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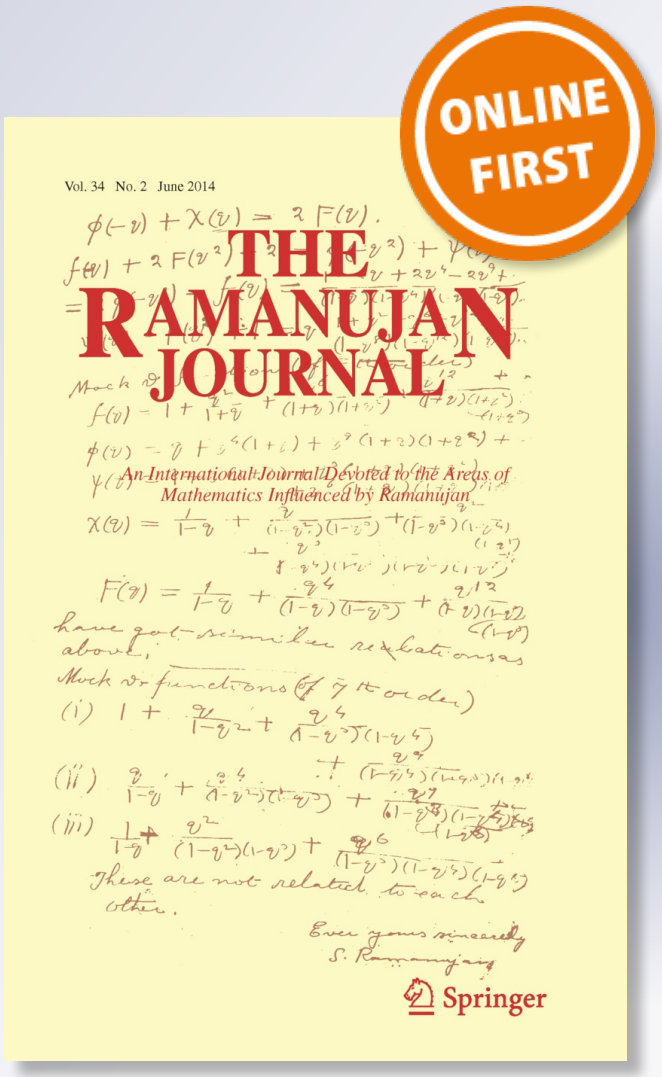


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# Continued fraction expansions with variable numerators

**Karma Dajani · Cor Kraaikamp · Niels Daniël Simon Langeveld**

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**Abstract** A new continued fraction expansion algorithm, the so-called $a/b$-expansion, is introduced and some of its basic properties, such as convergence of the algorithm and ergodicity of the underlying dynamical system, have been obtained. Although seemingly a minor variation of the regular continued fraction (RCF) expansion and its many variants (such as Nakada's $\alpha$-expansions, Schweiger's odd- and even-continued fraction expansions, and the Rosen fractions), these $a/b$-expansions behave very differently from the RCF and many important question remains open, such as the exact form of the invariant measure, and the "shape" of the natural extension.

**Keywords** Continued fractions · Ergodicity · Invariant measures

**Mathematics Subject Classification** Primary 28D05 · 11K50

## 1 Introduction

In 1940, Leighton introduced in [16] as a generalization of the regular continued fraction (RCF) expansion the so-called proper continued fractions. Given any sequence of positive integers $(a_n)_{n\geq 1}$, Leighton showed that for any real number $x$ there exists

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a (finite or infinite) sequence of integers $(b_n)_{n\geq 0}$ with $b_i \geq a_i$ if $i \geq 1$ and $b_0 = \lfloor x \rfloor$, such that $x$ can be written as a *proper* continued fraction of the form

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}}. \tag{1}$$

Note that the case $a_n = 1$ (for $n \geq 1$) is the RCF case. In case $x$ is rational, Leighton's algorithm yields that the continued fraction expansion (1) of $x$ is finite, while it is infinite when $x$ is irrational. In both cases, taking finite truncations yield rational number $p_n/q_n$, which converge to $x$. The numerators $(p_n)$ and denominators $(q_n)$ of these convergents satisfy well-known recurrence relations, viz.

$$\begin{aligned} p_{-1} &:= 1,\ p_0 := b_0,\ p_n = b_n p_{n-1} + a_n p_{n-2},\quad n \geq 1 \\ q_{-1} &:= 0,\ q_0 := 1,\quad q_n = b_n q_{n-1} + a_n q_{n-2},\quad n \geq 1. \end{aligned}$$

In case (1) is the proper continued fraction expansion of $x$, we write

$$x = [b_0; a_1/b_1, \ldots, a_n/b_n, \ldots], \quad \text{and} \quad p_n/q_n = [b_0; a_1/b_1, \ldots, a_n/b_n], \ \text{for}\, n \geq 0.$$

Periodic proper continued fraction expansions were studied by Bankier and Leighton in 1942 [4] and by Oppenheim in 1960 [19]. Oppenheim showed that for arbitrary quadratic irrational numbers $x$, infinitely many periodic proper expansions exist with a period of length 1. A conjecture was formulated by Oppenheim, which was disproved in [10].

More recently, Edward Burger and his co-authors showed in [3] that a similar result also holds if $a_n = N$ for all $n \geq 1$ and for infinitely many positive integers $N$. However, in their result, the continued fraction expansion (which is now a so-called "$N$-expansion") is not necessarily *proper*, i.e., we need not have that $b_n \geq N$ for all $n \geq 1$. In fact, Anselm and Weintraub showed in [1] that if we drop the demand that the expansion is proper, *every* $x$ between 0 and $N$ has infinitely many $N$-expansions, i.e., expansions of the form

$$x = \cfrac{N}{b_1 + \cfrac{N}{b_2 + \ddots + \cfrac{N}{b_n + \ddots}}}. \tag{2}$$

See also [9] for a different proof of this remarkable result.

Proper $N$-expansions are called *best expansions* by Anselm and Weintraub [1]. In [9] it is shown that these best expansions for (fixed) $N \in \mathbb{N}$ can be obtained via the transformation $T_N : [0, N] \to [0, N]$, defined by

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \ x \neq 0; \quad T_N(0) = 0. \tag{3}$$

Setting $d_1 = d_1(x) = \lfloor N/x \rfloor$, and $d_n = d_n(x) = d_1\left(T_N^{n-1}(x)\right)$, whenever $T_N^{n-1}(x) \neq 0$, we find

$$x = \cfrac{N}{d_1 + \cfrac{N}{d_2 + \ddots + \cfrac{N}{d_n + T_N^n(x)}}}.$$

Furthermore, in [9] it is shown that the underlying dynamical system for these best $N$-expansions is very similar to that of the classical RCF-case $N = 1$.

In this paper we propose a new proper continued fraction algorithm, which can be seen as a variation on the idea behind best $N$-expansions and the way Leighton defined his proper expansions in [16]. In Sect. 3 this new proper continued fraction expansion is introduced, and some of its basic properties are studied, while in Sect. 4 it is shown that there exists an invariant measure, and that the underlying dynamical system has strong mixing properties. Unfortunately, the explicit form of this invariant measure still escapes us; in Sect. 5 we report on a simulation which yields an approximation of the invariant measure.

Before discussing the new proper expansion, we first investigate in Sect. 2 continued fractions of the form (1) corresponding to a sequence $(a_n)_{n\geq 1}$ which is *non-proper*; i.e., for which we have that $b_n < a_n$ for at least one $n \geq 1$. As in [1] we show that if we drop the demand that the expansion is proper, and if $a_n \geq 2$ infinitely often, *every* $x$ between 0 and $a_1$ has infinitely many expansions of the form (1), of which at least one is infinite. Furthermore, if infinitely often we have that $a_n \geq 3$ we have that *every* $x$ between 0 and $a_1$ has infinitely many infinite expansions of the form (1).

