The wave equation on black hole interiors

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The wave equation on black hole interiors

De golfvergelijking binnen zwarte gaten

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Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof. dr. G. J. van der Zwaan, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op dinsdag 1 december 2015 des avonds te 6.00 uur

door

Anne Theresia Franzen

geboren op 5 maart 1980 te Neuss, Duitsland

Promotoren: Prof. dr. G. 't Hooft
Prof. dr. M. Dafermos

To all in love with
physics and mathematics

Wenn nicht mehr Zahlen und Figuren
Sind Schlüssel aller Kreaturen
Wenn die, so singen oder küssen,
Mehr als die Tiefgelehrten wissen,
Wenn sich die Welt ins freye Leben
Und in die Welt wird zurück begeben,
Wenn dann sich wieder Licht und Schatten
Zu ächter Klarheit werden gatten,
Und man in Mährchen und Gedichten
Erkennt die wahren Weltgeschichten,
Dann fliegt vor Einem geheimen Wort
Das ganze verkehrte Wesen fort [a].

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[a](Novalis: Schriften, Paul Kluckhohn u. Richard H. Samuel, W. Kohlhammer, 1960, S. 344)
When numbers and diagrams
are no longer the key to all creatures,
when those who sing or kiss
know more than the wisest scholars,
when the world returns to free life
and back into the world,
when light and shadow unite
once more to engender genuine clarity
and people recognize in fairy tales and poems
the true stories of the world,
then the whole topsy-turvy business
will take flight in the face of a single secret word.

Publications

This thesis is based on the following publication:

- Anne Franzen, (2014). Boundedness of massless scalar waves on Reissner-Nordström interior backgrounds. *To appear in Comm. Math. Phys. arXiv:gr-qc:1407.7093*

Other publications to which the author has contributed:

- Edward Anderson and Anne Franzen, (2010). Quantum cosmological metroland model. *Class. Quantum Grav.* **27** *045009. arXiv:gr-qc/0909.2436*

- Anne Franzen, Sashideep Gutti and Claus Kiefer, (2010). Classical and quantum gravitational collapse in the Lemaître–Tolman–Bondi model with positive cosmological constant. *Class. Quantum Grav.* **27** *015011. arXiv:gr-qc/0908.3570*

- Anne Franzen, Payal Kaura, Aalok Misra and Rajyavardhan Ray, (2006). Uplifting the Iwasawa. *Fortschritte der Physik* **54**, *207. hep-th/0506224*

Preface

This thesis is devided in two parts which can be read independently. The first part is concerned with Cauchy horizon stability and instability in general. This topic has been a subject of research for the past 40 years and we will review the most important models that have been investigated and conclude what insights were gained from them. This part is intended to not only shed light on the conceptual ideas but also state mathematically precise results that have been obtained related to the problem. Part II, the main part of this thesis deals with the analysis of scalar waves on fixed Reissner-Nordström backgrounds. This is our specific approach towards investigating stability under perturbations in the interior of charged black holes. In this part we will focus on the detailed proofs that led to our main Theorem which is concerned with boundedness of solutions to the scalar wave equation up to and including the Cauchy horizon on fixed Reissner-Nordström backgrounds.

Part I puts our work into context to preceding investigations related to the stability of the Cauchy horizon. The first part can be read as a historical summary to readers interested in the blueshift effect at the Cauchy horizon and its consequences. The second part may provide a first reading guide to young researchers interested in partial differential equations and in particular the vector field method. The reader is assumed to have familiarity with general relativity and knowledge of Penrose diagrams will be favorable in particular for the study of part II.

Cologne, Anne Franzen
August 2015

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Part I

Reviewing Cauchy horizon instability of charged and rotating black holes

Chapter 1

Introduction to part I

It is widely assumed that the *Cauchy horizon* $\mathcal{CH}$ of black hole interiors is classically and quantum mechanically unstable. In this review we will focus on the classical aspect of the instability and give a brief exposition of investigations leading to its dicovery. Moreover, we will mainly discuss subextremal and two-ended asymptotically flat spacetimes and only briefly mention results within the analyses of extremal black holes.

In the literature, which discusses exact black hole solutions of the Einstein field equations, the Cauchy horizon $\mathcal{CH}$ is sometimes mentioned in the same vein as the event horizon $\mathcal{H}$ by simply stating that charged and rotating black holes possess two horizons. In this thesis we will see that this description is neglecting crucial features of Cauchy horizons. The importance of Cauchy horizons becomes evident by considering Cauchy problems for general relativity. Roughly speaking a Cauchy problem allows for decomposing spacetime and evolving data to instances in time. The initial data of a Cauchy problem is given on a *Cauchy hypersurface* Σ, which is a hypersurface that by definition is intersected exactly once by every inextendible timelike curve on the underlying spacetime manifold $\mathcal{M}$. The time function is then chosen such that each point of spacetime is passed by a Cauchy hypersurface and all these Cauchy hypersurfaces are diffeomorphic to each other. The initial data is given by a triple $(\Sigma, \bar{g}, K)$, with $\bar{g}$ the induced metric on Σ and K the second fundamental form[1]. Further, the data has to satisfy certain constraints and is then evolved to the future Cauchy hypersurfaces via evolution equations derived from the Einstein equations. Due to the finite speed of

[1]The second fundamental form messures the rate of change of the geometry of the initial hypersurface under normal displacement.

propagation the data on a Cauchy hypersurface is completely determined by the data of its *past domain of dependence* $\mathcal{D}^-(\Sigma)$ and in turn determines all events contained in the *future domain of dependence* $\mathcal{D}^+(\Sigma)$. To be more precise the future (past) domain of dependence, is defined by the set of all points through which every past (future) inextendible causal curve intersects Σ. The (full) domain of dependence is given by $\mathcal{D}(\Sigma) = \mathcal{D}^-(\Sigma) \cup \mathcal{D}^+(\Sigma)$. That means that the domain of dependence of the subset Σ is the set of points getting all information of that subset in its future and its past including the subset itself. The maximal domain of dependence is the largest possible development. The fact that a solution to relativistic equations depends only on its past is an important property for general relativity revealing its causal structure. The initial data set $(\Sigma, \bar{g}, K)$ determines the entire development $\mathcal{D}(\Sigma)$. If $\mathcal{D}(\Sigma) = \mathcal{M}$, then the spacetime is called *globally hyperbolic.* The Cauchy horizon $\mathcal{CH}$ is defined as the boundary of $\mathcal{D}(\Sigma)$. It marks the limit of the region, which is determined by the initial data on Σ. For non-globally hyperbolic spacetimes predictability breaks down since evolution outside $\mathcal{CH}$ is not uniquely determined anymore. The uniqueness of solutions in the maximal domain of dependence was first proven by Choquet-Bruhat and Geroch, [15]. It was suggested by Penrose that all physically reasonable spacetimes should be globally hyperbolic. This proposal is also known as *the Cosmic Censorship Conjecture.* One approach to verify if global hyperbolicity is a reasonable physical requirement is to show that non-globally hyperbolic spacetimes are not stable under perturbations. The work presented in this thesis is along these lines and will conclude that global hyperbolicity might be a too strong requirement.

Before we elaborate more on Cauchy horizon instabilities and *Cosmic Censorship* let us first elaborate more on asymptotically flat spacetimes containing a *black hole.* The conformal boundary of an asymptotically flat spacetime at infinity is usually denoted by $\mathcal{I}^+$ for *future null infinity* and $\mathcal{I}^-$ for *past null infinity*, see Figure 1.1 and also Figure 1.2. The black hole region, which is in both figures denoted by the darker shaded region II, of an asymptotically flat spacetime given by a 4 dimensional Lorentzian manifold $\mathcal{M}$ and a metric g is then given by the complement of the causal past of $\mathcal{I}^+$, provided that $\mathcal{I}^+$ is complete[2]. We will denote the causal past/ future by $J^{\mp}(\cdot)$, respectively. In the figures the completeness is indicated by the depiction of $i^{\pm}$ which represents *future/past timelike infinity*, respectively. Similarly, the symbol i^0 represents *spatial infinity.* Thus,

[2]Completeness of $\mathcal{I}^+$ implies that far away observers live for infinite proper time.

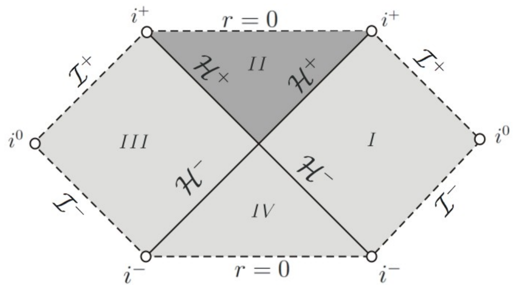


Figure 1.1: Penrose diagram of Schwarzschild spacetime. Each point of the diagram represents a two-sphere. The dashed lines as well as i^0 and $i^\pm$ are not to be interpreted as part of the spacetime manifold.

we express the black hole region $\mathcal{B}$ as

$$\mathcal{B} = \mathcal{M} \backslash J^-(\mathcal{I}^+).$$

In case of two asymptotic ends as in Figures 1.1 and 1.2, $J^-(\mathcal{I}^+)$ is the union of the past of each future null infinities and therefore the union of regions I, III and IV. When we later refer to the *exterior* of the black hole we always mean region I and by *interior* we mean $\mathcal{B}$ itself, which is region II. Regions III and IV are time reversed to regions I and IV, respectively, but share apart from that the same properties. In particular, region III can also be thought of as an exterior region but with its timelike Killing flow in reversed direction compared to region I. Region IV is sometimes referred to as *white hole* region since objects cannot fall into it but will be radiated out. The past boundary of $\mathcal{B}$, basically the hypersurface separating exterior and interior regions, is exactly the future *event horizon* $\mathcal{H}^+$. The event horizon is the boundary from which not even light can escape the black hole region. To understand the future boundary of $\mathcal{B}$ we have to explain a bit more. The rotating black hole solution to the vacuum Einstein field equations

$$R_{\mu\nu} = 0, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, is the axialsymmetric Kerr solution. The special case of zero angular momentum was discovered first and is known as the Schwarzschild solution. Schwarzschild spacetime has two special features, first of all it is the only existing spherically symmetric exterior vacuum solution to the field equations. This fact was first proven in 1921 by Jebson, cf. [50] and two years later by Birkhoff, cf. [7] and goes by the name *Birkhoff's theorem.* The second interesting feature of Schwarzschild black holes is, that an observer traveling in its interior would be torn apart by the growing tidal forces and the spacetime eventually ends in a strong spacelike singularity at $r = 0$ from which it cannot be extended[3].

Although various researchers made effort to provide a uniqueness proof for Kerr, no axialsymmetric analog of Birkhoff's theorem could be shown yet. Further, the inextendibility of Schwarzschild spacetime is special since the Kerr solution remains regular up to a null hypersurface, the *Cauchy horizon* $\mathcal{CH}$, from which the spacetime can in fact be extended. As mentioned before, the drawback auf these extensions are that they are not uniquely determined from initial data and evolution of (1.1). Therefore, predictability for the fate of an observer traveling in these regions is lost. The future boundary of $\mathcal{B}$ for Kerr spacetime is the future Cauchy horizon $\mathcal{CH}^+$, separating the interior and a region to which the spacetime extends. Reissner-Norström spacetime which is a spherically symmetric solution to the Einstein-Maxwell-field equations shares this feature of exhibiting a Cauchy horizon with Kerr spacetime, see Figure 1.2. Note that despite of lack of spherical symmetry, for representation purposes Kerr spacetime is usually depicted as Figure 1.2 but for fixed angles. Further, since Reissner-Norström is a spherically symmetric black hole solution its analysis is much simpler compared to Kerr and there even is a Birkhoff's theorem analog. Due to the similarity of the causal structure while at the same time having simpler geometric properties, Reissner-Norström black holes are popular proxies for the astrophysically more realistic Kerr black holes. For that reason a big share of this review will be dedicated to Reissner-Norström black holes.

In 1968 Penrose [78] first suggested Cauchy horizon instability[4] for perturbations that do not decay fast enough. A few years later, together with Simpson by numerical results he argued that small perturbations of Reissner-Nordström would lead to a spacetime whose boundary would be a genuine spacetime curvature singularity, [90]. The reason that a small perturbation could possibly grow into an irregularity of the spacetime is the well known blueshift effect, which we are going to discuss more in Section 2. The blueshift effect at the Cauchy horizon can be thought of a time reversed analog of the redshift effect at the event horizon. This implies the puzzling consequence that an observer at the Cauchy horizon would be able to see the entire past in finite time. Further, this means

[3] The inextendibility of Schwarzschild spacetime is widely discussed in the physics literature and was recently rigorously proven by Sbierski [87].

[4] To determine stability means to determine whether an initially small disturbance of the solution will remain small or grow large with time.

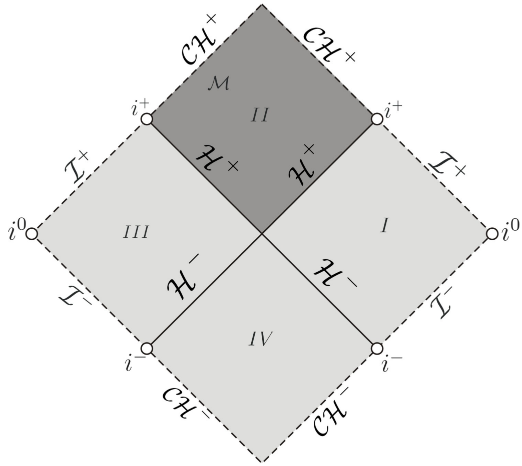


Figure 1.2: Penrose diagram of the maximal domain of dependence of Reissner-Nordström spacetime. Each point of the diagram represents a two-sphere. The dashed lines as well as i^0 and $i^\pm$ are not to be interpreted as part of the spacetime manifold.

that a small perturbation at the Cauchy horizon accumulates into an infinite energy density. Mathematically this can be investigated by analyzing specific mass quantities, e.g. the *Hawking mass*, which can be shown to blow up in certain models, as we will see in Sections 4.1.2 and 4.2.2. This result of the blueshift effect goes by the name *mass inflation* effect. The possible scenarios arising from the exponential blueshift effect are then that the Cauchy horizon could either evolve into a strong spacelike singularity such as the Schwarzschild singularity or into a null singularity along the Cauchy horizon. This suggests that the smoothness of the Cauchy horizon of Reissner-Nordström and Kerr spacetimes might not be a generic property that would be stable under small perturbations.

With this idea Penrose coined *the Cosmic Censorship Conjecture* in 1978, [79]. The conjecture stated that *generically*[5] a gravitational collapse should lead to a genuine curvature singularity and not produce naked singularities. Exact solutions to the Einstein field equations, such as the Reissner-Nordström and the Kerr solution therefore do not falsify Penrose's conjecture, since these spacetimes are not likely to arise from generic initial data. Validity of the conjecture implies that the production of locally

[5]Roughly, by "generically" we mean "not assuming any special properties". To be more precise, it means that "generic initial data" forms a dense subset in the set of all possible data. In this sense initial data leading to Reissner-Nordström or Kerr spacetimes are not generic but of measure zero within the set of all data.

visible naked singularities in a collapse is not a stable property. Capturing this idea in a definite precise statement is not an easy task. In fact formulating the conjecture was developed and refined over the years simply by finding counter examples to the statement previously made. In particular, it was also rendered more precise by splitting the vague earlier statement into two conjectures of separate concern, namely *the Strong Cosmic Censorship Conjecture* and *the Weak Cosmic Censorship Conjecture*. The names "strong" and "weak" are slight misnomers in the sense that the latter is not implied by the former and also not vica versa. *The Weak Cosmic Censorship Conjecture* is widely known as the statement that singularities should be hidden behind an event horizon. More mathematically speaking *the Weak Cosmic Censorship Conjecture* demands that $\mathcal{I}^+$ is complete, so that in case of its validity it is ensured that far away observers live forever. In the language of partial differential equations this translates into demanding *global existence* of the solution. The main concern of this review is *the Strong Cosmic Censorship Conjecture* which originally captured the idea that locally visible singularities, although not accessible to observers from infinity should be prevented. Christodoulou suggested the following mathematical formulation in [17]:

> "<u>Generic</u> asymptotically flat initial data for Einstein spacetimes have a maximal future development which is inextendible as a suitably regular Lorentzian manifold." (1.2)

In the language of partial differential equations this statement proposes *global uniqueness* for solutions to generic suitable initial data. The conjecture is mathematically not fully understood. In order to approach the problem, it is important to understand that there are two stages of evolution concerning black hole formation from axialsymmetric collapse.

1. The process of formation of the black hole, the decay of perturbations and the settling of the black hole to Kerr spacetime[a].

2. The evolution of the decaying small perturbations[b] inside the black hole from the event horizon to the formation of the curvature singularity at the Cauchy horizon due to the infinite blueshift effect. (1.3)

[a] Despite of the lack of an analog Birkhoff's theorem for axialsymmetric vacuum solutions, it is widely assumed that rotating collapsing objects will eventually settle down to stationary Kerr black holes. For further discussion on that see [73] and §32.7 of [68].

[b] We will explain in Section 3.1 why at this stage of evolution it is sufficient to consider small perturbations according to Price's law [83].

Restricting to the neighborhood of explicit solutions such as Schwarzschild, Reissner-Nordström and Kerr, regarding both stages of evolution we may ask:

(i) *Are the exteriors of Schwarzschild and Kerr spacetimes stable under evolution of* (1.1) *to perturbation of data?*

(ii) *Are the smooth Reissner-Nordström and Kerr Cauchy horizons unstable under perturbation?* (1.4)

(iii) *In case of positive finding of (ii), does the singular nature of the Reissner-Nordström and Kerr Cauchy horizons allow for extensions?*

As the title of part I of the thesis suggests this review aims to mainly shed light on understanding the answer to question (ii). But since (i) is a necessary prerequisite and (iii) an almost immediate follow up question of (ii), we will elaborate on all of them.

Aiming to gain a better understanding of stability and instability properties of black hole Cauchy horizons it is suggestive to neglect the backreaction of the gravitational field in first investigations. We will therefore distinguish between the fate of the singularity that is expected from investigations of linear models and the outcome from non-linear models. Hereby, the first serve as proxies for the latter. Moreover, investigations of the two-ended case could be considered as toy-models for the more realistic one-ended case. In this review we abstain from considering complicated realistic matter models and instead focus on perturbations within and of two-ended exact black hole solutions.

Further, we are not aiming to give a review in chronological historic order but instead start from explaining the most simple models and then go further with growing complexity.

The goal of this review is to show that from various models it follows that a singularity forms at the Cauchy horizon which is a result of the blueshift mechanism. One could think now that the instability caused by the blueshift can be prevented if infalling radiation would again decay exponentially in v and thus cancel the blueshift factor. However, the infalling radiation is expected to decay much slower. The blueshift instability can therefore not be prevented by the decay of radiation.

1.1 Outline of part I

In chapter 2 of part I of this thesis we will elaborate on the redshift and blueshift effect which are the underlying mechanisms for stability and instability, respectively. Therefore, these effects constitute the physical reason for all decay or blow-up results of the models we will discuss in the following chapters. As mentioned in the introduction of part I, we start from surveying linear models with symmetry in Section 3 and add some degree of complexity with each section. We will see that already the simplest models indicate the weak singularity and this conclusion withstands when looking at more elaborate models. In Section 4 we go further by discussing non-linear models under symmetry. In particular, we inspect null-fluid models in Section 4.1 which mimic the blueshift effect and also reveal the mass inflation effect via in- and outgoing null fluxes. Further, we review scalar field perturbations with symmetry in Section 4.2. In Section 5 we will elaborate on studies of linear perturbations but without symmetry to then finally review non-linear perturbation analysis without symmetry in Section 6. In Section 7 we briefly discuss open problems concerning the Cauchy horizon singularity and close with some concluding remarks in Section 8.

Chapter **2**

The redshift and blueshift effect

The redshift effect is the fundamental stability mechanism responsible for decay of perturbations along the event horizon, while the blueshift effect is the mechanism taking charge for the anticipated blow-up instability at the Cauchy horizon. Therefore, we will briefly review the redshift effect derived in the conventional way by considering proper times of local observers, e.g. presented in [58]. Thereafter, similar considerations for the blueshift effect in the vicinity of $\mathcal{CH}^+$ of Reissner-Nordström spacetime will be discussed. Let us further mention that Dafermos and Rodnianski explained in [35] as well as [30] how the redshift effect can be used to obtain decay in the exterior and even along $\mathcal{H}^+$, the event horizon by using *the divergence theorem* and *the vector field method* which we are going to introduce in Section 10.1. The interested reader is referred to Appendix A for a sketch of Dafermos' and Rodnianski's proof. We will see that the redshift effect is a striking feature for obtaining decay when using an appropriate vector field which is not null along $\mathcal{H}^+$.

Suppose two observers A and C start off from the exterior Reissner-Nordström region. A who is crossing $\mathcal{H}^+$ sends signals to C, located at infinity, in constant time intervals and at constant frequency according to his own proper time. C will measure an ever-increasing time gap between the signals, or equivalently she will notice that the frequency of the signals is shifted to the red, i.e. she observes lower frequencies compared to A. This redshift effect is *global* and is due to the fact that the proper time of C is infinite while the proper time of A until he crosses the event horizon is finite. Since we want to inquire about outgoing photons it is convenient to work in retarded Eddington-Finkelstein coordinates (u, r, θ, ϕ). As defined in Section 12.1.3 the Regge-Wheeler tortoise coordinate is given by

$$\mathrm{d}r^* = \frac{\mathrm{d}r}{1-\mu} \quad \Rightarrow r^* = r + \frac{1}{\kappa_+}\ln|\frac{r-r_+}{r_+}| + \frac{1}{\kappa_-}\ln|\frac{r-r_-}{r_-}| + \tilde{C},$$

with $\tilde{C}$ an arbitrary constant, $\mu = \frac{2M}{r} - \frac{e^2}{r^2}$, and the surface gravities $\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}$ and $r_\pm = M \pm \sqrt{M^2 - e^2}$, where the index "+" labels surface gravity and area radius at the event horizon and a "−" denotes the same but at the Cauchy horizon. We further define the null coordinate[1]

$$u = t - r^*. \tag{2.1}$$

In (u,r) coordinates the line element is given by

$$\mathrm{d}s^2 = -(1-\mu)\mathrm{d}u^2 - 2\mathrm{d}u\mathrm{d}r + r^2\mathrm{d}\sigma_{\mathbb{S}}^2. \tag{2.2}$$

We can now set $\mathrm{d}s^2 = 0$ and also constitute that the outgoing photons travel along $u = const$ lines, which is why we made this coordinate choice. The frequency shift can be calculated by the time interval between reception of photons and emission of photons.

$$\frac{\lambda_C}{\lambda_A} = \frac{(\Delta t)_C}{(\Delta\tau)_A} = \frac{(\Delta u)_C}{(\Delta\tau)_A} = \frac{(\Delta u)_A}{(\Delta\tau)_A} = \frac{\mathrm{d}u}{\mathrm{d}\tau}\Big|_A.$$

Remember that A is the emitter and C is sitting at infinity. With Δt and $\Delta\tau$ we mean the time interval between two signals in the proper time of C and A respectively. The first equality holds just by expressing wavelength into frequency and thus into time. The second holds since C is at fixed radius at infinity and therefore t can easily be related to u via (2.1). Since u is a coordinate and not a quantity related to one particular spacetime traveler the third equation follows and thus the last. $\frac{\mathrm{d}u}{\mathrm{d}\tau}\Big|_A$ is the component U^u of the emitters four-velocity $\mathbf{U}$ that we need to find. Since ∂_t is a Killing vector field we know that the component U_t is a constant of motion along a geodesic. Similarly in (u, r, θ, ϕ) coordinates the metric does not depend on u so that we may choose

$$U_u = -E. \tag{2.3}$$

[1] Note that we define the retarded coordinate with different sign compared to Section 12.1.3 since here we are working in the exterior region whereas in part II of this thesis we always consider the interior region.

The fact that the four-velocity is timelike implies $\mathbf{U}\cdot\mathbf{U} = -1$ which we can solve for the r component

$$U_r \overset{(2.3)}{=} \frac{-E - \sqrt{E^2 - 1 + \mu}}{1-\mu}. \tag{2.4}$$

We can then write

$$U^u = g^{uu}U_u + g^{ur}U_r = 0 + \frac{E + \sqrt{E^2 - 1 + \mu}}{1-\mu}, \tag{2.5}$$

$$U^r = g^{ur}U_u + g^{rr}U_r = -\sqrt{E^2 - 1 + \mu}, \tag{2.6}$$

and therefore

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{\mathrm{d}u}{\mathrm{d}\tau}\cdot\frac{\mathrm{d}\tau}{\mathrm{d}r} = \frac{U^u}{U^r} = -\frac{E + \sqrt{E^2 - 1 + \mu}}{\sqrt{E^2 - 1 + \mu}(1-\mu)}. \tag{2.7}$$

Near the event horizon at $r = r_+$ this gives us

$$\mathrm{d}u \approx -\frac{2}{1-\mu}\mathrm{d}r = \frac{\kappa_+}{r_+ - r}\mathrm{d}r. \tag{2.8}$$

Integration yields

$$u \approx -\kappa_+ \ln(r_+ - r) + const \qquad \Rightarrow (r_+ - r) \sim e^{-\frac{u}{\kappa_+}}. \tag{2.9}$$

Plugging this back in (2.5) for r close to r_+, using that $(1-\mu) \approx -\frac{2(r_+ - r)}{\kappa_+}$ we obtain

$$U^u \sim e^{\frac{u}{\kappa_+}}. \tag{2.10}$$

Now note that the advanced coordinate $v = t + r^*$ can be expressed as $v = u - 2r^*$ and r^* is fixed for the observer C at infinity. So finally we obtain the expected shift to the red

$$\frac{\lambda_A}{\lambda_C} \sim e^{-\frac{v}{\kappa_+}}.$$

Similarly, we obtain a *local redshift effect* when the signals are measured by another observer B who also crosses the horizon but at advanced time later than A, cf. Figure 2.1. One may just repeat the above calculation for observer B and by dividing the

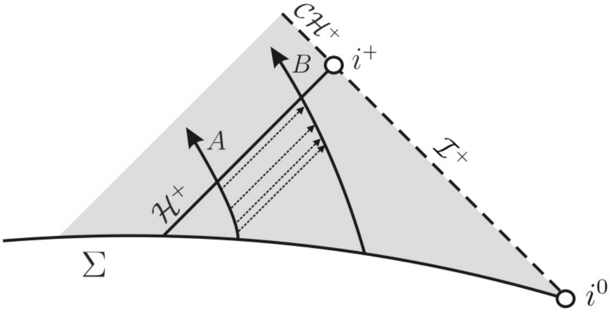


Figure 2.1: Penrose diagram showing observer A sending signals to observer B.

frequency shift of B, C by the frequency shift of A, C we obtain the desired shift of B, A and constitute that it is exponential in advanced time v, namely

$$\frac{\lambda_A}{\lambda_B} \sim e^{-\frac{\Delta v}{\kappa_+}}.$$

Analogous, to the redshift effect we can consider observer A traveling to $\mathcal{CH}^+$ while observer C sitting at infinity sends signals in constant time intervals and at constant frequency. Again the proper time of A is finite while the proper time of C is infinite. Note however that this time C is sending the signals to A who consequently measures signals with enhanced frequency compared to the observation of C, i.e. blueshifted signals. A will observe an ever-decreasing time gap between the signals. This effect also leads to the bizarre property that an observer crossing $\mathcal{CH}^+$ would see the entire remaining history of the exterior region in finite time. We refer to *global blueshift effect* when observer C sending the signals is located at infinity. Letting B, the emitter travel to $\mathcal{CH}^+$instead but at retarded time earlier we again constitute an exponential shift but this time with the surface gravity of the Cauchy horizon as a factor, namely

$$\frac{\lambda_B}{\lambda_A} \sim e^{-\frac{\Delta u}{\kappa_-}}.$$

Note that κ_- is negative and thus we have exponential increase. This is denoted as a *local blueshift effect*, cf. Figure 2.2.

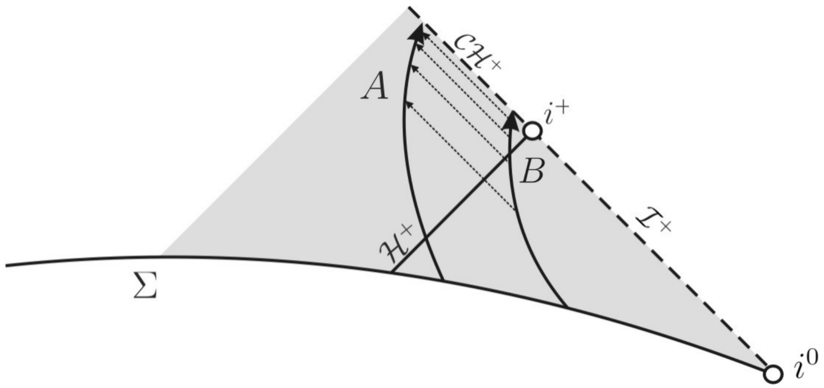


Figure 2.2: Penrose diagram showing observer B sending signals to observer A.

Chapter 3

Linear perturbations under symmetry

Having explained the underlying stability and instability mechanism in the previous chapter we will now carry on by explaining the models in which the effect of these mechanisms can be investigated. One of the simplest linear problems on black hole backgrounds is the analysis of solutions to the massless scalar wave equation

$$\Box_g \psi = 0, \tag{3.1}$$

where g is the background metric. Equation (3.1) can be considered a "poor man's" linearization of the Einstein field equations, in which not only the lower order terms but also the tensorial structure of the equations are dismissed. Therefore, analysis of properties of the solutions ψ to (3.1) is the precursor to full linear and finally non-linear investigations of the Einstein field equations. Moreover, heuristic arguments, cf. Section 4.2.1, indicate that the stability mechanism may indeed already be comprised in linear order, which renders analysis of (3.1) an important problem. Analogous to the Cauchy problem in general relativity described in the introduction we can determine the entire solution to (3.1) from initial data on a hypersurface and evolving it in time. The initial data for the Cauchy problem of (3.1) is prescribed by the value of ψ and $\partial_t \psi$ on a spacelike $t = constant$ hypersurface, where t is a standard timelike coordinate. In the following we will see what conclusions for solutions of (3.1) in the exterior as well as in the interior regions were reached in the 70's already.

3.1 Spherically symmetric wave equation on *exterior* black hole backgrounds

It was shown by Price, [83], that the gravitational radiation produced during collapse will decay like v^{-6}, (where v is the advanced Eddington-Finkelstein time), which is due to the fact that gravitational radiation will be dominated by the quadrupol modes. More precisely, generically the decay rate along constant r hypersurfaces is $2l+2$ for nonvanishing initial static perturbation, with l the multipolar number. If there is no initial static l pole but one develops during collapse, which is to say we assume compact support on the initial data, then it will fall of with a decay rate $2l+3$. The result was later confirmed by Dafermos and Rodnianski [29] remarkably for a non-linear setting in spherical symmetry, cf. Section 4.2.2, and also by Tataru [91] and Metcalfe et al. [67] under stronger assumptions of the spacetime geometry but proving a sharper decay-rate for the linear problem.

The proven inverse power law with advanced time can be considered as remaining tail after the collapse has settled down to a black hole. Therefore, the perturbation of interest for evolution stage 2, see (1.3) of the introduction, can be considered small. Further, this fact that generic collapse leads to decaying tails along the event horizon implies that the exterior field is characterized merely by mass, charge and angular momentum. The characterization of black holes by just these three parameters is known as the famous *no-hair theorem*.

3.2 Spherically symmetric wave equation on *interior* black hole backgrounds

In [66] McNamara investigates the qualitative behavior of scalar field perturbations under spherical symmetry between and along event and Cauchy horizon. In particular, since the perturbing field is not defined on the Cauchy horizon he questions whether the field can be extended in a non-singular manner along $\mathcal{CH}^+$. He has answered part of the question in his previous paper [65], by showing that for fixed modes certain partial derivatives of the field cannot be extended when considering massless scalar fields on Reissner-Nordström interior backgrounds. To be more specific, there exist perturbations with polynomial decay along $\mathcal{H}^+$ that have an unbounded partial derivative along $\mathcal{CH}^+$. This instability result depends on the global structure of the underlying spacetime, namely its causal properties as well as the global symmetries, but not on the nature of the perturbing field and can be stated as follows.

Theorem 3.2.1. *("Poor man's" linear instability under symmetry, McNamara, [65]). For solutions ψ of* (3.1) *on Reissner-Nordström backgrounds with fixed individual modes ψ_l there exists initial data on $\mathcal{I}^-$ such that the partial derivative $\partial_v \psi_l$ in the neighborhood of the bifurcate two-sphere on $\mathcal{CH}^+$ is unbounded.*

Further, McNamara was able to prove the following.

Theorem 3.2.2. *("Poor man's" linear stability under symmetry, McNamara, [66]). Given bounded initial data on $\mathcal{I}^-$ each individual mode ψ_l of solutions ψ of* (3.1) *on Reissner-Nordström backgrounds remains bounded on $\mathcal{CH}^+$.*

Concluding from less rigorous analysis Gürsel et al. [46, 47] and Chandrasekhar-Hartle [14] suggest similar singular scenarious. In particular, the latter conclude that waves arising from exterior perturbations will cross the event horizon directly and after secondary scattering by the spacetime's curvature. The fluxes of radiation (derived from fixed mode analysis) measured by an observer crossing $\mathcal{CH}^+$ therefore turn out infinite in the Chandrasekhar-Hartle setup. Moreover, Ori developed a strategy for linear considerations of the wave equation but since this non-rigorous investigation extends to non-linear models we will postpone the discussion to Section 4.2.1.

Chapter **4**

Non-linear perturbations under symmetry

In this chapter we will review two different approaches to non-linear perturbations under symmetry. In Section 4.1 we will discuss null-fluid models which were constructed to gain deeper insight into the consequences of the blueshift effect close to the Cauchy horizon. Section 4.2 deals with scalar field perturbations in the exterior as well as the interior regions.

4.1 Null-fluids for *interior* investigations

4.1.1 Infalling uncharged null fluid

In 1981 Hiscock [49] made a first attempt to model the backreaction of the infalling power-law tail, cf. discussion of Section 3.1, of radiation on the geometry and in particular on the regularity of the Cauchy horizon by investigating the Reissner-Nordström-Vaidya metric. This metric constitutes an exact spherically symmetric solution to the Einstein-Maxwell equations with an infalling uncharged null fluid which mimics the classical instability in a self-consistent way. In (v, r, φ, θ) coordinates, where v is the advanced Eddington-Finkelstein coordinate, the metric is given by

$$g = -\left(1 - \frac{2M(v)}{r} + \frac{e^2}{r^2}\right) \mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}r + r^2(\sin^2\theta \mathrm{d}\varphi^2 + \mathrm{d}\theta^2), \tag{4.1}$$

where e is the charge of the black hole and v is gauged so that $v = \infty$ at $\mathcal{CH}^+$. The function $M(v)$ is chosen such that for small enough values of v it is constant and represents the mass of an ordinary Reissner-Nordström black hole. For bigger v the function $M(v)$ inherits the inverse power law decay exponent predicted by Price [83].

Within this setup it is immediate that the scalar curvature invariants are finite for all values of v including $v = \infty$, approaching $\mathcal{CH}^+$. To gain better insight on the expected singular behavior at $\mathcal{CH}^+$, Hiscock considers the local tidal forces experienced by a radially infalling observer. As can be seen from the geodesic deviation equation, the tidal forces are encapsulated by the curvature contracted with the four velocities and the relative acceleration of the neighbouring geodesics. For the details of the calculation see [49]. In fact, from the more complicated models it will become apparent that the tidal forces increase with increase of the mass parameter which therefore has an important local meaning. Since obtaining an exact analytic form of the four velocities is difficult, a sequence of inequalities is used to show divergence of the tidal forces measured in a parallelly propagated frame as v approaches ∞.

Interestingly, in this model with the estimates made, the divergence does not hold for extremality $|e| = \tilde{M}$, where $\tilde{M} > M(v)$ is the mass of a Reissner-Nordström black hole used for achieving the required inequalities[1]. This already indicated that it could be worthwhile to look for certain stability properties on extremal black hole spacetimes. Later investigations of Aretakis [3–5] confirmed that specific stability properties can be proven as we will discuss in the end of Section 5.1. Hiscock states further that the tidal forces integrated along the path of the infalling observer also diverge which eliminates the possibility that the effect is a mere discontinuity or shock in the test field. Since clearly there exist a frame in which the curvature tensors are well behaved when approaching $\mathcal{CH}^+$, the occurring singularity at $\mathcal{CH}^+$ was denoted as *nonscalar-polynomial curvature singularity*, *intermediate singularity* or *whimper*. As the name suggests *whimpers* are of much weaker nature than the well known Schwarzschild singularity, at which the Lorentzian manifold ceases to be meaningful.

We have seen that within this simple model we can gain good insights into behavior of the tidal forces and the strength of the singularity. As explained in the introduction, the blueshift effect leads to infinite energy densities close to $\mathcal{CH}^+$. In some models this becomes manifest via certain quantities associated to the mass that can be shown to

[1] The reader might expect a specific physical interpretation clarifying the meaning of the black hole of mass $\tilde{M}$ within the given setting, but it's mass has to be understood merely as a tool for deriving the desired mathematical inequalities.

blow up. The mass inflation effect at $\mathcal{CH}^+$ is not visible in Hiscock's model. Therefore, in the next section we will have a closer look into models accounting for it.

4.1.2 In- and outgoing uncharged null fluids accounting for mass inflation

Another null fluid model was studied by Israel and Poisson [81, 82] in 1989, motivated by the aim to analyze the perturbations beyond linear order and interpreting how their growth deforms the background geometry. From this analysis they intended to answer the question whether the expected singularity at $\mathcal{CH}^+$ could be shown to be strong enough to effectively stop the evolution of spacetime at the Cauchy horizon. An affirmative result would destroy extendibility and thus conveniently remove the problem of non-uniqueness beyond $\mathcal{CH}^+$.

It is emphasized throughout the paper [82] that the mass inflation phenomenon depends on two underlying effects:

1. The presence of a highly blueshifted influx.
2. A separation between the Cauchy and the inner apparent horizon caused by an arbitrarily small outflux.

