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# Games on Networks: Direct Complements and Indirect Substitutes


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# Games on Networks: Direct Complements and Indirect Substitutes*

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**Abstract**

We study linear quadratic games played on a network where strategies are complements between neighbors and substitutes between agents at distance-two. We provide micro-founded problems where this pattern of interaction is due to a local congestion effect. Equilibrium behavior systematically differs from a model of peer effects only. First, the ranking of equilibrium actions may not follow that of network centralities, with large behavior prevailing at the periphery of the network. Second, network density affects aggregate behavior in a non-monotonic way. Third, segregating agents according to their preferences has a non-monotonic effect on the polarization of behavior. We relate these patterns to evidence from smoking networks, industrial districts and ethnically fragmented societies. We conclude by discussing the implications for the identification of peer effects.

**Keywords**: Games on Networks, Peer Effects, Key-player, Centrality, Congestion.
**JEL Class**: C7, D85, I1, H23.

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# 1 Introduction

Socio-economic decisions are typically taken in relational networks, reflecting interpersonal, institutional and technological ties. Within the network, neighbors jointly consume and produce goods, discuss political opinions, share information. As a consequence, they tend to display correlation in behavior. Positive correlation (peer effects) has commanded substantial attention in economics, partly because of its pervasiveness in social interaction, and because it amplifies individual shocks acting as a "social multiplier" (Glaeser et al., 2003). Peer effects may stem from emulation, shared identity, and conformity, as in risky behavior (such as smoking, drinking, and drug use),[1] or in technological complementarities in production.[2]

In this paper we consider problems where the actions that generate peer effects between neighbors are also responsible for local congestion. More precisely, we assume that aggregate behavior in the neighborhood of a given agent negatively affects her neighbors' incentives to act. Thus, alongside with strategic complementarities with neighbors, agents also experience *strategic substitution* with agents who are at distance-two in the network (i.e. agents with whom they share a common neighbor).

There are various instances of social and economic problems where this particular pattern of interaction has relevance. Congestion at the neighborhood level may be due, for instance, to the accumulation of stocks of negative externalities. As an illustration, consider smoking behavior, characterized by both strong peer effects between neighbors (Christakis and Fowler, 2008) and negative externalities in the form of passive smoke. Local congestion occurs when an agent's incentive to smoke is affected by the passive smoke experienced by her neighbors (friends and relatives). This may due to the agent's concern for her friends' health; alternatively, the health condition experienced by neighbors may affect an agent's awareness of her own health risk, increasing the perceived damage from smoking. A similar mechanism is found in problems when the stocks of pollutant accumulated in a given site leaks into neighboring locations (e.g., trans-boundary pollution).

Market based mechanisms, such as negative pecuniary externalities, can also result in similar interaction patterns. Consider, for instance, a network of firms linked by relations

[1] Evans et al. (1992); Gaviria and Raphael (2001); Kirke (2004); Christakis and Fowler (2007); Clark and Loheac (2007); Christakis and Fowler (2008); Poutvara and Siemers (2008); Fowler and Christakis (2008); Calvó-Armengol et al. (2009); Fletcher (2010).

[2] Krugman (1991); Porter (2001); Wheeler (2001).

of mutual input supply. Neighbor firms' output decisions are strategic complements, since a higher output by one results in a higher demand for the other. In addition, firms are subject to strategic substitution through common neighbors: when a firm's output increases, its neighbors' prices increase, and so does the marginal cost of its neighbours' neighbors, whose incentives to produce decrease as a result.

Local congestion may also result from pure competition effects. In the co-author model by Fafchamps et al. (2006), agents compete for the time and effort of a common co-author, and the incentives to collaborate with a given agent decrease in the number of projects this agent is engaged in. Similar considerations apply to job market networks (Calvó-Armengol and Jackson, 2004), where agents compete for information about job vacancies passed on by common (employed) neighbors. In the short run, agents face stronger incentives to stay on the labor market the more connected they are, and the less connected their neighbors are. Thus, labor market participation has distance-one complementarity and distance-two substitution.

Finally, local congestion is found in military and international alliances, where allies play a deterrence game (characterized by complementarities) against a common enemy, and free ride on each other's deterrence effort - in all respects a public good for the alliance (Olson, 1965; Olson and Zeckhauser, 1966; Hartley and Sandler, 2001). As in the previous examples, sharing a common neighbor generates strategic substitution.

The co-presence of local congestion (with the associated indirect substitutability) and peer effects poses a series of theoretical and policy relevant questions. How, and how systematically, do equilibrium predictions differ from a case of peer effects only? And can we invoke this difference to help account for empirical evidence which is inconsistent with a model of peer effects only, despite the presence of strong complementarities? To address these questions, we study the equilibrium relation between agents' behavior and the topology of the relational network from which both complementarities (*via* direct links) and substitutabilities (*via* distance-two relations) originate.

We adopt the framework with linear best replies used in Ballester et al. (2006) and in Bramoullé et al. (2014), adding the assumption that the strategic interdependence of any two agents is the sum of an element of complementarity - if they are linked in the network - and an element of substitutability - if they share one or more common neighbors. Being primarily interested in how this model predictions differ from a model of peer effects only, we focus on positive and interior equilibria, by assuming, as in Ballester et al. (2006), that substitution effects are of small magnitude. A general analysis of indirect substitution effects of arbitrary magnitude (as studied in Bramoullé et al., 2014) in this framework is certainly of great relevance and it is left for future research.

We first provide sufficient conditions on the topology of the relational network for the existence of a positive equilibrium. We find that, while in a model with peer effect alone an internal equilibrium exists only if the network is not too densely connected (relative to the strength of peer effects), the presence of indirect substitution effects allows for positive equilibria also in densely connected networks. On the contrary, equilibria may fail to exist in networks with average connectedness. We trace this property to the different rate at which complementarity and substitution channels increase as we add links to a given network. We then show that the unique interior equilibrium is proportional to a *weighted version of Bonacich centralities* for the relational network, with weights being themselves centrality measures for the same network.

Next, we find that the presence of small substitution effects has important implications for how both individual and aggregate behavior relate to the network topology. First, while in a model with peer effects alone more connected networks always generate larger aggregate behavior, in the presence of indirect substitution the relation between network density and aggregate behavior is non-monotonic. In particular, this relation is positive in sparse networks and negative in dense networks. The inverted bell pattern is consistent with some evidence of diseconomies of aggregation in firms' districts with sufficiently large size (Shaver and Flyer, 2000; Folta et al., 2006).

Second, as the magnitude of the substitution effects increases, behavior tends to move towards the periphery of the relational network. More precisely, while in a model of peer effects alone central agents display larger behaviors, the ranking of agents' behaviors is affected by the introduction of indirect substitution effects, and a reversal of the order occurs for sufficiently large magnitudes of the substitution effects. This result seems consistent with the fact, recorded by Christakis and Fowler (2008) in their study of smoking in social networks, that smoking behavior progressively turns stronger (weaker) at the periphery (center) of the network.

Third, when preferences are heterogeneous, the presence of indirect substitution affects the relation between network segregation and polarization of behavior. In particular, while peer effects alone would imply that segregating agents according to their propensity to act generates a stronger polarization of behavior, here the effect is non monotonic. Increased segregation sharpens the polarization of behavior when segregation is low, and weakens it when segregation is high. The peak in polarization is reached at moderate levels of segregation, well before the complete homogenization of neighborhoods. We discuss this finding and relate it to some recent theoretical and empirical evidence on the adoption of identity enhancing behavior in segregated social networks (Fryer Jr. and Torelli, 2010; Bisin et al., 2010, 2011).

In the final part of the paper, we derive some implications for the empirical estimation of peer effects. We find that neglecting the presence of congestion and indirect substitution causes a systematic underestimation of the peer effects. We then derive conditions for the identification of peer effects both when these are defined as the sum (Liu and Lee, 2010) and as the average (Bramoullé et al., 2009) of peers' actions. Finally, we discuss the choice of the instruments needed to estimate both the endogenous peer effect (defined as in Manski, 1993) and the endogenous indirect substitution effect.

The paper is organized as follows. Section 2 describes the model and its applications. Section 3 characterizes equilibrium, and relates equilibrium predictions to empirical evidence from networks of smokers and industrial districts. Section 4 presents implications for empirical estimation of peer effects and section 5 concludes.

# 2 The Model

We consider a set $N$ of $n$ agents, organized in a relational network $\mathbf{g}$ defined by a $n \times n$ matrix $\mathbf{G}$ whose generic element $g_{ij} \in \{0, 1\}$ measures the presence of a social tie between agents $i$ and $j$. We limit our analysis to symmetric networks, where $g_{ij} = g_{ji}$ for all $i, j \in N$. Agents $i$ and $j$ are "neighbors" in $\mathbf{g}$ whenever $g_{ij} = 1$, and the degree $d_i$ of agent $i$ in the network $\mathbf{g}$ denotes the number of neighbors of $i$ in $\mathbf{g}$. We use the convention $g_{ii} = 0$, $\forall i$. We define a *walk* between $i$ and $j$ in $\mathbf{g}$, as a series of agents $i_1, i_2, ...., i_m$ such that $i_1 = i$, $i_m = j$ and $g_{i_p i_{p-1}} = 1$ for all $p = 2, 3, ..., m$. Let $g^{[2]}_{ij}$ denote the generic term of the squared matrix $\mathbf{G^2}$, counting the number of walks of length two from node $i$ to node $j$ in $\mathbf{G}$. Agent $i$ derives the following utility from the vector $\bar{x} \in \mathbb{R}^n_+$ of actions chosen in the network:

$$U_i = \alpha_i x_i - \frac{\sigma}{2} x_i^2 + \phi \sum_{j \in N} g_{ij} x_i x_j - \gamma \sum_k g^{[2]}_{ik} x_i x_k \quad (1)$$

The first two terms of the function $U_i$ capture the private benefits from one's own action. These benefits are the sum of a linear increasing part and a quadratic decreasing part, with intensity measured respectively by parameters $\alpha_i$ and $\sigma$. The third term captures the peer effect: the marginal incentive to act increases with the aggregate of the actions taken by neighbours. The intensity of such complementarity is measured by the parameter $\phi > 0$. The fourth term describes the indirect substitution effects: if $\gamma > 0$, the marginal incentives to act are decreasing in the aggregate level of the actions taken by neighbors' neighbors.

In the introductory section we have mentioned various economic problems where the above strategic pattern applies. Here we present two applications in detail.

**Complementarities in production.** Let the network $\mathbf{g}$ represent mutual supply relations between firms in a district. Each node of the network is a monopolistic firm, whose product is both demanded by consumers and used as input by its neighbors.[3] An increase in firm $i$'s output increases the demand for the products of $i$'s neighbors' and, therefore, their output (strategic complementarities). Substitution at distance-two arises since an increase in firm $i$'s output increases the prices of $i$'s neighbors' products, therefore rising the marginal cost of firms at distance two from $i$'s (see McCann and Folta, 2008, 2009). Higher marginal costs decrease the incentives of these firms to produce, thus creating an indirect substitution effect.
Formally, let firm $i$'s production technology be Leontief with constant returns to scale, transforming the set of employed inputs $\{y_j : g_{ij} = 1\}$ into $i$'s production level $x_i$:

$$f_i(\{y_j : g_{ij} = 1\}) = \frac{1}{k} min\{y_j : g_{ij} = 1\} \quad (2)$$

Denoting by $p_j$ the price for commodity $j$ for $j = 1, 2, ..., n$, the marginal cost of each firm $i$ is constant and equal to:

$$c_i = k \sum_j g_{ij} p_j \quad (3)$$

Demand for commodity $i$ is given by the following function:

$$x_i = A_i + D_i - p_i \quad (4)$$

where $A_i$ is the size of $i$'s consumers' market and where $D_i$ is the demand for input $i$

[3]Since we are assuming that links are undirected, when a firm provides an input to another firm, also the latter provides an input to the former.

coming from $i$'s neighbors. From the Leontief technology specification, it follows that:

$$D_i = k \sum_j g_{ij} x_j. \tag{5}$$

Each firm maximizes its profit as a monopolist:

$$\pi_i = (A_i + D_i - x_i - c_i)\, x_i = (A_i - x_i + k \sum_j g_{ij} x_j - k \sum_j g_{ij} p_j) x_i \tag{6}$$

Substituting the expression of each price $p_j$ from the appropriate demand function $p_j = A_j - x_j + \sum_k g_{jk} x_k$, we obtain:

$$\pi_i = (A_i - k \sum_j g_{ij} A_j - x_i + 2k \sum_j g_{ij} x_j - k^2 \sum_j g_{ij} \sum_k g_{jk} x_k) x_i \tag{7}$$

which can be written as (1) once we set $\alpha_i = (A_i - k \sum_j g_{ij} A_j)$, $\sigma = 1$, $\phi = 2k$ and $\gamma = k^2$. Note how firm $i$'s production is increasing in $i$'s neighbors' production (strategic complementarities) and linearly decreasing in the production of firms that share a common input provider with $i$ (substitution at distance-two in the network).