Finally, in Sect. 6 we study the so-called *approximation coefficients* $\theta_n = \theta_n(x)$ of $x$ for $n \geq 1$, defined by

$$\theta_n = \theta_n(x) = \frac{q_n^2}{\prod_{i=1}^n a_i}\left|x - \frac{p_n}{q_n}\right|,$$

where $p_n/q_n$ is the $n$th convergent of the continued fraction expansion (1) of $x$. In particular we study the properties of these approximation coefficients in case $a_n = N$ and $b_n \geq N$ for some (fixed) $N \in \mathbb{N}$, so if (1) is a *proper* (i.e., *best*) $N$-expansion of $x$. The results we obtain for the $N$-expansions are variations of the (classical) results for the RCF expansion (which is the case $N = 1$). For the $a/b$-expansion, similar results have not yet been obtained.

## 2 Infinitely many expansions

### 2.1 Proper expansions

Leighton's proper continued fraction expansions can be obtained from $N$-expansions if we vary the $N$'s according to the sequence $(a_n)_{n\geq 1}$. To be more precise, given any

sequence of positive integers $(a_n)_{n\geq 1}$ and any $x \in \mathbb{R}, x \neq 0$, let $b_0 = \lfloor x \rfloor, t_0 = x - b_0$, and define inductively for $n \geq 1$,

$$t_n = T_{a_n}(t_{n-1}) \in [0, 1),$$

whenever

$$t_{n-1} := T_{a_{n-1}}\left(T_{a_{n-2}}\left(\cdots\left(T_{a_1}(x - b_0)\right)\cdots\right)\right) \neq 0,$$

and $t_n = 0$ whenever $t_{n-1} = 0$. For $n \geq 1$ and $t_{n-1} \neq 0$, the partial quotients $b_n$ are given by

$$b_n = \left\lfloor \frac{a_n}{t_{n-1}} \right\rfloor.$$

In case $x$ is irrational this obviously yields an infinite sequence $(b_n)_{n\geq 0}$ for which $b_n \geq a_n$ for $n \geq 1$ and

$$T_{a_n}\left(\ T_{a_{n-1}}\left(\cdots\left(T_{a_1}(x - b_0)\right)\cdots\right)\ \right) = \frac{a_n}{T_{a_{n-1}}\left(T_{a_{n-2}}\left(\cdots\left(T_{a_1}(x - b_0)\right)\cdots\right)\right)} - b_n,$$

from which we find that

$$x = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + t_n}}}. \tag{4}$$

In case $x \in \mathbb{Q}$, it is not difficult to show that there exists an $n \geq 1$ such that $t_n = 0$, so in (4) we have a finite continued fraction expansion of $x$; see Theorem 1.2 in [16]. Since $b_n \geq a_n$ for all $n \geq 1$, we thus find Leighton's proper continued fraction expansion of $x$ via the maps $T_{a_n}$.

## 2.2 Non-proper expansions

Note that for $n \in \mathbb{N}$, $N \geq 2$ fixed, we have for every $x \in (0, N)$ that there is at least one and that there are at most $N$ maps of the form

$$T(x) = \frac{N}{x} - d, \qquad d \in \mathbb{N},$$

for which $T(x) \in [0, N)$. The "extreme" cases are when $x \in (N/2, N)$ (in that case the digit (partial quotient) $d$ must be equal to 1) and when $x \in (0, 1)$ (for these $x$ $d$ can range from $\lfloor \frac{N}{x} \rfloor$ to $\lfloor \frac{N}{x} \rfloor - N + 1$). Since

$$T(x) = \frac{N}{x} - 1 \in (0, 1),$$

for $x \in (N/2, N)$, Theorem 1.8 from [1] immediately follows; see also [9]. Clearly, a similar result can also be obtained in the case a sequence of numerators $(a_n)_{n\geq 1}$ where $a_n \in \mathbb{N}$ is given. In case $a_n = N$ for all $n \geq 1$ and $N \in \mathbb{N}$ fixed, we are back in the case of $N$-expansions. In case $a_n \geq 2$ infinitely often, we have the following result.

**Theorem 2.1** *Let $(a_n)_{n\geq 1}$ be a sequence of positive integers, for which $a_n \geq 2$ for infinitely many $n \in \mathbb{N}$. Then for every $x \in \mathbb{R} \setminus \{0, a_1\}$ there exist infinitely many sequences $(b_n)_{n\geq 0}$ with $b_0 \in \mathbb{Z}$, $b_n \in \mathbb{N}$, and at least one of these sequences is infinite, such that (1) is a continued fraction expansion of $x$. In case $a_n \geq 3$ infinitely often, then there are infinitely many infinite expansions of the form (1).*

*Proof* First, let $b_0 \in \mathbb{Z}$ be such that $t_0 = x - b_0 \in [0, a_1)$. In case $a_1 \neq 1$ and $t_0 \in (0, a_1/2)$, we have more than one option for the next partial quotient $b_1 \in \mathbb{N}$; choose $b_1$ such that

$$t_1 = \frac{a_1}{t_0} - b_1 \in (0, a_2).$$

In all other cases (i.e., $a_1 = 1$ or $t_0 \in (a_1/2, a_1)$ when $a_1 \geq 2$) we must choose $b_1 = 1$. The partial quotients $b_2, b_3, \ldots$ and the "incomplete quotients" $t_2, t_3, \ldots$ are constructed in a similar way by induction. In case $a_n \geq 2$ for all $n$ from some index $n_0$ on, it is clear that at least each other time after $n_0$ we can choose $b_n$ from at least two possible positive integers. So without loss of generality we may assume that $a_n = 1$ infinitely often. Since we also assumed that $a_n \geq 2$ infinitely often it follows that there exist infinitely many $n$ for which $a_n = 1$ and $a_{n+1} \geq 2$. But then we must have that

$$t_n \in (0, 1) \quad \text{and} \quad t_{n+1} = \frac{1}{t_n} - \left\lfloor \frac{1}{t_n} \right\rfloor \in [0, 1) \subset \big[0, a_{n+1}\big),$$

so for $b_{n+1}$ there are $a_{n+1}$ possible values to choose from. For at least one of these choices $t_{n+2}$ is different from zero. If $a_n \geq 3$ infinitely often we have that for at least two of these choices $t_{n+2} \neq 0$. This proves the theorem.

## 3 An interesting variation: the $a/b$-continued fraction expansion

### 3.1 Introduction and definition

In view of definition (2) of the proper $N$-expansion map $T_N$, we define the continued fraction map $T : [0, 1) \to [0, 1)$ by $T(0) = 0$ and

$$T(x) = \frac{n}{x} - \left\lfloor \frac{n}{x} \right\rfloor, \qquad \text{for } x \neq 0,$$

where $n \in \mathbb{N}$ is such that $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$; see Fig. 1.