Therefore, in order to analyze the perturbations they mathematically formulated the most simple model such that these two mechanisms are kept. That is, they investigate spherically symmetric models in which the black hole is initially described by a Reissner-Nordström black hole. Further, the gravitational radiation is modeled by an ingoing null flux intersecting an outgoing flux while not interacting with it. This is reasonable since due to the infinite blueshift the high frequency modes will dominate close to $\mathcal{CH}^+$ and thus the geometric optics approximation can be applied.

The idea of in- and outgoing null fluxes was inspired by earlier investigations of Dray, 't Hooft [38] and independently Redmount [84] within Schwarzschild black holes. In particular, in Dray and 't Hooft's work from 1985 they investigated in- and outfalling colliding spherical thin shells within Schwarzschild black holes in order to gain deeper insight into quantum gravitational effects. The in- and outgoing shell separate the spacetime into four sectors of different mass functions which can be related via the product of metric coefficients for each sector evaluated at the intersection point. Using a generalization of this relation Poisson, and Israel showed that the mass difference along the infalling flux blows up. In fact the mass inflation effect is exponential in advanced time, which is infinite at $\mathcal{CH}^+$.

Note that in their model of two crossing flows the explicit solution is not known. Therefore, it would be hard to draw more specific conclusions on the fate of an observer trying to cross the Cauchy horizon and on extendibility of the spacetime itself.

Extendibility of the modeled spacetime

These questions were investigated further in 1991 by Ori [71]. Inspired by [81, 82] he constructed an explicit mass-inflation solution by gluing two Vaidya solutions along an outgoing null hypersurface. This construction constitutes a very short outgoing pulse as a null layer of energy with vanishing thickness. As a matching condition of the two Vaidya solutions Ori requires that the metric tensor is continuous along the connecting null hypersurface. To model an electrically neutral and pressureless outflux, the null layer has to have vanishing charge and vanishing surface tension[2].

This setup implies that the charge is the same in both Vaidya regions. Further, the affine parameter along the connecting null hypersurface is the same. Ori then derives suitable matching equations across the two regions in explicit form. These equations prescribe the mass $m(\lambda)$, the advanced time $v(\lambda)$ and the function $z(\lambda) = \frac{R}{\partial_\lambda v}$, where $R(\lambda)$ gives back the value of r along the outgoing null hypersurface. Each of these functions are dependent on the affine parameter λ. The problem is then fully determined once the function for the value of r along the outgoing null hypersurface and the integration constants are known. Making a suitable assumption for the radiative tail the value of the functions can be approximated at $\mathcal{CH}^+$.

Like in the work of Poisson and Israel the mass can be shown to grow almost exponentially in advanced time. Subsequent investigation of the properties of the mass-inflation singularity reveals that in Poisson and Israel's model some curvature scalars do blow up. Considering the tidal forces Ori was able to show that the distortion is finite in all three directions and thus the singularity has to be considered weak, with a C^0 metric. The possibility of classically extending this particular spacetime can therefore not be excluded.

Furthermore, Ori points out that from purely classical investigations it cannot be concluded how to extend the spacetime beyond $\mathcal{CH}^+$. Instead he suggests that this should follow from a more fundamental Quantum Gravitational theory. To be more specific from such a theory it should become evident if there are classical extensions, what is the right continuation or what other kind of existence could be there beyond $\mathcal{CH}^+$. It could then be possible that a quantum state along $\mathcal{CH}^+$ would bridge two classical states, namely

[2]That is to say the induced stress energy tensor has vanishing non-radial components.

the interior of the black hole and the universe beyond the Cauchy horizon. Ori's result suggests that the Strong Cosmic Censorship may not be valid in its originally formulated sense that no observer could cross $\mathcal{CH}^+$. The question how generic the mass-inflation effect is and whether it is limited to spherically symmetric solutions remains open from this work but was tackled by Ori in 1992, see [72], where he extends his result to rotating black holes. We will discuss this further in the following section.

4.2 Scalar field perturbations

In the first part of this section about scalar field perturbations we will discuss some early heuristic arguments given by Ori, which gave –despite of being non-rigorous– a good indication what to expect for scalar wave solutions in the interior. The second part then reviews mathematically precise results derived within an Einstein-Maxwell-scalar field system.

4.2.1 Heuristic arguments for scalar field perturbations on black hole *interiors*

As mentioned in Section 3.2, Ori developed a heuristic method of dealing with linear perturbations that extends to the analyses of the more realistic case of rotating black holes and in a very non-rigorous sense to non-linear, non-symmetric metric perturbations. He calls this method the *late-time-expansion*. For the non-linear, non-symmetric and spinning case his considerations suggest that the singularity remains weak at the early portion of $\mathcal{CH}^+$, so by the time the curvature blows up the deformation arising from the tidal forces is still not strong enough to destroy or even damage a physical object at a sufficiently early portion of the weak singular remaining part of $\mathcal{CH}^+$. Although Ori did already present his proposal for non-linear perturbations of spinning black holes in 1992, [72], in a later series of papers [73]-[77] he gives more details of his strategy.

The primary goal of Ori's work was to investigate the structure of the inner horizon singularity. In order to do this he differentiates between the two stages of evolution that we already mentioned in the introduction, see (1.3). The first paper of this series [73] aims to qualitatively explain the stage 1 of evolution and suggests that the perturbations propagated further inside during stage 2 of evolution can be considered small.

Since Ori aimed to consider a characteristic problem, with data along the event horizon and a transverse null segment, he required a generalization of Price's law [83] from Schwarzschild to Kerr which constitutes a non-trivial task and was approached by Barack and him with non-rigorous methods in [6]. Their work indicated that it is reasonable to assume the same late-time asymptotic behavior along $\mathcal{H}^+$ for the Kerr case as was found for Schwarzschild.

In the latter papers [74]-[77] the perturbations are then assumed to be small and explicit examples of perturbative toy models –gradually extending from linear and symmetric models to the non-linear, non-symmetric and spinning case– for the black hole interior are presented.

Ori's method does not make use of Fourier decomposition but of separation into spherical harmonics. The complication for the extension to Kerr is then that the spheroidal harmonics depend on the temporal frequency. He resolves this by first expanding for late times and then separating the angular dependence. Why this strategy extends to non-linear perturbations can be understood as follows. Ori calls a perturbation of the form

$$\psi \approx \sum_{k=0}^{\infty} \psi_k(r,\theta,\varphi) t^{-(2l+2+k)}. \tag{4.2}$$

semi-stationary. Since the first-order linear perturbation is semi-stationary it follows that the source term of the second-order perturbation is semi-stationary which renders the second-order perturbation itself semi-stationary. Subsequently applied to each orders this means that non-linear perturbations of all orders are semi-stationary. Unfortunately expansion (4.2) breaks down at $\mathcal{CH}^+$ since the functions ψ_k diverge at the inner horizon although ψ remains finite. In order to overcome this problem Ori applies another expansion in small parameters of $\frac{1}{u}$ and $\frac{1}{v}$. Matching both yields

$$\psi \approx \sum_{k=0}^{\infty} \left[A_k(\theta) u^{-(2l+1+k)} + B_k(\theta) v^{-(2l+1+k)} \right], \tag{4.3}$$

for axially symmetric perturbations.

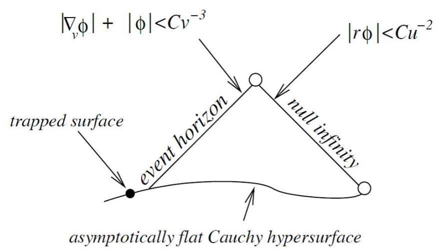
He was also able to show that the metric perturbations are regular for late times, implying that the singularity is weak and the tidal distortion is bounded. Although the derivation given in [72]-[77] is not rigorous –due to (amongst other reasons) the lack of proof of convergence in the expansion– it provided an intuitively clear picture paving the way for more rigorous mathematical analyzes.

4.2.2 Partial Differential Equation analysis of scalar field perturbations

Exterior scalar field perturbations

As already explained in Section 3.1, heuristic arguments led Price to formulate a polynomial decay law for the advanced time along the event horizon. Since his arguments were based on Fourier analysis and spectral theory they were not robust enough to carry over to non-linear considerations. The extension to non-linear analysis was then achieved by Dafermos and Rodnianski via using properties of the global geometry, the redshift effect and local energy conservation. Their result is given in the following theorem.

Theorem 4.2.1. *(Price's Law for Einstein-Maxwell-scalar field equations, Dafermos and Rodnianski, [29]). Consider spherically symmetric asymptotically flat initial data for the Einstein-Maxwell-scalar field equations, where the scalar field and its gradient are initially of compact support, and assume the data contains a trapped surface*[3]*. Then the maximal development of initial data contains a domain of outer communications possessing a complete future null infinity, as depicted: Defining a natural advanced time*



coordinate v on the event horizon, and a retarded time coordinate u on null infinity, then the decay rates as depicted above hold.

For sharper decay rates under stronger assumptions on the spacetime geometry see [67, 91].

[3]By a *trapped surface* we mean the following, consider a sphere $\mathcal{S}$ surrounding a mass from which future-directed in-and outgoing null geodesics are emerging orthogonally. At a later time t the wave fronts form surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ from the in-and outgoing null geodesics, respectively. Usually the area of $\mathcal{S}_1$ from the ingoing wave should be smaller than the area of the initial sphere $\mathcal{S}$ while the area of $\mathcal{S}_2$ from the outgoing wave will be bigger. For a trapped region both arising areas will be smaller or equal than the initial one, given that gravity remains attractive.

***Interior* scalar field perturbations**

A more rigorous interior analysis by Dafermos [23] but in a simpler model is concordant with Ori's result and described in the following. Considering the *spherically symmetric* Einstein-Maxwell-scalar field equations as a toy model, Dafermos proved that the solution indeed exists up to the Cauchy horizon and moreover is extendible as a C^0 metric but generically fails to be extendible as a C^1 metric beyond $\mathcal{CH}^+$, cf. [22, 23]. To be more specific as summarized in [25] the following theorem was proven.

Theorem 4.2.2. *(C^0-stability of a piece of the Cauchy horizon, Dafermos, [22, 23]). For all two-ended asymptotically flat spherically symmetric initial data for spherically symmetric Einstein-Maxwell-scalar field system with non-vanishing charge, the maximal development can be extended through a non-empty Cauchy horizon $\mathcal{CH}^+$ as a spacetime with C^0 metric.*

Remark. This statement includes the possibility that the future boundary of the spacetime may contain non-empty spacelike pieces. In order to prove his statement Dafermos used upper polynomial decay rates for the scalar field ψ along $\mathcal{H}^+$ obtained from *exterior* analysis in collaboration with Rodnianski [29] as stated in Theorem 4.2.1. Further, Dafermos makes a more specific statement about the nature of the evolving singularity.

Theorem 4.2.3. *(Weak null singularities, Dafermos, [23]). For spherically symmetric initial data as above where a pointwise lower bound on $\partial_v \psi$ is assumed to hold asymptotically along the event horizon $\mathcal{H}^+$ that forms, then the above Cauchy horizon $\mathcal{CH}^+$ is singular: The Hawking mass[4] (thus the curvature) diverges and, moreover, the extension of Theorem 4.2.2 fails to have locally square integrable Christoffel symbols.*

Since the metric extends continuously while the Christoffel symbols fail to be square integrable, the singularity is considered a weak singularity. This result is in agreement with the conclusions drawn from the null-fluid models [49, 71, 82], that we explained in Section 4.1 of this chapter as well as the numerical studies [11, 12]. Note further, that the special case of small perturbations of Reissner-Nordström spacetime is included in the above theorems but as proven in [24] this case does not contain a spacelike part and instead the singular Cauchy horizon pieces merge in a bifurcate two-sphere. Kommemi extended Dafermos' model further to Einstein-Maxwell-*charged* scalar fields, [57], and Costa et al. extended it further for Einstein-Maxwell-scalar fields with cosmological

[4] In spherical symmetry the Hawking mass takes the simple form $m = \frac{r}{2}\left[1 - g(\nabla r, \nabla r)\right]$.

constant [19–21].

Chapter 5

Linear perturbations without symmetry

In this chapter we will discuss solutions to the scalar wave equation as a toy model for the Einstein field equation. In particular, in Section 5.2 the work explained in detail in part II of the thesis is put into context to previous and latter investigations.

5.1 Wave equation on *exterior* black hole backgrounds

As already explained in Chapter 3, the first step towards linear stability investigations is the "poor man's" linear analysis, namely analysis of (3.1). Regge and Wheeler first derived mode stability within Schwarzschild spacetime for non-spherical perturbations and concluded that the Schwarzschild solution is stable against small departures from sphericity in the exterior. For solutions of (3.1) on Schwarzschild background, uniform boundedness of ψ in the exterior up to and including $\mathcal{H}^+$ was first proven by Wald and Kay-Wald [51, 94]. They used classic application of vector field commutators and multipliers, together with elliptic estimates and the Sobolev inequality. For a survey on the vector field method see Klainerman's [56] and find the original introduction in [52–55]. Moreover, the vector field method will be introduced in Section 10.1. The result of Kay-Wald was further improved from proving not just boundedness but also decay and extended from Schwarzschild backgrounds to first slow and then fast subextremal Kerr backgrounds. Various researchers have contributed to this progress, [29], [9], [30], [10], [40], [31], [92], [1], which eventually led to the following statement.

Theorem 5.1.1. *("Poor man's" linear stability of Kerr, Dafermos et al., [33, 36]). For Kerr exterior backgrounds in the full subextremal range $|a| < M$, general solutions ψ of* (3.1) *arising from regular localised data remain bounded and decay at a sufficiently fast polynomial rate through a hyperboloidal foliation of spacetime.*

Let us briefly mention that in order to prove the above theorem there are three main features of subextremal Kerr spacetime that enter the analysis, namely the *redshift effect*[1] close to the horizon, *superradiance*[2] and *trapping* of null geodesics[3]. In particular, the three properties are coupled to each other and thus have to be treated in a specific way. For slow Kerr $|a| \ll M$ it can be inferred from continuity –coming from Schwarzschild and adding small angular momentum– that the superradiant part is not trapped. In fact it even disperses which can be shown via the redshift effect when $|a|$ is small. Suprinsingly, it turns out that for fast Kerr $|a| < M$ the superradiant part remains not trapped which is a key ingredient for the proof.

For results on mode stability on Kerr background see the work of Whiting, remarkably already from 1989, [95] and the later but more rigorous investigation of Shlapentokh-Rothman [89].

Similar investigations of (3.1) on extremal black holes spacetimes were carried out. In particular, Aretakis proved stability and instability properties for the evolution of a massless scalar field on a fixed extremal Reissner-Nordström exterior background, [3]. He showed that first transverse derivatives of ψ generically *do not decay* along the event horizon for late times. He further proved that higher derivatives *blow up* along the event horizon. Aretakis has further considered (3.1) on a fixed extremal Kerr background, [4]. Analogously to the extremal Reissner-Nordström case, he showed decay up to and including the event horizon for axialsymmetric solutions ψ and in [5] showed instability properties when considering transverse derivatives. The occurring instability properties can be understood by recalling that in subextremal black holes the redshift effect close to $\mathcal{H}^+$ acts as a stability mechanism while the blueshift effect at $\mathcal{CH}^+$ acts as an instability

[1] See Section 2.

[2] Consider an infalling wave penetrating the ergoregion of a rotating black hole which is partly absorbed and partly be scattered off this potential barrier of the black hole. Naively the amplitude of the scattered wave would be expected to be smaller than the initial wave amplitude. Investigations in rotating black holes have shown that this is not neccessarily the case and the scattered wave can even be amplified through the energy of the black hole. This phenomenon is called *superradiance*. For further explanation see [44] and references therein.

[3] A well known example for trapped null geodesics is the photon sphere arising in Schwarzschild spacetime outside the event horizon for the radius $r = 3M$. Kerr spacetime reveals several such orbits along which the photons are forced to travel due to the strength of the gravitational force. Note that, although the name suggests different, this phenomenon should not be confused with the notion of trapped surfaces.

mechanism. In the extremal case $\mathcal{H}^+$ and $\mathcal{CH}^+$ coincide and both effects are canceled so that from a purely classical point of view it may a priori not be clear what to expect. Aretakis' work has been further generalized by Lucietti et al. in [59, 60].

Now going from "poor man's" linear stability and instability results to full linear considerations the following theorem was proven for Schwarzschild background.

Theorem 5.1.2. *(Full linear stability of Schwarzschild, Dafermos et al., [27]). Solutions for the linearisation of the Einstein equations around Schwarzschild arising from regular admissible*[4] *data remain bounded in the exterior and decay (with respect to a hyperboloidal foliation) to a linearised Kerr solution.*

5.2 Wave equation on the *interior* black hole background

The result stated in Theorem 4.2.1 and Theorem 5.1.1 can be used to propagate decay estimates further inside the interior of Reissner-Nordström and Kerr black holes in order to investigate stability and instability properties up to the Cauchy horizon. Weigthed energy[5] estimates, commutation with angular momentum operators[6] and Sobolev embedding lead to the following stability statement.

Theorem 5.2.1. *("Poor man's" linear stability without symmetries, [42] and [43]). On subextremal Kerr spacetime* $(\mathcal{M}, g)$*, with mass* M *and angular momentum per unit mass* a *and* $M > |a| \neq 0$*, (or subextremal Reissner-Nordström spacetime* $(\mathcal{M}, g)$*, with mass* M *and charge* e *and* $M > |e| \neq 0$*), let* ψ *be a solution of* (3.1) *arising from smooth compactly supported Cauchy data on a two-ended asymptotically flat Cauchy surface* Σ*. Then*

$$|\psi| \leq C \tag{5.1}$$

globally in the black hole interior, in particular up to and including the Cauchy horizon $\mathcal{CH}^+$*, to which* ψ *extends in fact continuously.*

[4]For Schwarzschild this means that the Cauchy hypersurface does not intersect the white hole of the solution.

[5]By energy here we simply mean the integral of squared first derivatives of ψ on a specific hypersurface arising from the scalar field energy-momentum-tensor contracted with a suitable multiplier. The naming *energy* should not be misinterpreted as an energy that a local observer would measure. For a more detailed explanation see Section 10.1.

[6]For a more detailed explanation of the angular momentum operators see Section 10.3.2.

The crux of the proof given in [42] is the analysis of a characteristic rectangle Ξ in the neighborhood of timelike infinity, with one past boundary along $\mathcal{H}^+$ and one future boundary along $\mathcal{CH}^+$, as will be explained in detail in part II of the thesis. Further, in order to derive statement (5.1) the characteristic rectangle Ξ has to be separated into different regions with spacelike future boundaries along which we propagated the decay-rate further inside, see Section 9.2. This was done by exploiting specific properties of each region and choosing suitable vector fields accordingly. The analysis of [43] works along the same lines together with certain strategies employed to account for the lack of spherical symmetry of the background geometry.

In further investigations of (3.1) on fixed subextremal Reissner-Nordström backgrounds, Luk and Oh were able to prove certain instability properties along $\mathcal{CH}^+$.

Theorem 5.2.2. *("Poor man's" linear instability without symmetries, Luk and Oh, [64]). Generic smooth and compactly supported initial data to* (3.1) *on a two-ended asymptotically flat Cauchy surface* Σ *give rise to solutions that are not in* $W^{1,2}_{loc}$ *in a neighborhood of any point on the future Cauchy horizon* $\mathcal{CH}^{+}$[7].

They showed that due to the blueshift effect generic smooth and compactly supported initial data on a Cauchy hypersurface indeed give rise to solutions with infinite non-degenerate[8] energy near the Cauchy horizon in the interior of the black hole. In particular, the solution generically does not belong to $W^{1,2}_{\mathrm{loc}}$. A version of the Strong Cosmic Censorship Conjecture demanding that solutions should be inextendible in $W^{1,2}_{\mathrm{loc}}$ beyond $\mathcal{CH}^+$, cf. (1.2), would therefore imply that generically the Lorentzian manifold cannot be extended such that the Einstein equations are still satisfied. This was already pointed out in [18] by Christodoulou.

For investigations on fixed extremal *interior* black hole spacetimes see also work of Gajic [45].

[7] A function ψ belonging to the Sobolev space $W^{1,2}_{\mathrm{loc}}$ would have the properties that locally ψ and all of its first weak derivatives exist and are square integrable. Taking (3.1) seriously as a model for the full Einstein field equations and therefore considering ψ as an agent for the metric two tensor g, the result of Luk and Oh suggests that in the full theory the Christoffel symbols would fail to be square integrable. For an introduction to Sobolev spaces refer for example to [39], [93] or to Appendix B.

[8] By non-degeneracy we mean that the multiplier is constructed such that it does not become null on the hypersurfaces of interest.

Chapter 6

Non-linear perturbations without symmetry

In this chapter we will review the recently achieved remarkable non-linear result in the interior. The non-linear analyzes without symmetry of the exterior is still one of the most important open problems and its discussion is therefore postponed to Chapter 7.

6.1 Non-linear perturbations without symmetry on *interior* backgrounds

The motivation of analyzing all the above described models is to gain a better understanding of the dynamics of the full Einstein field equations. In this review we are particularly interested in the stability and instability properties of the Cauchy horizon that will be predicted from the full non-linear equations. A first non-linear step toward investigation of null singularities in vacuum spacetime was achieved by Luk. In [63] he managed to construct examples of local patches of vacuum spacetimes with weak null singular boundary. In subsequent work of Dafermos and Luk [28] it was shown that fully non-linear perturbations of Kerr spacetime are not expected to form a strong spacelike singularity but instead singular behaviour along $\mathcal{CH}^+$ is revealed. Putting together the investigations described in Sections 4.2.2 and 5.2 they were able to prove the following Theorem as already stated in [25].

Theorem 6.1.1. *(Global stability of the Kerr Cauchy horizon, Dafermos and Luk, [28]). Consider characteristic initial data for* (1.1) *on a bifurcate null hypersurface* $\mathcal{H}^+ \cup \mathcal{H}^+$*, where* $\mathcal{H}^\pm$ *have future-affine complete null generators and their induced geometry dynamically approaches that of the event horizon of Kerr with* $0 < |a| < M$ *at a sufficiently fast polynomial rate. Then the maximal development can be extended beyond a bifurcate Cauchy horizon* $\mathcal{CH}^+$ *as a Lorentzian manifold with* C^0 *metric. All finitely-living observers pass into the extension.*

As already mentioned exterior Kerr stability is still an open problem, which we will discuss more in Section 7.1. The proof of Conjecture 7.1.1 together with the above Theorem 6.1.1 would result in the conclusion that the Cauchy horizon of Kerr spacetime is globally stable and a C^0-formulation of the Strong Cosmic Censorship Conjecture would be false. As suggested from the "poor man's" linear analysis, see Theorem 5.2.2 a possible formulation could be that the spacetime be inextendible as a Lorentzian manifold with locally square integrable Christoffel symbols.

Chapter 7

Open problems

We have discussed Kerr Cauchy horizon stability in the last chapter. An important prerequisite to verify stability in the interior is however to first derive conclusions about the exterior. In the following we will clarify what exact statements are desired to be proven. Further, we will briefly discuss open aspects of extendibility of the spacetime.

7.1 Non-linear perturbations without symmetry on *exterior* backgrounds

In fact the significance of Theorem 6.1.1 depends on exterior Kerr stability since it was used as an assumption for proving the C^0-stability of the Kerr Cauchy horizon. Investigation of (1.1) in the neighborhood of the Kerr family in the exterior region is one of the central open questions of general relativity. The importance of this questions lies in the current working assumption that realistic astrophysical objects can be described by the Kerr metric, see footnote a of the introduction. Therefore, the following conjecture was proposed in [26] and also elaborated on further in [25].

Conjecture 7.1.1. *(Non-linear stability of the Kerr family). For all vacuum initial data sets* $(\Sigma, \bar{g}, K)$ *sufficiently "near" data*[1] *corresponding to a subextremal* ($|a_0| < M_0$) *Kerr metric* g_{a_0,M_0}*, the maximal vacuum Cauchy development spacetime* (M, g) *satisfies:*

[1] The smallness assumption of the data depends on the difference of the paramters a_0 and M_0 from extremality.

1. *(M, g) possesses a complete null infinity $\mathcal{I}^+$ whose past $J^-(\mathcal{I}^+)$ is bounded in the future by a smooth affine complete event horizon $\mathcal{H}^+ \subset \mathcal{M}$,*

2. *(M, g) stays globally close to g_{a_0,M_0} in $J^-(\mathcal{I}^+)$,*

3. *(M, g) asymptotically settles down in $J^-(\mathcal{I}^+)$ to a nearby subextremal member of the Kerr family g_a, M with parameters $a \approx a_0$ and $M \approx M_0$.*

In the language of partial differential equations statement 1 can be interpreted as *global existence.* Statement 2 represents *orbital stability* and statement 3 comprises *asymptotic stability.*

7.2 Extendibility of non-linearly perturbed interior backgrounds

Given that the Kerr exterior is stable there was great progress in showing stability properties along $\mathcal{CH}^+$. There is still work left to be done in understanding the occurring instability better. If it could be shown that generically non-linearly the instability is such that the Christoffel symbols in the extension would not be locally square integrable, then there would be no weak solutions of (1.1) beyond the Cauchy horizon. The following conjecture suggested in [26] is therefore still open to be proven.

Conjecture 7.2.1. *Given a suitable assumption on the data on $\mathcal{H}^+$ in Theorem 6.1.1, then $\mathcal{CH}^+$ is a weak null singularity, accross which the metric is inextendible as a Lorentzian manifold with locally square integrable Christoffel symbols.*

Validity of the above conjecture would imply that on a purely classical level there are no physically meaningful extensions. Generically there occurs a mechanism taking care of the fate of our astronaut at the Cauchy horizon. Completely understanding this mechanism mathematically and physically remains an open problem.

Chapter 8

Conclusions of Part I

The Cauchy horizon singularity of charged as well as spinning black holes was found to be essentially linear in the sense that the more complicated non-linear investigations equally result in a weak null singularity. This conclusion derives from completely rigorous but purely classical partial differential equation analyzes.

In the "poor" linear toy model of Luk and Oh a strict C^0 formulation of the Strong Cosmic Censorship Conjecture was falsified, see [64]. For non-linear investigations, given initial data which is close to Kerr geometry along the event horizon, it was shown that the solution exists up to and including $\mathcal{CH}^+$ and can be extended with C^0 metric, see [28]. This means that even in the full non-linear analysis a C^0 formulation of the Strong Cosmic Censorship Conjecture cannot hold. The assumption of closeness of data to Kerr geometry along $\mathcal{H}^+$, that was used in the above mentioned investigation, implies that the Kerr exterior is stable. It still remains open to show that this premise is justified, by showing that the induced geometry on null generators in the Kerr exterior approaches the geometry of the event horizon of another Kerr spacetime at a sufficiently fast polynomial rate.

Suppose now, the "poor man's" linear result of Luk and Oh, that solutions cannot be in $W^{1,2}_{\mathrm{loc}}$ would extend to the full Einstein equations. Then, as a minimal requirement for weak solutions to the Einstein field equations the Christoffel symbols should not be square integrable. This implies that weak extensions of the Einstein field equations beyond the maximal domain of dependence would generically not exist. Any possible extension (of lower regularity) would cease to be physically meaningful since notions such as curvature would no longer be well defined.

Recall the questions (1.4) stated in the introduction and let us summarize what answers were found:

(i.) *Stability of exteriors:*
Full non-linear exterior Kerr stability is still an important open problem in general relativity. On a linear level there is good indication in favor of Kerr stability coming from "poor man's" linear investigations, see Theorem 5.1.1. For Schwarzschild exterior full linear stability has successfully been shown, see Theorem 5.1.2.

(ii.) *Instability of Cauchy horizons:*
It was shown "poor" linearly that Reissner-Nordström and Kerr Cauchy horizons exhibit stable and unstable properties. While the solution itself remains bounded, see Theorem 9.1.1, generically the transverse first derivatives were shown to grow infinitely for fixed Reissner-Nordström backgrounds, see Theorem 5.2.2. This result was extended to non-linear investigations yielding stable behavior up to and including $\mathcal{CH}^+$, see Theorem 6.1.1.

(iii.) *Extendibility of spacetimes:*
According to "poor" linear investigations of Luk and Oh, solutions do generically not belong to $W^{1,2}_{\text{loc}}$ close to the Cauchy horizon. Therefore, the solutions cannot be extended with this regularity. It was shown non-linearly for Einstein-Maxwell-scalar fields under symmetry, that C^0 extensions are possible, see Theorem 4.2.2, while the spacetime is inextendable with a C^1 metric, see Theorem 4.2.3. Similarly, non-linear considerations for Kerr spacetime yielded C^0 extendibility, see Theorem 6.1.1. The question for which regularity Kerr is non-linearly inextendible remains presently open. The combination of all reviewed investigations suggests that generically Einstein spacetimes should not be extendible as Lorentzian manifolds with square integrable Christoffel symbols.

We have seen that the instability mechanism is not strong enough to advocate that all physically reasonable spacetimes should always be globally hyperbolic given that we consider C^0 extensions of the metric. However, a $W^{1,2}_{\text{loc}}$-formulation of the "poor" Strong Cosmic Censorship Conjecture could translate to generically not having extensions with square integrable Christoffel symbols on a full non-linear level. This could then restrict classically meaningful and well defined physics to the maximal domain of dependence. The fate of the finitely living observers outside the domain of dependence would remain uncertain. Investigating mechanisms deciding on life and death of an observer traveling beyond the Cauchy horizon is still an important open topic of research.

Part II

Boundedness of massless scalar waves on Reissner-Nordström interior backgrounds

Chapter **9**

Introduction to part II

The Reissner-Nordström spacetime $(\mathcal{M}, g)$ is a fundamental 2-parameter family of solutions to the Einstein field equations coupled to electromagnetism, cf. Figure 1.2 for the conformal representation of the subextremal case, $M > |e| \neq 0$, with e the charge and M the mass of the black hole. Figure 9.1 represents the future maximal development from a Cauchy hypersurface Σ. The problem of analysing the scalar wave equation

$$\Box_g \phi = 0 \tag{9.1}$$

on a Reissner-Nordström background is intimately related to the stability properties of the spacetime itself and to the celebrated Strong Cosmic Censorship Conjecture, as discussed in Chapter 3. The analysis of (9.1)[1] in the exterior region $J^-(\mathcal{I}^+)$ has been

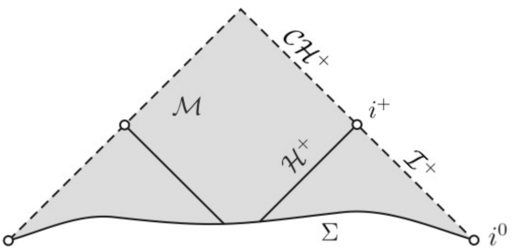


Figure 9.1: Maximal development of Cauchy hypersurface Σ in Reissner-Nordström spacetime $(\mathcal{M}, g)$.

accomplished already, cf. [9] and [35] for an overview and references therein for more details, as well as Sections 5.1 and 11.1. The purpose of part II of this thesis is to extend the investigation to the interior of the black hole, up to and including the Cauchy horizon $\mathcal{CH}^+$.

9.1 Main result

The main result of this thesis can be stated as follows.

Theorem 9.1.1. *On subextremal Reissner-Nordström spacetime $(\mathcal{M}, g)$, with mass M and charge e and $M > |e| \neq 0$, let ϕ be a solution of the wave equation $\Box_g \phi = 0$, arising from smooth compactly supported Cauchy data on a two-ended asymptotically flat Cauchy surface Σ. Then*

$$|\phi| \leq C \tag{9.2}$$

globally in the black hole interior, in particular up to and including the Cauchy horizon $\mathcal{CH}^+$, to which ϕ extends in fact continuously.

The constant C is explicitly computable in terms of parameters e and M and a suitable norm on initial data. The above theorem will follow, after commuting (9.1) with angular momentum operators and applying Sobolev embedding, from the following theorem, expressing weighted energy boundedness.

Theorem 9.1.2. *On subextremal Reissner-Nordström spacetime $(\mathcal{M}, g)$, with mass M and charge e and $M > |e| \neq 0$, let ϕ be a solution of the wave equation $\Box_g \phi = 0$, arising from smooth compactly supported Cauchy data on a two-ended asymptotically flat Cauchy surface Σ. Then*

$$\begin{aligned}
&\int\limits_{\mathbb{S}^2} \int\limits_{v_{fix}}^{\infty} \left[v^p (\partial_v \phi)^2(u, v, \theta, \varphi) + |\nabla\!\!\!\!/\, \phi|^2(u, v, \theta, \varphi) \right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \leq E, \\
&\qquad\qquad\qquad\qquad \textit{for } v_{fix} \geq 1,\ u > -\infty \qquad (9.3) \\
&\int\limits_{\mathbb{S}^2} \int\limits_{u_{fix}}^{\infty} \left[u^p (\partial_u \phi)^2(u, v, \theta, \varphi) + |\nabla\!\!\!\!/\, \phi|^2(u, v, \theta, \varphi) \right] r^2 \mathrm{d}u \mathrm{d}\sigma_{\mathbb{S}^2} \leq E,
\end{aligned}$$

[1] As opposed to ψ satisfying (3.1) on any fixed background, we use the symbol ϕ to indicate that we mean solutions to (9.1) where g is the Reissner-Nordström metric.

$$\textit{for } u_{fix} \geq 1,\ v > -\infty \tag{9.4}$$

where $p > 1$ *is an appropriately chosen constant, and* (u, v) *denote Eddington-Finkelstein coordinates in the black hole interior, where by* $\mathrm{d}\sigma_{\mathbb{S}^2}$ *we denote the volume element of the unit two-sphere and* $|\nabla\!\!\!/ \phi|^2 = \frac{1}{r^2}\left[(\partial_\theta\phi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\phi)^2\right]$.

9.2 A first look at the analysis

The proof of Theorem 9.1.1 and 9.1.2 involves first considering a characteristic rectangle Ξ within the black hole interior, whose future *right* boundary coincides with the Cauchy horizon $\mathcal{CH}^+$ in the vicinity of i^+, cf. Figure 9.2.

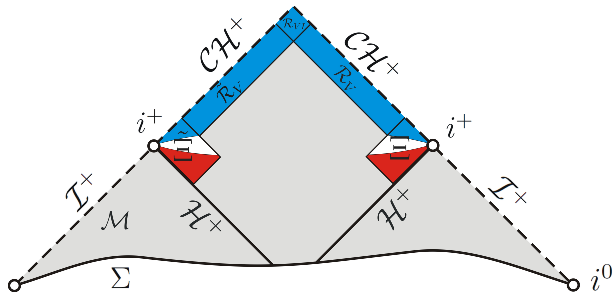


Figure 9.2: Penrose diagram depicting all regions considered in the entire proof.

Establishing boundedness of weighted energy norms in Ξ is the crux of the entire proof. Once that is done, analogous results hold for a characteristic rectangle $\tilde{\Xi}$ to the *left* depicted in Figure 9.2. Hereafter, boundedness of the energy is easily propagated to regions $\mathcal{R}_V$, $\tilde{\mathcal{R}}_V$ and $\mathcal{R}_{VI}$ as depicted, giving Theorem 9.1.2. Commutation by angular momentum operators and application of Sobolev embedding then yields Theorem 9.1.1.

Let us return to the discussion of Ξ since that is the most involved part of the proof.

In order to prove Theorem 9.1.2 (and hence Theorem 9.1.1) restricted to Ξ we will begin with an upper decay bound for $|\phi|$ and its derivatives on the event horizon $\mathcal{H}^+$, which can be deduced by putting together preceding work of Blue-Soffer, cf. [9], Dafermos-Rodnianski, cf. [29] and Schlue, cf. [88]. The precise result from previous work that we shall need will be stated in Section 11.1.

In Ξ the proof involves distinguishing redshift $\mathcal{R}$, noshift $\mathcal{N}$ and blueshift $\mathcal{B}$ regions, as shown in Figure 9.3.

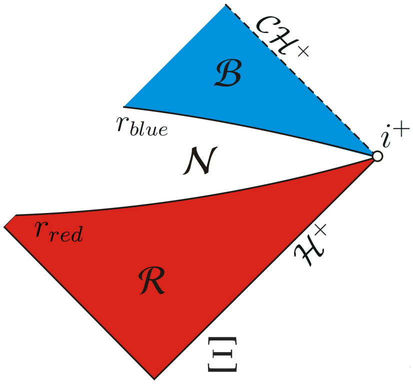


Figure 9.3: Conformal representation of a characteristic regtangle Ξ with redshift $\mathcal{R}$, noshift $\mathcal{N}$ and blueshift regions $\mathcal{B}$.

Some of these regions have appeared in previous analysis of the wave equation, especially $\mathcal{R} = \{r_{red} \leq r \leq r_+\}$. Region $\mathcal{N} = \{r_{blue} \leq r \leq r_{red}\}$ and region $\mathcal{B} = \{r_- \leq r \leq r_{blue}\}$ were studied in [23] in the *spherically symmetric self-gravitating* case, but using techniques which are very special to $1+1$ dimensional hyperbolic equations.[2] We will discuss this separation into $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{B}$ regions further in Section 11.2. One of the main analytic novelties of this thesis is the introduction of a new vector field energy identity constructed for analyses in region $\mathcal{B}$. In particular, the weighted vector field is given in Eddington-Finkelstein coordinates (u, v) by

$$S = |u|^p \partial_u + v^p \partial_v,$$

for $p > 1$ as appearing in Theorem 9.1.2. This vector field associated to region $\mathcal{B}$ will allow to prove uniform boundedness despite the blueshift instability.

9.3 Outline of part II

Part II of this thesis is organized as follows.

In the remaining Section 9.4 of the introduction we will briefly summarize aspects given in part I of the thesis about Strong Cosmic Censorship and its relation to this work.

[2]Let us note that the result of Theorem 9.1.1 for *spherically symmetric* solutions ϕ can be obtained by specializing [23, 24, 29] to the uncoupled case. Restricted results for fixed spherical harmonics can be in principle also inferred from [66].

In Section 10 we introduce the basic tools needed to derive energy estimates from the energy momentum tensor associated to (9.1) and an appropriate vector field. A review of the Reissner-Nordström solution and the coordinates used in this part of the thesis will be given. Moreover, we will discuss further features of Reissner-Nordström geometry.

