Journal of Algebra 434 (2015) 65–71

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Extrapolating an Euler class

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ARTICLE INFO

Article history:
Received 10 February 2015
Available online 14 April 2015
Communicated by Luchezar L. Avramov

Keywords:
Orbit set of unimodular rows
Euler class group

ABSTRACT

Let R be a noetherian ring of dimension d and let n be an integer so that $n \leq d \leq 2n - 3$. Let $(a_1, \dots, a_{n+1})$ be a unimodular row so that the ideal $J = (a_1, \dots, a_n)$ has height n. Jean Fasel has associated to this row an element $[(J, \omega_J)]$ in the Euler class group $E^n(R)$, with $\omega_J : (R/J)^n \to J/J^2$ given by $(\bar{a}_1, \dots, \bar{a}_{n-1}, \bar{a}_n\bar{a}_{n+1})$. If R contains an infinite field F then we show that the rule of Fasel defines a homomorphism from $WMS_{n+1}(R) = \mathrm{Um}_{n+1}(R)/E_{n+1}(R)$ to $E^n(R)$. The main problem is to get a well defined map on all of $\mathrm{Um}_{n+1}(R)$. Similar results have been obtained by Das and Zinna [5], with a different proof. Our proof uses that every Zariski open subset of $SL_{n+1}(F)$ is path connected for walks made up of elementary matrices.

1. Recollections

1.1. The group of orbits

Let $n \geq 3$. Let R be a commutative noetherian ring of Krull dimension d, $d \leq 2n - 2$. As usual $E_m(R)$ denotes the subgroup of $SL_m(R)$ generated by elementary matrices $e_{ij}(r)$ and $\mathrm{Um}_m(R)$ denotes the set of unimodular rows of length m over R. Then [7, Theorem 4.1] provides an abelian group structure on the orbit set $\mathrm{Um}_{n+1}(R)/E_{n+1}(R)$. The abelian group that is obtained is called $WMS_{n+1}(R)$. As ex-

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http://dx.doi.org/10.1016/j.jalgebra.2015.04.001
0021-8693/

plained in [8, §3] the group law may be characterized as follows. If $\alpha, \beta \in WMS_{n+1}(R)$, one may choose representatives $(a_1, \dots, a_{n+1}) \in \alpha$, $(b_1, \dots, b_{n+1}) \in \beta$ so that $a_1 + b_1 = 1$ and $a_i = b_i$ for $i > 1$. Then $(a_1b_1, a_2, \dots, a_{n+1})$ is a representative of $\alpha + \beta$. This rule reflects the homotopic join of Borsuk [3].

1.2. The Euler class group

From now on let $3 \le d \le 2n - 3$. In [2] the authors introduce an Euler class group $E^n(R)$ generalizing the Euler class group of [1]. The latter corresponds with the case $n = d$. The Euler class group is an abelian group given by a presentation. Generators are pairs (J, ω_J) where J is a height n ideal in R equipped with a surjective map $(R/J)^n \to J/J^2$. Think of a codimension n subvariety with trivial conormal bundle together with a trivialization of said bundle. Relations are

Disconnected sum

Let (J, ω_J) be a generator. If $J = KL$ with K, L comaximal ideals of height n, then $R/J = R/K \times R/L$ and $J/J^2 = K/K^2 \times L/L^2$, so that $\omega_J = \omega_k \times \omega_L$. The relation is

$$(J, \omega_J) = (K, \omega_K) + (L, \omega_L).$$

Complete intersection

Let (J, ω_J) be a generator such that ω_J lifts to a surjection $R^n \to J$. Then

$$(J, \omega_J) = 0.$$

Elementary action

Let (J, ω_J) be a generator and let $g \in E_n(R/J)$. Then

$$(J, \omega_J) = (J, \omega_J \circ g).$$

One may define $E^n(R)$ by taking the disconnected sum relations and the complete intersection relations as defining relations, cf. [5, Proposition 2.2]. We denote the class of (J, ω_J) in $E^n(R)$ by $[(J, \omega_J)]$. One shows with [2, Corollary 2.4, Proposition 3.1] that every element of $E^n(R)$ can be written in the form $[(J, \omega_J)]$. And one shows as in [5, Proposition 2.2] that the elementary action relations also hold. (Use [2, Corollary 2.4] and use that $E_n(R) \to E_n(R/K) \times E_n(R/L)$ is surjective in the disconnected sum setting.) Note that [2, Corollary 2.4] only needs $d \le 2n - 1$.

