Codimension-one Symplectic Foliations: Constructions and Examples

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Codimension-one Symplectic Foliations: Constructions and Examples

Symplectische Foliaties van Codimensie één: Constructies en Voorbeelden

(met een samenvatting in het Nederlands)

Proefschrift

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A mi familia

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Introduction

This thesis addresses the following question:

Which manifolds admit codimension-one symplectic foliations?

A codimension-one symplectic foliation is, intuitively, a decomposition of the manifold by hypersurfaces where each hypersurface is endowed with a symplectic structure and the symplectic structures vary smoothly from one hypersurface to the other. The symplectic structures form together a "foliated symplectic structure", called a *leafwise-symplectic structure*.

This question belongs to a general class of problems of existence of differential-geometric structures on manifolds. Solving these problems often gives some insight into the topology of the space where the structure lives, to name a few examples: closed manifolds with symplectic structures have non-zero second cohomology; manifolds with complete Riemannian metrics of positive Ricci curvature bounded below by positive constant are compact and have finite fundamental groups ([Mye41]), etc. In the case of codimension-one symplectic foliations, however, we believe that the only obstructions for its existence is the existence of symplectic hyperplane distribution.

These existence problems have been on the spotlight in the last years; partly because of the plethora of new geometric structures: generalised complex structures, folded symplectic structures, near-symplectic structures, log-symplectic structures; and partly because of the renewed attention that foliated structures have received: contact foliations ([CPP14]), symplectic foliations (see [FF12, MT13, Mit11]), holomorphic foliations ([MV02, MV11, Bru04]); each new structure of foliated structure bringing along a natural question of existence.

Why symplectic foliations? Foliations are sometimes too abundant (e.g. all manifolds with Euler characteristic zero have a codimension-one foliation [Thu76a]) and therefore, to be able to say something about them, it is necessary to endow them with an extra structure. Symplectic structures are a natural choice to endow foliations with due to its increasing significance in differential geometry, e.g., in the study of four-manifolds. Symplectic foliations are also relevant from the point of view of Poisson geometry, since they form an important class of "non-singular" Poisson structures.

Why codimension-one? Codimension-one foliations are the ones that are best understood. Furthermore, codimension-one symplectic foliations are related with contact structures: they both define symplectic codimension-one distributions, which has led to putting them in the same framework. In three dimensions, for example, the theory of confoliations ([ET98]) is such a framework. Moreover, the ideas used to make some constructions of contact structures can be used to construct codimension-one symplectic foliations. It is worth mentioning that contact structures are by nature less rigid than codimension-one symplectic foliations, and we expect that there are many more of the former than of the latter, but when it comes to the existence problem of both structures, we expect them to have a similar answer, namely that the only obstruction for the existence of such structures is the existence of a codimension-one distribution with a symplectic form on it.

The question of existence of codimension-one symplectic foliations can be split in two parts: a) Which manifolds admit codimension-one foliations? b) When does a given foliation admit a leafwise-symplectic structure? As mentioned before, the first question was answered by Thurston [Thu76a]: he proved that it is enough for a manifold to have zero Euler characteristic to admit such a foliation. The second question is hard. One of the most important attempts to answer this question was made by Bertelson [Ber00], who exhibited examples of foliations for which there is not a leafwise-symplectic structure, even though all the leaves admit symplectic structures and there is a leafwise non-degenerate two-form.

There are several features common to the problem of existence of symplectic foliations and to the one of symplectic structures. First of all, all symplectic foliations are locally isomorphic [Wei83], making the problem of existence of symplectic foliations a global problem. Second, there is a foliated version of Moser's lemma for symplectic foliations [HMS89] that helps study the deformations of symplectic foliations. However, the cohomology involved in the symplectic-foliations case is the foliated cohomology, which is more difficult to handle than the de Rham cohomology that appears in the symplectic case.

Furthermore, as it also happens in symplectic geometry, the problem of existence of symplectic foliations is very different in closed manifolds than in open manifolds (non-compact manifolds or manifolds with boundary). For open manifolds the problem has been fairly well understood using Gromov's h-principle for open manifolds. These h-principles for symplectic foliations were studied, among others, by [FF12, Ber02]. One of the most remarkable results is:

Theorem ([FF12]). *Given a foliation and a foliated non-degenerate two-form $(\mathcal{F}_0, \omega_0)$ on an open manifold, there is a path of foliations and leafwise non-degenerate two-forms $(\mathcal{F}_t, \omega_t)$, starting at $(\mathcal{F}_0, \omega_0)$ and ending at a symplectic foliation $(\mathcal{F}_1, \omega_1)$.*

There is no h-principle for closed manifolds and the problem of existence of codimension-one symplectic foliations on closed manifolds has still many basic questions unanswered. For many basic examples (such as S^7, $S^3 \times S^4$) it is not known if they admit a codimension-one symplectic foliation. The main goal of this thesis is to develop some methods to construct codimension-one symplectic foliations on closed manifolds.

<u>Outline of the content</u>

This thesis is divided into seven chapters. The first two chapters form the introductory part and give the basic definitions. The following two chapters develop the basic tools that will be used to construct codimension-one symplectic foliations. The following two apply these tools to construct these foliations in some interesting manifolds, and the last chapter studies the deformation of a certain type of Poisson structures.

More in detail, in Chapter 1, we give the main notions and results that will be used throughout the thesis: we start by giving the main definitions and results on foliations, foliated cohomology and symplectic foliations, and stating precisely the driving problem of the thesis (explained roughly above) and related problems. Then, as a simple example where the main problem can be solved very explicitly, we study invariant codimension-one symplectic foliations on Lie groups and construct invariant ones on five-dimensional nilpotent Lie groups.
After this we discuss Poisson geometry and Lie algebroids. A *Poisson structure* on a manifold is a generalised symplectic foliation and therefore Poisson geometry provides a natural framework to study symplectic foliations. A Poisson structure endows the cotangent bundle with the structure of a "generalised tangent bundle", also known as Lie algebroid structure. Lie algebroids provide then a useful framework to study Poisson geometry and they will be discussed in the following section. Lie algebroids will be specially useful when studying deformations of Poisson structures in Chapter 7.
In the last part of the chapter, we give some background material on maps between manifolds with boundary. This is of interest because we will often deal with maps between manifold with boundary when constructing codimension-one symplectic foliations on closed manifolds.