**Local negative externalities.** Consider a set of agents whose actions produce local negative externalities (at the neighborhood level) that accumulate in stocks. In particular, each agent's stock is given by the sum of her neighbors' actions. Assume also that the utility of each agent depends both on her stock and on her neighbors' stocks. This assumption is appropriate in problems where stocks of pollutant leak into neighbors' locations or, alternatively, in problems where agents care about the damage caused by their neighbors' accumulated stocks. As an illustration of such problems, consider the decision of how intensively to smoke within a network of friends and family members. Assume that agents enjoy smoking, but suffer a quadratic health damage from the stock of active and passive smoke they are exposed to. Assume also that each agent cares about her own health and about her neighbors' health. Define the stock

$$Q_i \equiv \left( \sum_{k \in N} g_{ik} x_k + x_i \right)$$

as the sum of all actions taken by agent $i$ and by her neighbors. The utility function takes the following form:

$$U_i = \alpha_i x_i - \gamma_0 \frac{x_i^2}{2} + \theta \sum_{j \in N} g_{ij} x_i x_j - \gamma_1 \delta \frac{Q_i^2}{2} - (1-\gamma_1) \sum_{j \in N} g_{ij} \delta \frac{Q_j^2}{2} \tag{8}$$

The parameter $\delta$ can be interpreted as one's awareness of the health risks coming from smoke. The parameter $\gamma_1$ is the weight each agent assigns to her own health, so that the term $1-\gamma_1$ can be interpreted as a measure of altruism. The case $\gamma_1 = 1$ is the limit case when agents do not care about their neighbors, and the substitution effect at distance-two disappears. Expanding the squared terms, we obtain the following expression:

$$\alpha_i x_i - \frac{\gamma_0 + \delta\gamma_1}{2} x_i^2 + (\theta - \delta) \sum_{j \in N} g_{ij} x_i x_j - (1-\gamma_1)\delta \sum_{k} g_{ik}^{[2]} x_i x_k + h_{-i} \tag{9}$$

that can be rewritten as (1) by setting $\sigma = \gamma_0 + \delta\gamma_1$, $\phi = \theta - \delta$ and $\gamma = (1-\gamma_1)\delta$, and where $h_{-i}$ is a term independent of $x_i$.

# 3 Behavior on the Network

In this section we study equilibrium behavior and its properties. The adjacency matrix $\tilde{\mathbf{G}}$ involved in the first order conditions for an internal equilibrium (see (11) below) describes the pattern of strategic interaction among players. This matrix keeps track of both the direct complementarities that are active on the links of $\mathbf{G}$ and of the indirect substitution effects at distance-two in $\mathbf{G}$. We will address both the existence of an internal equilibrium and its characterization in terms of the topological properties of $\mathbf{G}$. We will then study the role of the parameter $\gamma$, measuring the strengths of indirect substitution effects, for three structural properties of equilibrium: (i) the ranking of agents' behavior; (ii) the relation between network density and aggregate behavior; (iii) the effect of network segregation on the polarization of behavior.

## 3.1 Existence and Characterization of a Positive Interior Equilibrium

We study the game with set of players $N$, strategy set $\mathbb{R}_+$ for each player, and payoff functions given by (1). To simplify the analysis we assume that the parameter $\alpha$ is homogeneous across agents; the role of heterogeneity is studied in section 3.4. The first order conditions characterizing an interior equilibrium $\bar{x}$ are written in the following matrix form, each row referring to a specific agent:

$$\alpha \cdot \bar{1} = \left[\sigma \mathbf{I} - \phi \tilde{\mathbf{G}}\right] \bar{x}, \tag{10}$$

where the adjacency matrix of strategic interaction is defined as follows:

$$\tilde{\mathbf{G}} \equiv \mathbf{G} - \frac{\gamma}{\phi} \mathbf{G^2}. \tag{11}$$

In the symmetric matrix $\mathbf{G^2}$, the generic element $g_{ij}^{[2]}$, counting the walks of lenght two, also counts the number of common neighbors between $i$ and $j$ when $i \neq j$; $g_{ii}^{[2]}$ is simply the degree of $i$ in $\mathbf{G}$. The generic element of $\tilde{\mathbf{G}}$ is given by:

$$\tilde{g}_{ij} = \begin{cases} 0 & \text{if } g_{ij} = 0 \text{ and } g_{ij}^{[2]} = 0 \\ 1 & \text{if } g_{ij} = 1 \text{ and } g_{ij}^{[2]} = 0 \\ -\frac{\gamma}{\phi} g_{ij}^{[2]} & \text{if } g_{ij} = 0 \text{ and } g_{ij}^{[2]} > 0 \\ 1 - \frac{\gamma}{\phi} g_{ij}^{[2]} & \text{if } g_{ij} = 1 \text{ and } g_{ij}^{[2]} > 0 \end{cases}$$

Note that $\tilde{\mathbf{G}}$ always contains negative elements, since $\tilde{g}_{ii} = -\frac{\gamma}{\phi} d_i$ for all $i$. Note finally that the network $\tilde{\mathbf{G}}$ is symmetric being the sum of symmetric matrices, and that $\mathbf{G}$ and $\tilde{\mathbf{G}}$ coincide for $\gamma = 0$. As in Ballester et al. (2006), we impose a bound on the magnitude of substitution effects:

$$\gamma \cdot \max_{ij} \{g_{ij}^{[2]}\} < \sigma. \tag{12}$$

A key notion for the characterization of equilibrium behavior is Bonacich centrality.

**Definition 1 (Bonacich Centrality)** *Let* $\mathbf{A}$ *be an adjacency matrix, and let* $a \in \mathbf{R}_+$ *be a discount parameter. i) The Bonacich centrality matrix is given by* $\mathbf{M}(\mathbf{A}, a) \equiv (I - a\mathbf{A})^{-1}$*; ii) The vector of Bonacich centralities is given by* $\bar{b}(\mathbf{A}, a) \equiv \mathbf{M}(\mathbf{A}, a) \cdot \bar{1}$*; iii) The vector of weighted Bonacich centralities with weights vector* $\bar{w}$ *is given by* $\bar{b}_{\bar{w}}(\mathbf{A}, a) = \mathbf{M}(\mathbf{A}, a) \cdot \bar{w}$*.*

The matrix $\mathbf{M}(\mathbf{A}, a)$ is well defined if and only if $\mu_1(\mathbf{A}) < \frac{1}{a}$, where $\mu_1(\mathbf{A})$ is the largest eigenvalue associated with the matrix $\mathbf{A}$.

An internal equilibrium $\bar{x}$, solving equation (10), can be characterized by direct application of Ballester et al. (2006). Start by denoting by $\theta$ the absolute value of the maximal substitutability in $\tilde{\mathbf{G}}$, by $\delta$ the maximal complementarity in $\tilde{\mathbf{G}}$, and let $\lambda = \delta + \theta$ be the range between the maximal and minimal elements in $\tilde{\mathbf{G}}$. It is straightforward, from the definition of $\tilde{\mathbf{G}}$ that $\theta = \frac{\gamma}{\phi} \max\{d_i | i \in N\}$. Define then the normalized matrix $\mathbf{C}$, where $c_{ij} = \frac{\tilde{g}_{ij} + \theta}{\lambda} \in [0, 1]$. Ballester et al. (2006) have shown that if $\mu_1(\mathbf{C}) < \frac{\sigma}{\phi\lambda}$, then the unique internal equilibrium behavior is proportional to Bonacich centralities in $\mathbf{C}$ (see Appendix B for details):

$$\bar{x} = \frac{\alpha \bar{b}(\mathbf{C}, \frac{\phi\lambda}{\sigma})}{\sigma + \phi\theta b(\mathbf{C}, \frac{\phi\lambda}{\sigma})}, \tag{13}$$

where $b(\mathbf{C}, \frac{\phi\lambda}{\sigma})$ denotes the sum of all agents' centralities in $\mathbf{C}$.

Form the above characterization, however, it is difficult to track the role of the relational network $\mathbf{G}$ in shaping equilibrium behaviour. In particular, the above analysis in terms of $\mathbf{C}$ (that normalizes $\tilde{\mathbf{G}}$) does not allow for explicit comparative statics in terms of the relational network $\mathbf{G}$. In Proposition 1 we identify a sufficient condition for the existence of an internal equilibrium that bears on the eigenvalues of the network $\mathbf{G}$. We then provide in Proposition 2 a new equilibrium characterization that relates behavior to a weighted version of Bonacich centralities for $\mathbf{G}$. In the following analysis we will refer to $\mu_i(\mathbf{G})$ as the $i$-th eigenvalue of the matrix $\mathbf{G}$.

**Proposition 1** *Let either i) $\phi \leq -2\gamma\theta + 2\sqrt{\gamma(\gamma\theta^2 + \sigma)}$ or ii) $\phi > -2\gamma\theta + 2\sqrt{\gamma(\gamma\theta^2 + \sigma)}$ and for each $i \in N$ either $\mu_i(\mathbf{G}) < \frac{\phi - \sqrt{4\gamma\theta\phi + \phi^2 - 4\gamma\sigma}}{2\gamma}$ or $\mu_i(\mathbf{G}) > \frac{\phi + \sqrt{4\gamma\theta\phi + \phi^2 - 4\gamma\sigma}}{2\gamma}$. Then the unique interior Nash equilibrium of the game is given by (13).*

Let us examine the main insights behind proposition 1. Condition *i)* defines a relation between $\phi$ and $\gamma$ under which existence of an internal equilibrium is guaranteed. Larger values of $\gamma$ (indicating stronger substitution effects) allow for larger values of $\phi$ (more intense peer effects). This happens because indirect substitution weakens the positive (and possibly explosive) feedbacks due to peer effects. When peer effects are too strong compared to $\gamma$ - and condition *i)* is violated - an interior equilibrium still exists when all eigenvalues lie either in a low or in a high region of values, defined by the root expressions in *ii)*. Note that the bound we imposed in (12) on the magnitude of strategic substitution implies that $\phi\theta < \sigma$, so that the upper bound $\frac{\phi - \sqrt{4\gamma\theta\phi + \phi^2 - 4\gamma\sigma}}{2\gamma}$, defining the lower region in *ii)*, is strictly positive.

The positiveness of the upper bound provides an intuitive interpretation of point *ii)*: an internal equilibrium exists when the network $\mathbf{G}$ is either sparse (low largest eigenvalue) or dense (high largest eigenvalue). In the case of $\gamma = 0$, existence simply requires that the network is not too dense relative to the strength of complementarities (see the proof in Appendix A). On the contrary, when $\gamma > 0$ equilibrium is possible in densely connected networks because the large number of complementarity channels is paralleled by the large number of substitution channels (the paths of length two). The reason why equilibrium may fail to exist for intermediate density levels has to do with the different paces at which paths of lengths one and two expand as the density of $\mathbf{G}$ increases. To fix ideas, consider the class of regular networks, where the maximal eigenvalue coincides with the average degree. As the latter increases, distance-two substitution channels increase at the square of the degree, dominating distance-one relations when the average degree is large enough.