1.3. The old homomorphism and the new one

If our R is a regular ring containing an infinite field then Bhatwadekar and Sridharan define a homomorphism $WMS_{n+1}(R) \to E^n(R)$ with useful properties when n is even

([2, Theorem 5.7]). But it vanishes when n is odd. Jean Fasel noticed that in $\mathbb{A}^1$ homotopy one can do better. He proposed a formula that would also be useful when n is odd. In fact the same formula was discussed by Bhatwadekar and Sridharan after [1, Theorem 7.3], in the case of even n. It is already known that the formula of Fasel works for $3 \leq n = d$ [5, Theorem 3.6]. That is, it defines a homomorphism $WMS_{n+1}(R) \to E^n(R)$. If R is a domain it is also known to work [5, Remark 3.11] (always assuming $3 \leq d \leq 2n-3$). Our purpose is to show that his formula works when R contains an infinite field F. The main difference between this note and [5] is in the proof strategy. Rather than studying $E^n(R)$ more closely, as is done in [5], we concentrate on $\mathrm{Um}_{n+1}(R)$. We use paths made up of elementary matrices in $SL_{n+1}(F)$ to walk back and forth between general unimodular rows and rows for which we already know what to do.

2. Elementary paths

The group $E_{n+1}(R)$ is generated by elementary matrices $e_{ij}(r)$. We call a sequence $g_1, \dots, g_m$ of elements of $E_{n+1}(R)$ an *elementary path* if the $g_i^{-1}g_{i+1}$ are elementary for $i = 1, \dots, m-1$. We call it an F-path if moreover all g_i are in $SL_{n+1}(F)$. Notice that if $g_1, \dots, g_m$ is a path, then the reverse sequence $g_m, \dots, g_1$ is also a path.

We provide $SL_{n+1}(F)$ with the topology induced by the Zariski topology on the algebraic group SL_{n+1} defined over F. We say that a subset U of $SL_{n+1}(F)$ is path connected if any two elements of U can be joined by an F-path that stays within U.

Proposition 2.1. *Any nonempty open subset of $SL_{n+1}(F)$ is path connected.*

Proof. We give two proofs. Let Ω be the big cell of $SL_{n+1}(F)$. Then by [6, Proposition 2.6] every open subset of Ω is path connected. But then for any $g \in SL_{n+1}(F)$ every open subset of $g\Omega$ is path connected. So if U is a nonempty open subset of $SL_{n+1}(F)$, then it is covered by mutually intersecting path connected subsets of type $U \cap g\Omega$.

For the second proof recall that an element $g \in SL_{n+1}(F)$ that is in general position may be reduced to the identity matrix with just $N = (n+1)^2 - 1$ elementary operations. It goes like this. Add a multiple of the last column to the first to achieve that g_{11} becomes equal to one. Then clear the first row in n steps. Add a multiple of the last column to the second to achieve that g_{22} becomes equal to one. Then clear the second row in n steps. Keep going until the second last row has been cleared. Notice that $g_{n+1,n+1}$ has become equal to one. Clear the last row in n steps. Reversing this procedure one finds a sequence $\alpha_1, \dots, \alpha_N$ of roots so that the map $(t_1, \dots, t_N) \mapsto e_{\alpha_1}(t_1) \cdots e_{\alpha_N}(t_N)$ defines a birational map from F^N to $SL_{n+1}(F)$. Now if p, q are elements of the open subset U, then the condition that the path p, $pe_{\alpha_1}(t_1)$, $pe_{\alpha_1}(t_1)e_{\alpha_2}(t_2)$, … stays inside U defines an open condition on F^N. So there is a nonempty open subset of U of elements that can be reached by an F-path within U that starts at p. Similarly there is a nonempty open subset of U that can be reached by an F-path starting at q. These two subsets intersect. □

Remark 2.2. Of course there is no such result for finite fields.

2.3. Prime avoidance

We will tacitly use variations on the proof of the following classical lemma.