So, $T$ is defined as

$$T(x) = \frac{\lfloor \frac{1}{x} \rfloor}{x} - \left\lfloor \frac{\lfloor \frac{1}{x} \rfloor}{x} \right\rfloor, \qquad x \neq 0, \tag{5}$$

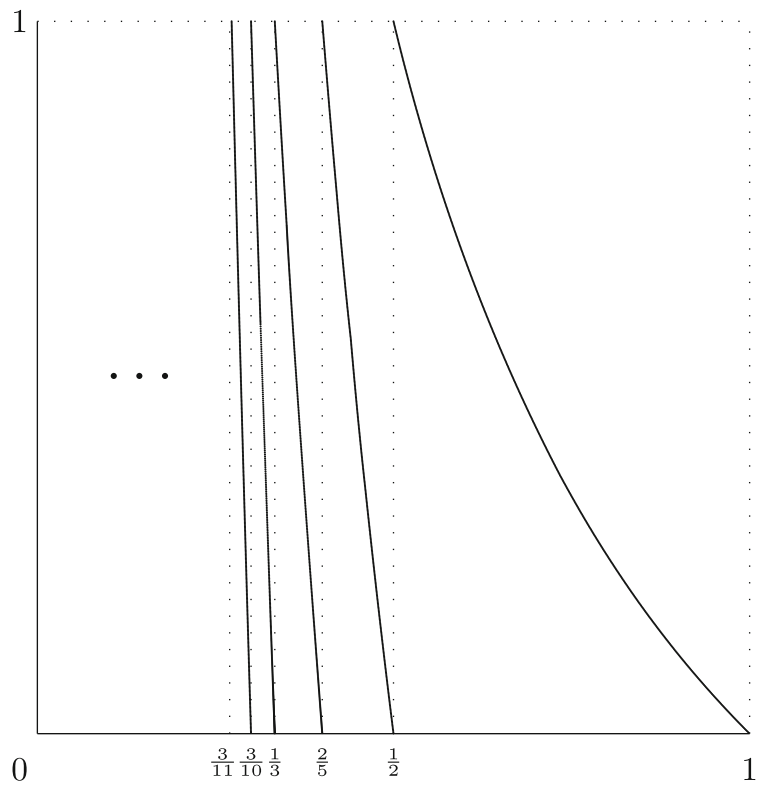


**Fig. 1** The map $T$

and writing $a_1(x) = \lfloor 1/x \rfloor$ and $b_1(x) = \lfloor a_1/x \rfloor$, it follows that

$$x = \frac{a_1}{b_1 + T(x)}.$$

Setting $a_n = a_n(x) = a_1\left(T^{n-1}(x)\right)$ and $b_n = b_n(x) = b_1\left(T^{n-1}(x)\right)$, whenever $T^{n-1}(x) \neq 0$, we find

$$x = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + T^n(x)}}}.$$

For $n \geq 1$, we define the continued fraction convergents $c_n$ of $x$ by

$$c_n = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n}}}, \quad n \geq 1. \tag{6}$$

Since this new continued fraction expansion of $x$ is proper (see (8) below), it immediately follows from Theorem 1.3 of [16] that $x = \lim_{n\to\infty} c_n$. This is expressed by writing that

$$x = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}},$$

or in short-hand notation,

$$x = [0; a_1/b_1, a_2/b_2, \ldots, a_n/b_n, \ldots]. \tag{7}$$

### 3.2 Elementary properties of the $a/b$-expansion

Clearly, if there exists an $n \in \mathbb{N}$ for which $T^n(x) = 0$, then $x$ has a finite continued fraction expansion, and we see that $x \in \mathbb{Q}$. It is also clear that if $x \in [0, 1) \setminus \mathbb{Q}$, then $T(x) \in [0, 1) \setminus \mathbb{Q}$ (so $T(x)$ is irrational if $x$ is).

**Proposition 3.1** *Let $x \in [0, 1)$, then the continued fraction expansion (7) of $x$ is finite if and only if $x \in \mathbb{Q}$.*

*Proof* Let $t_1, n_1 \in \mathbb{N}$ be such that $1 \leq t_1 < n_1$, $\gcd(t_1, n_1) = 1$, and $x = t_1/n_1$. Then it follows from the definition of the map $T$ that $T(x) \in [0, 1) \cap \mathbb{Q}$, so we can find $t_2, n_2 \in \mathbb{N}$ be such that $0 \leq t_2 < n_2$, $\gcd(t_2, n_2) = 1$, and $T(x) = t_2/n_2$. But then we have that $n_2 \leq t_1 < n_1$. Now suppose that $T^n(x) \neq 0$ for all $n \geq 1$. Then, there exist sequences of positive integers $(t_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$, for which $\gcd(t_k, n_k) = 1$,

$$T^{k-1}(x) = \frac{t_k}{n_k},$$

and

$$n_1 > n_2 > n_3 > \cdots > n_k > n_{k+1} > \cdots,$$

which is impossible. □

Note that the interval $\left[\frac{1}{n+1}, \frac{1}{n}\right)$ is sent by the map $S(x) = n/x$ to the interval $(n^2, n^2 + n]$, which is the union of the intervals

$$(n^2, n^2 + 1],\ (n^2 + 1, n^2 + 2],\ \ldots, (n^2 + n - 1, n^2 + n],$$

and from this we see immediately that for $T^{k-1}(x) \in \left[\frac{1}{n+1}, \frac{1}{n}\right)$ we have that

$$a_k = n \quad \text{and that } b_k \in \left\{a_k^2, a_k^2 + 1, \ldots, a_k^2 + a_k - 1\right\}. \tag{8}$$

In other words, for $k \geq 1$ we have that $a_k = \lfloor \sqrt{b_k} \rfloor$, and that (7) is a *proper* expansion of $x$.

**Proposition 3.2** *If $(a_n, b_n)_{n\geq 1}$ is a (finite or infinite) sequence of pairs of positive integers satisfying (8), then the sequence of positive rationals $(p_n/q_n)_{n\geq 1}$, given by*

$$\frac{p_n}{q_n} = [0; a_1/b_1, a_2/b_2, \ldots, a_n/b_n], \quad n \geq 1,$$

*converges to a unique number* $x \in [0, 1)$.

For a proof, see Theorem 2.1 in [16].

## 4 Ergodic properties of the $a/b$-expansion

### 4.1 Introduction

There are many ways to prove ergodicity of maps like the map $T$ from (5). A "general way" is to use Fritz Schweiger's *fibred systems* approach. Let $B \subset \mathbb{R}^n$ be a set and $T : B \to B$ be a map. The pair $(B, T)$ is called a *fibred system* if the following three conditions are satisfied:

(a) There is a finite or countable set $I$ (called the digit set).
(b) There is a map $k : B \to I$. Then the sets

$$B(i) = k^{-1}\{i\} = \{x \in B : k(x) = i\}$$

form a partition of $B$.
(c) The restriction of $T$ to $B$ is an injective map.

See also [23], Definition 1.1.1.

In order to show that the continued fraction map $T$ from (5) is ergodic and has a unique $T$-invariant measure $\mu$ which is absolutely continuous with respect to Lebesgue measure $\lambda$ on $[0, 1)$, one only needs to show that $T$ satisfies the conditions of Adler's *Folklore Theorem*; see Theorem 15.2.1 in [23]. Once the conditions of this theorem are satisfied, we know that the so-called *Rényi Condition* holds, which is condition (c) of Rényi's Theorem 9.5.3 in [23]. Since the new continued fraction converges for every $x \in [0, 1)$, Corollary 9.5.4 (from [23]) now immediately yields that $T$ is ergodic.

Furthermore, since the conditions of Rényi's Theorem 9.5.3 in [23] are satisfied, Theorem 15.1.2 in [23] now yields that there exists a unique invariant probability measure $\mu$ on $[0, 1)$ such that for some positive constant $C$

$$C^{-1}\lambda(E) \leq \mu(E) \leq C\lambda(E),$$

where $E$ is any Borel measurable subset of $[0, 1)$.

The conditions in Adler's Folklore Theorem are conditions on *cylinder sets* (or: *fundamental intervals*), defined for pairs $(a, b)$ with $a \in \mathbb{N}$ and $b \in \{a^2, a^2+1, \ldots, a^2+a-1\}$ by

$$\Delta_{(a,b)} = \{x \in [0, 1]; a_1(x) = a, b_1(x) = b\}.$$

So

$$\Delta_{(1,1)} = \left(\tfrac{1}{2}, 1\right),$$

and for $a \geq 1$, $b \geq 2$ (and $b \in \{a^2, a^2+1, \ldots, a^2+a-1\}$) we have that

$$\Delta_{(a,b)} = \left(\tfrac{a}{b+1}, \tfrac{a}{b}\right].$$

Obviously, the digit set $I$ consists now of pairs of digits, and is given by

$$I = \left\{(a,b);\ a \in \mathbb{N},\ b \in \{a^2, a^2+1, \ldots a^2+a-1\}\right\}.$$

The first three conditions of Adler's Folklore Theorem are easily satisfied; see also [23], p. 107.