Our strategy for constructing codimension-one symplectic foliations will be the following: first, decompose the closed manifold into compact manifolds with boundary; second, construct a codimension-one symplectic foliation on each one of the pieces and third, glue the pieces together along their boundary. The gluing procedure is discussed in Chapter 2. We start this chapter introducing the notion of *tameness*: intuitively speaking, a leafwise symplectic form on a foliation is called *tame* if it does not change in a normal direction to the leaves. This notion has appeared in the literature in different forms and ensures that, when we glue them along the common boundary, the symplectic foliations glue smoothly. We discuss this in the last part of the chapter and prove some simple gluing results (Theorem 2.3.4 and Corollary 2.3.5) that are by now folklore in the theory of symplectic foliations.

In Chapter 3 we develop the main tool of this thesis to construct tame codimension-one symplectic foliations on manifolds with boundary. To do that we first discuss the notion of *cosymplectic structure* and define a *cosymplectic behaviour* of symplectic structures. A symplectic structure on a manifold with boundary M^{2n} is called of *cosymplectic type at the boundary* if there is a transversal direction to the boundary in which the symplectic structure does not change. Such a structure induces a cosymplectic structure on the boundary ∂M, namely a pair of closed forms $(\eta, \theta) \in \Omega^2(\partial M) \times \Omega^1(\partial M)$ such that $\eta^{n-1} \wedge \theta$ is a volume form. Using these notions, we will prove the Turbulisation Theorem, that roughly states:

Theorem (Symplectic Turbulisation, Thm 3.4.8)**.** *Let $f : M \to S^1$ be a submersion of a manifold with boundary such that $f|_{\partial M}$ is also a submersion. If there exists a closed two-form $\omega \in \Omega^2(M)$ symplectic on the fibres of f and displaying "a suitable" cosymplectic behaviour around ∂M, then M admits a codimension-one symplectic foliation tame at the boundary.*

Where by "a suitable" cosymplectic behaviour at the boundary it is meant, roughly, that there is no variation along a transversal direction to the boundary (Definition 3.4.1).

In Chapter 4 we discuss a class of Poisson structures related with symplectic structures of cosymplectic type. These structures are called *log-symplectic structures* and are Poisson structures on even dimensional manifolds M^{2n} which are invertible everywhere except along a hypersurface, where they drop rank only by two. These structures can be used to construct codimension-one symplectic foliations on manifolds of the form $M \times S^1$ (Theorem 4.2.2) and this process is related to the Turbulisation Theorem discussed in the previous chapter.

With all these tools, in the following two chapters we embark on the enterprise of constructing codimension-one symplectic foliations on closed five-dimensional manifolds. We start in Chapter 5, where we prove the following:

Theorem (Thm 5.0.6)**.** *If M is a closed oriented four-manifold that admits either a genuine fibration or an achiral Lefschetz fibration onto a two-dimensional manifold, then $M \times S^1$ admits a codimension-one symplectic foliation.*

Achiral Lefschetz fibrations are maps to S^2 with a special type of isolated singularities that can be seen as complex versions of Morse singularities. In that Chapter, we first use the given fibration to find a suitable decomposition of the closed manifold M into pieces M_i, $i = 1, \ldots, k$. Then we construct codimension-one symplectic foliations on each one of the pieces $M_i \times S^1$ using the Turbulisation Theorem and finally, since everything was done in a way compatible with the fibration, we check that the pieces can be glued together. At the end of the chapter we discuss a failed attempt to reproduce these techniques for *broken Lefschetz fibrations*, a more general type of fibration. While proving that this does not work, we prove a non-existence result for symplectic structures (Theorem 5.3.8) that is interesting on its own.

In Chapter 6, we discuss the construction of codimension-one symplectic foliations on *open book decompositions*. An open book decomposition on a manifold is a decomposition of the manifold into a codimension-two submanifold with trivial normal bundle, called the *binding*, and its complement, such that the complement has a submersion to S^1 that is compatible with the normal bundle of the binding. The closure of the generic fibres of the map to S^1 is called the *page* and in this section we will prove that manifolds admitting open book decompositions with good enough pages admit codimension-one symplectic foliations. In the second part of the chapter we use these ideas and the tools from Chapter 3 to construct a codimension-one symplectic foliation on S^5. Mitsumatsu [Mit11] constructed this structure on S^5 and part of the work in this thesis is inspired by his construction.

In the final chapter, Chapter 7, we depart from the main problem of constructing symplectic foliations to study log-symplectic structures, which appeared initially in Chapter 4 as structures that are useful to construct codimension-one symplectic foliations. The geometry of log-symplectic structures is interesting in itself, having alluring non trivial features but still being tractable and suitable for a thorough study. In this chapter we study the deformations of these structures and as the main result of the chapter, we characterise the space of Poisson structures near a log-symplectic structure, up to small diffeomorphisms (Theorem 7.3.1).

Chapter 1

Symplectic Foliations

In this chapter we give an overview of the main notions and fix the notation and conventions that will be used throughout this thesis. This chapter is split in three parts.

In the first part (the first three sections) we recall the notions of foliation, foliated cohomology and symplectic foliation, discuss the leading problems in the thesis and study, as an example, invariant codimension-one symplectic foliations on five-dimensional nilpotent Lie groups.

In the second part (the following two sections), symplectic foliations are discussed as a special type of Poisson structures and then Poisson structures are framed in the more general framework of Lie algebroids.

In the third part (the last two sections) we discuss some background material. First we go more in detail about symplectic fibrations, which we had started discussing in the first part of this chapter as an example of symplectic foliations. This will be useful in Chapter 5. Second, we discuss normal forms for maps between manifolds with boundary. This will be useful throughout the whole thesis, since we will have to deal with manifolds with boundary and maps between them when constructing symplectic foliations.

Conventions. Throughout the thesis, all manifolds and maps are smooth and fibrations are locally trivial submersions.

1.1 Foliations

In this section we recall the notions of foliation and of foliated cohomology. For a more detailed introduction to the theory of foliations, see for instance [Moe03].

Generalities

Definition 1.1.1. *A **foliation** $\mathcal{F}$ of dimension k on a manifold M^n is a decomposition of $M = \cup_x L_x$ into a union of disjoint connected immersed submanifolds of dimension k conforming to the following local model: for every point $x \in M$ there is an open neighborhood $U \subset M$ of x and local coordinates $(x_1, \ldots, x_n)$ on U such that, for each element of the decomposition L_y,* *each connected component of $L_y \cap U$ is described by $x_{k+1}, \cdots, x_n = const$. The* ***codimension*** *of $\mathcal{F}$ is $n-k$.*

For each $x \in M$, the submanifold of the decomposition that contains x is denoted by L_x and is called the *leaf* through x. The union of the tangent spaces T_xL_x for all $x \in M$ forms a distribution (subbundle) $T\mathcal{F} \subset TM$ of dimension k. The **normal bundle** of the foliation is the quotient bundle $\nu := TM/T\mathcal{F}$ and the **co-normal** bundle is the dual bundle ν^*, which is the same as the annihilator $\nu^* = \mathrm{Ann}(T\mathcal{F})$.