Lemma 2.4. *Let $(a_1, \ldots, a_{m+1})$ be a unimodular row over R. There are $\lambda_i \in R$ so that the ideal $(a_1 + \lambda_1 a_{m+1}, \ldots, a_m + \lambda_m a_{m+1})$ has height m or equals R.*

Proof. We argue by induction on m. For $m = 1$ it is a prime avoidance exercise [4, Exercise 3.19]. Let $m > 1$. The set of ideals of the form $(a_1 + \lambda_1 a_{m+1}, \ldots, a_m + \lambda_m a_{m+1})$ does not change when we add multiples of $a_2, \ldots, a_{m+1}$ to a_1. So by the same exercise we may assume a_1 avoids all minimal primes. Apply the inductive hypothesis to the unimodular row $(\bar{a}_2, \ldots, \bar{a}_{m+1})$ over $R/(a_1)$. □

2.5. Generic unimodular rows

We define $\mathrm{Um}_{\mathrm{gen}}(R)$ to be the set of unimodular rows $\mathbf{a} = (a_1, \ldots, a_{n+1})$ over R for which the ideal $(a_n a_{n+1})$ has height one. We call such rows generic. So in a generic row the last two entries avoid every minimal prime ideal of R.

If $\mathbf{a} \in \mathrm{Um}_{n+1}(R)$ and $g_1, \ldots, g_m$ is an elementary path in $E_{n+1}(R)$, then we call the sequence $\mathbf{a}g_1, \ldots, \mathbf{a}g_m$ a path in $\mathrm{Um}_{n+1}(R)$. If moreover $g_1, \ldots, g_m$ is an F-path, then we also call $\mathbf{a}g_1, \ldots, \mathbf{a}g_m$ an F-path.

Proposition 2.6. *Generic rows detect orbits:*

1. *Every $SL_{n+1}(F)$-orbit in $\mathrm{Um}_{n+1}(R)$ intersects $\mathrm{Um}_{\mathrm{gen}}(R)$ in a nonempty path connected subset.*
2. *Every $E_{n+1}(R)$-orbit in $\mathrm{Um}_{n+1}(R)$ intersects $\mathrm{Um}_{\mathrm{gen}}(R)$ in a nonempty path connected subset.*

Proof. For $\mathbf{a} \in \mathrm{Um}_{n+1}(R)$ consider the set of $g \in SL_{n+1}(F)$ for which $\mathbf{a}g \in \mathrm{Um}_{\mathrm{gen}}(R)$. It is a nonempty open subset of $SL_{n+1}(F)$. (The complement is closed because minimal primes are linear subspaces.) Therefore part 1 follows from Proposition 2.1.

To prove the second part, fix a path component P of $\mathrm{Um}_{\mathrm{gen}}(R)$ and let X be the set of $\mathbf{a} \in \mathrm{Um}_{n+1}(R)$ for which there is an F-path starting at $\mathbf{a}$ and ending in P. Clearly X is invariant under the action by $SL_{n+1}(F)$. Notice that by part 1, if $\mathbf{a} \in X$, then every F-path from $\mathbf{a}$ to $\mathrm{Um}_{\mathrm{gen}}(R)$ lands in P. If we show that X is also invariant under the action by $e_{21}(r)$ for $r \in R$, then it will follow that X is an $E_{n+1}(R)$-orbit with $X \cap \mathrm{Um}_{\mathrm{gen}}(R) = P$. So let $\mathbf{a} \in X$. We need to show that $\mathbf{a}e_{21}(r) \in X$. We may replace $\mathbf{a}$ with $\mathbf{a}g$ for any $g \in SL_{n+1}(F)$ that commutes with $e_{21}(r)$. Therefore we may assume $\mathbf{a} = (a_1, \ldots, a_{n+1})$ is such that there are λ, $\mu \in F$ with $\mathbf{a}e_{1,n}(\lambda)e_{1,n+1}(\mu) \in$