(i) For every $k = (a,b) \in I$ we have that

$$\left(\tfrac{a}{b+1}, \tfrac{a}{b}\right) \subset B(k) = \Delta_{(a,b)} \subset \left[\tfrac{a}{b+1}, \tfrac{a}{b}\right] =: I_k.$$

(ii) The map $T : B(k) \to B =: [0,1]$ can be extended to a function of class $C^2$ on $I(k)$.

(iii) The map $T$ restricted to $B(k)$ is *full*, i.e., $TI(k) = B$.

Also the fifth condition holds; take $M = 2$ in

(v) There exists a constant $M > 0$ such that

$$\left|\frac{T''(x)}{T'(x)^2}\right| \leq M, \quad x \in I(k),\ k \in I.$$

What remains to show is that the fourth condition in Adler's Folklore Theorem holds for the map $T$ from (5), viz.

(iv) There exists a constant $\theta > 1$ such that

$$\left|T'(x)\right| \geq \theta, \quad x \in I(k),\ k \in I.$$

Obviously, this condition is **not** guaranteed for the continued fraction map $T$ from (5). In fact, Schweiger remarks that the RCF map does not satisfy condition $(iv)$. In order to remedy this situation he shows that condition $(iv)$ can be replaced by

(iv*) There is an $N \geq 1$ such that

$$\left|(T^N)'(x)\right| \geq \theta > 1,$$

for all $x \in I(k_1, \ldots, k_N)$, which is the closure of

$$B(k_1, \ldots, k_N) = \{x \in [0,1);\ k_1(x) = k_1, \ldots, k_N(x) = k_N\};$$

see [23], p. 109. In [6] piecewise differential transformations with this property are called *eventually expensive*, and a transformation like $T$ from (5) is called a *generalized Gauss transformation*.

For the RCF expansion $(iv^*)$ is satisfied for $N = 2$. This is also the case for the continued fraction map $T$ from (5). To see this, define for $a_1, a_2 \in \mathbb{N}$, $b_i \in \{a_i^2, a_i^2+1, \ldots, a_i^2+a_i-1\}$ (for $i = 1, 2$) the cylinder set $\Delta_{(a_1,b_1),(a_2,b_2)}$ of order 2 by

$$\Delta_{(a_1,b_1),(a_2,b_2)} = \{x \in (0, 1) \ ; \ a_i(x) = a_i, b_i(x) = b_i, i = 1, 2\}.$$

For all $x \in \Delta_{(a_1,b_1),(a_2,b_2)}$ one has that

$$T^2(x) = \frac{a_2 x}{a_1 - b_1 x} - b_2,$$

from which we see that

$$(T^2)'(x) = \frac{a_1 a_2}{(a_1 - b_1 x)^2}.$$

Now define the function $h : \Delta_{(a_1,b_1)} \to \mathbb{R}$ by

$$h(x) = \frac{1}{(a_1 - b_1 x)^2}.$$

Then

$$h'(x) = \frac{2b_1}{(a_1 - b_1 x)^3} > 0, \qquad \text{for } x \in \Delta_{(a_1,b_1)},$$

and we find (since $b_1 \in \{a_1^2, a_1^2 + 1, \ldots, a_1^2 + a_1 - 1\}$ and $a_2 \in \mathbb{N}$) that

$$(T^2)'(x) \geq \frac{(b_1+1)^2}{a_1^2} \geq \frac{a_1^4 + 2a_1^2 + 1}{a_1^2} > a_1^2 + 2 \geq 3;$$

i.e., condition $(iv^*)$ is satisfied.

We have obtained the following theorem; see also [6], Lemma 2.2.

**Theorem 4.1** *For the continued fraction map $T$ from (5) there exists a $T$-invariant probability measure $\mu$ with density $\rho$ on* $[0, 1]$, *such that $T$ is ergodic, and for some constant $C > 0$ we have that*

$$\frac{1}{C} \leq \rho(x) \leq C, \quad \textit{for } x \in [0, 1].$$

*Remark 4.1* Although we know that a $T$-invariant measure $\mu$ exists, it should be stressed that this is almost everything we know about $\mu$. In Theorem 2.4 of [6] some constraints on generalized Gauss transformations are given, which yield that the density $\rho$ of the $T$-invariant measure satisfies

$$\rho(0) = \left(1 - \frac{1}{\phi'(1)}\right)\rho(1). \tag{9}$$

Here $\phi(x)$ is such that the generalized Gauss transformation is given by $T(x) = \phi(x) - \lfloor \phi(x) \rfloor$. Although not all the demands of Theorem 2.4 from [6] are met for

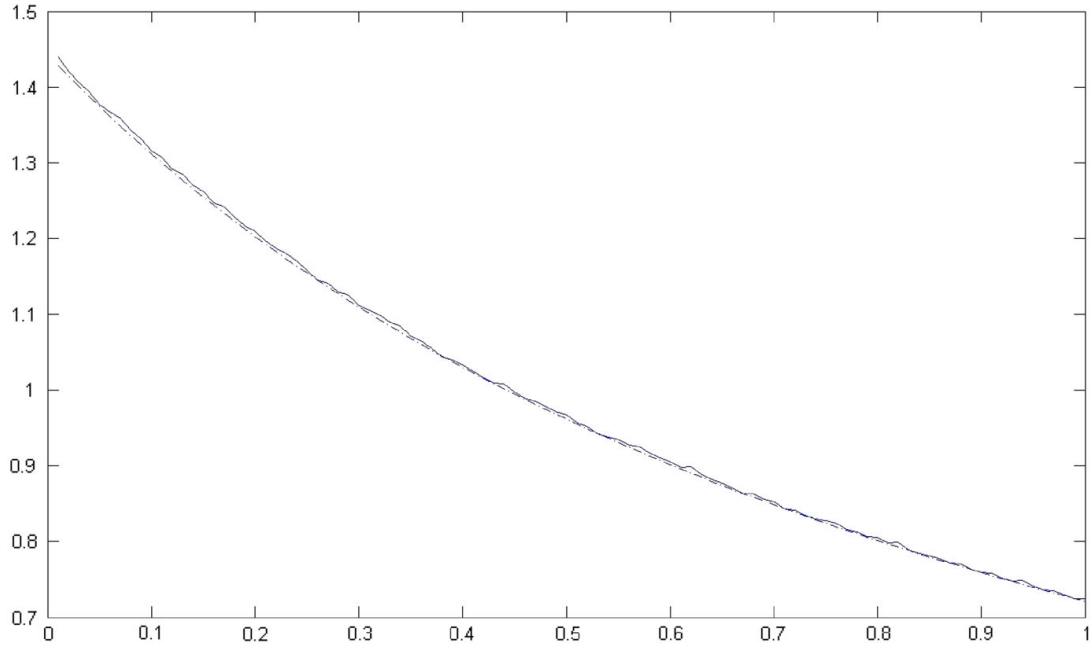


**Fig. 2** Density of the Gauss map $T_1$ and a simulation of the density based on 20 million points

the RCF expansion (in particular one does not have that $\phi''(x) < 0$), (9) does hold for the RCF expansion.

## 5 An estimate of the invariant measure of $a/b$-expansions

In general, to get an idea of the density of the invariant measure, we could use Geon Ho Choe's computational approach; see [6,7]. Essential in Choe's approach is that a randomly selected point becomes typical after iterating the continued fraction map sufficiently often. Unfortunately, this approach did not yield satisfactory results in our case; see Sect. 7 in [14]. For this reason, Choe's method has been adapted in the following manner. Instead of selecting one $x$ for which we iterate the continued fraction map $T$ many times we randomly select many (viz. 2 000) points which we iterate very few (to be precise: 20) times under $T$. We repeat this procedure 500 times and then we take the average density. We run this process one more time but instead of sampling points uniformly we sample from the density just obtained.