In Section 11.1 we give a brief review of estimates obtained along $\mathcal{H}^+$ from previous work, [9], [29] and [88], for ϕ arising from sufficiently regular initial data on a Cauchy hypersurface. This is stated as Theorem 11.1.1. Section 11.2 states our main result specialized to the rectangle Ξ (see Theorem 11.2.1) and gives an outline of its proof. The investigation is divided into considerations within the redshift $\mathcal{R}$, noshift $\mathcal{N}$, and blueshift $\mathcal{B}$ regions.

The decay bound for the energy flux of ϕ given on the event horizon $\mathcal{H}^+$, cf. Theorem 11.1.1, will be propagated through the redshift region $\mathcal{R}$ up to the hypersurface $r = r_{red}$ in Section 12.1.1.

Thereafter, in Section 12.1.2 we propagate the decay bound further into the black hole interior through the noshift region $\mathcal{N}$ up to the hypersurface $r = r_{blue}$.

In Section 12.1.3 a decay bound for the energy flux of ϕ is proven on a well chosen hypersurface γ that separates the blueshift region into a region in the past of γ, $J^-(\gamma)\cap\mathcal{B}$, and a region to the future of γ, $J^+(\gamma) \cap \mathcal{B}$. In Section 12.1.4 we will derive pointwise estimates on Ω^2 to the future of γ (in particular implying finiteness of the spacetime volume, $\mathrm{Vol}(J^+(\gamma)) < C$). This will allow us to propagate our estimates into $J^+(\gamma) \cap \mathcal{B}$ up to $\mathcal{CH}^+$, yielding finally Corollary 12.1.13.

Section 12.2 reveals how commutation with angular momentum operators and applying Sobolev embedding will return us pointwise boundedness for $|\phi|$. The necessary higher order boundedness statement is given in Theorem 12.2.1. This completes the proof of Theorem 11.2.1.

We now must extend our result to the full interior region.

In Section 12.3 we will state the analog of Theorem 11.2.1 restricted to the rectangle $\tilde{\Xi}$ to the *left*. In Section 12.4 and Section 12.5 we propagate the energy estimates further along $\mathcal{CH}^+$ in the depicted regions $\mathcal{R}_V$ and $\tilde{\mathcal{R}}_V$. Eventually, in Section 12.6 we propagate the estimate to the region $\mathcal{R}_{VI}$ up to the bifurcation two-sphere, and thus obtain a bound for the energy flux globally in the black hole interior (see Corollary 12.6.3) completing the proof of Theorem 9.1.2.

In Section 12.7 we prove Theorem 12.7.1, stating boundedness of the weighted higher order energies. Using the conclusion of this theorem, we apply again Sobolev embedding as before (using also the result of Section 12.2.3) and thus obtain the boundedness statement of Theorem 9.1.1. Finally, in Section 12.8 we show continuous extendibility of ϕ to the Cauchy horizon.

We conclude in Section 13 by discussing what our results together with other analyses explained in part I of the thesis suggests about the nonlinear dynamics of the Einstein equations themselves.

9.4 Motivation and Strong Cosmic Censorship

In this section we briefly summarize facts relevant to our work that were already explained in part I of the thesis. As mentioned before our motivation for proving Theorem 9.1.1 is the Strong Cosmic Censorship Conjecture. We have stated the mathematical formulation of this conjecture, applied to electrovacuum, given in [17] by Christodoulou in (1.2). Reissner-Nordström spacetime serves as a counterexample to the inextendibility statement since it is (in fact smoothly) extendable beyond the Cauchy horizon $\mathcal{CH}^+$.[3] Thus, for the above conjecture to be true, this property of Reissner-Nordström must in particular be unstable.

Originally it was suggested by Penrose and Simpson that small perturbations of Reissner-Nordström would lead to a spacetime whose boundary would be a spacelike singularity as in Schwarzschild and such that the spacetime would be inextendable as a C^0 metric, cf. [90]. On the other hand, a heuristic study of a spherically symmetric but fully nonlinear toy model by Israel and Poisson, cf. [82], led to an alternative scenario, which suggested that spacetimes resulting from small perturbations would exist up to a Cauchy horizon, which however would be singular in a weaker sense, see also [71] by Ori. Considering the *spherically symmetric* Einstein-Maxwell-scalar field equations as a toy model, Dafermos proved that the solution indeed exists up to a Cauchy horizon and moreover is extendible as a C^0 metric but generically fails to be extendible as a C^1 metric beyond $\mathcal{CH}^+$, cf. [22, 23]. For more recent extensions see [19–21, 57].

In this work, as a first attempt towards investigation of the stability of the Cauchy horizon under perturbations *without symmetry*, we employ (9.1) on a fixed Reissner-Nordström background $(\mathcal{M}, g)$ as a toy model for the full nonlinear Einstein field equations, cf. (10.7). The result of uniform pointwise boundedness of ϕ and continuous extension to $\mathcal{CH}^+$ is concordant with the work of Dafermos [22]. This suggests that

[3] Outside the future maximal domain of dependence $\mathcal{D}^+(\mathcal{M})$ in the future of the Cauchy horizon $J^+(\mathcal{CH}^+)$ the spacetime shows the peculiar feature that uniqueness of the solutions of the initial value problem is lost *without loss of regularity*. It is precisely the undesirability of this feature that motivates the conjecture.

the non-spherically symmetric perturbations of the astrophysically more realistic Kerr spacetime may indeed exist up to $\mathcal{CH}^+$. See the conclusions of part I of the thesis.

Chapter 10

Preliminaries

10.1 Energy currents and vector fields

The essential tool used throughout this work is the so called vector field method. Let $(\mathcal{M}, g)$ be a Lorentzian manifold. Let ϕ be a solution to the wave equation $\Box_g \phi = 0$. A symmetric stress-energy tensor can be identified from variation of the massless scalar field action by

$$T_{\mu\nu}(\phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi, \tag{10.1}$$

and this satisfies the energy-momentum conservation law

$$\nabla^\mu T_{\mu\nu} = 0. \tag{10.2}$$

By contracting the energy-momentum tensor with a vector field V, we define the current

$$J^V_\mu(\phi) \doteq T_{\mu\nu}(\phi) V^\nu. \tag{10.3}$$

In this context we call V a multiplier vector field. If the vector field V is timelike, then the one-form J^V_μ can be interpreted as the energy-momentum density. When we integrate J^V_μ contracted with the normal vector field over an associated hypersurface we will often refer to the integral as energy flux. Note that $J^V_\mu(\phi) n^\mu_\Sigma \geq 0$ if V is future directed timelike and Σ spacelike, where n^μ_Σ is the future directed normal vector on the hypersurface Σ.

We will frequently use versions of the divergence theorem, often refered to as Stokes' Theorem. Consider a spacetime region $\mathcal{S}$ which is bound by the homologous hypersurfaces Σ_τ in the future and Σ_0 in the past as shown in Figure 10.1, then the by the

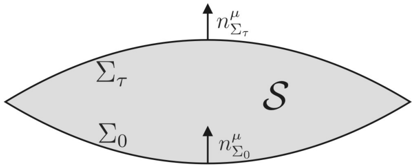


Figure 10.1: Spacetime region $\mathcal{S}$, bound by the hypersurfaces Σ_τ and Σ_0, with normal vectors $n^\mu_{\Sigma_\tau}$ and $n^\mu_{\Sigma_0}$.

divergence theorem we obtain

$$\int_{\Sigma_\tau} J^V_\mu(\psi) n^\mu_{\Sigma_\tau} \mathrm{dVol}_{\Sigma_\tau} + \int_{\mathcal{S}} \nabla^\mu J_\mu(\psi) \mathrm{dVol} = \int_{\Sigma_0} J^V_\mu(\psi) n^\mu_{\Sigma_0} \mathrm{dVol}_{\Sigma_0}. \tag{10.4}$$

n^μ denotes the normal to the subscript hypersurface oriented according to Lorentzian geometry convention and dVol denotes the volume element over the entire spacetime region and the volume elements on the subscript hypersurfaces, respectively. A proof of the divergence theorem is for example given in [41] and [93]. Because of the second term of (10.4), which we will call the spacetime integral or sometimes bulk term, we are interested in the divergence of the current (10.3). Defining

$$K^V(\phi) \doteq T(\phi)(\nabla V) = (\pi^V)^{\mu\nu} T_{\mu\nu}(\phi), \tag{10.5}$$

by (10.1) it follows that

$$\nabla^\mu J^V_\mu(\phi) = K^V(\phi). \tag{10.6}$$

Further, $(\pi^V)^{\mu\nu} \doteq \frac{1}{2}(\mathcal{L}_V g)^{\mu\nu}$ is the so called deformation tensor of V. Therefore, $\nabla^\mu J^V_\mu(\phi) = 0$ if V is Killing. If that is the case it is immediate from (10.4) that the future integrated flux can be controlled by the past integrated flux. If for some reason non-Killing multipliers have to be used, we further need to bound the bulk term in order to gain control over the future integrated flux.

For a Killing vector field W we have in addition the commutation relation $[\Box_g, W] = 0$.

In that context W is called a commutation vector field. In particular, we note already that in Reissner-Nordström spacetime we have $\Box_g T\phi = 0$ and $\Box_g \Omega_i \phi = 0$, where T and Ω_i with $i = 1, 2, 3$ are Killing vector fields that will be defined in Section 10.3.1 and 10.3.2, respectively.

For a more detailed discussion see [35] by Dafermos and Rodnianski, [56] by Klainerman and [16] by Christodoulou.

10.2 The Reissner-Nordström solution

In the following we will briefly recall the Reissner-Nordström solution[1] which is a family of solutions to the Einstein-Maxwell field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2T^{EM}_{\mu\nu}, \tag{10.7}$$

with $R_{\mu\nu}$ the Ricci tensor, R the Ricci scalar and the units chosen such that $\frac{8\pi G}{c^4} = 2$. The Maxwell equations are given by

$$\nabla^{\alpha} F_{\alpha\beta} = 0, \qquad \nabla_{[\lambda} F_{\alpha\beta]} = 0, \tag{10.8}$$

and the energy-momentum tensor by

$$T^{EM}_{\mu\nu} \;=\; F^{\alpha}_{\mu} F_{\alpha\nu} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}. \tag{10.9}$$

The system (10.7)-(10.9) describes the interaction of a gravitational field with a source free electromagnetic field.

The Reissner-Nordström solution represents a charged black hole as an isolated system in an asymptotically Minkowski spacetime. The causal structure is similar to the structure of the astrophysically more realistic axisymmetric Kerr black holes. Since spherical symmetry can often simplify first investigations, Reissner-Nordström spacetime is a popular proxy for Kerr.

[1] The reader unfamiliar with this solution may for example consult [48] for a more detailed review.

10.2.1 The metric and ambient differential structure

To set the semantic convention, whenever we refer to the Reissner-Nordström solution $(\mathcal{M}, g)$ we mean the maximal domain of dependence $\mathcal{D}(\Sigma) = \mathcal{M}$ of complete two-ended asymptotically flat data Σ. The manifold $\mathcal{M}$ can be expressed by $\mathcal{M} = \mathcal{Q} \times \mathbb{S}^2$, and $\mathcal{Q} = (-1,1) \times (-1,1)$ with coordinates $U, V \in (-1,1)$ and thus

$$\mathcal{M} = (-1,1) \times (-1,1) \times \mathbb{S}^2. \tag{10.10}$$

The metric in global double null coordinates then takes the form

$$g = -\Omega^2(U,V)\mathrm{d}U\mathrm{d}V + r^2(U,V)\left[\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right], \tag{10.11}$$

where Ω^2 and r will be described below.

As a gauge condition we choose the hypersurface $U = 0$ and $V = 0$ to coincide with what will be the event horizons and we set

$$\Omega^2(0,V) = \frac{1}{1-V^2}, \tag{10.12}$$

$$\Omega^2(U,0) = \frac{1}{1-U^2}, \tag{10.13}$$

consistent with the fact that these hypersurfaces are to have infinite affine length. Fix parameters $M > |e| \neq 0$. The Reissner-Nordström metric (10.11) in our gauge is uniquely determined from (10.7)-(10.9) by setting

$$r(0,V) = r|_{\mathcal{H}_A^+} = M + \sqrt{M^2 - e^2} = r_+, \tag{10.14}$$

$$r(U,0) = r|_{\mathcal{H}_B^+} = M + \sqrt{M^2 - e^2} = r_+. \tag{10.15}$$

Rearranging the Einstein-Maxwell equations (10.7) using (10.11) we obtain the following Hessian equation

$$\partial_U\partial_V r = \frac{e^2\Omega^2}{4r^3} - \frac{\Omega^2}{4r} - \frac{\partial_U r \partial_V r}{r}, \tag{10.16}$$

from the U, V component. From the θ,θ or equivalently ϕ,ϕ component we obtain

$$\partial_U\partial_V \log\Omega^2 = -\frac{2\partial_U\partial_V r}{r} \tag{10.17}$$

$$\stackrel{(10.16)}{=} \quad -\frac{e^2\Omega^2}{2r^4} + \frac{\Omega^2}{2r^2} + \frac{2\partial_U r \partial_V r}{r^2}, \tag{10.18}$$

In fact, all relevant information about Reissner-Nordström geometry can be understood directly from (10.12) to (10.17) without explicit expressions for $\Omega^2(U,V)$ and $r(U,V)$. In particular, one can derive the Raychaudhuri equations

$$\partial_U\left(\frac{\partial_U r}{\Omega^2}\right) = 0, \tag{10.19}$$

$$\partial_V\left(\frac{\partial_V r}{\Omega^2}\right) = 0, \tag{10.20}$$

from the above.

We can illustrate the 2-dimensional quotient spacetime $\mathcal{Q}$ as a subset of an ambient $\mathbb{R}^{1+1}$: Identifying U, V with ambient null coordinates of $\mathbb{R}^{1+1}$, the boundary

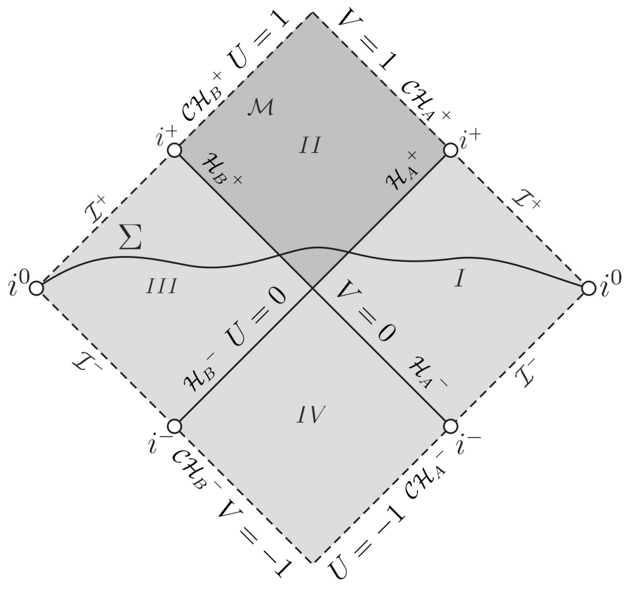


Figure 10.2: Conformal diagram of the maximal domain of dependence of Reissner-Nordström spacetime.

of $\mathcal{Q} \subset \mathbb{R}^{1+1}$ is given by $\pm 1 \times [-1,1] \cup [-1,1] \times \pm 1$. Let us further define the darker shaded region II of Figure 10.2 by $\mathcal{Q}|_{II} = [0,1) \times [0,1)$. Particularly important is $\mathcal{CH}^+ = \mathcal{CH}_A^+ \cup \mathcal{CH}_B^+ = 1 \times (0,1] \cup (0,1] \times 1$, which is the future boundary of the interior of region II. We define $\mathcal{M}|_{II} = \pi^{-1}(\mathcal{Q}|_{II})$, where π is the projection $\pi : \mathcal{M} \to \mathcal{Q}$.

10.2.2 Eddington-Finkelstein coordinates

It will be convenient to rescale the global double null coordinates and define

$$u = f(U) = \frac{2r_+}{{r_+}^2 - e^2} \ln \left| \ln \left| \frac{1+U}{1-U} \right| \right|, \qquad v = h(V) = \frac{2r_+}{{r_+}^2 - e^2} \ln \left| \ln \left| \frac{1+V}{1-V} \right| \right|. \tag{10.21}$$

Note that u is the retarded and v is the advanced Eddington-Finkelstein coordinate. These coordinates are both regular in the interior of $\mathcal{Q}|_{II}$, cf. Figure 10.2. Nonetheless, we can view the whole of $\mathcal{Q}|_{II}$ as

$$\mathcal{Q}|_{II} = [-\infty, \infty) \times [-\infty, \infty), \tag{10.22}$$

where we have formally parametrized by

$$\begin{aligned} {\mathcal{H}_A}^+ &= \{-\infty\} \times [-\infty, \infty), \\ {\mathcal{H}_B}^+ &= [-\infty, \infty) \times \{-\infty\}, \end{aligned}$$

as depicted in Figure 10.3, see also (10.21).

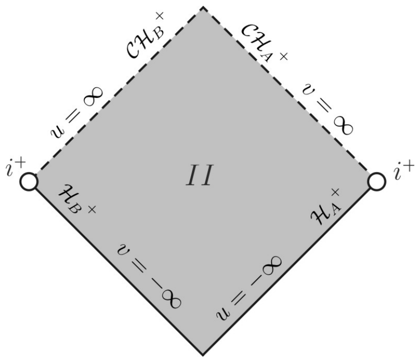


Figure 10.3: Conformal diagram of darker shaded region II, compare Figure 10.2, with the ranges of (u, v) depicted.

In u, v coordinates the metric is given by

$$g = -\Omega^2(u,v)\mathrm{d}u\mathrm{d}v + r^2(u,v)\left[\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right], \tag{10.23}$$

with

$$\Omega^2(u,v) = \frac{\Omega^2(U,V)}{\partial_U f \partial_V h} = -\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right), \tag{10.24}$$

where the unfamiliar minus sign on the right hand side arises since all definitions have been made suitable for the interior. We will often make use of the fact that by the choice of Eddington-Finkelstein coordinates (10.21) for the interior we have scaled our coordinates such that

$$\frac{\partial_u r}{\Omega^2} = -\frac{1}{2}, \qquad \frac{\partial_v r}{\Omega^2} = -\frac{1}{2}. \tag{10.25}$$

(The fact that the above expressions are constants follows from the Raychaudhuri equations (10.19) and (10.20).) Taking the derivatives of (10.24) with respect to u and v and using (10.25) it follows that

$$\frac{\partial_u \Omega}{\Omega}(u,v) = \frac{1}{2r^2}\left(M - \frac{e^2}{r}\right), \tag{10.26}$$

$$\frac{\partial_v \Omega}{\Omega}(u,v) = \frac{1}{2r^2}\left(M - \frac{e^2}{r}\right). \tag{10.27}$$

10.3 Further properties of Reissner-Nordström geometry

10.3.1 $(t, r^\star)$ and (t, r) coordinates

It is useful to define the function $t : \mathring{\mathcal{M}}|_{II} \to \mathbb{R}$ by

$$t(u,v) = \frac{v-u}{2}, \tag{10.28}$$

where $\mathring{\mathcal{M}}|_{II} = \mathcal{M}|_{II} \setminus \partial\mathcal{M}|_{II}$ is the interior of $\mathcal{M}|_{II}$. Moreover, we define the function $r^\star : \mathring{\mathcal{M}}|_{II} \to \mathbb{R}$ by

$$r^\star(u,v) = \frac{v+u}{2}, \tag{10.29}$$

where $r^\star$ is usually referred to as the Regge-Wheeler coordinate. Note that for coordinates $(t, r^\star, \varphi, \theta)$ defined in $\mathring{\mathcal{M}}|_{II}$ we have that $\frac{\partial}{\partial t}$ is a spacelike Killing vector field which extends to the globally defined Killing vector field T on $\mathcal{M}$. By φ_τ we denote a 1-

parameter group of diffeomorphisms generated by the Killing field T. We can moreover relate the functions r and $r^\star$ by

$$
\begin{aligned}
\mathrm{d}r^\star &= \frac{\mathrm{d}r}{1-\frac{2M}{r}+\frac{e^2}{r^2}} && (10.30)\\
\Rightarrow r^\star &= r+\frac{1}{\kappa_+}\ln|\frac{r-r_+}{r_+}|+\frac{1}{\kappa_-}\ln|\frac{r-r_-}{r_-}|+C, && (10.31)
\end{aligned}
$$

where C is constant which is implicitly fixed by previous definitions,

$$r_- = M-\sqrt{M^2-e^2}, \tag{10.32}$$

and the surface gravities are given by

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2r_\pm^2}. \tag{10.33}$$

Note that κ_+ is the surface gravity at $\mathcal{H}^+$ and κ_- is the surface gravity at $\mathcal{CH}^+$. The function $r(u,v)$ extends continuously and is monotonically decreasing in both u and v towards $\mathcal{CH}^+$ such that we have

$$
\begin{aligned}
r(u,\infty) &= r|_{\mathcal{CH}_A{}^+} = r_-, && (10.34)\\
r(\infty,v) &= r|_{\mathcal{CH}_B{}^+} = r_-. && (10.35)
\end{aligned}
$$

10.3.2 Angular momentum operators

We have already mentioned the generators of spherical symmetry Ω_i, $i=1,2,3$, in Section 10.1. They are explicitly given by

$$
\begin{aligned}
\Omega_1 &= \sin\varphi\partial_\theta+\cot\theta\cos\varphi\partial_\varphi, && (10.36)\\
\Omega_2 &= -\cos\varphi\partial_\theta+\cot\theta\sin\varphi\partial_\varphi, && (10.37)\\
\Omega_3 &= -\partial_\varphi, && (10.38)
\end{aligned}
$$

which satisfy

$$\sum_{i=1}^{3}(\Omega_i\phi)^2 = r^2|\nabla\!\!\!/\phi|^2, \tag{10.39}$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\Omega_i\Omega_j\phi\right)^2 \quad = \quad r^4|\nabla\!\!\!/^2\phi|^2, \tag{10.40}$$

where we define

$$|\nabla\!\!\!/\phi|^2 = \frac{1}{r^2}\left[(\partial_\theta\phi)^2 + \frac{1}{\sin^2\theta}(\partial_\varphi\phi)^2\right]. \tag{10.41}$$

10.3.3 The redshift, noshift and blueshift region

As we have already mentioned in the introduction, in the interior we can distinguish

$$\text{redshift} \quad \mathcal{R} = \{r_{red} \leq r \leq r_+\} \quad , \tag{10.42}$$
$$\text{noshift} \quad \mathcal{N} = \{r_{blue} \leq r \leq r_{red}\} \quad , \tag{10.43}$$
$$\text{and blueshift} \quad \mathcal{B} = \{r_- \leq r \leq r_{blue}\} \tag{10.44}$$

subregions, as shown in Figure 10.4, for values r_{red}, r_{blue} to be defined immediately below.

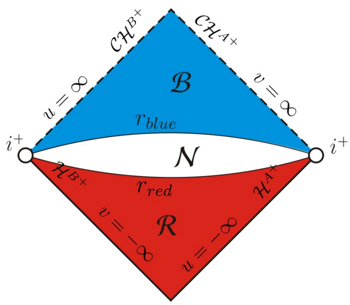


Figure 10.4: Region II with distinction into redshift $\mathcal{R}$, noshift $\mathcal{N}$ and blueshift $\mathcal{B}$ regions.

In the redshift region $\mathcal{R}$ we make use of the fact that the surface gravity κ_+ of the event horizon is positive. The region is then characterized by the fact that there exists a vector field N such that its associated current $J^N_\mu n^\mu_{v=const}$ on a $v = const$ hypersurface can be controlled by the related bulk term K^N, cf. Proposition 12.1.1. This positivity of the bulk term K^N is only possible sufficiently close to $\mathcal{H}^+$. In particular we shall define

$$r_{red} = r_+ - \epsilon, \tag{10.45}$$

with $\epsilon > 0$ and small enough such that Proposition 12.1.1 is applicable. (Furthermore, note that the quantity $M - \frac{e^2}{r}$ is always positive in $\mathcal{R}$.)

As defined in (10.43) the r coordinate in the noshift region $\mathcal{N}$ ranges between r_{red} defined by (10.45) and r_{blue}, defined below, strictly bigger than r_-. In $\mathcal{N}$ we exploit the fact that $J^{-\partial_r}$ and $K^{-\partial_r}$ are invariant under translations along ∂_t. For that reason we can uniformly control the bulk by the current along a constant r hypersurface. This will be explained further in Section 12.1.2.

The blueshift region $\mathcal{B}$ is characterized by the fact that the bulk term K^{S_0} associated to the vector field S_0 to be defined in (11.10) is positive. We define

$$r_{blue} = r_- + \tilde{\epsilon}, \tag{10.46}$$

with $\tilde{\epsilon} > 0$ for an $\tilde{\epsilon}$ such that $M - \frac{e^2}{r}$ carries a negative sign and such that (for convenience)

$$r^\star(r_{blue}) > 0. \tag{10.47}$$

In particular, in view of (10.26) and (10.27) for $\tilde{\epsilon}$ sufficiently small the following lower bound holds in $\mathcal{B}$

$$0 < \beta \quad \leq \quad -\frac{\partial_u \Omega}{\Omega}, \tag{10.48}$$

$$0 < \beta \quad \leq \quad -\frac{\partial_v \Omega}{\Omega}, \tag{10.49}$$

with β a positive constant.

10.4 Notation

We will describe certain regions derived from the hypersurfaces $r = r_{red}$, $r = r_{blue}$ and in addition the hypersurface γ which will be defined in Section 12.1.3. For example given the hypersurface $r = r_{red}$ and the hypersurface $u = \tilde{u}$ we define the v value at which these two hypersurfaces intersect by a function $v_{red}(\tilde{u})$ evaluated for $\tilde{u}$. Let us therefore introduce the following notation:

$$\begin{aligned}
v_{red}(\tilde{u}) &\quad \text{is determined by} \quad r(v_{red}(\tilde{u}), \tilde{u}) = r_{red}, \\
v_{\gamma}(\tilde{u}) &\quad \text{is determined by} \quad (v_{\gamma}(\tilde{u}), \tilde{u}) \in \gamma, \\
v_{blue}(\tilde{u}) &\quad \text{is determined by} \quad r(v_{blue}(\tilde{u}), \tilde{u}) = r_{blue},
\end{aligned}$$

and similarly we will also use

$$\begin{aligned} u_{red}(\tilde{v}) &\quad \text{is determined by} \quad && r(u_{red}(\tilde{v}), \tilde{v}) = r_{red}, \\ u_{\gamma}(\tilde{v}) &\quad \text{is determined by} \quad && (u_{\gamma}(\tilde{v}), \tilde{v}) \in \gamma, \\ u_{blue}(\tilde{v}) &\quad \text{is determined by} \quad && r(u_{blue}(\tilde{v}), \tilde{v}) = r_{blue}. \end{aligned} \tag{10.50}$$

For a better understanding the reader may also refer to Figure 10.5 and Figure 10.6.

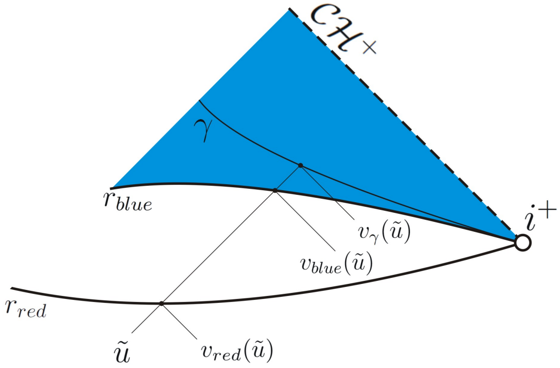


Figure 10.5: Sketch of blueshift region $\mathcal{B}$ with quantities depicted dependent on $\tilde{u}$.

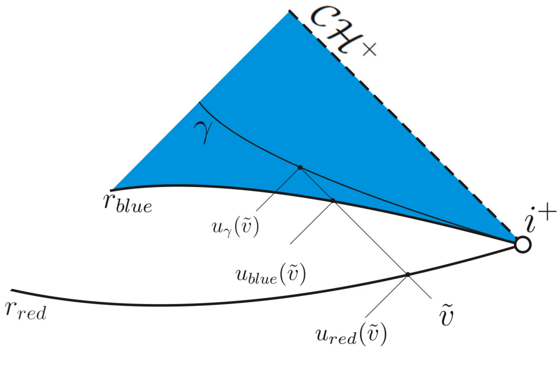


Figure 10.6: Sketch of blueshift region $\mathcal{B}$ with quantities depicted dependent on $\tilde{v}$.

Note that the above functions are well defined since $r = r_{red}$, $r = r_{blue}$ and γ are spacelike hypersurfaces terminating at i^+.

Chapter 11

The setup

11.1 Horizon estimates and Cauchy stability

Our starting point will be previously proven decay bounds for ϕ and its derivatives in the black hole *exterior* up to and including the event horizon; in particular we can state:

Theorem 11.1.1. *Let ϕ be a solution of the wave equation* (9.1) *on a subextremal Reissner-Nordström background $(\mathcal{M}, g)$, with mass M and charge e and $M > |e| \neq 0$, arising from smooth compactly supported initial data on an arbitrary Cauchy hypersurface Σ, cf. Figure 11.1. Then, there exists $\delta > 0$ such that*

$$\int_{\mathbb{S}^2}\int_{v}^{v+1} \left[(\partial_v\phi)^2(-\infty,v) + |\nabla\!\!\!/\phi|^2(-\infty,v)\right] r^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \leq C_0 v^{-2-2\delta}, \quad (11.1)$$

$$\int_{\mathbb{S}^2}\int_{v}^{v+1} \left[(\partial_v\Omega\phi)^2(-\infty,v) + |\nabla\!\!\!/\Omega\phi|^2(-\infty,v)\right] r^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \leq C_1 v^{-2-2\delta}, \quad (11.2)$$

$$\int_{\mathbb{S}^2}\int_{v}^{v+1} \left[(\partial_v\Omega^2\phi)^2(-\infty,v) + |\nabla\!\!\!/\Omega^2\phi|^2(-\infty,v)\right] r^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \leq C_2 v^{-2-2\delta}, \quad (11.3)$$

on ${\mathcal{H}_A}^+$, for all v and some positive constants C_0, C_1 and C_2 depending on the initial data.[1]

[1] The notation Ω and Ω^2 is explained in Section 12.2.1 and simply denotes summation over angular momentum operators Ω_i and $\Omega_i\Omega_j$.

Proof. The Theorem follows by putting together work of P. Blue and A. Soffer [9] on integrated local energy decay, M. Dafermos and I. Rodnianski [29] on the redshift and V. Schlue [88] on improved decay using the method of [31] in the exterior region. The assumption of smoothness and compact support can be weakened. Moreover, we can in fact take δ arbitrarily close to $\frac{1}{2}$, but $\delta > 0$ is sufficient for our purposes and allows in principle for a larger class of data on Σ. □

On the other hand, trivially from Cauchy stability, boundedness of the energy along the second component of the past boundary of the characteristic rectangle Ξ, cf. Section 9.2, which we have picked to be $v = 1$, can be derived. More generally we can state the following proposition.

Proposition 11.1.2. *Let $u_\diamond, v_\diamond \in (-\infty, \infty)$. Under the assumption of Theorem 11.1.1, the energy at advanced Eddington-Finkelstein coordinate $\{v = v_\diamond\} \cap \{-\infty \leq u \leq u_\diamond\}$ is bounded from the initial data*

$$\int_{\mathbb{S}^2}\int_{-\infty}^{u_\diamond}\left[\Omega^{-2}(\partial_u\phi)^2(u,v_\diamond)+\frac{\Omega^2}{2}|\nabla\!\!\!/\phi|^2(u,v_\diamond)\right]r^2\mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_0(u_\diamond,v_\diamond), \tag{11.4}$$

$$\int_{\mathbb{S}^2}\int_{-\infty}^{u_\diamond}\left[\Omega^{-2}(\partial_u\Omega\!\!\!/\phi)^2(u,v_\diamond)+\frac{\Omega^2}{2}|\nabla\!\!\!/\Omega\!\!\!/\phi|^2(u,v_\diamond)\right]r^2\mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_1(u_\diamond,v_\diamond), \tag{11.5}$$

$$\int_{\mathbb{S}^2}\int_{-\infty}^{u_\diamond}\left[\Omega^{-2}(\partial_u\Omega\!\!\!/^2\phi)^2(u,v_\diamond)+\frac{\Omega^2}{2}|\nabla\!\!\!/\Omega\!\!\!/^2\phi|^2(u,v_\diamond)\right]r^2\mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_2(u_\diamond,v_\diamond), \tag{11.6}$$

and further

$$\sup_{-\infty\leq u\leq u_\diamond}\int_{\mathbb{S}^2}(\phi)^2(u,v_\diamond)\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_0(u_\diamond,v_\diamond), \tag{11.7}$$

$$\sup_{-\infty\leq u\leq u_\diamond}\int_{\mathbb{S}^2}(\Omega\!\!\!/\phi)^2(u,v_\diamond)\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_1(u_\diamond,v_\diamond), \tag{11.8}$$

$$\sup_{-\infty\leq u\leq u_\diamond}\int_{\mathbb{S}^2}(\Omega\!\!\!/^2\phi)^2(u,v_\diamond)\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_2(u_\diamond,v_\diamond), \tag{11.9}$$

with $D_0(u_\diamond,v_\diamond)$, $D_1(u_\diamond,v_\diamond)$ and $D_2(u_\diamond,v_\diamond)$ positive constants depending on the initial data on Σ.

Proof. This follows immediately from local energy estimates in a compact spacetime region. Note the Ω^{-2} and Ω^2 weights which arise since u is not regular at $\mathcal{H}_A^+$. □

11.2 Statement of the theorem and outline of the proof in the neighbourhood of i^+

The most difficult result of this paper can now be stated in the following theorem.

Theorem 11.2.1. *On subextremal Reissner-Nordström spacetime with $M > |e| \neq 0$, let ϕ be as in Theorem 11.1.1, then*

$$|\phi| \leq C$$

locally in the black hole interior up to $\mathcal{CH}^+$ in a "small neighbourhood" of timelike infinity i^+, that is in $(-\infty, u_{\text{✂}}] \times [1, \infty)$ for some $u_{\text{✂}} > -\infty$.

Remark. We will see that C depends only on the initial data.

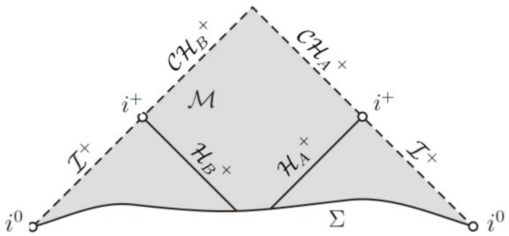


Figure 11.1: Maximal development of Cauchy hypersurface Σ in Reissner-Nordström spacetime $(\mathcal{M}, g)$.

We will consider a characteristic rectangle Ξ extending from ${\mathcal{H}_A}^+$ as shown in Figure 11.2. We pick the characteristic rectangle to be defined by $\Xi = \{(-\infty \leq u \leq u_{\text{✂}}), (1 \leq v < \infty)\}$, where $u_{\text{✂}}$ is sufficiently close to $-\infty$ for reasons that will become clear later on, cf. Proposition 12.1.11. As described in Section 11.1, from bounds of data on Σ bounds on the solution on the lower segments follow according to Theorem 11.1.1 and Proposition 11.1.2.

In order to prove Theorem 11.2.1 we distinguish the redshift $\mathcal{R}$, the noshift $\mathcal{N}$ and the blueshift $\mathcal{B}$ region, with the properties as explained in Section 10.3.3, cf. Figure 11.3. This distinction is made since different vector fields have to be employed in the different

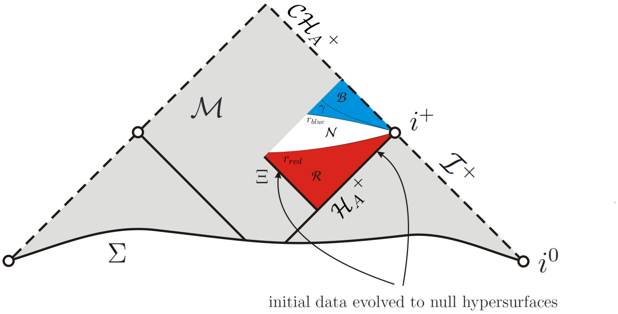


Figure 11.2: Characteristic rectangle Ξ in the interior of Reissner-Nordström spacetime $(\mathcal{M}, g)$, for Ξ zoomed in see Figure 11.3.

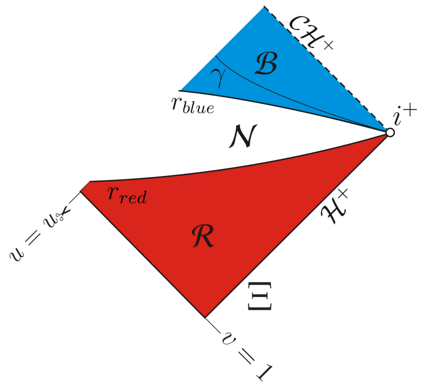


Figure 11.3: Characteristic rectangle Ξ with redshift $\mathcal{R}$, noshift $\mathcal{N}$ and blueshift $\mathcal{B}$ regions.

regions[2].

In the redshift region $\mathcal{R}$ we will make use of the redshift vector field N of [35] on which we will elaborate more in Section 12.1.1. Proposition 12.1.1 gives the positivity of the bulk K^N which thus bounds the current $J^N_\mu n^\mu_{v=const}$ from above. Applying the divergence theorem, decay up to $r = r_{red}$ will be proven.

In the noshift region $\mathcal{N}$ we can simply appeal to the fact that the future directed timelike vector field $-\partial_r$ is invariant under the flow of the spacelike Killing vector field ∂_t. It is for that reason that the bulk term $K^{-\partial_r}$ can be uniformly controlled by the energy flux $J^{-\partial_r}_\mu n^\mu_{r=\bar{r}}$ through the $r = \bar{r}$ hypersurface. Decay up to $r = r_{blue}$ will be

[2]The reader may wonder why the noshift region $\mathcal{N}$ is introduced instead of just separating the red- and the blueshift regions along the r hypersurface whose value renders the quantity $M - \frac{e^2}{r}$ equal zero. This was to ensure strict positivity/negativity of the quantity in the redshift/blueshift region.

proven by making use of this together with the uniform boundedness of the v length of $\mathcal{N}$.