Definition 1.1.2. *A foliation is called* ***oriented*** *if the vector bundle $T\mathcal{F}$ is oriented and* ***co-oriented*** *if the normal bundle ν is oriented.*

The foliation $\mathcal{F}$ can be recovered from the distribution $T\mathcal{F}$: its leaves are the maximal integral submanifolds of $T\mathcal{F}$. The main property of distributions ξ arising from foliations is that through each point of M there passes an integral manifold of dimension equal to the rank of ξ, which is locally unique. Distributions $\xi \subset TM$ with this property are called **integrable**. Hence foliations on M can be identified with integrable distributions $\xi \subset TM$. The integrability condition may be difficult to check directly but fortunately, Frobenius theorem shows that it can be checked infinitesimally:

Theorem 1.1.3 (Frobenius)**.** *A subbundle $\xi \subset TM$ is integrable if and only if it is* ***involutive*** *in the sense that:*

$$[\Gamma(\xi), \Gamma(\xi)] \subset \Gamma(\xi),$$

i.e. the Lie bracket of any two vector fields tangent to ξ is tangent to ξ.

There is then a 1-1 correspondence

$$\{\text{foliations } \mathcal{F} \text{ on } M\} \overset{1-1}{\longleftrightarrow} \{\text{involutive sub-bundles } T\mathcal{F} \subset TM\},$$

and thus we will often identify the foliation $\mathcal{F}$ with the associated tangent bundle $T\mathcal{F}$.

The approach to foliations via sub-bundles of TM has a dual version in terms of T^*M. Given a distribution $\xi \subset TM$ of codimension k, one can consider its annihilator $\mathrm{Ann}(\xi) \subset T^*M$. The sections of $\mathrm{Ann}(\xi)$ generate a graded ideal $\mathcal{J}$ of the graded algebra $\Omega^\bullet(M)$. This ideal $\mathcal{J}$ has the property that for any $x \in M$, there is an open neighborhood such that $\mathcal{J}|_U$ is the ideal in $\Omega(M)|_U$ generated by k linearly independent one-forms. A graded ideal satisfying this last property is called a *locally trivial ideal* of rank k. Conversely, any locally trivial ideal of rank k defines a codimension-k distribution by taking the intersection of the kernels of the one-forms generating the ideal. An equivalent version of Frobenius theorem tells us when a distribution defined by an ideal comes from a foliation.

Theorem 1.1.4. *A distribution $\xi \subset TM$ is integrable if and only if its associated graded ideal $\mathcal{J} \subset \Omega^\bullet(M)$ satisfies:*

$$d(\mathcal{J}) \subset \mathcal{J}.$$

For a foliated manifold $(M, \mathcal{F})$, the associated graded ideal $\mathcal{J}$ is usually denoted by

$$\Omega^\bullet_{\mathcal{F}}(M) \subset \Omega^\bullet(M).$$

The best description of $\Omega^\bullet_{\mathcal{F}}(M)$ (and the fact that it must be closed under de Rham differential) arises when looking at the complex of **foliated forms**:

$$\Omega^\bullet(\mathcal{F}) := \Gamma(\Lambda^\bullet T^*\mathcal{F}).$$

Intuitively, $\Omega^\bullet(\mathcal{F})$ puts together, in a smooth fashion, the de Rham complexes $\Omega^\bullet(L)$ of all the leaves L of $\mathcal{F}$. These complexes fit into a short exact sequence:

$$0 \to \Omega^\bullet_{\mathcal{F}}(M) \to \Omega^\bullet(M) \xrightarrow{r} \Omega^\bullet(\mathcal{F}) \to 0, \tag{1.1}$$

where $\Omega^\bullet_{\mathcal{F}}(M)$ is the kernel of the restriction (from TM to $T\mathcal{F}$) map r.

Remark 1.1.5 (Associated cohomologies)**.** All these complexes give rise to several cohomologies associated to $(M, \mathcal{F})$:

- The **$\mathcal{F}$-relative cohomology**, denoted $H^\bullet_{\mathcal{F}}(M)$, is the cohomology associated to the sub-complex $(\Omega^\bullet_{\mathcal{F}}(M), d)$ of the de Rham complex of M.
- The **foliated cohomology**, denoted $H^\bullet(\mathcal{F})$, is the cohomology associated to the quotient complex $(\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}})$ of the de Rham complex of M.

Of course, $d_{\mathcal{F}}$ is just the leafwise de Rham differential and can be described explicitly by the usual Koszul formula:

$$\begin{aligned}(d_{\mathcal{F}}\alpha)(X_1 \dots, X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}\mathcal{L}_{X_i}(\alpha(X_1, .., \hat{X}_i, .., X_{k+1})) \\ &+ \sum_{i<j}(-1)^{i+j}\alpha([X_i, X_j], X_1, .., \hat{X}_i, .., \hat{X}_j, .., X_{k+1}),\end{aligned} \tag{1.2}$$

for any $\alpha \in \Omega^k(\mathcal{F})$ and any $(k+1)$-tuple $X_1, \dots, X_{k+1} \in \Gamma(T\mathcal{F})$.

Remarks 1.1.6. The short exact sequence of complexes (1.1) induces a long exact sequence in cohomology

$$\cdots \to H^k_{\mathcal{F}}(M) \to H^k(M) \to H^k(\mathcal{F}) \xrightarrow{\mathfrak{d}} H^{k+1}_{\mathcal{F}}(M) \to H^{k+1}(M) \to \cdots, \tag{1.3}$$

where the connecting homomorphism $\mathfrak{d} : H^\bullet(\mathcal{F}) \to H^{\bullet+1}_{\mathcal{F}}(M)$ is given by:

$$\mathfrak{d}([\alpha]) = [d\tilde{\alpha}],$$

where $\alpha \in \Omega^k(\mathcal{F})$ is a closed foliated form, $\tilde{\alpha} \in \Omega^k(M)$ is any extension of α (the form $d\tilde{\alpha}$ lies on $\Omega^{k+1}_{\mathcal{F}}(M)$ because α is $d_{\mathcal{F}}$-closed).