Having discussed the existence of an internal equilibrium, let us now turn to the problem of characterizing it in terms of agents' centrality in the network $\mathbf{G}$. We start by defining the following two scalars:

$$a_1 = \frac{\phi \pm \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma}, \; a_2 = \frac{\phi \mp \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma}. \tag{14}$$

**Proposition 2** *Let $\phi > 2\sqrt{\gamma\sigma}$ and $\mu_1(\mathbf{G}) < \frac{2\sigma}{\phi+\sqrt{\phi^2-4\gamma\sigma}}$. Then the unique interior Nash equilibrium of the game is given by:*

$$\bar{x} = \frac{\alpha}{\sigma} \bar{b}_{\bar{b}(\mathbf{G},a_2)}(\mathbf{G}, a_1) \tag{15}$$

Proposition 2 shows that, in order to express equilibrium behavior as a function of centralities in $\mathbf{G}$, we need to look at the family of weighted centrality measures. In particular, weights are themselves Bonacich centrality measures for the network $\mathbf{G}$, computed at a discount factor that crucially depends on $\gamma$. The condition on the maximal Eigenvalue $\mu_1(\mathbf{G})$, under which this characterization is valid, rules out very dense networks because Bonacich centralities of $\mathbf{G}$ enter the characterization of equilibrium, calling for a positive inverse for both the matrices $[\mathbf{I} - a_1\mathbf{A}]$ and $[\mathbf{I} - a_2\mathbf{A}]$. We refer to the proof in Appendix A for details.[4]

## 3.2 Centrality and Marginalization of behaviour

Let us now turn to the effect of small positive values of $\gamma$ on various features of equilibrium. In this section we examine the effect of $\gamma$ on the ranking of equilibrium actions. If $\gamma = 0$,

[4]See also Appendix C for a comparison of the thresholds in propositions 1 and 2.

direct application of Ballester et al. (2006) implies that agents' behavior is proportional to their Bonacich centrality in $\mathbf{G}$. We will show that positive values of $\gamma$ can reverse this relation.

Empirically, the failure of behaviour to reflect agents' centralities in the relational network has been recorded by Christakis and Fowler (2008) in their analysis of smoking behaviour in the Framingham Heart Study. Despite the robust evidence of a positive correlation of smoking between family members and close friends, Christakis and Fowler (2008) observe an inverse relationship between eigenvalue centrality and smoking, contrary to what would result from the peer effects alone (see Figure 1).

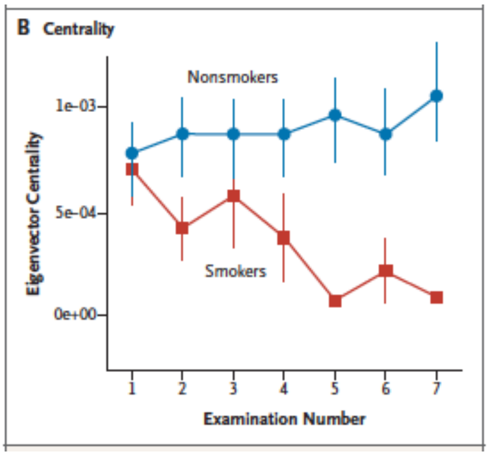


Figure 1: Centralities of Smokers and Nonsmokers in Christakis and Fowler (2008).

In what follows we argue that such reversal in the ranking of actions can be obtained in equilibrium from to the presence of indirect substitution effects. Let us first rewrite the system of FOCs (10) as follows (see Appendix D for derivations):

$$\bar{x} = \frac{\alpha}{\sigma}\bar{b}\left(\mathbf{G}, \frac{\phi}{\sigma}\right) - \frac{\gamma}{\sigma}\mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma}) \cdot \mathbf{G^2}\bar{x} \tag{16}$$

The first term on the right hand side of (16) is the equilibrium vector of actions when $\gamma = 0$, and it is proportional to agents' centralities in $\mathbf{G}$. This serves as a benchmark to analyze what happens when $\gamma > 0$. The second term measures the correction of equilibrium behavior due to $\gamma$. This correction is proportional to the Bonacich matrix $\mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma})$ multiplied by the vector $\bar{z} \equiv \mathbf{G^2}\bar{x}$, whose generic element $z_i = \sum_j g_{ij}^{[2]} x_j$ measures the aggregate equilibrium actions at distance-two from $i$. This correction has important implications for the ranking of equilibrium actions. First, although $\gamma > 0$ reduces the actions of all agents, the reduction is not uniform across agents. It is in fact stronger for those agents who are connected through numerous and short paths with agents in whose neighborhood behavior is large. This is the case for very central agents in **G**, typically linked to other central agents in whose neighborhoods actions are large.

To obtain an explicit measure of the equilibrium correction due to $\gamma$, we look at small increases from $\gamma = 0$.

**Proposition 3** *The marginal effect of the introduction of $\gamma$ on equilibrium behavior is given by:*

$$\frac{\partial \bar{x}^*}{\partial \gamma}|_{\gamma=0} = -\frac{\partial}{\partial \phi}\bar{b}_{\bar{d}}(\mathbf{G}, \frac{\phi}{\sigma}), \tag{17}$$

Proposition 3 considers equilibrium actions when $\gamma = 0$, and provides an explicit measure of the reduction due to small indirect substitution effects. Equation (17) provides two important insights.

First, central agents tend to display larger decrease on equilibrium behaviour than non central ones. In particular, the reduction due to $\gamma$ is larger for those agents who would suffer (as a result of a decrease in $\phi$) larger decreases in the value of their discounted paths towards agents with a large degree in **G**. Since this decrease is larger for short paths, sharp reductions in equilibrium behavior will occur for agents who have short paths towards agents with a large degree. These agents will be typically central in **G**, and may well end up choosing smaller actions compared to less central agents in **G**, for whom the reduction is milder.

Second, the introduction of small substitution effects may change the ranking of actions even when a marginal decrease of $\phi$ would not. Indeed, we note that the right hand side of (17) generally fails to be proportional to the reduction in equilibrium actions that would follow a marginal reduction of the complementarity parameter $\phi$, due to the weighting vector $\bar{d}$.

A general analysis (for arbitrary values of $\gamma$) of the topological conditions under which an inversion of the ordering would occur seems complex due to the strong non linearities of centrality measures. However, the inversion can be observed by means of simulations in the context of simple network architectures where central and peripheral players in **G** are clearly identified (see figure 2 and Table 1). The comparison between equilibrium behavior for strictly positive values of $\gamma$ and with $\gamma = 0$ show how equilibrium actions are reduced for all players. Moreover, although the impact of $\gamma$ is non uniform across agents, an inversion in the ranking occurs only in the connected star, where the central agent is connected to agents with sufficiently large degree.

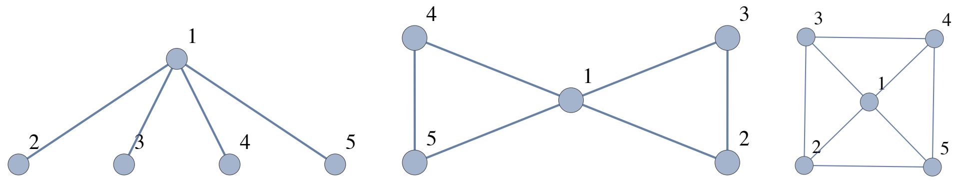


Figure 2: Star, Papillon and Connected Star Networks

Table 1: Effect of $\gamma$ on equilibrium actions

| **Network** | **Players** | | $\gamma = 0$ | $\gamma = 0.8$ | $\lvert x_0 - x_{0.8} \rvert$ | $\lvert x_0 - x_{0.8} \rvert / x_0$ |
|---|---|---|---|---|---|---|
| **Star** | 1 | Center | 0.0439 | 0.0403 | 0.0036 | 0.0820 |
| | 2-5 | Periphery | 0.0401 | 0.0371 | 0.0030 | 0.0748 |
| **Papillon** | 1 | Center | 0.0440 | 0.0367 | 0.0073 | 0.1659 |
| | 2-5 | Periphery | 0.0415 | 0.0363 | 0.0052 | 0.1253 |
| **Connected Star** | 1 | Center | 0.0442 | 0.0334 | 0.0108 | 0.2443 |
| | 2-5 | Periphery | 0.0429 | 0.0341 | 0.0088 | 0.2000 |

Parametrization: $\gamma_0 = 15$, $\alpha = 0.6$, $\delta = 0.5$, $\theta = 1$ in equation (8)

We also observe the inversion in the inter-linked cliques architecture used in Ballester et al. (2006) reported in Figure 3. There are basically three types of agents in this network, types 1, 2 and 3, from the names of the corresponding representative nodes. Table 2 records the equilibrium actions in this network. The most central agent for low

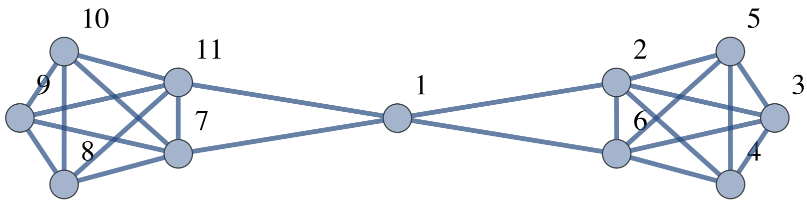


Figure 3: Inter-linked Cliques (Ballester et al. (2006)).

levels of $\gamma$ is type 2, and switches to type 3 for higher levels of $\gamma$.[5]

[5]Appendix E also shows how similar results hold for the analysis of the key player.

Table 2: Effect of $\gamma$ on equilibrium actions.

| Network | Players | | $\gamma = 0$ | $\gamma = 0.7$ | $\lvert x_0 - x_{0.8}\rvert$ | $\lvert x_0 - x_{0.8}\rvert / x_0$ |
|---|---|---|---|---|---|---|
| **Inter-linked Cliques** | 1 | Middle | 1.1531 | 0.0294 | 1.1236 | 0.9745 |
| | 2 | Center | 1.1868 | 0.0299 | 1.1561 | 0.9748 |
| | 3 | Periphery | 1.1508 | 0.0309 | 1.1191 | 0.9731 |

Parametrization: $\gamma_0 = 15$, $\alpha = 0.6$, $\delta = 0.5$, $\theta = 1$ in equation (8)

## 3.3 Network Density and behavior

In a model with peer effects only, the creation of additional connections always results in larger equilibrium actions, due to the more intense positive equilibrium feedbacks. When $\gamma > 0$, things become more complex, as the new connections channel both strategic complementarities and, through the creation of new paths of length two, strategic substitutabilities. Indeed, the simple centralized architectures of Figure 2 suggest that adding links to the network $\mathbf{G}$ may decrease aggregate behavior (see Table 1). To allow for a more systematic analysis, we focus on the class of regular networks, where the common degree $d$ acts as a measure of network density. Let $d^* \equiv \frac{\phi}{2\gamma}$. The following proposition characterizes the relation between network density and aggregate equilibrium actions.

**Proposition 4** *Let* $\mathbf{g}$ *be a regular network and* $\sigma - \phi d + \gamma d^2 > 0$. *The symmetric equilibrium is given by*

$$x^* = \frac{\alpha}{\sigma - \phi d + \gamma d^2}. \tag{18}$$

*This is a non-monotonic function of the degree, increasing for low degrees* ($d < d^*$) *and decreasing for high degrees* ($d > d^*$)*. The threshold* $d^*$ *is decreasing with* $\gamma$*, and* $d^* \to \infty$ *when* $\gamma \to 0$.

The non-monotonic relation between density and behavior has an intuitive explanation. As the common degree $d$ increases, direct connections grow with $d$, while distance-two connections grow with $d^2$. The effect of indirect substitution takes over for $d$ large enough, causing a decrease in overall behavior.[6]

[6] The logic behind the results of this section seems to extend to the framework of "network games", studied in Galeotti et al. (2010), where agents have an incomplete knowledge of the network, and share a common prior in terms of the degree distribution (we are thankful to Andrea Galeotti for pointing this out to us). An increase in the density of the network would take the form of a FOSD shift in the degree distribution. If neighbors' degrees come from independent draws from the same distribution, the FOSD shift would replicates the same effects on behavior described in this section where we study increases in

The possibility that network density may affect behavior according to a non linear trend has been recently recognized in the industrial organization literature. Industrial districts are motivated by economies of agglomeration, that is, increasing returns that generate complementarities between firms. As Folta et al. (2006) and McCann and Folta (2008) observe, however, diseconomies of agglomeration are likely to play a key role after some critical district size, mainly due to increased marginal costs of production. Such increased cost may arise because of increased competition for - and increased prices of - valuable inputs (Venables, 1996; Ottaviano and Puga, 1998). Pouder and StJohn (1996), Folta et al. (2006) and McCann and Folta (2008) theorize a non-monotonic relation between firms' performance and district size, with an inverted U-shape consistent with the one in figure 4.

