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MASTER THESIS

Higher Spin Black Holes in Three Dimensions

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Abstract

Higher spin gauge theories are consistent interacting theories of higher spin fields in AdS spacetimes wherein they can evade the usual no-go theorems. With these theories one can study physics beyond the supergravity regime and it is believed that they belong to some parameter space corner of M-theory. Higher Spin gauge theory in (2+1)-dimensions is a toy model for Higher Spin theories in arbitrary dimensions in AdS. The advantage is that in three dimensions the theory becomes topological, due to the lack of propagating degrees of freedom. As a result, it is interesting to work out the details of this theory in the black hole regime. In this thesis, the basic aspects of three-dimensional Higher Spin theories are reviewed in both metric-like and gauge-like formulations. More emphasis is put on the Chern-Simons formulation, where the gauge connections are valued in $SL(N, \mathbb{R})$. Specifically, a discussion about asymptotic symmetries, proper and improper transformations is included, inspired by the original work on central charges by Brown and Henneaux, and later from application of their results in the Chern-Simons formulation. In addition, different proposals for higher spin black holes are reviewed and studied, including their thermodynamics. Also, an attempt to show that there is no ergosphere for the rotating cases has been made, but in a specific gauge. It is important to point out that higher spin theories are included in a specific family of theories that do not keep the metric invariant under diffeomorphisms, because these transformations braid the spin-2 field with all the higher spin fields. Therefore, concepts such as geodesics and horizon of a black hole are not well defined or need refinement. However, a proposal to use Wilson lines and Wilson loops was made by A. Castro, M. Ammon and N. Iqbal, that allows the calculation of the entanglement entropy using the Ryu and Takayanagi formula, which has been revised in this thesis. Finally, an attempt to extract information about the horizon shape and radius, using the Wilson line approach, has been made and a comparison between different black holes regimes of the same temperature has been included.

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Chapter 1

Introduction

The motivation for studying higher spin fields has varied in time. In the start, Fierz and Pauli [1] studied the field equations for massive higher spin fields, already in 1939, just as an academic exercise. They found that there exist higher spin representations of the Poincare group and therefore it is natural to seek field theories which would describe particles that carry these representations. It took almost forty years but their equations in flat space-time were obtained [2], despite all the concerns about potential difficulties in constructing interactions, e.g. no-go theorems.

In the same period, the massless limit was obtained by Fronsdal [3, 4] and soon after, during the seek of interacting theories the difficulties that were anticipated a decade ago, were better understood [5, 6], which led to no-go theorems for flat space-time. However, the discovery of supergravity in the mid-seventies, led to a boost in the interest in the search for consistent interactions of massless higher spin fields. The motivation of this renewed interest was the better understanding of supergravity. The inclusion of higher spin gauge fields opened the possibility of a better quantum behaviour of supergravity theories, which would be a promising step towards the construction of a consistent quantum gravity. Also, the development of a deeper understanding of gauge theories that goes beyond Yang-Mills was an intriguing prospect.

In 1978, Flato and Fronsdal [7] made the first attempt to construct a higher spin theory in Anti-de Sitter space-time, where most of the no go theorems would be resolved, because they hold in flat space-times. Motivated by this result, the free field equation for massless fields of arbitrary spin in AdS_4 was constructed the same year [8, 9].

Almost a decade later, Fradkin and Vasiliev [10, 11] make an important breakthrough towards the solution of the problem of interacting higher spin gauge theories. They showed that the gravitational interaction of massless higher spin fields does exist and

they provided the construction based on an infinite dimensional extension of the AdS_4 algebra. The key to this development is the recognition of the importance of a suitable choice of higher spin symmetry algebra. An important requirement is that once a generator with spin higher than two is introduced its closure will require the introduction of an infinite set of higher spin generators.

Until the late eighties, the massless higher spin theories were mainly considered in their own right. The motivation of the research was to demonstrate that higher spin gauge theories can be self consistently constructed in a unified theory involving gravity. However, until that point no connection with string theory was made yet, despite the fact that the theory contains an infinite set of fields which resembles the spectrum of string theory. However, after the discovery of supermembranes the situation changed. In 1988, Bergshoeff et al. [12] proposed that the spectrum of the supermembrane in $AdS_4 \times S^7$ background contains the massless higher spin states contained in the symmetric product of the two $N = 8$ Fronsdal superfields. It was also pointed out in [12] that the massive multiplets contained in the products of three or more Fronsdal fields would appear in the spectrum. It was also suggested that the resulting theory in AdS_4 could provide a field theoretic realization of the infinite dimensional higher spin algebra of the kind considered by Fradkin and Vasiliev [10]. This gave the motivation to the connection between higher spin gauge theories and string theory. Regardless of the consideration of a possible connection with strings or membranes, the theory of a consistent, interacting higher spin gauge theory initiated by Fradkin and Vasiliev [10, 11] was developed further by Vasiliev in a series of papers, e.g. [13].

Later developments shed light to different aspects of the theory. In fact, Maldacena, in his famous paper [14], was the first to anticipate the correspondence between physics in the bulk of AdS space and conformal field theory on its boundary. This development motivated researchers to revisit higher spin gauge theories in the framework of AdS/CFT correspondence, i.e. [15]. Until then, four- and five-dimensional higher spin theories were the main interest of the community. After the AdS/CFT correspondence started to develop in extreme detail, research in three dimensional higher spin theories reserved more attention by the community. There are various of reasons for this. Firstly, it is a rather simple toy model, due to the fact that neither gravity nor higher spin fields admit propagating degrees of freedom (see Chapter 2 and 5 for a detailed discussion). However the most important aspect is the connection with string theory, a reason that motivated the higher spin research for decades. In particular, Gabardiel and Gopakumar are the

first who worked towards this goal starting in 2010, [16, 17]. In their recent work [18], it was proven that a subsector of the CFT duals of the tensionless limit of string theory on $AdS_3 \times S^3 \times T^4$ describes the CFT dual of higher spin gravity in AdS_3 .

Their research was simplified by the fact that higher spin gauge fields admit an easier formulation than in higher dimensions. In the late eighties, Witten [19] made the connection between three dimensional gravity and Chern-Simons theory. Therefore it was a natural extension to search for a Chern-Simons formulation for gravity coupled to higher spin gauge fields in three dimensions. The proof came by Campoleoni et al [20] almost two decades later, due to the lack of interest in the three dimensional case the prior period. Since then, Chern-Simons formulation in the higher spin gravity attracted a lot of interest. Although the arbitrary higher spin theory is far from well understood, the spin-3 case is well developed and insightful for understanding problems, such as geodesic configurations and higher spin black holes, in the arbitrary spin realization.

Nevertheless, AdS/CFT correspondence does not only hold for pure AdS spacetimes. Demanding that a space-time is only asymptotically AdS is sufficient to make sure that the correspondence is present. Black hole solutions are an example of such an asymptotic AdS_3 space and are formed by global identification of global AdS, corresponding to CFT theories with a finite temperatures. Moreover, black holes play a central role in both classical and quantum gravitational physics. In classical general relativity, black holes constitute an important class of exact solution of Einstein's equations.

The discovery of Hawking [21], in 1975, that black holes radiate as black bodies leads to the black hole information paradox, whereby in the process of black hole formation and evaporation pure states seem to evolve into mixed states and hence information is not preserved. Any theory of quantum gravity needs to resolve this clash between classical gravity and quantum mechanics. For instance, the AdS/CFT correspondence [22] provides a conceptual resolution of the information paradox for black holes in asymptotically anti-de Sitter spacetimes. As time evolution on the CFT side is unitary, the time evolution on the dual bulk gravity side must be unitary too. These arguments give credence to the point of view that the information of an evaporating black hole is somehow encoded in the outgoing Hawking radiation. Therefore, it is useful to understand black hole sufficient enough, if the ultimate goal is to develop a self consistent quantum theory of gravity.

Thus, there has been many attempts to describe black hole solutions in higher spin gravity in four dimensions, [23, 24], but more extensively in the three dimensional toy

model [25]. However, despite the fact that Euclidean black hole solutions are well defined as an extension of the black hole solution of pure gravity in three dimensions (BTZ black hole), it is not clear how to define higher spin black holes in the Lorentzian signature. The reason is that higher spin fields do not couple with Einstein gravity the usual way, to form a gravitational theory. Usually, in gravitational theories, objects that are defined using the metric, i.e. scalars, construct invariants of the theory. As it is discussed in Chapter 7, this is not the case for higher spin gravity in three dimensions. Therefore, there has been an extensive effort the last years to find a generalized way of describing geodesics, which was first encountered in 2013 by two independent groups [26, 27]. Their goal was to calculate the entanglement entropy of higher spin gravity in AdS_3 .

In this thesis, we will review important aspects of pure gravity in three dimensions, higher spin theories and black holes. In Chapter 2, there will be a review of the solutions of three dimensional gravity, i.e pure AdS_3 , thermal AdS_3 and BTZ black hole, by focusing on the construction of the solutions, their thermodynamics and the geodesics in metric like formulation. Moreover, there will be a proof of the non existence of ergosphere in the rotating case of BTZ black hole. In Chapter 3, we will review Chern-Simons formulation in the framework of pure AdS and BTZ black hole, with a reference in the properties of the Euclidean theories. There will be also a proposal of a specific gauge where the thermodynamic properties of thermal AdS_3 manifest. However, not all the gauge transformations leave the physical state invariant as it is explained in Chapter 4. There is also a detailed review of Brown and Henneaux work, which was the first evidence of the correspondence in AdS_3/CFT_2 , in the same Chapter. In Chapters 5 and 6, we review the higher spin solutions together with their asymptotic symmetries. In more detail, in Chapter 6, we mention the necessary requirements in order for a higher spin solution to be a Euclidean black hole, and mention the most recent proposals for black hole configurations. In addition we point out their thermodynamic properties and show that in the particular gauges of rotating higher spin black holes there is no ergosphere. Finally in Chapter 7, we review the proposal of describing geodesics -and as extension entanglement entropy- by Wilson lines. Therefore, we calculate the geodesic length with the same end points between different solutions from pure and higher spin gravity, and try to realize if some properties of a generalized notion of black hole horizon manifest.

Chapter 2

Metric-like formulation in AdS_3

2.1 Pure AdS_3 space

Let us consider the standard vacuum Einstein Hilbert action

$$I = \frac{1}{16\pi G} \int \sqrt{-g} (R + 2\Lambda) + I_{bdy}, \quad (2.1)$$

where I_{bdy} is a term that is needed in order to ensure a proper variational principle, and will be discussed in the next chapters. Λ is the cosmological constant and was inserted in the action by Einstein -for positive values-, who wanted to achieve a static universe. Λ gives the space time specific local properties and there are three possible values. If it equals zero the corresponding action is considered to be of a flat space, if it is positive, the spacetime is called de Sitter and for negative cosmological constant the vacuum solutions are called Anti-de Sitter. In Anti-de Sitter spacetime (AdS), it is more convenient to define the cosmological constant as $\Lambda = -l^{-2}$ where l is the AdS radius which measures the curvature of spacetime.

Extremization of the action with respect to the spacetime metric $g_{\mu\nu}(x, t)$, gives the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2}(R + 2l^{-2})g_{\mu\nu} = 0. \quad (2.2)$$

These general equations, in the 2 + 1 dimensional case, determine the full Riemann tensor to be

$$R_{\mu\nu\lambda\rho} = -l^{-2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}), \quad (2.3)$$

which represents a symmetric space with constant negative curvature $R = -\frac{6}{l^2}$.

Three-dimensional AdS space has another important feature, it does not contain any propagating degrees of freedom, except some possible topological degrees of freedom

that live on the boundary, as will be discussed in the next section [28]. This absence of degrees of freedom can be verified by counting the independent components of the Weyl tensor, by using symmetries of the Riemann tensor, following the reasoning of [29]. Thus, the number of the independent components of the Weyl tensor reads,[30]

$$\frac{1}{12}d^2(d^2 - 1) - \frac{1}{2}d(d + 1),$$

which vanishes for $d = 3$. Another way to see it is by counting the physical degrees of freedom per spacetime point, [31].

One solution of the equations of motion (2.1) is AdS in global coordinates

$$ds^2 = -(1 + r^2/l^2)dt^2 + \frac{dr^2}{1 + r^2/l^2} + r^2d\phi^2. \quad (2.4)$$

In order to obtain (2.4), AdS_3 can be visualized as an embedding in four dimensional flat space with (2,2) signature, [32]

$$-u^2 - v^2 + x^2 + y^2 = -l^2$$

Consequently, the corresponding embedded metric is

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2,$$

and in order to find the metric of AdS_3 without reference to an embedding in a higher dimensional hypersurface, the following transformations will be useful

$$\begin{aligned} u &= l \cosh \mu \sin \lambda, & x &= l \sinh \mu \cos \theta \\ v &= l \cosh \mu \cos \lambda, & y &= l \sinh \mu \sin \theta \end{aligned}$$

which transforms the line element to

$$ds^2 = l^2(-\cosh^2 \mu d\lambda^2 + d\mu^2 + \sinh^2 \mu d\theta^2). \quad (2.5)$$

As it can be seen, λ is an angle and as a result there are closed timelike curves in AdS. Then, in order to obtain only causal solutions, one does not identify λ with $\lambda + 2\pi$, but "unwraps" it with the following coordinate transformation

$$\lambda = \frac{t}{l}, \quad r = l \sinh \mu.$$

The obtained line element is (2.4) which is the metric on the universal covering space.

2.1.1 Symmetries and Killing Vectors

The fact that AdS_3 can be defined as a hyperboloid in $\mathbb{R}^{2,2}$ makes it easy to see that the metric is invariant under $SO(2,2)$ transformations and, as a result, $SO(2,2)$ is the isometry group of AdS_3 . However, in general relativity, the Killing vectors are defined to be isometries of the considered manifold. The Killing vectors are

$$J_{ab} = x_b \frac{\partial}{\partial x^a} - x_a \frac{\partial}{\partial x^b},$$

where $x^a = (u, v, x, y)$ and in detail

$$\begin{aligned} J_{01} &= u\partial_v - v\partial_u & J_{02} &= x\partial_u + u\partial_x \\ J_{03} &= y\partial_u + u\partial_y & J_{12} &= x\partial_v + v\partial_x \\ J_{13} &= y\partial_v + v\partial_y & J_{23} &= y\partial_x - x\partial_y \end{aligned}$$

It is trivial to see that vector J_{01} generates time displacements ($J_{01} = \partial_\lambda$), and J_{23} generates rotations in the $x - y$ plane ($J_{23} = \partial_\theta$). The rest of the Killing vectors are generators of rotations and boosts.

2.1.2 Geodesics

From the above it follows that the asymptotic Killing vectors are $\xi^a = (\frac{\partial}{\partial t})^a$ and $\psi^a = (\frac{\partial}{\partial \theta})^a$. Then the corresponding conserved energy E and angular momentum J , per unit mass for geodesics, are

$$E = -u^a \xi_a = -g_{tt} u^t = (1 + r^2/l^2) \dot{t}, \quad (2.6)$$

$$J = u^a \psi_a = -g_{\phi\phi} u^\phi = r^2 \dot{\phi}, \quad (2.7)$$

where $u^a = \dot{x}^a = dx^a/d\lambda$ is the tangent to the curve parametrized by λ . The tangent vector is normalized by the condition

$$u^a u_a = -m^2, \quad (2.8)$$

where $m = 0$ for null geodesics and $m = 1$ for timelike geodesics. It has to be stressed that E cannot be interpreted as the local energy of the particle at infinity as in flat space, but as the initial value of energy for the particle, and the reason will be explained below.

Massless Particles

In the case of massless particles Eq.(2.8), for parametrization by proper time, becomes

$$-(1 + r^2/l^2) \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{1 + r^2/l^2} \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\phi}{d\tau} \right)^2 = 0 \quad (2.9)$$

$$\Rightarrow r^2 \left(\frac{dr}{d\tau} \right)^2 + J^2 = -r^2 \left(\frac{J^2}{l^2} - E \right), \quad (2.10)$$

where it is obvious that, by requiring that $r \in \mathbb{R}$, the right hand side of (2.10) must be positive. This means that

$$|J| \leq lE \quad (2.11)$$

In addition, the solution of this ODE (ordinary differential equation) is a second order polynomial over τ with a positive second derivative, because of (2.11):

$$r^2 \sim a\tau^2 \quad \text{where } a \geq 0$$

Finally, from this, one can find the additional geodesic equations by using

$$\frac{d\phi}{d\tau} = \frac{J}{r^2} \quad (2.12)$$

$$\frac{dt}{d\tau} = \frac{E}{1 + r^2/l^2} \quad (2.13)$$

It is not necessary for them to be solved in order to confirm that all three of them cover the whole spacetime.

Massive Particles

The next step is to find the geodesics for massive particles using the same procedure. Then,

$$-(1 + r^2/l^2) \left(\frac{dt}{d\tau} \right)^2 + \frac{1}{1 + r^2/l^2} \left(\frac{dr}{d\tau} \right)^2 + r^2 \left(\frac{d\phi}{d\tau} \right)^2 = -m^2 \quad (2.14)$$

$$\Rightarrow r^2 \left(\frac{dr}{d\tau} \right)^2 = -\frac{m^2 r^4}{l^2} - r^2 \left(\frac{J^2}{l^2} - E + m^2 \right) - J^2 \quad (2.15)$$

Using the same argument as before, the left hand side of (2.15) is positive for $r \in \mathbb{R}$ and consequently the right hand side has to be positive. However, in this situation the right hand side is a second order polynomial with respect to r^2 with negative second derivative. There are two important consequences due to this fact.

Firstly, this equation maps to real values only for a specific spectrum of the parameters of the problem. Because of the negative curvature of the polynomial, there is a valid range of radii only if there is at least one trivial solution of the polynomial ($m^2 = 1$):

$$-\frac{r^4}{l^2} - r^2 \left(\frac{J^2}{l^2} - E + 1 \right) - J^2 = 0 \quad (2.16)$$

This is possible only if :

$$\left(\frac{J^2}{l^2} - E + 1\right)^2 - 4\frac{J^2}{l^2} \geq 0, \quad (2.17)$$

which translates into two possibilities for the spectrum of $|J|$

$$\frac{|J|}{l} \leq |1 - E| \quad \text{or} \quad \frac{|J|}{l} \geq 1 + E. \quad (2.18)$$

The second comment has to do with the fact that the second order polynomial in r^2 has negative curvature. Equation (2.15) can be integrated directly. The solution for timelike geodesics for $m = 1$ is

$$r^2(\tau) = \frac{1}{2}[\alpha + \gamma \sin 2(\tau - \tau_0)] \quad , \quad (2.19)$$

with $\alpha = E^2 - L^2 + 1$, $\beta = L^2 - \frac{1}{4}$ and $\gamma = \sqrt{\alpha^2 + 4\beta}$. From this solution, one can see that the radius has an upper bound and for specific values of $L < 1/2$ it has a lower bound as well.

2.1.3 Thermal *AdS*

One can consider thermodynamics of the global *AdS* space by Wick rotating and then compactifying in the time direction. In this way it is possible to define temperature in order to compare characteristics of the global *AdS* space with BTZ or generalized black holes in three dimensions.

The first step is to get the Euclidean signature by Wick-rotating Lorentzian *AdS*₃, so that the metric is an analytic continuation of (2.4). Thus by transforming $t \rightarrow it$ one obtains

$$ds^2 = (1 + r^2/l^2)dt^2 + \frac{dr^2}{1 + r^2/l^2} + r^2d\phi^2, \quad (2.20)$$

where $r \in (0, \infty)$, $\phi \in [0, 2\pi]$ and $t \in [0, \infty)$. Nevertheless, in order to define temperature one needs to compactify time, $t \in [0, \beta)$, where β can be interpreted as the inverse temperature.

2.2 BTZ Black Hole

So far, we have only discussed the geometry, or the local aspects, of *AdS*₃. But a manifold is not only defined by its geometry, but also by its topology. For instance, a flat surface locally looks the same as a cylinder, but globally they are not the same because two of its edges have been glued together. As we shall see, this can drastically change the physics

of a spacetime. Identifying points with each other in a topological space is forming a quotient space. Points in the space are identified which is why a quotient space is sometimes called an identification space.

If the identification is performed by using points on Killing orbits (an orbit along which the metric is symmetric), the quotient space will inherit the structure from the original space. A Killing vector gives us a subgroup of the symmetry group of the manifold. By exponentiating the Killing vector and using it to transform the manifold (identifying along the Killing vector's direction), we get a manifold that is still locally isomorphic to AdS_3 . In 1992 it was discovered in [33] that, if we parametrize the metric in a certain way and then identify a coordinate, we obtain a black hole metric. In this section we will obtain this metric following [32].

Any Killing vector ξ defines a one parameter subgroup of isometries of anti-de Sitter space, by taking,

$$P \Rightarrow e^{n\xi} P \quad , \quad n = 2k\pi, \quad (2.21)$$

where ξ is a Killing vector, we get an identification subgroup. Because the Killing orbits we identify in AdS_3 are isometries, the quotient space remains a manifold with a well-defined metric with constant negative curvature. This way it remains a solution to Einstein's equations.

To ensure preservation of causality and to exclude closed timelike lines, ξ must be spacelike $\xi \cdot \xi > 0$. Some patches of AdS_3 contain parts where the identification Killing vectors used are null or timelike. These patches need to be excluded from our solution, which does not seem logical before the identifications are made. However, they will make sense after the identifications are done, because we will see that the Killing horizon (defined as $\xi \cdot \xi = 0$) coincides with the singularity in the quotient space and becomes the black hole horizon.

We do the identification using the following combination of $J_{\mu\nu}$,

$$\xi = \frac{r_+}{l} J_{ux} - \frac{r_-}{l} J_{vy} \quad (2.22)$$

in which the r_{\pm} can be seen as parametrizing constants for now. We will describe their physical interpretation later on. As said earlier, $\xi \cdot \xi$ must be larger than 0 to exclude closed timelike lines in the quotient space. We thus find that

$$\xi \cdot \xi = \frac{r_+^2 - r_-^2}{l^2} (u^2 - x^2) + r_-^2 \quad , \quad (2.23)$$

which means that in the region $-\frac{r_+^2}{r_+^2 - r_-^2} < u^2 - x^2 < 0$ the Killing vector is spacelike.