Example 1.1.7 (Simple foliations)**.** Any surjective submersion $\pi : M \to B$ with connected fibres induces a foliation $\mathcal{F}(\pi)$ on M whose leaves are the fibres of π. In this case $T\mathcal{F} = \ker d\pi$ and $\mathrm{Ann}(T\mathcal{F}) = \pi^*(T^*B)$. Foliations of this form are called **simple foliations**. With this in mind, foliations may be seen as generalizations of fibrations, where the "base space" is no longer smooth. For general foliations $\mathcal{F}$ on M, the role of the base space is played by the space $M/\mathcal{F}$ of leaves, endowed with the quotient topology. With this terminology, a foliation is simple if and only if the space of leaves is smooth, in the sense that it admits a smooth structure which makes the canonical projection $p : M \to M/\mathcal{F}$ a smooth submersion. Such a structure is unique, if it exists.

While, in general, foliated cohomology is difficult to compute, the situation is better in the case of simple foliations, at least when M is compact. In this case π is locally trivial. If it is a product $M = B \times F$, then $H^k(\mathcal{F}) = C^\infty(B, H^k(F))$. In general, the k-th cohomology groups of the fibres $M_b = \pi^{-1}(b)$ fit together into a vector bundle $\mathcal{H}^k$ over B:

$$\mathcal{H}^k_b = H^k(M_b).$$

With this, one has an identification

$$H^k(\mathcal{F}) \cong \Gamma(B, \mathcal{H}^k),$$

which identifies the cohomology class of a foliated form ω with the section

$$B \ni b \mapsto [\omega|_{M_b}] \in H^k(M_b) = \mathcal{H}^k_b.$$

The vector bundle $\mathcal{H}^k$ can be endowed with a natural flat connection. We discuss it in more detail in Section 5.1.

Codimension-one case

We will be mainly interested in foliations of codimension one and we will often make the simplifying assumption of co-orientability. The first interesting examples are related to one of the main questions in foliations theory (open until Thurston's work from 1976 [Thu76a]):

Former central problem: which compact manifolds admit codimension-one foliations?

There are versions of this problem for higher codimensions, or for open manifolds; and several of those problems are by now quite well understood. The reason that we restrict to the codimension-one case is that, when looking at the symplectic version of such problems (which is our main interest), even this "simpler case" is very far from being understood.

Example 1.1.8 (Closed one-forms and unimodular foliations)**.** Any nowhere-zero closed one-form

$$\alpha \in \Omega^1(M)$$

induces a foliation $\mathcal{F}_\alpha$ whose involutive sub-bundle is the kernel of α (involutivity follows from $d\alpha = 0$). The foliation $\mathcal{F}_\alpha$ is of codimension one and co-orientable since the normal bundle is actually trivialized by α.

The codimension-one, co-orientable foliations of this type are known under the name of **unimodular foliations**. They are not very far from simple foliations. Indeed, the main theorem of Tischler from [Tis70] says that:

Theorem 1.1.9 (Tischler)**.** *If a compact manifold M admits a unimodular (codimension-one, co-orientable) foliation, then M fibres over S^1.*

If we denote by θ the angle coordinate in S^1 and α the closed one-form defining the unimodular foliation, the proof from [Tis70] shows that the fibration $\pi : M \to S^1$ can be chosen so that $\ker \pi^*(d\theta)$ is arbitrarily close to $\ker \alpha$. Hence, the original foliation $\mathcal{F}_\alpha$ can be approximated by simple foliations $\mathcal{F}(\pi)$ and therefore any unimodular (codimension-one, co-orientable) foliation on a compact manifold can be approximated by a simple one.

Integrable one-forms and the modular class

In general, any codimension-one co-oriented foliation can be described as the kernel of a nowhere-zero one-form

$$\alpha \in \Omega^1(M).$$

A trivialisation of ν can be seen as a nowhere vanishing one-form on M, when combined with the projection map $TM \mapsto \nu$. In terms of α, the integrability condition arising from Frobenius' theorem reads

$$\alpha \wedge d\alpha = 0,$$

usually referred to as "α is integrable". This integrability condition is equivalent to

$$d\alpha = \alpha \wedge \beta \quad \text{for some } \beta \in \Omega^1(M).$$

For any such β we have that $d_{\mathcal{F}}(\beta|_{\mathcal{F}}) = 0$, since $d\beta \wedge \alpha = 0$. Hence $\beta|_{\mathcal{F}}$ defines a cohomology class in $H^1(\mathcal{F})$. A simple computation shows that this class depends only on the foliation, and not on the choice of α or β.

Definition 1.1.10. *The* ***modular class*** *of a codimension-one, co-orientable foliation $\mathcal{F}$ is the resulting cohomology class*

$$\mathrm{mod}_{\mathcal{F}} := [\beta|_{\mathcal{F}}] \in H^1(\mathcal{F}).$$

The notion of a modular class can be defined for foliations of higher codimension and also for non co-oriented foliations, see Definition 1.5.13. In our context, one obtains the following characterization of unimodularity (which is sometimes taken as definition):

Lemma 1.1.11. *The foliation $\mathcal{F}$ is unimodular if and only if* $\mathrm{mod}_{\mathcal{F}} = 0$.

Proof. Let us see the non-trivial implication. Assume the modular class vanishes, i.e., there is a one-form α defining the foliation, a one-form β satisfying $d\alpha = \alpha \wedge \beta$ and a function $f \in C^\infty(M)$ such that $\beta|_{\mathcal{F}} = df|_{\mathcal{F}}$. Then the one-form $e^f\alpha$ also defines the foliation $\mathcal{F}$ and $d(e^f\alpha) = e^f\alpha \wedge (\beta - df) = 0$. □

Some Examples

Example 1.1.12 (The Reeb foliation on S^3)**.** While foliation theory has roots in various studies of partial differential equations, it was considered a subject on his own when Reeb constructed a codimension-one foliation on S^3. This is the first non-trivial example of a codimension-one foliation (since S^3 does not fibre over S^1, the existence of such a foliation on S^3 came as a surprise at that time). This foliation arises from the standard decomposition of S^3 into two solid tori; in turn, each of the (closed) solid tori $S^1 \times D^2$ is foliated by the boundary torus $S^1 \times S^1$ and by interior planes that spiral towards the boundary $S^1 \times S^1$. More details will be given later on. It is worth keeping in mind that the "spiraling" is a general phenomena/construction, known under the name of **turbulisation**, which is extremely useful in building new examples. The precise construction will be discussed in Section 3.3. Moreover, one of the main tools of this thesis is precisely a generalization of this construction to the setting of symplectic foliations (Chapter 3).