$\mathrm{Um_{gen}}(R)$ and $\mathbf{a}e_{21}(r)e_{1,n}(\lambda)e_{1,n+1}(\mu) \in \mathrm{Um_{gen}}(R)$. Note that $\mathbf{a}e_{1,n}(\lambda)e_{1,n+1}(\mu) \in P$. We can walk inside $\mathrm{Um_{gen}}(R)$ from $\mathbf{a}e_{21}(r)e_{1n}(\lambda)e_{n+1}(\mu)$ to $\mathbf{a}e_{1n}(\lambda)e_{n+1}(\mu)$ by way of $(a_1, a_2, \ldots, a_{n-1}, a_n + \lambda(a_1 + ra_2), a_{n+1} + \mu(a_1 + ra_2))$ and $(a_1, a_2, \ldots, a_{n-1}, a_n + \lambda a_1, a_{n+1} + \mu(a_1 + ra_2))$. $\square$

3. The map

Before defining a map $\phi : \mathrm{Um}_{n+1}(R) \to E^n(R)$ we will define one on generic rows. But first we define $\phi_0(\mathbf{a})$ when $\mathbf{a} = (a_1, \ldots, a_{n+1})$ is such that the ideal $J = (a_, \ldots, a_n)$ has height at least n. If $J = R$ we put $\phi_0(\mathbf{a}) = 0$. Remains the case that J has height n. Then we follow Fasel and put

$$\phi_0(\mathbf{a}) = [(J, \omega_J)]$$

with $\omega_J : (R/J)^n \to J/J^2$ given by $(\bar{a}_1, \ldots, \bar{a}_{n-1}, \bar{a}_n\bar{a}_{n+1})$.

Now let $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathrm{Um_{gen}}(R)$ be generic. Choose λ_i, μ_i in R so that $J_1 = (a_1 + \lambda_1 a_n, \ldots, a_{n-1} + \lambda_{n-1}a_n, a_{n+1})$ and $J_2 = (a_1 + \mu_1 a_{n+1}, \ldots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n)$ have height at least n.

Lemma 3.1. *The sum of*

$$[(J_1, \omega_{J_1})] := \phi_0((a_1 + \lambda_1 a_n, \ldots, a_{n-1} + \lambda_{n-1}a_n, a_{n+1}, a_n))$$

and

$$[(J_2, \omega_{J_2})] := \phi_0((a_1 + \mu_1 a_{n+1}, \ldots, a_{n-1} + \mu_{n-1}a_{n+1}, a_n, a_{n+1}))$$

vanishes.

Therefore $[(J_1, \omega_{J_1})]$ does not depend on the choice of the λ_i and $[(J_2, \omega_{J_2})]$ does not depend on the choice of the μ_i.

Proof. (Compare [5, Proposition 3.4].) As $\bar{a}_n$ is invertible in R/J_1 we get that $[(J_1, \omega_{J_1})]$ equals

$$\phi_0((a_1 + \lambda_1 a_n + \mu_1 a_{n+1}, \ldots, a_{n-1} + \lambda_{n-1}a_n + \mu_{n-1}a_{n+1}, a_{n+1}, a_n)).$$

Similarly $[(J_2, \omega_{J_2})]$ equals

$$\phi_0((a_1 + \lambda_1 a_n + \mu_1 a_{n+1}, \ldots, a_{n-1} + \lambda_{n-1}a_n + \mu_{n-1}a_{n+1}, a_n, a_{n+1})).$$

As J_1, J_2 are comaximal, $[(J_1, \omega_{J_1})] + [(J_2, \omega_{J_2})]$ equals

$$\phi_0((a_1 + \lambda_1 a_n + \mu_1 a_{n+1}, \ldots, a_{n-1} + \lambda_{n-1}a_n + \mu_{n-1}a_{n+1}, a_{n+1}a_n, 1)). \qquad \square$$

3.2. The map on generic rows

For $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathrm{Um}_{\mathrm{gen}}(R)$ we choose the μ_i as above and put

$$\phi(\mathbf{a}) = \phi_0((a_1 + \mu_1 a_{n+1}, \dots, a_{n-1} + \mu_{n-1} a_{n+1}, a_n, a_{n+1})).$$

Note that $\phi((a_1, \dots, a_{n+1})) = -\phi((a_1, \dots, a_{n-1}, a_{n+1}, a_n))$. Note also that, if $i \neq j$, $j < n$, then $\phi(\mathbf{a})$ does not change if we add a multiple of a_i to a_j.

Proposition 3.3. *The map ϕ is constant on path components of* $\mathrm{Um}_{\mathrm{gen}}(R)$.