We can divide this region into three different types of subregions separated by the null surfaces $y^2 - x^2 = 0$.

We can classify these regions into:

- $u^2 - x^2 > l^2$ with $r_-^2 < \xi \cdot \xi < \infty$
- $0 < u^2 - x^2 < l^2$ with $r_-^2 < \xi \cdot \xi < r_+^2$
- $-\frac{r_-^2 l^2}{r_-^2 + r_+^2} < u^2 - x^2 < 0$ with $0 < \xi \cdot \xi < r_+^2$

Thus, each of the regions can then be parametrized as,

$$r_- < r$$

$$\begin{aligned} u &= \sqrt{A(r)} \cosh \chi(t, \phi) \\ x &= \sqrt{A(r)} \sinh \chi(t, \phi) \\ y &= \sqrt{B(r)} \cosh \tau(t, \phi) \\ v &= \sqrt{B(r)} \sinh \tau(t, \phi) \end{aligned} \tag{2.24}$$

$$r_- < r < r_+$$

$$\begin{aligned} u &= \sqrt{A(r)} \cosh \chi(t, \phi) \\ x &= \sqrt{A(r)} \sinh \chi(t, \phi) \\ y &= -\sqrt{-B(r)} \cosh \tau(t, \phi) \\ v &= -\sqrt{-B(r)} \sinh \tau(t, \phi) \end{aligned} \tag{2.25}$$

$$r > r_+$$

$$\begin{aligned} u &= \sqrt{-A(r)} \cosh \chi(t, \phi) \\ x &= \sqrt{-A(r)} \sinh \chi(t, \phi) \\ y &= -\sqrt{-B(r)} \cosh \tau(t, \phi) \\ v &= -\sqrt{-B(r)} \sinh \tau(t, \phi), \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} A(r) &= l^2 \left(\frac{r^2 - r_-^2}{r_+^2 - r_-^2} \right), & B(r) &= l^2 \left(\frac{r^2 - r_+^2}{r_+^2 - r_-^2} \right) \\ \tau &= \frac{1}{l} \left(\frac{r_+ t}{l} - r_- \phi \right), & \chi &= \frac{1}{l} \left(\frac{-r_- t}{l} - r_+ \phi \right). \end{aligned}$$

These three patches, give the same metric

$$ds^2 = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2} dt^2 + \frac{l^2 r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} dr^2 + r^2 \left(d\phi - \frac{r_+ r_-}{l r^2} dt \right). \quad (2.27)$$

Here, all coordinates take values in $[-\infty, \infty]$. However, to interpret it as a black hole, the ϕ -coordinate must be angular, otherwise it would be a boosted portion of AdS_3 .

Then one can write the line element in the more convenient form as,

$$ds^2 = -N^2(r) dt^2 + N^{-2}(r) dr^2 + r^2 (d\phi + N^\phi(r) dt)^2, \quad (2.28)$$

which is the line element for a rotating black hole with no charge. The properties of this black hole will be analyzed in the next section.

2.2.1 Singularity and event horizons in the glued patches

Usually black holes have a true singularity in the manifold, however in this case, BTZ solution is a quotient space of AdS_3 . As it was already mentioned, since quotient space BTZ is locally AdS_3 , and the only things that change are the global properties. This can make one wonder if BTZ has a true singularity, in the meaning of a ripped space time.

In order to investigate that, one can look at curvature invariants that are coordinate independent. One of these invariants is the Kretschman scalar

$$K = R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \quad (2.29)$$

which for the BTZ black hole reads

$$K = \frac{12}{l^2}. \quad (2.30)$$

This shows that there is no curvature singularity in this case, as it was suspected due to the previous reasoning. However, the argument for the BTZ solution to be called a black hole is as follows [32]. In order to avoid closed timelike loops, the three different patches are glued together and are separated by null surfaces at $r = r_\pm$. A causal curve that goes through one of these surfaces can never return. This is called a causality singularity and is then equivalent to a black hole since the latter is also defined by the irreversibility of world lines that enter through its event horizons which are, in this case, $r = r_\pm$.

2.2.2 Mass, Angular Momentum and Fefferman-Graham coordinates

The goal of this subsection is to rewrite the Eq.(2.27) in the most convenient form for generalizing it to higher spins in the next chapters.

Following the steps of [34] one can use the quasilocal stress tensor associated with a spacetime region that has been defined by Brown and York to be [35],

$$T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta I_{EH}}{\delta \gamma_{\mu\nu}}, \quad (2.31)$$

where $\gamma_{\mu\nu}$ is the induced metric at the boundary.

Then, one can find the mass and angular momentum by,

$$M = \int_0^{2\pi} T_{tt} l d\phi = \int_0^{2\pi} \frac{r_+^2 + r_-^2}{2\pi l^3} l d\phi \quad (2.32)$$

$$J = \int_0^{2\pi} T_{t\phi} l d\phi = \int_0^{2\pi} \frac{r_+ r_-}{\pi l^2} l d\phi. \quad (2.33)$$

In addition, in order to transform the metric into Fefferman- Graham coordinates, it needs to be written in the following way,

$$ds^2 = d\rho^2 + g_{ij}(x^k, \rho) dx^i dx^j, \quad i, j = t, \phi. \quad (2.34)$$

This can be accomplished by redefining the coordinates as [36]

$$r^2 = r_+^2 \cosh^2(\rho - \rho_0) - r_-^2 \sinh^2(\rho - \rho_0), \quad e^{2\rho_0} \equiv \frac{r_+^2 - r_-^2}{4}, \quad (2.35)$$

such that the metric reads

$$ds^2 = d\rho^2 + \frac{2\pi\mathcal{L}}{k} (dx^+)^2 + \frac{2\pi\bar{\mathcal{L}}}{k} (dx^-)^2 - \left(e^{2\rho} + (2\pi)^2 \frac{\mathcal{L}\bar{\mathcal{L}}e^{-2\rho}}{k^2} \right) dx^+ dx^-, \quad (2.36)$$

where, $x^\pm = t \pm \phi$ are defined to be the light-cone coordinates and

$$\frac{2\pi\mathcal{L}}{k} = \frac{(r_+ + r_-)^2}{2}, \quad \frac{2\pi\bar{\mathcal{L}}}{k} = \frac{(r_+ - r_-)^2}{2}. \quad (2.37)$$

The parameters $\mathcal{L}, \bar{\mathcal{L}}$ are related to the zero modes of the boundary stress tensor, as it can be seen in Eq.(2.32), and one obtains that¹

$$Ml = \mathcal{L} + \bar{\mathcal{L}}, \quad J = \mathcal{L} - \bar{\mathcal{L}} \quad (2.38)$$

Finally, one can rewrite the line element (2.28) using the above results for the mass and the angular momentum of the black hole. Then the lapse and shift terms are obtained [36]

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \quad (2.39)$$

$$N^\phi(r) = -\frac{J}{2r^2} \quad (2.40)$$

¹The convention that is been used for the *AdS* radius is $l = 1$

2.3 Black hole properties

Horizon revisited

The rotating black hole that was defined above is the three dimensional analogue of the Kerr solution in four dimensions. Thus we expect it to have two horizons, which were defined above. They are the null surfaces where the three patches were glued together. However, another way to realize these surfaces is by finding the surface where the radial component of the metric blows up, which happens for vanishing lapse function $N(r)$. This again gives the two values of the inner and out horizons

$$r_{\pm} = l \left[\frac{M}{2} \left(1 \pm \sqrt{1 - \left(\frac{J}{Ml} \right)^2} \right) \right]^{1/2}. \quad (2.41)$$

In order for the horizons to exist one must have

$$M \geq 0, \quad |J| \leq Ml. \quad (2.42)$$

In the case of the extremal BTZ black hole, $|J| = Ml$, both roots of $N^2 = 0$ coincide. In addition, in the non rotating case, $J = 0$, there is the maximum value of the outer horizon radius, for a given mass, $r_+^2 = Ml^2$. On the other hand the inner horizon has shrunk to a point.

Ergosphere

Due to the fact that BTZ is an analogue of the Kerr solution, one should figure out if there is an ergosphere in the rotating BTZ solution, as there is one in the Kerr black hole.

Ergosphere is the region between the outer horizon and the stationary limit surface. Inside the ergosphere any object must move in the direction of the rotation of the black hole. This means that there is no stationary observer allowed in this region. More interesting phenomena occur in the ergosphere. One of the most important results is the Penrose process, [37, 38].

To show the existence of an ergosphere, an asymptotic timelike Killing vector ζ^μ has to be defined first. Then we want to find the region where this timelike killing vector becomes spacelike. However, we should not make the mistake and take a specific timelike killing vector, for example $\zeta^\mu = \left(\frac{\partial}{\partial t} \right)^\mu$. In this case, we would find that the Killing vector becomes null and then spacelike at the root $g_{tt} = 0$, $r^2 = Ml^2$ [32]. Furthermore, one needs to consider that the existence of the ergosphere is a coordinate invariant notion. As a result, in order to ensure that, all the asymptotically timelike Killing vectors have

to become spacelike. Thus, let the most general killing vector to be a linear combination $\tilde{\zeta}^\mu = \left(\frac{\partial}{\partial t}\right)^\mu + \Omega \left(\frac{\partial}{\partial \phi}\right)^\mu$, then using (2.27)

$$\tilde{\zeta}^\mu \tilde{\zeta}_\mu = g_{\mu\nu} \tilde{\zeta}^\mu \tilde{\zeta}^\nu = g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}. \quad (2.43)$$

Note that for our purposes $g(\tilde{\zeta}, \tilde{\zeta}) < 0$ asymptotically, which gives Ω a constraint,

$$g(\tilde{\zeta}, \tilde{\zeta}) \Big|_{\text{bdry}} < 0 \implies r^2 \left(\Omega^2 - \frac{1}{l^2} \right) + \frac{r_+^2 + r_-^2}{l^2} - \frac{2r_+ r_- \Omega}{l} \Big|_{r \rightarrow \infty} < 0 \quad (2.44)$$

$$\implies \Omega^2 < \frac{1}{l^2} \quad \text{for an asymptotically timelike killing vector.} \quad (2.45)$$

Furthermore, we want to find the point where $g(\tilde{\zeta}, \tilde{\zeta})$ flips sign, so we will look for the roots of $g(\tilde{\zeta}, \tilde{\zeta}) = 0$. The solution reads

$$r_e^2 = \frac{\frac{r_+^2 + r_-^2}{l^2} - \frac{2r_+ r_- \Omega}{l}}{\frac{1}{l^2} - \Omega^2}. \quad (2.46)$$

Note that the numerator is positive for the positive values of $\Omega < \frac{1}{l}$. However, the ergosphere is a region outside the horizon r_+ , thus it is logical to search for solutions such that $r_e^2 > r_+^2$. Using (2.46)

$$r_e^2 > r_+^2 \implies \left(r_+ \Omega - \frac{r_-}{l} \right)^2 > 0, \quad (2.47)$$

which is satisfied for every Ω except $\Omega = \frac{r_-}{lr_+}$, where the surface that the timelike killing vector becomes spacelike coincides with the outer horizon.

As a result, we have proven that one can always find a Killing vector that is timelike outside the horizon of BTZ black hole. This means that there is no region where an observer cannot stay stationary, thus there is no ergosphere despite the general belief that there is one, e.g. [32].²

Euclidean signature - Conical Singularities Resolution

In order to obtain the BTZ metric in Euclidean signature from (2.27), we use the usual transformations,

$$\begin{aligned} t &\rightarrow it & r_- &\rightarrow ir_- \\ M &\rightarrow M_E & J &\rightarrow iJ_E. \end{aligned}$$

Then the Euclidean metric is

$$ds^2 = \frac{(r^2 - r_+^2)(r^2 + r_-^2)}{l^2 r^2} dt^2 + \frac{l^2 r^2}{(r^2 - r_+^2)(r^2 + r_-^2)} dr^2 + r^2 \left(d\phi + \frac{r_+ r_-}{l r^2} dt \right)^2. \quad (2.48)$$

²I would like to thank Dr. Castro who brought this problem to my attention.

It is important to mention at this point that this is well defined only for the patch $r > r_+$ because r_- becomes purely imaginary. However, one needs to impose that the horizon is smooth in order to find the temperature of the black hole. This is an important point that we will re-investigate later on again using the Chern-Simons formulation in the next chapter.

To avoid conical singularities in metric-like formulation, we examine the near-outer horizon geometry, following [39]. Performing the following transformation

$$t'_E = r_+ t_E + r_- \phi \quad , \quad \phi' = r_+ \phi - r_- t_E \quad , \quad r'^2 = \frac{r^2 - r_+^2}{r_-^2 + r_+^2} \quad , \quad (2.49)$$

one obtains the Euclidean metric

$$ds_E^2 = \frac{r'^2}{l^2} (dt'_E)^2 + \frac{l^2}{1 + r'^2} dr'^2 + (1 + r'^2) d\phi'^2. \quad (2.50)$$

Therefore, at the near-horizon limit, $r' \rightarrow 0$, the non-angular part is similar to the metric of plane,

$$ds_{plane}^2 = r'^2 (dt'_E)^2 + dr'^2. \quad (2.51)$$

This means that in order for it not to be singular, t'_E needs to be an angle, with identification $t'_E \simeq t'_E + 2\pi$. Using this and the fact that ϕ' does not need to be periodic, transforming back to the original coordinates one finds that, if β and Φ are the periodicities in the t_E and ϕ direction respectively, then

$$\beta = \frac{2\pi l r_+}{r_+^2 + r_-^2} \quad , \quad \Phi = \frac{2\pi l^2 r_+}{r_+^2 + r_-^2}. \quad (2.52)$$

It is important to be noted that after this identification, the geometry of the Euclidean BTZ black hole manifold has a solid torus topology.

The Hawking temperature that can thus be assigned to the Black hole is given by

$$T = \frac{1}{\beta} = \frac{r_+^2 + r_-^2}{2\pi l r_+}. \quad (2.53)$$

The Bekenstein-Hawking entropy gives the entropy of the black hole

$$S = \frac{A}{4G} = \frac{2\pi r_+}{4G} \quad (2.54)$$

where A is the area of the black hole. The entropy could be derived in metric like formulation in several ways: e.g. from the Euclidean path integral or with the Wald formula (see [40] for an overview of these derivations).

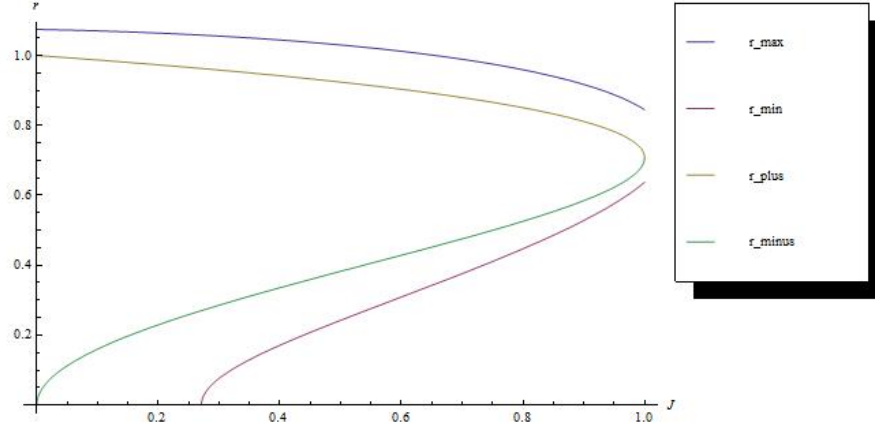


Figure 2.1: The figure presents the lower and upper bound (r_{\min}, r_{\max}) as functions of the rotation of the black hole, J . The radius of the inner and outer horizon (r_-, r_+) are also provided in order to show that the bounds are always outside the horizon.

2.3.1 Geodesics

Following [subsection 2.1.2](#), one obtains the constants of motion along the geodesic

$$E = -g_{ab}\zeta^a u^b = \left(-M + \frac{r^2}{l^2}\right) \left(\frac{dt}{d\tau}\right) + \frac{J}{2} \left(\frac{d\phi}{d\tau}\right), \quad (2.55)$$

$$J = -g_{ab}\psi^a u^b = r^2 \left(\frac{d\phi}{d\tau}\right) - \frac{J}{2} \left(\frac{dt}{d\tau}\right). \quad (2.56)$$

Using the following rescaling

$$r \rightarrow l\sqrt{M}r, \quad \phi \rightarrow \frac{\phi}{\sqrt{M}}, \quad t \rightarrow \frac{lt}{\sqrt{M}}, \quad \lambda \rightarrow l\lambda, \quad (2.57)$$

$$E \rightarrow \sqrt{M}E, \quad L \rightarrow l\sqrt{M}L, \quad J \rightarrow lMJ, \quad (2.58)$$

we obtain the geodesic equations [\[41\]](#),

$$r^2 \left(\frac{dr}{d\tau}\right)^2 = -m^2 \left(r^4 - r^2 + \frac{J^2}{4}\right) + (E^2 - L^2)r^2 + L^2 - JEL, \quad (2.59)$$

$$\frac{d\phi}{d\tau} = \frac{(r^2 - 1)L + \frac{1}{2}JE}{(r^2 - r_+^2)(r^2 - r_-^2)}, \quad (2.60)$$

$$\frac{dt}{d\tau} = \frac{Er^2 - \frac{1}{2}JL}{(r^2 - r_+^2)(r^2 - r_-^2)}. \quad (2.61)$$

They can be integrated directly. The solution for the timelike radial geodesic ($m = 1$) is

$$r^2(\tau) = \frac{1}{2}[\alpha + \gamma \sin 2(\tau - \tau_0)] \quad , \quad (2.62)$$

with $\alpha = E^2 - L^2 + 1$, $\beta = L^2 - JEL - \frac{J^2}{4}$ and $\gamma = \sqrt{\alpha^2 + 4\beta}$.

Using the same reasoning with [subsection 2.1.2](#), it is clear that there always exists a finite upper bound r_{\max} for the radial coordinate, in contrast to massless particles that can

travel until the boundary. However, as we can obtain from the Figure 2.1, for the upper bound $r_{max} > r_+$, which means that the massive particles are allowed to be outside of the horizon of the black hole.

Chapter 3

Chern-Simons Formulation

Chern-Simons theory is a quantum theory in (2+1) dimensions that computes only topological invariants. It can be defined on any manifold, and the metric does not need to be specified as it is a topological theory. Thus the physical quantities do not depend on the local geometry. Chern-Simons is also a gauge theory, which means that in a given gauge group G and on a manifold \mathcal{M} , the theory is defined by a principal G -bundle on \mathcal{M} . A principle G - bundle is a formalism defining the action of the group G on \mathcal{M} , and the projection of this action on the manifold. For greater investigation of the properties of Chern-Simons theories one can read [42]. This holographic realization will be a convenient way to generalize black holes with higher spin coupling in the next chapter.

3.1 Pure Gravity as a Chern Simons Theory

In this section we will show that the vacuum Einstein (2+1)-dimensional AdS gravity is equivalent to a Chern-Simons gauge theory, as it was first proposed by Achucarro and Tounsend in [43] and developed by Witten in [44]. In order to do that one needs to work on a coordinate independent system which is the first order formalism. Following [44] one can start by defining a d -dimensional space-time manifold \mathcal{M} of Lorentzian signature and its associated tangent bundle T . Then one can also introduce a d -dimensional vector bundle V with a structure group $SO(d-1, 1)$, that has flat metric. A vector bundle is a topological construction of how one or more vector spaces are parametrized by another space- \mathcal{M} . It is assumed that V is of the same topological type as \mathcal{M} , so that there exists an isomorphism between the vector bundle V and the tangent bundle of \mathcal{M} . This isomorphism is the vielbein which has to be invertible for our theory to be consistent. Thus, the vielbein gives us the transformation from the tangent space (where the metric

$g_{\mu\nu}$ lives) to the vector bundle. The basic properties of vielbein and spin connection are discussed in Appendix B.

In the case of three-dimensional vacuum space with a negative cosmological constant, the isometry group is $SO(2,2)$, as we have mentioned before. However, $SO(2,2)$ is isomorphic to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. Thus, Witten, [44], constructed the Chern-Simons action to be invariant under this group and then showed that is equivalent to Einstein-Hilbert action in three dimensions which, in terms of the Cartan formalism, reads

$$I_{EH} = \frac{1}{16\pi G} \int_{\mathcal{M}} \left(e^a \wedge \left(2d\omega_a + \epsilon_{abc}\omega^b \wedge \omega^c \right) + \frac{\Lambda}{3} \epsilon_{abc} e^a \wedge e^b \wedge e^c \right), \quad (3.1)$$

where $\Lambda = -l^2$ is the cosmological constant. Using the conventions in [28], the Chern-Simons actions can be written in the form,

$$I_{CS}[A^\pm] = \frac{k}{4\pi} \int_{\mathcal{M}} Tr \left(A^\pm \wedge dA^\pm + \frac{2}{3} A^\pm \wedge A^\pm \wedge A^\pm \right). \quad (3.2)$$

A^\pm are two connection one-forms that are valued in the gauge group's Lie algebra of $SL(2, \mathbb{R})$, and the trace is taken over the group generators. The Chern-Simons potential can be written in terms of the vielbein and the spin connection,

$$A^\pm = \omega \pm \frac{1}{l} e \quad \rightarrow \quad A^\pm = \left(\omega_\mu^a \pm \frac{1}{l} e_\mu^a \right) L_a dx^\mu, \quad (3.3)$$

where L_a are the generators of $SL(2, \mathbb{R})$, see Appendix A.1. On the other hand, k is a parameter that will be found by the identification of the Chern-Simons action with the Einstein-Hilbert action in Eq.(3.1)

$$I_{EH} = I_{CS}[A^+] - I_{CS}[A^-]. \quad (3.4)$$

After using the Killing metric for the fundamental representation of $SL(2, \mathbb{R})$ one can find that the above identification holds if

$$k = \frac{l}{4G} \quad (3.5)$$

Another way to see this identification is by thinking of A^\pm as right and left moving fields and then one can say that the level k is related to the level of the coset CFT on the boundary, see [17].