Example 1.1.13 (Three-dimensional manifolds)**.** The existence of codimension-one foliations on all three-dimensional compact manifolds was established by Lickorish [Lic65] (in the orientable case) and Wood [Woo69] (general case). Their approach is to decompose a three-dimensional compact manifold into various pieces (almost all of them solid tori) on which turbulisation can be performed. The turbulisation procedure was clearly stated (e.g. Lemma 2.3 in [Woo69]). Simplifications of the argument (based on improvements on the type of decompositions used) were given later on by Lawson [Law74] who noticed that it is enough to start with any "Alexander decomposition". Such decompositions (in arbitrary dimensions) are well-known nowadays under the name of open book decompositions and they will enter this thesis in Chapter 6. It is interesting to see that, for three-manifolds, the existence of codimension-one foliations went hand in hand with the existence of contact structures, e.g. in [TW75], Thurston and Winkelnkemper remarked that both structures can be constructed in a very similar manner. These later evolved into the notion of confoliation of Eliashberg and Thurston [ET98] which provides the framework within which the interaction of these two types of structures can be studied.

Example 1.1.14 (The Lawson foliation on S^5)**.** The next interesting example of codimension-one foliation is Lawson's foliation on S^5 [Law71]. First of all, Lawson describes the general process of turbulisation for all manifolds that fibre over a circle (Lemma 1 in [Law71]). Then, his example on S^5 is very similar to Reeb's in the sense that S^5 is divided into two pieces, each one of them being foliated by turbulisation. The two pieces that Lawson uses arise from the study of the singularity

$$p : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0), \quad (z_1, z_2, z_3) \mapsto z_1^3 + z_2^3 + z_3^3.$$

In a little more detail: the intersection of S^5 with the zero-locus of p is a smooth manifold B, with trivial normal bundle, hence with a tubular neighborhood diffeomorphic to $B \times D^2$. This is one of the pieces. The other piece is the closure of its complement, denoted by C. The piece that is easier to handle is C, due to Milnor's fibration theorem, which gives rise to a fibration of C over S^1 (namely $p/|p|$). The resulting foliation on C can then be turbulised to a foliation which has the boundary as a leaf. The specific form of p becomes important when looking at the tube piece $B \times D^2$; indeed, it follows that B fibres even over the 2-torus and this gives rise, in particular, to a fibration of $B \times D^2$ over S^1 that can be turbulised. Full details of this construction, in the more general context of symplectic foliations, will be given in Chapter 6.

Lawson also remarks that Reeb's example can be obtained similarly, by using the polynomial $p = z_1 + z_2$. Later on, Lawson [Law74] remarks that the usefulness of the Milnor fibrations comes from the fact that it gives rise to specific open book decompositions (cf. Example 1.1.13).

Example 1.1.15 (Higher dimensional spheres)**.** Lawson used his techniques to prove that all spheres of dimensions $n = 2^k + 1$ or $n = 2^k + 3$ with $k \geq 1$ admit codimension-one foliations. In order to handle all odd-dimensional spheres using these foliations and a complete induction argument, there is an extra step needed regarding the existence of certain open book decompositions on spheres of type S^{4m+1}. This step was solved by Durfee and Tamura, see the overview [Law74]). At the time of the overview it was actually believed that, on an arbitrary compact manifold, the Euler characteristic was the only obstruction to the existence of codimension-one foliations. While this may have sounded too optimistic at that time, it was soon afterwards proven to be true:

Theorem 1.1.16 (Thurston [Thu76a]). *A compact manifold admits a codimension-one foliation if and only if it has zero Euler characteristic.*

The (transversal) variation of leafwise forms

In the next section we will deal with foliated two-forms that are symplectic along the leaves; interpreting them as families of leafwise-symplectic forms, we would like to give a precise meaning to "their variation in the direction transversal to the leaves". This can be carried out for all leafwise closed k-forms on a given foliated manifold $(M, \mathcal{F})$; the key point is the presence of a canonical variation map, or "transversal differentiation",

$$d_\nu : H^k(\mathcal{F}) \to H^k(\mathcal{F}, \nu^*).$$

In this section we describe the map d_ν.

Foliated cohomology with coefficients

To describe the map d_ν in general, let us start by recalling the definition of the foliated cohomology groups with coefficients in the conormal bundle, $H^k(\mathcal{F}, \nu^*)$. The starting point is the so-called **Bott connection** on the normal bundle ν of the foliation; it is a partial connection, in the sense that it is defined only along vectors tangent to the foliation (a $T\mathcal{F}$-connection); explicitly, it is the operation defined by

$$\nabla : \Gamma(T\mathcal{F}) \times \Gamma(\nu) \to \Gamma(\nu), \qquad \nabla_X \overline{N} = \overline{[X, N]}, \ \ \forall N \in \mathfrak{X}(M), \ X \in \Gamma(T\mathcal{F}),$$

where $\overline{N}$ is the class of N modulo $T\mathcal{F}$. This operation is $C^\infty(M)$-linear in the first component, which makes ∇ a genuine partial connection. This connection induces a similar (dual) $T\mathcal{F}$-connection ∇^* on ν^* by the Leibniz-type formula

$$\mathcal{L}_X(\alpha(\overline{N})) = (\nabla^*_X \alpha)(\overline{N}) + \alpha(\nabla_X \overline{N}).$$

This connection induces a differential $d_{\mathcal{F}}$ defined on $\Omega^\bullet(\mathcal{F}, \nu^*)$ (whose elements are foliated forms with values in the conormal bundle), whose formula is similar to the one defined in Equation (1.2): for every $\alpha \in \Omega^k(\mathcal{F}, \nu^*)$ and every $X_1, \dots, X_{k+1} \in \Gamma(T\mathcal{F})$,

$$\begin{aligned}(d_{\mathcal{F}}(\alpha))(X_1 \dots, X_{k+1}) = &\sum_{i=1}^{k+1} (-1)^{i+1} \nabla^*_{X_i}(\alpha(X_1, .., \hat{X}_i, .., X_{k+1})) \\ &+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_1, .., \hat{X}_i, .., \hat{X}_j, .., X_{k+1}).\end{aligned} \tag{1.4}$$

$H^\bullet(\mathcal{F}, \nu^*)$ is defined as the cohomology of the complex $(\Omega^\bullet(\mathcal{F}, \nu^*), d_{\mathcal{F}})$. Note that, by the usual Leibniz-rule identities, the Bott connection induces similar connections on $\wedge^k \nu^*$ for all $k \in \mathbb{Z}_{\geq 0}$, giving rise to cohomologies $H^\bullet(\mathcal{F}, \wedge^k \nu^*)$.