Proof. We must show that $\phi(\mathbf{a}) = \phi(\mathbf{a}e_{ij}(r))$ if $\mathbf{a} = (a_1, \dots, a_{n+1}) \in \mathrm{Um}_{\mathrm{gen}}(R)$ and $e_{ij}(r)$ is an elementary matrix so that $\mathbf{a}e_{ij}(r) \in \mathrm{Um}_{\mathrm{gen}}(R)$. We already know it when $j < n$. Remain cases with $j = n$ or $j = n + 1$.

For instance, let $i = n$, $j = n + 1$. We may add multiples of a_n, a_{n+1} to a_1 through a_{n-1} to reduce to the case that $\phi_0(\mathbf{a})$ is defined. But then $\phi_0(\mathbf{a})$ and $\phi_0(\mathbf{a}e_{ij}(r))$ are computed as the class of the same (J, ω_J).

Next try $i = n - 1$, $j = n + 1$. Choose $\lambda \in R$ so that $(a_{n-1} + \lambda a_n)$ and $(a_{n+1} + r(a_{n-1} + \lambda a_n))$ both have height one. Then $\phi(\mathbf{a}) = \phi((a_1, \dots, a_{n-2}, a_{n-1} + \lambda a_n, a_n, a_{n+1}))$ and $\phi(\mathbf{a}e_{n-1,n+1}(r)) = \phi((a_1, \dots, a_{n-2}, a_{n-1} + \lambda a_n, a_n, a_{n+1} + r(a_{n-1} + \lambda a_n)))$ by the cases that have already been treated. So replacing a_{n-1} with $a_{n-1} + \lambda a_n$ we may further assume that (a_{n-1}) has height one. Adding multiples of a_{n-1}, a_n, a_{n+1} to a_1 through a_{n-2} we may further arrange that $(a_1, \dots, a_{n-1})$ has height $n - 1$. And adding a multiple of a_n to a_{n+1} we may further assume that $(a_1, \dots, a_{n-1}, a_{n+1})$ and $(a_1, \dots, a_{n-1}, a_{n+1} + r a_{n-1})$ have height n. Choose μ so that $(a_n + \mu a_{n+1})$ has height one and so that $\phi(\mathbf{a}e_{n-1,n+1}(r)) = \phi_0((a_1, \dots, a_{n-1}, a_n + \mu(a_{n+1} + r a_{n-1}), a_{n+1} + r a_{n-1}))$. Then $\phi(\mathbf{a}e_{n-1,n+1}(r)) = \phi_0((a_1, \dots, a_{n-1}, a_n + \mu(a_{n+1} + r a_{n-1}), a_{n+1}))$, which equals $-\phi_0((a_1, \dots, a_{n-1}, a_{n+1}, a_n + \mu(a_{n+1} + r a_{n-1})))$. Therefore $\phi(\mathbf{a}e_{n-1,n+1}(r)) = -\phi_0((a_1, \dots, a_{n-1}, a_{n+1}, a_n + \mu a_{n+1})) = \phi(\mathbf{a})$.

Other cases are analogous or easier. □

It is now clear how to extend ϕ to all of $\mathrm{Um}_{n+1}(R)$. For $\mathbf{a} \in \mathrm{Um}_{n+1}(R)$ take a path to $\mathrm{Um}_{\mathrm{gen}}(R)$ and define $\phi(\mathbf{a})$ to be the value of ϕ at the end of the path. Note that ϕ now also extends the map ϕ_0.

Theorem 3.4. *Let $3 \leq d \leq 2n - 3$ and let R be a commutative noetherian ring of Krull dimension d. If R contains an infinite field then ϕ_0 induces a homomorphism $WMS_{n+1}(R) \to E^n(R)$.*

Proof. To see that ϕ defines a homomorphism $WMS_{n+1}(R) \to E^n(R)$ consider $\mathbf{a} = (a_1, \dots, a_{n+1})$, $\mathbf{b} = (b_1, \dots, b_{n+1})$ so that $a_1 + b_1 = 1$ and $a_i = b_i$ for $i > 1$. We may add multiples of a_{n+1} to the other entries to arrange that moreover $\phi_0(\mathbf{a})$, $\phi_0(\mathbf{b})$ are defined. Now use that the ideals $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ are comaximal. □

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