To understand the blueshift region $\mathcal{B}$, we will partition it by the hypersurface γ admitting logarithmic distance in v from $r = r_{blue}$, cf. Section 12.1.3. We will then separately consider the region to the past of γ, $J^-(\gamma) \cap \mathcal{B}$ and the region to the future of γ, $J^+(\gamma) \cap \mathcal{B}$. The region to the future of γ is characterized by good decay bounds on Ω^2 (implying for instance that the spacetime volume is finite, $\mathrm{Vol}(J^+(\gamma)) < C$).

In $J^-(\gamma) \cap \mathcal{B}$ we use a vector field

$$S_0 = r^q \partial_{r^\star} = r^q(\partial_u + \partial_v), \tag{11.10}$$

where q is sufficiently large, cf. Section 12.1.3. We will see that for the right choice of q we can render the associated bulk term K^{S_0} positive which is the "good" sign when using the divergence theorem.

In order to complete the proof, we consider finally the region $J^+(\gamma) \cap \mathcal{B}$ and propagate the decay further from the hypersurface γ up to the Cauchy horizon in a neighbourhood of i^+. For this, we introduce a new timelike vector field S defined by

$$S = |u|^p \partial_u + v^p \partial_v, \tag{11.11}$$

for an arbitrary p such that

$$1 < p \leq 1 + 2\delta, \tag{11.12}$$

where δ is as in Theorem 11.1.1. We use pointwise estimates on Ω^2 in $J^+(\gamma)$ as a crucial step, cf. Section 12.1.4.

Putting everything together, in view of the geometry and the weights of S, we finally obtain for all $v_* \geq 1$

$$\int_{\mathbb{S}^2} \int_1^{v_*} v^p (\partial_v \phi)^2 r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \leq \text{Data}, \tag{11.13}$$

for the weighted flux. Using the above, the uniform boundedness for ϕ stated in Theorem 11.2.1 then follows from an argument that can be sketched as follows.

Let us first see how we get an integrated bound on the spheres of symmetry. By the fundamental theorem of calculus and the Cauchy-Schwarz inequality one obtains

$$\int_{\mathbb{S}^2} \phi^2(u, v_*, \theta, \varphi) \mathrm{d}\sigma_{\mathbb{S}^2} \quad \leq \quad C \int_{\mathbb{S}^2} \left(\int_1^{v_*} v^p (\partial_v \phi)^2 \mathrm{d}v \right) \left(\int_1^{v_*} v^{-p} \mathrm{d}v \right) r^2 \mathrm{d}\sigma_{\mathbb{S}^2} + \mathrm{data},$$

where the first factor of the first term is controlled by (11.13). Therefore, we further get

$$\begin{aligned} \int_{\mathbb{S}^2} \phi^2 \mathrm{d}\sigma_{\mathbb{S}^2} \quad &\overset{(11.13)}{\leq} \quad \mathrm{Data} \int_{\mathbb{S}^2} \int_1^{v_*} v^{-p} \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} + \mathrm{data} \\ &\leq \quad \mathrm{Data} + \mathrm{data}, \end{aligned} \tag{11.14}$$

where we have used $\int_1^{\infty} v^{-p} \mathrm{d}v < \infty$ which followed from the first inequality of (11.12).

Obtaining a pointwise statement from the above will be achieved by commuting (9.1) with symmetries as well as applying Sobolev embedding. As outlined in Section 10.1 in Reissner-Nordström geometry we have $\Box_g \Omega_i \phi = 0$, where Ω_i with $i = 1, 2, 3$ are the 3 spacelike Killing vector fields resulting from the spherical symmetry. Thus one obtains the analogue of (11.14) but with $\Omega_i \phi$ and $\Omega_i \Omega_j \phi$ in place of ϕ. Using Sobolev embedding on $\mathbb{S}^2$ thus leads immediately to the desired bounds. See Section 12.2.3. This will close the proof of Theorem 11.2.1.

Chapter 12

Energy and pointwise estimates in the interior

In this section we will derive the proof of Theorem 9.1.2. For this we will first state the decay bound for the energy flux of ϕ given on the event horizon $\mathcal{H}^+$, cf. Theorem 11.1.1. Using this we propagate the decay rate through the entire interior up to the Cauchy horizon. As outlined in Section 9.2, we first consider a characteristic rectangle Ξ in the neighbourhood of i^+. Within Ξ we need to separate the interior into different subregions. We then apply suitable vector fields according to the specific properties of the underlying subregion, which will be described further in the following subsections.

12.1 Energy estimates in the neigbourhood of i^+

12.1.1 Propagating through $\mathcal{R}$ from $\mathcal{H}^+$ to $r = r_{red}$

The estimates in this and the following section are motivated by work of Luk [61]. He proves that any polynomial decay estimate that holds along the event horizon of Schwarzschild black holes can be propagated to any constant r hypersurface in the black hole interior. This followed a previous spherically symmetric argument of [23]. See also Dyatlov [37].

As outlined in Section 11.1, we will first propagate energy decay from $\mathcal{H}^+$ up to the $r = r_{red}$ hypersurface.

The rough idea can be understood with the help of Figure 12.1. By Theorem 11.1.1

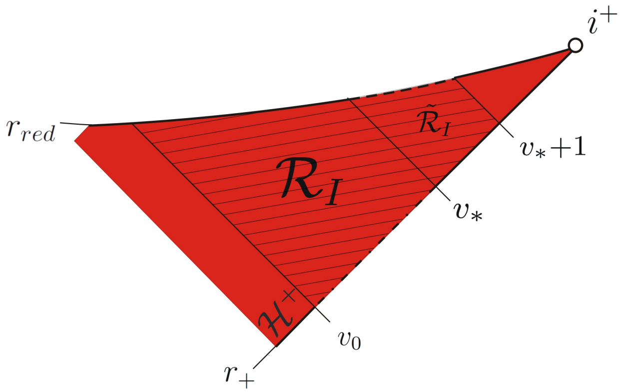


Figure 12.1: Regions $\mathcal{R}_I$ and $\tilde{\mathcal{R}}_I$.

we are given energy decay on the event horizon $\mathcal{H}^+$, see dash-dotted line. By using the energy identity for the vector field N in region $\mathcal{R}_I = \{r_{red} \leq r \leq r_+\} \cap \{1 < v \leq v_*\}$, the coarea formula etc., we obtain decay of the flux through constant v hypersurfaces throughout the entire region. Using this result and considering the energy identity once again in region $\tilde{\mathcal{R}}_I = \{r_{red} \leq r \leq r_+\} \cap \{v_* \leq v \leq v_* + 1\}$ we eventually obtain decay on the $r = r_{red}$ hypersurface, note the dashed line.

The redshift vector field was already introduced by Dafermos and Rodnianski in [30] and elaborated on again in [35], see also Appendix A. The existence of such a vector field in the neighbourhood of a Killing horizon $\mathcal{H}^+$ depends only on the positivity of the surface gravity, in this case κ_+. Thus by (10.33) the following proposition follows by Theorem 7.1 of [35].

Proposition 12.1.1. *(M. Dafermos and I. Rodnianski) For r_{red} sufficiently close to r_+ there exists a φ_τ-invariant*[1] *smooth future directed timelike vector field N on $\{r_{red} \leq r \leq r_+\} \cap \{v \geq 1\}$ and a positive constant b_1 such that*

$$b_1 J^N_\mu(\phi) n^\mu_v \leq K^N(\phi), \tag{12.1}$$

for all solutions ϕ of $\Box_g \phi = 0$.

The decay bound along $r = r_{red}$ can now be stated in the following proposition.

[1] See Section 10.3

Proposition 12.1.2. *Let ϕ be as in Theorem 11.1.1. Then, for all $\tilde{r} \in [r_{red}, r_+)$, with r_{red} as in Proposition 12.1.1 and for all $v_* > 1$,*

$$\int\limits_{\{v_* \leq v \leq v_*+1\}} J^N_\mu(\phi) n^\mu_{r=\tilde{r}} dVol_{r=\tilde{r}} \leq C v_*^{-2-2\delta},$$

with C depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$.

Remark 1. The decay in Proposition 12.1.2 matches the decay on $\mathcal{H}^+$ of Theorem 11.1.1.

Remark 2. $n^\mu_{r=r_{red}}$ denotes the normal to the $r = r_{red}$ hypersurface oriented according to Lorentzian geometry convention. dVol denotes the volume element over the entire spacetime region and $\mathrm{dVol}_{r=r_{red}}$ denotes the volume element on the $r = r_{red}$ hypersurface. Similarly for all other subscripts.[2]

Proof. Applying the divergence theorem, see e.g. [35] or [93], in region $\mathcal{R}_I = \{r_{red} \leq r \leq r_+\} \cap \{v_0 \leq v \leq v_*\}$, with $v_0 \geq 1$, we obtain

$$\begin{aligned} &\int\limits_{\mathcal{R}_I} K^N(\phi)\mathrm{dVol} + \int\limits_{\{v_0 \leq v \leq v_*\}} J^N_\mu(\phi) n^\mu_{r=r_{red}} \mathrm{dVol}_{r=r_{red}} \\ + &\int\limits_{\{r_{red} \leq r \leq r_+\}} J^N_\mu(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} \\ = &\int\limits_{\{r_{red} \leq r \leq r_+\}} J^N_\mu(\phi) n^\mu_{v=v_0} \mathrm{dVol}_{v=v_0} + \int\limits_{\{v_0 \leq v \leq v_*\}} J^N_\mu(\phi) n^\mu_{\mathcal{H}^+} \mathrm{dVol}_{\mathcal{H}^+}. \end{aligned}$$

We immediately see that the second term on the left hand side is positive since $r = r_{red}$ is a spacelike hypersurface and N is a timelike vector field. Therefore, we write

$$\begin{aligned} &\int\limits_{\mathcal{R}_I} K^N(\phi)\mathrm{dVol} + \int\limits_{\{r_{red} \leq r \leq r_+\}} J^N_\mu(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} \\ \leq &\int\limits_{\{r_{red} \leq r \leq r_+\}} J^N_\mu(\phi) n^\mu_{v=v_0} \mathrm{dVol}_{v=v_0} + \int\limits_{\{v_0 \leq v \leq v_*\}} J^N_\mu(\phi) n^\mu_{\mathcal{H}^+} \mathrm{dVol}_{\mathcal{H}^+}. \end{aligned} \tag{12.2}$$

[2] Refer to Appendix C.1 for further discussion of the volume elements.

By Theorem 11.1.1 we have

$$\int\limits_{\{v_0 \le v \le v_*\}} J^N_\mu(\phi) n^\mu_{\mathcal{H}^+} \mathrm{dVol}_{\mathcal{H}^+} \le C_0 \max\{v_* - v_0, 1\}\, v_0^{-2-2\delta}.$$

Using that the energy current associated to the timelike vector field N is controlled by the deformation K^N as shown in (12.1) and substituting

$$E(\phi;\tilde{v}) = \int\limits_{\{r_{red} \le r \le r_+\}} J^N_\mu(\phi) n^\mu_{v=\tilde{v}} \mathrm{dVol}_{v=\tilde{v}} \tag{12.3}$$

into (12.2) as well as using the coarea formula

$$\int\limits_{\mathcal{R}_I} J^N_\mu(\phi) n^\mu_{v=\tilde{v}} \mathrm{dVol} \sim \int\limits_{v_0}^{v_*} \int\limits_{\{r_{red} \le r \le r_+\}} J^N_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \mathrm{d}\bar{v}, \tag{12.4}$$

for the bulk term,[3] we obtain for all $v_0 \ge 1$ and $v_* > v_0$, the relation

$$E(\phi; v_*) + \tilde{b}_1 \int\limits_{v_0}^{v_*} E(\phi;\bar{v}) \mathrm{d}\bar{v} \le E(\phi; v_0) + C_0 \max\{v_* - v_0, 1\}\, v_0^{-2-2\delta}. \tag{12.5}$$

Note by Proposition 11.1.2, applied to $u_\diamond$ defined through the relation $r_{red} = r(u_\diamond, 1)$, we have

$$E(\phi;1) \le C D_0(u_\diamond, 1), \tag{12.6}$$

since the vector field N is regular at $\mathcal{H}^+$ and thus $E(\phi;1)$ is comparable to the left hand side of (11.4). In order to obtain estimates from (12.5) we appeal to the following lemma.

Lemma 12.1.3. *Let f be a continuous function, $f : [1,\infty) \to \mathbb{R}^+$,*

$$f(t) + b \int\limits_{\tilde{t}}^{t} f(\bar{t}) \mathrm{d}\bar{t} \le f(\tilde{t}) + C_0 (t - \tilde{t} + 1) \tilde{t}^{-\tilde{p}}, \tag{12.7}$$

[3] where $f \sim g$ means that there exist constants $0 < b < B$ with $bf < g < Bf$

for all $\tilde{t} \geq 1$*, where* C_0*,* $\tilde{p}$ *are positive constants. Then for any* $t \geq 1$ *we have*

$$f(t) \leq \tilde{C} t^{-\tilde{p}}, \tag{12.8}$$

where $\tilde{C}$ *depends only on* $f(1)$*,* b *and* C_0*.*

Proof. For $t > t_0$, we will show (12.8) by a continuity (bootstrap) argument. It suffices to show that

$$f(\tilde{t}) \leq 2\tilde{C}\tilde{t}^{-\tilde{p}}, \quad \text{for } \tilde{t} \leq t, \tag{12.9}$$

leads to

$$\Rightarrow f(\tilde{t}) \leq \tilde{C}\tilde{t}^{-\tilde{p}}, \quad \text{for } \tilde{t} \leq t, \tag{12.10}$$

for some large enough constant $\tilde{C}$.

We note first that given any t_0, from (12.7) we obtain, $\forall\, 1 \leq t \leq t_0$

$$\begin{aligned} f(t) &\leq f(1) + C_0\, t, \\ &\leq \left[f(1){t_0}^{\tilde{p}} + C_0\, {t_0}^{\tilde{p}+1}\right] t^{-\tilde{p}}. \end{aligned} \tag{12.11}$$

Given $t \geq t_0$, choose $\tilde{t} = t - L$ for an L to be determined later. Moreover, t_0 will have to be chosen large enough so that $\forall\, t \geq t_0$,

$$(t-L)^{-\tilde{p}} = \tilde{t}^{-\tilde{p}} < 2t^{-\tilde{p}}. \tag{12.12}$$

Given a t satisfying (12.9) applying (12.7) yields

$$\begin{aligned} f(t) + b\int_{\tilde{t}}^{t} f(\bar{t})\mathrm{d}\bar{t} &\leq \left[2\tilde{C} + C_0(L+1)\right]\tilde{t}^{-\tilde{p}} \\ &\overset{(12.12)}{\leq} \left[4\tilde{C} + 2C_0(L+1)\right] t^{-\tilde{p}}. \end{aligned} \tag{12.13}$$

Further, by the pigeonhole principle, there exists $t_{in} \in \left[\tilde{t}, t\right]$ such that

$$f(t_{in}) \leq \frac{1}{L}\int_{\tilde{t}}^{t} f(\bar{t})\mathrm{d}\bar{t}. \tag{12.14}$$

Since $f(t)$ is a positive function (12.13) also leads to

$$b\int_{\tilde{t}}^{t} f(\bar{t})\mathrm{d}\bar{t} \leq \left[4\tilde{C} + 2C_0(L+1)\right] t^{-\tilde{p}}. \tag{12.15}$$

Thus, (12.14) and (12.15) yield

$$f(t_{in}) \leq \frac{1}{bL}\left[4\tilde{C} + 2C_0(L+1)\right] t^{-\tilde{p}}. \tag{12.16}$$

Now let $\tilde{t} = t_{in}$ and use (12.16) in (12.7), then

$$\begin{aligned} f(t) \leq f(t) + b\int_{t_{in}}^{t} f(\bar{t})\mathrm{d}\bar{t} \quad &\leq \quad \left(\frac{1}{bL}\left[4\tilde{C} + 2C_0(L+1)\right] t^{-\tilde{p}} + C_0(L+1)\right) t_{in}^{-\tilde{p}}, \\ &\overset{(12.12)}{\leq} \quad \left[\frac{4\tilde{C}}{bL} + \frac{2C_0(L+1)}{bL} + 2C_0(L+1)\right] t^{-\tilde{p}}. \end{aligned} \tag{12.17}$$

If $1 - \frac{4}{bL} > 0$ and

$$\tilde{C} \geq \left(1 - \frac{4}{bL}\right)^{-1}\left[\frac{2C_0(L+1)}{bL} + 2C_0(L+1)\right] \tag{12.18}$$

then (12.10) follows.

Thus picking first L such that $1 - \frac{4}{bL} > 0$, and then t_0 such that $t_0 \geq L+1$ and satisfying (12.12), and finally choosing $\tilde{C}$ as $\tilde{C} = \max\left\{\left[f(1)t_0 + C_0 {t_0}^{\tilde{p}+1}\right], \left(1 - \frac{4}{bL}\right)^{-1}\left[\frac{2C_0(L+1)}{bL} + 2C_0(L+1)\right]\right\}$ (12.10) and thus (12.8) follows by continuity.

□

By Lemma 12.1.3 we obtain from (12.5) together with (12.6)

$$E(\phi; v_*) = \int_{\{r_{red} \leq r \leq r_+\}} J^N_\mu(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} \leq \tilde{\tilde{C}} {v_*}^{-2-2\delta}, \tag{12.19}$$

with $\tilde{\tilde{C}}$ depending on $\tilde{b}_1$ and $D_0(u_\diamond, 1)$.

Finally, in order to close the proof of Proposition 12.1.2 we perform again the divergence theorem but for region $\tilde{\mathcal{R}}_I = \{r_{red} \le r \le r_+\} \cap \{v_* \le v \le v_* + 1\}$:

$$\begin{aligned}
&\int_{\tilde{\mathcal{R}}_I} K^N(\phi)\mathrm{dVol} + \int_{\{v_* \le v \le v_*+1\}} J^N_\mu(\phi) n^\mu_{r=r_{red}} \mathrm{dVol}_{r=r_{red}} \\
&+ \int_{\{r_{red} \le r \le r_+\}} J^N_\mu(\phi) n^\mu_{v=v_*+1} \mathrm{dVol}_{v=v_*+1} \\
= &\int_{\{r_{red} \le r \le r_+\}} J^N_\mu(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} + \int_{\{v_* \le v \le v_*+1\}} J^N_\mu(\phi) n^\mu_{\mathcal{H}^+} \mathrm{dVol}_{\mathcal{H}^+}.
\end{aligned} \tag{12.20}$$

In view of the signs we obtain

$$\begin{aligned}
\Rightarrow \int_{\{v_* \le v \le v_*+1\}} J^N_\mu(\phi) n^\mu_{r=r_{red}} \mathrm{dVol}_{r=r_{red}} \le &\int_{\{r_{red} \le r \le r_+\}} J^N_\mu(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} \\
&+ \int_{\{v_* \le v \le v_*+1\}} J^N_\mu(\phi) n^\mu_{\mathcal{H}^+} \mathrm{dVol}_{\mathcal{H}^+}.
\end{aligned}$$

Due to (12.19) and Theorem 11.1.1 we are left with the conclusion of Proposition 12.1.2. □

Note that the above also implies the following statement.

Corollary 12.1.4. *Let ϕ be as in Theorem 11.1.1 and for r_{red} as in Proposition 12.1.1. Then, for all $v_* \ge 1$, $v_* + 1 \le v_{red}(\tilde{u})$ and for all $\tilde{u}$ such that $r(\tilde{u}, v_* + 1) \in [r_{red}, r_+)$, we have*

$$\int_{\{v_* \le v \le v_*+1\}} J^N_\mu(\phi) n^\mu_{u=\tilde{u}} dVol_{u=\tilde{u}} \le C v_*^{-2-2\delta}, \tag{12.21}$$

with C depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$ and $v_{red}(\tilde{u})$ as in (10.50)*.*

Proof. The conclusion of the statement follows by applying again the divergence theorem and using the results of the proof of Proposition 12.1.2. □

12.1.2 Propagating through $\mathcal{N}$ from $r = r_{red}$ to $r = r_{blue}$

Now that we have obtained a decay bound along the $r = r_{red}$ hypersurface in the previous section, we propagate the estimate further inside the black hole through the noshift region $\mathcal{N}$ up to the $r = r_{blue}$ hypersurface. In order to do that we will use the future directed timelike vector field

$$-\partial_r = \frac{1}{\Omega^2}(\partial_u + \partial_v). \tag{12.22}$$

Using (12.22) in (C.11) of Appendix C.2 we obtain

$$\begin{aligned} K^{-\partial_r} &= \frac{4}{\Omega^4}\left[\frac{\partial_u\Omega}{\Omega}(\partial_v\phi)^2 + \frac{\partial_v\Omega}{\Omega}(\partial_u\phi)^2\right] \\ &\quad -\frac{4}{r\Omega^2}(\partial_u\phi\partial_v\phi), \end{aligned} \tag{12.23}$$

for the bulk current. It has the property that it can be estimated by

$$|K^{-\partial_r}(\phi)| \leq B_1 J_\mu^{-\partial_r}(\phi) n^\mu_{r=\bar{r}}, \tag{12.24}$$

where B_1 is independent of v_*. Validity of the estimate can in fact be seen without computation from the fact that timelike currents, such as $J_\mu^{-\partial_r}(\phi)n^\mu_{r=\bar{r}}$ contain all derivatives. The uniformity of B_1 is given by the fact that $K^{-\partial_r}$ and $J^{-\partial_r}$ are invariant under translations along ∂_t, cf. Section 10.3.1 for definition of the t coordinate. Therefore, we can just look at the maximal deformation on a compact $\{t = const\} \cap \{r_{blue} \leq r \leq r_{red}\}$ hypersurface and get an estimate for the deformation everywhere.

Proposition 12.1.5. *Let ϕ be as in Theorem 11.1.1, r_{blue} as in* (10.46) *and r_{red} as in Proposition 12.1.1. Then, for all $v_* > 1$ and $\tilde{r} \in [r_{blue}, r_{red})$, we have*

$$\int_{\{v_* \leq v \leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=\tilde{r}} dVol_{r=\tilde{r}} \leq C{v_*}^{-2-2\delta},$$

with C depending on the initial data or more precisely depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$.

Proof. Given v_*, we define regions $\mathcal{R}_{II}$ and $\tilde{\mathcal{R}}_{II}$ as in Figure 12.2, where we use (10.50) and

$$v(\tilde{r}, v_*) \qquad \text{is determined by} \qquad r(u_{blue}(v_*), v(\tilde{r}, v_*)) = \tilde{r}. \tag{12.25}$$

Thus the depicted regions are given by $\mathcal{R}_{II} \cup \tilde{\mathcal{R}}_{II} = \mathcal{D}^+(\{v_1 \leq v \leq v_* + 1\} \cap \{r = r_{red}\}) \cap \mathcal{N}$, where region $\mathcal{R}_{II}$ is given by $\mathcal{R}_{II} = \mathcal{D}^+(\{v_1 \leq v \leq v_*\} \cap \{r = r_{red}\})$.

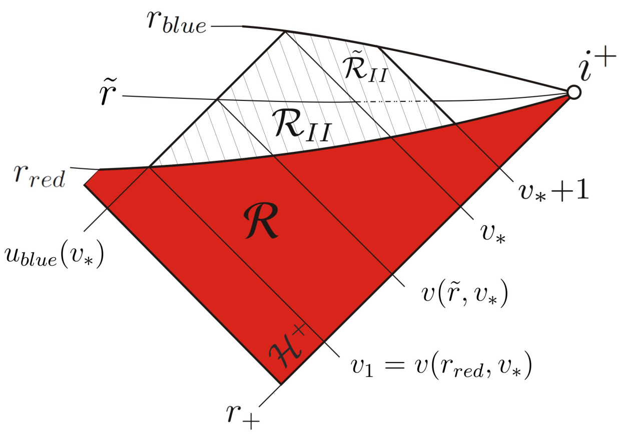


Figure 12.2: Region $\mathcal{R}_{II} \cup \tilde{\mathcal{R}}_{II}$ represented as the hatched area.

In the following we will apply the divergence theorem in region $\mathcal{R}_{II} \cup \tilde{\mathcal{R}}_{II}$ to obtain decay on an arbitrary $r = \tilde{r}$ hypersurface, dash-dotted line, for $\tilde{r} \in [r_{blue}, r_{red})$, from the derived decay on the $r = r_{red}$ hypersurface.

$$\int_{\mathcal{R}_{II} \cup \tilde{\mathcal{R}}_{II}} K^{-\partial_r}(\phi) \mathrm{dVol}$$
$$+ \int_{\{r_{blue} \leq r \leq r_{red}\}} J_\mu^{-\partial_r}(\phi) n^\mu_{u=u_{blue}(v_*)} \mathrm{dVol}_{u=u_{blue}(v_*)}$$
$$+ \int_{\{v_* \leq v \leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{blue}} \mathrm{dVol}_{r=r_{blue}}$$

$$
\begin{aligned}
&+ \int\limits_{\{r_{blue}\leq r\leq r_{red}\}} J_\mu^{-\partial_r}(\phi) n^\mu_{v=v_*+1} \mathrm{dVol}_{v=v_*+1} \\
&= \int\limits_{\{v_1\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{red}} \mathrm{dVol}_{r=r_{red}}.
\end{aligned}
$$

The second integral of the left hand side represents the current through the $u = u_{blue}(v_*)$ hypersurface, defined by (10.50). As $u = u_{blue}(v_*)$ is a null hypersurface and $-\partial_r$ is timelike, the positivity of that second term is immediate. Similarly, the fourth term of the left hand side of our equation is positive and we obtain

$$
\begin{aligned}
&\int\limits_{\{v_*\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{blue}} \mathrm{dVol}_{r=r_{blue}} \\
\leq &\int\limits_{\mathcal{R}_{II}\cup\tilde{\mathcal{R}}_{II}} |K^{-\partial_r}(\phi)| \mathrm{dVol} \\
&+ \int\limits_{\{v_1\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{red}} \mathrm{dVol}_{r=r_{red}}.
\end{aligned}
$$

Further, we use that the deformation $K^{-\partial_r}$ is controlled by the energy associated to the timelike vector field $-\partial_r$ as stated in (12.24). Thus we obtain

$$
\begin{aligned}
&\int\limits_{\{v_*\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{blue}} \mathrm{dVol}_{r=r_{blue}} \\
\leq\ & B_1 \int\limits_{\mathcal{R}_{II}\cup\tilde{\mathcal{R}}_{II}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=\bar{r}} \mathrm{dVol} \\
&+ \int\limits_{\{v_1\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{red}} \mathrm{dVol}_{r=r_{red}}.
\end{aligned} \tag{12.26}
$$

By the coarea formula we obtain

$$
\begin{aligned}
&\int\limits_{\{v_*\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=r_{blue}} \mathrm{dVol}_{r=r_{blue}} \\
\leq\ & \tilde{B}_1 \int\limits_{r_{blue}}^{r_{red}} \int\limits_{\{v(\bar{r},v_*)\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi) n^\mu_{r=\bar{r}} \mathrm{dVol}_{r=\bar{r}} \mathrm{d}\bar{r}
\end{aligned}
$$

$$+\int\limits_{\{v_1\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi)n^\mu_{r=r_{red}}\mathrm{dVol}_{r=r_{red}}. \tag{12.27}$$

Now let

$$E(\phi;\tilde{r},\tilde{v})=\int\limits_{\{\tilde{v}\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi)n^\mu_{r=\tilde{r}}\mathrm{dVol}_{r=\tilde{r}}, \tag{12.28}$$

with $\tilde{r}\in[r_{blue},r_{red})$. Replacing r_{blue} with $\tilde{r}$ in the above, considering the future domain of dependence of $\{v_1\leq v\leq v_*+1\}\cap\{r=r_{red}\}$ up to the $r=\tilde{r}$ hypersurface we obtain similarly to (12.27)

$$E(\phi;\tilde{r},v(\tilde{r},v_*))\leq\tilde{B}_1\int\limits_{\tilde{r}}^{r_{red}} E(\phi;\bar{r},v(\bar{r},v_*))\mathrm{d}\bar{r}+E(\phi;r_{red},v_1). \tag{12.29}$$

Using Grönwall's inequality in (12.29) yields

$$\begin{aligned} E(\phi;\tilde{r},v(\tilde{r},v_*)) &\leq E(\phi;r_{red},v_1)\left[1+\tilde{B}_1(r_{red}-\tilde{r})e^{\tilde{B}_1(r_{red}-\tilde{r})}\right] \\ \Rightarrow E(\phi;\tilde{r},v(\tilde{r},v_*)) &\leq \tilde{C}E(\phi;r_{red},v_1). \end{aligned} \tag{12.30}$$

Finally, note that

$$[v_*+1]-v(r_{red},v_*)=[v_*+1]-v_1=k<\infty, \tag{12.31}$$

where $k=2\left[r^\star(r_{blue})-r^\star(r_{red})\right]+1$. This can be seen since (10.25) and (10.30) yields

$$\begin{aligned} &\frac{\partial_v r}{\Omega^2}=-\partial_v r^\star=-\frac{1}{2} \\ \Rightarrow\quad &r^\star(u_{blue}(v_*),v_*)-r^\star(u_{blue}(v_*),v_1)=r^\star(r_{blue})-r^\star(r_{red})\overset{(10.29)}{=}\frac{1}{2}(v_*-v_1). \end{aligned}$$

Further, by using the conclusion of Proposition 12.1.2 and (12.31) we have

$$\begin{aligned} E(\phi;r_{red},v_1) &= \int\limits_{\{v_1\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi)n^\mu_{r=r_{red}}\mathrm{dVol}_{r=r_{red}} \\ &= \int\limits_{\{v_1\leq v\leq v_1+k\}} J_\mu^{-\partial_r}(\phi)n^\mu_{r=r_{red}}\mathrm{dVol}_{r=r_{red}} \end{aligned}$$

$$\leq \quad C\max\{k,1\}\, {v_1}^{-2-2\delta} \sim C{v_*}^{-2-2\delta}. \tag{12.32}$$

We thus infer

$$\begin{aligned}\int_{\{v_*\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi)n^\mu_{r=\tilde{r}}\mathrm{dVol}_{r=\tilde{r}} &\leq \tilde{C}\int_{\{v_1\leq v\leq v_*+1\}} J_\mu^{-\partial_r}(\phi)n^\mu_{r=r_{red}}\mathrm{dVol}_{r=r_{red}}\\ &\leq \tilde{C}C{v_*}^{-2-2\delta}.\end{aligned}$$

□

The above now also implies the following statement.

Corollary 12.1.6. *Let ϕ be as in Theorem 11.1.1, r_{blue} as in (10.46) and r_{red} as in Proposition 12.1.1. Then, for all $v_*>1$ and all $\tilde{u}$ such that $r(\tilde{u},v_*)\in[r_{blue},r_{red})$*

$$\int_{\{v_{red}(\tilde{u})\leq v\leq v_{blue}(\tilde{u})\}} J_\mu^N(\phi)n^\mu_{u=\tilde{u}}dVol_{u=\tilde{u}} \leq Cv_*^{-2-2\delta}, \tag{12.33}$$

with C depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond,1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red}=r(u_\diamond,1)$ and $v_{red}(\tilde{u})$, $v_{blue}(\tilde{u})$ are as in (10.50).

Proof. The conclusion of the statement follows by considering the divergence theorem for a triangular region $J^-(x)\cap\mathcal{N}$ with $x=(\tilde{u},v_{blue}(\tilde{u}))$, $x\in J^-(r=r_{blue})$ and using the results of Proposition 12.1.5. Note that $v_*\sim v_{blue}(\tilde{u})\sim v_{red}(\tilde{u})$. □

By the previous proposition we have successfully propagated the energy estimate further inside the black hole, up to $r=r_{blue}$. To go even further will be more difficult and we will address this in the next section.

12.1.3 Propagating through $\mathcal{B}$ from $r=r_{blue}$ to the hypersurface γ

In the following we want to propagate the estimates from the $r=r_{blue}$ hypersurface further into the blueshift region to a hypersurface γ which is located a logarithmic distance in v from the $r=r_{blue}$ hypersurface, cf. Figure 12.3. We will define the hypersurface γ and its most basic properties in Section 12.1.3 and propagate the decay bound to γ in Section 12.1.3.

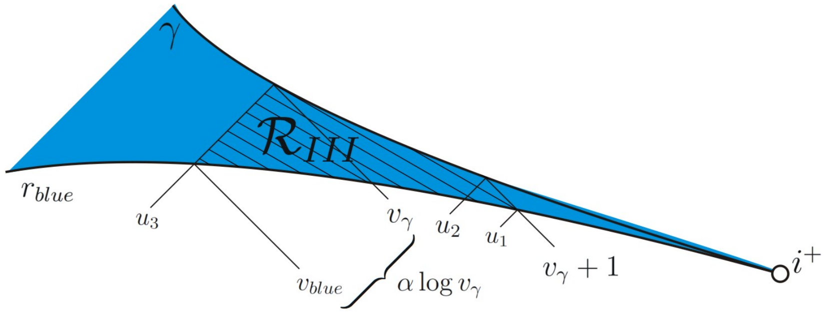


Figure 12.3: Logarithmic distance of hypersurface $r = r_{blue}$ and hypersurface γ depicted in a Penrose diagram.

The hypersurface γ

The idea of the hypersurface γ was already entertained in [23] by Dafermos and basically locates γ a logarithmic distance in v from a constant r hypersurface living in the blueshift region.

Let α be a fixed constant satisfying

$$\alpha > \frac{p+1}{\beta}, \tag{12.34}$$

with β as in (10.48) and (10.49). (The significance of the bound (12.34) will become clear later.) Let us for convenience also assume that

$$\alpha \quad > \quad 1, \tag{12.35}$$

and

$$\alpha(2 - \log 2\alpha) \quad > \quad 2r^\star_{blue} + 1. \tag{12.36}$$

We define the function $H(u, v)$ by

$$H(u, v) = u + v - \alpha \log v - 2r^\star(r_{blue}) = u + v - \alpha \log v - 2r^\star_{blue}, \tag{12.37}$$

were $r^\star(r_{blue}) = r^\star_{blue}$ is the $r^\star$ value evaluated at r_{blue} according to (10.31), and $r^\star_{blue} > 0$

according to the choice (10.47). We then define the hypersurface γ as the levelset

$$\gamma = \{H(u,v) = 0\} \cap \{v > 2\alpha\}. \tag{12.38}$$

Since

$$\frac{\partial H}{\partial u} = 1, \qquad \frac{\partial H}{\partial v} = 1 - \frac{\alpha}{v}, \tag{12.39}$$

we see that γ is a spacelike hypersurface and terminates at i^+, cf. Appendix C.1. (In the notation (10.50), $u_\gamma(v) \to -\infty$ as $v \to \infty$.) Note that by our choices $u < -1$ and $v > |u|$ in $\mathcal{D}^+(\gamma)$.

Recall that in Section 10.3.1 we have defined the Regge-Wheeler tortoise coordinate $r^\star$ depending on u, v by (10.29). Using this for $r^\star_{blue}$ we have

$$r^\star(r_{blue}) = \frac{v_{blue}(u) + u}{2}, \tag{12.40}$$

with $v_{blue}(u)$ as in (10.50). Plugging this into (12.38) recalling $v_\gamma(u)$ defined in (10.50), we obtain the relation

$$v_\gamma(u) - v_{blue}(u) = \alpha \log v_\gamma(u). \tag{12.41}$$

As we shall see in Section 12.1.4 the above properties of γ will allow us to derive pointwise estimates of Ω^2 in $J^+(\gamma) \cap \mathcal{B}$. We first turn however to the region $J^-(\gamma) \cap \mathcal{B}$.

Energy estimates from $r = r_{blue}$ to the hypersurface γ

Now we are ready to propagate the energy estimates further into the blueshift region $\mathcal{B}$ up to the hypersurface γ. We will in this part of the proof use the vector field

$$S_0 = r^q \partial_{r^\star} = r^q(\partial_u + \partial_v),$$

which we have defined in (11.10). Let us now consider the bulk term and derive positivity properties which are needed later on. Plugging (11.10) in (C.11) of Appendix C.2 yields

$$\begin{aligned} K^{S_0} = \; & + \quad q r^{q-1}\left[(\partial_v\phi)^2 + (\partial_u\phi)^2\right] \\ & - \quad \left[\frac{q r^{q-1}}{2}\left[\partial_v r + \partial_u r\right] + r^q\left(\frac{\partial_u\Omega}{\Omega} + \frac{\partial_v\Omega}{\Omega}\right)\right]|\nabla\!\!\!/\,\phi|^2 \end{aligned}$$

$$- \quad 4r^{q-1}(\partial_u\phi\partial_v\phi). \tag{12.42}$$

Our aim is to show that K^{S_0} is positive. All terms multiplied by the angular derivatives are manifestly positive in $\mathcal{B}$, cf. (10.48), (10.49) together with (10.25) to (10.27). Therefore, it is only left to show that the first term on the right hand side dominates the last term. Since by the Cauchy-Schwarz inequality

$$2qr^{q-1}(\partial_u\phi\partial_v\phi) \leq qr^{q-1}\left[(\partial_v\phi)^2 + (\partial_u\phi)^2\right], \tag{12.43}$$

K^{S_0} is positive in $\mathcal{B}$ for all $q \geq 2$.

We show now that at the expense of one polynomial power, we can extend the local energy estimate on $r = r_{blue}$ to an energy estimate along γ which is valid for a dyadic length.

Proposition 12.1.7. *Let ϕ be as in Theorem 11.1.1. Then, for all $v_* > 2\alpha$*

$$\int\limits_{\{v_*\leq v\leq 2v_*\}} J^{S_0}_\mu(\phi)n^\mu_\gamma dVol_\gamma \leq C{v_*}^{-1-2\delta}, \tag{12.44}$$

on the hypersurface γ, with C depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$.

Remark. n^μ_γ denotes the normal vector on the hypersurface γ which is a levelset $\gamma = \{H(u,v) = 0\}$ of the function $H(u,v)$ defined in (12.37). For calculation of n^μ_γ and $J^{S_0}_\mu(\phi)n^\mu_\gamma$ refer to (C.1) and (C.6) of Appendix C.1.

Proof. In the following we will again make use of notation (10.50).