Example 1.1.17 (Simple Foliations). Before we explain the map d_ν (and the cohomology $H^k(\mathcal{F}, \nu^*)$), it is instructive to have in mind the case of simple foliations $\mathcal{F}(\pi)$ induced by a submersion $\pi : M \to B$ as in Example 1.1.7. We assume that M is compact. If it is a product bundle $M = B \times F$ (with the first projection),

$$H^k(\mathcal{F}) = C^\infty(B, H^k(F)),\ \ H^k(\mathcal{F}, \nu^*) = \Omega^1(B, H^k(F))$$

and d_ν is the usual de Rham differential (with coefficients in the finite dimensional vector space $H^k(F)$). When M is not a product, we use the flat vector bundle $\mathcal{H}^k$ over B made of the k-th homology groups of the fibres (see Example 1.1.7). Similar to the identification of $H^k(\mathcal{F})$ with the space of sections of $\mathcal{H}^k$, $H^k(\mathcal{F}, \nu^*)$ can be identified with the space of sections of $\mathcal{H}^k \otimes T^*B$, hence

$$H^k(\mathcal{F}) \simeq \Omega^0(B, \mathcal{H}^k), \quad H^k(\mathcal{F}, \nu^*) \simeq \Omega^1(B, \mathcal{H}^k).$$

Remark 1.1.18 (The relationship with the relative cohomology)**.** The cohomology $H^\bullet(\mathcal{F}, \nu^*)$ is closely related to the relative cohomology $H^\bullet_{\mathcal{F}}(M)$ from Remark 1.1.5. Indeed, there is an obvious map $p : \Omega^{k+1}_{\mathcal{F}}(M) \longrightarrow \Omega^k(\mathcal{F}, \nu^*)$ characterised by

$$p(\alpha)(X_1, \ldots, X_k)(\overline{N}) = \alpha(X_1, \ldots, X_k, N)$$

for any k-tuple of vector fields $X_1, \ldots, X_k \in \Gamma(T\mathcal{F})$ and any vector field $N \in \mathfrak{X}(M)$. A direct computation shows that $d_{\mathcal{F}} \circ p = p \circ d$. In particular, there is a canonical map

$$p : H^{k+1}_{\mathcal{F}}(M) \to H^k(\mathcal{F}, \nu^*).$$

The variation map d_ν

The desired map d_ν is now the composition of the map p from the previous remark with the boundary map $\mathfrak{d}$ from the long exact sequence (1.3):

$$H^k(\mathcal{F}) \xrightarrow{\mathfrak{d}} H^{k+1}_{\mathcal{F}}(M) \xrightarrow{p} H^k(\mathcal{F}, \nu^*). \tag{1.5}$$

Explicitly, for any closed foliated form $\alpha \in \Omega^k(\mathcal{F})$, choose any extension $\tilde{\alpha} \in \Omega^k(M)$ of α and note that $d\tilde{\alpha} \in \Omega^{k+1}(M)$ lies in $\Omega^{k+1}_{\mathcal{F}}(M)$. Then,

$$d_\nu[\alpha] = [p(d\tilde{\alpha})]. \tag{1.6}$$

1.2 Symplectic Foliations and the Leading Problem

Roughly speaking, a symplectic foliation is a foliation together with symplectic structures on its leaves, which "vary smoothly" from leaf to leaf. The symplectic structures on the leaves fit together into a section of $\Lambda^2 T^*\mathcal{F}$, and the "smooth variation from leaf to leaf" means that the resulting section is smooth:

$$\omega \in \Gamma(\Lambda^2 T^*\mathcal{F}) = \Omega^2(\mathcal{F}).$$

Definition 1.2.1. *A* ***symplectic foliation*** *on M is a foliation $\mathcal{F}$ together with a* ***leafwise-symplectic form*** *ω along the leaves of $\mathcal{F}$, i.e. a foliated two-form $\omega \in \Omega^2(\mathcal{F})$ that is non-degenerate on the leaves and such that $d_{\mathcal{F}}\omega = 0$.*

An *isomorphism* between the symplectic foliations $(M_1, \mathcal{F}_1, \omega_1)$ and $(M_2, \mathcal{F}_2, \omega_2)$ is a diffeomorphism $\varphi : M_1 \xrightarrow{\sim} M_2$ such that $\varphi^{-1}(\mathcal{F}_2) = \mathcal{F}_1$ and $\varphi^*(\omega_2) = \omega_1$.

Although some of the notions and constructions that we will present are valid for arbitrary symplectic foliations, we will be interested mainly in the codimension-one case. As in the case of ordinary foliations, we will often make the simplifying assumption that the symplectic foliations are co-orientable. However, since the leafwise-symplectic forms induce an orientation on $T\mathcal{F}$, the co-orientability of $\mathcal{F}$ is equivalent to the orientability of M. Hence we will concentrate on orientable manifolds. Therefore, the main question underlying this thesis:

Leading problem: which compact (orientable) manifolds admit codimension-one symplectic foliations?

We focus on compact manifolds because the h-principle techniques ([Gro86]) that allow us to see if an open manifold admits a codimension-one foliation or a symplectic structure have been extended to the setting of symplectic foliations [FF12] on open manifolds.

Remark 1.2.2. While a codimension-one, co-oriented foliation $\mathcal{F}$ can be represented by the kernel of a nowhere-vanishing one-form α satisfying the integrability condition $\alpha \wedge d\alpha = 0$, codimension-one symplectic foliations $(\mathcal{F}, \omega_{\mathcal{F}})$ can be represented by

$$\text{pairs } (\alpha, \omega), \quad \text{where } \ \alpha \in \Omega^1(M),\ \omega \in \Omega^2(M),$$

satisfying the conditions $\mathcal{F} = \ker(\alpha)$, $\omega_{\mathcal{F}} = \omega|_{\mathcal{F}}$. The integrability conditions that ensure that $(\mathcal{F}, \omega_{\mathcal{F}})$ is a symplectic foliation become:

$$\alpha \wedge d\alpha = 0, \qquad d\omega \wedge \alpha = 0, \qquad \omega^n \wedge \alpha \neq 0, \tag{1.7}$$

where $\dim(M) = 2n + 1$.

Remark 1.2.3. A symplectic foliation on an n-dimensional manifold M can be described locally as follows: the manifold locally looks like $\mathbb{R}^n = \mathbb{R}^{n-q} \times \mathbb{R}^q$, with the leaves of the foliation given by $\mathbb{R}^{n-q} \times \{t\}$, $t \in \mathbb{R}^q$, and the leafwise-symplectic structure given by a family of symplectic structures $\omega_t \in \Omega^2(\mathbb{R}^{n-q})$, $t \in \mathbb{R}^q$. The symplectic structures do not have to be symplectomorphic. Indeed, the variation in the t-direction will be described by the variation (see Definition 1.2.14).