The equations of motion for the fields A^\pm obtained from Eq.3.2 read

$$F^\pm \equiv dA^\pm + A^\pm \wedge A^\pm = 0. \quad (3.6)$$

where F is the curvature tensor of the Chern-Simons potential. The above equation shows that Chern-Simons is regarded as a theory of flat connection, which can be easily shown

to be equivalent to the requirement that the connection be torsion-free and that the metric has a constant negative cosmological constant.

The advantage of this theory is that gravity becomes an ordinary gauge theory and, in particular, on shell diffeomorphisms are now equivalent to ordinary gauge transformations. We now obtain a particular representation of diffeomorphisms in terms of global and gauge transformations. The Lie derivative of the connection is

$$\mathcal{L}_{\tilde{\zeta}}A = d(\tilde{\zeta} \cdot A) + \tilde{\zeta} \cdot dA = \tilde{\zeta} \cdot F + D_A(\tilde{\zeta} \cdot A) \quad , \quad (3.7)$$

where D_A is the gauge-covariant exterior derivative. Then on shell, which means $F = 0$, the equation (3.7) is an infinitesimal gauge transformation with gauge parameter $\lambda^a = \tilde{\zeta}^\mu A_\mu^a$.

Finally, the general gauge transformation of the gauge field in a Chern-Simons theory is

$$A_\mu \rightarrow \bar{A}_\mu = GA_\mu G^{-1} + G\partial_\mu G^{-1} \quad , \quad (3.8)$$

where G is a transformation under the gauge group. Thus, if x^a is a constant parameters, then $G(x) = e^{x^a L_a}$. Therefore, equivalently, the gauge transformation for the right and left moving gauge fields are

$$A^+ \rightarrow \bar{A}^+ = L(A^+ + d)L^{-1} \quad A^- \rightarrow \bar{A}^- = R^{-1}(A^- + d)R \quad , \quad (3.9)$$

where L and R are the gauge transformations that take values in the first and the second copy of $SL(2, \mathbb{R})$, respectively.

3.2 The BTZ black hole

Although the BTZ black hole is well understood in a metric like formulation it will be useful to understand it from the holographic point of view, which is the purpose of this section.

One can use the metric element that we found in [section 2.2](#) and using that

$$g_{\mu\nu} = \frac{1}{2} \text{Tr}(e_\mu e_\nu) \quad , \quad (3.10)$$

one can obtain the corresponding Chern-Simons fields, in light-cone coordinates $A_t dt + A_\phi d\phi = A_{x^+} dx^+ + A_{x^-} dx^-$

$$A^+ = \begin{pmatrix} \frac{1}{2}d\rho & -\frac{4G}{l}\mathcal{L}^+ e^{-\rho} dx^+ \\ -e^\rho dx^+ & -\frac{1}{2}d\rho \end{pmatrix} \quad , \quad A^- = \begin{pmatrix} -\frac{1}{2}d\rho & -e^\rho dx^- \\ -\frac{4G}{l}\mathcal{L}^- e^{-\rho} dx^- & \frac{1}{2}d\rho \end{pmatrix} . \quad (3.11)$$

This result is obtained after imposing the correct asymptotic behaviour, which will be analyzed in more details in the next chapter. It is important to point out that, firstly, the lightlike components here, $A_{x^-}^+$ and $A_{x^+}^-$ are set to zero asymptotically. Secondly, $A_{x^-}^-$ and $A_{x^+}^+$ are not functions of the variables t and ϕ . At the same time, A_ρ^\pm are set to vanish. These points are going to be important in order to show that Chern Simons action corresponds to the $SL(2, \mathbb{R})$ no-chiral Wess-Zumino-Witten model, [45].

Therefore, we can write the light-like components of the gauge fields in a more compact way using the commutation relations for $SL(2, \mathbb{R})$, see Appendix A.1, as

$$A^+ = b^{-1}(\rho)a(x^+)b(\rho) \quad (3.12)$$

$$A^- = b(\rho)\bar{a}(x^-)b^{-1}(\rho), \quad (3.13)$$

with $b(\rho) = e^{\rho L_0}$. Then Eq.(3.5) obtains

$$a(x^+) = \left(L_{+1} - \frac{2\pi\mathcal{L}^+}{k}L_{-1} \right) dx^+ \quad (3.14)$$

$$\bar{a}(x^-) = - \left(L_{-1} - \frac{2\pi\mathcal{L}^-}{k}L_{+1} \right) dx^- \quad (3.15)$$

We can always parametrize the solutions following the trivial gauge, i.e. $A^+ = A^- = 0$, in the transformation (3.9), leading to

$$A^+ = LdL^{-1} \quad , \quad A^- = R^{-1}dR. \quad (3.16)$$

Here L and R are, for solutions that do not depend on z and \bar{z} ,

$$R(x^+, x^-, \rho) = \exp \left(\int_0^{x^-} dx'^- \bar{a} \right) b^{-1}(\rho) \quad (3.17)$$

$$L(x^+, x^-, \rho) = b^{-1}(\rho) \exp \left(\int_0^{x^+} dx'^+ a \right). \quad (3.18)$$

In particular, for the metric found in section 2.2, when \mathcal{L}^\pm are the constants in (2.37) the two functions take the form

$$R = \exp \left[- \left(L_{-1} - \frac{2\pi\mathcal{L}^-}{k}L_{+1} \right) x^- \right] b^{-1}(\rho) \quad (3.19)$$

$$L = b^{-1}(\rho) \exp \left[\left(L_{+1} - \frac{2\pi\mathcal{L}^+}{k}L_{-1} \right) x^+ \right]. \quad (3.20)$$

In addition, we give the form for the vacuum solution. The reason why we did not calculate it before is that the gauge field has the same form as Eq.(3.12) for different values of the asymptotic charges. If we follow the argument in section 2.2 and Ref.[32], the AdS

vacuum can be obtained, by setting $M = -1$ and $J = 0$. Thus, by using [section 2.2](#) one obtains $\frac{2\pi\mathcal{L}^\pm}{k} = -\frac{1}{4}$, and as a result, [\[20\]](#),

$$a(x^+) = \left(L_{+1} + \frac{1}{4}L_{-1} \right) dx^+ \quad (3.21)$$

$$\bar{a}(x^-) = - \left(L_{-1} + \frac{1}{4}L_{+1} \right) dx^-. \quad (3.22)$$

Finally, one can relate the AdS_3 Killing vectors and tensors with the isometries in the Chern-Simons formalism, [\[20\]](#). Thus, the gauge transformations that are generated by the parameters

$$\zeta_a = L^{-1}L_aL = \left(L^{-1}L_aL \right)^b L_b \quad (3.23)$$

$$\bar{\zeta}_a = R^{-1}L_aR = \left(R^{-1}L_aR \right)^b L_b \quad (3.24)$$

leave the AdS_3 connections A^\pm invariant, respectively. If we construct the gauge parameters using the inverse of the AdS vielbein, the complete set of isometries give rise to the Killing vectors.

3.3 Thermodynamics of BTZ black hole

In order to realize the thermodynamic properties of a black hole one transforms it into the Euclidean signature. The geometry of a Euclidean 2+1 black hole was investigated in [\[46\]](#). It is shown that the topology induced by the metric on the three dimensional Euclidean space is that of a solid torus or equivalently $\mathbb{R}^2 \times S^1$.

Temperature

The Euclidean metric can also be written in Chern-Simons formulation. Using the coordinates $z = \phi + i\frac{t}{l}$ and its complex conjugate, in addition to the ϕ periodicity, we have

$$(z, \bar{z}) \equiv (z + 2\pi\tau, \bar{z} + 2\pi\bar{\tau}). \quad (3.25)$$

The relation of the identification of τ with β, Φ found in [\(2.52\)](#) in [Section 2.3](#), is given by $\tau = i(\Phi + \frac{i}{l}\beta)$, and for the anti-holographic part $\bar{\tau} = -i(\Phi - \frac{i}{l}\beta)$. As a result the connections can be written the same way as in the Lorentzian signature [\(3.14\)](#)

$$a(z) = \left(L_{+1} - \frac{2\pi\mathcal{L}^+}{k}L_{-1} \right) dz \quad (3.26)$$

$$\bar{a}(\bar{z}) = - \left(L_{-1} - \frac{2\pi\mathcal{L}^-}{k}L_{+1} \right) d\bar{z}. \quad (3.27)$$

Furthermore we provide a Chern-Simons interpretation of requiring the solution to be smooth by avoiding the conical singularity. The equivalent statement is that the holonomy around the contractible direction is equal to the central of $SL(2, \mathbb{R})$, i.e. [47]

$$\text{Hol}_{\tau, \bar{\tau}}(a) = e^{2\pi(\tau a_z - \bar{\tau} a_{\bar{z}})} = e^{2\pi i L_0} = -\mathbb{1} \quad (3.28)$$

$$\text{Hol}_{\tau, \bar{\tau}}(\bar{a}) = e^{2\pi(\tau \bar{a}_z - \bar{\tau} a_{\bar{z}})} = e^{2\pi i L_0} = -\mathbb{1}, \quad (3.29)$$

where we used the fundamental representation of $SL(2, \mathbb{R})$, see Appendix A.1. In the non rotating case, (3.28) reduces to demanding that the holonomy around the time circle has to be trivial,

$$\text{Hol}_C(a_t) = e^{2\pi \tau a_t}. \quad (3.30)$$

The solution to (3.28) reduces to the connection between temperature and asymptotic charges

$$\tau = \frac{il}{r_+ - r_-} = \frac{ik}{2} \sqrt{\frac{2\pi}{k\mathcal{L}^+}} \quad (3.31)$$

$$\bar{\tau} = \frac{-il}{r_+ + r_-} = \frac{-ik}{2} \sqrt{\frac{2\pi}{k\mathcal{L}^-}}, \quad (3.32)$$

where we have used (2.37).

An important fact that will be useful in the next section is that we can also reproduce empty AdS_3 in global coordinates by picking $2\pi i \mathcal{L}^+ / k = 2\pi i \mathcal{L}^- / k = -1/4$. For this choice of parameters one can check that the holonomy around the ϕ -cycle is trivial

$$\text{Hol}_C(a_\phi) = e^{2\pi a_\phi} = e^{2\pi i L_0}, \quad (3.33)$$

and the same for the anti-holomorphic component. This is consistent, since there is no singularity in global AdS that prevents us from shrinking a circle around the ϕ -cycle to zero.

If we take a combination of τ and $\bar{\tau}$ we again find a relation with the inverse Hawking temperature found in the metric formulation, (2.52),

$$T_H = \frac{1}{\pi l(\tau - \bar{\tau})} = \frac{r_+^2 + r_-^2}{2\pi l r_+}. \quad (3.34)$$

Entropy and the integrability condition

A useful way to calculate the entropy for a black hole is to write the partition function and calculate the entropy by assuming that the first law of thermodynamics $dM = TdS + \Omega dJ$ holds. The black hole partition function can be defined as

$$Z(\tau, \bar{\tau}) = \text{Tr} \left[e^{i2\pi(\tau \mathcal{L}^+ - \bar{\tau} \mathcal{L}^-)} \right] = e^{S + i2\pi\tau \mathcal{L}^+ - i2\pi\bar{\tau} \mathcal{L}^-}, \quad (3.35)$$

where in the last equality the thermodynamical limit is taken. From this expression of the partition function the charges can be written in the following way

$$\mathcal{L}^+ = -\frac{i}{2\pi} \frac{\partial \ln Z}{\partial \tau}, \quad \mathcal{L}^- = \frac{i}{2\pi} \frac{\partial \ln Z}{\partial \bar{\tau}}. \quad (3.36)$$

Moreover, the entropy of the black hole can be written in terms of the partition function, the charges and the chemical potential (equivalently instead of $\ln(Z)$, we could have written the free energy $F = -T \ln Z$)

$$S = \ln(Z) - i2\pi\tau\mathcal{L}^+ + i2\pi\bar{\tau}\mathcal{L}^-. \quad (3.37)$$

Now, since the dependence of \mathcal{L} in terms of τ is known, (3.31), $\ln Z$ can be found by integrating (3.36)

$$\ln Z = i2\pi \left(\int \mathcal{L}^+ d\tau - \int \mathcal{L}^- d\bar{\tau} \right) \quad (3.38)$$

$$= 2\pi i \frac{k}{4} \left(\frac{1}{\tau} - \frac{1}{\bar{\tau}} \right) \quad (3.39)$$

$$= \pi\sqrt{k\mathcal{L}^+} + \pi\sqrt{k\mathcal{L}^-}. \quad (3.40)$$

Then one obtains the thermal entropy of BTZ black hole

$$S_{\text{th}} = 2\pi\sqrt{k\mathcal{L}^+} + 2\pi\sqrt{k\mathcal{L}^-}, \quad (3.41)$$

which is equal to the Bekenstein-Hawking entropy.

We could in general add a charge to the black hole which gives an extra component to the first law. We can then write a new partition function

$$Z(\tau, \bar{\tau}, \alpha, \bar{\alpha}) = e^{S+i2\pi\tau\mathcal{L}^+-i2\pi\bar{\tau}\mathcal{L}^-+\alpha Q^+-\bar{\alpha} Q^-} \quad (3.42)$$

where $\mathcal{L}^+, \mathcal{L}^-, Q^+, Q^-$ are the charges and $\tau, \bar{\tau}, \alpha, \bar{\alpha}$ are the conjugate potentials associated to these charges. Until now this is a straightforward generalization of the BTZ black hole. However, in order for the partition function to exist, an extra restriction on the charges is given [48]. Consider the analogue of (3.36),

$$\mathcal{L}^+ = -\frac{i}{2\pi} \frac{\partial \ln Z}{\partial \tau}, \quad Q^+ = -\frac{i}{2\pi} \frac{\partial \ln Z}{\partial \bar{\alpha}}, \quad (3.43)$$

and similar for the anti-holomorphic part. Relating these two expressions implies a condition on the charges

$$\left. \frac{\partial \mathcal{L}^+}{\partial \alpha} \right|_{\tau} = \left. \frac{\partial Q^+}{\partial \tau} \right|_{\alpha} \quad (3.44)$$

We will refer to this relation as the integrability condition, and it will show to be important in defining black holes in higher spin theories.

3.4 Thermal AdS_3

In Section 2.1.3, we considered the description of thermal AdS space and as we realized, the line element (2.20), remains the same for every value of temperature. The goal of this section is to try to write the thermal AdS connection in a gauge in which the thermodynamical properties can be interpreted. The first step is to write the most general connection with only diagonal elements, along the same lines as for the global AdS and the (Euclidean) BTZ black hole, allowing the same boundary conditions,

$$\begin{aligned} a_\phi &= aL_{+1} + bL_{-1} \\ a_t &= cL_{+1} + dL_{-1} \end{aligned} \quad (3.45)$$

$$\begin{aligned} \bar{a}_\phi &= \bar{a}L_{+1} + \bar{b}L_{-1} \\ \bar{a}_t &= \bar{c}L_{+1} + \bar{d}L_{-1}, \end{aligned} \quad (3.46)$$

where $\phi \in [0, 2\pi)$ and $t \in [0, 2\pi\beta)$, as mentioned in Section 2.1.3.

Moreover, as it was mentioned in the previous section, AdS space should have contractible ϕ -cycle, (3.33). Furthermore, since t is compactified and behaves as an angle, the t -cycle should be also contractible,

$$\text{Hol}_{C'}(a_t) = e^{2\pi\beta a_t} = e^{2\pi i L_0}. \quad (3.47)$$

One finds that the constraints obtained are

$$ab = \frac{1}{4} \quad \text{and} \quad cd = \frac{1}{4\beta^2}, \quad (3.48)$$

with similar equations for the anti-holomorphic part.

An other important feature of empty AdS is that the line element is diagonal, thus an additional constraint is obtained by demanding $g_{t\phi} = 0$. Using (3.10) and (3.3), one can write some of the parameters as a function of the remaining ones. Then, the gauge fields read

$$\begin{aligned} a_\phi &= aL_{+1} + \frac{1}{4a}L_{-1} \\ a_t &= cL_{+1} + \frac{1}{4c\beta^2}L_{-1} \end{aligned} \quad (3.49)$$

$$\begin{aligned} \bar{a}_\phi &= \frac{1}{4\bar{b}}L_{+1} + \bar{b}L_{-1} \\ \bar{a}_t &= -\frac{a}{4\bar{b}c\beta^2}L_{+1} - \frac{\bar{b}c}{a}L_{-1}. \end{aligned} \quad (3.50)$$

Furthermore, this gauge has a radial dependence that is not explicitly shown. One way to find it is to realize the parameters as functions of the radial component and find the diffeomorphism between them. One can find the radial dependence of empty AdS by using (3.21) and (3.12). Thus, the connection reads

$$a(x^+) = \left(e^\rho L_{+1} + e^{-\rho} \frac{1}{4} L_{-1} \right) dx^+ \quad (3.51)$$

$$\bar{a}(x^-) = - \left(e^\rho L_{-1} + e^{-\rho} \frac{1}{4} L_{+1} \right) dx^-. \quad (3.52)$$

In order to show this map we calculate the metric elements $g_{tt}, g_{\phi\phi}$ for both connections (3.51) and (3.21). One then finds that for $a = e^\rho$

$$\bar{b} = e^\rho \quad (3.53)$$

$$c = \frac{1}{8} (-4 - e^{-\rho} - e^{-2\rho} \sqrt{1 + 8e^{2\rho} + 16e^{4\rho} - 16e^{2\rho} T^2}), \quad (3.54)$$

where $T = \frac{1}{\bar{b}}$. By substituting these result in (3.49) one obtains a connection for thermal AdS_3 that has dependence on the temperature.

Chapter 4

Global and Local Transformations

It has become obvious that there are two important aspects of (2+1)-dimensional gravity that have to be taken into account if we want to proceed to the thermodynamical properties of the BTZ black hole or to generalize the theory to include couplings to higher spin fields. The purpose of this chapter is to take a closer look to the work of Brown and Henneaux and present it in the Chern-Simons language. In addition, we look at the symmetries and distinguish the ones that are symmetries of the system, which correspond to proper transformations, and the transformations that change the physical state, improper ones.

4.1 Asymptotic Symmetries and Surface Charges

One can start by pointing out that the BTZ solution is locally and asymptotically AdS_3 , as mentioned before. Therefore, before we can talk about how deformations act on the asymptotic structure of a space, we need to realise what it actually means by "a space has the asymptotic structure of AdS ". We follow the definition given by Henneaux and Teitelboim, [49]. In their paper, they give a natural definition of a four-dimensional asymptotically Anti-de Sitter space by posing invariance under the isometry group of AdS_4 , $O(2,3)$. They figured out that the most lenient boundary conditions are the ones that close under $O(2,3)$. A natural generalization is the application of this strategy to other spaces, by simply replacing the AdS_4 isometry group with the isometry group of the space in question.

The boundary conditions are basically restrictions on the allowed finite deformations of some background geometry. Let us denote the metric on the background geometry by $\bar{g}_{\mu\nu}$ and the deformation by $h_{\mu\nu}$, so that the deformed metric is $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. The

boundary conditions can be formulated in powers of a well defined radial coordinate

$$h_{\mu\nu} \sim O(r^n), \quad (4.1)$$

where $O(r^n)$ are arbitrary functions in t and ϕ . Also n is an integer and different for each component.

We consider an asymptotic deformation ζ^μ that acts on the metric as $g_{\mu\nu} \rightarrow g_{\mu\nu} + \mathcal{L}_\zeta g_{\mu\nu}$. Such a deformation can be generated by a corresponding charge denoted Q_ζ . In order to find an explicit expression for such a charge, one needs to work in the Hamiltonian formulation, which is introduced in [50].

The Hamiltonian formulation for general relativity generally has the form,

$$H = \int_{\Sigma_t} d^d y \sqrt{q} (N \hat{\mathcal{H}} + N^a \hat{\mathcal{H}}_a) + \oint_{\Sigma_t} d^{d-1} \theta \sqrt{\gamma} (N \hat{\mathcal{H}}^{bdy} + N^a \hat{\mathcal{H}}_a^{bdy}). \quad (4.2)$$

The Hamiltonian generates time translations in the standard canonical formalism. The above Hamiltonian is a slight generalization of this idea, because it actually generates a flow along the flow vector $t^\mu = N n^\mu + N^a p_a^\mu$. This becomes more obvious when we go from the normal/tangent basis to the full-spacetime basis. We thus write $\hat{\mathcal{H}} = n^\mu \mathcal{H}_\mu$ and $\hat{\mathcal{H}}_a = p_a^\mu \mathcal{H}_\mu$ as the normal and tangent components of the same $(d+1)$ -dimensional object \mathcal{H}_μ . We can do the same for the boundary quantities

$$\hat{\mathcal{H}}^{bdy} = n^\mu \mathcal{Q}_\mu \quad \text{and} \quad \hat{\mathcal{H}}_a^{bdy} = p_a^\mu \mathcal{Q}_\mu. \quad (4.3)$$

One should keep in mind that the latter two quantities are not constrains. Writing the Hamiltonian in this basis

$$H = \int_{\Sigma_t} d^d y \sqrt{q} \mathcal{H}_\mu t^\mu + \oint_{\Sigma_t} d^{d-1} \theta \sqrt{\gamma} \mathcal{Q}_\mu t^\mu. \quad (4.4)$$

Naturally, one can define the generators of the Lie transportation that we defined above by replacing the flow vector t^μ by the generic vector ζ^μ ,

$$Q_\zeta = \int_{\Sigma_t} d^d y \sqrt{q} \mathcal{H}_\mu \zeta^\mu + \oint_{\Sigma_t} d^{d-1} \theta \sqrt{\gamma} \mathcal{Q}_\mu \zeta^\mu. \quad (4.5)$$

It is obvious that $Q_t = H$. This charge Q_ζ depends on the fields and their canonically conjugated momenta. An interesting solution is for $\mathcal{H}_\mu = 0$, where the only contribution is coming from the boundary in which case Q_ζ is called surface charge and it is obtained, on-shell,

$$Q_\zeta = \oint_{\Sigma_t} d^{d-1} \theta \sqrt{\gamma} \mathcal{Q}_\mu \zeta^\mu \quad (4.6)$$

The purpose of this analysis is to make sure that we have a well defined variational principle of the Hamiltonian (4.4). More precisely, the \mathcal{Q}_μ is defined in such a way that

its variation exactly cancels the unwanted surface terms from the variation of the bulk constraints \mathcal{H}_μ . In order to identify what comprises this Q_μ , however, we cannot blindly assume that $\delta g_{ab} = 0$. We wish to be less restrictive and roughly allow for $\delta g_{ab} \sim h_{ab}$, which is dictated by the boundary conditions (4.1). This means that the surface terms in the Hamiltonian will not cancel all the surface terms coming from the variation of the bulk constraints \mathcal{H}_μ with respect to the metric. Thus, we need to vary the bulk piece and keep all surface terms that emerge. The variation of the bulk piece of (4.5) will be computed in the following section. However, the bulk variation will give the constraints and a boundary term due to the use of partial integration. Then one can define the surface term Q_μ through its variation, which must precisely cancel the boundary term coming from the variation of the bulk term. We will use the same technique in the relatively simpler Chern-Simons formulation in the following sections.