Of course, this will only give a rough idea of the invariant density. However, it still seems to be a valuable method. To test the correctness of the code and see how well the algorithm works, we first check it on the RCF expansion, before we apply it on the new $a/b$-continued fraction expansion.

### 5.1 A simulation for what is known

The results of the simulation applied to the Gauss map are given in Fig. 2, where the curves drawn in the graphics represent, respectively, the Gauss density (which is the dotted curve) and its approximation.

**Table 1** Asymptotic frequencies of RCF-partial quotients, and numerically approximations of these

| $a_n$ | Theoretical | Numerical | Difference |
|---|---|---|---|
| 1 | 0.415037 | 0.414725 | 0.000312 |
| 2 | 0.169925 | 0.169634 | 0.000291 |
| 3 | 0.093109 | 0.093245 | 0.000136 |
| 4 | 0.058894 | 0.058829 | 0.000064 |
| 5 | 0.040642 | 0.040545 | 0.000096 |
| 6 | 0.029747 | 0.029835 | 0.000087 |
| 7 | 0.022720 | 0.022942 | 0.000221 |
| 8 | 0.017922 | 0.018025 | 0.000103 |
| 9 | 0.014500 | 0.014577 | 0.000078 |
| 10 | 0.011972 | 0.012002 | 0.000029 |

Visually, the Gauss density and its approximation are very close. This could be tested in a Kolmogorov-Smirnov test. However, in order to validate our method, we use this approximation of the density of the Gauss map to obtain approximations of various classical constants. These constants can be obtained for the RCF using Birkhoff's Ergodic Theorem; see Sect. 3.5 in [8].

### *5.1.1 Lévy constants*

For the RCF we have the quite well-known 1929 result by Paul Lévy that for almost all $x$ (with respect to Lebesgue measure)[1] the asymptotic frequency that a partial quotients (digit) $a_n$ is equal to $a$, for $a \in \mathbb{N}$ equals

$$\frac{1}{\log 2} \int_{\frac{1}{a+1}}^{\frac{1}{a}} \frac{1}{1+x}\, \mathrm{d}x = \frac{1}{\log 2} \log\left(\frac{(a+1)^2}{a(a+2)}\right);$$

see [15], or Exercise 3.5.4 in [8]. So for almost all $x \in [0, 1)$ we have that about 41.5% of all partial quotients (digits) are equal to 1, and about 17% of all digits are equal to 2. From our simulation we find that the frequencies are quite well approximated; see Table 1.

### *5.1.2 Khintchine constants*

In 1935, A.Ya. Khintchine showed in [13] that for almost all $x$ one has that

$$\lim_{n\to\infty} \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}} = 1.7454056\ldots,$$

[1] All *almost sure* statements in this paper are wrt. Lebesgue measure $\lambda$.

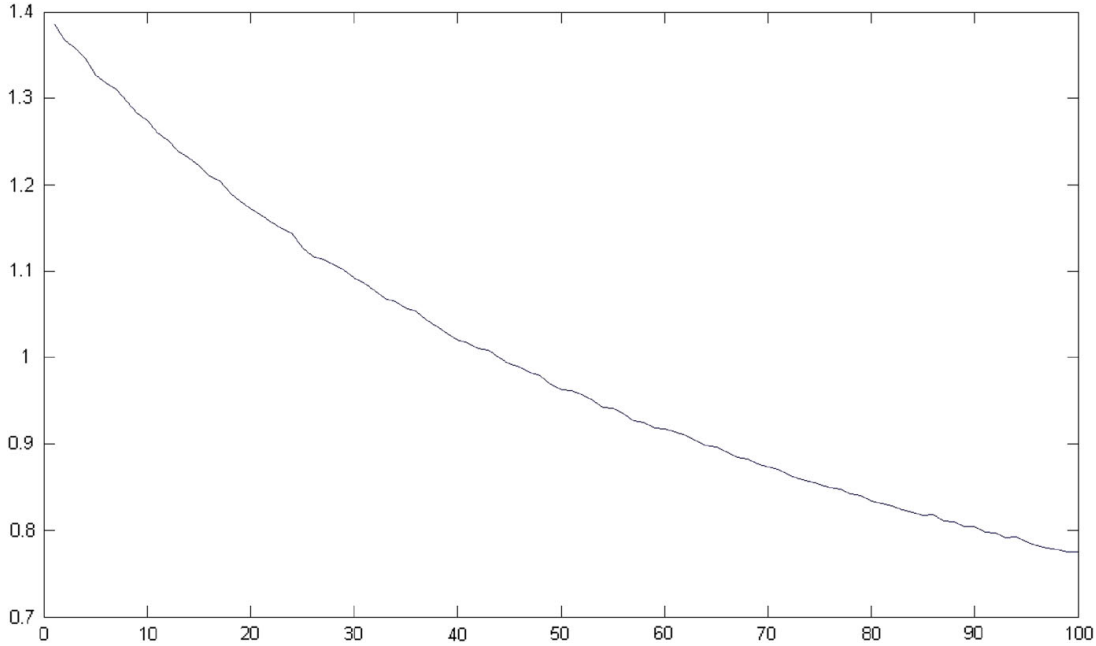


**Fig. 3** Approximation $\hat{f}$ of the density $f$ of $T$ based on 30 million points

see also Exercises 3.5.6 and 3.5.8 in [8]. Using our approximation of the Gauss density, we find the following numerical approximations $\hat{K}$ of $K = 1.7454056\ldots$

$$\hat{K} = 1.7644052.$$

The approximation is based on the frequencies of the first 10 numbers.

## 5.2 ... and a simulation for what is unknown

The results of the simulation for the $a/b$-continued fraction map $T$ from (5) are given in Fig. 3. In this simulation, we took 3000 random points iterated those 20 times and repeated that 500 times.

Before we give our approximations of the Lévy and Khintchine constants for the $a/b$-continued fraction expansion based on this approximation $\hat{f}$ of the density $f$ of $T$, we first give some numerical "evidence" that our approximation indeed is close to the (unknown) density $f$ of the $a/b$-expansion. Let $[c, d] \subset (0, 1)$ be any subinterval of the unit interval and $\mu$ be the $T$-invariant measure. Then by definition we must have for the $a/b$-expansion map $T$ that

$$\mu([c, d]) = \mu(T^{-1}([c, d]),$$

from which we see that

$$\int_c^d f(x)\,\mathrm{d}x = \sum_{i=0}^{\infty} \sum_{j=i^2}^{i^2+i-1} \int_{\frac{i}{d+j}}^{\frac{i}{c+j}} f(x)\,\mathrm{d}x.$$

**Table 2** Results of testing the invariant measure

| $[c, d]$ | $\hat{\mu}([c, d])$ | $\hat{\mu}(T^{-1}([c, d]))$ | Difference | Relative difference in % |
|---|---|---|---|---|
| [0.01, 1] | 0.9861 | 0.9792 | 0.0069 | 0.7 |
| [0.01, 0.7] | 0.7407 | 0.7354 | 0.0053 | 0.72 |
| [0.01, 0.5] | 0.5578 | 0.5536 | 0.0042 | 0.75 |
| [0.01, 0.3] | 0.3533 | 0.3504 | 0.0029 | 0.82 |
| [0.01, 0.1] | 0.1188 | 0.1175 | 0.0013 | 1.09 |
| [0.33, 1] | 0.6005 | 0.5951 | 0.0054 | 0.9 |
| [0.33, 0.8] | 0.4403 | 0.4361 | 0.0042 | 0.95 |
| [0.33, 0.6] | 0.2659 | 0.2630 | 0.0029 | 1.09 |
| [0.33, 0.4] | 0.0730 | 0.0719 | 0.0012 | 1.64 |
| [0.5, 1] | 0.4284 | 0.4239 | 0.0045 | 1.05 |
| [0.5, 0.8] | 0.2681 | 0.2651 | 0.0030 | 1.12 |