Some examples and a second central problem

Example 1.2.4 (Three-dimensional manifolds)**.** The leading problem is not so interesting in the three-dimensional case because the leaves will be two-dimensional hence a symplectic form on a leaf is just a volume form. So, in that case, the problem is about the existence of orientable codimension-one foliations. Thurston's work mentioned above implies then that all oriented three-dimensional manifolds admit codimension-one symplectic foliations.

Example 1.2.5 (Product five-dimensional manifolds)**.** The leading problem becomes much harder in higher dimensions. This is related to the fact that, in dimensions ≥ 4, the question of whether a manifold admits a symplectic structure is a very difficult one. Even more, the simple cohomological obstructions to the existence of symplectic structures, such as the non-vanishing of the second de Rham cohomology group, is lost when passing to symplectic foliations.

Searching for interesting examples in dimension five, it is natural to first start with the simplest case, that of product manifolds:

- For a product $M^2 \times N^3$ of a two and a three-dimensional (orientable) manifold, the answer is clearly positive because of the previous example.

- However, for products $M^4 \times S^1$ of an oriented four-dimensional manifold M with a circle, the situation is far from clear unless M^4 is symplectic. Hence already the case $S^4 \times S^1$ does not have an obvious answer. However, with the techniques that we will develop in Chapter 5, we will be able to prove that many manifolds of type $M^4 \times S^1$ (including $S^4 \times S^1$) do admit codimension-one symplectic foliations; even more, we think that our methods can be used to prove such a result for all oriented four-dimensional manifolds M.

Example 1.2.6 (The five-sphere)**.** Moving away from products, the simplest case to consider is that of S^5. While S^5 admits a codimension-one foliation (the Lawson foliation mentioned in the previous section), the question of whether this foliation can be made into a symplectic one is rather non-trivial; the answer is positive, as was proven recently by Mitsumatsu [Mit11].

As an application of the methods we will develop in this thesis, in Chapter 6, we will present a new proof of the existence of a codimension-one symplectic foliation on S^5.

Example 1.2.7 (Higher dimensional spheres)**.** Lawson [Law71] constructed codimension-one foliations on the spheres S^{2^k+3}, $k \geq 1$ (recall Example 1.1.15). In the case of S^3 and S^5 the foliations coincide with the foliations described in the previous examples and therefore, we know that Lawson's foliations on S^3 and S^5 admit leafwise-symplectic structures. We will see in the next proposition that the foliations on higher dimensional spheres do not admit such structures. The only piece of information we use of Lawson's foliations is that the foliation on S^{2^k+3} for $k > 1$ has a compact leaf diffeomorphic to $(SO(n+1)/SO(n-1)) \times S^1$, where $2n+1 = 2^k+3$. It is worth mentioning that $SO(n+1)/SO(n-1)$ can be identified with the unit tangent bundle of S^n by noting that $SO(n+1)$ acts transitively with stabiliser $SO(n-1)$.

Proposition 1.2.8. *For $k > 1$, Lawson's foliation of S^{2^k+3} given in [Law71] does not admit a leafwise-symplectic structure.*

The proof of this proposition relies on the following observation regarding the cohomological obstructions for closed product manifolds to admit symplectic structures

Lemma 1.2.9. *Let $M = A \times B$ be a closed $2n$ dimensional product manifold with $\dim A \geq 1$ and $\dim B \geq 1$. Assume $H^2(B) = \{0\}$. Then, if either $H^1(A) = \{0\}$ or $H^1(B) = \{0\}$ or $\dim A \leq 2n-2$, then M does not admit symplectic structures.*

Proof. We use the Künneth formula to express de Rham cohomology of M in terms of the ones of A and B. This formula in degree 2 reads:

$$H^2(M) \simeq (H^2(A) \otimes H^0(B)) \oplus (H^1(A) \otimes H^1(B)) \oplus (H^0(A) \otimes H^2(B)).$$

Since $H^2(B)$ is trivial, the cohomology class of a closed two-form ω on M can be decomposed as $[\omega] = [\omega_A + \eta_A \wedge \eta_B]$, with ω_A a closed two-form on A and η_A, η_B closed one-forms on A and B respectively. Taking the n-th power of the equation, we get $[\wedge^n \omega] = [\wedge^n \omega_A + n(\wedge^{n-1}\omega_A) \wedge \eta_A \wedge \eta_B]$. Now, $\wedge^n \omega_A = 0$ since $\dim A < 2n$. On the other hand the second term always vanishes since either $H^1(A) = \{0\}$ and $[\eta_A] = 0$ or $H^1(B) = \{0\}$ and $[\eta_B] = 0$ or $\dim A < 2n - 2$ and $\wedge^{n-1}\omega_A \wedge \eta_A = 0$. In any case we get $[\wedge^n \omega] = 0$. If ω were symplectic, then $\wedge^n \omega$ would be a volume form, but closed manifolds do not admit exact volume forms. □

Proof. Proposition 1.2.8. Let $2n + 1 = 2^k + 3$. As we mentioned before, Lawson's foliation on S^{2^k+3} has a compact leaf diffeomorphic to $SO(n+1)/SO(n-1) \times S^1$. We prove here that this compact leaf does not admit symplectic structures. To do that, we prove that $H^1(SO(n+1)/SO(n-1)) = \{0\}$ and use the previous lemma with $A = SO(n+1)/SO(n-1)$ and $B = S^1$. Note that $SO(n+1)$ is a $SO(n-1)$ bundle over A. The last part of the long exact sequence in homotopy groups of the $SO(n-1)$-bundle $SO(n+1) \to A$ reads

$$\cdots \to \pi_2(A) \to \pi_1(SO(n-1)) \to \pi_1(SO(n+1)) \to \pi_1(A) \to 0.$$

Since $k > 1$ implies $n = 2^{k-1} + 1 > 3$, we have $\pi_1(SO(n+1)) = \mathbb{Z}_2$ which in turn implies that $\pi_1(A)$ is finite. $H_1(A, \mathbb{Z})$ is then finite and therefore $H^1(A) = H^1(A, \mathbb{R}) = \{0\}$. □

Example 1.2.10 (Five-dimensional nilmanifolds)**.** Nilmanifolds form another interesting class of compact manifolds. Because of their simplicity, they are usually the first step when studying geometric structures in homogeneous spaces (see [Kut14, CG04]). Given the classification of five-dimensional nilpotent Lie algebras [Gra07], one can check directly that all five-dimensional nilpotent Lie groups admit invariant codimension-one symplectic foliations; in turn, this implies that all five-dimensional nilmanifolds admit such symplectic foliations. A detailed description of this will be given separately in Section 1.3.

Example 1.2.11 (Fibrewise-symplectic fibrations)**.** Let us first consider symplectic foliations $(\mathcal{F}, \omega_\mathcal{F})$ for which the foliation $\mathcal{F}$ is simple, i.e. induced by a submersion $\pi : M \to B$ as in Example 1.1.7.