In addition, the deformation ζ^μ whose surface charge vanishes is called a trivial deformation and acts trivially at ‘infinity’. An asymptotic symmetry is defined to be a non-trivial deformation that respects the boundary conditions. The asymptotic symmetries form an algebra, which can be found in the bibliography as ‘asymptotic symmetry group’. If we denote the asymptotic symmetry group of a space \mathcal{M} as $G_{AS}(\mathcal{M})$, then

$$G_{AS}(\mathcal{M}) := \{\zeta^\mu | \mathcal{L}_\zeta \text{ respects the b.c. and } Q_\zeta \neq 0\} \quad (4.7)$$

which clearly depends on the asymptotic boundary conditions. Notice that the asymptotic symmetry group contains the isometry group as a subgroup.

According to Brown and Henneaux, [51], the surface charges form a projective representation of the asymptotic symmetry algebra $\{\zeta\}$ which allows a central extension. In other words, their Poisson brackets (or Dirac brackets to be more precise) are

$$\{Q_\zeta[g], Q_\eta[g]\} = Q_{[\zeta, \eta]}[g] + C_{\zeta\eta}, \quad (4.8)$$

where $C_{\zeta\eta}$ is the central charge and the surface charge generators are defined such that $Q_\zeta[\bar{g}] = 0$ and $Q_0[g] = 0$ for any ζ^μ and $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$.

We find that the Brown-Henneaux central charges by evaluating (4.8) on the background, i.e. $h_{\mu\nu} = 0$. Thus, using that the variation of the metric is a Lie derivative, $\delta_\eta \bar{g}_{\mu\nu} = \mathcal{L}_\eta \bar{g}_{\mu\nu}$ and the canonical relations $\{Q_\zeta, Q_\eta\} = \delta_\eta Q_\zeta$, $\delta_\eta \zeta = [\zeta, \eta]$, one obtains

$$C_{\zeta\eta} = Q_\zeta[\mathcal{L}_\eta \bar{g}] = -Q_\eta[\mathcal{L}_\zeta \bar{g}] \quad (4.9)$$

The introduction of surface charges has been rather formal so far. In the next section we will move to the most well known example, and we will explicitly calculate an algebra of surface charges.

4.2 Brown and Henneaux

In 1986, Brown and Henneaux wrote one of the most famous papers on three dimensional gravity. [51] The main idea of the paper is to study the asymptotic symmetry group of AdS_3 and to find the surface charge representation explicitly.

We start by considering the global AdS metric (2.4), which is the background on which we define our surface charges and the BTZ metric (2.28). The most general boundary conditions for the BTZ metric are obtained by using all the possibly $SO(2,2)$ transformations and then demand that it have the right asymptotic behaviour

$$[(g_{\mu\nu})_{BTZ} - (g_{\mu\nu})_{AdS}]_{r \rightarrow \infty} = O(1). \quad (4.10)$$

The following boundary conditions are generated

$$g_{tt} = -\frac{r^2}{l^2} + O(1) \quad (4.11)$$

$$g_{tr} = O(1/r^3) \quad (4.12)$$

$$g_{t\phi} = O(1) \quad (4.13)$$

$$g_{rr} = \frac{l^2}{r^2} + O(1/r^4) \quad (4.14)$$

$$g_{t\phi} = O(1/r^3) \quad (4.15)$$

$$g_{\phi\phi} = r^2 + O(1) \quad (4.16)$$

and can be written in the compact form

$$(h_{\mu\nu}) = \begin{pmatrix} h_{tt} & h_{t\phi} & h_{tr} \\ h_{t\phi} & h_{\phi\phi} & h_{\phi r} \\ h_{tr} & h_{\phi r} & h_{rr} \end{pmatrix} = \begin{pmatrix} O(1) & O(1) & O(1/r^3) \\ O(1) & O(1) & O(1/r^3) \\ O(1/r^3) & O(1/r^3) & O(1/r^4) \end{pmatrix}. \quad (4.17)$$

Geometries are said to be (locally) asymptotically AdS_3 when they respect these boundary conditions. It is more convenient to go back to the normal/tangent basis, when doing actual calculations in the Hamiltonian formalism. The bulk term of (4.5) for pure gravity is

$$\mathcal{H}_\mu \zeta^\mu = \hat{\mathcal{H}} \hat{\zeta} + \hat{\mathcal{H}}_a \hat{\zeta}^a \quad (4.18)$$

where we have written the deformation on the normal/tangent basis as well, $\zeta^\mu = \hat{\zeta} n^\mu + \hat{\zeta}^a p_a^\mu$. From the Hamiltonian (4.4), the Hamilton and momentum constraints are obtained

$$\hat{\mathcal{H}} = -\frac{R - 2l^2}{2} + 2(p_{ab} p^{ab} - \frac{1}{d-1} p^2) \quad \text{and} \quad \hat{\mathcal{H}}_a = -2\nabla^b p_{ab}. \quad (4.19)$$

The variation of the bulk piece of the surface charge can be straightforwardly computed and is given by

$$\int_{\Sigma_t} d^d y \sqrt{g} \left\{ (\dots)^{ab} \delta q_{ab} + (\dots)^{ab} \delta p_{ab} \right\} - \oint_{\partial \Sigma_t} d\sigma_c \left\{ \frac{1}{2} G^{abcd} (\hat{\zeta} \nabla_d \delta q_{ab} - \nabla_d \hat{\zeta} \delta q_{ab}) + \left(2 \hat{\zeta}^a p^{bc} \hat{\zeta}^c p^{ab} \right) \delta q_{ab} + 2 \hat{\zeta}_a \delta p^{ac} \right\},$$

where the surface element is $d\sigma_a = d^{d-1} \theta \sqrt{\gamma} r_a$ and we introduce $G_{abcd} \equiv q_{c(a} q_{b)d} - q_{ab} q_{cd}$. The surface charge density \mathcal{Q}_μ is then defined in such a way that the variation of the surface piece in (4.4) precisely cancels the surface term that emerges from varying the bulk piece, so that

$$\delta \mathcal{Q}_\zeta = - \oint_{\partial \Sigma_t} d\sigma_c \left\{ \frac{1}{2} G^{abcd} (\hat{\zeta} \nabla_b \delta q_{cd} - \nabla_b \hat{\zeta} \delta q_{cd}) + \left(2 \hat{\zeta}^b p^{ac} \hat{\zeta}^a p^{bc} \right) \delta q_{bc} + 2 \hat{\zeta}_b \delta p^{ab} \right\}. \quad (4.20)$$

One then can use the above boundary conditions and find [49]

$$\mathcal{Q}_\zeta = \oint_{\partial \Sigma_t} d\sigma_a \left\{ \frac{1}{2} \bar{G}^{abcd} (\bar{\nabla}_b h_{cd} - h_{cd} \bar{\nabla}_b) \hat{\zeta} + 2 \hat{\zeta}_b \delta p^{ab} \right\} + O(h^2) \quad (4.21)$$

for any $h_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ that respects (4.17). The barred quantities depend on the background metric $\bar{g}_{\mu\nu}$. The momentum p^{ab} is the canonical conjugated of the induced metric $q_{ab} = p_a^\mu p_b^\nu g_{\mu\nu}$.

Now, solving the Killing equations $\mathcal{L}_\zeta g_{\mu\nu} = \zeta^\rho \partial_\rho g_{\mu\nu} + g_{\mu\rho} \partial_\nu \zeta^\rho + g_{\rho\nu} \partial_\mu \zeta^\rho = 0$ asymptotically gives the general form of the asymptotic symmetries $\zeta = \zeta^\mu \partial_\mu$

$$\zeta^t = l(T + \bar{T}) + \frac{l^2}{2r^2} (\partial^2 T + \bar{\partial}^2 \bar{T}) + O(r^{-4}) \quad (4.22)$$

$$\zeta^\phi = (T - \bar{T}) + \frac{l^2}{2r^2} (\partial^2 T - \bar{\partial}^2 \bar{T}) + O(r^{-4}) \quad (4.23)$$

$$\zeta^r = -r(\partial T + \bar{\partial} \bar{T}) + O(r^{-1}), \quad (4.24)$$

where T, \bar{T} are generic functions and everything is expressed in light-cone coordinates. The asymptotic Killing vector $\zeta^\mu \partial_\mu$ can be decomposed into a T -dependent term and a \bar{T} -dependent one, i.e.

$$\zeta = \lambda[T] + \bar{\lambda}[\bar{T}] + O(1/r), \quad (4.25)$$

where we introduced the following notions

$$\lambda[T] = \left(2T + \frac{l^2}{r^2} \partial^2 T \right) \partial - \partial T r \partial_r \quad (4.26)$$

$$\bar{\lambda}[\bar{T}] = \left(2\bar{T} + \frac{l^2}{r^2} \bar{\partial}^2 \bar{T} \right) \bar{\partial} - \bar{\partial} \bar{T} r \bar{\partial}_r \quad (4.27)$$

It is convenient to write out the deformations on a Fourier basis. We define, for all $n \in \mathbb{Z}$,

$$\lambda \equiv \sum_n \lambda_n e^{in(t/l+\phi)}, \quad \bar{\lambda} \equiv \sum_n \bar{\lambda}_n e^{in(t/l-\phi)}, \quad (4.28)$$

which must span the asymptotic symmetry algebra. These modes λ_n and $\bar{\lambda}_n$ obey the conformal algebra

$$[\lambda_m, \lambda_n] = i(m-n)\lambda_{m+n}, \quad [\bar{\lambda}_m, \bar{\lambda}_n] = i(m-n)\bar{\lambda}_{m+n}, \quad [\lambda_m, \bar{\lambda}_n] = 0 \quad (4.29)$$

The algebra of global symmetries of AdS_3 consists of two copies of the Mbius algebra $sl(2, \mathbb{R}) = \lambda_{-1}, \lambda_0, \lambda_1$. Thus, the asymptotic symmetry algebra that corresponds to the boundary conditions (4.17) is an infinite-dimensional extension of the isometry algebra. Let us denote the surface charge that generates $\lambda_n(\bar{\lambda}_n)$ at infinity by $L_n(\bar{L}_n)$, i.e.

$$L_n \equiv Q_{\lambda_n}, \quad \bar{L}_n \equiv Q_{\bar{\lambda}_n}. \quad (4.30)$$

From [52], we know that the algebra of surface charges is isomorphic to the algebra of asymptotic deformations up to a central extension, only the central charge remains to be computed via (4.9). We find that

$$C_{\lambda_m \lambda_n} = \frac{l}{8G} m(m^2 - 1) \delta_{m+n,0} \quad (4.31)$$

$$C_{\bar{\lambda}_m \bar{\lambda}_n} = \frac{l}{8G} m(m^2 - 1) \delta_{m+n,0} \quad (4.32)$$

$$C_{\lambda_m \bar{\lambda}_n} = 0, \quad (4.33)$$

which means that we now end up with the full Virasoro algebra

$$\{L_m, L_n\} = (m-n)L_{m+n} + \frac{l}{8G} m(m^2 - 1) \delta_{m+n,0} \quad (4.34)$$

$$\{\bar{L}_m, \bar{L}_n\} = (m-n)\bar{L}_{m+n} + \frac{l}{8G} m(m^2 - 1) \delta_{m+n,0} \quad (4.35)$$

$$\{L_m, \bar{L}_n\} = 0, \quad (4.36)$$

where the central charge is

$$c = \frac{3l}{2G}. \quad (4.37)$$

Finally, it is important to point out that a widely used convention for the zero modes of the asymptotic charges is shifted by $c/24$, i.e. we redefine $L_0 \equiv L_0^{B-H} - \frac{c}{24}$. Specifically this means that $L_0 = -\frac{c}{24}$ for global AdS space.

4.3 Proper and Improper transformations

It has been mentioned above that in (2+1)-dimensional anti-de Sitter gravity does not contain any propagating degrees of freedom. However, for spacetimes with boundaries or asymptotic regions, this picture becomes a bit more complex. For the action principle to hold one must typically introduce boundary conditions on the field and add boundary terms to the action, as we calculated in Section 4.1. These generally break the gauge and diffeomorphism symmetries of the theory. Furthermore, configurations that are gauge equivalent in the absence of a boundary may not be connected by transformations that behave properly at the boundary. Moreover, the remaining transformations at the boundary are properly viewed as symmetries not gauge invariances, [53], [54]. Thus, most of the degrees of freedom can be viewed as excitations that would naively be considered 'pure gauge', but they become physical at the conformal boundary.

As a first step in obtaining these degrees of freedom in Chern-Simon formulation, one must understand the distinction between gauge invariances and symmetries on manifolds with timelike boundaries. The difference between 'proper' and 'improper' transformations was first studied by Benguria et al. [54], and it was made more explicit in [53]. However the analysis in Abelian and non-Abelian Chern-Simons action was carried out later on, [55], which is the approach that we will follow.

Let us consider the non-Abelian Chern-Simons action (3.2) on a manifold with the topology $\mathbb{R} \times \Sigma$, where Σ is a two manifold with boundary $\partial\Sigma$. The canonical action is

$$I_{SC} = \frac{k}{4\pi} \int dt \int_{\Sigma} d^2x \epsilon^{ij} \text{Tr} (\dot{A}_i A_j + A_t F_{ij}), \quad (4.38)$$

where $i, j = 1, 2$ the coordinate basis in Σ . The general definition for the connection (3.3) is $A = A_{\mu}^a T_a dx^{\mu}$, where T_a are the generators that correspond to the isometries of the manifold. It is well known that the phase space of the action is described by the equal time Poisson brackets

$$\{A_i^a(x), A_j^b(x')\} = \frac{2\pi}{k} \epsilon_{ij} g^{ab} \delta^2(x - x'), \quad \epsilon_{12} = -\epsilon_{21} = 1 \quad (4.39)$$

where $g^{ab} = \text{Tr}(T^a T^b)$ is the Cartan-Killing metric on the gauge group.

It is apparent from the canonical form of the action that A_t is a Lagrange multiplier. The corresponding first class constraints

$$G_a^{(0)} = \frac{k}{2\pi} g_{ab} \epsilon^{ij} F_{ij}^b \quad (4.40)$$

generate gauge transformations with generators

$$G^{(0)}[\eta] = \int_{\Sigma} d^2x \eta^a G_a^{(0)}. \quad (4.41)$$

Using the Poisson brackets for the gauge fields (4.39), we obtain

$$\left\{ G^{(0)}[\eta], A_k^a \right\} = \partial_k \eta^a + f_{bc}^a A_k^b \eta^c = D_k \eta^a = \delta_\eta A_k^a, \quad (4.42)$$

where D_k is the gauge-covariant derivative that is defined using the structure constants f_{bc}^a of the gauge group. Now we are ready to calculate the Poisson brackets of the generators

$$\left\{ G^{(0)}[\eta], G^{(0)}[\xi] \right\} = G^{(0)}[\zeta], \quad \zeta^c = f_{ab}^c \eta_a \xi_b. \quad (4.43)$$

These generators satisfy a Poisson algebra isomorphic to the gauge algebra. However, because these generators are the analogue of the central charges in the dual gauge theory, we can suspect from (4.8) that these generators are not differentiable. Indeed the functional derivative of the generator $G^{(0)}[\eta]$ involves an ill-defined surface term. A simple calculation shows that

$$\delta G^{(0)}[\eta] = \frac{k}{2\pi} \int_\Sigma d^2 x \epsilon^{ij} \eta_a D_i \delta A_j^a = -\frac{k}{2\pi} \int_\Sigma d^2 x \epsilon^{ij} D_i \eta_a \delta A_j^a + \frac{k}{2\pi} \int_{\partial\Sigma} \eta_a \delta A_k^a dx^k \quad (4.44)$$

Thus, if $\eta \neq 0$ on the boundary, an additional term has to be added to the generators in order to make the Poisson algebra well defined again. Inspired by the previous section, one adds a boundary term $Q[\eta]$ to the generator, with a variation

$$\delta Q[\eta] = -\frac{k}{2\pi} \int_{\partial\Sigma} \eta_a \delta A_k^a dx^k, \quad (4.45)$$

in order to cancel out the last term of (4.44). As a result, for the full propagator $G[\eta] = G^{(0)}[\eta] + Q[\eta]$ it is straight forward to find

$$\{G[\eta], G[\xi]\} = G[[\eta, \xi]] + K[\eta, \xi], \quad (4.46)$$

with $K[\eta, \xi] = \frac{k}{2\pi} \int_{\partial\Sigma} \eta_a \delta \xi^a$ being the central charge,[56].

This Poisson algebra can be recognized as a central extension of the original algebra of gauge transformations, the same way as (4.8) in the previous section. A useful and insightful calculation is to make the connection between the central charges in the previous section with $Q[\eta]$, [28], using the relation between diffeomorphisms in metric-like formulation and gauge transformations $\eta^a = \xi^\mu A_\mu^a$. However that lies outside of the scope of this project.

Moreover, let us consider the implications for the symmetries of our Chern-Simons theory. The quantity $G^{(0)}[\eta]$ vanishes by virtue of the field equations, and its Poisson bracket with any physical observable \mathcal{O} must therefore also vanish, $\{G^{(0)}[\eta], \mathcal{O}\} = 0$. In the quantum theory, the Poisson brackets become commutators, and the corresponding

statement is that matrix elements of $[G^{(0)}[\eta], \mathcal{O}] = 0$ between physical states, in order for \mathcal{O} to be an operator that describes an observable. This translates to the statement that physical observables must be gauge-invariant. Moreover, the ‘pure gauge’ can not have any physical meaning because it correspond to a non invertible metric. This implies that

$$G^{(0)}[\eta] |phys\rangle = 0 \quad (4.47)$$

If Σ has a boundary, the generator of gauge transformations is not $G^{(0)}[\eta]$ but $G[\eta]$ and the boundary contribution $Q[\eta]$ does not need to vanish. Consequently, in $\eta \neq 0$ at $\partial\Sigma$, it is not consistent to set $G[\eta]$ to zero due to (4.46). Hence physical observables do not need to be invariant under gauge transformations at the boundary, but it is enough that they transform under some representation of the algebra (4.46). Gauge transformations are thus very different in the bulk and at a boundary: in the bulk they are true invariances, but at a boundary they are only symmetries, as we pointed out in the beginning of the section. The transformations that leave also the boundary invariant can be found in the bibliography as ‘proper’ transformations and the ones that do not leave it invariant, so they change the physical state, ‘improper’.

In addition, it is useful to see how the gauge transformations act on the action (3.2) in order to derive an action that also describes the dynamics of the boundary degrees of freedom. In fact, Witten was the first to suggest [57] that this dynamics can be described by a Wess-Zumino-Witten (WZW) model, [58].

In order to understand this relationship we consider the manifold $M = \mathbb{R} \times D^2$, which corresponds to the topology of the AdS_3 space. This implies that the potential A is ‘pure gauge’

$$\bar{A} = g^{-1}dg, \quad (4.48)$$

as it can also be understood from (3.6). However, it is expected that the gauge parameter g has non trivial dynamic on the boundary. One way to see this is by noticing that the action is not invariant under a gauge transformation of the form

$$\bar{A} = g^{-1}dg + g^{-1}Ag. \quad (4.49)$$

Then the action (3.2) transforms as

$$I_{CS}[\bar{A}] = I_{CS}[A] - \frac{k}{4\pi} \int_{\partial M} \text{Tr} \left((dgg^{-1} \wedge A) \right) - \frac{k}{12\pi} \int_M \text{Tr}(g^{-1}dg)^3. \quad (4.50)$$

This result is obtained as follows. The first term of (3.2) transforms as

$$\text{Tr}[\bar{A}d\bar{A}] = \text{Tr}[-dgg^{-1}dgg^{-1}dgg^{-1} - 2dgg^{-1}dgg^{-1}A - 2dgg^{-1}AA + AdA - d(dgg^{-1}A)] \quad (4.51)$$

where the cyclic property of the trace, partial integration and $d^2 = 2$, $dg^{-1} = -g^{-1}dg g^{-1}$ have been used. Moreover, the gauge transformation of the last term of (3.2) gives

$$\text{Tr}[\bar{A}\bar{A}\bar{A}] = \text{Tr}[dgg^{-1}dgg^{-1}dgg^{-1} + 3dgg^{-1}dgg^{-1}A + 3dgg^{-1}AA + AAA]. \quad (4.52)$$

When we add them up and integrate, most of terms cancel out and the last term of (4.52) gives the boundary term.

For a closed manifold, the last term in (4.50) is proportional to a winding number, for k an integer, I_{CS} shifts by $2\pi kn$, so the $e^{iI_{CS}}$ of the partition function is invariant. In general, however, this term cannot be discarded for a manifold with boundary.

Moreover, as it is already mentioned, a surface term has to be added to the action when M is not compact in order to make the variation principle well defined

$$\delta I_{CS}[A] = \frac{k}{2\pi} \int_M \text{Tr}[\delta(dA + A \wedge A)] - \frac{k}{4\pi} \int_{\partial M} \text{Tr}[A \wedge \delta A] \quad (4.53)$$

For a manifold with boundary the last term does not vanish, thus one has to add a boundary contribution to the action to cancel it out. This term depends on the boundary conditions. Thus, in order to define the boundary conditions, one needs to find the canonical conjugates of the potential A and fix half of the total information. However, as it was shown above, (4.39), A is self-conjugate.

Typically, in order to make sure that the boundary conditions are not over determined, we can resort to choose a complex structure on ∂M . If A_z is the fixed boundary value, the appropriate boundary term, that needs to be added to the action, is given by

$$I_{bdry}[A] = \frac{k}{4\pi} \int_{\partial M} \text{Tr}[A_z A_{\bar{z}}], \quad (4.54)$$

which transforms under (4.49) as

$$I_{bdry}[\bar{A}] = I_{bdry}[A] + \frac{k}{4\pi} \int_{\partial M} \text{Tr}(\partial_z g g^{-1} \partial_{\bar{z}} g g^{-1} + \partial_z g g^{-1} A_z + \partial_z g g^{-1} A_z). \quad (4.55)$$

Combining (4.50) and (4.55), we realize that the total action is not invariant under gauge transformations,

$$(I_{CS} + I_{bdry})[\bar{A}] = (I_{CS} + I_{bdry})[A] + k I_{WZW}^+[g^{-1}, A] \quad (4.56)$$

where [59]

$$I_{WZW}^+[g^{-1}, A_z] = \frac{1}{4\pi} \int_{\partial M} \text{Tr}(g^{-1} \partial_z g g^{-1} \partial_{\bar{z}} g - 2g^{-1} \partial_z g A_z) + \frac{1}{12\pi} \int_M \text{Tr}(g^{-1} dg)^3 \quad (4.57)$$

is the chiral WZW action for g couples to a background field A_z . The same procedure is followed also for the other potential in (3.4). Then, for specific boundary conditions, we

obtain a sum of two chiral WZW models with opposite chiralities which, according to the Polyakov-Wiegman formula, combine naturally to a single nonchiral WZW action [45].

As a result, g becomes dynamical at the boundary. One can understand that by looking at the partition function, for example. In this case, one usually can split the integral into one over A and one over g and integrate out the second one. That would happen if the additional term was only the last term of I_{WZW}^+ , which is just the number of times the gauge parameter is wrapped around a sphere -for specific values of k , as it was pointed out earlier. However, in the case that is considered here, there is a coupling term between the gauge parameter and the gauge field. Because of this coupling term, the action is no longer gauge invariant for every gauge transformation g , but instead for 'improper' transformations we recover an action that describes the dynamics of the boundary at conformal infinity. This appearance of a conformal field theory at the conformal boundary of an asymptotically anti-de Sitter space is perhaps the simplest example of the famous AdS/CFT correspondence of string theory.

Chapter 5

Higher Spin Gravity in (2+1)-dimensions

The study of field theories for particles of arbitrary spin has a long history as it is realized from the Introduction. However, the last years there has been a particular interest in three dimensional space-time with negative cosmological constant. One of them is that it can be described by a Chern Simons theory by only promoting the $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry group of pure gravity to $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$. In this chapter we will provide the most important characteristics of Vasiliev theory and the advantages in the three dimensional case. In the second section, we will make the connection with the Chern-Simons Formulation. Finally in the last section we will focus on the relatively simple example of $N = 3$, where we will analyse its properties, such as the asymptotic symmetry.

5.1 Introduction to Vasiliev Theory

The free propagator of a bosonic massless spin- s particle in a Minkowski background of arbitrary dimension $d \geq 4$ can be described by a tensor $\phi_{\mu_1 \dots \mu_s}$, which is of rank s , totally symmetric and double traceless. The field equation of this tensor is [3]

$$\mathcal{F}_{\mu_1 \dots \mu_s} \equiv \square \phi_{\mu_1 \dots \mu_s} - \partial_{(\mu_1} \partial^\lambda \phi_{\mu_2 \dots \mu_s) \lambda} + \partial_{(\mu_1} \partial_{\mu_2} \phi_{\mu_3 \dots \mu_s) \lambda}{}^\lambda = 0, \quad (5.1)$$

where the parenthesis denotes a complete symmetrisation of the indices it encloses. The (5.1) is invariant under the gauge transformation

$$\delta \phi_{\mu_1 \dots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \dots \mu_s)} \quad (5.2)$$

and because the Fronsdal field is double traceless $\phi_{\mu_1 \dots \mu_{s-4} \lambda \rho}{}^{\lambda \rho} = 0$, the gauge parameter should be single traceless

$$\tilde{\xi}_{\mu_1 \dots \mu_{s-3} \lambda}{}^\lambda = 0. \quad (5.3)$$

Imposing the double-trace constraint one can build a second order Lagrangian that is invariant under (5.2), up to total derivatives. The resulting action, first identified by Fronsdal [3], is

$$S = \frac{1}{2} \int d^d x \phi^{\mu_1 \dots \mu_s} \left(\mathcal{F}_{\mu_1 \dots \mu_s} - \frac{1}{2} \eta_{(\mu_1 \mu_2} \mathcal{F}_{\mu_3 \dots \mu_s) \lambda}{}^\lambda \right) \quad (5.4)$$

and it leads to the field equations (5.1). We need to keep in mind that, especially in AdS_3 , the total derivatives mentioned above are important symmetries as in the pure gravity case.

In the AdS background one can follow the same logic by substituting the partial derivatives with covariant derivatives. Then the gauge transformation will be

$$\delta \phi_{\mu_1 \dots \mu_s} = \nabla_{(\mu_1} \tilde{\xi}_{\mu_2 \dots \mu_s)}, \quad (5.5)$$

where ∇_μ is the AdS covariant derivative. However, covariant derivatives do not commute as the ordinary ones, but the commutator acting on a vector field results in

$$[\nabla_\mu \nabla_\nu] V_\rho = \frac{1}{l^2} (g_{\nu\rho} V_\mu - g_{\mu\rho} V_\nu) \quad (5.6)$$

Therefore, one has to take into account additional terms to the field equation in order to keep the gauge invariance. As a result, the Fronsdal equation for AdS space becomes

$$\hat{\mathcal{F}} \equiv \mathcal{F}_{\mu_1 \dots \mu_s} - \frac{1}{l^2} \left\{ [s^2 + (d-6)s - 2(d-3)] \phi_{\mu_1 \dots \mu_s} + 2g_{(\mu_1 \mu_2} \phi_{\mu_3 \dots \mu_s) \lambda}{}^\lambda \right\} = 0. \quad (5.7)$$

The \mathcal{F} denotes (5.1), where we promoted the ordinary partial derivatives to covariant ones, [8]. Thus, only by imposing gauge invariance, we obtain the Fronsdal action for curved spacetimes,

$$S = \frac{1}{2} \int d^d x \sqrt{-g} \phi^{\mu_1 \dots \mu_s} \left(\hat{\mathcal{F}}_{\mu_1 \dots \mu_s} - \frac{1}{2} g_{(\mu_1 \mu_2} \hat{\mathcal{F}}_{\mu_3 \dots \mu_s) \lambda}{}^\lambda \right). \quad (5.8)$$

In $d = 3$ the little group of massless particles is a direct product of the multiplicative group $\{1, -1\}$ with \mathbb{R} [60]. Therefore, excluding representations with continuous spin, one is left only with the two inequivalent representations of $\{1, -1\}$, bosons and fermions [60]. Nevertheless, one can still consider the above field equations for tensor of arbitrary rank. In general, the field equations force, on-shell, the number of local propagating degrees of freedom to be equal to the number of components of a traceless tensor of the same rank in $d - 2$ dimensions [61]. Therefore, in the three dimensional case, this leads to the

same result as the pure three dimensional gravity: no local degrees of freedom exist for the higher spin fields. However, even if the bulk dynamics is trivial, fields with different rank lead to different boundary dynamics in the presence of a cosmological constant.

A relatively simpler description of the free dynamics, that was first used by Vasiliev [62], is to generalize the frame formulation of gravity to a 1-form field $e_\mu^{a_1 \dots a_{s-1}}$. In this standard approach this vielbein-like field is traceless and fully symmetric in its fiber indices. The gauge transformation of this new field is local Lorentz-like,

$$\delta e_\mu^{a_1 \dots a_{s-1}} = D_\mu \zeta^{a_1 \dots a_{s-1}} + \bar{e}_{\mu,b} \Lambda^{b, a_1 \dots a_{s-1}}, \quad (5.9)$$

where D_μ is the Lorentz-covariant derivative while \bar{e}_μ^a is the background vielbein. This leads to the introduction of a gauge connection $\omega_\mu^{b_1 \dots b_t, a_1 \dots a_{s-1}}$ for the new gauge transformation parameter. It is the higher-spin analogue of the spin connection of gravity and it is traceless. As in the gravity case, it must be an auxiliary field and it will be expressed in terms of $e_\mu^{a_1 \dots a_{s-1}}$ and its first derivative.

As a result, a spin- s field which is freely propagating in a constant curvature background of arbitrary dimension, can be described by the pair of one-forms

$$e_\mu^{a_1 \dots a_{s-1}}, \omega_\mu^{b, a_1 \dots a_{s-1}},$$

which are irreducible Lorentz tensors in the flat indices.

One can recover the Fronsdal formulation by considering the Lorentz-like invariant combination

$$\phi_{\mu_1 \dots \mu_s} \equiv \frac{1}{s} \bar{e}_{(\mu_1}^{a_1} \dots \bar{e}_{\mu_{s-1}}^{a_{s-1}} e_{\mu_s) a_1 \dots a_{s-1}}. \quad (5.10)$$

The original theory was obtained in the four dimensional model, which was shortly extended in spaces with constant curvature and arbitrary space-time dimensions [63]. An important observation is that in general the resulting action is invariant under an enlarged set of gauge transformations. For example, in a Minkowski background the free action is invariant under

$$\delta e_\mu^{a_1 \dots a_{s-1}} = \partial_\mu \zeta^{a_1 \dots a_{s-1}} + \bar{e}_{\mu,b} \Lambda^{b, a_1 \dots a_{s-1}} \quad (5.11)$$

$$\delta \omega_\mu^{b, a_1 \dots a_{s-1}} = \partial_\mu \Lambda^{b, a_1 \dots a_{s-1}} + \bar{e}_{\mu,c} \Theta^{bc, a_1 \dots a_{s-1}}, \quad (5.12)$$

where $\Theta^{bc, a_1 \dots a_{s-1}}$ is an additional traceless gauge parameter. The same holds in the case of *AdS* backgrounds [63]. The existence of a new gauge parameter leads us to the introduction of an extra gauge connection with the same properties. Consequently one needs to introduce a tower of extra gauge connections

$$\omega_\mu^{b_1 \dots b_t, a_1 \dots a_{s-1}}, \quad 2 \leq t \leq s-1. \quad (5.13)$$

They can be represented by Young diagrams with two rows, the first has $(s - 1)$ boxed and the second one t boxes [64]. They are called extra fields and they are necessary in order to rewrite the field equations in terms of relations between gauge invariant objects. Even if the extra fields do not enter the free action they are important in the interacting theory. Nevertheless, in the three dimensional case, the gauge parameter $\Theta^{bc,a_1\dots a_{s-1}}$ vanishes and as a result all the extra fields as well. This is the second important fact about the tree dimensional toy model and we will revise it when the CS- formulations are introduced. Therefore, using that in the three-dimensional context the spin connection can be rewritten, $\omega_\mu^a = \frac{1}{2}\epsilon^{abc}\omega_{\mu,b,c}$, and the higher spin fields are described by the pair of gauge potentials $e_\mu^{a_1\dots a_{s-1}}, \omega_\mu^{a_1\dots a_{s-1}}$, which have the same index structure. This will allow us to reformulate the theory using CS- gauge fields. However, for a detailed review of higher spin gauge fields, we refer the reader to [64].

5.2 Chern-Simons formulation

In this section we will discuss the description of gravitational theory with negative cosmological constant coupled to the (bosonic) higher spin fields that were described above. Nevertheless, the first connection between CS action and higher spin theories was made by Blencowe, [65], when he proposed an interacting theory for higher spin fields in $d = 3$ based on a CS action. In particular he considered a gauge group which is the product of two copies of an infinite-dimensional extension of $SL(2, \mathbb{R})$, thus mimicking the Fradkin-Vasiliev higher spin algebra in a four-dimensional AdS background [10].

In order to reformulate Einstein gravity in $d = 3$ as a CS theory, one defines the linear combinations of vielbein and spin connection as it was mentioned in (3.3)

$$j_\mu^a = \omega_\mu^a + \frac{1}{l}e_\mu^a, \quad \tilde{j}_\mu^a = \omega_\mu^a - \frac{1}{l}e_\mu^a, \quad (5.14)$$

where we interpret j and \tilde{j} , as the $SL(2, \mathbb{R})$ from (3.3). In similar fashion one defines the linear combinations

$$t_\mu^{a_1\dots a_{s-1}} = (\omega + \frac{1}{l}e)_\mu^{a_1\dots a_{s-1}}, \quad \tilde{t}_\mu^{a_1\dots a_{s-1}} = (\omega - \frac{1}{l}e)_\mu^{a_1\dots a_{s-1}} \quad (5.15)$$

of the fields $e_\mu^{a_1\dots a_{s-1}}, \omega_\mu^{a_1\dots a_{s-1}}$, mentioned in the previous section. As in the pure gravity case, one can contract them with some generators $T_{a_1\dots a_{s-1}}$ of the higher spin algebra, and add them to the $sl(2, \mathbb{R})$ ones. Consequently, we obtain the one-forms

$$A = (j_\mu^a J_a + t_\mu^{a_1\dots a_{s-1}} T_{a_1\dots a_{s-1}}) dx^\mu \quad (5.16)$$

$$\tilde{A} = (\tilde{j}_\mu^a J_a + \tilde{t}_\mu^{a_1\dots a_{s-1}} T_{a_1\dots a_{s-1}}) dx^\mu. \quad (5.17)$$

Since no local degrees of freedom should be involved, it is natural to identify the equations of motion for a spin- s gauge field coupled to gravity with the flatness condition for A and \tilde{A} , (3.6). This leads, at the level of the action, to the CS theory. One can check that the resulting field equations reduce to the Fronsdal one from the previous section. This, however, lies outside of the scope of this thesis. We refer the reader to [20] for the details of the proof.

Thus, if the J_a and $T_{a_1 \dots a_{s-1}}$ generate a Lie algebra \mathfrak{g} admitting a non-degenerate bilinear form denoted by tr one can then consider the CS action (3.2), with all the same characteristics as the pure gravity one. In [20], it is also proven that \mathfrak{g} is $SL(N, \mathbb{R})$. Specifically this implies that a $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$ CS theory can be interpreted as describing the coupling of a tower of fields with increasing spin $2, 3, \dots, n$, where each value of the spin appears only once. In the limit $n \rightarrow \infty$ the present construction leads to the higher spin theory based on the algebra of area-preserving diffeomorphisms on a two-dimensional hyperboloid, $Sdiff^{1,1}$, [66]. Nevertheless, since including the total tower of higher spin fields is quite complicated, the main subject of this thesis is the spin-3 example, which is the simplest of these theories.

5.2.1 $SL(3, \mathbb{R})$

The spin-3 case is given by promoting $SL(2, \mathbb{R})$ to $SL(3, \mathbb{R})$, whose explicit algebra can be found in the Appendix A.2. The properties of the additional fields are actually determined by how $sl(2, \mathbb{R})$, associated to pure gravity, is embedded into $sl(3, \mathbb{R})$. In $sl(3, \mathbb{R})$, there are two different possible embeddings, which cannot be related to each other by conjugation and represent two inequivalent extensions of pure gravity, but only one of them describes a spin-3 field coupled to gravity.

Principle embedding

To construct an embedding we simply have to pick three generators from the set of $sl(3, \mathbb{R})$ generators that respect $sl(2, \mathbb{R})$ algebra, which in the case of the principal embedding will be L_1, L_0, L_{-1} . The decomposition of the other part of $sl(3, \mathbb{R})$ can be readily understood by considering the adjoint representation [67]:

$$\text{adj}_3 \cong \mathfrak{3}_2 \oplus \mathfrak{5}_2 \tag{5.18}$$

which means that there are three generators in the adjoint representation that can be decomposed in a 3×3 and a 5×5 block structure, both satisfying the $sl(2, \mathbb{R})$ algebra.

From the perspective of the bulk, this corresponds to the metric field and a spin-3 field:

$$g_{\mu\nu} = \frac{1}{2} \text{tr}_f(e_\mu e_\nu), \quad \phi_{\mu\nu\rho} = \frac{1}{3!} \text{tr}_f(e_\mu e_\nu e_\rho). \quad (5.19)$$

Diagonal embedding

The diagonal embedding is found by choosing W_2, L_0, W_2 as a sub-algebra. The name of the embedding comes from the fact that the adjoint representation decomposes into a block-diagonal structure:

$$\text{adj}_3 \cong \mathbf{3}_2 \oplus \mathbf{2} \cdot \mathbf{2}_2 \oplus \mathbf{1}_2, \quad (5.20)$$

reflecting a spectrum consisting of the graviton, one spin-1 and two spin-3/2 fields. It turns out that this embedding includes negative norm states and will therefore not be possible to quantize, [47].

5.3 Asymptotic symmetries

In this section, we will impose boundary conditions on the connections and derive the asymptotic symmetries and the surface charges for the spin-3 case.

We have seen in previous chapters that the action, in order to have a well defined variation needs an additional boundary term, (4.54). This vanishes if we impose $A_- = 0$ and $\tilde{A}_+ = 0$ at the boundary. In [20] it was showed that using the gauge freedom, the radial dependence can be fixed and the connection can always be written as

$$A(\phi, t, \rho) = b(\rho)^{-1} a(\phi, t) b(\rho) + b(\rho)^{-1} db(\rho) \quad (5.21)$$

$$\tilde{A}(\phi, t, \rho) = b(\rho) \tilde{a}(\phi, t) b(\rho)^{-1} + b(\rho) db(\rho)^{-1}, \quad (5.22)$$

where $b(\rho) = e^{\rho L_0}$, the same as in the BTZ case, (3.12), and ρ is the radial component of the torus. We will work on the A chiral component, although the same can be done in the other one as well. Due to the boundary conditions, the most general way to write the connection is

$$a(x^+) = \left(\sum_{i=-1}^1 l^i(x^+) L_i + \sum_{j=-2}^2 w^j(x^+) W_j \right) dx^+, \quad (5.23)$$

where l^i, w^j are general functions of the x^+ lightcone coordinate and L_i, W_j the generators of the principle embedding of $SL(3, \mathbb{R})$ algebra, see Appendix A.2.1.

The crucial point in this description is that this theory has to be asymptotically Anti-de Sitter, as we required for the BTZ black hole (4.10),

$$(A - A_{AdS}) \Big|_{\text{boundary}} = \mathcal{O}(1). \quad (5.24)$$

The connection, including the radial components, can be found by noticing, due to the commutation relations, Appendix A.2.1,

$$L_j e^{\rho L_0} = e^{\rho L_0} e^{j\rho} L_j \quad (5.25)$$

$$W_j e^{\rho L_0} = e^{\rho L_0} e^{j\rho} W_j. \quad (5.26)$$

Therefore, the lightlike component of the gauge connection reads

$$A_+ = \left(\sum_{i=-1}^1 e^{i\rho} l^i(x^+) L_i + \sum_{j=-2}^2 e^{j\rho} w^j(x^+) W_j \right) dx^+. \quad (5.27)$$

By imposing the boundary condition (5.24), and the use of (3.21), one can find the following conditions on the components l^i and w^j

$$l^1 = 1, \quad w^1 = w^2 = 0. \quad (5.28)$$

Also one can use the gauge freedom of the theory [20] to set additional parameters to zero

$$l^0 = 0, \quad w^0 = w^{-1} = 0. \quad (5.29)$$

This completely fixes the gauge freedom [68]. The degrees of freedom that remain are the components l^{-1} and w^{-2} , which can be referred to as the higher weight gauge, since L_{-1}, W_{-2} are the higher weight generators. This leaves us with the following gauge connections^{1,2}

$$a(x^+) = \left(L_1 - \frac{2\pi}{k} \mathcal{L}(x^+) L_{-1} - \frac{\pi}{2k} \mathcal{W}(x^+) W_{-2} \right) dx^+ \quad (5.30)$$

$$\tilde{a}(x^-) = - \left(L_{-1} - \frac{2\pi}{k} \tilde{\mathcal{L}}(x^-) L_1 - \frac{\pi}{2k} \tilde{\mathcal{W}}(x^-) W_2 \right) dx^- \quad (5.31)$$

CFT on the boundary

Now let us see if these connections indeed have an extended Virasoro algebra as an asymptotic symmetry group. Let us consider the residual gauge symmetry that remains after gauging out the radial dependence by picking (5.21). The infinitesimal gauge transformations that preserve this gauge choice are of the form

$$\delta A = d\Lambda + [A, \Lambda], \quad \Lambda = b^{-1}(\rho) \lambda(x^+, x^-) b(\rho), \quad (5.32)$$

where $\lambda(x^+, x^-)$ is an arbitrary function. However, these transformations are incompatible with the boundary conditions that we have imposed since the beginning, unless

¹Note that different choices for them distinguish physically inequivalent solutions.

²A similar analysis leads to the anti-holomorphic part.

$\partial_{x^-}\lambda(x^+, x^-) = 0$. Thus the allowed gauge transformations are those whose parameters are of the form $\lambda(x^+)$ and are valued in $SL(3, \mathbb{R})$.