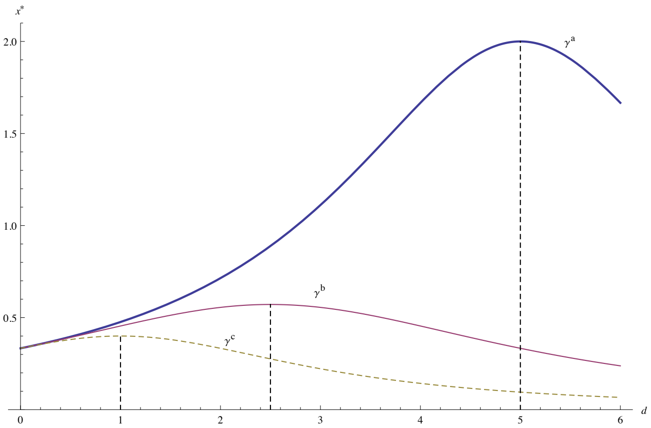


Figure 4: Non monotonic relation between density and behaviour ($\gamma^a = 0.1$, $\gamma^b = 0.2$, $\gamma^c = 0.5$, $\alpha = 1$, $\sigma = 3$, $\phi = 1$)

Our network model, and in particular its specific application to technological complementarities presented in (7), allows us to identify the source of the non-monotonic pattern presented in figure 4. In our stylized model, diseconomies come from increased input prices in the cluster, but similar arguments would apply if congestion effects originated from longer queuing times, shortages of inputs and production delays etc. . .

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the degree of a regular network. In details, calling  the prior on neighbors' degree, the equilibrium is given by

$$x_i^* = \frac{\alpha}{\sigma - \phi d_i + \gamma \bar{d}^2}. \tag{19}$$

Outside the class of regular networks it is very difficult to relate behavior to a measure of network density. Here, we address a simpler problem, that is identifying what changes in the topology of the network would unambiguously increase, or decrease, aggregate behavior.

**Proposition 5** *Consider the network* $\mathbf{G}'$ *obtained from* $\mathbf{G}$ *by fully connecting an independent set of* $Z$ *nodes in* $\mathbf{G}$*. If* $\phi \leq (z-2)\gamma$*, then* $\bar{x}(\tilde{\mathbf{G}}') < \bar{x}(\tilde{\mathbf{G}})$*.*

We see that a sufficient condition to reduce aggregate behavior is the presence of sets of agents who are not connected in $\mathbf{G}$; the number of such agents is inversely related to the intensity of indirect substitution $\gamma$. Aggregate behavior is thus reduced by creating very dense relations among these sparse agents, so that new direct ties come with enough new indirect connections.

## 3.4 Network Segregation and Polarization of Behaviour

In this section we allow agents to be heterogeneous in their private benefit $\alpha$ from their own action. We are primarily interested in how polarization of behavior between agents of different types depends on the degree of segregation of agents in $\mathbf{G}$. By *polarization* of behavior we mean the amplification, due to social effects, of differences in behavior associated with differences in preferences.

In a model of peer effects alone, network segregation increases the polarization of behavior, by sharpening the differential effect of equilibrium feedbacks. This effect in turn widens the gap between agents with high and low willingness to act. The effect of societal segregation on types of social behaviors that exacerbate cultural differences has been studied, both theoretically and empirically, by a vast literature (Bisin et al., 2004, 2010, 2011; Fryer Jr. and Torelli, 2010). Despite the robust evidence of the presence of peer effects, however, these papers show that the relation between segregation and polarization is complex and possibly non monotonic, and may depend on various aspects of the socialization technology.

The effective design of policies that target segregation as a way to affect behavior (e.g., favoring contact across ethnic groups in schools or clustering agents with similar habits) would benefit from a thorough understanding of these issues. In the context of our model, we find that when substitution effects are strong enough, polarization of behavior decreases with segregation of social contacts, and that a non-monotonic pattern may arise for large degrees.

To keep the problem tractable, and to focus on segregation only, we consider regular networks. We assume that agents come in two types: high marginal benefits $\alpha_h$ and low marginal benefit $\alpha_l$. Populations of the two types are assumed of equal sizes. The level of segregation in a given regular network with degree $d$ is captured by the parameter $q$, common to all agents and measuring the fraction of neighbors of the same type that each agent has.[7] Figure 5 provides the examples of three networks composed by the same number of agents and with the same degree, but with different degrees of segregation $q$.

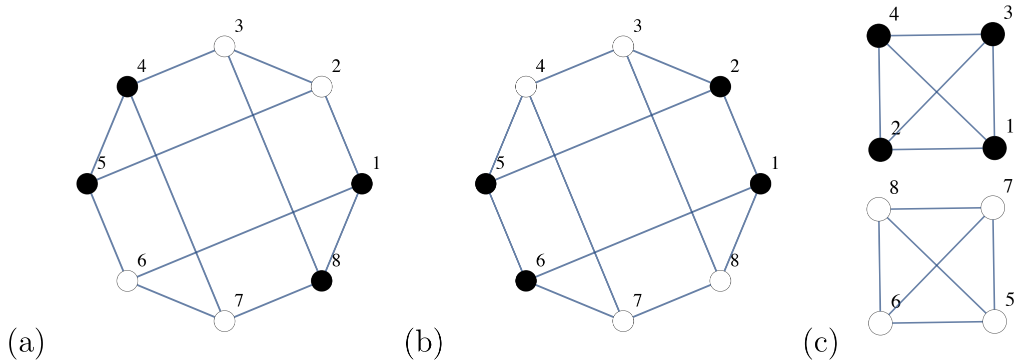


Figure 5: Three networks with increasing degrees of segregation ($q = 1/3,\ 2/3,\ 1$).

The type-symmetric equilibrium levels for high and low types are (see proof of proposition 6 for derivations):

$$\begin{cases} x_h &= \frac{1}{2}\left[\frac{\alpha_h+\alpha_l}{\sigma-\phi d+\gamma d^2} + \frac{\alpha_h-\alpha_l}{\sigma+\phi(1-2q)d+\gamma(1-2q)^2d^2}\right] \\ x_l &= \frac{1}{2}\left[\frac{\alpha_h+\alpha_l}{\sigma-\phi d+\gamma d^2} + \frac{\alpha_l-\alpha_h}{\sigma+\phi(1-2q)d+\gamma(1-2q)^2d^2}\right] \end{cases} \tag{20}$$

In this two-type case, a sufficient condition for a positive solution in regular networks is $\gamma < 1/d$. The equilibrium behavior of each type is the sum of two terms. The first common term coincides with the equilibrium behavior if $\alpha = \frac{\alpha_h+\alpha_l}{2}$ for all agents. The second term measures how types' actions are spread around this mean, and captures the extent of polarization. Symmetry of this spread implies that a change in segregation $q$ does not affect the average behavior. Note first that if $\gamma = 0$, polarization always increases with $q$, as an effect of pure complementarities. When $\gamma > 0$, we obtain the following result, where $d^* \equiv \frac{\phi}{2\gamma}$ and $\bar{q} \equiv \frac{\phi+2\gamma d}{4\gamma d} > \frac{1}{2}$:

[7] This measure is common in the literature on homophily in social networks, where the share of similar "friends" is compared to the relative weight of that type in the whole population (see Currarini et al., 2009). This is also a specific case of a more general measure of segregation used in Vorsatz and Ballester (2010), when agents are assumed not to move along the links of the network.

**Proposition 6** *When $d < d^*$, the spread between $x_h$ and $x_l$ is monotonically increasing in $q$. When $d > d^*$, the spread is non monotonic in $q$, reaching its maximum at $\bar{q} \in (1/2, 1]$. Moreover, the maximal spread is independent of the degree.*

The first part of Proposition 6 states that if the network density is low ($d < d^*$) segregation has a monotonic effect on behavioural polarization; this non monotonicity arises because of the low number of distance-two relations. On the contrary, in dense networks an increase in segregation is first followed by an increase in polarization and, as segregation increases further, by a progressive re-homogenization of behaviour. Figure 6 provides a numerical example of this non-monotonic relation.

The crucial role of indirect substitutes becomes clear once we consider the forces at

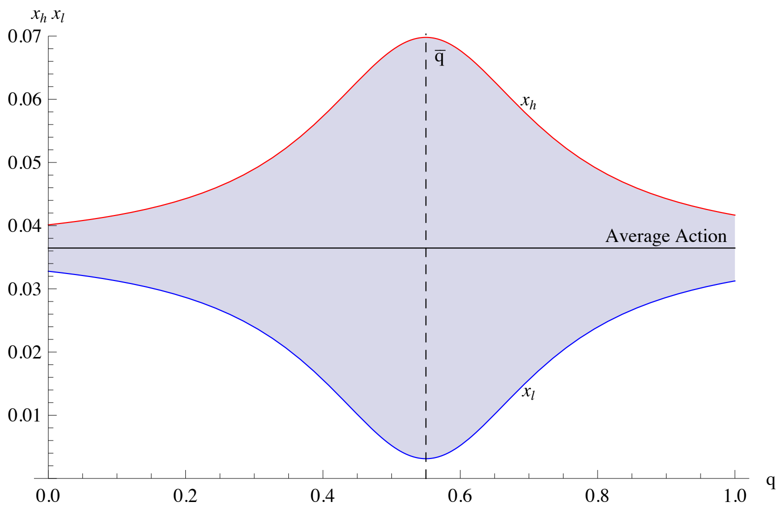


Figure 6: Segregation and Polarization ($\sigma = 8, \phi = 1$, $\gamma = 0.5$, $\alpha^H = 1.5$, $\alpha^L = 1$, $d = 10$).

work as $q$ increases. At low levels of $q$, high types are mainly surrounded by low types and *viceversa*. Thus, given that high types always choose higher actions than low types, the aggregate action is larger in the neighborhood of high types than of low types. To appreciate the effect of a change in $q$, let us now focus on high types only. An increase in $q$ replaces low types with high types in the neighborhood of high types. This naturally tends to drive high types' actions up *via* the peer effects; moreover, at low $q$, an increase in $q$ has the effect of replacing distance-two neighbors of high type with distance-two neighbors of low type, driving the high action further up through a weaker strategic substitution effect. This explains the initial steep increase of the curve. As $q$ increases further, high types tend to have more and more high type neighbors; for large enough $q$, replacing low type neighbors with high type neighbors increases the indirect substitution effects for low types (this occurring the sooner, the larger $\gamma$). Eventually, when this substitution outweighs the peer effect, high types' actions start decreasing. Key to the above argument is the fact that, while peer effects apply to flows of individual actions, distance-two substitutability comes from the stocks of actions in each agent's neighborhood. While high types always play a larger action than low types, the stock of actions in high types' neighborhoods is smaller (larger) than the stock in low types' neighborhoods, for small (large) values of the parameter $q$ . Hence the non-monotonic pattern.

Our findings can be read as the deterrence effect of congestion in clustered and homogeneous groups with strong propensity to take actions. There exists a number of empirical studies that record a similar non monotonic pattern for behavior that are relevant for group identity and that are likely to generate local negative externalities and congestion. We think, for instance, of "oppositional identities" in ethnically fragmented societies, documented in Bisin et al. (2010, 2011), and Fryer Jr. and Torelli (2010). Behaviors relevant for group identity are particularly intense in more fragmented societies, where the ethnic group composition is quite balanced (an intermediate $q$ in our model). Our mechanism provides a key of interpretation based on the congestion that results from the negative external effects coming from such behaviors, and on how these behaviors affect agents' incentives as the ethnic composition varies.

For the sake of illustration, let us consider the act of skipping school assuming that and agents from different ethnic groups face different costs from doing it (different values of the parameter $\alpha$). Assume also that skipping school generates peer effects and leads to bad school performance. In addition, agents' performance benefits from the contact with agents that perform well. In very segregated societies, agents from the disadvantaged group face neighborhoods with diffused poor school performance. This may weaken their incentives to skip school by increasing their awareness of the negative effects of school skipping (here measured by the parameter $\gamma$). This pattern would be consistent with the observed reduction of polarization at high levels of segregation.