Let $v_* > 2\alpha$, such that γ is spacelike for $v > v_*$, cf. Section 12.1.3. Define u_3 by $(u_3, v_*) \in \gamma$, i.e. $u_\gamma(v_*) = u_3$ and define v_{blue} as the intersection of u_3 with r_{blue}, i.e. $v_{blue}(u_3) = v_{blue}$. And similarly the hypersurfaces $u = u_1$ and $u = u_2$ as shown in Figure 12.3 are given by $u_{blue}(2v_*) = u_1$ and $u_\gamma(2v_*) = u_2$. Having defined the relations between all these quantities we can now carry out the divergence theorem for region $\mathcal{R}_{III}$:

$$\int\limits_{\mathcal{R}_{III}} K^{S_0}(\phi)\mathrm{dVol} + \int\limits_{\{v_{blue}\leq v\leq v_*\}} J^{S_0}_\mu(\phi)n^\mu_{u=\tilde{u}}\mathrm{dVol}_{u=\tilde{u}}$$

$$+ \int_{\{v_* \leq v \leq 2v_*\}} J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma + \int_{\{u_1 \leq u \leq u_2\}} J^{S_0}_\mu(\phi) n^\mu_{v=2v_*} \mathrm{dVol}_{v=2v_*}$$
$$= \int_{\{v_{blue} \leq v \leq 2v_*\}} J^{S_0}_\mu(\phi) n^\mu_{r=r_{blue}} \mathrm{dVol}_{r=r_{blue}}. \tag{12.45}$$

Positivity of the flux along the $u = u_3$ segment and the flux along the $v = 2v_*$ segment, as well as positivity of K^{S_0} for the choice $q \geq 2$, which was derived in (12.42) and (12.43), leads to

$$\begin{aligned}
\int_{\{v_* \leq v \leq 2v_*\}} J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma &\leq \int_{\{v_{blue} \leq v \leq 2v_*\}} J^{S_0}_\mu(\phi) n^\mu_{r=r_{blue}} \mathrm{dVol}_{r=r_{blue}}, \\
&\leq C \max\{2v_* - v_{blue}, 1\} v_{blue}^{-2-2\delta}, \\
&\overset{(12.41)}{\leq} C\left(v_* + \alpha \log v_*\right) v_{blue}^{-2-2\delta}, \\
&\leq \tilde{C} C v_*^{-1-2\delta},
\end{aligned} \tag{12.46}$$

where the second step is implied by Proposition 12.1.5 and the last step follows from the inequality $v_* \leq C v_{blue}$ which is implied by (12.41). □

We have already mentioned in the introduction that we will use the vector field S, cf. (11.11) in the region $J^+(\gamma) \cap \mathcal{B}$. To control the initial flux term of S we require a weighted energy estimate along the hypersurface γ.

Corollary 12.1.8. *Let ϕ be as in Theorem 11.1.1. Then, for all $v_* > 2\alpha$*

$$\int_{\{v_* \leq v < \infty\}} v^p J^{S_0}_\mu(\phi) n^\mu_\gamma dVol_\gamma \leq C v_*^{-1-2\delta+p}, \tag{12.47}$$

on the hypersurface γ, with C depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$ and p as in (11.12).

Proof. The proof is obtained by summing dyadically and weighting the result with v_*^p as follows. We introduce a dyadic sequence $v_i \in [v_*, \infty)$, with $i \in \mathbb{N}_0$, such that

$$v_{i+1} = 2v_i. \tag{12.48}$$

With this sequence we can picture a series of regions shaped such as region $\mathcal{R}_{III}$ and partly overlapping each other going up to i^+ as shown in Figure 12.4.

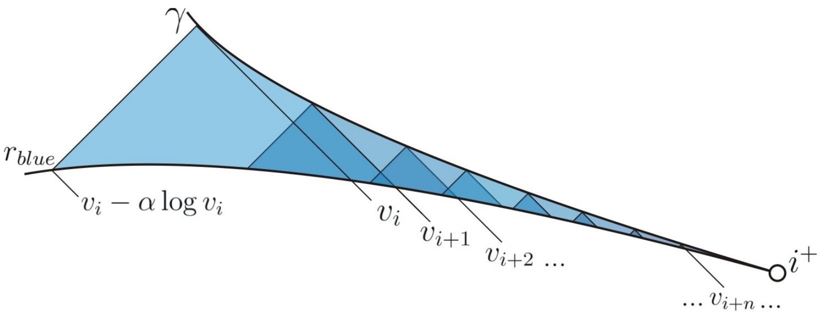


Figure 12.4: Qualitative sketch of the sequence of regions.

Then with the divergence theorem we can state

$$\begin{aligned}
\int\limits_{\{v_1 \leq v < \infty\}} J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma &\leq \sum_{i=1} \int\limits_{\{v_i \leq v \leq v_{i+1}\}} J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma \\
&\leq \sum_{i=1} C v_i^{-1-2\delta} \\
&\leq C v_1^{-1-2\delta} \sum_{i=1} 2^{(i-1)(-1-2\delta)} \\
&\leq \tilde{C} C v_1^{-1-2\delta},
\end{aligned} \tag{12.49}$$

where the second step follows by the conclusion of Proposition 12.1.7 and the last step follows since all v_i are comparable and the sequence converges for $\delta > 0$. To introduce the weight we will again make use of sequence (12.48). Let us first state

$$\int\limits_{\{v_1 \leq v < \infty\}} v^p J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma \leq \int\limits_{\{v_1 \leq v < \infty\}} g(v) J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma, \tag{12.50}$$

where $g(v)$ is a piecewise constant function

$$g(v) = \sum_{i=1} v^p_{i+1} \chi_i(v) \overset{(12.48)}{=} \sum_{i=1} 2^p v_i^p \chi_i(v), \tag{12.51}$$

with

$$\chi_i(v) = \begin{cases} 1, & v \in (v_i, v_{i+1}) \\ 0, & v \notin (v_i, v_{i+1}) \end{cases},$$

so that $g(v)$ is everywhere bigger or equal to a function v^p.

We then obtain

$$
\begin{aligned}
\int\limits_{\{v_1\leq v<\infty\}} v^p J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma &\leq \sum_{i=1} 2^p v_i^p \int\limits_{\{v_i\leq v\leq v_{i+1}\}} J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma \\
&\leq C\sum_{i=1} 2^p v_i^{-1-2\delta+p} \\
&\leq C v_1^{-1-2\delta+p} \sum_{i=1} 2^{(i-1)(-1-2\delta+p)+p} \\
&\leq \tilde{C} C v_1^{-1-2\delta+p}, \qquad (12.52)
\end{aligned}
$$

where the sum in the last step converges for $p < 1+2\delta$. And the Corollary follows for $v_1 = v_*$. □

Further, we can state the following.

Corollary 12.1.9. *Let ϕ be as in Theorem 11.1.1, r_{blue} as in (10.46) and γ as in (12.38). Then, for all $v_* > 2\alpha$ and for all $\tilde{u} \in [u_{blue}(v_*), u_\gamma(v_*))$*

$$\int\limits_{\{v_{blue}(\tilde{u})\leq v\leq v_\gamma(\tilde{u})\}} v^p J^{S_0}_\mu(\phi) n^\mu_{u=\tilde{u}} dVol_{u=\tilde{u}} \leq C v_*^{-1-2\delta+p}, \qquad (12.53)$$

with C depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$ and $v_\gamma(\tilde{u})$, $v_{blue}(\tilde{u})$ as in (10.50).

Proof. The proof is similar to the proof of Corollary 12.1.6 by considering the divergence theorem for a triangular region $J^-(x)\cap\mathcal{B}$ with $x = (\tilde{u}, v_\gamma(\tilde{u}))$, $x \in J^-(\gamma)$ and using the results of the proof of Proposition 12.1.7. □

12.1.4 Propagating through $\mathcal{B}$ from the hypersurface γ to $\mathcal{CH}^+$ in the neighbourhood of i^+

In order to prove our Theorem 11.2.1 and close our estimates up to the Cauchy horizon in the neighbourhood of i^+ we are interested in considering a region $\mathcal{R}_{IV}$ within the trapped region whose boundaries are made up of the hypersurface γ, a constant u and a constant v segment, which can reach up to the Cauchy horizon, cf. Figure 12.5. Let $v_* > 2\alpha$ and let $\hat{v} > v_*$. We may write $\mathcal{R}_{IV} = J^+(\gamma)\cap J^-(x)$ with $x = (u_\gamma(v_*), \hat{v})$, $x \in J^+(\gamma)\cap\mathcal{B}$. Note that $\mathcal{R}_{IV}$ lies entirely in the blueshift region, which was characterized by the fact

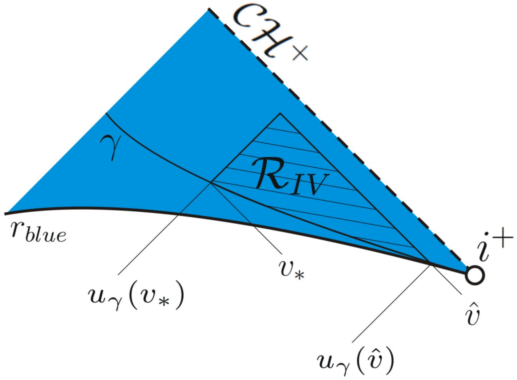


Figure 12.5: Blueshift region of Reissner-Nordström spacetime from hypersurface γ onwards.

that the quantity $M - \frac{e^2}{r}$ takes the negative sign, cf. (10.48), (10.49) and (10.25) to (10.27).

In Section 12.1.4 we will derive pointwise estimates for Ω^2 in the future of the hypersurface γ. With this estimate, the bulk term will be bounded in terms of the currents through the null hypersurfaces. Consequently, we will be able to absorb the bulk term and to show that the currents through the null hypersurfaces can be bounded by the current along the hypersurface γ, cf. Section 12.1.4.

Pointwise estimates on Ω^2 in $J^+(\gamma)$

In the following we will derive pointwise estimates on Ω^2 in $J^+(\gamma)$. We note that these will imply that the spacetime volume to the future of the hypersurface γ is finite, $\mathrm{Vol}(J^+(\gamma)) < C$.

We first derive a future decay bound along a constant u hypersurface for the function $\Omega^2(u,v)$ for $(u,v) \in \mathcal{B}$. Let $x = (u_{fix}, v_{fix})$, $x \in \mathcal{B}$, then, from (10.27) we can immediately see that

$$\begin{aligned} \log\left(\Omega^2(u_{fix}, v)\right)\Big|_{v_{fix}}^{\bar{v}} &= \int\limits_{v_{fix}}^{\bar{v}} \frac{1}{2r^2}\left(M - \frac{e^2}{r}\right) \mathrm{d}v, \\ &\overset{(10.49)}{\leq} -\beta[\bar{v} - v_{fix}]. \end{aligned} \tag{12.54}$$

It then immediately follows that

$$\Omega^2(\bar{u},\bar{v})\leq\Omega^2(\bar{u},v_{fix})e^{-\beta[\bar{v}-v_{fix}]},\quad \text{for all } (\bar{u},v_{fix})\in\mathcal{B} \text{ and } \bar{v}>v_{fix}. \tag{12.55}$$

Analogously, we obtain

$$\Omega^2(\bar{u},v_{fix})\leq\Omega^2(u_{fix},v_{fix})e^{-\beta[\bar{u}-u_{fix}]},\quad \text{for all } (\bar{u},v_{fix})\in J^+(x), \tag{12.56}$$

and plugging (12.55) into (12.56) it yields

$$\Omega^2(\bar{u},\bar{v})\leq\Omega^2(u_{fix},v_{fix})e^{-\beta[\bar{u}-u_{fix}+\bar{v}-v_{fix}]},\quad \text{for all } (\bar{u},\bar{v})\in J^+(x). \tag{12.57}$$

From (12.55) and (12.41) we obtain a relation for $\Omega^2(u,v)$ on the hypersurface γ as follows

$$\begin{aligned}\Omega^2(\bar{u},\bar{v}) &\leq \Omega^2(\bar{u},v_{blue}(\bar{u}))e^{-\beta\alpha\log v_\gamma(\bar{u})}\\ &= \Omega^2(\bar{u},v_{blue}(\bar{u}))v_\gamma(\bar{u})^{-\beta\alpha},\quad \text{for } (\bar{u},\bar{v})\in\gamma.\end{aligned} \tag{12.58}$$

For $J^+(\gamma)$, using (12.55) we further get

$$\Omega^2(\bar{u},\bar{v})\leq Cv_\gamma(\bar{u})^{-\beta\alpha}e^{-\beta[\bar{v}-v_\gamma(\bar{u})]},\quad \text{for } (\bar{u},\bar{v})\in J^+(\gamma), \tag{12.59}$$

where we have used $\Omega^2(\bar{u},v_{blue}(\bar{u}))\leq C$. Moreover, we may think of a parameter $\bar{v}$ which determines the associated u value via intersection with γ, we denote this value by the evaluation the function $u_\gamma(\bar{v})$ which was introduced in (10.50), cf. Figure 10.6.

Moreover, by (10.26) we can also state

$$\Omega^2(\bar{u},\bar{v})\leq C|u_\gamma(\bar{v})|^{-\beta\alpha}e^{\beta[u_\gamma(\bar{v})-\bar{u}]}\quad \text{for } (\bar{u},\bar{v})\in J^+(\gamma). \tag{12.60}$$

Note that the choice (12.34) of α implies that $\beta\alpha>1$. From (12.60), the fact that $|u_\gamma(\bar{v})|\sim\bar{v}$, and the extra exponential factor, finiteness of the spacetime volume to the future of γ follows,

$$\mathrm{Vol}(J^+(\gamma))<C. \tag{12.61}$$

See also [57].

Bounding the bulk term K^S

To derive energy estimates in $\mathcal{R}_{IV}$ we use the timelike vector field multiplier

$$S = |u|^p \partial_u + v^p \partial_v,$$

which we have given before in (11.11). The weights of S are chosen such that they will allow us to derive pointwise estimates from energy estimates; see Section 12.2.3.

In order to obtain our desired estimates first of all we need a bound on the scalar current K^S, in terms of $J^S_\mu(\phi)n^\mu_{v=\bar{v}}$ and $J^S_\mu(\phi)n^\mu_{u=\bar{u}}$. In the following we will bound the occurring (u,v)-dependent weight functions by functions that depend on either u or v, respectively. Plugging the vector field S, cf. (11.11), into (C.11) of Appendix C.2 we obtain

$$\begin{aligned} K^S = & - \frac{2}{r}\left[v^p + |u|^p\right](\partial_u\phi\partial_v\phi) \\ & - \left[\frac{\partial_u\Omega}{\Omega}|u|^p + \frac{\partial_v\Omega}{\Omega}v^p + \frac{p}{2}(v^{p-1} + |u|^{p-1})\right]|\nabla\!\!\!/\phi|^2. \end{aligned} \tag{12.62}$$

Recall (10.48) and (10.49). For large absolute values of v and u the first two terms multiplying the angular derivatives of ϕ dominate the last two terms, so in total the term multiplying the angular derivatives is always positive in $\mathcal{D}^+(\gamma)$. Consequently we will be able to use this property to derive an inequality by using the divergence theorem in the proof of Proposition 12.1.11. Let us therefore define

$$\tilde{K}^S = - \frac{2}{r}\left[v^p + |u|^p\right](\partial_u\phi\partial_v\phi), \tag{12.63}$$

and state

$$-K^S \leq |\tilde{K}^S| \quad \text{for } v > \tfrac{p}{2\beta} \text{ and } |u| > \tfrac{p}{2\beta}, \text{ cf. (10.49), (10.48).} \tag{12.64}$$

(Note that $\tilde{K}^S$ coincides with the bulk term for spherically symmetric ϕ.) We have the following

Lemma 12.1.10. *Let ϕ be an arbitrary function. Then, for all $v_* > 2\alpha$ and all $\hat{v} > v_*$, the integral over region $\mathcal{R}_{IV} = J^+(\gamma) \cap J^-(x)$ with $x = (u_\gamma(v_*), \hat{v})$, $x \in \mathcal{B}$, cf. Figure*

12.5, of the current $\tilde{K}^S$, defined by (12.63)*, can be estimated by*

$$\begin{aligned}\int\limits_{\mathcal{R}_{IV}} |\tilde{K}^S| dVol \leq\ & \delta_1 \sup_{u_\gamma(\hat{v})\leq \bar{u}\leq u_\gamma(v_*)} \int\limits_{\{v_\gamma(\bar{u})\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\bar{u}} dVol_{u=\bar{u}} \\ +\ & \delta_2 \sup_{v_*\leq \bar{v}\leq \hat{v}} \int\limits_{\{u_\gamma(\hat{v})\leq u\leq u_\gamma(\bar{v})\}} J^S_\mu(\phi) n^\mu_{v=\bar{v}} dVol_{v=\bar{v}}, \end{aligned} \tag{12.65}$$

where δ_1 and δ_2 are positive constants, with $\delta_1 \to 0$ and $\delta_2 \to 0$ as $v_* \to \infty$.

Remark. In the proof of Proposition 12.1.11 we will see that the above proposition determines $u_{\text{✂}}$ of Theorem 11.2.1, depicted in Figure 11.3. We have to choose $u_{\text{✂}} = u_\gamma(v_*)$, with v_* such that δ_1 is small.

Proof. Using the Cauchy-Schwarz inequality twice for the remaining part of the bulk term we obtain

$$|\tilde{K}^S| \quad \leq \quad \frac{1}{r}\left[\left(1+\frac{|u|^p}{v^p}\right) v^p(\partial_v\phi)^2 + \left(1+\frac{v^p}{|u|^p}\right)|u|^p(\partial_u\phi)^2\right], \tag{12.66}$$

with the related volume element

$$\mathrm{dVol} \quad = \quad r^2\frac{\Omega^2}{2}\mathrm{d}u\mathrm{d}v\mathrm{d}\sigma^2_{\mathbb{S}}. \tag{12.67}$$

Note that the currents related to the vector field S with their related volume elements are given by

$$J^S_\mu(\phi)n^\mu_{v=\bar{v}} \quad = \quad \frac{2}{\Omega^2}\left[|u|^p(\partial_u\phi)^2 + \frac{\Omega^2}{4}\bar{v}^p|\nabla\!\!\!/\phi|^2\right], \tag{12.68}$$

$$\mathrm{dVol}_{v=\bar{v}} = r^2\frac{\Omega^2}{2}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}u, \tag{12.69}$$

$$J^S_\mu(\phi)n^\mu_{u=\bar{u}} \quad = \quad \frac{2}{\Omega^2}\left[v^p(\partial_v\phi)^2 + \frac{\Omega^2}{4}|\bar{u}|^p|\nabla\!\!\!/\phi|^2\right], \tag{12.70}$$

$$\mathrm{dVol}_{u=\bar{u}} = r^2\frac{\Omega^2}{2}\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}v, \tag{12.71}$$

cf. Appendix C.1. Taking the integral over the spacetime region yields

$$\int\limits_{\mathcal{R}_{IV}} |\tilde{K}^S(\phi)|\mathrm{dVol} \quad \leq \quad \int\limits_{u_\gamma(\bar{v})}^{u_\gamma(v_*)} \int\limits_{\{v_*\leq v\leq \hat{v}\}} \frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{|\bar{u}|^p}{\bar{v}^p}\right) J^S_\mu(\phi)n^\mu_{u=\bar{u}}\mathrm{dVol}_{u=\bar{u}}\mathrm{d}\bar{u}$$

$$+\int_{\bar{v}}^{\hat{v}} \int_{\{u_\gamma(\hat{v})\leq u\leq u_\gamma(v_*)\}} \frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{v}^p}{|\bar{u}|^p}\right) J^S_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \mathrm{d}\bar{v}, \tag{12.72}$$

with $u_\gamma(v)$ in the integration limits as defined in (10.50).

Note the following general relation for positive functions $f(\bar{u},\bar{v})$ and $g(\bar{u},\bar{v})$

$$\begin{aligned}
&\int_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \int_{\bar{v}}^{\hat{v}} f(\bar{u},\bar{v}) g(\bar{u},\bar{v}) \mathrm{d}\bar{v}\mathrm{d}\bar{u} \\
\leq\ & \int_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \int_{\bar{v}}^{\hat{v}} \left[\sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} f(\bar{u},\bar{v})\right] g(\bar{u},\bar{v}) \mathrm{d}\bar{v}\mathrm{d}\bar{u} \\
\leq\ & \int_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \left[\sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} f(\bar{u},\bar{v})\right] \int_{\bar{v}}^{\hat{v}} g(\bar{u},\bar{v}) \mathrm{d}\bar{v}\mathrm{d}\bar{u} \\
\leq\ & \int_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} f(\bar{u},\bar{v}) \mathrm{d}\bar{u} \sup_{u_\gamma(\hat{v})\leq\bar{u}\leq u_\gamma(v_*)} \int_{\bar{v}}^{\hat{v}} g(\bar{u},\bar{v}) \mathrm{d}\bar{v} \\
\leq\ & \int_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} f(\bar{u},\bar{v}) \mathrm{d}\bar{u} \sup_{u_\gamma(\hat{v})\leq\bar{u}\leq u_\gamma(v_*)} \int_{v_*}^{\hat{v}} g(\bar{u},\bar{v}) \mathrm{d}\bar{v}.
\end{aligned} \tag{12.73}$$

Similarly, it immediately follows that

$$\begin{aligned}
&\int_{v_*}^{\hat{v}} \int_{u_\gamma(\bar{v})}^{u_\gamma(v_*)} f(\bar{u},\bar{v}) g(\bar{u},\bar{v}) \mathrm{d}\bar{u}\mathrm{d}\bar{v} \\
\leq\ & \int_{v_*}^{\hat{v}} \sup_{u_\gamma(\bar{v})\leq\bar{u}\leq u_\gamma(v_*)} f(\bar{u},\bar{v}) \mathrm{d}\bar{v} \sup_{v_*\leq\bar{v}\leq\hat{v}} \int_{u_\gamma(\bar{v})}^{u_\gamma(v_*)} g(\bar{u},\bar{v}) \mathrm{d}\bar{u}.
\end{aligned} \tag{12.74}$$

Using (12.73) and (12.74) in (12.72) we obtain

$$\begin{aligned}\int\limits_{\mathcal{R}_{IV}} |\tilde{K}^S(\phi)|\mathrm{dVol} \quad \leq \quad & \int\limits_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{|\bar{u}|^p}{\bar{v}^p}\right)\right]\mathrm{d}\bar{u} \\ & \times \sup_{u_\gamma(\hat{v})\leq\bar{u}\leq u_\gamma(v_*)} \int\limits_{\{v_*\leq v\leq\hat{v}\}} J^S_\mu(\phi)n^\mu_{u=\bar{u}}\mathrm{dVol}_{u=\bar{u}} \\ & + \int\limits_{v_*}^{\hat{v}} \sup_{u_\gamma(\bar{v})\leq\bar{u}\leq u_\gamma(v_*)} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{v}^p}{|\bar{u}|^p}\right)\right]\mathrm{d}\bar{v} \\ & \times \sup_{v_*\leq\bar{v}\leq\hat{v}} \int\limits_{\{u_\gamma(\bar{v})\leq u\leq u_\gamma(v_*)\}} J^S_\mu(\phi)n^\mu_{v=\bar{v}}\mathrm{dVol}_{v=\bar{v}}.\end{aligned} \tag{12.75}$$

It remains to show finiteness and smallness of $\int\limits_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{|\bar{u}|^p}{\bar{v}^p}\right)\right]\mathrm{d}\bar{u}$ and $\int\limits_{v_\gamma(\bar{u})}^{\hat{v}} \sup_{u_\gamma(\bar{v})\leq\bar{u}\leq u_\gamma(v_*)} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{v}^p}{|\bar{u}|^p}\right)\right]\mathrm{d}\bar{v}$. Earlier we obtained the relation (12.60) for Ω^2 in region $\mathcal{R}_{IV}$. Therefore, we can write

$$\begin{aligned} & \int\limits_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{|\bar{u}|^p}{\bar{v}^p}\right)\right]\mathrm{d}\bar{u} \\ \leq \quad & C\int\limits_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} \sup_{v_\gamma(\bar{u})\leq\bar{v}\leq\hat{v}} \left[|u_\gamma(\bar{v})|^{-\beta\alpha}e^{\beta[u_\gamma(\bar{v})-\bar{u}]}\left(1+\frac{|\bar{u}|^p}{\bar{v}^p}\right)\right]\mathrm{d}\bar{u} \\ \leq \quad & \tilde{C}\int\limits_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} |\bar{u}|^{-\beta\alpha}\left(1+\frac{|\bar{u}|^p}{v_*^p}\right)\mathrm{d}\bar{u} \\ \leq \quad & \tilde{\tilde{C}}\int\limits_{u_\gamma(\hat{v})}^{u_\gamma(v_*)} |\bar{u}|^{-\beta\alpha+p}\mathrm{d}\bar{u} \\ \leq \quad & \frac{\tilde{\tilde{C}}}{|-\beta\alpha+p+1|}\left[|\bar{u}|^{-\beta\alpha+p+1}\right]^{u_\gamma(v_*)}_{u_\gamma(\hat{v})}\end{aligned}$$

$$\leq \quad \delta_1, \tag{12.76}$$

where $\delta_1 \to 0$ for $|u_\gamma(v_*)| \to -\infty$ and thus for $v_* \to \infty$. Note that we have here used (12.34).

For finiteness of the second term in (12.75) we follow the same strategy and use (12.59) for the second term to obtain

$$\begin{aligned}
&\int_{v_*}^{\hat{v}} \sup_{u_\gamma(\bar{v}) \leq \bar{u} \leq u_\gamma(v_*)} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r} \left(1 + \frac{\bar{v}^p}{|\bar{u}|^p}\right)\right] d\bar{v} \\
\leq \quad & C \int_{v_*}^{\hat{v}} \sup_{u_\gamma(\bar{v}) \leq \bar{u} \leq u_\gamma(v_*)} \left[\bar{v}^{-\beta\alpha} \left(1 + \frac{\bar{v}^p}{|\bar{u}|^p}\right)\right] d\bar{v} \\
\leq \quad & \tilde{C} \int_{v_*}^{\hat{v}} \bar{v}^{-\beta\alpha} \left(1 + \frac{\bar{v}^p}{|u_\gamma(v_*)|^p}\right) d\bar{v} \\
\leq \quad & \frac{\tilde{\tilde{C}}}{|-\beta\alpha + p + 1|} \left[|\bar{v}|^{-\beta\alpha+p+1} \right]_{v_*}^{\hat{v}} \\
\leq \quad & \delta_2,
\end{aligned} \tag{12.77}$$

where $\delta_2 \to 0$ for $v_* \to \infty$. Therefore, we obtain the statement of Lemma 12.1.10. □

Energy estimates from γ up to $\mathcal{CH}^+$ in the neighbourhood of i^+

Now we come to the actual proof of weighted energy boundedness up to the Cauchy horizon.

Proposition 12.1.11. *Let ϕ be as in Theorem 11.1.1 and p as in* (11.12). *Then, for $u_{✄}$ sufficiently close to $-\infty$, for all $v_* \geq v_\gamma(u_{✄})$ and $\hat{v} > v_*$*

$$\begin{aligned}
&\int_{\{u_\gamma(\hat{v}) \leq u \leq u_\gamma(v_*)\}} J^S_\mu(\phi) n^\mu_{v=\hat{v}} dVol_{v=\hat{v}} \\
+ \quad & \int_{\{v_* \leq v \leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=u_\gamma(v_*)} dVol_{u=u_\gamma(v_*)} \leq C {v_*}^{-1-2\delta+p},
\end{aligned} \tag{12.78}$$

where C is a positive constant depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$.

Remark. Refer to (10.50) for the definition of $u_\gamma(v)$ and see Figure 10.6 for further clarification.

Proof. In Section 12.1.3, Corollary 12.1.8, we have obtained the global estimate (12.47) for the weighted S_0 current which follows from Proposition 12.1.7. Recall that in $\mathcal{D}^+(\gamma)$ we have $|u|^p \leq v^p$, cf. Section 12.1.3, which immediately leads to

$$\int\limits_{\{v_*\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma \leq \tilde{C} \int\limits_{\{v_*\leq v\leq \hat{v}\}} v^p J^{S_0}_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma, \tag{12.79}$$

cf. Appendix C.1 for explicit expressions of $J^{S_0}_\mu(\phi)n^\mu_\gamma$ and $J^S_\mu(\phi)n^\mu_\gamma$. From (12.79) we see that

$$\int\limits_{\{v_*\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma \leq C {v_*}^{-1-2\delta+p} \quad \text{for all } v_* > \alpha \text{ and } p \text{ as in (11.12)}, \tag{12.80}$$

is implied by Corollary 12.1.8.

Let $v_* > 2\alpha$ and $\hat{v} > v_*$. In order to obtain (12.78) we consider a region $\mathcal{R}_{IV} = J^+(\gamma) \cap J^-(x)$ with $x = (u_\gamma(v_*), \hat{v})$, $x \in \mathcal{B}$, as shown in Figure 12.5. Applying the divergence theorem we obtain

$$\begin{aligned}
&\int\limits_{\mathcal{R}_{IV}} K^S(\phi)\mathrm{dVol} + \int\limits_{\{u_\gamma(\hat{v})\leq u\leq u_\gamma(v_*)\}} J^S_\mu(\phi) n^\mu_{v=\hat{v}} \mathrm{dVol}_{v=\hat{v}} \\
&+ \int\limits_{\{v_*\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=u_\gamma(v_*)} \mathrm{dVol}_{u=u_\gamma(v_*)} \\
&= \int\limits_{\{v_*\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma.
\end{aligned} \tag{12.81}$$

In Section 12.1.4 we found that the angular part of $K^S(\phi)$ is positive in $\mathcal{R}_{IV}$ and we called the remaining part $\tilde{K}^S(\phi)$ given in (12.63). Using (12.64) we can therefore write

$$\begin{aligned}
&\int\limits_{\{u_\gamma(\hat{v})\leq u\leq u_\gamma(v_*)\}} J^S_\mu(\phi) n^\mu_{v=\hat{v}} \mathrm{dVol}_{v=\hat{v}} + \int\limits_{\{v_*\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=u_\gamma(v_*)} \mathrm{dVol}_{u=u_\gamma(v_*)} \\
&\leq \int\limits_{\mathcal{R}_{IV}} |\tilde{K}^S(\phi)|\mathrm{dVol} + \int\limits_{\{v_*\leq v\leq \hat{v}\}} J^S_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma.
\end{aligned} \tag{12.82}$$

Using Lemma 12.1.10 we obtain

$$\begin{aligned}
&\int_{\{u_\gamma(\hat{v})\le u\le u_\gamma(v_*)\}} J^S_\mu(\phi) n^\mu_{v=\hat{v}} \mathrm{dVol}_{v=\hat{v}} + \int_{\{v_*\le v\le \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=u_\gamma(v_*)} \mathrm{dVol}_{u=u_\gamma(v_*)} \\
\le\ & \delta_1 \sup_{u_\gamma(\hat{v})\le \bar{u}\le u_\gamma(v_*)} \int_{\{v_\gamma(\bar{u})\le v\le \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \\
&+\delta_2 \sup_{v_*\le \bar{v}\le \hat{v}} \int_{\{u_\gamma(\hat{v})\le u\le u_\gamma(\bar{v})\}} J^S_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} + \int_{\{v_*\le v\le \hat{v}\}} J^S_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma \qquad (12.83)
\end{aligned}$$

Repeating estimate (12.83) with $\bar{u}$, $\bar{v}$ in place of $u_\gamma(v_*)$, $\hat{v}$ and taking the supremum we have

$$\begin{aligned}
& \sup_{u_\gamma(\hat{v})\le \bar{u}\le u_\gamma(v_*)} \int_{\{v_\gamma(\bar{u})\le v\le \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \\
+\ & \sup_{v_*\le \bar{v}\le \hat{v}} \int_{\{u_\gamma(\hat{v})\le u\le u_\gamma(\bar{v})\}} J^S_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \\
\le\ & \delta_1 \sup_{u_\gamma(\hat{v})\le \bar{u}\le u_\gamma(v_*)} \int_{\{v_\gamma(\bar{u})\le v\le \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \\
&+\delta_2 \sup_{v_*\le \bar{v}\le \hat{v}} \int_{\{u_\gamma(\hat{v})\le u\le u_\gamma(\bar{v})\}} J^S_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \\
&+ \int_{\{v_*\le v\le \hat{v}\}} J^S_\mu(\phi) n^\mu_\gamma \mathrm{dVol}_\gamma. \qquad (12.84)
\end{aligned}$$

Recalling $\delta_1 \to 0$, $\delta_2 \to 0$ as $v_* \to \infty$, choose $u_{✂}$ sufficiently close to $-\infty$, such that for $v_* > v_\gamma(u_{✂})$, say

$$\delta_1, \delta_2 \le \frac{1}{2} \qquad (12.85)$$

holds. The conclusion of Proposition 12.1.11 then follows by absorbing the first two terms of the right hand side of (12.84) by the two terms on the left and estimating the third from (12.80).

□

Energy estimates globally in the rectangle Ξ up to $\mathcal{CH}^+$ in the neighbourhood of i^+

In the previous Sections 12.1.1 to 12.1.4 we have proven energy estimates for each region with specific properties separately. Putting all results together we can state the following proposition.

Proposition 12.1.12. *Let ϕ be as in Theorem 11.1.1 and p as in* (11.12)*. Then, for $u_{\ast}$ sufficiently close to $-\infty$, for all $v_* > 1$, $\hat{v} > v_*$ and $\tilde{u} \in (-\infty, u_{\ast})$.*

$$\int_{\{v_* \le v \le \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} dVol_{u=\tilde{u}} \le C {v_*}^{-1-2\delta+p}, \tag{12.86}$$

where C is a positive constant depending on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$.

Proof. First of all we partition the integral of the statement into a sum of integrals of the different regions. That is to say

$$\begin{aligned}
&\int_{\{v_* \le v \le \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} \mathrm{dVol}_{u=\tilde{u}} \\
= &\int_{\{v_* \le v \le \hat{v}\}\cap\mathcal{R}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} \mathrm{dVol}_{u=\tilde{u}} + \int_{\{v_* \le v \le \hat{v}\}\cap\mathcal{N}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} \mathrm{dVol}_{u=\tilde{u}} \\
&+ \int_{\{v_* \le v \le \hat{v}\}\cap J^-(\gamma)\cap\mathcal{B}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} \mathrm{dVol}_{u=\tilde{u}} + \int_{\{v_* \le v \le \hat{v}\}\cap J^+(\gamma)\cap\mathcal{B}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} \mathrm{dVol}_{u=\tilde{u}}.
\end{aligned}$$

For the integral in $\mathcal{R}$ and the integral in $\mathcal{N}$ we use Corollaries 12.1.4 and 12.1.6. (Note that they have to be summed dyadically resulting in the loss of one polynomial power.) Further, for the integral in region $J^-(\gamma)\cap\mathcal{B}$ we apply Corollary 12.1.9 and for the integral in region $J^+(\gamma)\cap\mathcal{B}$ we use Proposition 12.1.11. Putting all this together we arrive at the conclusion of Proposition 12.1.12. $\square$

In particular, we have

Corollary 12.1.13. *Let ϕ be as in Theorem 11.1.1 and p as in* (11.12)*. Then, for $u_{✂}$ sufficiently close to $-\infty$, for all $v_{fix} \geq 1$, and $\tilde{u} \in (-\infty, u_{✂})$,*

$$\int\limits_{\mathbb{S}^2} \int\limits_{v_{fix}}^{\infty} \left[v^p (\partial_v \phi)^2 (\tilde{u}, v) + |\slashed{\nabla} \phi|^2 (\tilde{u}, v) \right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \leq C, \tag{12.87}$$

where C is a positive constant dependent on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, 1)$ of Proposition 11.1.2, where $u_\diamond$ is defined by $r_{red} = r(u_\diamond, 1)$.

Proof. The conclusion of the proposition follows immediately examining the weights in Proposition 12.1.12. □

12.2 Pointwise estimates from higher order energies

12.2.1 The Ω notation and Sobolev inequality on spheres

Recall that we had stated the expressions for the generators of spherical symmetry Ω_i, $i = 1, 2, 3$, in Section 10.3.2. They were explicitly given by (10.36) to (10.38). Further, having expressions (10.39) and (10.40) in mind we introduce the following notation

$$\sum_{k=0}^{2} \left(\Omega^k \phi \right)^2 = |\phi|^2 + \sum_{i=1}^{3} \left(\Omega_i \phi \right)^2 + \sum_{i=1}^{3} \sum_{j=1}^{3} \left(\Omega_i \Omega_j \phi \right)^2, \tag{12.88}$$

where k has to be understood as the order of an exponent and not as an index. By Sobolev embedding on the standard spheres we have in this notation

$$\sup_{\theta, \varphi \in \mathbb{S}^2} |\phi(u, v, \theta, \varphi)|^2 \leq \tilde{C} \sum_{k=0}^{2} \int\limits_{\mathbb{S}^2} \left(\Omega^k \phi \right)^2 (u, v, \theta, \varphi) \mathrm{d}\sigma_{\mathbb{S}^2}, \tag{12.89}$$

which means that we can derive a pointwise estimate from an estimate of the integrals on the spheres, see e.g. [30]. More generally, in the following we will also use the notation

$$J^X_\mu (\Omega \phi) = \sum_{i=1}^{3} J^X_\mu (\Omega_i \phi), \tag{12.90}$$

for any J-current related to an arbitrary vector field X, and similarly for other quadratic expressions, e.g. (12.92), (12.93).

12.2.2 Higher order energy estimates in the neighbourhood of i^+

We will need the following extension of Corollary 12.1.13 for higher order energies.

Theorem 12.2.1. *On subextremal Reissner-Nordström spacetime $(\mathcal{M}, g)$, with mass M and charge e and $M > |e| \neq 0$, let ϕ be a solution of the wave equation $\Box_g \phi = 0$ arising from sufficiently regular Cauchy data on a two-ended asymptotically flat Cauchy surface Σ. Then, for $v_{fix} \geq 1$ and $u_{fix} > -\infty$*

$$\begin{aligned}
\int_{\mathbb{S}^2} \int_{v_{fix}}^{\infty} \left[v^p (\partial_v \phi)^2 (u_{fix}, v, \theta, \varphi)\right. &+ \left.|\nabla\!\!\!/ \phi|^2 (u_{fix}, v, \theta, \varphi)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \\
&\leq E_0, \qquad (12.91)\\
\int_{\mathbb{S}^2} \int_{v_{fix}}^{\infty} \left[v^p (\partial_v \Omega\!\!\!/ \phi)^2 (u_{fix}, v, \theta, \varphi)\right. &+ \left.|\nabla\!\!\!/ \Omega\!\!\!/ \phi|^2 (u_{fix}, v, \theta, \varphi)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \\
&\leq E_1, \qquad (12.92)\\
\int_{\mathbb{S}^2} \int_{v_{fix}}^{\infty} \left[v^p (\partial_v \Omega\!\!\!/^2 \phi)^2 (u_{fix}, v, \theta, \varphi)\right. &+ \left.|\nabla\!\!\!/ \Omega\!\!\!/^2 \phi|^2 (u_{fix}, v, \theta, \varphi)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \\
&\leq E_2, \qquad (12.93)
\end{aligned}$$

where p is as in (11.12).