If we expand $\lambda(x^+)$ in the $sl(3, \mathbb{R})$ generator basis, as we did also for the connection, then

$$\lambda(x^+) = \left(\sum_{i=-1}^1 \epsilon^i(x^+)L_i + \sum_{j=-2}^2 \chi^j(x^+)W_j \right). \quad (5.33)$$

We are interested in the transformations that preserve the structure of (5.21). Under an infinitesimal gauge transformation with gauge parameter $\lambda(x^+)$, $a(x^+)$ transforms as

$$\delta a = d\lambda + [a, \lambda]. \quad (5.34)$$

Thus, we have to impose that all terms proportional to $L_1, L_0, W_2, W_1, W_0, W_1, W_2$ vanish. These constraints can be solved to find $\epsilon_0, \epsilon_1, \chi_1, \chi_0, \chi_1, \chi_2$ in terms of ϵ_1, χ_2 and their derivatives, where we suppressed the dependence on x^+ for notational convenience. Writing $\epsilon_1 = \epsilon$, $\chi_2 = \mu$ and denoting derivatives with respect to x^+ as primes for convenience, one finds

$$\epsilon_0 = -\epsilon', \quad (5.35)$$

$$\epsilon_{-1} = -\frac{1}{2}\epsilon'' + \frac{2\pi}{k}\epsilon\mathcal{L} + \frac{4\pi}{k}\mu\mathcal{W}, \quad (5.36)$$

$$\chi_1 = -\mu', \quad (5.37)$$

$$\chi_0 = +\frac{1}{2}\mu'' + \frac{4\pi}{k}\mu\mathcal{L}, \quad (5.38)$$

$$\chi_{-1} = -\frac{1}{6}\mu''' - \frac{10\pi}{3k}\mu'\mathcal{L} - \frac{4\pi}{3k}\mu\mathcal{L}', \quad (5.39)$$

$$\chi_{-2} = \frac{1}{24}\mu'''' + \frac{4\pi}{3k}\mu''\mathcal{L} + \frac{7\pi}{6k}\mu'\mathcal{L}' + \frac{\pi}{3k}\mu\mathcal{L}'' + \frac{4\pi^2}{k^2}\mu\mathcal{L}^2 - \frac{\pi}{2k}\epsilon\mathcal{W}, \quad (5.40)$$

which we will refer to as the 6 auxiliary equations. To understand how \mathcal{L}, \mathcal{W} transform under the gauge transformations, we will use the six auxiliary equations to write $\delta\mathcal{L}, \delta\mathcal{W}$ in terms of $\epsilon, \mu, \mathcal{L}, \mathcal{W}$ and their derivatives. Now the transformations of \mathcal{L} and \mathcal{W} are given by

$$\mathcal{L} \rightarrow \mathcal{L} + \delta_\epsilon\mathcal{L} + \delta_\mu\mathcal{L}, \quad \mathcal{W} \rightarrow \mathcal{W} + \delta_\epsilon\mathcal{W} + \delta_\mu\mathcal{W}, \quad (5.41)$$

where

$$\delta_\epsilon\mathcal{L} = 2\epsilon'\mathcal{L} + \epsilon\mathcal{L}' + \frac{k}{2\pi}\epsilon''', \quad (5.42)$$

$$\delta_\mu\mathcal{L} = 3\mu'\mathcal{W} + 2\mu\mathcal{W}', \quad (5.43)$$

$$\delta_\epsilon\mathcal{W} = \epsilon\mathcal{W}' + 3\epsilon'\mathcal{W}, \quad (5.44)$$

$$\delta_\mu\mathcal{W} = -\frac{1}{3} \left(2\mu\mathcal{L}''' + 9\mu'\mathcal{L}'' + 15\mu''\mathcal{L}' + \frac{k}{4\pi}\mu^{(5)} + \frac{64\pi}{k}(\mu\mathcal{L}\mathcal{L}' + \mu'\mathcal{L}^2) \right). \quad (5.45)$$

Here ϵ is the gauge parameter related to the $SL(2, \mathbb{R})$ subgroup of $SL(3, \mathbb{R})$, which generates conformal transformations.

The charges that generate these transformations, c.f. (4.45) are then given by

$$Q(\lambda) = \int dx^+ (\epsilon \mathcal{L} + \mu \mathcal{W}). \quad (5.46)$$

The Poisson brackets can be written down by using that $\delta a = \{Q(\lambda), a\}$. Also, we can expand \mathcal{L}, \mathcal{W} into Fourier modes,

$$\mathcal{L}(x^+) = -\frac{1}{2\pi} \sum_p \mathcal{L}_p e^{-ipx^+}, \quad \mathcal{W}(x^+) = -\frac{1}{2\pi} \sum_p \mathcal{W}_p e^{-ipx^+}, \quad (5.47)$$

and following the same Bown and Henneaux approach again, [51], we also shift the vacuum energy,

$$\mathcal{L}_p \rightarrow \mathcal{L}_p + \frac{k}{4} \delta_{p,0}. \quad (5.48)$$

A W_3 algebra can then be identified³, [20],

$$i\{\mathcal{L}_p, \mathcal{L}_q\} = (p-q)\mathcal{L}_{p+q} + \frac{c}{12}(p^3-p)\delta_{p+q,0} \quad (5.49)$$

$$i\{\mathcal{W}_p, \mathcal{W}_q\} = (2p-q)\mathcal{W}_{p+q} \quad (5.50)$$

$$i\{\mathcal{W}_p, \mathcal{W}_q\} = \frac{1}{3}[(p-q)(2p^2+2q^2-pq-8)\mathcal{L}_{p+q} \quad (5.51)$$

$$+ \frac{96}{c}(p-q)\Lambda_{p+q} + \frac{c}{12}p(p^2-1)(p^2-4)\delta_{p+q,0}], \quad (5.52)$$

where $\Lambda_{p+q} = \sum_{q \in \mathbb{Z}} \mathcal{L}_{p+q} \mathcal{L}_q$. The central charge is again related to the level of the Chern-Simons theory as

$$c = 6k = \frac{3l}{2G}. \quad (5.53)$$

\mathcal{W} can be identified as a weight (3,0) operator and \mathcal{L} as the stress energy tensor with weight (2,0). Note that (5.49) represents a Virasoro algebra. Hence, after turning off the \mathcal{W} charges, one obtains the Virasoro algebra, as expected.

The similar procedure for the diagonal embedding resulting in a asymptotic symmetry algebra $\mathcal{W}_3^{(2)}$ [69].

³Refer also to the Appendix A.2 for the classical version without central charges

Chapter 6

Higher Spin Black Holes

There were many attempts to define higher spin black holes in the past few years. However, despite the fact that none has yet figured out how to interpret higher spin black holes in Lorentzian signature, there are many proposals about Euclidean higher spin black hole, which are well developed.

Just as the Euclidean BTZ solution can be thought of as a contribution to the partition function (3.35), [70], one can think of the higher spin black hole solutions as a contribution to a generalized partition function which includes chemical potentials conjugate to the higher spin currents. Therefore, we consider the generalized partition function that we examined in (3.42). One can relate the potentials $\alpha, \bar{\alpha}$ with the chemical potentials of the asymptotic charges \mathcal{W}^\pm at the CFT boundary (5.46), [48]. As a result one can generalize the concept of Euclidean BTZ black hole by imposing three conditions.

- The Euclidean geometry is smooth and the higher spin fields are non singular at the horizon.
- In the limit where the the higher spin potential are vanishing the solution goes smoothly over to the BTZ black hole. In particular, if one takes the limit where all the asymptotic charges of the higher spin fields go to zero then the entropy should reduce to the BTZ one.
- The integrability conditions (3.44) should be satisfied .

There are several proposals for higher spin black hole solutions that satisfy the above conditions. The most developed one is for the $SL(3, \mathbb{R})$ gauge group. The purpose of this chapter is to review them.

6.1 Black holes admitting $SL(3, \mathbb{R})$ symmetries

6.1.1 GK Black Hole

This black hole was first constructed by Gutperle and Kraus [70], also see [69], by adding a chemical potential to the connection from a Lagrangian point of view. One can also find it in the literature referred to as a holographic black hole. Note that, by construction, the GK black holes have the same boundary conditions that we considered in Section 5.3, and also \mathcal{L}, \mathcal{W} are constants. The gauge connection in the radial independent gauge then reads

$$a^\pm = \pm \left(L_\pm - \mathcal{L}^\pm L_{\mp 1} - \frac{\mathcal{W}^\pm}{4} W_{\mp 2} \right) dx^\pm \\ \pm \mu^\pm (W_{\pm 2} - 2\mathcal{L}^\pm W_0 + (L^\pm)^2 W_{\mp 2} + 2\mathcal{W}^\pm L_{\mp 1}) dx^\mp, \quad (6.1)$$

where we have substituted $\frac{2\pi}{k}\mathcal{L}^\pm, \frac{2\pi}{k}\mathcal{W}^\pm \rightarrow \mathcal{L}^\pm, \mathcal{W}^\pm$ for simplicity and the generators are in the principle embedding, see Appendix A.2.1. The corresponding connections are

$$A^\pm = \pm \left(e^\rho L_{\pm 1} - \mathcal{L}^\pm e^{-\rho} L_{\mp 1} - \frac{\mathcal{W}^\pm}{4} e^{-2\rho} W_{\mp 2} \right) dx^\pm \\ \pm \mu^\pm (e^{2\rho} W_{\pm 2} - 2\mathcal{L}^\pm W_0 + (L^\pm)^2 e^{-2\rho} W_{\mp 2} + 2\mathcal{W}^\pm e^{-\rho} L_{\mp 1}) dx^\mp. \quad (6.2)$$

6.1.2 Canonical Black Hole

A more recent proposal for a black hole containing spin-2 and spin-3 fields was described by Henneaux et al in [70]. Their approach uses the Hamiltonian formulation. The authors claim that rather than introducing the chemical potential in the x^- -component of the connection, one should keep the ϕ -component fixed and alter the t -component of the connection. Thus, In the Hamiltonian formalism, one works on a constant time slice at some time t and it is more natural to work with the coordinates ϕ, t , and not the light-cone coordinates that are usually used in the Lorentzian formalism. The connection in the Lorentzian continuation [71] reads,

$$a_\phi^\pm = L_{\pm 1} - \mathcal{L}^\pm L_{\mp 1} - \frac{\mathcal{W}^\pm}{4} W_{\mp 2} \quad (6.3)$$

$$a_t^\pm = \pm \left[\xi^\pm \left(L_{\pm 1} - \mathcal{L}^\pm L_{\mp 1} - \frac{\mathcal{W}^\pm}{4} W_{\mp 2} \right) \right. \\ \left. - \eta^\pm (W_{\pm 2} - 2\mathcal{L}^\pm W_0 + (L^\pm)^2 W_{\mp 2} + 2\mathcal{W}^\pm L_{\mp 1}) \right], \quad (6.4)$$

where we used the same conventions as in the GK black hole. Note that both these black holes are valued in the principal embedding of $SL(3, \mathbb{R})$, see discussion in Subsection

5.2.1. Moreover, the full connection if we take into account the radial contribution is obtained

$$A_\phi^\pm = e^\rho L_{\pm 1} - \mathcal{L}^\pm e^{-\rho} L_{\mp 1} - \frac{\mathcal{W}^\pm}{4} e^{-2\rho} W_{\mp 2} \quad (6.5)$$

$$A_t^\pm = \pm \left[\hat{\xi}^\pm \left(e^\rho L_{\pm 1} - \mathcal{L}^\pm e^{-\rho} L_{\mp 1} - \frac{\mathcal{W}^\pm}{4} e^{-2\rho} W_{\mp 2} \right) - \eta^\pm \left(e^{2\rho} W_{\pm 2} - 2\mathcal{L}^\pm W_0 + (L^\pm)^2 e^{-2\rho} W_{\mp 2} + 2\mathcal{W}^\pm e^{-\rho} L_{\mp 1} \right) \right]. \quad (6.6)$$

Note that in this black hole the boundary conditions are $A_-^+ = 0, A_+^- = 0$ of [20]. However one can generalize the boundary conditions of the BTZ black hole in order to allow chemical potentials [71]. The most general boundary conditions for $SL(2, \mathbb{R})$ have to be formulated in the following way

$$A_\phi^\pm(\rho, \phi) \xrightarrow[\rho \Rightarrow \infty]{} L_{\pm 1} - \mathcal{L}^\pm(\rho, \phi) L_{\mp 1} \quad (6.7)$$

$$\mathcal{L}^\pm(\rho, \phi) \xrightarrow[\rho \Rightarrow \infty]{} \mathcal{L}^\pm(\rho) + \mathcal{O}(e^{-\rho}) \quad (6.8)$$

$$A_\rho^\pm \xrightarrow[\rho \Rightarrow \infty]{} \mathcal{O}(e^{-\rho}). \quad (6.9)$$

In an analogous way, when switching on the spin-3 chemical potentials the most general boundary conditions are of the form

$$A_\phi^\pm(\rho, \phi) \xrightarrow[\rho \Rightarrow \infty]{} L_{\pm 1} - \mathcal{L}^\pm(\rho, \phi) L_{\mp 1} - \frac{1}{4} \mathcal{W}^\pm(\rho, \phi) W_{\mp 2} \quad (6.10)$$

$$\mathcal{L}^\pm(\rho, \phi) \xrightarrow[\rho \Rightarrow \infty]{} \mathcal{L}^\pm(\rho) + \mathcal{O}(e^{-\rho}) \quad (6.11)$$

$$\mathcal{W}^\pm(\rho, \phi) \xrightarrow[\rho \Rightarrow \infty]{} \mathcal{W}^\pm(\rho) + \mathcal{O}(e^{-\rho}) \quad (6.12)$$

$$A_\rho^\pm \xrightarrow[\rho \Rightarrow \infty]{} \mathcal{O}(e^{-\rho}). \quad (6.13)$$

6.1.3 Black hole in the diagonal embedding

The first black hole in the diagonal embedding was proposed in [47] and was generalized in [70]. The generators are in another basis, and the explicit matrices are written in the Appendix A.2.2. The connections of the Euclidean black hole are given by

$$A_\phi = \hat{L}_1 - \frac{8\pi}{k} \left[\left(\hat{\mathcal{L}} - \frac{6\pi}{k} \mathcal{U}^2 \right) \hat{L}_{-1} + \frac{3}{2} \mathcal{U} J_0 + \psi_{[a]} G_{-1/2}^{[a]} \right], \quad (6.14)$$

$$A_t = -i \left[\hat{\xi} \left(\hat{L}_1 - \frac{8\pi}{k} \left[\left(\hat{\mathcal{L}} - \frac{6\pi}{k} \mathcal{U}^2 \right) \hat{L}_{-1} + \frac{3}{2} \mathcal{U} J_0 + \psi_{[a]} G_{-1/2}^{[a]} \right] \right) \right. \quad (6.15)$$

$$\left. + \nu J_0 + \theta_{[a]} \left(a G_{1/2}^{[-a]} - \frac{12\pi}{k} \mathcal{U} G_{-1/2}^{[-a]} + \frac{4\pi}{k} \psi_{[a]} \hat{L}_{-1} \right) \right], \quad (6.16)$$

and similarly for the anti-holographic part. As it is mentioned in the Appendix A.2.2 the basis elements, \hat{L}_i , generate the $sl(2, \mathbb{R})$ subalgebra and describe a spin-2 field. Additionally, the elements $G_{\pm 1/2}^{[a]}$ with $a = \pm 1$, describe two independent spinors, and J_0 a scalar field. Therefore, the corresponding black hole is endowed with spin 2 and lower spin charges, namely $U(1)$ and spin $\frac{3}{2}$ charges. We should point out that the corresponding generators in the asymptotic conformal field theory describe only bosons due to the fact that their algebra involves only commutators.

The charges $\mathcal{L}, \mathcal{U}, \psi_{[a]}$ are respectively assigned to the chemical potentials $\hat{\xi}, \nu, \theta_{[a]}$ which could be justified by explicitly writing the thermodynamics of the black hole. Additional constraints are given such that the flatness condition, $F_{t\phi} = 0$, in (3.6) is conserved,

$$a\theta_{[a]}\psi_{[a]} = 0 \quad \text{and} \quad \theta_{[-a]} \left(\frac{24\pi}{k} \mathcal{U}^2 - \hat{\mathcal{L}} \right) + \nu\psi_{[a]} = 0. \quad (6.17)$$

As a final remark, one should mention that GK black hole belongs to the diagonal embedding. It can be related to (6.14) under a proper gauge transformation (3.8) for $G^\pm = e^{\lambda^\pm}$, where λ^\pm map to the specific gauge transformation given by [70]

$$\lambda^\pm = \pm \frac{1}{2} \log(4\mu) \left(L_0 \mp \frac{2\mu + \sqrt{\mu}}{2\mu(1 - 4\mu)} W_{\mp 1} \right). \quad (6.18)$$

6.2 Thermodynamics

6.2.1 Holonomy condition

We will focus on the non rotating case for simplicity, where $\mathcal{L}^- = \mathcal{L}^+, \mathcal{W}^- = -\mathcal{W}^+, \mu^- = -\mu^+, \tau = \bar{\tau} = i\beta$. In that case the connections depend on the three parameters $\mathcal{L}, \mathcal{W}, \mu$ and the inverse temperature β .

Due to the second condition of the definition of a generalized Euclidean black hole, one has to impose the same trivial holonomy condition around the t-cycle as in BTZ black hole, (3.28). The easiest way to solve the holonomy condition is by solving the characteristic polynomial where $\lambda = 0, \pm 2\pi, \lambda^3 - \frac{1}{2}\text{Tr}(\omega^2) - \det(\omega) = 0$, where ω is defined as $\omega = 2\pi\beta a_t = 2\pi(\tau a_z - \bar{\tau} a_{\bar{z}})$, the same way as in the thermodynamics of BTZ black hole in Section 3.3. The characteristic polynomial is solved by

$$\det(\omega) = 0 \quad \text{and} \quad \text{Tr}(\omega^2) + 8\pi^2 = 0. \quad (6.19)$$

Thus the goal of this section is to find the constraints that need to be satisfied in order for black hole solutions not to have conical singularities.

GK Black Hole

The two equations (6.19) can be explicitly written in terms of the four parameters of (6.1),

$$-512\mu^3\mathcal{L}^3 + 288k\mu\mathcal{L}^2 - 432k\mu^2\mathcal{W}\mathcal{L} + 432k\mu^3\mathcal{W}^2 - 27k^2\mathcal{W} \quad (6.20)$$

$$64\mu^2\mathcal{L}^2 + 12k\mathcal{L} - 36k\mu\mathcal{W} = 12k^2\pi^2\frac{1}{\beta^2}. \quad (6.21)$$

The second equations is easily solved

$$\mathcal{W} = -\frac{\pi^2k}{3\mu\beta^2} + \frac{\mathcal{L}}{3\mu} + \frac{16\mu\mathcal{L}^2}{9k}. \quad (6.22)$$

Therefore, by plugging it back to (6.20) one obtains a fourth order polynomial in \mathcal{L}

$$4096\beta^4\mu^6\mathcal{L}^4 - 2304k\beta^4\mu^4\mathcal{L}^3 + 48k^2\beta^2\mu^2(9\beta^2 - 32\pi^2\mu^2)\mathcal{L}^2 - 9k^3\beta^2(3\beta^2 - 16\pi^2\mu^2)\mathcal{L} + 9k^4(3\pi^2\beta^2 + 16\pi^4\mu^2) = 0. \quad (6.23)$$

It is easy to check the integrability condition (3.44) by simply differentiate (6.22) with respect to $\tau = \frac{i\beta}{2\pi}$ and (6.23) with respect to $a = \mu\tau$.

An important fact is that (6.23) is a fourth order polynomial, as a result it admits four solutions [72], in contrast with the pure gravity case where we found one unique solution.

Furthermore, there exists a convenient parametrization for \mathcal{L}, \mathcal{W} introduced by Gutperle and Kraus [48]. The parametrization introduces a dimensionless variable C which is related to the sources and charges as

$$\mathcal{L} = \frac{k\pi^2}{\beta^2} \frac{C(3-2C)^2}{(C-3)^2(4C-3)}, \quad \mathcal{W} = \frac{4(C-1)}{\sqrt{C^3}} \sqrt{\frac{\mathcal{L}^3}{k}}, \quad \frac{\mu}{\beta} = \frac{3}{4\pi} \frac{(C-3)\sqrt{4C-3}}{(2C-3)^2}. \quad (6.24)$$

The dimensionless parameter C runs from 3 to ∞ . However, for a detailed review on the values of C in the different branches of solutions, [72] is quite useful.

Moreover, similar to the discussion of the BTZ black hole, it is possible to find an expression for the entropy of the black hole through the partition function (3.42). Following the derivation in [48], we write the entropy as

$$S = \ln Z - i2\pi\tau\mathcal{L} - i2\pi\alpha\mathcal{W}. \quad (6.25)$$

However, due to the first condition of the black hole requirements, the entropy should have the appropriate BTZ limit. Then, a convenient way to write the entropy is

$$S = 4\pi\sqrt{L}f(x(C)), \quad \text{where} \quad x(C) = \frac{27(C-1)^2}{2C^3} \quad (6.26)$$

It has to be pointed out that due to (6.24) the BTZ limit is obtained when $C \rightarrow \infty$. Then for the BTZ limit $x(C) = 0$, as a result it is required that $f(0) = 1$.

In order to obtain the thermal entropy explicitly, we insert (6.22) to (3.36) to obtain the differential equation

$$36x(2-x)(f'(x))^2 + f^2(x) - 1 = 0, \quad (6.27)$$

where we used (6.24) and $f'(x) = \frac{d}{dx}f(x)$. The solutions of this differential equation reads

$$f(x) = \cos\theta, \quad \text{where} \quad \theta = \frac{1}{6} \arctan\left(\frac{\sqrt{x(2-x)}}{1-x}\right) \quad (6.28)$$

The range of x is $x \in [0, 2]$ thus we choose the branch of the arc-tangent where $\theta \in [0, \frac{\pi}{6}]$. Nevertheless, when we plug in (6.26) in order to write $f(x)$ in C -parametrization, it takes the simple form

$$f(C) = \sqrt{1 - \frac{3}{4C}}. \quad (6.29)$$

Hence, the thermal entropy for the non rotating case reads

$$S = 4\pi\sqrt{L}\sqrt{1 - \frac{3}{4C}}, \quad (6.30)$$

where the BTZ limit $C \rightarrow \infty$ is satisfied.

Finally before we conclude the review of the GK black hole, we want to argue that in the rotating case, the holonomy conditions will give four independent equations. One would obtain two additional equations to (6.20) and (6.22), that would have the same structure but only bared quantities. As it is becoming clear, the same procedure will apply to the bared equations independently, therefore, the entropy will have to terms with identical structure, one for each copy of $SL(3, \mathbb{R})$. As a result, the final thermal entropy or the general rotating case reads

$$S = 2\pi\sqrt{L^+}\sqrt{1 - \frac{3}{4C}} + 2\pi\sqrt{L^-}\sqrt{1 - \frac{3}{4\bar{C}}}, \quad (6.31)$$

where \bar{C} is the dimensional parameter for the parameters of the A^- gauge connection.