# 4 Implications for Empirical Work on Peer Effects

In this final section we discuss how the introduction of indirect substitution modifies the procedure for the estimation of peer effects in social networks (see Bramoullé et al., 2009;

Lee et al., 2010; Liu and Lee, 2010; Liu et al., 2012).[8] We start by considering the FOC derived by (1), allowing for a possibly heterogeneity in $\alpha_i$:[9]

$$\alpha_i - \sigma x_i + \phi \sum_{j \in N} g_{ij} x_j - \gamma \sum_{k \in N} g_{ik}^{[2]} x_k = 0, \tag{21}$$

where $\alpha_i$ accounts for a set of observable personal characteristics ($z_i$),[10] average friends' characteristics ($\frac{1}{d_i} \sum g_{ij} z_i$) and a random error term $\epsilon$. The FOC to be estimated can now be written as:

$$x_i = \frac{\xi}{\sigma} z_i + \frac{\kappa}{\sigma} \frac{1}{d_i} \sum_{j \in N} g_{ij} z_i + \frac{\phi}{\sigma} \sum_{j \in N} g_{ij} x_j - \frac{\gamma}{\sigma} \sum_{k \in N} g_{ik}^{[2]} x_k + \epsilon. \tag{22}$$

Note that the parameters are identified up to a normalization, since every coefficient is divided by a factor $\sigma$ measuring the concavity of agent's utility function. Which parameters of the utility function we are able to identify will depend on the model we want to estimate.[11]

Define $\mathbf{G}^*$ as the row normalized matrix $\mathbf{G}$, with $g_{ij}^* = \frac{1}{d_i} g_{ij}$. As previously, $\mathbf{G^2}$ is the matrix counting the number of distance-two walks between agents, with diagonal terms $d_i$. Calling $\beta_1 = \frac{\phi}{\sigma}$, $\beta_2 = -\frac{\gamma}{\sigma}$, $\rho = \frac{\xi}{\sigma}$, $\zeta = \frac{\kappa}{\sigma}$ we obtain the following matrix form specification.

$$x = \beta_1 \mathbf{G} x + \beta_2 \mathbf{G^2} x + \rho z + \zeta \mathbf{G}^* z + \epsilon \tag{23}$$

The actions' vector $x$ is determined by the sum of the actions chosen by peers ($\mathbf{G}x$), the actions chosen by distance-two neighbors ($\mathbf{G^2}x$), own demographics ($z$), own neighbors' demographics ($\mathbf{G}^*z$) and a random error term $\epsilon$.

In section 4.1 we characterize the bias that arises in the estimation of peer effects; in section 4.2 we derive new conditions for identification of the model with indirect substitutes and the optimal set of instruments we need to solve the reflection problem (see Manski, 1993).

[8] To keep the model simple and to make it comparable to the previous literature we do not include in the specification neither a constant term nor a network fixed effect

[9] Calvó-Armengol et al. (2009) provide conditions for the existence of an unique and interior equilibrium for this case.

[10] Without loss of generality, we include in the model just one demographic characteristic.

[11] In the framework of smoking behavior presented in section 2, if $\delta = 1$, we have $\sigma = \gamma_0 + \gamma_1$, $\phi = \theta - 1$, $\gamma = 1 - \gamma_1$. Under the appropriate identification conditions, to be discussed in section 4.2, when $\gamma_0 = 0$, the model identifies both $\theta$ and $\gamma_1$, capturing the pure net peer effects and the effect of externalities.

## 4.1 Bias in the Estimation of Peer Effects

Suppose that $\gamma > 0$, but the variable $\mathbf{G^2}x$ is not included in (23) (as in Bramoullé et al., 2009; Lee et al., 2010; Liu and Lee, 2010):

$$x = \beta_1 \mathbf{G}x + \rho z + \zeta \mathbf{G}^* z + \epsilon \tag{24}$$

Using the usual omitted variable bias formula, the coefficient of the peer effect $\hat{\beta_1}$ can be written as the sum of the real effect $\beta_1$ and a bias derived from the correlation between the omitted variable $\mathbf{G^2}x$ and the included explanatory one $\mathbf{G}x$:

$$\frac{Cov(x, \mathbf{G}x)}{Var(\mathbf{G}x)} = \beta_1 + \beta_2 \pi_{\mathbf{G^2}x, \mathbf{G}x} \tag{25}$$

where $\pi_{\mathbf{G^2}x, \mathbf{G}x}$ is the coefficient from a regression of $\mathbf{G^2}x$ on $\mathbf{G}x$. The theoretical model suggests the patterns of substitutability and complementarity between the actions of the agents in the network. In particular, we expect $\beta_2$ to be negative due to second order substitutabilities. However, the actions chosen by friends and by second order neighbors are, between them, strategic complements and $\pi_{\mathbf{G^2}x, \mathbf{G}x}$ is positive. Thus, the omitted variable bias $\beta_2 \pi_{\mathbf{G^2}x, \mathbf{G}x}$ is always negative and the peer effects in (24) systematically underestimated. Moreover, the larger the complementarities between first and second order neighbors' choices, the larger the bias. This result could help explaining why in some cases peer effects are positive, but not statistically different from zero (see Fletcher, 2010).

## 4.2 Identification

As shown by Manski (1993), identification in a model with peer effects is difficult due to the reflection problem. However, when networks are not complete so that people do not interact in groups and data on the network interaction is available, identification can be achieved under some conditions. In existing literature adjacency matrices associated with neighbors' actions or neighbors' demographics have been modeled either as row normalized or not. We consider two relevant cases. Case 1: both matrices are row normalized and equal to $\mathbf{G}^*$ (Bramoullé et al., 2009; Lee et al., 2010), or both matrices are not row normalized and equal to $\mathbf{G}$ (Liu and Lee, 2010); Case 2: one is row normalized and one is not. In what follows we consider the identification conditions for these two cases, when

$\gamma = 0$ and $\gamma > 0$.

**Case 1.**

**Proposition 7 (Bramoullé et al. (2009))** *Let $\gamma = 0$. If $\zeta + \beta_1 \rho \neq 0$ and $\mathbf{I}, \mathbf{G}$ and $\mathbf{G^2}$ are linearly independent then $\beta_1$ in (24) is identified.*

This sufficient condition states that, when demographics have some explanatory value ($\zeta + \beta_1 \rho \neq 0$), the peer effect cannot be identified in fully connected networks.

**Proposition 8** *Let $\gamma > 0$. If $\mathbf{I}, \mathbf{G}, \mathbf{G^2}, \mathbf{G^3}$ are linearly independent, the net peer effect $\beta_1$ and $\beta_2$ in (23) are identified if $\beta_1 \zeta + \frac{\zeta^2}{\rho} + \beta_2 \rho \neq 0$ and $\rho \neq 0$.*[12]

Notice that, with respect to the previous case with $\gamma = 0$, when $\beta_2 \neq 0$ more restrictive conditions are required in order to identify parameters $\beta_1$ and $\beta_2$.

**Case 2.**

Adapting to our framework the results from Bramoullé et al. (2009), the following holds:

**Proposition 9** *Let $\gamma = 0$. If $\mathbf{I}, \mathbf{G}, \mathbf{G}^*, \mathbf{GG}^*$ are linearly independent and if $\rho \neq 0$ or $\zeta \neq 0$, the net peer effect $\beta_1$ in (24) is identified .*

It is now important to identify which classes of networks are ruled out by the above sufficient conditions for identification. Note first that $\mathbf{I}$ is linearly dependent with $\mathbf{G}$ and also with $\mathbf{G}^*$ only in the empty network. Note then that $\mathbf{G}$ and $\mathbf{G}^*$ are linearly dependent only in regular networks, where $d$ is the common degree and $\mathbf{G} = \frac{1}{d}\mathbf{G}^*$. Finally, let us consider when $\mathbf{GG}^*$ is linearly independent from both $\mathbf{G}$ and $\mathbf{G}^*$. Since $\mathbf{GG}^*$ keeps track of weighted distance-two paths, while $\mathbf{G}$ and $\mathbf{G}^*$ just consider distance-one neighbors, a necessary condition for linear dependence is that all triangles in $\mathbf{G}$ close, leading to the complete network. However, a complete network is a special case of regular network. It follows that, in order to identify the peer effect in model (24), all regular networks must be excluded.

**Proposition 10** *Let $\gamma > 0$. If $\mathbf{I}, \mathbf{G}, \mathbf{G^2}, \mathbf{G}^*, \mathbf{GG}^*, \mathbf{G^2 G}^*$ are linearly independent and if $\rho \neq 0$ or $\zeta \neq 0$, the peer effect $\beta_1$ and the coefficient $\beta_2$ in (23) are identified.*

[12]Note that this condition can be written as $\zeta + \beta_1 \rho + \frac{\beta_2 \rho^2}{\zeta}$, which is a modification of the one found by Bramoullé et al. (2009)

The introduction of second order interactions restricts the set of networks that enable the identification of both $\beta_1$ and $\beta_2$. In fact, together with regular networks, other classes of networks must be excluded. These include the star, where $\mathbf{G^2G}^* = (n-1)\mathbf{G}^*$, and the papillon network in Figure 2, in which $\mathbf{GG}^* = \frac{1}{2}\mathbf{G^2}$.

The case covered in proposition 10 needs also a new set of instruments. Let us write the complete model to be estimated as follows:

$$x_r = \beta_1 \mathbf{G_r} x_r + \beta_2 \mathbf{G_r^2} x_r + \mathbf{Z_r}^* \delta + \epsilon_r \tag{26}$$

where $r$ is the number of networks in the dataset, $nr$ the number of individuals in the network, $x_r = (x_{1,r}, ......, x_{nr,r})'$, $z_r = (z_{1,r}, ......, z_{nr,r})'$, $\epsilon_r$=($\epsilon_{1,r}$ ,......, $\epsilon_{nr,r}$), $\mathbf{Z}_r^* = (z_r, \mathbf{G}^* z_r)$ and $\delta = (\zeta, \rho)'$, $\beta_1$ captures the peer effect, $\beta_2$ the effect of indirect substitution.

It is easy to see that the variables $\mathbf{G_r} x_r$ and $\mathbf{G_r^2} x_r$ are both endogenous because they are the result of the same maximization process. Following Liu et al. (2012), we derive the explicit expression for the two endogenous variables and the corresponding sets of instruments (see Appendix F for the choice of the instruments and the discussion about their relevance and exogeneity). We can instrument $\mathbf{G}x_r$ with $\mathbf{G}z_r, \mathbf{G^2}z_r, \mathbf{GG}^*z_r$ and their higher terms (see also Liu et al., 2012), and $\mathbf{G^2}x_r$ with $\mathbf{G^2}z_r, \mathbf{G}^3 z_r, \mathbf{G^2G}^*z_r$ and their higher terms.

# 5 Conclusions

When social contact generates both peer effects and local congestion, the network of strategic interaction may differ from the relational one. We have studied a model where congestion at the neighborhood level creates strategic interdependence of the substitute type at distance-two. We have shown that equilibrium is affected by distance-two substitution in a systematic way. First, individual behavior tends to move towards the periphery of the network. Second, aggregate behavior tends to decrease with network density in very dense networks. Third, social segregation affects polarization of behavior according to a non monotonic pattern. Given the widespread relevance of local congestion in social networks, our results provide valuable insights on the relation between social position and behavior, that should be taken into account in designing network based policies.