Proof. Statement (12.91) was already derived in Corollary 12.1.13. Recall that $\Omega\!\!\!/_i \phi$, $\Omega\!\!\!/_i \Omega\!\!\!/_j \phi$ also satisfy the massless scalar wave equation, cf. Section 10.1. Summing over all angular momentum operators, keeping in mind notation (12.88), etc., we therefore obtain (12.92) and (12.93). $\square$

12.2.3 Pointwise boundedness in the neighbourhood of i^+

We turn the discussion to the derivation of pointwise boundedness from energy estimates. In particular we prove Theorem 11.2.1 from Theorem 12.2.1.

By the fundamental theorem of calculus it follows for all $v_* > 1$, $\hat{v} > v_*$ and $u \in (-\infty, u_{\text{✂}})$ that

$$\phi(u, \hat{v}, \theta, \varphi) = \int_{v_*}^{\hat{v}} (\partial_v \phi)(u, v, \theta, \varphi) \mathrm{d}v + \phi(u, v_*, \theta, \varphi),$$

$$
\begin{aligned}
&\leq \int_{v_*}^{\hat{v}} (\partial_v \phi)(u, v, \theta, \varphi) v^{\frac{p}{2}} v^{-\frac{p}{2}} \mathrm{d}v + \phi(u, v_*, \theta, \varphi), \\
&\leq \left(\int_{v_*}^{\hat{v}} v^p (\partial_v \phi)^2 (u, v, \theta, \varphi) \mathrm{d}v \right)^{\frac{1}{2}} \left(\int_{v_*}^{\hat{v}} v^{-p} \mathrm{d}v \right)^{\frac{1}{2}} + \phi(u, v_*, \theta, \varphi),
\end{aligned} \tag{12.94}
$$

where we have used the Cauchy-Schwarz inequality in the last step. Squaring the entire expression, using Cauchy-Schwarz again and integrating over $\mathbb{S}^2$ we obtain the expression that we had sketched in Section 11.2 already

$$
\begin{aligned}
\int_{\mathbb{S}^2} \phi^2(u, \hat{v}) \mathrm{d}\sigma_{\mathbb{S}^2} \quad \leq \quad \tilde{C} \Bigg[\int_{\mathbb{S}^2} \left(\int_{v_*}^{\hat{v}} v^p (\partial_v \phi)^2 (u, v) \mathrm{d}v \int_{v_*}^{\hat{v}} v^{-p} \mathrm{d}v \right) r^2 \mathrm{d}\sigma_{\mathbb{S}^2} \\
+ \int_{\mathbb{S}^2} \phi^2(u, v_*) \mathrm{d}\sigma_{\mathbb{S}^2} \Bigg],
\end{aligned} \tag{12.95}
$$

with p as in (11.12) and the first term on the right hand side controlled by the flux for which we derived boundedness in Section 12.1.4. Therefore, by using Theorem 12.2.1 we obtain

$$
\begin{aligned}
\int_{\mathbb{S}^2} \phi^2(u, \hat{v}) \mathrm{d}\sigma_{\mathbb{S}^2} &\leq \tilde{C} \left[E_0 \int_{\mathbb{S}^2} \int_{v_*}^{\hat{v}} v^{-p} \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\mathbb{S}^2} \phi^2(u, v_*) \mathrm{d}\sigma_{\mathbb{S}^2} \right] \\
&\leq \tilde{C} \left[\tilde{\tilde{C}} E_0 + \int_{\mathbb{S}^2} \phi^2(u, v_*) \mathrm{d}\sigma_{\mathbb{S}^2} \right].
\end{aligned} \tag{12.96}
$$

It is here that we have used the requirement $p > 1$ of (11.12). Applying all our estimates to $\Omega_i \phi$, $\Omega_i \Omega_j \phi$ and summing, we obtain in the notation of Section 12.2.1 the following:

$$
\int_{\mathbb{S}^2} (\Omega \phi)^2 (u, \hat{v}) \mathrm{d}\sigma_{\mathbb{S}^2} \leq \tilde{C} \left[\tilde{\tilde{C}} E_1 + \int_{\mathbb{S}^2} (\Omega \phi)^2 (u, v_*) \mathrm{d}\sigma_{\mathbb{S}^2} \right], \tag{12.97}
$$

$$
\int_{\mathbb{S}^2} (\Omega^2 \phi)^2 (u, \hat{v}) \mathrm{d}\sigma_{\mathbb{S}^2} \leq \tilde{C} \left[\tilde{\tilde{C}} E_2 + \int_{\mathbb{S}^2} (\Omega^2 \phi)^2 (u, v_*) \mathrm{d}\sigma_{\mathbb{S}^2} \right]. \tag{12.98}
$$

Let us now use (11.7) to (11.9) of Proposition 11.1.2 to estimate the right hand sides of (12.96) to (12.98) with $v_* = 1$. Adding all equations up we derive pointwise boundedness according to (12.89)

$$
\begin{aligned}
& \sup_{\mathbb{S}^2} |\phi(u,\hat{v},\theta,\varphi)|^2 \\
\leq \quad & \tilde{C}\left[\int_{\mathbb{S}^2}(\phi)^2(u,\hat{v})\mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\mathbb{S}^2}(\Omega\phi)^2(u,\hat{v})\mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\mathbb{S}^2}(\Omega^2\phi)^2(u,\hat{v})\mathrm{d}\sigma_{\mathbb{S}^2}\right], \quad (12.99)\\
\leq \quad & \tilde{C}\left[\tilde{\tilde{C}}\left(E_0+E_1+E_2\right)+D_0(u_\diamond,1)+D_1(u_\diamond,1)+D_2(u_\diamond,1)\right], \quad (12.100)\\
\leq \quad & C, \quad (12.101)
\end{aligned}
$$

with C depending on the initial data on Σ. We therefore arrive at the statement given in Theorem 11.2.1.

□

12.3 *Left* neighbourhood of i^+

We now turn to establish boundedness in the neighbourhood of the *left* timelike infinity i^+. For this we simply repeat the entire proof carried out in the characteristic rectangle Ξ in Section 12.1.1 to Section 12.1.4 but this time for the region $\tilde{\Xi}$ at the *left end* of $\mathcal{Q}|_{II}$, cf. (10.22), as shown in the Penrose diagram 12.6.

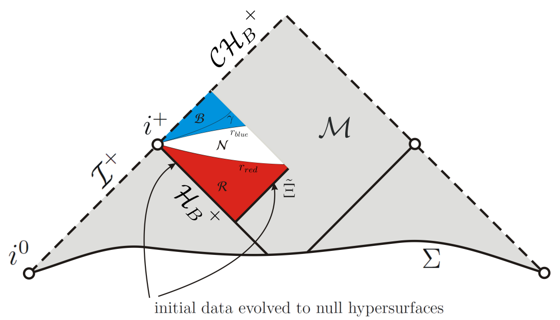


Figure 12.6: Penrose diagram with characteristic rectangle $\tilde{\Xi}$ depicted on the *left* side.

Since everything is completely analogous to the derivation for the *right* side, we will merely state the main Theorems and Propositions and remind the reader that u and v

are interchanged now. Recall that $u \to \infty$ on $\mathcal{CH}_B^+$ and $v \to -\infty$ on $\mathcal{H}_B^+$ for the *left end*. The rectangle under consideration is now $\tilde{\Xi} = \{(1 \le u < \infty), (-\infty \le v \le v_{✂})\}$

Theorem 12.3.1. *Let ϕ be a solution of the wave equation (9.1) on a subextremal Reissner-Nordström background $(\mathcal{M}, g)$, with mass M and charge e and $M > |e| \neq 0$, arising from smooth compactly supported initial data on an arbitrary Cauchy hypersurface Σ, cf. Figure 12.6. Then, there exists $\delta > 0$ such that*

$$\int_{\mathbb{S}^2}\int_u^{u+1} \left[(\partial_u \phi)^2(u,-\infty) + |\nabla\!\!\!\!/\, \phi|^2(u,-\infty)\right] r^2 \mathrm{d}u \mathrm{d}\sigma_{\mathbb{S}^2} \quad \le \quad C_0 u^{-2-2\delta}, \tag{12.102}$$

$$\int_{\mathbb{S}^2}\int_u^{u+1} \left[(\partial_u \Omega\!\!\!\!/\,\phi)^2(u,-\infty) + |\nabla\!\!\!\!/\, \Omega\!\!\!\!/\,\phi|^2(u,-\infty)\right] r^2 \mathrm{d}u \mathrm{d}\sigma_{\mathbb{S}^2} \quad \le \quad C_1 u^{-2-2\delta}, \tag{12.103}$$

$$\int_{\mathbb{S}^2}\int_u^{u+1} \left[(\partial_u \Omega\!\!\!\!/\,^2\phi)^2(u,-\infty) + |\nabla\!\!\!\!/\, \Omega\!\!\!\!/\,^2\phi|^2(u,-\infty)\right] r^2 \mathrm{d}u \mathrm{d}\sigma_{\mathbb{S}^2} \quad \le \quad C_2 u^{-2-2\delta}, \tag{12.104}$$

on ${\mathcal{H}_B}^+$, for all u and some positive constants C_0, C_1 and C_2 depending on the initial data.

Proposition 12.3.2. *Let $u_\diamond, v_\diamond \in (-\infty, \infty)$. Under the assumption of Theorem 12.3.1, the energy at retarded Eddington-Finkelstein coordinate $\{u = u_\diamond\} \cap \{-\infty \le v \le v_\diamond\}$ is bounded from the initial data*

$$\begin{aligned}\int_{\mathbb{S}^2}\int_{-\infty}^{v_\diamond} \left[\Omega^{-2}(\partial_v \phi)^2(u_\diamond, v)\right. \quad &+ \quad \left.\frac{\Omega^2}{2}|\nabla\!\!\!\!/\,\phi|^2(u_\diamond, v)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \\ &\le \quad D_0(u_\diamond, v_\diamond),\end{aligned} \tag{12.105}$$

$$\begin{aligned}\int_{\mathbb{S}^2}\int_{-\infty}^{v_\diamond} \left[\Omega^{-2}(\partial_v \Omega\!\!\!\!/\,\phi)^2(u_\diamond, v)\right. \quad &+ \quad \left.\frac{\Omega^2}{2}|\nabla\!\!\!\!/\,\Omega\!\!\!\!/\,\phi|^2(u_\diamond, v)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \\ &\le \quad D_1(u_\diamond, v_\diamond),\end{aligned} \tag{12.106}$$

$$\begin{aligned}\int_{\mathbb{S}^2}\int_{-\infty}^{v_\diamond} \left[\Omega^{-2}(\partial_v \Omega\!\!\!\!/\,^2\phi)^2(u_\diamond, v)\right. \quad &+ \quad \left.\frac{\Omega^2}{2}|\nabla\!\!\!\!/\,\Omega\!\!\!\!/\,^2\phi|^2(u_\diamond, v)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \\ &\le \quad D_2(u_\diamond, v_\diamond),\end{aligned} \tag{12.107}$$

and further

$$\sup_{-\infty\leq v\leq v_\diamond}\int_{\mathbb{S}^2}(\phi)^2(u_\diamond,v)\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_0(u_\diamond,v_\diamond), \tag{12.108}$$

$$\sup_{-\infty\leq v\leq v_\diamond}\int_{\mathbb{S}^2}(\Omega\!\!\!/\phi)^2(u_\diamond,v)\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_1(u_\diamond,v_\diamond), \tag{12.109}$$

$$\sup_{-\infty\leq v\leq v_\diamond}\int_{\mathbb{S}^2}(\Omega\!\!\!/^2\phi)^2(u_\diamond,v)\mathrm{d}\sigma_{\mathbb{S}^2} \leq D_2(u_\diamond,v_\diamond), \tag{12.110}$$

with $D_0(u_\diamond,v_\diamond)$, $D_1(u_\diamond,v_\diamond)$ and $D_2(u_\diamond,v_\diamond)$ positive constants depending on the initial data.

Note the Ω^{-2} weights which arise since v is not regular at $\mathcal{H}_B^+$. Analogous to the Proposition 12.1.12 obtained for the *right* side we can state the following for the *left*.

Proposition 12.3.3. *Let ϕ be as in Theorem 12.3.1 and p as in* (11.12)*. Then, for $v_{\times}$ sufficiently close to $-\infty$, for $u_* > 1$, $\hat{u} > u_*$ and $\tilde{v} \in (-\infty, v_{\times})$.*

$$\int_{\{u_*\leq u\leq\hat{u}\}} J^S_\mu(\phi)n^\mu_{v=\tilde{v}}\,dVol_{v=\tilde{v}} \leq C{u_*}^{-1-2\delta+p}, \tag{12.111}$$

where C is a positive constant depending on C_0 of Theorem 12.3.1 and $D_0(u_\diamond,1)$ of Proposition 12.3.2, where $v_\diamond$ is defined by $r_{red} = r(1,v_\diamond)$.

Proof. The proof is analogous to the proof of Proposition 12.1.12 with u and v interchanged. □

Having obtained Proposition (12.3.3) we can derive higher order estimates analogous to Section 12.2.2. The pointwise estimate is then obtained via the same strategy as in Section 12.2.3 but integrated in u and not in v, and can be stated as follows.

Theorem 12.3.4. *Let ϕ be as in Theorem 12.3.1, then*

$$|\phi| \leq C$$

locally in the black hole interior up to $\mathcal{CH}^+$ in a "small neighbourhood" of left *timelike infinity i^+, that is in $[1,\infty)\times(-\infty,v_{\times}]$ for some $v_{\times} > -\infty$.*

12.4 Energy along the future boundaries of $\mathcal{R}_V$

Let $u_\diamond > u_{\scissors}$ and $v_* \geq v_\gamma(u_{\scissors})$. Define $\mathcal{R}_V = \{u_{\scissors} \leq u \leq u_\diamond\} \cap \{v_* \leq v \leq \hat{v}\}$, cf. Figure 12.7, and note that $\mathcal{R}_V \subset \mathcal{B}$. We will apply the vector field

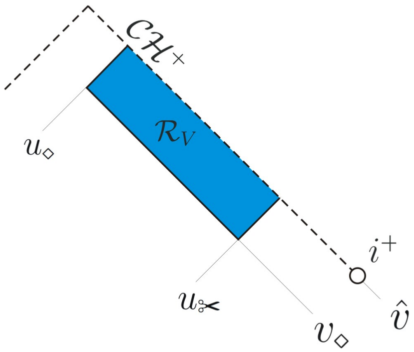


Figure 12.7: Penrose diagram depicting region $\mathcal{R}_V$.

$$W = v^p \partial_v + \partial_u \tag{12.112}$$

as a multiplier. The bulk can be calculated as

$$\begin{aligned} K^W = & -\frac{2}{r}[v^p + 1](\partial_v \phi \partial_u \phi) \\ & - \left[\frac{1}{2} p v^{p-1} + \frac{\partial_v \Omega}{\Omega} v^p + \frac{\partial_u \Omega}{\Omega}\right] |\slashed{\nabla} \phi|^2. \end{aligned} \tag{12.113}$$

Let us define

$$\tilde{K}^W = -\frac{2}{r}[v^p + 1](\partial_v \phi \partial_u \phi), \tag{12.114}$$

and

$$K^W_{\slashed{\nabla}} = -\left[\frac{1}{2} p v^{p-1} + \frac{\partial_v \Omega}{\Omega} v^p + \frac{\partial_u \Omega}{\Omega}\right] |\slashed{\nabla} \phi|^2, \tag{12.115}$$

with $K^W_{\slashed{\nabla}}$ positive since the second term in (12.115) dominates over the first for $v_* > 2\alpha$ and $\frac{\partial_u \Omega}{\Omega}$, $\frac{\partial_v \Omega}{\Omega}$ are negative in the blueshift region. We have therefore

$$-K^W \leq |\tilde{K}^W| \tag{12.116}$$

in $\mathcal{R}_V$. We aim for estimating it via the currents along $v = constant$ and $u = constant$ hypersurfaces.

Lemma 12.4.1. *Let ϕ be an arbitrary function. Then, for all $v_* \geq v_\gamma(u_{✂})$, $\hat{v} > v_*$, for $u_\diamond \geq u_2 > u_1 \geq u_{✂}$ and $\epsilon \geq u_2 - u_1 > 0$*

$$
\begin{aligned}
\int\limits_{\mathcal{R}_{V_1}} |\tilde{K}^W| dVol \leq \quad & \delta_1 \sup_{u_1 \leq \bar{u} \leq u_2} \int\limits_{\{v_* \leq v \leq \hat{v}\}} J^W_\mu(\phi) n^\mu_{u=\bar{u}} dVol_{u=\bar{u}} \\
+ \quad & \delta_2 \sup_{v_* \leq \bar{v} \leq \hat{v}} \int\limits_{\{u_1 \leq u \leq u_2\}} J^W_\mu(\phi) n^\mu_{v=\bar{v}} dVol_{v=\bar{v}}, \qquad (12.117)
\end{aligned}
$$

where $\mathcal{R}_{V_1} = \{u_1 \leq u \leq u_2\} \cap \mathcal{R}_V$ and δ_1, δ_2 are positive constants, depending only on v_* and ϵ such that $\delta_1 \to 0$ for $\epsilon \to 0$ and $\delta_2 \to 0$ as $v_* \to \infty$.

Proof. Using the Cauchy-Schwarz inequality for equation (12.114) we obtain

$$
|\tilde{K}^W| \quad \leq \quad \frac{1}{r}\left[\left(1 + v^{-p}\right) v^p (\partial_v \phi)^2 + (1 + v^p)\,(\partial_u \phi)^2\right], \qquad (12.118)
$$

with the related volume element

$$
\mathrm{dVol} \quad = \quad r^2 \frac{\Omega^2}{2} \mathrm{d}u \mathrm{d}v \mathrm{d}\sigma^2_{\mathbb{S}}. \qquad (12.119)
$$

Note that the currents related to the vector field W with their related volume elements are given by

$$
\begin{aligned}
J^W_\mu(\phi) n^\mu_{v=\bar{v}} \quad &= \quad \frac{2}{\Omega^2}\left[(\partial_u \phi)^2 + \frac{\Omega^2}{4} \bar{v}^p |\nabla\!\!\!\!/\,\phi|^2\right], \\
& \qquad \mathrm{dVol}_{v=\bar{v}} = r^2 \frac{\Omega^2}{2} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}u, \qquad (12.120) \\
J^W_\mu(\phi) n^\mu_{u=\bar{u}} \quad &= \quad \frac{2}{\Omega^2}\left[v^p (\partial_v \phi)^2 + \frac{\Omega^2}{4} |\nabla\!\!\!\!/\,\phi|^2\right], \\
& \qquad \mathrm{dVol}_{u=\bar{u}} = r^2 \frac{\Omega^2}{2} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}v, \qquad (12.121)
\end{aligned}
$$

cf. Appendix C.1. Taking the integral over the spacetime region therefore yields

$$
\int\limits_{\mathcal{R}_{V_1}} |\tilde{K}^W(\phi)| \mathrm{dVol} \leq \int\limits_{u_1}^{u_2} \int\limits_{\{v_* \leq v \leq \hat{v}\}} \frac{\Omega^2(\bar{u}, \bar{v})}{2r} \left(1 + \bar{v}^{-p}\right) J^W_\mu(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \mathrm{d}\bar{u}
$$

$$
\begin{aligned}
&+\int_{v_*}^{\hat{v}} \int_{\{u_1 \leq u \leq u_2\}} \frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^p\right) J^W_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \mathrm{d}\bar{v},\\
\leq\;& \int_{u_1}^{u_2} \sup_{v_* \leq \bar{v} \leq \hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^{-p}\right)\right] \mathrm{d}\bar{u} \sup_{u_1 \leq \bar{u} \leq u_2} \int_{\{v_* \leq v \leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}}\\
&+\int_{v_*}^{\hat{v}} \sup_{u_1 \leq \bar{u} \leq u_2} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^{p}\right)\right] \mathrm{d}\bar{v} \sup_{v_* \leq \bar{v} \leq \hat{v}} \int_{\{u_1 \leq u \leq u_2\}} J^S_\mu(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}}.
\end{aligned}
\tag{12.122}
$$

It remains to show finiteness and smallness of $\int_{u_1}^{u_2} \sup_{v_* \leq \bar{v} \leq \hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^{-p}\right)\right] \mathrm{d}\bar{u}$ and $\int_{v_*}^{\hat{v}} \sup_{u_1 \leq \bar{u} \leq u_2} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^{p}\right)\right] \mathrm{d}\bar{v}$. Recall the properties of the hypersurface γ shown in Section 12.1.3. Since $v_* > v_\gamma(u_{\text{✂}})$, (12.60) implies that

$$
\Omega^2(\bar{u},\bar{v}) \leq C\Omega^2(u_{\text{✂}}, v_*), \quad \text{for any } (\bar{u},\bar{v}) \in J^+(x), \text{ with } x=(u_{\text{✂}}, v_*),\ x \in \mathcal{B}, \tag{12.123}
$$

so that we obtain

$$
\begin{aligned}
\int_{u_1}^{u_2} \sup_{v_* \leq \bar{v} \leq \hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^{-p}\right)\right] \mathrm{d}\bar{u} \quad &\overset{(12.123)}{\leq} \quad C\int_{u_1}^{u_2} \sup_{v_* \leq \bar{v} \leq \hat{v}} \Omega^2(u_{\text{✂}}, v_*)\left(1+\bar{v}^{-p}\right) \mathrm{d}\bar{u},\\
&\leq \quad \tilde{C}\int_{u_1}^{u_2} |u_{\text{✂}}|^{-\beta\alpha}\left(1+{v_*}^{-p}\right) \mathrm{d}\bar{u},\\
&\leq \quad \tilde{\tilde{C}}\,|u_2-u_1|\\
&\leq \quad \delta_1,
\end{aligned}
\tag{12.124}
$$

and moreover $\delta_1 \to 0$ for $\epsilon \to 0$.

Further, in Section 12.1.4 we derived that similarly

$$
\Omega^2(\bar{u},\bar{v}) \leq C\bar{v}^{-\beta\alpha}, \quad \text{for any } (\bar{u},\bar{v}) \in J^+(x), \text{ with } x=(u_{\text{✂}}, v_*),\ x \in \mathcal{B}, \tag{12.125}
$$

where $v_* > v_\gamma(u_{\ltimes})$.

$$\int_{v_*}^{\hat{v}} \sup_{u_1 \le \bar{u} \le u_2} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\bar{v}^p\right)\right] \mathrm{d}\bar{v} \overset{(12.125)}{\le} C \int_{v_*}^{\hat{v}} \bar{v}^{-\beta\alpha}\left(1+\bar{v}^p\right)\mathrm{d}\bar{v}$$
$$\le \frac{\tilde{C}}{|-\beta\alpha+p+1|}\left[\bar{v}^{-\beta\alpha+p+1}\right]_{v_*}^{\hat{v}}$$
$$\le \delta_2, \tag{12.126}$$

where $\delta_2 \to 0$ for $v_* \to \infty$. Thus the conclusion of Lemma 12.4.1 is obtained. □

From the above we obtain

Proposition 12.4.2. *Let ϕ be as in Theorem 11.1.1 and p as in* (11.12)*. For all $v_* > v_\gamma(u_{\ltimes})$ sufficiently large, $\hat{v} \in (v_*, \infty)$, for $u_\diamond \ge u_2 > u_1 \ge u_{\ltimes}$ and $\epsilon \ge u_2 - u_1 > 0$. Then for ϵ sufficiently small, the following is true. If*

$$\int_{\{v_* \le v \le \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=u_1} dVol_{u=u_1} \le \tilde{C}_1, \tag{12.127}$$

then

$$\int_{\{v_* \le v \le \hat{v}\}} J_\mu^W(\phi) n^\mu_{u_2} dVol_{u_\diamond} + \int_{\{u_1 \le u \le u_2\}} J_\mu^W(\phi) n^\mu_{v=\hat{v}} dVol_{v=\hat{v}} \le \tilde{C}_2(\tilde{C}_1, u_\diamond, v_*), \tag{12.128}$$

where $\tilde{C}_2$ depends on $\tilde{C}_1$, C_0 of Theorem 11.1.1 and $D_0(u_\diamond, v_*)$ of Proposition 11.1.2.

Remark. Note already that the hypothesis (12.127) is implied by the conclusion of Proposition 12.1.12 for $u_1 = u_{\ltimes}$.

Proof. By the divergence theorem and (12.116) we can state

$$\int_{\{v_* \le v \le \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=u_2} \mathrm{dVol}_{u=u_2} + \int_{\{u_1 \le u \le u_2\}} J_\mu^W(\phi) n^\mu_{v=\hat{v}} \mathrm{dVol}_{v=\hat{v}}$$
$$\le \int_{\mathcal{R}_{V_1}} |\tilde{K}^W| \mathrm{dVol} + \int_{\{u_1 \le u \le u_2\}} J_\mu^W(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*}$$

$$+ \int\limits_{\{v_* \leq v \leq \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=u_1} \mathrm{dVol}_{u=u_1}. \tag{12.129}$$

We can replace the hypersurfaces $u = u_2$ and $v = \hat{v}$ with $u = \bar{u}$ and $v = \bar{v}$ hypersurfaces and therefore obtain

$$\begin{aligned}
& \sup_{u_1 \leq \bar{u} \leq u_2} \int\limits_{\{v_* \leq v \leq \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \\
+ \quad & \sup_{v_* \leq \bar{v} \leq \hat{v}} \int\limits_{\{u_1 \leq u \leq u_2\}} J_\mu^W(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \\
\leq \quad & \int\limits_{\mathcal{R}_{V_i}} |\tilde{K}^W| \mathrm{dVol} + \int\limits_{\{u_{\ast} \leq u \leq u_\diamond\}} J_\mu^W(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} \\
& + \int\limits_{\{v_* \leq v \leq \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=u_1} \mathrm{dVol}_{u=u_1}, \\
\overset{\text{Lem.12.4.1}}{\leq} \quad & \delta_1 \sup_{u_1 \leq \bar{u} \leq u_2} \int\limits_{\{v_* \leq v \leq \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \\
& + \delta_2 \sup_{v_* \leq \bar{v} \leq \hat{v}} \int\limits_{\{u_1 \leq u \leq u_2\}} J_\mu^W(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \\
& + \int\limits_{\{u_{\ast} \leq u \leq u_\diamond\}} J_\mu^W(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*} \\
& + \int\limits_{\{v_* \leq v \leq \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=u_1} \mathrm{dVol}_{u=u_1}.
\end{aligned} \tag{12.130}$$

Thus, we have

$$\begin{aligned}
\Rightarrow \quad & \sup_{u_1 \leq \bar{u} \leq u_2} \int\limits_{\{v_* \leq v \leq \hat{v}\}} J_\mu^W(\phi) n^\mu_{u=\bar{u}} \mathrm{dVol}_{u=\bar{u}} \\
+ \quad & \sup_{v_* \leq \bar{v} \leq \hat{v}} \int\limits_{\{u_1 \leq u \leq u_2\}} J_\mu^W(\phi) n^\mu_{v=\bar{v}} \mathrm{dVol}_{v=\bar{v}} \\
\leq \quad & \frac{1}{1 - \max\{\delta_1, \delta_2\}} \Bigg[\int\limits_{\{u_{\ast} \leq u \leq u_\diamond\}} J_\mu^W(\phi) n^\mu_{v=v_*} \mathrm{dVol}_{v=v_*}
\end{aligned}$$

$$
\left. + \int\limits_{\{v_* \leq v \leq \hat{v}\}} J^W_\mu(\phi) n^\mu_{u=u_1} \mathrm{dVol}_{u=u_1} \right] .
$$

$$
\leq \quad \tilde{C} D_0(u_\diamond, v_*) + \tilde{C}_1, \tag{12.131}
$$

where the last step follows by statement (11.4) of Proposition 11.1.2 and by (12.127), and where we have chosen ϵ sufficiently small and v_* sufficiently close to ∞, such that δ_1 and δ_2 satisfy say

$$
\delta_1, \delta_2 \leq \frac{1}{2}. \tag{12.132}
$$

The conclusion of Proposition 12.4.2 is obtained. □

We are now ready to make a statement for the entire region $\mathcal{R}_V$.

Proposition 12.4.3. *Let ϕ be as in Theorem 11.1.1 and p as in* (11.12)*. Then, for all $v_* > v_\gamma(u_{\scissors})$ sufficiently large, $\hat{v} > v_*$, and $u_\diamond > \hat{u} > u_{\scissors}$,*

$$
\int\limits_{\{u_{\scissors} \leq u \leq u_\diamond\}} J^W_\mu(\phi) n^\mu_{v=\hat{v}} dVol_{v=\hat{v}}
+ \int\limits_{\{v_* \leq v \leq \hat{v}\}} J^W_\mu(\phi) n^\mu_{u=\hat{u}} dVol_{u=\hat{u}} \leq C(u_\diamond, v_*), \tag{12.133}
$$

where C depends on C_0 of Theorem 11.1.1 and $D_0(u_\diamond, v_*)$ of Proposition 11.1.2.

Proof. Let ϵ be as in Proposition 12.4.2. We choose a sequence $u_{i+1} - u_i \leq \epsilon$ and $i = \{1, 2, .., n\}$ such that $u_1 = u_{\scissors}$ and $u_n = \hat{u}$. Denote $\mathcal{R}_{V_i} = \{u_i \leq u \leq u_{i+1}\} \cap \{v_* \leq v \leq \hat{v}\}$, cf. Figure 12.8. Iterating the conclusion of Proposition 12.4.2 from u_1 up to u_n then completes the proof. Note that n depends only on the smallness condition on ϵ from Proposition 12.4.2, since $n \lesssim \frac{u_\diamond - u_{\scissors}}{\epsilon}$. □

12.5 Energy along the future boundaries of $\tilde{\mathcal{R}}_V$

Again we also need the estimates on the *left* side, therefore, we repeat the derivation of Section 12.4 for region $\tilde{\mathcal{R}}_V = \{u_* \leq u \leq \hat{u}\} \cap \{v_{\scissors} \leq v \leq v_\diamond\}$, according to Figure 12.9, which is located in the blueshift region $\tilde{\mathcal{R}}_V \subset \mathcal{B}$. Note that for region $\tilde{\mathcal{R}}_V$ not only do

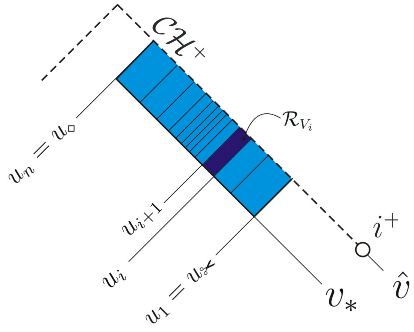


Figure 12.8: Penrose diagram depicting regions $\mathcal{R}_{V_i}$.

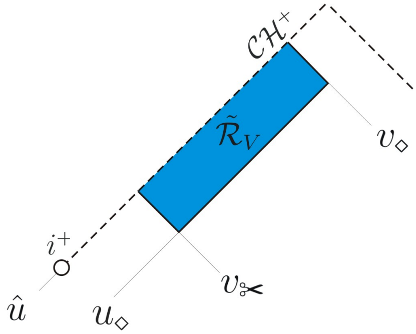


Figure 12.9: Penrose diagram depicting region $\tilde{\mathcal{R}}_V$.

we have to interchange u and v, we also have to use the vector field

$$Z = \partial_v + u^p \partial_u \tag{12.134}$$

instead of W, cf. (12.112). Therefore, we can immediately state the following proposition about the energy along the horizon away from $v_{✂}$.

Proposition 12.5.1. *Let ϕ be as in Theorem 12.3.1 and p as in* (11.12)*. Then, for $v_{✂}$ sufficiently close to $-\infty$, for all $u_* > u_\gamma(v_{✂})$ sufficiently large, $\hat{u} > u_*$, and for $v_\diamond > v_{✂}$*

$$\int_{\{v_{✂} \leq v \leq v_\diamond\}} J^Z_\mu(\phi) n^\mu_{u=\hat{u}} dVol_{u=\hat{u}} + \int_{\{u_* \leq u \leq \hat{u}\}} J^Z_\mu(\phi) n^\mu_{v=v_\diamond} dVol_{v=v_\diamond} \quad \leq \quad C(u_*, v_\diamond),$$

where C depends on C_0 of Theorem 12.3.1 and $D_0(u_*, v_\diamond)$ of Proposition 12.3.2.

Proof. The proof is analogous to the proof of Proposition 12.4.3. □

12.6 Propagating the energy estimate up to the bifurcation sphere

In this section we will use both results from the right and left side on $\mathcal{CH}^+$. Fix $u_\diamond = v_\diamond$, such that moreover Proposition 12.4.3 holds with $v_\diamond = v_*$, and such that Proposition 12.5.1 holds with $u_\diamond = u_*$. We will consider a region $\mathcal{R}_{VI} = \{u_\diamond \leq u \leq \hat{u}, v_\diamond \leq v \leq \hat{v}\}$, with $\hat{u} \in (u_\diamond, \infty)$ and $\hat{v} \in (v_\diamond, \infty)$, cf. Figure 12.10. Recall, that in Section 12.1.4 we

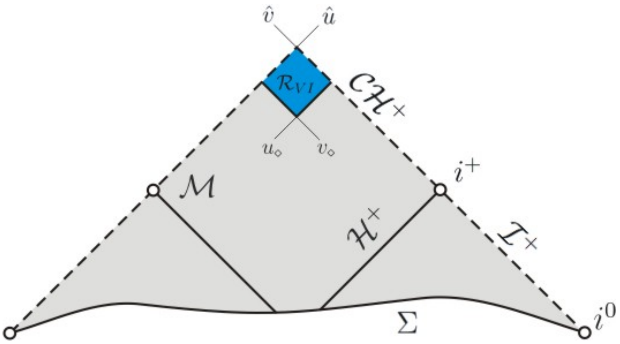


Figure 12.10: Penrose diagram depicting region $\mathcal{R}_{VI}$.

have defined the weighted vector field[4]

$$S = v^p \partial_v + u^p \partial_u, \tag{12.135}$$

which we are going to use again to obtain an energy estimate up to the bifurcate two-sphere. Recall K^S given in (12.62), where the terms multiplying the angular derivatives are positive since $\mathcal{R}_{VI}$ is located in the blueshift region. We further defined $\tilde{K}^S$ in (12.63) and stated (12.64) which will be useful to state the following proposition.

Lemma 12.6.1. *Let ϕ be an arbitrary function. Then, for all $(u_\diamond, v_\diamond) \in J^+(\gamma) \cap \mathcal{B}$ and all $\hat{u} > u_\diamond$, all $\hat{v} > v_\diamond$, the integral over $\mathcal{R}_{VI}$, cf. Figure 12.10 of the current $\tilde{K}^S$, defined*

[4]Since u is always positive in the remaining region under consideration $\mathcal{R}_{VI}$, we have omitted the absolute value in the u-weight.

by (12.63), *can be estimated by*

$$\begin{aligned}\int\limits_{\mathcal{R}_{VI}} |\tilde{K}^S| dVol \leq \quad & \delta_1 \sup_{u_\diamond \leq \bar{u} \leq \hat{u}} \int\limits_{\{v_\diamond \leq v \leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\bar{u}} dVol_{u=\bar{u}} \\ + \quad & \delta_2 \sup_{v_\diamond \leq \bar{v} \leq \hat{v}} \int\limits_{\{u_\diamond \leq u \leq \hat{u}\}} J^S_\mu(\phi) n^\mu_{v=\bar{v}} dVol_{v=\bar{v}}, \end{aligned} \quad (12.136)$$

where δ_1 *and* δ_2 *are positive constants, with* $\delta_1 \to 0$ *as* $u_\diamond \to \infty$ *and* $\delta_2 \to 0$ *as* $v_\diamond \to \infty$.

Proof. The proof is similar to the proof of Lemma 12.1.10 of Section 12.1.4 and Lemma 12.4.1 of Section 12.4. We still need to show finiteness and smallness of $\int\limits_{u_\diamond}^{\hat{u}} \sup_{v_\diamond \leq \bar{v} \leq \hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{u}^p}{\bar{v}^p}\right)\right] \mathrm{d}\bar{u}$ and $\int\limits_{v_\diamond}^{\hat{v}} \sup_{u_\diamond \leq \bar{u} \leq \hat{u}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{v}^p}{\bar{u}^p}\right)\right] \mathrm{d}\bar{v}$. In Section 12.1.4 we derived (12.57) which we will use now for all $\bar{u}, \bar{v} \in J^+(u_\diamond, v_\diamond)$ Therefore, we can write

$$\begin{aligned} & \int\limits_{u_\diamond}^{\hat{u}} \sup_{v_\diamond \leq \bar{v} \leq \hat{v}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{u}^p}{\bar{v}^p}\right)\right] \mathrm{d}\bar{u} \\ \leq \quad & C \int\limits_{u_\diamond}^{\hat{u}} \sup_{v_\diamond \leq \bar{v} \leq \hat{v}} \left[\Omega^2(u_\diamond, v_\diamond) e^{-\beta[\bar{v}-v_\diamond+\bar{u}-u_\diamond]}\left(1+\frac{\bar{u}^p}{\bar{v}^p}\right)\right] \mathrm{d}\bar{u}, \\ \leq \quad & \tilde{C} \int\limits_{u_\diamond}^{\hat{u}} \Omega^2(u_\diamond, v_\diamond) e^{-\beta[\bar{u}-u_\diamond]}\left(1+\frac{\bar{u}^p}{v_\diamond^p}\right) \mathrm{d}\bar{u}, \\ \leq \quad & \delta_1, \end{aligned} \quad (12.137)$$

where $\delta_1 \to 0$ as $u_\diamond = v_\diamond \to \infty$ (since $\Omega^2(u_\diamond, v_\diamond) \to 0$, cf. (10.24)). Similarly, for finiteness of the second term we obtain

$$\begin{aligned} & \int\limits_{v_\diamond}^{\hat{v}} \sup_{u_\diamond \leq \bar{u} \leq \hat{u}} \left[\frac{\Omega^2(\bar{u},\bar{v})}{2r}\left(1+\frac{\bar{v}^p}{\bar{u}^p}\right)\right] \mathrm{d}\bar{v} \\ \leq \quad & C \int\limits_{v_\diamond}^{\hat{v}} \sup_{u_\diamond \leq \bar{u} \leq \hat{u}} \left[\Omega^2(u_\diamond, v_\diamond) e^{-\beta[\bar{v}-v_\diamond+\bar{u}-u_\diamond]}\left(1+\frac{\bar{v}^p}{\bar{u}^p}\right)\right] \mathrm{d}\bar{v}, \end{aligned}$$

$$
\begin{aligned}
&\leq \tilde{\tilde{C}} \int_{v_\diamond}^{\hat{v}} \Omega^2(u_\diamond, v_\diamond) e^{-\beta[\bar{v}-v_\diamond]} \left(1 + \frac{\bar{v}^p}{u_\diamond^p}\right) \mathrm{d}\bar{v}, \\
&\leq \delta_2, \qquad (12.138)
\end{aligned}
$$

where $\delta_2 \to 0$ as $u_\diamond = v_\diamond \to \infty$. Thus we obtain the statement of Lemma 12.1.10 by fixing $u_\diamond = v_\diamond$ sufficiently large.