Another way to calculate the thermal entropy is reviewed in [73], but it has been proven that the procedures are equivalent. They use the fact that in the semi-classical we can approximate the path indegral by the on-shell action, such that

$$Z = e^{-S_{\text{on-shell}}^E}, \quad (6.32)$$

where $S_{\text{on-shell}}^E$ Firstly, a boundary term is introduced, such that the variation of the action is the Euclidean signature is well-defined

$$S^E = S_{CS}^E + S_B^E \quad (6.33)$$

The boundary term that needs to be included depends on how one includes the chemical potential. In this case, the Euclidean boundary term reads [74]

$$S_B^E = -\frac{k}{2\pi} \int_{\partial M} d^2x \text{Tr}[(a_+^+ - a_-^+ - 2L_1)a_+^+ + (a_+^- - a_-^- - 2L_{-1})a_+^-] \quad (6.34)$$

Thus using (6.33), (6.32) and the periodicities of the connections $\tau, \bar{\tau}$ [73] one obtains the thermal entropy

$$S = \ln Z - 2\pi k \text{Tr} \left[\sum_{q=\{+,-\}} \left(\frac{1}{2} (a_+^q + a_-^q)^2 \tau^q + (a_+^q + a_-^q - L_{q1})(\bar{\tau} - \tau) a_-^q \right) \right] \quad (6.35)$$

$$= -2\pi k \text{Tr} \left[\sum_{q=\{+,-\}} ((a_+^q + a_-^q)(\tau a_+^q + \bar{\tau} a_-^q)) \right]. \quad (6.36)$$

If we change to C-parametrization we obtain exactly the same result as in (6.31).

Canonical Black Hole

Following the same strategy for the non rotating case of the canonical black hole. One needs to impose the trivial holonomy condition (6.19). The conditions obtained are the same constrains as in (6.20) and the same holds for the thermal entropy.

Furthermore, some solutions can be obtained by expanding \mathcal{L}, \mathcal{W} around $T = \beta^{-1} = 0$ and $\mu = 0$.¹ The result is four different branches [72], where the first one and the third one are the stable states. We refer to the first one as the "BTZ-branch", because it is the one that smoothly connects to the BTZ black hole at zero spin-3 chemical potential.

Branch	\mathcal{L}	\mathcal{W}	
I	$\pi^2 k T^2 + \frac{80}{4} \pi^4 k \mu^2 T^4 + \mathcal{O}(T^5)$	$\frac{32}{3} \pi^4 k \mu T^4 + \frac{3k}{16\mu^2} - \frac{3\pi l T}{4\mu} + \mathcal{O}(T^6)$	
II	$\frac{3k}{16\mu^2} - \frac{3\pi l T}{4\mu} + \mathcal{O}(T^2)$	$\frac{k}{8\mu^3} - \frac{3\pi l T}{4\mu} + \mathcal{O}(T^2)$	(6.37)
III	$\frac{3k}{16\mu^2} - \frac{1}{3} \pi^2 k T^2 + \mathcal{O}(T^3)$	$\frac{k}{8\mu^3} - \frac{32T^4}{243} \pi^4 k \mu + \mathcal{O}(T^5)$	
IV	$\frac{3k}{16\mu^2} + \frac{3\pi l T}{4\mu} + \mathcal{O}(T^2)$	$\frac{k}{8\mu^3} + \frac{3\pi l T}{4\mu} + \mathcal{O}(T^2)$	

These expressions will be useful in the last Chapter, because they will help us to obtain the final results.

6.3 Ergosphere

In this subsection, we will consider the possibility of the existence of an ergosphere for higher spin black holes. As it was discussed in Section 2.3, in order to show that there

¹This expansion will give the same results in GK black hole and the reason that we mention it here is the historic significance.

is an ergosphere in a three-dimensional black hole one needs to prove that there is no asymptotically light-like killing vector that remains light-like at least until the horizon. For this purpose, we need to turn to the metric like formulation. Thus we formulate the higher spin black holes in metric like formulation and define a generalized Killing vector $\zeta^\mu = \left(\frac{\partial}{\partial t}\right)^\mu + \Omega \left(\frac{\partial}{\partial \phi}\right)^\mu$. Then it is possible to explore the possibility of ergosphere existence using (2.43).

We consider the Canonical black hole (6.5) as an example, but the same implies to the rest of the solutions. The metric element is obtained by using (3.10),

$$g_{\phi\phi} = \frac{3}{4}e^{2\rho} + \mathcal{L}^- + \frac{\mathcal{L}^+}{2} + \frac{\mathcal{L}^+}{4} (2\mathcal{L}^- + \mathcal{L}^+) e^{-2\rho} - \frac{\mathcal{W}^+\mathcal{W}^-}{4} e^{-4\rho} \quad (6.38)$$

$$g_{t\phi} = \frac{1}{4}(2\zeta^+ - \zeta^-)e^{2\rho} + \mathcal{O}(1) \quad (6.39)$$

$$g_{tt} = 4\mu^+\mu^- e^{4\rho} + \mathcal{O}(e^{2\rho}). \quad (6.40)$$

Due to the higher order polynomial for the radial dependence in g_{tt} , $g(\zeta, \zeta) \sim 4\mu^+\mu^- e^{4\rho}$ asymptotically. Thus, in order for ζ^μ to be asymptotically timelike, we only need to consider the cases where $\mu^+\mu^- < 0$. The rest of the cases do not have the right asymptotic behaviour and they do not describe a black hole. However, this constrain implies that every branch in (6.37) gives you the constrain that $\mathcal{W}^+\mathcal{W}^- < 0$. In that case one observes that from (6.38) this component of the metric is always positive, $g_{\phi\phi} > 0$. This is quite expected due to the fact that $g_{\phi\phi} = r^2$ in the BTZ case.

Moreover, one needs to notice that the asymptotic behaviour does not depend on the value of the parameter Ω , in contrast to the BTZ case. As a result, this makes it easy to realize that there is no ergosphere for higher spin black holes. In addition, one can define an always timelike Killing vector not with a specific value of Ω but for a large spectrum.² Specifically, for large absolute values of $\Omega < 0$, $g_{\phi\phi}$ will give the main contribution for $g(\zeta, \zeta)$, which is always negative. Thus, when we assume that $\Omega \rightarrow -\infty$, which is an accepted value, then this Killing vector is always timelike. However this is just a limit, so we need to take into account that for large, but not infinite values of $|\Omega|$, there will be a radius that the Killing vector can become space-like. However, because there is not an upper bound in that value, we can always pick an Ω such that this radius is inside the horizon.

Finally we conclude that there is no ergosphere in higher spin black holes due to the different asymptotic behaviour that they admits, which is not AdS_3 but is has a steeper

²The review of the calculations is not necessary in this case because one the one hand it can be proven qualitatively and secondly the solutions of the equations are extremely large.

potential, thus the upper bound for the geodesics of the massive particles is twice smaller from the BTZ case, Section [2.1.2](#).

Chapter 7

Wilson Lines and Geodesics in Chern-Simons Formulation

It is true that not every gravitational theory is an Einstein theory coupled to fields. There is a class of gravitational theories that more resemble a gauge theory, as detailed in the introduction. This means that in these theories Einstein basic properties as the curvature invariant $R_{\mu\nu}R^{\mu\nu}$ or the line element are not observables. Therefore, they are not well defined objects to describe the theory because they are not invariant under the total gauge symmetry that defines the theory.

This class of gravitational theories includes Vasiliev higher-spin theories. The way to understand this, in the case that we are interested in, is by considering the transformation of the spin-2 and spin-3 fields under the gauge group $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$, [75]

$$\delta\phi_{\mu\nu\rho} = 3\nabla_{(\mu}\xi_{\nu\rho)} + \mathcal{O}(\phi^2) \quad (7.1)$$

$$\begin{aligned} \delta g_{\mu\mu} = 12\tilde{\zeta}^{\rho\sigma} & \left[\nabla_{\rho}\phi_{\mu\nu\sigma} - 2\nabla_{(\mu}\phi_{\nu)\rho\sigma} + 2g_{\rho(\mu} \left[\nabla \cdot \phi_{|\nu)\sigma} - \nabla_{\sigma}\phi_{|\nu)} - \nabla_{|\nu)}\phi_{\sigma} \right] \right. \\ & \left. + \frac{1}{2}g_{\rho\mu}g_{\sigma\nu}\nabla \cdot \phi - g_{\mu\nu} \left[\nabla \cdot \phi_{\rho\sigma} - 2\nabla_{\rho}\phi_{\sigma} \right] \right] + \mathcal{O}(\phi^3), \end{aligned} \quad (7.2)$$

generated by a traceless $\tilde{\zeta}^{\rho\sigma}$. As it can be noticed, the spin-2 and spin-3 fields are entangled under the symmetry transformations. As a result, notions that are defined using the metric or the line element are not invariant under these transformations and cannot be used to describe the theory. Such quantities are the horizon of a black hole or more general the geodesics of test particles.

This problem prompted physicists to try to use observables of the theory to generalize the notion of the geodesics in this case, as well as to extend the notion of the entanglement entropy, due to the usual prescription of Ryu and Takayanagi [76, 77].

The original claim was made by two independent groups in [26, 27]. In [26], the authors are using composite Wilson lines to calculate the entanglement entropy for higher spin gravity. On the other hand, an alternative formula was proposed in [27] using an open interval of higher spin gravity and calculating the Wilson line for it. However, recent work [78] showed that both propositions are equivalent, thus we only use the formulation by Ammon et al., which we will review in the next sections

7.1 First Steps in Pure Gravity

The Wilson line is defined by evaluating the trace of the holonomy of the gauge connection

$$W_{\mathcal{R}}(C) = \text{Tr}_{\mathcal{R}} \left(\mathcal{P} e^{\int_C A} \right), \quad (7.3)$$

where \mathcal{R} is the representation of the gauge group, C is the path that is taken and \mathcal{P} denotes the taking into account of the normal ordering.

If C is a closed path, a Wilson line is invariant under the global gauge transformations (3.8). The Wilson line is aimed to be a Chern-Simons analogue of the geodesics, i.e. the proper distance between two points. Because a geodesic can be interpreted as the probe of a massive particle coupled to gravity, one has to encode this information in the Wilson line description. The choice of representation \mathcal{R} is crucial to this consideration. In order to describe a massive particle with continuous mass spectrum, an infinite-dimensional representation has to be chosen [27]. An additional motivation is that there is no unitary finite dimensional representation of $SL(2, \mathbb{R})$. The simplest infinite dimensional representation is the highest weight representation, which can be defined via the highest weight state $|h\rangle$, such that

$$L_1 |h\rangle = 0, \quad L_0 |h\rangle = h |h\rangle. \quad (7.4)$$

Moreover, due to the commutator $[L_{-1}, L_0] = L_{-1}$ we have $L_0 L_{-1}^n = L_{-1} (L_0 + n)$. It is then trivial to prove that all the other states can be formed by applying $(L_{-1})^n, n \in \mathbb{N}$, because they are eigenstates of L_0 with different eigenvalues

$$L_0 (L_{-1})^n |h\rangle = (h + n) |h\rangle. \quad (7.5)$$

This representation is unitary and can be labelled by the quadratic Casimir operator, see Appendix A.1,

$$C_2 = 2L_0^2 - (L_{-1}L_1 + L_1L_{-1}), \quad (7.6)$$

where, when applied to the highest-weight state, one finds its eigenvalue

$$c_2 = 2h(h - 1). \quad (7.7)$$

Thus we can also specify the representation by requiring the Casimir operator to have eigenvalues $h(h - 1)$, which in the end will refer to the mass.

Now, one needs to compute the trace of the Wilson line in this infinite dimensional representation. However, the highest-weight representations are convenient because they are commonly used as Hilbert spaces used in quantum mechanics. This suggests that the trace can be computed performing a path integral over an auxiliary field U . If the dynamics of U are chosen to recover the states of the representation \mathcal{R} after quantization, the description of a Wilson line (7.3) is equivalent to

$$W_{\mathcal{R}}(C) = \int \mathcal{D}U e^{-S[U, A^+, A^-]_C}. \quad (7.8)$$

The action can be decomposed in

$$S[U, A^+, A^-]_C = S[U]_{C, \text{free}} + S[U, A^+, A^-]_{C, \text{int}}, \quad (7.9)$$

where the free action will have a $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ global symmetry and will be coupled to the gauge fields A^\pm in $S[U, A^+, A^-]_{C, \text{int}}$ by promoting the global symmetry to a local one. Moreover, we should carefully choose the initial and final state of U , because we want the Wilson line to compute the geodesic length. Therefore, we need the open Wilson line to be Lorentz invariant. Since (7.8) in the endpoints of the Wilson line depends on U , we need boundary conditions for U that are Lorentz invariant and this requirement completely determines the values for $U_{i, f}$, [27].

Appropriate Action

The free action $S[U]_{C, \text{free}}$ in (7.8) can be constructed as follows. One can start from the action of a relativistic particle of mass m on a manifold that is endowed with a metric $g_{\mu\nu}$

$$\begin{aligned} S_{\text{rel}} &= -\sqrt{m} \int ds \\ &= -\sqrt{m} \int d\lambda \sqrt{\dot{x}^\mu g_{\mu\nu} \dot{x}^\nu}, \end{aligned} \quad (7.10)$$

where $\dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda}$. Now we consider \mathcal{M} to be the manifold of $SL(2, \mathbb{R})$ with metric $g_{\mu\nu} = \frac{1}{2} \text{Tr}[U^{-1} \partial_\mu U U^{-1} \partial_\nu U]$, which is the natural metric on a manifold of a group if U lies in the group. The action can be written as

$$S[U] = -\sqrt{\frac{m}{2}} \int d\lambda \sqrt{\text{Tr}[(U^{-1} \dot{U})^2]}, \quad (7.11)$$

where we used the chain rule $\dot{x}^\mu \partial_\mu U = \dot{U}$. We can introduce a Langrange multiplier σ in order to get rid of the square root and the momenta $P \in sl(2, \mathbb{R})$. The action then reads [79]

$$e^{-S[U]} = \int \mathcal{D}\sigma \mathcal{D}P e^{-S[U,P,\sigma]}, \quad \text{where}$$

$$S[U,P,\sigma] = \int d\lambda \left(\text{Tr}[PU^{-1}\dot{U}] + \sigma(\text{Tr}P^2 - \frac{m}{2}) \right). \quad (7.12)$$

As can be realized, the last term will play the roll of constant to the appropriate representation. Therefore the mass is actually the Casimir (7.7), $c_2 = \frac{m}{2}$.

Through variation of the action (7.12) with respect to U and P one finds the equations of motion

$$-U^{-1}\dot{U}PU^{-1} = \frac{d}{d\lambda}(PU^{-1}) = \dot{P}U^{-1} - PU^{-1}\dot{U}U^{-1}$$

$$\implies \dot{P} = [P, U^{-1}\dot{U}], \quad (7.13)$$

$$U^{-1}\dot{U} + 2\sigma P = 0, \quad (7.14)$$

where we used that $\frac{d}{d\lambda}U^{-1} = -U^{-1}\dot{U}U^{-1}$. Note that (7.13) implies that $[P, U^{-1}\dot{U}] = 0$. The equations of motion thus read

$$\dot{P} = 0, \quad U^{-1}\dot{U} + 2\sigma P = 0. \quad (7.15)$$

Note that, if $R, L \in SL(2, \mathbb{R})$, the action (7.12) is invariant under the global symmetry that acts as

$$U(\lambda) \rightarrow LU(\lambda)R, \quad P(\lambda) \rightarrow R^{-1}P(\lambda)R. \quad (7.16)$$

However, as was mentioned above, one should promote this global symmetry to a local symmetry by coupling U to the gauge fields A_\pm . Therefore, the action has to be invariant under

$$U(\lambda) \rightarrow L(x^\mu(\lambda))U(\lambda)R(x^\mu(\lambda)), \quad P(\lambda) \rightarrow R^{-1}(x^\mu(\lambda))P(\lambda)R(x^\mu(\lambda)). \quad (7.17)$$

In order to achieve this, the ordinary derivatives are replaced by covariant derivatives such that

$$D_\lambda U = \frac{dU}{d\lambda} + A_\lambda^+ U - UA_\lambda^-, \quad D_\lambda P = \frac{dP}{d\lambda} + A_\lambda^- P - PA_\lambda^-, \quad (7.18)$$

where $A_\lambda = A_\mu \dot{x}^\mu$. The gauge fields are transforming in a covariant manner under the group

$$A_\lambda^+ \rightarrow L(x^\mu(\lambda))(A_\lambda^+ + d)L^{-1}(x^\mu(\lambda)), \quad A_\lambda^- \rightarrow R^{-1}(x^\mu(\lambda))(A_\lambda^- + d)R(x^\mu(\lambda)). \quad (7.19)$$

As a result, the final action, which is invariant under (7.17) and (7.19), reads

$$S[U, P, \sigma, A^\pm] = \int d\lambda \text{Tr}[PU^{-1}\dot{U}] + \text{Tr}[PU^{-1}A_\lambda^+U] - \text{Tr}[PA_\lambda^-] + \sigma \left(\text{Tr}P^2 - \frac{m}{2} \right). \quad (7.20)$$

However, due to the appearance of the gauge fields A^\pm in the action, the equations of motion with respect to U and P are slightly altered

$$\dot{P} = [P, A_\lambda^-] \implies D_\lambda P = 0, \quad U^{-1}D_\lambda U + 2\sigma P = 0. \quad (7.21)$$

Now that we have the interacting action, we can investigate if it also corresponds to geodesics of a massive particle, on-shell. It turns out that it does if one chooses the specific gauge of $U(t) = \mathbb{I}$, [27]. As a result, we understand that calculating Wilson lines using this prescription is equivalent to calculating the geodesic length between two point.

7.1.1 Evaluation Of The On-Shell Action

We evaluate the path integral with a saddle point approximation. The largest contribution will come from the classical action. Using the equations of motion that we derived for the interacting theory (7.21), the on-shell action is obtained

$$\begin{aligned} S_{\text{on-shell}} &= \int_C d\lambda \text{Tr}[PU^{-1}D_\lambda U] = \int_C d\lambda (-\sigma(\lambda) \text{Tr}P^2) \\ \implies S_{\text{on-shell}} &= -2c_2 \int_C d\lambda \sigma(\lambda), \end{aligned} \quad (7.22)$$

where in the last equality we used the equation of motion that we inserted using Lagrange multiplier $\text{Tr}P^2 = c_2$. However, because finding a solution for general connections is not trivial, it is simpler to find solutions for the unphysical non-interacting theory, $A^\pm = 0$ and then switch on the interactions by using the gauge transformation (4.49). The solutions for the unphysical equations of motion are obtained by (7.14)

$$U_0(\lambda) = u_0 e^{-a(\lambda)P_0} \quad \text{and} \quad P_0(\lambda) = P_0, \quad (7.23)$$

where $\sigma(\lambda) = \frac{d}{d\lambda} a(\lambda)$, u_0 is an arbitrary constant matrix valued in $SL(2, \mathbb{R})$ and P_0 a constant matrix in $sl(2, \mathbb{R})$ such that the original momentum has the mass constraint $\text{Tr}P_0 = c_2$. Therefore, using the gauge transformations (7.17) and (7.19), we obtain the physical gauge

$$U = L(x^\mu(\lambda))U_0(\lambda)R(x^\mu(\lambda)), \quad P(\lambda) = R^{-1}(x^\mu(\lambda))P_0R(x^\mu(\lambda)), \quad (7.24)$$

$$A^+ = L(x^\mu(\lambda))dL^{-1}(x^\mu(\lambda)), \quad A^- = R^{-1}(x^\mu(\lambda))dR(x^\mu(\lambda)). \quad (7.25)$$

This solution is the physical solution that we will use in order to calculate the geodesics. Therefore, plugging (7.24) into (7.22), the on-shell action reads

$$S_{\text{on-shell}} = -2c_2 \int_{\lambda_i}^{\lambda_f} d\lambda \sigma(\lambda) = -2c_2(a(\lambda_f) - a(\lambda_i)). \quad (7.26)$$

However, one can calculate the $\Delta a \equiv (a(\lambda_f) - a(\lambda_i))$ by considering the matrix

$$M \equiv U_0^{-1}(\lambda_i)U_0(\lambda_f) = e^{-2\Delta a P_0}, \quad (7.27)$$

which in the physical gauge (7.24) becomes

$$M = R(\lambda_i)U_i L(\lambda_i)L^{-1}(\lambda_f)U_f R^{-1}(\lambda_f), \quad (7.28)$$

where the boundary conditions of the auxiliary field are $U_{i,f} = U(x^\mu(\lambda_{i,f}))$.

This action needs to be invariant under proper transformations, see Section 4.3. As a result, the most natural boundary conditions are the trivial ones, $U_{i,f} = \mathbb{I}$. For more details about the matter, we refer the reader to Section III A of [27]. This simplifies the matrix M

$$M = R(\lambda_i)L(\lambda_i)L^{-1}(\lambda_f)R^{-1}(\lambda_f). \quad (7.29)$$

The problem of calculating the geodesic length will rely on finding the eigenvalues of M , as will become clear.

The on-shell action can be calculated by using the matrix M

$$S_{\text{on-shell}} = \text{Tr}(\text{Log}(M)P_0) = \text{Tr}(\text{Long}(\Lambda_M)\Lambda_{P_0}) \quad (7.30)$$

$$= \sqrt{2c_2} \text{Tr}(\text{Log} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}) L_0 = \sqrt{\frac{c_2}{2}} \text{Log} \frac{x_1}{x_2}, \quad (7.31)$$

where we used that M and P_0 are simultaneously diagonalizable (Λ_{M,P_0}) and $x_{1,2}$ are the eigenvalues of M . One additional simplification can be made since $M \in SL(2, \mathbb{R})$; the eigenvalues are connected by $x_2 = x_1^{-1}$. The final expression for the on-shell action is

$$S_{\text{on-shell}} = \sqrt{2c_2} \text{Log} x_1. \quad (7.32)$$

However, it is important to point out that there is an ambiguity in the choice of which of the two eigenvalues we use. This is resolved by the requirement that the geodesic length has to be positive definite, thus x_1 is the largest eigenvalue of M .

Finally, it is worth mentioning that in this case there is an alternative equation that will be more convenient for the purpose of this section. Using the identity

$$\text{Log} x = \cosh^{-1} \frac{x + x^{-1}}{2}, \quad \text{for } x > 1 \quad (7.33)$$

one finds that the geodesic length takes the form

$$\begin{aligned}
L &= \frac{S_{\text{on-shell}}}{\sqrt{2c_2}} = \frac{1}{2} \text{Log} x_1 \\
&= \frac{1}{2} \cosh^{-1} \frac{x_1 + x_1^{-1}}{2} \\
&= \frac{1}{2} \cosh^{-1} \frac{x_1 + x_2}{2} \\
\implies L &= \frac{1}{2} \cosh^{-1} \left[\frac{1}{2} \text{Tr} M \right], \tag{7.34}
\end{aligned}$$

where it was used that $x_2 = x_1^{-1}$.

7.1.2 Examples in pure gravity

Empty AdS_3

In order to find the eigenvalues of the matrix M we need to have the functions L, R . The worldline that we choose will have boundary points $x(\lambda_i) = \{\rho, t, \phi_i\}$ and $x(\lambda_f) = \{\rho, t, \phi_f\}$. Using (3.17) and (3.21), one obtains

$$L = b^{-1} e^{-\phi(L_{+1} + \frac{1}{4}L_{-1})} \tag{7.35}$$

$$R = e^{-\phi(L_{-1} + \frac{1}{4}L_{+1})} b^{-1} \tag{7.36}$$

The resulting matrix M is

$$M = e^{\bar{a}\phi_i} b^{-2} e^{\Delta\phi a} b^2 e^{-\bar{a}\phi_f} \tag{7.37}$$

The next step is to calculate the trace of M . One finds

$$\text{Tr} M = \frac{1}{8} e^{-4\rho} \left((1 + 4e^{4\rho})^2 \cos(\Delta\phi) - (1 - 4e^{4\rho})^2 \right) \tag{7.38}$$

as a result for the proper length between $x(\lambda_i)$ and $x(\lambda_f)$ results in

$$L = \frac{1}{2} \cosh^{-1} \left[\frac{1}{16} e^{-4\rho} \left((1 + 4e^{4\rho})^2 \cos(\Delta\phi) - (1 - 4e^{4\rho})^2 \right) \right] \tag{7.39}$$

It is important to point out that because we only used the ϕ components of the connection, i.e a_ϕ , there is no difference in the resulting length between empty and thermal AdS_3 . This is the case because in (3.49) and (3.50) the ϕ components are the same as in empty AdS_3 , which is expected because they both have the same metric element, thus the same proper length.

BTZ black hole

Repeating the same steps for the BTZ black hole solutions, one uses (3.19) in order to calculate the M matrix. In this case the trace results in

$$\text{Tr}M = \left(\sqrt{\mathcal{L}^+} \sqrt{\mathcal{L}^-} e^{-2\rho} + \frac{1}{\sqrt{\mathcal{L}^+} \sqrt{\mathcal{L}^-}} e^{2\rho} \right) \sinh(\sqrt{\mathcal{L}^+} \Delta\phi) \sinh(\sqrt{\mathcal{L}^-} \Delta\phi) \quad (7.40)$$

$$+ \cosh(\sqrt{\mathcal{L}^+} \Delta\phi) \cosh(\sqrt{\mathcal{L}^-} \Delta\phi). \quad (7.41)$$

Therefore, the geodesic length can be found by

$$L = \frac{1}{2} \cosh^{-1} \left[\frac{1}{2} \text{Tr}M \right] \quad (7.42)$$

for such value of the trace.

7.2 Generalized Geodesic Length in $SL(3, \mathbb{R})$

This generalized prescription was developed in order to be applied to higher spin theories. In this section, there will be a generalization in $SL(3, \mathbb{R})$ following [27] and [78].

Firstly, it has to be noted that the algebra has another non trivial Casimir, the quartic Casimir, see Appendix A.2.1. As a result the constraints that need to be inserted in the action via Langrange multiplier are two

$$\text{Tr}P^2 = c_2 \quad \text{and} \quad \text{Tr}P^3 = c_3. \quad (7.43)$$

As a result the interacting action reads

$$S[U, P, \sigma_{2,3}] = \int d\lambda \left(\text{Tr}[PU^{-1}D_\lambda U] + \sigma_2(\text{Tr}P^2 - c_2) + \sigma_3(\text{Tr}P^3 - c_3) \right), \quad (7.44)$$

where the traces denote using the killing forms of $SL(3, \mathbb{R})$ defined in the Appendix A.2.1, $\text{Tr}P^2 = P^i P^j \delta_{ij}$ and $\text{Tr}P^3 = P^i P^j P^k h_{ijk}$. Using the variational principle, one obtains the equations of motion

$$D_\lambda P = 0, \quad U^{-1} D_\lambda U + 2\sigma_2 P + 3\sigma_3 T^i P^j P^k h_{ijk} = 0, \quad (7.45)$$

where T^i is a generator of the algebra. Another generalization has to do with the highest weight representation. We pointed out that the theory has two independent Casimirs. As a result, the highest weight state is expected to be defined by two quantum numbers [78]

$$L_0 |h, w\rangle = h |h, w\rangle \quad \text{and} \quad W_0 |h, w\rangle = w |h, w\rangle \quad (7.46)$$

$$L_1 |h, w\rangle = 0, \quad W_1 |h, w\rangle = W_2 |h, w\rangle = 0. \quad (7.47)$$

The combinations of the other generators will create an infinite tower of states. However, the eigenvalues of the Casimirs, when applied to $|h, w\rangle$, are

$$c_2 = \frac{1}{2}h^2 + \frac{3}{8}w^2, \quad c_3 = \frac{3}{8}w(h^2 - \frac{1}{4}w^2). \quad (7.48)$$

The same holds for the anti-holomorphic parts as in the $SL(2, \mathbb{R})$ case.

Evaluation of the on-shell action

As in the pure gravity case (7.23), the solutions of the equations of motion for the non-physical gauge $A^\pm = 0$ read

$$U_0(\lambda) = u_0 e^{-2a_2 P_0 - 3a_3 T^i P_0^i P_0^k h_{ijk}} \quad (7.49)$$

$$P_0(\lambda) = P_0, \quad (7.50)$$

where $a_i = \frac{d}{d\lambda} \sigma_i$. Then, one can transform the solutions to the physical gauge (7.24). In this gauge, the on-shell action can be obtained using (7.49)

$$S_{\text{on-shell}} = -2\Delta a_2 c_2 - 3\Delta a_3 c_3, \quad (7.51)$$

which is an expected generalization of (7.26). By inverting (7.49), M can be written in terms of the boundary conditions of the auxiliary field (7.28).

However, the diagonal form of P_0 generally has a contribution of the higher spin quantum number. Nevertheless, we want to describe particles that have only mass, which means that we require that the Wilson line will only be the source of a non-zero mass particle.¹ In order to require that the Wilson line does not source the higher spin fields but only describe the quantum number associated with mass

$$h = \bar{h} \neq 0, w = \bar{w} = 0. \quad (7.52)$$

Thus the momentum has the same diagonal form as in the gravity case. The final on-shell action has the same structure as in (7.30),

$$S_{\text{on-shell}} = \text{Tr} \text{Log}(\Lambda_M) \Lambda_{P_0} = \sqrt{2c_2} \text{Tr} \text{Log}(\text{diag}(x_1, x_2, x_3)) L_0 = \sqrt{2c_2} \text{Log} \frac{x_1}{x_3}. \quad (7.53)$$

It is thus possible to find the geodesic length for different black hole solutions by using (3.17). However, we will not go into detail to write down an explicit formula for the geodesic length for the black hole proposals that we mentioned in the Section 6.1. The reason is that finding the eigenvalues of M is a non trivial task, where even with the use

¹It is advised to review backreaction of the Wilson line [78], because it is out of the scope of this thesis.

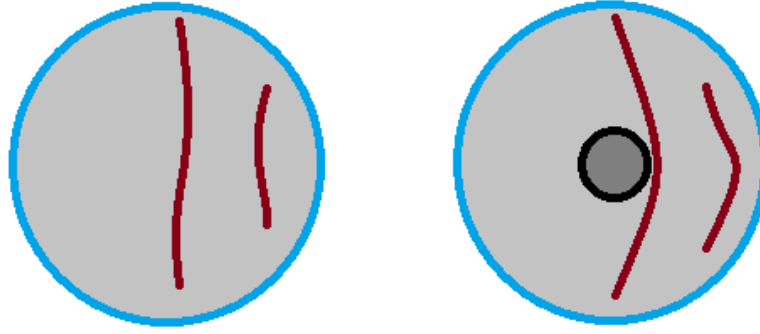


Figure 7.1: The figure presents a general abstract representation of the geodesic length in the absence (left) and the presence (right) of a black hole. In the presence of the black hole horizon, the test particle cannot enter the horizon and escape the black hole. Therefore the geodesics that have ending points outside the horizon have ruffly longer geodesics from the case of empty AdS .

of Mathematica the solutions are too large to be written down.² Nevertheless, in the next Section, we will find the geodesic length for specific values of boundary points in order to obtain some insight on properties of the different black holes.

7.3 Geodesic Comparison

In this final section, we will compare the geodesics of all the different solutions that we have found for different temperatures. The goal is a first attempt to understand the geometry of the different solutions. Specifically, the geodesic length between the same end-points is different if the relying geometry changes globally.

First of all, it needs to be pointed out that in order to compare the geodesic length between manifolds in different geometries, it is crucial that they all should have the same behaviour locally. If we consider cases that are locally AdS_3 , with different global behaviour, then the tangent space locally for every point at a considered manifold is auto-morphic to the tangent space of the corresponding point to another manifold with the same local behaviour. Thus the geodesics can map to each other locally and the difference between them relies only on the interaction with the black hole.

²These functions take a simpler form in the limit where the end points are on the boundary [80]. Note that in order to find the geodesic of a massive particle one should consider radius smaller than the upper bound.

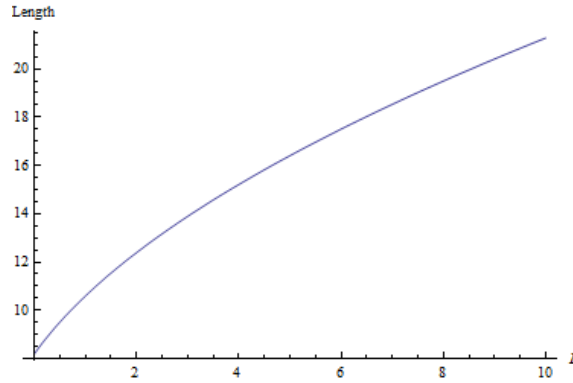


Figure 7.2: The graph shows the relation between the asymptotic charge of the non-rotating BTZ black hole and the geodesic length with fixed end point. These end points are outside the horizon and in a radius of smaller value than the upper bound that we found for the massive particle in Section 2.3.1. Finally this graph represents $\Delta\phi = \pi$.

Moreover, we pick a black hole solution. If we consider a test particle and we want to calculate its geodesic length between two points outside the horizon, then the test particle cannot cross the horizon because there is no causal geodesic that lets the test particle escape the black hole once it has fallen inside. Therefore, the resulting causal geodesic length will be larger than in the case of empty *AdS* space, Figure 7.1.

The goal of this simple example is to make clear that the existence of an event horizon will increase the geodesic length. Also, the difference of the proper distance with the same end points for different black holes, corresponds to a difference in the size of the horizon. A clear example is of the non-rotating BTZ black hole. Due to (2.41), it is clear that for larger values of $\mathcal{L} \sim M$ the black hole solution is characterized by a larger horizon. Moreover we have to be careful because for angles larger than π the geodesic length is not the proper one any more [81].

Unfortunately, the calculations are everything but trivial. Therefore, there has been an effort to find some approximate results by fixing some of the parameters and expand the rest in series of the temperature as in the (6.37). These preliminary results, Figure 7.3, show that the higher spin black hole solutions have larger geodesics compared to the BTZ solution. This is therefore a first evidence that there is a generalized notion of the event horizon that is independent of the gauge and it seems to be larger in the spin-3 case than the BTZ horizon.

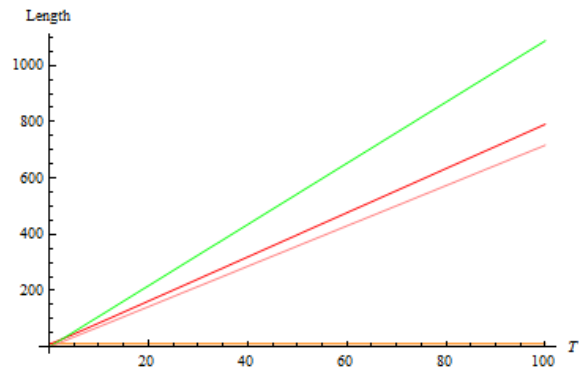


Figure 7.3: The graph shows the geodesic length between the geodesic length for different cases as a function of the temperature. As you can see the GK black hole (green) and the canonical black hole (red), for the BTZ branch, allow a larger proper length between two end-points, where $\Delta\phi = \pi$ and ρ large, than the BTZ solution (pink). Finally, it is important to note that the thermal AdS_3 is independent of the temperature and the geodesic length that relies on its geometry is comparably smaller than the others.

Chapter 8

Conclusion

In this thesis, we reviewed the relation between AdS_3 and the dual Chern-Simons theory with gauge group $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$. This dual theory makes it possible to describe higher spin theories coupled to gravity just by promoting the gauge group to $SL(N, \mathbb{R}) \times SL(N, \mathbb{R})$. We considered the simplest case of spin-3 field, where the gauge group is simplified to $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ due to the topological nature of AdS_3 . It has been shown that for the correct boundary conditions, the asymptotic symmetries form a conformal group. Moreover, in order to analyze the proposed solutions for Euclidean higher spin black holes one needs to generalize the notion of geodesics, due to the fact that the line element is not an observable of the Vasiliev theory. As a result, we tried to extract some information for the black holes using the proper length between two points, that has been generalized using Wilson lines.

Firstly, we reviewed (2+1)-dimensional gravity using metric-like and CS-like formulation. In the metric like formulation we reviewed the construction of BTZ black holes and the thermodynamics of BTZ black holes and the thermal AdS_3 . We also proved that there is no ergosphere in the black hole solutions. We followed the same steps in the CS-formulation and we analyzed the thermodynamics of the Euclidean solutions using holonomies. We also made a proposal for the choice of a specific gauge in thermal AdS_3 that manifests its thermodynamic properties.

However, due to the fact that AdS_3 is a compact manifold it has some additional properties. In Chapter 4, we revised these properties. We reviewed the famous Brown and Hanneaux paper [51] which was the first that showed that AdS_3 has an asymptotic conformal symmetry and it was the first evidence for the later on proven AdS/CFT correspondence.

In Chapter 5, we tried to give a brief review of Vasiliev theory in arbitrary dimensions,

and then we restricted ourselves to three dimensions. We connected Vasileiv theory to the dual CS action, that is values on $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$, for the spin-3 case. Then we calculated the asymptotic symmetries for a theory in that the spin-3 field is coupled to gravity. Furthermore, we mentioned the conditions that a gauge connection should have in order to allow Euclidean black hole conditions defined by a generalized partition function. There are several proposals for higher spin black holes that satisfy these requirements, and the most important ones have been reviewed, and prove that they do not contain an ergosphere.

However, it has been pointed out in Chapter 7 that the metric formulation is not appropriate to define these theories. The fact that the variation of the metric with respect to the symmetry group involves the higher spin fields shows that any usually invariant object that involves the metric in its definition is not well defined anymore.

In order to solve this problem, there was a proposal that connected massive probes with Wilson lines. We started by reviewing the proof that shows that the Wilson lines, when calculated in the highest weight infinite dimensional representation of the algebra, corresponds to the geodesic length between two end-points. Then we generalized the procedure in spin-3 solutions. Unfortunately, the equations are quite difficult and it is not possible to write an explicit solution. Therefore, a calculation for the proper length of spin-3 black holes was made using approximations. We found out that for these specific approximations, the higher spin black holes seem to have larger valued geodesic lengths than BTZ or thermal AdS solutions, when evaluated with the same end-points.

We claim that this is a preliminary result that proposes the existence of a generalized well defined notion of the event horizon. It is also argued that if this is true, then this horizon is larger for higher spin black holes than for the BTZ solution and thermal AdS, when evaluated at the same temperature. An interesting and ambitious continuation of these results is to use the Wilson lines in order to define a generalized horizon surface.

Moreover, we have used Wilson lines that have the end-points on the same circle, thus it would be interesting to explore different sets of end-points. An interesting example is to have only one of the end-points on the boundary and the other one hanging in the bulk. Furthermore, as we made clear in the procedure of the Wilson line calculation in the last chapter, we ignored the higher spin quantum numbers, $w_i = 0$, because we were looking for a massive probe. Therefore, one may explore the higher spin analogue of geodesics in these situations. Finally, there is no reason to expect that the calculations cannot be done also in higher symmetry groups or even different ones. It would be interesting to figure

out the details of the Wilson line prescription in the supersymmetric regime, such as the recently proposed hypergravity black hole [82]. In particular, in figuring out how to calculate the Wilson lines in this regime, we would succeed to calculate the entanglement entropy for the hypergravity black hole using the Ryu-Takayanagi formula.

Appendix A

Representations of Simple Lie algebras

A.1 $sl(2, \mathbb{R})$

We denote the three generators of $sl(2, \mathbb{R})$ as L_0, L_1, L_{-1} , which satisfy the algebra

$$[L_a, L_b] = (a - b)L_{a+b}. \quad (\text{A.1})$$

Then the non-vanishing Killing metric components are defined

$$\delta_{00} = \frac{1}{2}, \quad \delta_{+-} = \delta_{-+} = -1. \quad (\text{A.2})$$

Using this one can write the quadratic Casimir $C_2 = \delta_{ab}L_aL_b$. Moreover the generators in the fundamental representation are obtained

$$L_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (\text{A.3})$$

A.2 $sl(3, \mathbb{R})$

A.2.1 The principle embedding

We denote the three generators of $sl(3, \mathbb{R})$ as $T_a = \{L_0, L_1, L_{-1}, W_0, W_1, W_{-1}, W_2, W_{-2}\}$, which satisfy the algebra

$$[L_a, L_b] = (a - b)L_{a+b}, \quad (\text{A.4})$$

$$[L_a, W_b] = (2a - b)W_{a+b}, \quad (\text{A.5})$$

$$[W_a, W_b] = -\frac{1}{3}(a - b)(2a^2 + 2b^2 - ab - 8)L_{a+b}. \quad (\text{A.6})$$

The two Casimir operators are defined

$$C_2 = \delta^{ab} T_a T_b, \quad C_3 = h^{a,b,c} T_a T_b T_c, \quad (\text{A.7})$$

where the Killing forms are obtained by

$$\delta_{ab} = \text{Tr}(T_a T_b), \quad h_{a,b,c} = \text{Tr}(T_a T_b T_c). \quad (\text{A.8})$$

Furthermore we can define the fundamental representation

$$L_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix},$$

$$W_0 = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{4}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}, \quad W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad W_{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.9})$$

$$W_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}, \quad W_{-2} = \begin{pmatrix} 0 & 0 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.10})$$

A.2.2 The diagonal embedding

The generators of the diagonal embedding are obtained by the generators T_a of the principle embedding

$$\hat{L}_{\pm 1} = \pm \frac{1}{4} W_{\pm 2}, \quad \hat{L}_0 = \frac{1}{2} L_0, \quad J_0 = \frac{1}{2} W_0, \quad (\text{A.11})$$

$$G_{\pm 1/2}^{[+]} = \frac{1}{2\sqrt{2}} (\pm L_{\pm 1} - W_{\pm 1}), \quad G_{\pm 1/2}^{[-]} = \frac{1}{2\sqrt{2}} (L_{\pm 1} \pm W_{\pm 1}). \quad (\text{A.12})$$

The commutation relations are obtained

$$[\hat{L}_i, \hat{L}_j] = (i - j) \hat{L}_{i+j}, \quad [\hat{L}_i, J_0] = 0, \quad (\text{A.13})$$

$$[\hat{L}_i, G_m^{[a]}] = (i/2 - m) G_{i+m}^{[a]}, \quad [J_0, G_m^{[a]}] = a G_m^{[a]}, \quad (\text{A.14})$$

$$[G_m^{[+]}, G_n^{[-]}] = \hat{L}_{m+n} - \frac{3}{2} (m - n) J_0, \quad (\text{A.15})$$

where $i = -1, 0, 1, m = -1/2, +1/2$ and $a = \pm 1$. The basis element \hat{L}_i generates the $sl(2, \mathbb{R})$ subalgebra that is diagonally embedded, as it can be realized by the commutation relations. The $G_m^{[a]}$ transforms in $sl(2, \mathbb{R})$ -spin $\frac{1}{2}$ representations, while J_0 has $sl(2, \mathbb{R})$ -spin 0 representation. Note that all of the generator relations are commutators thus all of them are bossonic.

The fundamental representation can be written as

$$\hat{L}_0 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad \hat{L}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{L}_{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$J_0 = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad G_{+1/2}^{[+]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad G_{-1/2}^{[+]} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.16})$$

$$G_{+1/2}^{[-]} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad G_{-1/2}^{[-]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.17})$$

Appendix B

Frames and spin connections

It is often useful to introduce a local frame, that is, a set of orthonormal vectors $e^{\mu a}$ that satisfy

$$g^{\mu\nu} e_{\mu}^a e_{\nu}^b = \eta^{ab}, \quad e_{\mu}^a e_{\nu}^b \eta_{ab} = g_{\mu\nu} \quad (\text{B.1})$$

In general, the e_{μ}^a cannot have vanishing covariant derivatives, since this would imply flatness. Thus,

$$\nabla_{\mu} e_{\nu}^a + \omega_{\mu}^a{}_b e_{\nu}^b = 0, \quad (\text{B.2})$$

where $\omega_{\mu}^a{}_b$ is the spin connection. If the connection is torsion free, we obtain

$$D_{\omega} e^a = de^a + \omega^a{}_b \wedge e^b = 0, \quad (\text{B.3})$$

where $e^a = e_{\mu}^a dx^{\mu}$ is an one-form and D_{ω} is the gauge-covariant exterior derivative.

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