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# A Proofs

**Proof of Proposition 1.**

Let us start by proving that $\mu_i(\mathbf{G}) < \frac{\phi - \sqrt{4\gamma\theta\phi + \phi^2 - 4\gamma\sigma}}{2\gamma}$ or $\mu_i(\mathbf{G}) > \frac{\phi + \sqrt{4\gamma\theta\phi + \phi^2 - 4\gamma\sigma}}{2\gamma}$

Recall first that given a square matrix $\mathbf{A}$ and an associated polynomial $q(\mathbf{A})$, the eigenvectors $\mu(q(\mathbf{A})) = q(\mu)$. In addition

$$\mathbf{C} = \frac{1}{\lambda}\tilde{\mathbf{G}} + \frac{\phi}{\lambda}\mathbf{G}^0 = \frac{1}{\lambda}\mathbf{G} - \frac{\gamma}{\lambda\phi}\mathbf{G^2} + \frac{\theta}{\lambda}\mathbf{G^0}$$

so that any eigenvalue of C can be rewritten as:

$$\mu_i(\mathbf{C}) = \frac{1}{\lambda}\mu_i(\mathbf{G}) - \frac{\gamma}{\lambda\phi}\mu_i^2(\mathbf{G}) + \frac{\theta}{\lambda}\mu_i^0(\mathbf{G})$$

Now, the condition for the existence of an internal equilibrium is $\frac{\phi\lambda}{\sigma}\mu_i(\mathbf{C}) < 1$, $\forall i$. In terms of $\mu_i(\mathbf{G})$ this becomes

$$\frac{\phi\lambda}{\sigma}\mu_i(\mathbf{C}) = \frac{1}{\sigma}(\phi\mu_i(\mathbf{G}) - \gamma\mu_i^2(\mathbf{G}) + \phi\theta) < 1$$

satisfied exactly for the cases reported in the proposition. ■

**Proof of Proposition 2.**

To prove the proposition, we report here the FOC in the following complete form:

$$\frac{\alpha}{\sigma} \cdot \bar{1} = \left[\mathbf{I} - \frac{\phi}{\sigma}\left(\mathbf{G} - \frac{\gamma}{\phi}\mathbf{G^2}\right)\right]\bar{x}. \tag{A.1}$$

And rewrite them as

$$\frac{\alpha}{\sigma} \cdot \bar{1} = \left[\mathbf{I} - \frac{\phi}{\sigma}\mathbf{G} + \frac{\gamma}{\sigma}\mathbf{G^2}\right]\bar{x}. \tag{A.2}$$

Consider the matrix

$$\left[\mathbf{I} - \frac{\phi}{\sigma}\mathbf{G} + \frac{\gamma}{\sigma}\mathbf{G^2}\right] \tag{A.3}$$

This can be written as:

$$[\mathbf{I} - (a_1 + a_2)\mathbf{G} + a_1 a_2 \mathbf{G}] = [\mathbf{I} - a_1\mathbf{G}] \cdot [\mathbf{I} - a_2\mathbf{G}] \tag{A.4}$$

where

$$a_1 + a_2 = \frac{\phi}{\sigma} \tag{A.5}$$

and

$$a_1 a_2 = \frac{\gamma}{\sigma} \tag{A.6}$$

Solving the constraints in (A.5) and (A.6) we get only two couples $(a_1, a_2)$:

$$a_1 = \frac{\phi \pm \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma} \tag{A.7}$$

$$a_2 = \frac{\phi \mp \sqrt{\phi^2 - 4\gamma\sigma}}{2\sigma} \tag{A.8}$$

that exist if and only if $\phi > 2\sqrt{\gamma\sigma}$. The FOC now reads:

$$\frac{\alpha}{\sigma} \cdot \bar{1} = [\mathbf{I} - a_1\mathbf{G}] \cdot [\mathbf{I} - a_2\mathbf{G}]\, \bar{x}. \tag{A.9}$$

so that, if $\mu_1(\mathbf{G}) < \frac{1}{max\{a_1,a_2\}} = \frac{2\sigma}{\phi+\sqrt{\phi^2-4\gamma\sigma}}$, then

$$\bar{x} = \frac{\alpha}{\sigma}\,[\mathbf{I} - a_1\mathbf{G}]^{-1} \cdot [\mathbf{I} - a_2\mathbf{G}]^{-1} \cdot \bar{1} \tag{A.10}$$

Now

$$[\mathbf{I} - a_2\mathbf{G}]^{-1} \cdot \bar{1} = \bar{b}(\mathbf{G}, a_2) \tag{A.11}$$

and

$$[\mathbf{I} - a_1\mathbf{G}]^{-1} \cdot [\mathbf{I} - a_2\mathbf{G}]^{-1} \cdot \bar{1} = [\mathbf{I} - a_1\mathbf{G}]^{-1}\, \bar{b}(\mathbf{G}, a_2) \tag{A.12}$$

That can be rewritten as

$$[\mathbf{I} - a_1\mathbf{G}]^{-1} \cdot [\mathbf{I} - a_2\mathbf{G}]^{-1} \cdot \bar{1} = \bar{b}_{\bar{b}(\mathbf{G},a_2)}(\mathbf{G}, a_1) \tag{A.13}$$

so that

$$\bar{x} = \frac{\alpha}{\sigma}\bar{b}_{\bar{b}(\mathbf{G},a_2)}(\mathbf{G}, a_1) \tag{A.14}$$

Notice that, since solutions are symmetric, then

$$\bar{x} = \frac{\alpha}{\sigma}\bar{b}_{\bar{b}(\mathbf{G},a_2)}[\mathbf{G}, a_1] = \frac{\alpha}{\sigma}\bar{b}_{\bar{b}(\mathbf{G},a_1)}(\mathbf{G}, a_2) \tag{A.15}$$

■

**Proof of Proposition 3.** We start by considering the matrix:

$$\mathbf{M}\left(\mathbf{G} - \frac{\gamma}{\phi}\mathbf{G^2}, \frac{\phi}{\sigma}\right) \equiv [\mathbf{I} - \frac{\phi}{\sigma}(\mathbf{G} - \frac{\gamma}{\phi}\mathbf{G^2})]^{-1} \tag{A.16}$$

which by (10) determines equilibrium behaviour up to a proportionality factor. This can be rewritten as:

$$\sum_{k=0}^{\infty}\mathbf{G}^k[\frac{\phi}{\sigma}\mathbf{I} - \frac{\gamma}{\sigma}\mathbf{G}]^k = \sum_{k=0}^{\infty}\frac{1}{\sigma^k}\mathbf{G}^k[\phi\mathbf{I} - \gamma\mathbf{G}]^k \tag{A.17}$$

Applying the binomial expansion to the second term we get:

$$\sum_{k=0}^{\infty}\frac{1}{\sigma^k}\mathbf{G}^k\sum_{i=0}^{k}\binom{k}{i}\phi^i(-\gamma^{k-i})\mathbf{G}^{k-i} \tag{A.18}$$

from which

$$\sum_{k=0}^{\infty}\frac{1}{\sigma^k}\sum_{i=0}^{k}\binom{k}{i}\phi^i(-\gamma^{k-i})\mathbf{G}^{2k-i} \tag{A.19}$$

The derivative of this with respect to $\gamma$ evaluated at the point $\gamma = 0$ is:

$$\lim_{\gamma\to 0}\frac{\sum_{k=0}^{\infty}\frac{1}{\sigma^k}\sum_{i=0}^{k}\binom{k}{i}\phi^i(-\gamma^{k-i})\mathbf{G}^{2k-i} - \sum_{k=0}^{\infty}\frac{1}{\sigma^k}\phi^k\mathbf{G}^k}{\gamma} \tag{A.20}$$

$$\lim_{\gamma\to 0}\frac{\sum_{k=0}^{\infty}\frac{1}{\sigma^k}[(\sum_{i=0}^{k}\binom{k}{i}\phi^i(-\gamma)^{k-i}\mathbf{G}^{2k-i}) - \phi^k\mathbf{G}^k]}{\gamma} \tag{A.21}$$

Note now that: for $k = i$ we have $\binom{k}{i} = 1$, $\gamma^{k-i} = 1$ and $\mathbf{G}^{2k-i} = \mathbf{G}^k$, so that $\binom{k}{i}\phi^i(-\gamma)^{k-i}\mathbf{G}^{2k-i} = \phi^k\mathbf{G}^k$; for $k - i \geq 2$ we have $\lim_{\gamma\to 0}\frac{\gamma^{k-i}}{\gamma} = 0$; for $k - i = 1$ we have $\binom{k}{i} = k$, $(-\gamma^{k-i}) = -\gamma$ and $\mathbf{G}^{2k-i} = \mathbf{G}^{k+1}$. Summing up (A.21) now reads:

$$\begin{aligned} -\sum_{k\,=0}^{\infty}\frac{1}{\sigma^k}k\phi^{k-1}\mathbf{G}^{k+1} &= -\mathbf{G}\sum_{k=0}^{\infty}[\frac{\partial}{\partial\phi}\phi^k]\frac{1}{\sigma^k}\mathbf{G}^k \\ &= -\mathbf{G}\frac{\partial}{\partial\phi}\sum_{k=0}^{\infty}\frac{1}{\sigma^k}\phi^k\mathbf{G}^k \\ &= -\mathbf{G}\frac{\partial\mathbf{M}(\mathbf{G},\frac{\phi}{\sigma})}{\partial\phi}. \end{aligned} \tag{A.22}$$

Summing up we obtain:

$$\frac{\partial}{\partial\gamma}\mathbf{M}\left(\tilde{\mathbf{G}},\frac{\phi}{\sigma}\right)|_{\gamma=0} = -\mathbf{G}\frac{\partial\mathbf{M}(\mathbf{G},\frac{\phi}{\sigma})}{\partial\phi} = -\frac{\partial\mathbf{M}(\mathbf{G},\frac{\phi}{\sigma})}{\partial\phi}\mathbf{G} \tag{A.23}$$

where the last equality comes from symmetry of all involved matrices. Post multiplying the first and last term in the above equalities by $\bar{1}$ we finally get:

$$\frac{\partial}{\partial\gamma}\mathbf{M}\left(\tilde{\mathbf{G}},\frac{\phi}{\sigma}\right)|_{\gamma=0}\cdot\bar{1} = -\frac{\partial}{\partial\phi}\bar{b}_{\bar{d}}(\mathbf{G},\frac{\phi}{\sigma}), \tag{A.24}$$

which proves the proposition. ■

**Proof of Proposition 4.** Individual and aggregate behaviors are characterized by the following equilibrium first order conditions derived from (10):

$$\alpha - \sigma x_i + \phi\sum_{j\in N} g_{ij}x_j - \gamma\sum_k g_{ik}^{[2]}x_k = 0. \tag{A.25}$$

In a symmetric equilibrium, $x_i^* = x_j^*$ for all $i, j$. Moreover, in a regular graph of degree $d$ we have $\sum_{k\in N} g_{ik}^{[2]} = d^2$. We can then rewrite (A.25) as follows:

$$\alpha - x^*[\sigma - \phi d + \gamma d^2] = 0. \tag{A.26}$$

When $[\sigma - \phi d + \gamma d^2] < 0$, no positive action is consistent with equilibrium. When $[\sigma - \phi d + \gamma d^2] > 0$, the unique positive symmetric equilibrium is given by:

$$x^* = \frac{\alpha}{\sigma - \phi d + \gamma d^2}. \tag{A.27}$$

We express the effect of network density on behavior by means of the first derivative of

(A.27) with respect to $d$:

$$\frac{\partial x^*}{\partial d} = \frac{\alpha(\phi - 2\gamma d)}{\left[\sigma - \phi d + \gamma d^2\right]^2}. \tag{A.28}$$

The sign of the term $\frac{\partial x^*}{\partial d}$ is determined by the following regions:

$$\begin{cases} d < \frac{\phi}{2\gamma} & \Rightarrow & \frac{\partial x^*}{\partial d} > 0 \\ d = \frac{\phi}{2\gamma} & \Rightarrow & \frac{\partial x^*}{\partial d} = 0 \\ d > \frac{\phi}{2\gamma} & \Rightarrow & \frac{\partial x^*}{\partial d} < 0 \end{cases}$$

Since $\phi > 0$, we conclude that equilibrium behavior follows a non-monotonic pattern, reaching a maximum for $d^* \equiv \frac{\phi}{2\gamma}$. This immediately proofs the statement. ■

**Proof of Proposition 5.** Our result builds on Theorem 2 in Ballester et al. (2006), showing that increasing all entries in the network of social interactions unambiguously increases equilibrium behavior of all agents. Consider first a node $k \notin Z$ such that $g_{kz} = 0$ for all $z \in Z$. We have $\tilde{g}_{ki} = \tilde{g}'_{ki}$ for all $i \in N$. Consider then a node $k \notin Z$ such that $g_{ki} = 1$ for at least one $i \in Z$. We have that $\tilde{g}'_{ki} < \tilde{g}_{ki}$ and $\tilde{g}_{kz} \leq \tilde{g}_{kz}$ for all $z \in Z$. Consider now any two nodes $i, j \in Z$, for which, by construction, $g'_{ij} - g_{ij} = 1$. We also have $g'^{[2]}_{ij} - g^{[2]}_{ij} = Z - 2$, since all nodes in $Z$ are now linked with each other. Thus $\tilde{g}'_{ij} - \tilde{g}_{ij} = 1 - \frac{\rho\gamma}{\phi} \leq 0$ since we have assumed that $(\phi) \leq (Z-2)\gamma$. Thus, $\tilde{g}'_{ij} \leq \tilde{g}_{ij}$ for all $i, j \in Z$ with at least one strict inequality. ■