Definition 1.2.12. *A **fibrewise-symplectic fibration** $\xi = (\pi : M \to B, \omega_\mathcal{F})$ consists of a fibration $\pi : M \to B$ together with a fibrewise symplectic form $\omega_\mathcal{F}$. Hence making the foliation by the connected components of the fibres of π into a symplectic foliation.*

An *isomorphism* between fibrewise-symplectic fibrations $(M_i \to B, \omega_{\mathcal{F}_i})$, $i = 1, 2$, is a diffeomorphism $\varphi : M_1 \to M_2$ which is a bundle map and such that $\varphi^*(\omega_{\mathcal{F}_2}) = \omega_{\mathcal{F}_1}$. Also, there is a notion of trivial fibrewise-symplectic fibration: any symplectic manifold (F, ω) induces one over M, namely

$$M \times (F, \omega) \xrightarrow{\mathrm{pr}_1} M,$$

with the fibrewise symplectic structure constant equal to ω.

Note however that, for general fibrewise-symplectic fibrations, there is no local triviality and it may even happen that the fibres are not symplectomorphic. For instance, the first projection

$$\mathrm{pr}_1 : S^1 \times S^2 \times S^2 \to S^1,$$

endowed with the symplectic forms

$$\omega_\theta = (\cos(\theta) + 2)\mathrm{pr}_2^*(\sigma) + (\sin(\theta) + 2)\mathrm{pr}_3^*(\sigma)$$

(one for each fibre above $e^{i\theta} \in S^1$, and where σ is the area form on the 2-sphere) is such an example.

Example 1.2.13 (Symplectic fibrations)**.** Intuitively, a symplectic fibration is a fibrewise-symplectic fibration with "constant symplectic forms on the fibres". This can be made precise in several ways. Two of the conditions that can be found in the literature are:

- **Strict symplectic fibration**: a fibrewise-symplectic fibration $\xi = (M \xrightarrow{\pi} B, \eta)$ which is locally trivial: for every point $x \in B$, there is an open neighbourhood U of x such that $\xi|_U$ is isomorphic to a trivial fibrewise-symplectic fibration.

- **Symplectic fibrations**: a fibrewise-symplectic fibration $(M \xrightarrow{\pi} B, \eta)$ for which there is a two-form $\tilde{\eta}$ on M extending η (i.e, such that $\tilde{\eta}|_{\mathcal{F}} = \eta$), for which

$$\iota_{v_1 \wedge v_2} d\tilde{\eta} = 0 \tag{1.8}$$

  for all vector fields v_1, v_2 tangent to the fibres.

The first notion is the most natural one, at least at a first look. However,

- while the two notions are equivalent when M is compact (see Section 1.6), the second definition is better behaved in the general case. For instance, while [GLS96] starts with what we call strict symplectic fibrations, the definition that is ultimately adopted there (Definition 1.2.6 in *loc.cit*) is precisely our notion of symplectic fibration.

- as we will explain next, from the point of view of symplectic foliations, the most natural notion is, after all, the second one.

The last class of examples brings us to the notion of variation of leafwise-symplectic forms. For general symplectic foliations $(\mathcal{F}, \omega_{\mathcal{F}})$, the tool that we have available for measuring transversal variation is the transversal differentiation

$$d_\nu : H^2(\mathcal{F}) \to H^2(\mathcal{F}, \nu^*)$$

which was discussed in full generality at the end of the previous section.

Definition 1.2.14. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M. The* ***variation*** *of ω is the cohomology class*

$$\mathrm{var}_\omega := d_\nu[\omega] \in H^2(\mathcal{F}, \nu^*).$$

The symplectic foliation $(\mathcal{F}, \omega)$ is called ***tame*** *if $\mathrm{var}_\omega = 0$.*

The apparently strange condition for symplectic fibrations from the previous example (Equation (1.8) now comes to the surface:

Proposition 1.2.15. *A symplectic foliation $(M, \mathcal{F}, \omega)$ is tame if and only if there exists an extension of ω, $\tilde{\omega} \in \Omega^2(M)$, such that $i_{X\wedge Y} d\tilde{\omega} = 0$ for all $X, Y \in \Gamma(T\mathcal{F})$.*

Proof. Applying the definition of d_ν (see Equation (1.6)), we see that $\mathrm{var}_\omega = 0$ if and only if for any extension $\tilde{\omega} \in \Omega^2(M)$ of ω, there exists $\alpha \in \Omega^1(\mathcal{F}, \nu^*)$ such that

$$p(d\tilde{\omega}) = d^*_{\mathcal{F}}\alpha. \tag{1.9}$$

Choose a splitting σ of the sequence

$$\begin{tikzcd} 0 \arrow[r] & T\mathcal{F} \arrow[r] & TM \arrow[r] & \nu \arrow[l, bend right, "\sigma"'] \arrow[r] & 0 \end{tikzcd}$$

to decompose $TM = T\mathcal{F} \oplus \sigma(\nu)$. Define $\tilde{\alpha} \in \Omega^2(M)$ as

$$\tilde{\alpha}(X, Y) = 0, \quad \tilde{\alpha}(X, Z) = \alpha(X)(\overline{Z}) = -\tilde{\alpha}(Z, X), \quad \tilde{\alpha}(Z, W) = 0$$

for any $X, Y \in \Gamma(T\mathcal{F})$, $Z, W \in \Gamma(\sigma(\nu))$, where $\overline{Z}$ denotes the image of Z by the projection $TM \to \nu$. We claim that $\tilde{\omega}' := \tilde{\omega} - \tilde{\alpha}$ is an extension with the desired property. We fix $X, Y \in \Gamma(T\mathcal{F})$ and we check that $\iota_{X\wedge Y} d\tilde{\omega}' = 0$. First of all, the one-form $\iota_{X\wedge Y} d\tilde{\omega}'$ vanishes on vector tangent to $\mathcal{F}$ because $\tilde{\alpha}$ vanishes when restricted to the leaves (and hence, so does $d\tilde{\alpha}$) and ω is closed along the leaves. Secondly, for $Z \in \Gamma(\sigma(\nu))$, we have:

$$\begin{aligned} d\tilde{\omega}(X, Y, Z) &= (d^*_{\mathcal{F}}\alpha)(X, Y)(\overline{Z}) \\ &= \mathcal{L}_X(\alpha(Y)(\overline{Z})) - \alpha(Y)(\overline{[X, Z]}) - \mathcal{L}_Y(\alpha(X)(\overline{Z})) + \alpha(X)(\overline{[Y, Z]}) - \alpha([X, