complex numbers that must be raised to the $D/2$ power on the final line of (130). For $r > \Delta\eta$ that phase is zero, whereas it is $\pm\pi$ for $0 < r < \Delta\eta$,

$$r^2 - (\Delta\eta \mp i\delta)^2 = e^{\pm i\pi} \times \Bigl(\Delta\eta^2 - r^2\Bigr) \qquad \text{for} \qquad 0 < r < \Delta\eta \; . \qquad (131)$$

Hence the curly bracketed term of (130) becomes,

$$-\frac{1}{[r^2 - (\Delta\eta - i\delta)^2]^{\frac{D}{2}}} + \frac{1}{[r^2 - (\Delta\eta + i\delta)^2]^{\frac{D}{2}}} = \frac{2i\sin(\frac{\pi D}{2})}{[\Delta\eta^2 - r^2]^{\frac{D}{2}}} \quad \text{for} \quad 0 < r < \Delta\eta \; . \qquad (132)$$

It follows that steps 2-4 give,

$$I_1 = (D-2) \times \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \times \int_{-\frac{1}{H}}^{\eta} d\eta' \, (aa')^{2-\frac{D}{2}} \Delta\eta \times \int_0^{\Delta\eta} dr \, r^D \frac{2i\sin(\frac{\pi D}{2}) \, e^{-i\pi\frac{D}{2}}}{[\Delta\eta^2 - r^2]^D} \; , \qquad (133)$$

$$= (D-2) \times \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \times \int_{-\frac{1}{H}}^{\eta} d\eta' \, (aa')^{2-\frac{D}{2}} \Delta\eta^{-D+2} \times i\sin\Bigl(\frac{D\pi}{2}\Bigr) e^{-i\pi\frac{D}{2}} \int_0^1 dx \, x^{\frac{D-1}{2}} (1-x)^{-D} \; , \qquad (134)$$

$$= (D-2) \times \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \times H^{D-3} a \int_{\frac{1}{a}}^{1} dt \, t^{\frac{D}{2}-2} (1-t)^{-D+2} \times i\sin\Bigl(\frac{D\pi}{2}\Bigr) e^{-i\pi\frac{D}{2}} \times \frac{\Gamma(\frac{D+1}{2})\Gamma(-D+1)}{\Gamma(\frac{-D+3}{2})} \; , \qquad (135)$$

After step four the other three integrals are,

$$I_2 = \frac{1}{2}(D-4) \times \frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})} \times H^{D-3} a \int_{\frac{1}{a}}^{1} dt \, t^{\frac{D}{2}-3} (1-t)^{-D+3}$$

$$\times i\sin\Bigl(\frac{D\pi}{2}\Bigr)e^{-i\pi(\frac{D}{2}-1)}\times\frac{\Gamma(\frac{D+1}{2})\Gamma(-D+2)}{\Gamma(\frac{-D+5}{2})}\ , \quad (136)$$

$$I_3 \;=\; (D-1)\times\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}\times H^{D-3}a\int_{\frac{1}{a}}^{1}dt\,t^{\frac{D}{2}-2}(1-t)^{-D+2}$$

$$\times i\sin\Bigl(\frac{D\pi}{2}\Bigr)e^{-i\pi(\frac{D}{2}-1)}\times\frac{\Gamma(\frac{D-1}{2})\Gamma(-D+2)}{\Gamma(\frac{-D+3}{2})}\ , \quad (137)$$

$$I_4 \;=\; \frac{1}{2}(D-4)\times\frac{2\pi^{\frac{D-1}{2}}}{\Gamma(\frac{D-1}{2})}\times H^{D-3}a\int_{\frac{1}{a}}^{1}dt\,t^{\frac{D}{2}-2}(1-t)^{-D+3}$$

$$\times i\sin\Bigl(\frac{D\pi}{2}\Bigr)e^{-i\pi(\frac{D}{2}-1)}\times\frac{\Gamma(\frac{D+1}{2})\Gamma(-D+2)}{\Gamma(\frac{-D+5}{2})}\ . \quad (138)$$

Note that (for $D=4$) the t integrands of I_1, I_3 and I_4 are finite at $t=0$. This means we make only an error of order $1/a$ by extending the range of t down to $t=0$, at which point we get another beta function,

$$\int_{\frac{1}{a}}^{1}dt\,t^{\frac{D}{2}-2}(1-t)^{-D+2} \;=\; \frac{\Gamma(\frac{D}{2}-1)\Gamma(-D+3)}{\Gamma(-\frac{D}{2}+2)}+O\Bigl(\frac{1}{a}\Bigr)\ , \quad (139)$$

$$\int_{\frac{1}{a}}^{1}dt\,t^{\frac{D}{2}-2}(1-t)^{-D+3} \;=\; \frac{\Gamma(\frac{D}{2}-1)\Gamma(-D+4)}{\Gamma(-\frac{D}{2}+3)}+O\Bigl(\frac{1}{a}\Bigr)\ . \quad (140)$$