**Table 3** Numerical estimates for the asymptotic frequencies

| $a_n$ | $b_n$ | % | $a_n$ | $b_n$ | % | $a_n$ | $b_n$ | % |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 42.84 | 7 | 49 | 0.35 | 9 | 81 | 0.17 |
| 2 | 4 | 9.90 | 7 | 50 | 0.34 | 9 | 82 | 0.17 |
| 2 | 5 | 6.93 | 7 | 51 | 0.32 | 9 | 83 | 0.16 |
| 3 | 9 | 3.59 | 7 | 52 | 0.31 | 9 | 84 | 0.16 |
| 3 | 10 | 2.99 | 7 | 53 | 0.30 | 9 | 85 | 0.16 |
| 3 | 11 | 2.54 | 7 | 54 | 0.29 | 9 | 86 | 0.15 |
| 4 | 16 | 1.67 | 7 | 55 | 0.28 | 9 | 87 | 0.15 |
| 4 | 17 | 1.50 | 8 | 64 | 0.24 | 9 | 88 | 0.15 |
| 4 | 18 | 1.35 | 8 | 65 | 0.23 | 9 | 89 | 0.14 |
| 4 | 19 | 1.23 | 8 | 66 | 0.23 | 10 | 100 | 0.13 |
| 5 | 25 | 0.90 | 8 | 67 | 0.22 | 10 | 101 | 0.12 |
| 5 | 26 | 0.84 | 8 | 68 | 0.21 | 10 | 102 | 0.12 |
| 5 | 27 | 0.79 | 8 | 69 | 0.21 | 10 | 103 | 0.12 |
| 5 | 28 | 0.73 | 8 | 70 | 0.20 | 10 | 104 | 0.12 |
| 5 | 29 | 0.69 | 8 | 71 | 0.20 | 10 | 105 | 0.11 |
| 6 | 36 | 0.54 | | | | 10 | 106 | 0.11 |
| 6 | 37 | 0.52 | | | | 10 | 107 | 0.11 |
| 6 | 38 | 0.49 | | | | 10 | 108 | 0.11 |
| 6 | 39 | 0.47 | | | | 10 | 109 | 0.11 |
| 6 | 40 | 0.44 | | | | | | |
| 6 | 41 | 0.42 | | | | | | |

Replacing $\mu$ by its numerical estimate $\hat{\mu}$ (so $\hat{\mu}$ is a probability measure with density $\hat{f}$), we find the results in Table 2.

Using the numerical estimate $\hat{f}$ of the density $f$ of the $a/b$-expansion, we find estimates of the Lévy constants in Table 3.

Since Table 3 is essentially a table for the partial quotients $b_n$ we distil from this table the estimated asymptotic frequencies for the $a_n$; see Table 4.

**Table 4** Numerical estimates for the asymptotic frequencies of $a_n$

| $a_n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| % | 42.84 | 16.84 | 9.13 | 5.74 | 3.95 | 2.88 | 2.20 | 1.74 | 1.40 | 1.16 |

We can also estimate the Khintchine constants for the $a/b$-expansion. Not only for the $a_n$'s but also for the $b_n$'s as well. Using Table 4 we find

$$\lim_{n\to\infty} \frac{n}{\dfrac{1}{a_1} + \cdots + \dfrac{1}{a_n}} \approx 1.7296992\ldots.$$

From Table 3 we find

$$\lim_{n\to\infty} \frac{n}{\dfrac{1}{b_1} + \cdots + \dfrac{1}{b_n}} \approx 2.0710142\ldots.$$

## 6 Approximation coefficients

It is well-known that the RCF expansion, which is the case $a_n = 1$ (or $N = 1$) for all $n \geq 1$, has very good approximation properties. For example, if $x \in \mathbb{R} \setminus \mathbb{Q}$, with RCF-convergents $P_n/Q_n$, $n \geq 0$, then if we define the *approximation coefficients* $\Theta_n = \Theta_n(x)$, for $n \geq 0$ by

$$\Theta_n(x) = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|, \tag{10}$$

we have that $0 < \Theta_n(x) < 1$, $\min\{\Theta_{n-1}(x), \Theta_n(x)\} < 1/2$ (Vahlen, 1895), and $\min\{\Theta_{n-1}, \Theta_n(x), \Theta_{n+1}\} < 1/\sqrt{5}$ (Borel, 1903). Several authors also found that

$$\min\{\Theta_{n-1}, \Theta_n(x), \Theta_{n+1}(x)\} < \frac{1}{\sqrt{a_{n+1}^2 + 4}},$$

while in 1983 Jingcheng Tong showed that

$$\max\{\Theta_{n-1}(x), \Theta_n(x), \Theta_{n+1}(x)\} > \frac{1}{\sqrt{a_{n+1}^2 + 4}};$$

see e.g., [8], Sects. 5.1.2, 5.2, and 5.3 for proofs of these (and other properties).

### 6.1 A generalization

For the RCF it is rather straightforward to show that the convergents $P_n/Q_n$ are always in their lowest terms, i.e., $\gcd(P_n, Q_n) = 1$. For the convergents $p_n/q_n$ of a continued fraction expansion of $x$ coming either from a sequence $(a_n)_{n\geq 1}$ of positive integers or

from an $N$-expansion (with $N \in \mathbb{N}$, $N \geq 2$), this need not be the case. In view of (10) we would like to bring $p_n/q_n$ in their lowest terms, but this is rather cumbersome. We choose to generalize definition (10) of approximation coefficients in a different way.

Let $(a_n)_{n\geq 1}$ be a sequence of positive integers, and let $x \in \mathbb{R}$. Suppose that $(b_n)_{n\geq 0}$ is a sequence of integers with $b_n \geq 1$ for $n \geq 1$, such that (1) is a continued fraction expansion of $x$. Setting

$$A_0 = \begin{pmatrix} 1 & b_0 \\ 0 & 1 \end{pmatrix}, \quad A_n = \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}, \ n \geq 1,$$

and $M_n = \prod_{k=0}^{n} A_k$, for $n \geq 0$. Setting

$$M_n = \begin{pmatrix} r_n & p_n \\ s_n & q_n \end{pmatrix},$$

for appropriate values of $r_n$, $s_n$, $p_n$, and $q_n \in \mathbb{Z}$, it follows from $M_n = M_{n-1}A_n$ that

$$M_n = \begin{pmatrix} p_{n-1} & b_n p_{n-1} + a_n p_{n-2} \\ q_{n-1} & b_n q_{n-1} + a_n q_{n-2} \end{pmatrix},$$

i.e., we obtained the recurrence relations mentioned in the introduction. Note that $\det(M_n) = p_{n-1}q_n - p_n q_{n-1} = (-1)^n \prod_{k=1}^{n} a_k$.