**Proof of Proposition 6.** Each agent has $dq$ neighbors of own type and $d(1-q)$ neighbors of different type. Moreover, let $t \in L, H$ and consider an agents of type $t$. $dq(dq-1)$ is the number of agents of type $t$ (other then self) connected with neighbors of type $t$; $d(1-q)[d(1-q)-1]$ is the number of agents of type $t$ connected with neighbors of type different from $t$; $dqd(1-q)$ is the number of agents of type different from $t$ connected with neighbors of type $t$; $d(1-q)dq$ is the number of agents of different from $t$ connected with neighbors of different from $t$. Consequently, by imposing symmetry on the FOC of each type, we get

$$\begin{cases} \alpha_h - \sigma x_h + dq\phi x_h + d(1-q)\phi x_l & \\ -\gamma\{dq[dq-1] + d(1-q)[d(1-q)-1]\}x_h - \gamma\{dqd(1-q) + d(1-q)dq\}x_l & = & 0 \\ & & \\ \alpha_l - \sigma x_l + dq\phi x_l + d(1-q)\phi x_h & \\ -\gamma\{dq[dq-1] + d(1-q)[d(1-q)-1]\}x_l - \Gamma\{dqd(1-q) + d(1-q)dq\}x_h & = & 0 \end{cases} \tag{A.29}$$

and the equilibrium in (20) is derived.
Consider now the spread. The numerator is fixed, while the denominator depends on $q$. The first derivative of the denominator with respect to $q$ is given by $-2d[2d(1-2q)\gamma-\phi]$. Studying the sign we get that if $q>\bar{q}$ the spread is decreasing, while if $q<\bar{q}$ the spread is increasing. By noting that if $d<d^*$ then $\bar{q}>1$, so that the spread is always increasing in $q$.
In order to prove that the maximal spread in independent from $d$, simply note that $\bar{q}$ is independent from the level of segregation. Thus call the spread $S$ and notice that

$$S(q=\frac{\phi-2\gamma d}{4d\gamma})=\frac{4(\alpha_h-\alpha_l)\gamma}{4\sigma\gamma-\phi^2} \tag{A.30}$$

■

**Proof of Proposition 8.**

The reduced form is:

$$x=(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G^2})^{-1}(\rho\mathbf{I}+\zeta\mathbf{G})z+(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G^2})^{-1}\epsilon \tag{A.31}$$

Consider two sets of structural parameters $(\alpha,\beta_1,\rho,\zeta,\beta_2)$ and $(\alpha',\beta_1',\rho',\zeta',\beta_2')$. If they lead to the same reduced form, it means that $(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G^2})^{-1}(\rho\mathbf{I}+\zeta\mathbf{G})=(\mathbf{I}-\beta_1'\mathbf{G}-\beta_2'\mathbf{G^2})^{-1}(\rho'\mathbf{I}+\zeta'\mathbf{G})$. Premultiply the second equality by $(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G^2})(\mathbf{I}-\beta_1'\mathbf{G}-\beta_2'\mathbf{G^2})$ we get

$$(\mathbf{I}-\beta_1'\mathbf{G}-\beta_2'\mathbf{G^2})(\rho\mathbf{I}+\zeta\mathbf{G})=(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G^2})(\rho'\mathbf{I}+\zeta'\mathbf{G}) \tag{A.32}$$

which can be rewritten as

$$(\rho-\rho')\mathbf{I}+(\zeta-\zeta'+\beta_1\rho'-\beta_1'\rho)\mathbf{G}+(\beta_1\zeta'-\beta_1'\zeta+\beta_2\rho'-\beta_2'\rho)\mathbf{G^2}+(\beta_2\zeta'-\beta_2'\zeta)\mathbf{G^3}=0 \tag{A.33}$$

if $\mathbf{I},\mathbf{G},\mathbf{G^2},\mathbf{G^3}$ are linearly independent, then

$$\rho=\rho' \tag{A.34}$$

$$\zeta'-\beta_1\rho'=\zeta-\beta_1'\rho \tag{A.35}$$

$$\beta_1\zeta'-\beta_2\rho'=\beta_1'\zeta-\beta_2'\rho \tag{A.36}$$

$$\beta_2\zeta'=\beta_2'\zeta \tag{A.37}$$

Consider now $\beta_2'\zeta\neq 0$, thus $\zeta\neq 0$. From (A.37) define $\zeta'=\lambda\zeta$ and $\beta_2'=\lambda\beta_2$ so (A.36) becomes

$$\beta_1\lambda\zeta - \beta_2\rho = \beta_1'\zeta - \lambda\beta_2\rho \tag{A.38}$$

and (A.35) becomes

$$\lambda\zeta - \beta_1\rho = \zeta - \beta_1'\rho \tag{A.39}$$

from (A.39)

$$\beta_1' = \beta_1 - \frac{\lambda\zeta}{\rho} + \frac{\zeta}{\rho} \tag{A.40}$$

substituting in (A.38) we get

$$\beta_1\lambda\zeta - \beta_2\rho = (\beta_1 - \frac{\lambda\zeta}{\rho} + \frac{\zeta}{\rho})\zeta - \lambda\beta_2\rho \tag{A.41}$$

If $\beta_1\zeta + \frac{\zeta^2}{\rho} + \beta_2\rho \neq 0$ and $\rho \neq 0$,

$$\lambda(\beta_1\zeta + \frac{\zeta^2}{\rho} + \beta_2\rho) = \beta_1\zeta + \frac{\zeta^2}{\rho} + \beta_2\rho \tag{A.42}$$

i.e. $\lambda$=1, so the two sets of parameters are the same.
Consider now $\beta_2\zeta = 0$. This can be due to either $\zeta = 0$, or $\beta_2 = 0$ (or both).
Consider first the case of $\zeta = 0$, then the coefficients associated to $\mathbf{G}$ and $\mathbf{G^2}$ become

$$\beta_1\rho' = \beta_1'\rho \tag{A.43}$$

$$\beta_2\rho' = \beta_2'\rho \tag{A.44}$$

So $\beta_1$ and $\beta_2$ are identified if $\rho \neq 0$, and thus identified from (A.34).
If $\beta_2 = 0$ the problem collapses to the case of Bramoullé et al. (2009) and coefficients are identified if $\zeta + \beta_1\rho \neq 0$. ■

**Proof of Proposition 9.** We can write (24) as:

$$x = (\mathbf{I} - \beta_1\mathbf{G})^{-1}(\rho\mathbf{I} + \zeta\mathbf{G}^*)z + (\mathbf{I} - \beta_1\mathbf{G})^{-1}\epsilon \tag{A.45}$$

Consider two sets of parameters $(\beta_1, \rho, \zeta)$ and $(\beta_1', \rho', \zeta')$ that provide the same estimates. Then

$$(\mathbf{I} - \beta_1\mathbf{G})^{-1}(\rho\mathbf{I} + \zeta\mathbf{G}^*) = (\mathbf{I} - \beta_1'\mathbf{G})^{-1}(\rho'\mathbf{I} + \zeta'\mathbf{G}^*) \tag{A.46}$$

Multiplying both sides by $(\mathbf{I}-\beta_1\mathbf{G})(\mathbf{I}-\beta_1'\mathbf{G})$ we obtain

$$(\mathbf{I}-\beta_1'\mathbf{G})(\rho\mathbf{I}+\zeta\mathbf{G}^*) = (\mathbf{I}-\beta_1\mathbf{G})(\rho'\mathbf{I}+\zeta'\mathbf{G}^*) \tag{A.47}$$

This can be rewritten as

$$(\rho-\rho')\mathbf{I}+(\zeta-\zeta')\mathbf{G}^*-(\rho\beta_1'-\rho'\beta_1)\mathbf{G}-(\zeta\beta_1'-\zeta'\beta_1)\mathbf{G}\mathbf{G}^*=0 \tag{A.48}$$

Suppose $\mathbf{I},\mathbf{G},\mathbf{G}^*,\mathbf{G}\mathbf{G}^*$ to be linearly independent. Then $\rho=\rho'$ and $\zeta=\zeta'$. If $\rho\neq 0$ or $\zeta\neq 0$ then it immediately follows that $\beta_1=\beta_1'$. ■

**Proof of Proposition 10.** We can write (23) as

$$x=(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G}^{\mathbf{2}})^{-1}(\rho\mathbf{I}+\zeta\mathbf{G}^*)z+(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G}^{\mathbf{2}})^{-1}\epsilon \tag{A.49}$$

Consider two sets of parameters $(\beta_1,\beta_2,\rho,\zeta)$ and $(\beta_1',\beta_2',\rho',\zeta')$ that provide the same estimates. Then

$$(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G}^{\mathbf{2}})^{-1}(\rho\mathbf{I}+\zeta\mathbf{G}^*)=(\mathbf{I}-\beta_1'\mathbf{G}-\beta_2'\mathbf{G}^{\mathbf{2}})^{-1}(\rho'\mathbf{I}+\zeta'\mathbf{G}^*) \tag{A.50}$$

Premultiplying both sides by $(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G}^{\mathbf{2}})(\mathbf{I}-\beta_1'\mathbf{G}-\beta_2'\mathbf{G}^{\mathbf{2}})$ we obtain

$$(\mathbf{I}-\beta_1'\mathbf{G}-\beta_2'\mathbf{G}^{\mathbf{2}})(\rho\mathbf{I}+\zeta\mathbf{G}^*)=(\mathbf{I}-\beta_1\mathbf{G}-\beta_2\mathbf{G}^{\mathbf{2}})(\rho'\mathbf{I}+\zeta'\mathbf{G}^*) \tag{A.51}$$

This can be rewritten as

$$\begin{aligned}&(\rho-\rho')\mathbf{I}+(\zeta-\zeta')\mathbf{G}^*-(\rho\beta_1'-\rho'\beta_1)\mathbf{G}-(\zeta\beta_1'-\zeta'\beta_1)\mathbf{G}\mathbf{G}^*\\&\quad-(\rho\beta_2'-\rho'\beta_2)\mathbf{G}^{\mathbf{2}}-(\zeta\beta_2'-\zeta'\beta_2)\mathbf{G}^{\mathbf{2}}\mathbf{G}^*=0\end{aligned} \tag{A.52}$$

Suppose $\mathbf{I},\mathbf{G},\mathbf{G}^{\mathbf{2}},\mathbf{G}^*,\mathbf{G}\mathbf{G}^*,\mathbf{G}^{\mathbf{2}}\mathbf{G}^*$ to be linearly independent. Then $\rho=\rho'$ and $\zeta=\zeta'$. If $\rho\neq 0$ or $\zeta\neq 0$ then it immediately follows that $\beta_1=\beta_1'$ and $\beta_2=\beta_2'$. ■

# B Derivation of Equilibrium in terms of C

Let $\mathbf{U}$ denote the $n\times n$ matrix of ones. We can rewrite the first order conditions (10) as follows:

$$\alpha\bar{1}=[\sigma\mathbf{I}+\phi(\theta\cdot\mathbf{U}-\lambda\mathbf{C})]\,\bar{x}. \tag{B.1}$$

Rearranging terms we get:

$$\frac{\alpha}{\sigma}\bar{1} - \frac{\phi}{\sigma}\theta \mathbf{U}\bar{x} = \left[\mathbf{I} - \frac{\phi}{\sigma}\lambda \mathbf{C}\right]\bar{x}. \tag{B.2}$$

By writing $\mathbf{U}\bar{x} = \mathbf{x}\bar{1}$

$$\frac{\alpha}{\sigma}\bar{1} - \frac{\phi}{\sigma}\theta \mathbf{x}\bar{1} = \left[\mathbf{I} - \frac{\phi}{\sigma}\lambda \mathbf{C}\right]\bar{x}. \tag{B.3}$$

A necessary and sufficient condition for the matrix $\left[\mathbf{I} - \frac{\phi}{\sigma}\lambda \mathbf{C}\right]$ to admit a positive inverse is that $1 > \frac{\lambda\phi}{\sigma}\mu(\mathbf{C})$, with $\mu(\mathbf{C})$ being the largest eigenvalue of the $\mathbf{C}$ matrix. Under this restriction we write:

$$\left[\mathbf{I} - \frac{\phi}{\sigma}\lambda \mathbf{C}\right]^{-1} (\frac{\alpha}{\sigma} - \frac{\phi}{\sigma}\theta \cdot \mathbf{x})\bar{1} = \mathbf{I}\bar{x}. \tag{B.4}$$

Using the definition of Bonacich centrality vector, we can now write:

$$\frac{\alpha}{\sigma}\bar{b}\left(\mathbf{C}, \frac{\lambda\phi 2}{\sigma}\right) - \frac{\phi}{\sigma}\theta \mathbf{x}\bar{b}\left(\mathbf{C}, \frac{\lambda\phi}{\sigma}\right) = \mathbf{I}\bar{x} \tag{B.5}$$

In order to ease notation, from now on we drop the argument of the centrality vectors. Premultiplying by $\bar{1}'$ we get:

$$\frac{\alpha}{\sigma}b - \frac{\phi}{\sigma}\theta b\mathbf{x} = \mathbf{x} \tag{B.6}$$

and thus

$$\mathbf{x} = \frac{\alpha b}{\sigma + \phi\theta b} \tag{B.7}$$

substituting this into (B.5) we get the result of the proposition.