This allows us to evaluate I_1, I_3 and I_4. Setting $D=4-\epsilon$ and taking ϵ to zero gives the following results for these three integrals,

$$I_1 \;\longrightarrow\; 2\times4\pi\times aH\times-\frac{1}{2}\times-i\frac{\pi}{2}\epsilon\times\frac{1}{16\epsilon}=\frac{1}{8}\pi^2\times iaH\ , \quad (141)$$

$$I_3 \;\longrightarrow\; 3\times4\pi\times aH\times-\frac{1}{2}\times i\frac{\pi}{2}\epsilon\times-\frac{1}{8\epsilon}=\frac{3}{8}\pi^2\times iaH\ , \quad (142)$$

$$I_4 \;\longrightarrow\; -\frac{1}{2}\epsilon\times4\pi\times aH\times\frac{1}{\epsilon}\times i\frac{\pi}{2}\epsilon\times\frac{3}{8\epsilon}=-\frac{3}{8}\pi^2\times iaH\ . \quad (143)$$

This procedure is *not* valid for the t integral of I_2 because the integrand diverges at $t=0$. The right way to evaluate the t integral in (136) is to first add and subtract the t integral from (138),

$$\int_{\frac{1}{a}}^{1}dt\,t^{\frac{D}{2}-3}(1-t)^{-D+3}$$

$$=\int_{\frac{1}{a}}^{1}dt\,t^{\frac{D}{2}-2}(1-t)^{-D+3}+\int_{\frac{1}{a}}^{1}dt\,\Bigl\{\frac{1-t}{t}\Bigr\}t^{\frac{D}{2}-2}(1-t)^{-D+3}\ . \quad (144)$$

Term	Contribution from $\Big\langle\Omega\Big\vert\Big\{(2b), b^{\dagger}(\vec{k},s)\Big\}\Big\vert\Omega\Big\rangle$	Coef.
2b.II′.II′	$-\frac{i\kappa^2(D-1)}{4(D-3)}\int d^Dx'\partial_0\partial_0' i\delta\Delta^A_{++}(x;x')\gamma^0\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^0\Xi_0(x')$	$+0$
2b.II′.II	$\frac{i\kappa^2(D-1)}{4(D-3)}\int d^Dx'\partial_0\partial_k i\delta\Delta^A_{++}(x;x')\gamma^0\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^k\Xi_0(x')$	$-\frac{3}{4}$
2b.II.II′	$-\frac{i\kappa^2(D-1)}{4(D-3)}\int d^Dx'\partial_k\partial_0' i\delta\Delta^A_{++}(x;x')\gamma^k\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^0\Xi_0(x')$	$+\frac{3}{8}$
2b.II.II	$\frac{i\kappa^2(D-1)}{4(D-3)}\int d^Dx'\partial_k\partial_\ell i\delta\Delta^A_{++}(x;x')\gamma^k\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$-\frac{3}{8}$
2b.II′.I	$-\frac{i\kappa^2}{4(D-3)}\int d^Dx'\partial_0\partial_k i\delta\Delta^A_{++}(x;x')\gamma^0\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^k\Xi_0(x')$	$+\frac{1}{4}$
2b.II.I	$-\frac{i\kappa^2}{4(D-3)}\int d^Dx'\partial_k\partial_\ell i\delta\Delta^A_{++}(x;x')\gamma^k\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$+\frac{1}{8}$
2b.I.II′	$\frac{i\kappa^2}{4(D-3)}\int d^Dx'\partial_k\partial_0' i\delta\Delta^A_{++}(x;x')\gamma^k\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^0\Xi_0(x')$	$-\frac{1}{8}$
2b.I.II	$-\frac{i\kappa^2}{4(D-3)}\int d^Dx'\partial_k\partial_\ell i\delta\Delta^A_{++}(x;x')\gamma^k\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$+\frac{1}{8}$
2b.I.I.A	$-\frac{i\kappa^2}{16}\int d^Dx'\partial_k\partial_k i\delta\Delta^A_{++}(x;x')\gamma^\ell\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$+\frac{3}{32}$
2b.I.I.BC	$-\frac{i\kappa^2(D-5)}{16(D-3)}\int d^Dx'\partial_k\partial_\ell i\delta\Delta^A_{++}(x;x')\gamma^k\,\partial\!\!\!/ G_{\rm cf}(x-x')\gamma^\ell\Xi_0(x')$	$-\frac{1}{32}$
Total	$-\frac{i\kappa^2}{4}\int d^Dx'\partial^\alpha\partial'^\beta i\delta\Delta^A_{++}(x;x')[^{ij}T^{k\ell}_A]\gamma_i J_{j\alpha}\partial\!\!\!/ G_{\rm cf}(x-x')\gamma_k J_{\ell\beta}\Xi_0(x')$	$-\frac{5}{16}$

Table 5: The full contribution from each $(2b)$ term consists of its numerical coefficient times $\frac{i\kappa^2H^2}{16\pi^2}Ha\gamma^0\Xi_0(x)$.

Term	Contribution from $\Big\langle \Omega \Big\vert \Big\{ (4b), b^{\dagger}(\vec{k}, s) \Big\} \Big\vert \Omega \Big\rangle$	Coef.
4b.II′.II′	$-\frac{i\kappa^2 (D-1)}{4(D-3)} \int d^D x' \partial_0 \partial'_0 \delta G_A(x;x') \gamma^0 \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^0 \Xi_0(x')$	$+0$
4b.II′.II	$\frac{i\kappa^2 (D-1)}{4(D-3)} \int d^D x' \partial_0 \partial_k \delta G_A(x;x') \gamma^0 \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^k \Xi_0(x')$	$-\frac{3}{4}$
4b.II.II′	$-\frac{i\kappa^2 (D-1)}{4(D-3)} \int d^D x' \partial_k \partial'_0 \delta G_A(x;x') \gamma^k \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^0 \Xi_0(x')$	$+\frac{3}{8}$
4b.II.II	$\frac{i\kappa^2 (D-1)}{4(D-3)} \int d^D x' \partial_k \partial_\ell \delta G_A(x;x') \gamma^k \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^\ell \Xi_0(x')$	$-\frac{3}{8}$
4b.II′.I	$-\frac{i\kappa^2}{4(D-3)} \int d^D x' \partial_0 \partial_k \delta G_A(x;x') \gamma^0 \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^k \Xi_0(x')$	$+\frac{1}{4}$
4b.II.I	$-\frac{i\kappa^2}{4(D-3)} \int d^D x' \partial_k \partial_\ell \delta G_A(x;x') \gamma^k \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^\ell \Xi_0(x')$	$+\frac{1}{8}$
4b.I.II′	$\frac{i\kappa^2}{4(D-3)} \int d^D x' \partial_k \partial'_0 \delta G_A(x;x') \gamma^k \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^0 \Xi_0(x')$	$-\frac{1}{8}$
4b.I.II	$-\frac{i\kappa^2}{4(D-3)} \int d^D x' \partial_k \partial_\ell \delta G_A(x;x') \gamma^k \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^\ell \Xi_0(x')$	$+\frac{1}{8}$
4b.I.I.A	$-\frac{i\kappa^2}{16} \int d^D x' \partial_k \partial_k \delta G_A(x;x') \gamma^\ell \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^\ell \Xi_0(x')$	$+\frac{3}{32}$
4b.I.I.BC	$-\frac{i\kappa^2 (D-5)}{16(D-3)} \int d^D x' \partial_k \partial_\ell \delta G_A(x;x') \gamma^k \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma^\ell \Xi_0(x')$	$-\frac{1}{32}$
Total	$-\frac{i\kappa^2}{4} \int d^D x' \partial^\alpha \partial'^\beta \delta G_A(x;x') [^{ij} T^{k\ell}_A] \gamma_i J_{j\alpha} \partial\!\!\!/ i\Delta^{\rm cf}_{+-}(x-x') \gamma_k J_{\ell\beta} \Xi_0(x')$	$-\frac{5}{16}$

Table 6: The full contribution from each $(4b)$ term consists of its numerical coefficient times $\frac{i\kappa^2 H^2}{16\pi^2} H a \gamma^0 \Xi_0(x)$.

Now extend the range in the first integral and take $D = 4$ in the second,

$$\int_{\frac{1}{a}}^{1} dt\, t^{\frac{D}{2}-3} (1-t)^{-D+3} = \int_0^1 dt\, t^{\frac{D}{2}-2} (1-t)^{-D+3} + \int_{\frac{1}{a}}^{1} dt\, \frac{1}{t} + O\Big(\frac{1}{a}, D-4\Big) = \frac{1}{\epsilon} + O(1)\; . \qquad (145)$$

This gives,

$$I_2 \longrightarrow -\frac{1}{2}\epsilon \times 4\pi \times aH \times \frac{1}{\epsilon} \times i\frac{\pi}{2}\epsilon \times \frac{3}{8\epsilon} = -\frac{3}{8}\pi^2 \times iaH\; . \qquad (146)$$

Substituting our results for I_{1-4} into expressions (126-129) gives the entries for 2b.II'.I, 2b.II.I, 2b.I.II' and 2b.I.II in Table 5. Combining the totals

Term	Coef.
6	0
7	3
8	0

Table 7: The full contribution from each term consists of its numerical coefficient times $\frac{i\kappa^2 H^2}{16\pi^2} Ha\gamma^0 \Xi_0(x)$.

from Tables 3-7 gives a result in perfect agreement with our explicit computation (40),

$$\kappa^2 \Big\langle \Omega \Big| \Big\{ i\partial\!\!\!/ \Psi_2(x), b^\dagger(\vec{k}, s) \Big\} \Big| \Omega \Big\rangle \longrightarrow \frac{\kappa^2 H^2}{16\pi^2} \, iHa\gamma^0 \Xi_0(x) \\ \times \Big\{ -\frac{1}{8} - \frac{1}{8} - \frac{5}{16} - \frac{5}{16} + 3 \Big\} = \frac{\kappa^2 H^2}{16\pi^2} \, iHa\gamma^0 \Xi_0(x) \times \frac{17}{8} \, . \quad (147)$$

6 Discussion

We have taken a major step in developing a technique to sum the series of leading infrared logarithms of inflationary quantum gravity. Our technique was to employ a previous explicit computation [3, 4] as "data" in the search for a simple operator formalism for reproducing the leading infrared logarithms. We found that only gravitons with the A-type polarization contribute, and only a single term (85) in the infinite series expansion of their propagator matters. We do not yet know if our new rule (86-87) reproduces the leading logarithms of other quantities, or if it continues to work beyond one loop for the fermion effective mode function.

One can easily see that the old rule (7-8) fails to reproduce the leading logarithms of quantum gravity. For example, consider the term 1.b.II of Table 3,

$$\text{1b.II} \equiv -\frac{i\kappa^2}{2(D-3)} \int d^D x' \partial_k i\delta\Delta^A_{++}(x;x') \overline{\partial\!\!\!/} \partial\!\!\!/ \, G_{\rm cf}(x-x') \gamma^k \Xi_0(x') \, , \quad (148)$$

$$\longrightarrow \frac{\kappa^2 H^2}{16\pi^2} \, iaH\gamma^0 \Xi_0(x) \times -\frac{1}{4} \, . \quad (149)$$

The old replacement (7) corresponds to substituting a purely temporal function for $i\delta\Delta^A_{++}(x;x')$,

$$i\delta\Delta^A_{++}(x;x') \longrightarrow \frac{H^{D-2}}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(D-1)}{\Gamma(\frac{D}{2})} \, 2\ln\Bigl[\min(a,a')\Bigr] \; . \qquad (150)$$

Of course the spatial derivative of this would give *zero* for 1b.II!

There are also problems with the closely related contribution from 3b.II,

$$\begin{aligned} \text{3b.II} \quad &\equiv \quad -\frac{i\kappa^2}{2(D-3)} \int d^Dx' \partial_k \delta G_A(x;x') \overline{\partial\!\!\!/} \partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x') \gamma^k \Xi_0(x') \; , &(151)\\ &\longrightarrow \quad \frac{\kappa^2 H^2}{16\pi^2} \, iaH\gamma^0 \Xi_0(x) \times -\frac{1}{4} \; . &(152) \end{aligned}$$

The old replacement (8) corresponds to the substitution,

$$\delta G_A(x;x') \longrightarrow \frac{\theta(t-t')\delta^{D-1}(\vec{x}-\vec{x}')}{(D-1)Ha'^{D-1}} \; . \qquad (153)$$

This gives a nonzero result, but not the right one,

$$\begin{aligned} -\frac{i\kappa^2}{2(D-3)} \int d^Dx' \partial_k \Bigl\{ \frac{\theta(t-t')\delta^{D-1}(\vec{x}-\vec{x}')}{(D-1)Ha'^{D-1}} \Bigr\} \overline{\partial\!\!\!/} \partial\!\!\!/ \, i\Delta^{\rm cf}_{+-}(x-x') \gamma^k \Xi_0(x') \\ \longrightarrow \frac{i\kappa^2}{2\pi^{\frac{D}{2}} H} \frac{\Gamma(\frac{D}{2}+1)}{D-3} \gamma^0 \Xi_0(x) \int_{-\frac{1}{H}}^{\eta} \frac{d\eta' \, e^{i\pi\frac{D}{2}}}{a'^{D-1} \Delta\eta^{D+1}} \; , \quad &(154)\\ \longrightarrow \frac{\kappa^2 H^2}{16\pi^2} \, iaH\gamma^0 \Xi_0(x) \times -4 \; . \quad &(155) \end{aligned}$$

In fact the old rule (7-8) does not give correct results for any of the *thirty* distinct nonlocal contributions of Tables 3-6! The failure of this rule — which works for models without derivative couplings [10, 23, 24] —deserves comment. Massless, minimally coupled scalars and gravitons are active fields. In order to produce infrared logarithms a theory must possess interactions involving at least one undifferentiated active field. However, there is a hierarchy of increasingly complicated ways in which this can happen:

1. The theory may involve *only* undifferentiated active fields;

2. The theory may involve active and passive fields with non-derivative interactions; and

3. The theory may involve differentiated active fields, with or without passive fields.

The relation of our new rule (86-87) to the old rule (7-8) can be understood by considering how the expectation value of a given term in the free field expansion of some operator attains leading logarithm order in each case.

When only undifferentiated active fields are present, each pair of free fields and each vertex integration must contribute to an infrared logarithm [24]. Therefore only the infrared part of the free field mode sum matters and one can effect this truncation at the level of the Yang-Feldman equations. This is the case solved by Starobinskiĭ and Yokoyama [23]. At the level of expectation values of the free field expansion it corresponds to the replacements (7-8) with $D = 4$ because there are no ultraviolet divergences at leading logarithm order.

When passive fields are present, but the active fields are not differentiated, reaching leading logarithm order still requires every active field or active Green's function to contribute to an infrared logarithm. Passive fields cannot be infrared truncated because the order $[\ln(a)]^0$ contributions they make derive from all parts of the free field mode sum. However, *precisely because passive fields cannot produce infrared logarithms* we can perform passive vertex integrations without accounting for the spacetime dependence of active fields or Green's functions. This amounts to integrating out the passive fields and then evaluating the resulting, nonlocal effective action assuming the active fields are constant— which defines the effective potential. At the level of expectation values of the free field expansion it corresponds to the replacements (7-8) with dimensional regularization on because there can be ultraviolet divergences at leading logarithm order.

The situation is vastly more complicated when differentiated active fields are present. In this case the *vertex integration* of a differentiated active field propagator or Green's function can produce an infrared logarithm, even though the integrand contains no logarithm. For the fermion wave function, *every* infrared logarithm arises in this fashion. We cannot ignore differentiated active fields because they can still contribute infrared logarithms. Neither can we ignore their spacetime dependence in performing vertex integrations, and we must retain dimensional regularization in order to define these integrals. In view of this it seems doubtful that any infrared truncated formalism can correctly represent the theory, even at leading logarithm order. The replacements (86-87) of our new rule seem to represent the appropriate generalizations of the old rule (7-8) to this more singular environment.

In addition to showing that the new rule works, our analysis provides a deeper understanding of why the fermion mode function acquires a secular enhancement whereas the scalar mode function does not [36]. The reason is spin. At late times the kinetic energies of all quanta redshift to zero. This is why we could neglect the $\partial'_\rho \gamma_\sigma \Xi_0(x')$ contributions from terms $(1a)$, $(2a)$, $(3a)$ and $(4a)$ of Table 2. A massless scalar interacts with gravity only through its kinetic energy. Inflationary particle production immerses such a scalar in a sea of infrared gravitons but they do little because the interaction is so weak. In contrast, a massless fermion possesses an additional gravitational interaction through its spin, which does not redshift. That is why we found leading order contributions from the $\gamma_\rho J_{\sigma\beta} \Xi_0(x')$ terms of $(1b)$, $(2b)$, $(3b)$ and $(4b)$ on Table 2.

Gravitons also have spin and it is natural to wonder what the sea of infrared gravtions does to itself. One could answer this by using the known one loop graviton self-energy [11] to correct the graviton mode functions, just as we have done for fermions. It would also be interesting to understand in this way the null result that has been obtained at one loop order for the graviton 1-point function [37]. In particular, can the spin-spin interaction lead to significant quantum gravitational back-reaction?

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