□

Proposition 12.6.2. *Let ϕ be as in Theorem 11.1.1 and Theorem 12.3.1. Then, for $u_\diamond = v_\diamond$ sufficiently close to ∞ and $\hat{u} > u_\diamond$, $\hat{v} > v_\diamond$*

$$
\begin{aligned}
&\int_{\{v_\diamond \leq v \leq \hat{v}\}} J^S_\mu(\phi) n^\mu_{u=\tilde{u}} dVol_{u=\tilde{u}} \\
+ &\int_{\{u_\diamond \leq u \leq \hat{u}\}} J^S_\mu(\phi) n^\mu_{v=\tilde{v}} dVol_{v=\tilde{v}} \leq C(u_\diamond, v_\diamond), \qquad (12.139)
\end{aligned}
$$

where C depends on C_0 of Theorems 11.1.1, 12.3.1 and $D_0(u_\diamond, v_\diamond)$ of Propositions 11.1.2, 12.3.2.

Proof. The proof follows from applying the divergence theorem for the current $J^S_\mu(\phi)$ in the region $\mathcal{R}_{VI}$. The past boundary terms are estimated by Proposition 12.4.3 and Proposition 12.5.1. Note that the weights of $J^S_\mu(\phi)$ are comparable to the weights of $J^W_\mu(\phi)$ for fixed $u_\diamond$, and similarly the weights of $J^S_\mu(\phi)$ are comparable to the weights of $J^Z_\mu(\phi)$ for fixed $v_\diamond$. The bulk term is absorbed by Lemma 12.6.1. □

Now that we have shown boundedness for different subregions of the interior we can state the following proposition for the entire interior region $\mathring{\mathcal{M}}|_{II}$, cf. Section 10.3.1

Corollary 12.6.3. *Let ϕ be as in Theorem 11.1.1 and Theorem 12.3.1. Then*

$$
\int_{\mathbb{S}^2} \int_{v_{fix}}^{\infty} \left[(|v|+1)^p (\partial_v \phi)^2(u_{fix}, v, \theta, \varphi) + |\nabla\!\!\!\!/ \phi|^2(u_{fix}, v, \theta, \varphi)\right] r^2 \mathrm{d}v \mathrm{d}\sigma_{\mathbb{S}^2} \leq C, \qquad (12.140)
$$

for $v_{fix} \geq v_{\times}$, $u_{fix} > -\infty$,

$$
\int_{\mathbb{S}^2} \int_{u_{fix}}^{\infty} \left[(|u|+1)^p (\partial_u \phi)^2(u, v_{fix}, \theta, \varphi) + |\nabla\!\!\!\!/ \phi|^2(u, v_{fix}, \theta, \varphi)\right] r^2 \mathrm{d}u \mathrm{d}\sigma_{\mathbb{S}^2} \leq C, \qquad (12.141)
$$

for $u_{fix} \geq u_{\ltimes}$, $v_{fix} > -\infty$,

where p *is as in* (11.12) *and* C *depends on* C_0 *of Theorems 11.1.1, 12.3.1 and* $D_0(u_\diamond, v_\diamond)$ *of Propositions 11.1.2, 12.3.2, where* $u_\diamond = v_\diamond$ *is as in 12.6.2.*

Proof. This follows by examining the weights in Propositions 12.4.3, 12.5.1 and 12.6.2 together with Theorem 12.2.1 and its analog for the *left* side. □

12.7 Global higher order energy estimates and pointwise boundedness

To obtain pointwise bounds in analogy to Section 12.2 we first have to extend Corollary 12.6.3 to a higher order statement.

Theorem 12.7.1. *On subextremal Reissner-Nordström spacetime* $(\mathcal{M}, g)$, *with mass* M *and charge* e *and* $M > |e| \neq 0$, *let* ϕ *be a solution of the wave equation* $\Box_g \phi = 0$ *arising from sufficiently regular Cauchy data on a two-ended asymptotically flat Cauchy surface* Σ. *Then, for* $v_{fix} \geq v_{\ltimes}$, $u_{fix} > -\infty$

$$\int_{\mathbb{S}^2}\int_{v_{fix}}^{\infty} \left[(|v|+1)^p(\partial_v\phi)^2(u_{fix},v,\theta,\varphi) + |\nabla\!\!\!/\phi|^2(u_{fix},v,\theta,\varphi)\right] r^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \leq E_0, \tag{12.142}$$

$$\int_{\mathbb{S}^2}\int_{v_{fix}}^{\infty} \left[(|v|+1)^p(\partial_v\Omega\phi)^2(u_{fix},v,\theta,\varphi) + |\nabla\!\!\!/\Omega\phi|^2(u_{fix},v,\theta,\varphi)\right] r^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \leq E_1, \tag{12.143}$$

$$\int_{\mathbb{S}^2}\int_{v_{fix}}^{\infty} \left[(|v|+1)^p(\partial_v\Omega^2\phi)^2(u_{fix},v,\theta,\varphi) + |\nabla\!\!\!/\Omega^2\phi|^2(u_{fix},v,\theta,\varphi)\right] r^2 \mathrm{d}v\mathrm{d}\sigma_{\mathbb{S}^2} \leq E_2; \tag{12.144}$$

and for $u_{fix} \geq u_{\ltimes}$, $v_{fix} > -\infty$

$$\int_{\mathbb{S}^2}\int_{u_{fix}}^{\infty} \left[(|u|+1)^p(\partial_u\phi)^2(u,v_{fix},\theta,\varphi) + |\nabla\!\!\!/\phi|^2(u,v_{fix},\theta,\varphi)\right] r^2 \mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2}$$

$$\leq \quad E_0, \tag{12.145}$$

$$\int_{\mathbb{S}^2}\int_{u_{fix}}^{\infty} \left[(|u|+1)^p(\partial_u\Omega\phi)^2(u,v_{fix},\theta,\varphi) \quad + \quad |\nabla\!\!\!/\Omega\phi|^2(u,v_{fix},\theta,\varphi)\right] r^2 \mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2}$$

$$\leq \quad E_1, \tag{12.146}$$

$$\int_{\mathbb{S}^2}\int_{u_{fix}}^{\infty} \left[(|u|+1)^p(\partial_u\Omega^2\phi)^2(u,v_{fix},\theta,\varphi) \quad + \quad |\nabla\!\!\!/\Omega^2\phi|^2(u,v_{fix},\theta,\varphi)\right] r^2 \mathrm{d}u\mathrm{d}\sigma_{\mathbb{S}^2}$$

$$\leq \quad E_2, \tag{12.147}$$

where p is as in (11.12).

Proof. This follows immediately from Corollary 12.6.3 by commutation. □

Having proven Theorem 12.7.1, the pointwise boundedness of $|\phi|$ in all of $\mathring{\mathcal{M}}|_{II}$ follows analogously to Section 12.2.3. We estimate

$$\begin{aligned}
\int_{\mathbb{S}^2} \phi^2(\hat{u},v)\mathrm{d}\sigma_{\mathbb{S}^2} \quad &\leq \quad \tilde{C}\left[\int_{\mathbb{S}^2}\left(\int_{u_*}^{\hat{u}}(|u|+1)^p(\partial_u\phi)^2(u,v)\mathrm{d}u\int_{u_*}^{\hat{u}}(|u|+1)^{-p}\mathrm{d}v\right) r^2\mathrm{d}\sigma_{\mathbb{S}^2}\right.\\
&\qquad \left. + \int_{\mathbb{S}^2}\phi^2(u_*,v)\mathrm{d}\sigma_{\mathbb{S}^2}\right],\\
\int_{\mathbb{S}^2} \phi^2(\hat{u},v)\mathrm{d}\sigma_{\mathbb{S}^2} \quad &\leq \quad \tilde{C}\left[\tilde{\tilde{C}}E_0 + \int_{\mathbb{S}^2}\phi^2(u_*,v)\mathrm{d}\sigma_{\mathbb{S}^2}\right],
\end{aligned} \tag{12.148}$$

where $u_* \geq u_{✂}$, $\hat{u} \in (u_*,\infty)$ and $v \in (1,\infty)$. Commuting by angular momentum operators Ω_i and summing over them we obtain

$$\int_{\mathbb{S}^2} (\Omega\phi)^2(\hat{u},v)\mathrm{d}\sigma_{\mathbb{S}^2} \quad \leq \quad \tilde{C}\left[\tilde{\tilde{C}}E_1 + \int_{\mathbb{S}^2}(\Omega\phi)^2(u_*,v)\mathrm{d}\sigma_{\mathbb{S}^2}\right], \tag{12.149}$$

$$\int_{\mathbb{S}^2} (\Omega^2\phi)^2(\hat{u},v)\mathrm{d}\sigma_{\mathbb{S}^2} \quad \leq \quad \tilde{C}\left[\tilde{\tilde{C}}E_2 + \int_{\mathbb{S}^2}(\Omega^2\phi)^2(u_*,v)\mathrm{d}\sigma_{\mathbb{S}^2}\right]. \tag{12.150}$$

By using the result (12.100) in (12.148) we derive pointwise boundedness according to

(12.89)

$$\begin{aligned}
\sup_{\mathbb{S}^2} |\phi(\hat{u}, v, \theta, \varphi)|^2 &\leq \tilde{C}\left[\int_{\mathbb{S}^2} (\phi)^2(\hat{u}, v)\mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (\Omega\phi)^2(\hat{u}, v)\mathrm{d}\sigma_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (\Omega^2\phi)^2(\hat{u}, v)\mathrm{d}\sigma_{\mathbb{S}^2}\right], \\
&\leq \tilde{C}\left[\tilde{\tilde{C}}\left(E_0 + E_1 + E_2\right) + C\right], \\
&\leq C, \qquad (12.151)
\end{aligned}$$

with C depending on the initial data.

Inequalities (12.151) and (12.101) gives the desired (9.2) for all $v \geq 1$. Interchanging the roles of u and v, likewise (9.2) follows for all $u \geq 1$. The remaining subset of the interior has compact closure in spacetime for which (9.2) thus follows by Cauchy stability. We have thus shown (9.2) globally in the interior.

As we will see in the next section, the continuity statement of Theorem 9.1.1 follows easily by revisiting the Sobolev estimates.

12.8 Continuity statement of Theorem 9.1.1

In the previous section we have shown pointwise boundedness, $|\phi(u, v, \varphi, \theta)| \leq C$. In the following we prove that ϕ extends continuously to $\mathcal{CH}^+$, that is to say ϕ extends to $\{\infty\} \times (-\infty, \infty] \cup (-\infty, \infty] \times \{\infty\}$ so that ϕ is continuous as a function on $(-\infty, \infty] \times (-\infty, \infty] \times \mathbb{S}^2$. Showing the extension closes the proof of Theorem 9.1.1.

In order to first show continuous extendibility of ϕ to $(-\infty, \infty) \times \{\infty\}$, it suffices to show: Given $-\infty < u < \infty$ and $\varphi, \theta \in \mathbb{S}^2$, $\forall \epsilon > 0 \quad \exists \delta, v_*$, such that

$$|\phi(u, v, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\theta})| < 4\epsilon, \quad \text{for all} \begin{cases} v > \tilde{v}, \quad \tilde{v} \geq v_* \\ u - \tilde{u} < \delta \\ \varphi - \tilde{\varphi} < \delta \\ \theta - \tilde{\theta} < \delta. \end{cases} \qquad (12.152)$$

By the triangle inequality we obtain

$$\begin{aligned}
&|\phi(u, v, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\theta})| \\
\leq\ &|\phi(u, v, \varphi, \theta) - \phi(\tilde{u}, v, \varphi, \theta)| + |\phi(\tilde{u}, v, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \varphi, \theta)| \\
&+|\phi(\tilde{u}, \tilde{v}, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \theta)| + |\phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \theta) - \phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\theta})|. \qquad (12.153)
\end{aligned}$$

We will show that each term can be bounded by ϵ.

Considering first the u direction by the fundamental theorem of calculus we have

$$\phi(u,v,\varphi,\theta)-\phi(\tilde{u},v,\varphi,\theta) \quad = \quad \int_{\tilde{u}}^{u} \partial_{\bar{u}}\phi(u,v,\varphi,\theta)\mathrm{d}\bar{u}. \tag{12.154}$$

Applying Cauchy Schwarz, we obtain for fixed v,φ,θ

$$\begin{aligned}
& |\phi(u,v,\varphi,\theta)-\phi(\tilde{u},v,\varphi,\theta)|^2 \\
\leq\ & \left(\int_{\tilde{u}}^{u} |\partial_{\bar{u}}\phi(\bar{u},v,\varphi,\theta)|\mathrm{d}\bar{u}\right)^2 \\
\leq\ & \left(\int_{\tilde{u}}^{u} (|\bar{u}|+1)^p \left(\partial_{\bar{u}}\phi(\bar{u},v,\varphi,\theta)\right)^2 \mathrm{d}\bar{u}\right)\left(\int_{\tilde{u}}^{u} (|\bar{u}|+1)^{-p}\mathrm{d}\bar{u}\right) \\
\leq\ & \left(\int_{\tilde{u}}^{u} (|\bar{u}|+1)^p \left(\partial_{\bar{u}}\phi(\bar{u},v,\varphi,\theta)\right)^2 \mathrm{d}\bar{u}\right) \\
& \times\left(\frac{1}{-p+1}(|u|+1)^{-p+1}-\frac{1}{-p+1}(|\tilde{u}|+1)^{-p+1}\right) \\
\leq\ & \tilde{C}\left(\int_{\tilde{u}}^{u}\sum_{k=0}^{2}\int_{\mathbb{S}^2}(|\bar{u}|+1)^p\left(\Omega^k\partial_{\bar{u}}\phi\right)^2\mathrm{d}\sigma_{\mathbb{S}^2}\mathrm{d}\bar{u}\right) \\
& \times\frac{1}{p-1}\left(\left(|\tilde{u}|+1\right)^{-p+1}-\left(|u|+1\right)^{-p+1}\right) && (12.155) \\
\leq\ & \epsilon. && (12.156)
\end{aligned}$$

In the above p is as in (11.12) and (12.155) follows from (12.89) applied to $\partial_{\bar{u}}\phi$,

$$\sup_{\theta,\varphi\in\mathbb{S}^2}|(|\bar{u}|+1)^p\partial_{\bar{u}}\phi(\bar{u},v,\theta,\varphi)|^2 \leq \tilde{C}\sum_{k=0}^{2}\int_{\mathbb{S}^2}(|\bar{u}|+1)^p\left(\Omega^k\partial_{\bar{u}}\phi\right)^2(\bar{u},v,\theta,\varphi)\mathrm{d}\sigma_{\mathbb{S}^2}. \tag{12.157}$$

Further, the last step, (12.156), then follows from (12.145)-(12.147) for a suitable chosen δ in (12.152).

For the second term in (12.153), again by the fundamental theorem of calculus and the Cauchy Schwarz inequality we obtain

$$\begin{aligned}
& |\phi(\tilde{u}, v, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \varphi, \theta)|^2 \\
\leq\ & \left(\int_{\tilde{v}}^{v} |\partial_{\bar{v}} \phi(\tilde{u}, \bar{v}, \varphi, \theta)| \mathrm{d}\bar{v} \right)^2 \\
\leq\ & \left(\int_{\tilde{v}}^{v} \bar{v}^p \left(\partial_{\bar{v}} \phi(u, \bar{v}, \varphi, \theta)\right)^2 \mathrm{d}\bar{v} \right) \left(\int_{\tilde{v}}^{v} \bar{v}^{-p} \mathrm{d}\bar{v} \right) \\
\leq\ & \tilde{C} \left(\int_{\tilde{v}}^{v} \sum_{k=0}^{2} \int_{\mathbb{S}^2} \bar{v}^p \left(\Omega^k \partial_{\bar{v}} \phi \right)^2 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}\bar{v} \right) \frac{1}{p-1} \left(\tilde{v}^{-p+1} - v^{-p+1} \right) && (12.158) \\
\leq\ & \tilde{C} \left(\int_{\tilde{v}}^{v} \sum_{k=0}^{2} \int_{\mathbb{S}^2} \bar{v}^p \left(\Omega^k \partial_{\bar{v}} \phi \right)^2 \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}\bar{v} \right) \frac{v_*^{-p+1}}{p-1} && (12.159) \\
\leq\ & \epsilon, && (12.160)
\end{aligned}$$

where in the third step, (12.158), we have used (12.89) applied to $\partial_{\bar{v}}\phi$. Equation (12.159) follows since $v > \tilde{v}$, $\tilde{v} \geq v_*$ and the last step follows by using (12.142)-(12.144) and for v_* large enough.

In the φ direction for fixed $\tilde{u}, \tilde{v}$ and θ we can state

$$\begin{aligned}
& |\phi(\tilde{u}, \tilde{v}, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \theta)|^2 \\
\leq\ & \left(\int_{\tilde{\varphi}}^{\varphi} |\partial_{\bar{\varphi}} \phi(\tilde{u}, \tilde{v}, \bar{\varphi}, \theta)| \mathrm{d}\bar{\varphi} \right)^2 \\
\leq\ & C \left(\int_{\tilde{\varphi}}^{\varphi} \int_{0}^{\pi} \left[|\partial_{\bar{\varphi}} \phi(\tilde{u}, \tilde{v}, \bar{\varphi}, \theta)| + |\partial_\theta \partial_{\bar{\varphi}} \phi(\tilde{u}, \tilde{v}, \bar{\varphi}, \theta)| \right] \mathrm{d}\sigma_{\mathbb{S}^2} \right)^2 \\
\leq\ & \tilde{C} \left(\int_{\mathbb{S}^2} \left[|\Omega_3 \phi(\tilde{u}, \tilde{v}, \bar{\varphi}, \theta)|^2 + \left| \sum_i a_i \Omega_i \Omega_3 \phi(\tilde{u}, \tilde{v}, \bar{\varphi}, \theta) \right|^2 \right] \mathrm{d}\sigma_{\mathbb{S}^2} \right) \left(\int_{\tilde{\varphi}}^{\varphi} \int_{0}^{\pi} \mathrm{d}\sigma_{\mathbb{S}^2} \right) \\
\leq\ & \epsilon, && (12.161)
\end{aligned}$$

where in the second step we have used one dimensional Sobolev embedding. In the third step we have used (10.38) for $\partial_{\bar{\varphi}}$ and (10.36) to (10.38) for ∂_θ. Further, we applied the Cauchy Schwarz inequality twice. The last step then follows by using (12.148)-(12.150) for $\tilde{u} \geq u_{✂}$ and (12.96)-(12.98) for $\tilde{u} \leq u_{✂}$ and since the second integral term is arbitrarily small by suitable choice of δ.

Similarly, in θ direction for fixed $\tilde{u}, \tilde{v}$ and $\tilde{\varphi}$ we obtain

$$\begin{aligned}
& |\phi(\tilde{u},\tilde{v},\tilde{\varphi},\theta) - \phi(\tilde{u},\tilde{v},\tilde{\varphi},\tilde{\theta})|^2 \\
\leq \; & \left(\int_{\tilde{\theta}}^{\theta} |\partial_{\bar{\theta}} \phi(\tilde{u},\tilde{v},\tilde{\varphi},\bar{\theta})| \mathrm{d}\bar{\theta} \right)^2 \\
\leq \; & C \left(\int_{\tilde{\theta}}^{\theta} \int_0^{2\pi} \left[\left| \partial_{\bar{\theta}} \phi(\tilde{u},\tilde{v},\tilde{\varphi},\bar{\theta}) \right| + \left| \partial_{\varphi} \partial_{\bar{\theta}} \phi(\tilde{u},\tilde{v},\tilde{\varphi},\bar{\theta}) \right| \right] \mathrm{d}\sigma_{\mathbb{S}^2} \right)^2 \\
\leq \; & \tilde{C} \left(\int_{\mathbb{S}^2} \left[\left| \sum_i a_i \Omega_i \phi(\tilde{u},\tilde{v},\tilde{\varphi},\bar{\theta}) \right|^2 + \left| \Omega_3 \sum_i a_i \Omega_i \phi(\tilde{u},\tilde{v},\tilde{\varphi},\bar{\theta}) \right|^2 \right] \mathrm{d}\sigma_{\mathbb{S}^2} \right) \\
& \quad \times \left(\int_{\tilde{\varphi}}^{\varphi} \int_0^{2\pi} \mathrm{d}\sigma_{\mathbb{S}^2} \right) \\
\leq \; & \epsilon.
\end{aligned} \tag{12.162}$$

The second step follows by one dimensional Sobolev embedding and the third from (10.36)-(10.38) and using the Cauchy Schwarz inequality twice. In the last step we used (12.148)-(12.150) for $\tilde{u} \geq u_{✂}$ and (12.96)-(12.98) for $\tilde{u} \leq u_{✂}$ and a suitable choice of δ.

Using the above results (12.156), (12.160), (12.161) and (12.162) in (12.153) yields the desired result (12.152).

To show continuous extendibility of ϕ to $\{\infty\} \times (-\infty, \infty)$, it suffices to show: Given $-\infty < v < \infty$ and $\varphi, \theta \in \mathbb{S}^2$, $\forall \epsilon > 0 \quad \exists \delta, u_*$, such that

$$|\phi(u,v,\varphi,\theta) - \phi(\tilde{u},\tilde{v},\tilde{\varphi},\tilde{\theta})| < 4\epsilon, \quad \text{for} \begin{cases} u > \tilde{u}, \quad \tilde{u} \geq u_* \\ v - \tilde{v} < \delta \\ \varphi - \tilde{\varphi} < \delta \\ \theta - \tilde{\theta} < \delta. \end{cases} \tag{12.163}$$

This can be proven by substituting v with u and $\tilde{v}$ with $\tilde{u}$ and repeating all above steps.

Similarly, to show continuous extendibility to $\{\infty\} \times \{\infty\}$, it suffices to show: Given $\varphi, \theta \in \mathbb{S}^2$, $\forall \epsilon > 0 \quad \exists \delta, u_*, v_*$, such that

$$|\phi(u, v, \varphi, \theta) - \phi(\tilde{u}, \tilde{v}, \tilde{\varphi}, \tilde{\theta})| < 4\epsilon, \quad \text{for} \begin{cases} u > \tilde{u}, \quad \tilde{u} \geq u_* \\ v > \tilde{v}, \quad \tilde{v} \geq v_* \\ \varphi - \tilde{\varphi} < \delta \\ \theta - \tilde{\theta} < \delta. \end{cases} \tag{12.164}$$

This follows as in (12.159) and completes the proof of Theorem 9.1.1.

□

Chapter 13

Conclusions of part II

Our proof of *stability* properties, namely boundedness of solutions ϕ of (9.1) was motivated by the question what would happen to an object at the inner horizon of a black hole emerged from imploded matter.

To simplify the problem we looked at perturbations of exact two-ended black hole solutions. As a toy model for non-linear perturbations of the evolution of (10.7) we considered the "poor man's" linear version (9.1) in which the tensorial structure of the equations as well as all lower order terms are neglected. Due to the similarity of the causal structure Reissner-Norström spacetime serves as a spherically symmetric proxy for Kerr spacetime.

Even in this hugely simplified model the two stages of evolution (1.3) of a black hole from collapse, mentioned in the introduction of part I of the thesis are in a sense still incorporated. Namely, stage 1 of the evolution which is the decay of perturbations and the settling of the black hole to Kerr spacetime translates to finding an upper bound for the scalar wave decay rate in advanced time along $\mathcal{H}^+$. This was done by Dafermos and Rodnianski [29] whose result we have used as a starting point for our proof which tackles the second stage of evolution. Stage 2 of evolution, which deals with the decay of small perturbations inside the black hole from the event horizon to the Cauchy horizon was then analyzed by us via propagating the decay rate further inside the black hole using the divergence Theorem and the vector field method. The crux of the proof was showing boundedness of the solution ϕ within a characteristic rectangle Ξ in the vicinity of timelike infinity i^+[1]. We showed that ϕ is uniformely bounded up to the Cauchy horizon. Not only was this the first proof of boundedness for ϕ on fixed Reissner-Nordström interior backgrounds, the proof is also robust in that its strategy serves as a precursor for analyzing Kerr backgrounds and gave some indications for non-linear analysis.

A later "poor man's" linear proof of Luk and Oh [64] concerned with the *instability* properties of Reissner-Nordström interior showed that $\partial_v \phi$ will blow up in an integrable way along $\mathcal{CH}_A^+$. This feature can be understood by remembering that the blueshift of the right asymptotic end is like a Lorentz boost in u direction. Analogously to a Lorentz boost of the electromagnetic field in flat spacetime we expect that only the components transverse to the boost should amplify. Therefore, we expect $\partial_v \phi$ to diverge and $\partial_u \phi$ to be well behaved along the right end of the Cauchy horizon, $\mathcal{CH}_A^+$. Furthermore, in analogy to the vector potential, ϕ should be well behaved in v direction. But since it is a scalar we expect it to be bounded generally. In fact it was shown in Theorem 6.1.1 by Dafermos and Luk that even non-linearly the solutions are well behaved such that they can be extended as a Lorentzian manifold with C^0 metric beyond the Cauchy horizon. Another way to think of this problem is by anticipating the mass inflation. Because of the mass inflation, one could imagine a surface stress energy tensor on the $v = const$ hypersurfaces. This is due to the fact that as a first approximation the curvature scalar is proportional to the Hawking mass. We would then see that again the Christoffel symbols can only blow up across the hypersurface. The tangential derivatives of the induced metric have to be continuous while the transverse ones are possibly discontinuous.

Our finding explained in detail in part II of the thesis together with the results of closely related investigations that we have mostly described in part I of the thesis gives rise to the following conclusions. A C^0 formulation of *the Strong Cosmic Censorship Conjecture* is false since generically perturbed Reissner-Nordström and Kerr spacetimes can be extended with C^0 metric. Luk and Oh, [64], have shown "poor" linearly that ϕ generically does not belong to the Sobolev space $W^{1,2}_{\mathrm{loc}}$. This suggests a $W^{1,2}_{\mathrm{loc}}$-formulation of statement (1.2) for the Strong Cosmic Censorship Cenjecture on a "poor" linear level. Suppose now that this result carries over to the full non-linear case. Then, for the full non-linear case it would be plausible to demand that for Einstein spacetimes there exist no extensions to the maximal domain of dependence with weakly defined square integrable Christoffel symbols. Spacetimes could still be extended with lower regularity but would no longer serve the Einstein field equations due to the lack of well defined Christoffel symbols. In that sense, from a purely classical partial differential equation

[1] The difficulty of proving boundedness in this particular region is due to the fact that Ξ has infinite size no matter how large the past boundaries $v = v_0$ and $|u| = |u_0|$ are chosen.

perspective, solutions could not be extended in a physically meaningful way and the problem of lack of uniqueness is cured. This argument however suffers from the flaw of not providing a satisfactory answer what mechanism would prevent an astronaut from traveling further beyond $\mathcal{CH}^+$. Therefore, investigations along and beyond the interior would be required to mathematically and physically understand this mechanism more. On the other hand from a quantum gravitational perspective there is no obstacle towards matching two classically regular spacetime regions via a quantum tunnel, ensuring that an observer could either travel forever or has to be destroyed by growing tidal forces. From this point of view *the Strong Cosmic Censorship Conjecture* could be an artefact of applying a classical theory in a region outside its validity. If that was the case, then it is unclear which mechanism renders the classical theory invalid. An obvious indication that a quantum gravitational theory is needed would be radii of the spacetime geometry becoming as small as the characteristic strong interaction size of elementary particles (10^{-13}cm). As we have explained in part I of the thesis this is not the case close to the Cauchy horizon of Reissner-Nordström and Kerr spacetime. The singularity along $\mathcal{CH}^+$ is weak in nature and generically perturbations of Kerr and Reissner-Nordström do not lead to strong spacelike singularities in the interior. Moreover, the expected deformation from the tidal forces is so small that this cannot be taken as an indication that classical theory will have to break down.

Appendix A

The redshift effect derived from the vector field method

In this chapter we will briefly sketch how to prove the local redshift decay using the divergence theorem and the vector field method as it was done in [35] or in more detail for higher dimensional Schwarzschild black holes in [88]. Note that we are eventually interested in the interior region, in which we are also going to use the redshift multiplier, see Proposition 12.1.1. The same multiplier is used in the exterior region and for simplicity we will review the construction of the redshift multiplier which is derived in the exterior region.

The strategy is, to prove uniform lower boundedness for the spacetime integral of a region close to and including $\mathcal{H}^+$ from a current of which the decay rate is known, namely $b_1 J^N_\mu(\phi) N^\mu \leq K^N(\phi)$, as stated in Proposition 12.1.1. Using this in the divergence theorem will by Grönwall inequality, see Appendix D.2, lead to the desired exponential decay. The construction of the multiplier N will be done with the help of the vector fields T, Y and E_A which are required to obey certain properties. In the following we will list and explain these properties.

Before defining the needed multiplier vector fields let us first, for more transparency, introduce coordinates regular at the event horizon, namely

$$V_+ = V_+(v) = e^{\kappa_+ v}, \qquad U_+ = U_+(u) = -e^{-\kappa_+ u}, \tag{A.1}$$

with

$$u = t - r^*, \quad \text{and} \quad v = r^* + t,$$

and the surface gravitiy at the event horizon, κ_+, see (10.33). By this definition we have $V_+ = 0$ at the Cauchy horizon $\mathcal{CH}^+$ and $U_+ = 0$ at the event horizon $\mathcal{H}^+$. Therefore, we get

$$\begin{aligned} \partial_u &= -\kappa_+ U_+ \partial_{U_+}, \\ \partial_v &= \kappa_+ V_+ \partial_{V_+}. \end{aligned} \tag{A.2}$$

By this choice of coordinates and $\mathrm{d}r^* = \frac{\mathrm{d}r}{1-\mu}$, see Section 2, we have

$$\partial_{V_+} r = \frac{1-\mu}{2\kappa_+ V_+}, \qquad \partial_{U_+} r = \frac{1-\mu}{2\kappa_+ U_+}. \tag{A.3}$$

The metric with the above introduced coordinates is then given by $\mathrm{d}s^2 = -\mho^2 \mathrm{d}U_+ \mathrm{d}V_+ + r^2 \mathrm{d}\sigma_{\mathbb{S}}^2$ and for the metric coefficient we obtain

$$\mho^2(U_+, V_+) = \frac{\Omega^2(u,v)}{\partial_u U_+(u) \partial_v V_+(v)} = -\frac{1-\mu}{{\kappa_+}^2 U_+ V_+}, \tag{A.4}$$

where $\Omega^2(u,v)$ is explained more in Section 10.2.2 but with a different sign accounting for the fact that we consider the exterior region now while in part II of the thesis we always refer to the interior region.

We work with the Killing vector field $T = \partial_t$, which in the above defined coordinates reads as

$$T = \frac{\kappa_+}{2} \left(V_+ \partial_{V_+} - U_+ \partial_{U_+} \right). \tag{A.5}$$

The fact that T is Killing implies $\pi^T = 0$, see (C.10) of Appendix C.2. Further, as we see from

$$g(T,T) = \frac{\mho^2}{4} {\kappa_+}^2 U_+ V_+ = -\frac{1-\mu}{4} = \begin{cases} < 0, & r > r_+ \\ = 0, & r = r_+, r = r_- \\ > 0, & r_- < r < r_+ \end{cases}, \tag{A.6}$$

that T is timelike in the exterior, spacelike in the interior and null along the horizons.

From (A.5) it is immediate that the last term of T vanishes at the event horizon and both terms vanish at the bifurcation point, that is to say

$$T|_{\mathcal{H}^+} = \frac{\kappa_+}{2} V_+ \partial_{V_+}, \qquad T|_{\mathcal{H}^+ \cap \mathcal{H}^-} = 0.$$

Since the T vector field is at the event horizon tangent to $\mathcal{H}^+$, we cannot control the transverse derivatives by employing it.

This lead Dafermos and Rodnianski, cf. [30], to introduce a vector field Y which is also null at $\mathcal{H}^+$ but transverse to it. Boundary terms arising from Y are related to energies which a local observer close to the horizon would measure. We normalize Y by demanding

$$g(T, Y)|_{\mathcal{H}^+} = -2, \tag{A.7}$$

yielding

$$\Rightarrow Y|_{\mathcal{H}^+} \overset{(A.3)}{=} -\frac{4}{\partial_{U_+} r} \partial_{U_+}. \tag{A.8}$$

Further, it can be shown that the Lie-bracket of T and Y vanishes at the horizon, $[T, Y]_{\mathcal{H}^+} = [\nabla_T Y - \nabla_Y T]_{\mathcal{H}^+} = 0$, with ∇ the Levi-Civita connection associated to the metric. Moreover, we introduce local frame fields E_A, with $A = 1, 2$, which are tangential to the orbits of the spherical isometry, $SO(3)$, and orthonormal to each other as well as to T and Y along the event horizon. As we have mentioned earlier to obtain decay it is essential to prove that the surface gravity κ_+ is positive. Further one extends the vector field Y away from the horizon such that the Lie-bracket of T and Y still vanish. Then calculating K^Y according to (10.5) and using the Cauchy inequality we obtain the expression $K^Y \geq b\mathcal{T}(Y + T, Y + T) = b\mathcal{T}(N, N) = b(J^N, N)$, where $\mathcal{T}$ is the energy momentum tensor of the scalar field as defined by (10.1) and b is a uniform constant. The timelike vector field N equates T away from the horizon and $Y + T$ at the horizon. After all this preparation one can close with the divergence theorem

$$\int_{\{r_+ \leq r \leq r_+ + \epsilon\}} J^Y_\mu(\phi) n^\mu_{v_2} \mathrm{dVol}_{v_2} + b \int_{\mathcal{R}} J^Y_\mu(\phi) n^\mu_{\bar{v}} \mathrm{dVol}_{\bar{v}} = \int_{\{r_+ \leq r \leq r_+ + \epsilon\}} J^Y_\mu(\phi) n^\mu_{v_1} \mathrm{dVol}_{v_1},$$

where $\mathcal{R}$ is a region close to the event horizon. And then with Grönwall inequality, cf. Appendix D.2, this leads to

$$\int_{\{r_+ \leq r \leq r_+ + \epsilon\}} J^Y_\mu(\phi) n^\mu_{v_2} \mathrm{dVol}_{v_2} = -e^{b(v_2 - v_1)} \int_{\{r_+ \leq r \leq r_+ + \epsilon\}} J^Y_\mu(\phi) n^\mu_{v_1} \mathrm{dVol}_{v_1},$$

which agrees with what we found considering the proper times of the two observers in Section 2.

Appendix B

Sobolev spaces

When interested in the local existence of solutions to partial differential equations it is natural to consider Sobolev spaces rather than spaces of continuous functions and their derivatives. The reason is that usually it is impossible to derive good enough estimates to demonstrate affiliation of solutions to such more regular spaces. In the following we want to discuss the basic properties of these function spaces. For a more extensive treatment see for example [39], [93], [85].

The Sobolev space $H^k(\mathcal{U})$ is a Hilbert space with an inner product and is in Lebesgue space, $L^2(\mathcal{U})$ [1], with $\mathcal{U} \subset \mathbb{R}$. More general we can write $W^{k,2}_{\mathrm{loc}}$. Members of Sobolev spaces have weak derivatives.
Therefore, it is useful to first understand the notion of *weak derivative* which is a generalized version of a partial derivative.

Let $\phi \in C^k(\mathcal{U})$, with k a positive integer, and $\alpha = (\alpha_1, .., \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + .. + \alpha_n = k$, then

$$\int_{\mathcal{U}} \phi D^\alpha u \mathrm{d}x = (-1)^{|\alpha|} \int_{\mathcal{U}} D^\alpha \phi u \mathrm{d}x, \tag{B.1}$$

where u is an infinitely differentiable test function with compact support in $\mathcal{U}$ and $D^\alpha u = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} ... \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}$. The above equation follows from partially integrating $|\alpha|$ times. Now suppose that ϕ is not k times continuously differentiable. Then, the locally summable

[1] Investigating partial differential equations in general one might want to consider Sobolev spaces in $L^p(\mathcal{U})$, that is $W^{k,p}_{\mathrm{loc}}$. However, due to the hyperbolicity of the Einstein field equation the energy fluxes appear such that we are lead to consider square integrable function spaces.

function ψ is the α^{th} weak derivative of ϕ, i.e. $D^\alpha \phi = \psi$, provided that

$$\int\limits_{\mathcal{U}} \phi D^\alpha u \mathrm{d}x = (-1)^{|\alpha|} \int\limits_{\mathcal{U}} \psi u \mathrm{d}x. \tag{B.2}$$

Having defined weak derivatives we can render the above statement of Sobolev spaces more precise by stating $\phi \in H^k(\mathcal{U})$ if and only if all weak derivatives up to the k^{th} order are in $L^2(\mathcal{U})$. If $\phi \in H^k(\mathcal{U})$, we define its norm to be

$$||\phi||_{\mathrm{H}^k(\mathcal{U})} := \left(\sum_{|\alpha| \leq k} \int\limits_{\mathcal{U}} |D^\alpha \phi|^2 \mathrm{d}x \right)^{\frac{1}{2}}. \tag{B.3}$$

The idea of defining such a norm is now to conclude from its boundedness to the existence of weak solutions.
Let us further define the inner product

$$\langle \phi, \psi \rangle_{\mathrm{H}^k(\mathcal{U})} = \sum_{|\alpha| \leq k} \int\limits_{\mathcal{U}} D^\alpha \phi D^\alpha \psi \mathrm{d}x = \sum_{|\alpha| \leq k} \langle \phi, \psi \rangle_{\mathring{\mathrm{H}}^k(\mathcal{U})}, \tag{B.4}$$

where $\mathring{\mathrm{H}}^k(\mathcal{U})$ are the homogeneous Sobolev spaces. They are defined to consist of k times weakly differentiable functions ϕ such that $D^\alpha \phi \in L^2(\mathcal{U})$ for $|\alpha| = k$.