For any $A \in \mathrm{GL}_2(\mathbb{Z})$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we define (with a slight abuse of notation) the corresponding Möbius transformation $A : \mathbb{R} \to \mathbb{R}$ by

$$A(x) = \frac{ax + b}{cx + d}.$$

Since $(AB)(x) = A(B(x))$ for any pair $A, B \in \mathrm{GL}_2(\mathbb{Z})$, it immediately follows that

$$M_n(0) = \frac{p_n}{q_n} = [b_0; a_1/b_1, \ldots, a_n/b_n], \qquad \text{for } n \geq 0.$$

Furthermore, setting $t_n = [0; a_{n+1}/b_{n+1}, \ldots]$, and

$$M_n^* = M_n \begin{pmatrix} 0 & a_n \\ 1 & b_n + t_n \end{pmatrix}, \quad n \geq 1,$$

we have that

$$x = M_n^*(0) = \frac{p_n + t_n p_{n-1}}{q_n + t_n q_{n-1}}.$$

From this an easy calculation yields that

$$q_n^2 \left| x - \frac{p_n}{q_n} \right| = \frac{\left(\prod_{k=1}^n a_k\right) \cdot t_n}{1 + t_n \cdot \frac{q_{n-1}}{q_n}}. \tag{11}$$

Note that

$$\frac{q_{n-1}}{q_n} = \frac{1}{b_n + a_n \cdot \frac{q_{n-2}}{q_{n-1}}}, \tag{12}$$

so after finitely many steps we find that

$$\frac{q_{n-1}}{q_n} = [0; 1/b_n, a_n/b_{n-1}, a_{n-1}/b_{n-2}, \ldots, a_2/b_1].$$

In view of definition (10) of the RCF-approximation coefficients $\Theta_n(x)$, we define for $n \geq 1$ the approximation coefficients $\theta_n = \theta_n(x)$ of $x$ for sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 0}$ by

$$\theta_n = \theta_n(x) = \frac{q_n^2}{\prod_{i=1}^n a_i} \left| x - \frac{p_n}{q_n} \right|, \tag{13}$$

where $p_n/q_n$ is the $n$th convergent of the continued fraction expansion (1) of $x$. Note that if $a_n = 1$ for $n \geq 1$ we are back in the RCF-case.

### 6.2 Approximation coefficients for $N$-expansions

For $N$-expansions we have that $a_n = N$ for all $n \geq 1$ for some (fixed) $N \in \mathbb{N}$. Setting

$$v_n = N \cdot \frac{q_{n-1}}{q_n} = [0; N/b_n, N/b_{n-1}, \ldots, N/b_1],$$

it follows from (11) and (12) that

$$\theta_n = \frac{N t_n}{N + t_n v_n}, \quad n \geq 1. \tag{14}$$

In [9] we saw that $((t_n, v_n))_{n\geq 0}$ is a sequence in some subset $\mathcal{A}$ of $\Omega_N = [0, N) \times [0, N]$. For example, if we obtain the sequence $(b_n)_{n\geq 1}$ using the *greedy* $N$-expansion, so if we use the greedy/proper/best map

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \quad x \in (0, N); T_N(0) = 0,$$

then the sequence $((t_n, v_n)_{n\geq 1}$ is a sequence in $\mathcal{A} = \Omega_1 = [0, 1) \times [0, 1] \subset \Omega_N$; see also below. However, if we would use the *lazy* continued fraction map $T_\ell$ (for a precise definition, see Sect. 3 in [9]), then if $N \geq 2$ the set $\mathcal{A}$ consists of two disjoints

rectangles in $\Omega_N$. In order to keep the presentation simple, we only deal here with the *greedy* case, although all other cases can be obtained similarly.

Define for $N \in \mathbb{N}$ the natural extension map $\mathcal{T}_N : \Omega_1 \to \Omega_1$ by

$$\mathcal{T}_N(x, y) = \left(T_N(x), \frac{N}{b(x) + y}\right). \tag{15}$$

In [9] it was shown that the dynamical system

$$(\Omega_1, \mathcal{B}, \bar{\mu}, \mathcal{T}_N),$$

where $\mathcal{B}$ is the Borel $\sigma$-algebra of $\Omega_N$ and $\bar{\mu}$ is the $\mathcal{T}_N$-invariant probability measure with density $d_g(x, y)$, given by

$$\frac{1}{\log \frac{N+1}{N}} \frac{N}{(N + xy)^2} 1_{\Omega_1}(x, y),$$

has strong mixing properties. For our purposes here it is enough that this dynamical system is ergodic. Note that for all $(x, y) \in \Omega_N$ one has that $\mathcal{T}_N(x, y) \in \Omega_1$; this explains why the density $d_g$ has $\Omega_1$ as its support. Note that for $x \in (0, N)$ we have that

$$(t_n, v_n) = \mathcal{T}_N(x, 0), \quad \text{for } n \geq 1.$$

An immediate consequence of (11–14), and the fact that the dynamical system $(\Omega_1, \mathcal{B}, \bar{\mu}, \mathcal{T}_N)$ is ergodic, is the following result, which is a generalization of the so-called *Doeblin-Lenstra* conjecture.

**Theorem 6.1** *Let $N \in \mathbb{N}$, then for all $c \in [0, 1]$ and for a.e. $x$ we have that the limit*

$$\lim_{n\to\infty} \frac{1}{n} \#\{k;\ 1 \leq k \leq n,\ \theta_n(x) \leq c\}$$

*exists, and equals*

$$F_N(c) = \begin{cases} \dfrac{c}{N \log \frac{N+1}{N}}, & \text{if } 0 \leq c < \frac{N}{N+1} \\ \dfrac{1}{\log \frac{N+1}{N}} \left(1 - c + \log \dfrac{(N+1)c}{N}\right), & \text{if } \frac{N}{N+1} \leq c \leq 1. \end{cases}$$

*Furthermore, simply by calculating the first moment of $F_N$ we find for a.e. $x$ that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k \leq n} \theta_k(x) = \frac{1}{2(N+1) \log \frac{N+1}{N}}.$$

A proof of the original *Doeblin-Lenstra* conjecture for the RCF (so when $N = 1$) was given by Wieb Bosma, Henk Jager, and Freek Wiedijk, who in fact obtained in [2] a Doeblin-Lenstra result for all $\alpha$-expansions originally introduced by Hitoshi Nakada for $\alpha \in [1/2, 1]$. The (slightly technical) proof for general $N \in \mathbb{N}$ is very similar to that of the case $N = 1$ and is, therefore, omitted; for more details, see e.g., [8,11]. Note that the almost sure asymptotic mean of the approximation constant is a monotonically increasing function with minimum $\frac{1}{4\log 2}$ when $N = 1$ (a result already obtained by Bosma, Jager and Wiedijk in [2]), and with limit $1/2$ as $N$ tends to infinity.

In fact, much more can be said! From

$$t_n = \frac{N}{t_{n-1}} - b_n,$$

(where $b_n = \lfloor N/t_{n-1} \rfloor$, since we use the greedy map $T_N$), it follows that

$$t_{n-1} = \frac{N}{b_n + t_n}.$$

Since by definition $v_n = \dfrac{N}{b_n + v_{n-1}}$, we see that

$$v_{n-1} = \frac{N}{v_n} - b_n.$$

An easy calculation now yields that

$$\theta_{n-1}(x) = \frac{N t_{n-1}}{N + t_{n-1} v_{n-1}} = \frac{N v_n}{N + t_n v_n}; \tag{16}$$

see also Sect. 5.1.2 in [8] for the case $N = 1$.