# C Comparisons of Thresholds

To understand the relationship between the threshold in propositions 1 and 2, consider figure 7 that represents the threshold determining the areas in which an interior equilibrium exists.

Define $\mu_m^A$ and $\mu_M^A$ the thresholds on eigenvalues defined in proposition 1. An interior equilibrium exists if $\phi < \phi^A$ or, for any $\mu_i$, $\mu_i > \mu_M^A$ or $\mu_i < \mu^A$.

Define then $\mu^B$ the threshold on eigenvalues defined in proposition 2. An interior equilibrium exists if $\phi > \phi^B$ and $\mu_i < \mu^B$.

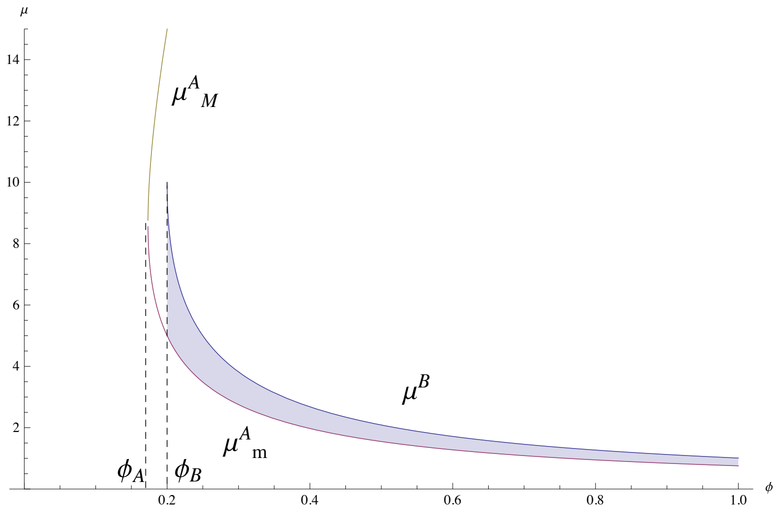


Figure 7: Comparison of Sufficient Conditions for Equilibrium Characterization. On the left conditions for equilibrium for $\gamma$ small, on the right conditions for $\gamma$ big.

Thus the shaded area represents the parameter space in which conditions in proposition 2 ensure an interior equilibrium, while conditions in 1 do not enable to write the equilibrium in terms of centralities of $\mathbf{C}$.

# D Derivation of condition (16)

Let us rewrite the FOC (10) as follows:

$$\alpha\mathbf{I}\bar{1} - \gamma\mathbf{G^2}\bar{x} = (\sigma\mathbf{I} - \phi\mathbf{G})\bar{x} \tag{D.1}$$

from which we obtain:

$$\alpha\mathbf{I}\bar{1} - \gamma\mathbf{G^2}\bar{x} = \sigma(\mathbf{I} - \frac{\phi}{\sigma}\mathbf{G})\bar{x} \tag{D.2}$$

Recalling that $(\mathbf{I} - \frac{\phi}{\sigma}\mathbf{G})^{-1}$ is the Bonacich centrality matrix $\mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma})$, we can write:

$$\frac{\alpha}{\sigma}\mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma})\bar{1} - \frac{\gamma}{\sigma}\mathbf{M}(\mathbf{G}, \frac{\phi}{\sigma})\mathbf{G^2}\bar{x} = \bar{x} \tag{D.3}$$

■

# E Change in Key Player

In this appendix we look at the effect of $\gamma$ on the identification of the *key-player*, that is the player who, if removed from the network, would trigger a maximal change in aggregate behavior. Such *key-player* might be of crucial importance in various health related policies and in policies that try to reduce crime (see Ballester et al., 2010). Ballester et al. (2006) show that the key-player is the node with the highest *intercentrality* in the network, defined as $c_i = b_i^2/m_{ii}$. As for the notion of Bonacich centrality, the ordering of inter-centralities is potentially affected by the presence of distance-two strategic substitution. In particular, the same marginalization of behavior observed in the above examples seems to characterize the ranking of inter-centralities. Consider, for instance, the case of a "line" network (Figure 8).

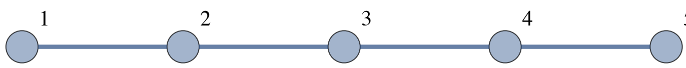


Figure 8: Line network

We can identify 3 types of agents depending on their position: agents 1 and 5, agents 2 and 4, and agent 3. Table 3 provides the ordering of centralities and inter-centralities for different values of $\gamma$ for the specific application to smoking. With $\gamma = 0$, the key-player

Table 3: Effect of $\gamma$ on central and key-player - Line network

| $\gamma$ | $b_i$ | $c_i$ |
|---|---|---|
| 0 | $3 > 2 > 1$ | $3 > 2 > 1$ |
| 0.8 | $2 > 1 > 3$ | $2 > 1 > 3$ |

Parametrization: $\gamma_0 = 15$, $\alpha = 0.6$, $\delta = 0.5$, $\theta = 1$

is 3. Consider now $\gamma > 0$. Player 3 is responsible for several distance-two relations, that keep aggregate behaviour low by generating strategic substitutability. For this reason, removing player 3 from the network has the effect of removing such indirect substitution effects, and is therefore little effective in decreasing aggregate behaviour. Player 1, in contrast, generates fewer distance-two relations, and her removal from the network is very effective in decreasing aggregate behaviour.

Similar considerations apply to the interconnected cliques network studied by Ballester et al. (2006) (see Figure 3 and Table 4). For positive levels of $\gamma$, type 3 agents have higher inter-centrality (and centrality) than type 1 agent. In fact, while agent 1 is critical for several distance-two relations, type 3 agents are not. It follows that removing agent 1 would delete several sources of substitutability, thereby offsetting the negative effect on behavior due to the removal of direct connections.

Table 4: Effect of $\gamma$ on central and key-player - Ballester et al. (2006)

| $\gamma$ | $b_i$ | $c_i$ |
|---|---|---|
| 0 | $2 > 1 > 3$ | $2 > 1 > 3$ |
| 0.7 | $3 > 2 > 1$ | $3 > 2 > 1$ |

Parametrization: $\gamma_0 = 15$, $\alpha = 0.6$, $\delta = 0.5$, $\theta = 1$

# F Choice of the instruments

$$E(\mathbf{G}x_r) = \beta_2 \mathbf{G_r M_{1r} G_r^2} x + \rho \mathbf{G_r} z_r + \rho\beta_1 \mathbf{G_r M_{1r} G_r} z_r + \zeta \mathbf{G_r M_{1r} G_r^*} z_r \tag{F.1}$$

and defining $\mathbf{M_{2r}} = (\mathbf{I} - \beta_2 \mathbf{M_{1r} G_r^2})^{-1}$, we get

$$E(\mathbf{G}x_r) = \rho \mathbf{G_r M_{2r}} z_r + \rho\beta_1 \mathbf{G_r M_{2r} M_{1r} G_r} z_r + \zeta \mathbf{G_r M_{2r} M_{1r} G_r^*} z_r \tag{F.2}$$

substituting $\mathbf{M_{1r}} = \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j$ we get

$$E(\mathbf{G}x_r) = \rho \mathbf{G_r M_{2r}} z_r + \rho\beta_1 \mathbf{M_{2r}} \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2} z + \zeta \mathbf{M_{2r}} \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r G_r^*} z \tag{F.3}$$

$$E(\mathbf{G^2}x_r) = \rho \mathbf{G_r^2 M_{2r}} z_r + \rho\beta_1 \mathbf{M_{2r}} \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^3} z_r + \zeta \mathbf{M_{2r}} \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2 G_r^*} z_r \tag{F.4}$$

Substituting $\mathbf{M_{2r}} = \sum_{j=0}^{\infty} [\beta_2 \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2}]^j$ into (F.3) and (F.4)

$$E(\mathbf{G}x_r) = \rho \mathbf{G_r} \sum_{j=0}^{\infty} [\beta_2 \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2}]^j z_r + \rho\beta_1 \sum_{j=0}^{\infty} [\beta_2 \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2}]^j \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2} z + \zeta \sum_{j=0}^{\infty} [\beta_2 \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r^2}]^j \sum_{j=0}^{\infty} (\beta_1 \mathbf{G_r})^j \mathbf{G_r G_r^*} z_r \tag{F.5}$$

$$E(\mathbf{G^2}x_r) = \rho\mathbf{G_r^2}\sum_{j=0}^{\infty}[\beta_2\sum_{j=0}^{\infty}(\beta_1\mathbf{G_r})^j\mathbf{G_r^2}]^j z_r + \rho\beta_1\sum_{j=0}^{\infty}[\beta_2\sum_{j=0}^{\infty}(\beta_1\mathbf{G_r})^j\mathbf{G_r^2}]^j\sum_{j=0}^{\infty}(\beta_1\mathbf{G_r})^j\mathbf{G_r^3}z_r$$
$$+ \zeta\sum_{j=0}^{\infty}[\beta_2\sum_{j=0}^{\infty}(\beta_1\mathbf{G_r})^j\mathbf{G_r^2}]^j\sum_{j=0}^{\infty}(\beta_1\mathbf{G_r})^j\mathbf{G_r^2}\mathbf{G_r^*}z_r \tag{F.6}$$

The endogenous variable $\mathbf{G}x_r$ is still correlated with $\mathbf{G}z_r, \mathbf{G^2}z_r, \mathbf{GG}^*z_r$ (and some higher terms) used in Liu et al. (2012). In addition, $\mathbf{G^2}x_r$ is correlated with $\mathbf{G^2}z_r, \mathbf{G}^3z_r, \mathbf{G^2G}^*z_r$ (and some higher terms) but not with $\mathbf{G}z_r$. Given that both $\mathbf{G_r}x_r$ and $\mathbf{G_r^2}x_r$ are endogenous variables, the rank condition valid for identification is modified with respect to the case in which just one endogenous variable is present. Call now W the total set of exogenous variables, i.e. exogenous variables Z included in the model (demographics and friends characteristics) and instruments Q, and V the set of all explanatory variables, i.e. W and the endogenous $\mathbf{G}x_r$ and $\mathbf{G}^2x_r$. Thus, the usual rank condition can be split in two parts:

1. rank $E(W'W) = l$
2. rank $E(W'V) = k$.

Where $l$ is the number of exogenous variables W and $k$ the total number of the explanatory variables.

Notice that identification is not achieved if the fitted values of the first stages $\hat{\mathbf{G}x_r}$ and $\hat{\mathbf{G}^2x_r}$ are perfectly collinear. Write both $\hat{\mathbf{G}x_r}$ and $\hat{\mathbf{G}^2x_r}$ as a linear combination of two instruments ($Q_1 = \mathbf{G}z_r$ and $Q_2 = \mathbf{G}^2z_r$) multiplied by the coefficients obtained in the two (different) first stages, $\hat{\mathbf{G}x_r} = b_1Q_1 + b_2Q_2$ and $\hat{\mathbf{G}^2x_r} = c_1Q_1 + c_2Q_2$. If $c_1 = 0$ after controlling for $Q_2$, and if $b_1$ and $b_2$ are both different from zero, then the fitted values of the two endogenous variables cannot be perfectly collinear and $\hat{\beta}_1$ and $\hat{\beta}_2$ are identified and consistent.[13] Note that the condition $c_1 = 0$ is not necessary for identification, but it just *ex ante* rules out the presence of multicollinearity in the set of instruments.

[13]However, if $Q_2$ is not strong enough and $Q_1$ and $Q_2$ are not jointly relevant (i.e they are weak instruments), the estimation of $\hat{\beta}_2$ could be severely biased in small samples.