We have argued above that for our purposes L^2-Sobolev spaces, which are together with (B.3) Hilbert spaces, are the natural function spaces to look at. Nevertheless, it might sometimes be useful to relate the degree of regularity obtained from considerations in Sobolev spaces with classical differentiability. Inequalities providing us with this relation are referred to as Sobolev embedding or Sobolev-type inequality. In other words we sacrifice differentiability to gain integrability. It is never the other way around since a locally integrable function need not have a well defined weak derivative.

APPENDIX C

The J and K-currents

C.1 The J-currents and normal vectors

In the following we will derive the J-currents on constant r, u and v hypersurfaces as well as the hypersurface γ, defined in (12.37).

We consider an arbitrary function $F(u,v)$ independent of the angular coordinates. Let ζ be a levelset $\zeta = \{F(u,v) = 0\}$. Then, the normal vector to the hypersurface ζ is given by

$$n^\mu_\zeta = \frac{1}{\sqrt{\Omega^2 |\partial_u F \partial_v F|}} (\partial_v F \partial_u + \partial_u F \partial_v). \tag{C.1}$$

In particular, for the future directed normal vector of an $r(u,v) = const$ hypersurface we obtain

$$n^\mu_{r=const} = \frac{1}{\sqrt{\Omega^2}} (\partial_u + \partial_v), \tag{C.2}$$

and on constant u and v null hypersurfaces with their related volume elements we have

$$n^\mu_{u=const} = \frac{2}{\Omega^2} \partial_v, \quad \mathrm{dVol}_{u=const} = r^2 \frac{\Omega^2}{2} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}v, \tag{C.3}$$

$$n^\mu_{v=const} = \frac{2}{\Omega^2} \partial_u, \quad \mathrm{dVol}_{v=const} = r^2 \frac{\Omega^2}{2} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}u. \tag{C.4}$$

For (C.3) and (C.4), note that since vectors orthogonal to null hypersurfaces cannot be normalized, their proportionality has to be chosen consistent with an associated volume form in the application of the divergence theorem. Further, the volume element of the 4 dimensional spacetime is given by

$$\mathrm{dVol} = r^2 \frac{\Omega^2}{2} \mathrm{d}\sigma_{\mathbb{S}^2} \mathrm{d}u \mathrm{d}v. \tag{C.5}$$

According to (10.3), using an arbitrary vector field $X = X^u\partial_u + X^v\partial_v$ we then obtain:

$$\begin{aligned} J^X_\mu(\phi) n^\mu_\zeta &= \frac{1}{\sqrt{\Omega^2}}\left[X^v\sqrt{\frac{\partial_u F}{\partial_v F}}(\partial_v\phi)^2 + X^u\sqrt{\frac{\partial_v F}{\partial_u F}}(\partial_u\phi)^2\right] \\ &\quad + \frac{\sqrt{\Omega^2}}{4}\left[X^v\sqrt{\frac{\partial_v F}{\partial_u F}} + X^u\sqrt{\frac{\partial_u F}{\partial_v F}}\right]|\nabla\!\!\!/\phi|^2, \end{aligned} \tag{C.6}$$

$$J^X_\mu(\phi) n^\mu_{r=const} = \frac{1}{\sqrt{\Omega^2}}\left[X^v(\partial_v\phi)^2 + X^u(\partial_u\phi)^2\right] + \frac{\sqrt{\Omega^2}}{4}\left[X^v + X^u\right]|\nabla\!\!\!/\phi|^2, \tag{C.7}$$

$$J^X_\mu(\phi) n^\mu_{v=const} = \frac{2}{\Omega^2}X^u(\partial_u\phi)^2 + \frac{1}{2}X^v|\nabla\!\!\!/\phi|^2, \tag{C.8}$$

$$J^X_\mu(\phi) n^\mu_{u=const} = \frac{2}{\Omega^2}X^v(\partial_v\phi)^2 + \frac{1}{2}X^u|\nabla\!\!\!/\phi|^2. \tag{C.9}$$

C.2 The K-current

In order to compute all scalar currents according to (10.5) in (u, v) coordinates we first derive the components of the deformation tensor which is given by

$$(\pi^X)^{\mu\nu} = \frac{1}{2}(g^{\mu\lambda}\partial_\lambda X^\nu + g^{\nu\sigma}\partial_\sigma X^\mu + g^{\mu\lambda}g^{\nu\sigma}g_{\lambda\sigma,\delta}X^\delta), \tag{C.10}$$

where X is an arbitrary vector field, $X = X^u\partial_u + X^v\partial_v$ without angular components.[1] From this we obtain

$$\begin{aligned} (\pi^X)^{vv} &= -\frac{2}{\Omega^2}\partial_u X^v, \\ (\pi^X)^{uu} &= -\frac{2}{\Omega^2}\partial_v X^u, \\ (\pi^X)^{uv} &= -\frac{1}{\Omega^2}(\partial_v X^v + \partial_u X^u) - \frac{2}{\Omega^2}\left(\frac{\partial_v\Omega}{\Omega}X^v + \frac{\partial_u\Omega}{\Omega}X^u\right), \end{aligned}$$

[1] Recall that all our multipliers N, $-\partial_r$, S_0 and S only contain u and v components.

$$
\begin{aligned}
(\pi^X)^{\theta\theta} &= \frac{1}{r^2}\left(\frac{\partial_v r}{r}X^v + \frac{\partial_u r}{r}X^u\right),\\
(\pi^X)^{\phi\phi} &= \frac{1}{r^2\sin^2\theta}\left(\frac{\partial_v r}{r}X^v + \frac{\partial_u r}{r}X^u\right).
\end{aligned}
$$

From (10.1) we calculate the components of the energy momentum tensor in (u, v) coordinates as

$$
\begin{aligned}
T_{vv} &= (\partial_v\phi)^2,\\
T_{uu} &= (\partial_u\phi)^2,\\
T_{uv} &= T_{vu} = \frac{\Omega^2}{4}|\nabla\phi|^2,\\
T_{\theta\theta} &= (\partial_\theta\phi)^2 + \frac{2r^2}{\Omega^2}(\partial_u\phi\partial_v\phi) - \frac{1}{2}r^2|\nabla\phi|^2,\\
T_{\phi\phi} &= (\partial_\phi\phi)^2 + \frac{2r^2\sin^2\theta}{\Omega^2}(\partial_u\phi\partial_v\phi) - \frac{1}{2}r^2\sin^2\theta|\nabla\phi|^2.
\end{aligned}
$$

Multiplying the components according to (10.5) and using the relations (10.25) we obtain

$$
\begin{aligned}
K^X = & -\frac{2}{\Omega^2}\left[\partial_u X^v(\partial_v\phi)^2 + \partial_v X^u(\partial_u\phi)^2\right]\\
& -\frac{2}{r}\left[X^v + X^u\right](\partial_u\phi\partial_v\phi)\\
& -\left[\frac{1}{2}\left(\partial_v X^v + \partial_u X^u\right) + \left(\frac{\partial_v\Omega}{\Omega}X^v + \frac{\partial_u\Omega}{\Omega}X^u\right)\right]|\nabla\phi|^2.
\end{aligned}
\tag{C.11}
$$

Appendix D

Grönwall inequality

In the following we will briefly review the derivation of the Grönwall inequality as can be found for example in [39].

D.1 Grönwall inequality in differential form

Assume $\eta(\cdot)$ is a nonnegative, absolutely continuous function on $[0,T]$, which satisfies a.e. t the differential inequality

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t), \tag{D.1}$$

with ϕ and ψ nonnegative, summable funtions on $[0,T]$, then

$$\eta(t) \leq e^{\int_0^t \phi(s)\mathrm{d}s} \left[\eta(0) + \int_0^t \psi(s)\mathrm{d}s\right], \tag{D.2}$$

for all $0 \leq t \leq T$.

Proof. We can manipulate (D.1) to

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\eta(s)e^{-\int_0^s \phi(r)\mathrm{d}r}\right) = e^{-\int_0^t \phi(s)\mathrm{d}s}(\eta'(s) - \phi(s)\eta(s)) \leq e^{-\int_0^t \phi(s)\mathrm{d}s}\psi(s), \tag{D.3}$$

for a.e. $0 \leq s \leq T$. Therefore we obtain

$$\eta(t)e^{-\int_0^t \phi(r)\mathrm{d}r} \leq \eta(0) + \int_0^t \eta(s)e^{-\int_0^s \phi(r)\mathrm{d}r}\psi(s)\mathrm{d}s \leq \eta(0) + \int_0^t \psi(s)\mathrm{d}s \tag{D.4}$$

for each $0 \leq t \leq T$, by integrating up to t and pulling the function evaluated at the lower limit to the right hand side. In the very last step we have used that ϕ and ψ are nonnegative and thus we have proven (D.2). □

D.2 Grönwall inequality in integral form

Let $\xi(t)$ be a nonnegative, summable funtion on $[0, T]$ satisfying the integral inequality

$$\xi(t) \quad \leq \quad C_1 \int_0^t \xi(s)\mathrm{d}s + C_2, \tag{D.5}$$

for a.e. t and C_1, C_2 nonnegative constants. Then,

$$\xi(t) \quad \leq \quad C_1 C_2 t e^{C_1 t} + C_2,$$

for a.e. $0 \leq t \leq T$.

Proof. Define $\eta(t) := \int_0^t \xi(s)\mathrm{d}s$, then

$$\begin{aligned} \eta'(t) \quad &\leq \quad C_1(\eta(t) - \eta(0)) + C_2 &\text{(D.6)} \\ &= \quad C_1\eta(t) + B. &\text{(D.7)} \end{aligned}$$

Using the differential form of the Grönwall inequality, c.f. Section D.1, we obtain

$$\Rightarrow \eta(t) \quad \leq \quad e^{C_1 t}(\eta(0) + Bt).$$

If $\eta(0) = 0$, then

$$\eta(t) \quad = \quad C_2 t e^{C_1 t}. \tag{D.8}$$

And if we plugg (D.8) back into (D.6), we obtain

$$\eta'(t) \quad \leq \quad C_1 C_2 t e^{C_1 t} + C_2.$$

□

English summary

Remarkably just a few months after Einstein published his famous theory of general relativity, Schwarzschild discovered the first non-trivial spacetime solution satisfying the field equations. Only a few decades later it was understood that this solution represented an asymptotically flat spacetime containing a black hole as an isolated system. Since their discovery, black holes have been a fascinating topic to both scientist and lay people and have therefore been subject to countless scientific investigations but also science fiction novels and movies. Thanks to the growing popularity of black holes it is almost common knowledge that from the exterior boundary of black holes, the so called event horizon, not even light can escape and any object crossing it will be swallowed by the black hole. The interior region of black hole spacetimes is considered mysterious and while in 70's stories the astronaut always smashes inside the black hole, in later material – for instance in the recent movie "Interstellar" – the rumor has spread that there might be a chance for survival.

Mathematically this can be understood as follows. There exist four known exact solutions to the Einstein field equations, which represent black hole spacetimes. The simplest of these is the Schwarzschild solution, which is a spherically symmetric, static solution without charge or angular momentum but the mass of the black hole as its only parameter. In Schwarzschild spacetimes it is indeed true that an observer entering the interior inevitably has to be destroyed by the growing tidal forces as he approaches the singularity of the black hole. The nature of the singularity in this spacetime is strong in the sense that a macroscopic object would suffer infinite deformation and also in the sense that the spacetime is not extendible at the singularity. In fact classically the singularity itself should not even be considered part of the spacetime since the theory of general relativity simply breaks down there.

Opposed to this, curvature does not blow up in the interior of charged, spherically symmetric static black holes, so called Reissner-Nordström black holes. Instead this solution shows a different peculiar feature: the solution is only unique up to a boundary which is called the Cauchy horizon or sometimes the inner horizon. Beyond the Cauchy horizon determinism is lost while the spacetime remains regular. Roughly speaking, neglecting the backreaction of the observer's mass onto the background, this implies that our astronaut could travel out of the black hole into another universe causally disconnected from ours. His further fate cannot by any means be predicted by classical theory. The same characteristic shows in the Kerr solution which is an axialsymmetric, stationary solution representing rotating black holes and in the Kerr-Newmann solution which represents a charged rotating black hole solution.

Note that general relativity is a dynamical theory. This implies that its exact solutions constitute only very specific cases. We may now wonder if the regularity in the interior of rotating and charged black holes is a stable feature or if under small perturbations singularities might emerge. In particular, a specific type of singularities evolving under perturbations –namely strong ones such as occurring in Schwarzschild spactime– would indicate that the doorway to the causally disconnected universe is closed. The result of this thesis does not advocate this possibility, which denies entry to an observer traveling close to the Cauchy horizon. The appearance of the non-deterministic regions cannot a priori be excluded. Instead we obtain certain stability results under perturbations which however leave some space for irregular behavior at the Cauchy horizon without completely closing the exit. In the following we will specify further which exact setup leads to this conclusion.

First of all we have to decide which spacetime we want to perturb. Of the above mentioned black hole solutions the Kerr solution is the astrophysically most relevant solution. Astrophysical objects are not expected to be significantly charged since the induced electric field is expected to balance itself out by accreting opposed charges. Nevertheless, due to the similarity of the causal structure, the Reissner-Nordström solution is a very important candidate as a toy model for the mathematically much more complicated Kerr solution. Therefore, in this thesis we will focus on the Reissner-Nordström solution.

The simplest perturbations one could consider are linear perturbations. Obviously, in order to finally analyze non-linear perturbations it is crucial to have a very good understanding of its linearized version first. On the other hand since general relativity is a non-linear theory the reader might be skeptical in how far conclusions derived within a linearized model can indicate anything physically relevant. Fortunately, for reasons too involved to explain here[1] there is good evidence to believe that the stability mechanism is indeed already captured by the linearized theory. Analyzing linear perturbations in full Einstein field equations is still a complicated problem due to the tensorial structure of the equations. Therefore, as a toy model for linear perturbations we consider solutions to scalar wave equations on fixed Reissner-Nordström backgrounds. Given this setup we prove boundedness of the wave equation in the entire interior up to and including the Cauchy horizon. This result is a new mathematical insight in its own right. Going one step further and taking it seriously as a model for the physical fate of the perturbed Reissner-Nordström spacetimes it seems to suggest that perturbations will not lead to a strong spacelike singularity. In fact it leaves space for at most a weak null singularity along the Cauchy horizon so that the travel of our astronaut to the causally disconnected and non-deterministic regions is not a priori prevented.

[1] We will catch up with this in part I of the thesis.

Nederlandse samenvatting

Maar een paar maanden nadat Einstein zijn beroemde theorie van de algemene relativiteit publiceerde, ontdekte Schwarzschild opmerkelijk genoeg al de eerste niet-triviale ruimtetijd oplossing van de veldvergelijkingen. Pas een paar decennia later werd begrepen dat deze oplossing een asymptotisch vlakke ruimtetijd voorstelde, dat een gesoleerd, zwart gat bevat. Sinds hun ontdekking zijn zwarte gaten een fascinerend thema voor zowel wetenschappers en leken en zijn daarom het onderwerp van talloze wetenschappelijke studies, maar ook van science fictie boeken en films. Met dank aan de groeiende populariteit van zwarte gaten is het nu vrijwel algemeen bekend dat zelfs licht niet kan ontsnappen van de buitenste rand van een zwart gat, de zogenaamde waarnemingshorizon, en dat elk voorwerp dat deze grens overschrijdt, opgeslokt zal worden door het zwarte gat. Het binnenste gebied van een zwart gat wordt als mysterieus gezien en terwijl in de jaren '70 de verhalen de astronaut altijd in het zwarte gat lieten verpletteren, gaat in later werk - niet in de laatste plaats vanwege de recente film "Interstellar" - het gerucht dat er een kans op ontsnapping is.

Wiskundig kan dit als volgt worden begrepen. Er zijn vier exacte oplossingen van Einsteins veldvergelijkingen bekend, die corresponderen met zwarte gaten. De eenvoudigste van deze is de Schwarzschild-oplossing, een bolvormige, statische oplossing, zonder lading of impulsmoment. De massa van het zwarte gat is de enige parameter van deze oplossing. Voor deze ruimtetijd-oplossing is het inderdaad zo dat een waarnemer, die het zwarte gat binnengaat, onvermijdelijk vernietigd zal worden door de groeiende getijdekrachten, als hij de singulariteit van het zwarte gat nadert. De aard van de singulariteit in deze ruimtetijd is sterk, in de zin dat een macroscopisch object oneindig vervormt, evenals dat de ruimtetijd niet uitbreidbaar is bij de singulariteit. Klassiek gezien zou de singulariteit zelfs niet als deel van de ruimtetijd gezien mogen worden, want de algemene relativiteitstheorie is hier niet meer geldig.

Daarentegen wordt de kromming in het binnenste van geladen, bolvormige, statische, zwarte gaten, de zogenaamde Reissner-Nordström zwarte gaten niet oneindig. In plaats daarvan vertoont deze oplossing een andere opmerkelijke eigenschap: de oplossing is uniek, op een rand na, de Cauchy-horizon. Voorbij de Cauchy-horizon gaat determinisme verloren, maar de ruimtetijd wordt niet singulier. Los gezegd, als we de terugkoppeling van de waarnemer op de achtergrond verwaarlozen, impliceert dit dat onze astronaut vanuit het zwarte gat in een ander universum kan reizen, dat causaal ontkoppeld is van dat van ons. Zijn verdere lot kan op geen enkele manier voorspeld worden door de klassieke theorie. De Kerr-oplossing, een axiaalsymmetrische, stationaire oplossing die een roterend zwart gat voorstelt, en de Kerr-Newmann oplossing, die een roterend, geladen zwart gat voorstelt, vertonen dezelfde eigenschap.

Algemene relativiteitstheorie is een dynamische theorie. Dit impliceert dat haar exacte oplossingen hele specifieke gevallen zijn. Nu kunnen we ons afvragen of de regulariteit van de ruimtetijd binnen een zwart gat een stabiele eigenschap is, of dat bij kleine verstoringen singulariteiten ontstaan. In het bijzonder zou het ontwikkelen van een specifiek type singulariteiten, namelijk de sterke singulariteiten, zoals die in de Schwarzschild-ruimtetijd, erop wijzen dat de deur naar het causaal ontkoppelde universum dicht zit. Het resultaat van dit proefschrift bepleit dit idee niet. In plaats daarvan verkrijgen wij bepaalde stabiliteitsresultaten onder verstoringen, welke ruimte laten voor singulier gedrag bij de Cauchy-horizon, zonder de uitgang volledig te sluiten. Het voorkomen van niet-deterministische regio's kan daarom niet a priori uitgesloten worden. In de volgende alinea bespreken we hoe we tot deze conclusie komen.

Eerst moeten we beslissen welke ruimtetijd we willen verstoren. Van de hierboven genoemde oplossingen met een zwart gat, is de Kerr-oplossing astrofysisch het meest relevant. Astrofysische objecten worden niet geacht significant geladen te zijn, want het genduceerde elektrische veld zal opgeheven worden door accretie van tegengestelde ladingen. Desalniettemin is de Reissner-Nordström oplossing, dankzij de overeenkomst van de causale structuur, een erg belangrijke kandidaat als speelgoedmodel voor de mathematisch veel ingewikkeldere Kerr oplossing. Daarom zullen we in dit proefschrift ons focussen op de Reissner-Nordström oplossing.

De eenvoudigste verstoringen die men kan beschouwen zijn lineaire verstoringen.

Het zal duidelijk zijn, dat om uiteindelijk niet-lineaire verstoringen te analyseren, het cruciaal is om eerst de lineaire versie goed te begrijpen. Aan de andere kant, aangezien algemene relativiteit een niet-lineaire theorie is, zou de lezer sceptisch kunnen zijn in hoeverre conclusies ontleent aan een lineair model überhaupt kunnen duiden op iets fysisch relevants. Gelukkig, om redenen die te gecompliceerd zijn om hier uit te leggen[2] zijn er goede bewijzen om aan te nemen dat het stabiliteitsmechanisme inderdaad ook al in de lineaire theorie optreedt. Het analyseren van lineaire verstoringen in de volledige Einstein-vergelijkingen is nog steeds een moeilijk probleem door de tensorstructuur van de vergelijkingen. Daarom zullen we, als speelgoedmodel voor lineaire verstoringen, oplossingen van scalaire golfvergelijkingen in een gefixeerde Reissner-Nordström-achtergrond beschouwen. Gegeven deze opzet, kunnen we de begrensdheid van de golfvergelijking in de gehele binnenste regio, tot en met de Cauchy-horizon bewijzen. Dit resultaat is op zichzelf een nieuw wiskundig inzicht. Als we een stap verder gaan en het speelgoedmodel serieus nemen als model voor het fysische lot van de verstoorde Reissner-Nordström-ruimtetijd, zinspeelt dit erop dat verstoringen niet leiden tot een sterke ruimtetijd-singulariteit. Daarentegen laat het alleen de mogelijkheid open voor een zwakke nul-singulariteit langs de Cauchy-horizon, zodat de weg naar de causaal ontkoppelde en niet-deterministische regio niet a priori versperd is voor onze astronaut.

[2]We zullen dit in part I van dit proefschrift behandelen.

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Paris, Anne Franzen
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Curriculum Vitae

The author was born on March 5, 1980 in Neuss, Germany. She attended Nelly-Sachs-Gymnasium, Neuss from 1990 to 1999. In 1999 she started studying mechanical engineering with specialization in space crafts at RWTH Aachen. Driven by curiosity for the underlying theories, she decided to switch to the physics course program after she had obtained her intermediate diploma in mechanical engineering. The author carried out her undergraduate studies of theoretical physics at RWTH Aachen, IIT Roorkee, TIFR Mumbai and Cologne University and obtained her diploma degree from the latter in 2008. In 2009 she started her PhD research at the Institute of theoretical physics at Utrecht University under the supervision of Prof. Gerard 't Hooft and Prof. Mihalis Dafermos. Amongst others she carried out her PhD research at Utrecht University, University of Cambridge and Princeton University and the results are presented in this thesis.

Bibliography

[1] Andersson, L. and Blue, P. (2009). *Hidden symmetries and decay for the wave equation on the Kerr spacetime. arXiv:0908.2265*

[2] Aretakis, S. (2011). Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations I. *Comm. Math. Phys.* **307***, 17-16. arXiv:gr-qc/1110.2007v1*

[3] Aretakis, S. (2011). Stability and Instability of Extreme Reissner-Nordström Black Hole Spacetimes for Linear Scalar Perturbations II. *Annales Henri Poincaré* **8***, 1491-1538. arXiv:gr-qc/1110.2009v1*

[4] Aretakis, S. (2012). Decay of Axisymmetric Solutions of the Wave Equation on Extreme Kerr Backgrounds. *J. Functional Analysis* **263***, 2770-2831. arXiv:gr-qc/1110.2006v1*

[5] Aretakis, S. (2012). *Horizon Instability of Extremal Black Holes. arXiv:gr-qc/1206.6598v2*

[6] Barack, L. and Ori, A. (1999). Late-time decay of scalar perturbationtside rotating black holes. *Phys. Rev. Lett. ,* **82**

[7] Birkhoff, G. D. (1923) Relativity and Modern Physics. (Cambridge, MA: Harvard University Press).

[8] Blue, P. and Soffer, A. (2009) Phase Space Analysis on some Black Hole Manifolds. *Journal of Functional Analysis* **256**, *1-90. arXiv 0511281 [math.AP]*

[9] Blue, P. and Soffer, A. (2003) Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates. *Adv. Differential Equations,* **8**, *no. 5, 595–614.*

[10] Blue, P. and Sterbenz, J. (2006) Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space. *Comm. Math. Phys. ,* **268**, *481–504.*

[11] Brady, P. and Smith, J. D. (1995). Black hole singularities: a numerical approach. *Phys. Rev. Lett.* **75**, *no. 7, 1256-1259.*

[12] Burko, L. M. (1997). Structure of the black holes Cauchy-horizon singularity *Phys. Rev. Lett.* **79**, *no. 25, 4958-4961.*

[13] Burko, L. M. and Ori, A. (1997). *Internal Structure of Black Holes and Spacetime Singularities: Proceedings.* (Annuals of the Israel Physical Society Series).

[14] Chandrasekar, S. and Hartle, J. B. (1982). On crossing the Cauchy Horizon of a Reissner-Nordström Black-Hole. *Proceedings of the Royal Society of London,* **384**,*301-315.*

[15] Choquet-Bruhat, Y. and Geroch, R. (1969). Global aspects of the Cauchy problem in general relativity. *Commun. Math. Phys.,* **14**, *329-335.*

[16] Christodoulou, D. (2000). *The Action Principle and Partial Differential Equations.* (Princeton University Press).

[17] Christodoulou, D. (1999). On the global initial value problem and the issue of singularities. *Class. Quantum Grav.* **16**, *A23-A35.*

[18] Christodoulou, D. (2009). The formation of black holes in general relativity. *EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich. arXiv:0805.3880*

[19] Costa, J. L., Girão, P. M., Natário, J. and Silva, J. (2014). *On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant. Part 1: Well posedness and breakdown criterion. arXiv:gr-qc/1406.7261*

[20] Costa, J. L., Girão, P. M., Natário, J. and Silva, J. (2014). *On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant. Part 2: Structure of the solutions and stability of the Cauchy horizon. arXiv:gr-qc/1406.7253*

[21] Costa, J. L., Girão, P. M., Natário, J. and Silva, J. (2014). *On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant. Part 3: Mass inflation and extendibility of the solutions. arXiv:gr-qc/1406.7245*

[22] Dafermos, M. (2003). Stability and Instability of the Cauchy Horizon for the Spherically Symmetric Einstein-Maxwell-Scalar Field Equations. *The Annales of Mathematics, Second Series* **158, 3**, *875–928.*

[23] Dafermos, M. (2005). The Interior of Charged Black Holes and the Problem of Uniqueness in General Relativity. *Communications on Pure and Applied Mathematics,* **LVIII**, *0445-0504.*

[24] Dafermos, M. (2012). *Black holes without spacelike singularities. arXiv:gr-qc/1201.1797v1*

[25] Dafermos, M. (2014). *The mathematical analysis of black holes in general relativity.* To appear in Proceedings of the ICM, Seoul.

[26] Dafermos, M., Holzegel, G. and Rodnianski, I. (2013) *A scattering theory construction of dynamical black hole spacetimes. arXiv:1306.5534*

[27] Dafermos, M., Holzegel, G. and Rodnianski, I. (2015). The full linear stability of Schwarzschild under gravitational perturbations. *(forthcoming)*

[28] Dafermos, M. and Luk, J. (2015). On the stability of the Kerr Cauchy horizon. *(forthcoming)*

[29] Dafermos, M. and Rodnianski, I. (2005). A proof of Price's law for the collapse of a self-gravitating scalar field *Invent. Math.* **162**, *381-457.*

[30] Dafermos, M. and Rodnianski, I. (2005). *The redshift effect and radiation decay on black hole spacetimes. arXiv:gr-qc/0512119*

[31] Dafermos, M. and Rodnianski, I. (2009). A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. *XVIth International Congress on Mathematical Physics, P. Exner (ed.), World Scientific, London, pp. 421-433. arXiv:0910.4957v1 [math.AP]*

[32] Dafermos, M. and Rodnianski, I. (2010). *Decay for solutions of the wave equation on Kerr exterior spacetimes I-II: The cases* $|a| \ll M$ *or axisymmetry. arXiv:1010.5132v1*

[33] Dafermos, M. and Rodnianski, I. (2010). *The black hole stability problem for linear scalar perturbations.* in Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, T. Damour et al (ed.), World Scientific, Singapore, 2011, pp. 132-189. *arXiv:1010.5137*

[34] Dafermos, M. and Rodnianski, I. (2011). A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. *Invent. Math.* **185**, *467–559. arXiv:gr-qc/0805.4309v1*

[35] Dafermos, M. and Rodnianski, I. (2013). Lectures on black holes and linear waves. *Clay Mathematics Proceedings, Amer. Math. Soc.* **17**, *97-205. arxiv:gr-qc/0811.0354*

[36] Dafermos, M., Rodnianski, I. and Shlapentokh-Rothman, Y. (2014). *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case* $|a| < M$*. arXiv:1402.7034v1*

[37] Dyatlov, S. (2011). Exponential energy decay for Kerr-de Sitter black holes beyond event horizons. *Math. Res. Lett.* **18**, *no. 5, 1023-1035.*

[38] Dray, T. and 't Hooft, G. (1985). The effect of spherical shells of matter on the Schwarzschild black hole. *Comm. Math. Phys.* **99**, *613–625.*

[39] Evans L. C., (1998). *Partial Differential Equations.* (American Mathematical Society).

[40] Finster, F., Kamran, N., Smoller, J. and Yau, S. T. (2006). Decay of solutions of the wave equation in Kerr geometry. *Comm. Math. Phys.,* **264**, *465-503.*

[41] Frankel, T. (2004). *The Geometry of Physics, an Introduction.* (Cambridge University Press).

[42] Franzen, A. (2014). *Boundedness of massless scalar waves on Reissner-Nordström interior backgrounds.To appear in Comm. Math. Phys. arxiv:gr-qc/1407.7093*

[43] Franzen, A. Boundedness of massless scalar waves on Kerr interior backgrounds. *(forthcoming)*

[44] Frolov, V. P. and Novikov, I. D. (1997) *Black Hole Physics: Basic Concepts and New Developments.* (Kluwer Academic Publishers).

[45] Gajic, D. (2015). *Linear waves in the interior of extremal black holes I. arXiv:gr-qc/1509.06568*

[46] Gürsel, Y., Novikov, I. D., Sandberg, V. D.and Starobinsky, A. ,A. (1979). Evolution of scalar perturbations near the Cauchy horizon of a charged black hole. *Phys. Rev. D,* **19**, *413–419.*

[47] Gürsel, Y., Novikov, I. D., Sandberg, V. D.and Starobinsky, A. ,A. (1979). Final state of the evolution of the interior of a charged black hole. *Phys. Rev. D,* **20**, *1260–1270.*

[48] Hawking, S. W. and Ellis, G. F. R. (1975) *The Large Scale Structure of Space-Time.* (Cambridge University Press).

[49] Hiscock, W. A. (1981). Evolution of the interior of a charged black hole *Phys. Lett. A,* **83**, *110-112.*

[50] Jebsen, J. T. (1921). On the general spherically symmetric solutions of Einstein's gravitational equations in vacuo. *Ark. Mat. Ast. Fys, 15. Reprinted Gen. Rel. Grav. 37, 12: 2253-2259 (2005).*

[51] Kay, B. and Wald, R. (1987). Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation 2-sphere. *Class. Quant. Grav.* **4**, *no. 4, 893-898.*

[52] Klainerman, S. (1980.) Global existence for nonlinear wave equations. *Comm. Pure Appl. Math.* **33** *, 43-101.*

[53] Klainerman, S. (1985). Global existence of small amplitude solutions to nonlinear KleinGordon equations in four spacetime dimensions. *Comm. Pure Appl. Math.* **38** *, 631-641.*

[54] Klainerman, S. (1986). The Null Condition and global existence to nonlinear wave equations. *Lect. in Appl. Math.,* **23**, *293–326.*

[55] Klainerman, S. (1987). Remarks on the global Sobolev inequalities in the Minkowski space $\mathcal{R}^{n+1}$. *Comm. Pure Appl. Math.* **40** *, 111-117.*

[56] Klainerman, S. (2010) *Brief history of the vector-field method.*

[57] Kommemi, J. (2014) The global structure of spherically symmetric charged scalar field spacetimes. *PhD thesis*

[58] Lightman, A. P., Press, W. H., Price, R. H. amd Teukolsky, S. A. (1975) *Problem book in Relativity and Gravitation.* (Princeton University Press).

[59] Lucietti, J., Murata, K. and Reall, H. S. and Tanahashi, N. (2013) *On the horizon instability of an extreme Reissner-Nordström black hole. arXiv:gr-qc/1212.2557v2*

[60] Lucietti, J. and Reall, H. S. (2012). Gravitational instability of an extreme Kerr black hole. *Phys. Rev. D,* **86**, *104030. arXiv:gr-qc/1208.1437*

[61] Luk, J. (2010). Improved decay for solutions to the linear wave equation on a Schwarzschild black hole. *Annales Henri Poincaré,* **11**, *805-880.*

[62] Luk, J. (2012). A vector field method approach to improved decay for solutions to the wave equation on a slowly rotating Kerr black hole. *Analysis & PDE,* **5, 3**, *553-625.*

[63] Luk, J. (2013). *On weak null singularities in general relativity. arXiv:1311.4970v1*

[64] Luk, J. and Oh, S.-J. (2015). *Proof of linear instability of Reissner-Nordström Cauchy horizon under scalar perturbations. arXiv:gr-qc/1501.04598*

[65] McNamara, J. M. (1978) Instability of Black Hole Inner Horizons. *Proc. R. Soc. Lon. A.* **358**, *499-517.*

[66] McNamara, J. M. (1978) Behaviour of Scalar Perturbations of a Reissner-Nordström Black Hole Inside the Event Horizon. *Proc. R. Soc. Lon. A.* **362**, *121-134.*

[67] Metcalfe, J., Tataru, D. and Tohaneanu, M. (2012). Price's law on nonstationary space-times. *Adv. Math.* **230**, *995–1028.*

[68] Misner, C. W., Thorne, K. S. and Wheeler, J. A. (1973). *Gravitation* (W. H. Freeman, San Francisco).

[69] Murata, K., Reall, H. S. and Tanahashi, N. (2013). *What happens at the horizon(s) of an extreme black hole? arXiv:gr-qc/1307.6800v1*

[70] O'Neill, B. (1983) *Semi-Riemannian geometry with apllications to relativity.* (Academic Press).

[71] Ori, A. (1991). Inner structure of a charged black hole: an exact mass-inflation solution. *Phys. Rev. Lett.,* **67**, *789-792.*

[72] Ori, A. (1992). Structure of the singularity inside a realistic rotating black hole. *Phys. Rev. Lett.,* **68**, *21172120.*

[73] Ori, A. (1997). Perturbative approach to the inner structure of a rotating black hole. *General Relativity and Gravitation,* **29***, no. 7.*

[74] Ori, A. (1997). Evolution of perturbations inside a charged black hole: Linear scalar field. *Phys. Rev. D,* **55***, no. 8.*

[75] Ori, A. (1998). Transmission and reflection coefficient for a scalar field inside a charged black hole. *Phys. Rev. D,* **57***, 2621–2623.*

[76] Ori, A. (1998). Evolution of scalar-field perturbations inside a Kerr black hole. *Phys. Rev. D,* **58***.*

[77] Ori, A. (1999). Evolution of linear gravitational and electromagnetic perturbations inside a Kerr black hole. *Phys. Rev. D,* **61***.*

[78] Penrose, R. (1968). Structure of Space-Time. *in . . DeWitt and J. A. Wheeler (eds.), Battelle Rencontres, Benjamin, N. Y.*

[79] Penrose, R. (1968). Singularities of spacetime. *in N. R. Leibowitz (eds.), Theoretical principles of astrophysics and general relativity, Univ. of Chicago Press*

[80] Penrose, R. (1979) Singularities and time-asymmetry. *General Relativity: An Einstein Centenary Survey*, S. W. Hawking and W. I. Israel (eds.), (Cambridge University Press), pp. 581-638.

[81] Poisson, E. and Israel, W. (1989). Inner-horizon instability and mass inflation in black holes. *Phys. Rev. Lett.,* **63**, 1663–1666.

[82] Poisson, E. and Israel, W. (1990). Internal structure of black holes. *Phys. Rev. D,* **41**, 1796–1809.

[83] Price, R. H. (1972). Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations. *Phys. Rev. D,* **5**.

[84] Redmount, I. H. (1985). Blue-sheet instability of Schwarzschild wormholes. *Prog. Theor. Phys.* **73***, 1401.*

[85] Ringström H. (2000). *The Cauchy Problem in General Relativity.* (European Mathematical Society).

[86] Sbierski, J. (2013). *Characterisation of the Energy of Gaussian Beams on Lorentzian Manifolds - with Applications to Black Hole Spacetimes. arXiv:gr-qc/1311.2477*

[87] Sbierski, J. (2015). *The C^0-inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian Geometry. arXiv:gr-qc/1507.00601*

[88] Schlue, V. (2010). Decay of linear waves on higher-dimensional Schwarzschild black holes. *Analysis & PDE,* **6, 3**, *515-600. arXiv:gr-qc/1012.5963*

[89] Shlapentokh-Rothman, Y. (2014). Quantitative Mode Stability for the Wave Equation on the Kerr Spacetime. *Annales Henri Poincaré, Online First. arXiv:gr-qc1302.6902*

[90] Simpson, M. and Penrose, R. (1973). Internal instability in a Reissner-Nordström black hole. *Inter. J. Th. Phys.* **17**, *183-197.*

[91] Tataru, D. (2013). Local decay of waves on asymptotically at stationary space-times. *Amer. J. Math.,* **17**, *2, 361–401.*

[92] Tataru, D. and Tohaneanu, M. (2011). Local energy estimate on Kerr black hole backgrounds. *IMRN, no. 2, 248-292.*

[93] Taylor, M. E. (2010). *Partial Differential Equations I, Basic Theory.* (Springer).

[94] Wald, R. M. (1979). Note on the stability of the Schwarzschild metric. *J. Math. Phys.* **20** , *1056-1058; J. Math. Phys.* **21** , *218 (erratum).*

[95] Whiting, B. F. (1989). Mode stability of the Kerr black hole. *J. Math. Phys.* **30**, *no. 6.*