In view of (14) and (16) we define the map $\Psi_N : \Omega_1 \to \mathbb{R}^2$ by

$$\Psi_N(t, v) = \left( \frac{N v}{N + t v}, \quad \frac{N t}{N + t v} \right).$$

Note that—apart from a set of Lebesgue measure zero – $\Psi_N$ maps $\Omega_1$ bijectively to the quadrangle $R_n$ with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(\frac{N}{N+1}, \frac{N}{N+1})$. Let $\mathcal{C} \in [0, 1]$, then easy calculations show that under $\Psi_N$ the horizontal line $y = \mathcal{C}$ is mapped to the line

$$\beta = -\frac{N}{\mathcal{C}^2} + \frac{N}{\mathcal{C}},$$

while the vertical line $x = \mathcal{C}$ is mapped under $\Psi_N$ to the line

$$\beta = -\frac{\mathcal{C}^2}{N} + \mathcal{C}.$$

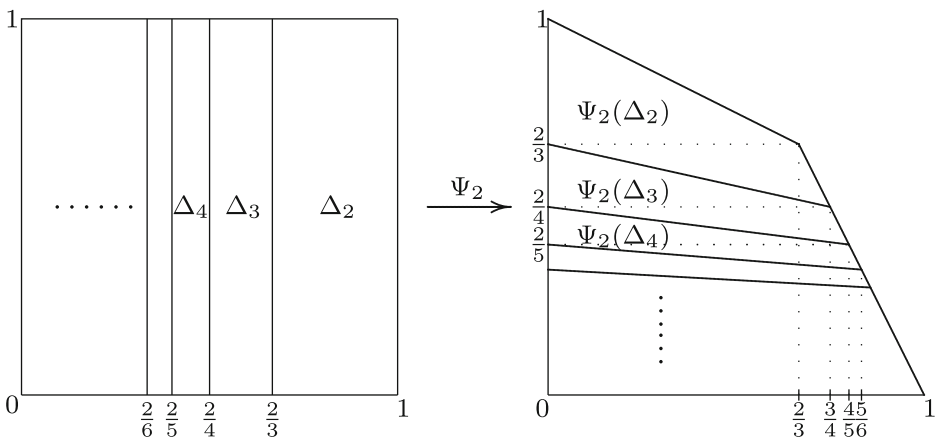


**Fig. 4** The $\Psi_2$-map for $N = 2$

Furthermore, for $k \in \mathbb{N} \cup \{0\}$, we have that

$$\Psi_N\left(\frac{N}{N+k}, 0\right) = \left(0, \frac{N}{N+k}\right), \text{ and}$$
$$\Psi_N\left(\frac{N}{N+k}, 1\right) = \left(\frac{N+k}{N+k+1}, \frac{N}{N+k+1}\right);$$

see Fig. 4.

Set for $b \in \{N, N+1, N+2, \dots\}$,

$$f_b = [0; \overline{N/b}] = \frac{-b + \sqrt{b^2 + 4N}}{2} \in \Delta(b) = \left[\frac{N}{b+1}, \frac{N}{b}\right),$$

(where the bar indicates the period), then obviously one has that $(f_b, f_b)$ is a fixed point for $\mathcal{T}_N$, i.e.,

$$\mathcal{T}_N(f_b, f_b) = (f_b, f_b), \quad \text{for all } b \geq N.$$

Furthermore, note that for $b \geq N$,

$$\Psi_N(f_b, f_b) = \left(\frac{Nf_b}{N + f_b^2}, \frac{Nf_b}{N + f_b^2}\right) = \left(\frac{N}{\sqrt{b^2 + 4N}}, \frac{N}{\sqrt{b^2 + 4N}}\right).$$

Following the approach from [12] (see also Exercise 5.3.3 in [8]) one finds that for all $x \in \mathbb{R} \setminus \mathbb{Q}$ and all $n \geq 1$ that

$$\Psi_N^{-1}(\alpha, \beta) = \left(\frac{N - \sqrt{N^2 - 4\alpha\beta N}}{2\alpha}, \quad \frac{N - \sqrt{N^2 - 4\alpha\beta N}}{2\beta}\right).$$

If we now define the map $\Phi_N : R_N \to R_N$ by $\Phi(\alpha, \beta) = \Psi_N(\mathcal{T}_N(\Psi_N^{-1}(\alpha, \beta)))$, then it follows from (16) and (14) that

$$\Psi_N(\alpha, \beta) = \left(\beta, \alpha + \frac{b_{n+1}}{N}\sqrt{N^2 - 4\alpha\beta N} - \frac{b_{n+1}^2}{N}\beta\right).$$

Since

$$\Phi_N(\theta_{n-1}(x), \theta_n(x)) = (\theta_n(x), \theta_{n+1}(x)), \qquad \text{for } n \geq 1,$$

we see that

$$\theta_{n+1}(x) = \theta_{n-1} + \frac{b_{n+1}}{N}\sqrt{N^2 - 4\theta_{n-1}\theta_n N} - \frac{b_{n+1}^2}{N}\theta_n, \qquad \text{for } n \geq 1. \tag{17}$$

This result was obtained for $N = 1$ (the RCF-case) in [12].

From (17) and the fact that the $f_b$'s are the fixed points of $\mathcal{T}_N$ we see that

$$\min\{\theta_{n-1}(x), \theta_n(x), \theta_{n+1}(x)\} < \frac{N}{\sqrt{b^2 + 4N}} \leq \frac{\sqrt{N}}{\sqrt{N+4}},$$

and the conjugate property $\max\{\theta_{n-1}(x), \theta_n(x), \theta_{n+1}(x)\} > \frac{N}{\sqrt{b^2+4N}}$.

### 6.3 Difficulties in the $a/b$-case

For many $N$-expansions it is fairly easy to "guess" what for the approximation coefficients a suitable version of the domain and the invariant measure of the natural extension should be, and it is straightforward to verify this guess; see also Sect. 3.1 in [9]. Of course, there are still many examples of $N$-expansions, in particular some of the examples with finitely many different partial quotients, for which the natural extension still escapes us; see, for example, Sects. 5.2 and 5.3 in [9].

Since we were not able to explicitly determine the invariant measure for the $a/b$-continued fraction map $T$ from (5), it follows that we cannot find a corresponding explicit natural extension for $T$. The problem is the second coordinate of such a map. In view of (15) we could define $\mathcal{T} : \Omega_1 \to \Omega_1$ by

$$\mathcal{T}(x, y) = \left(T(x), \frac{a(x)}{b(x) + y}\right),$$

where, as before, $a(x) = \lfloor 1/x \rfloor$ and $b(x) = \lfloor a(x)/x \rfloor$. Then $\mathcal{T}$ is bijective mod 0. However, no $\mathcal{T}$-invariant measure is known; it is easy to show for every positive $C$ that any probability measure with density of the form

$$\frac{1}{\log \frac{C+1}{C}} \frac{C}{(C + xy)^2} 1_{\Omega_1}(x, y)$$

is **not** a $\mathcal{T}$-invariant measure.

In view of (12) we could define $\mathcal{T} : \Omega_1 \to \Omega_1$ by

$$\mathcal{T}(x, y) = \left(T(x), \frac{1}{b(x) + a(x)y}\right).$$

In fact, this is the way the natural extension is given for the RCF expansion and its many variant (such as Nakada's $\alpha$-expansions [17], Schweiger's odd- and even-continued Fraction expansions [20,21], and the Rosen fractions [5,18]).

In this case (as in the $N$-expansion case) we have for every $x \in [0, 1)$ with $a/b$-expansion $[0; a_1/b_1, a_2/b_2, \dots]$ that

$$\mathcal{T}^n(x, 0) = (t_n, v_n), \quad n \geq 0,$$

where $t_n = T^n(x) = [0; a_{n+1}/b_{n+1}, \dots]$ and $v_n = [0; 1/b_n, a_n/b_{n-1}, \dots, a_2/b_1]$.

Unfortunately, the map $\mathcal{T} : \Omega_1 \to \Omega_1$ is neither injective nor surjective. Since for the $a/b$-expansion the approximation coefficients $\theta_n(x)$ are given by

$$\theta_n(x) = \frac{t_n}{1 + t_n v_n}, \quad n \geq 0, \tag{18}$$

the map $\mathcal{T}$; it is still an interesting map to study.

Applying $\mathcal{T}$ infinitely many times to $\Omega_1$ seems to "weed out" all mass from $\Omega_1$, it seems we are left with a set $\mathcal{C}$ of the form $\mathcal{C} = (0, 1) \times C$, where $C$ is a Cantor-set, i.e., an uncountable set of Lebesgue measure zero. In view of this and (18) it is likely that the Hurwitz and Lagrange spectra related to the $a/b$-expansion will exhibit a very different behavior when compared to the classical Hurwitz and Lagrange spectra related to the RCF.

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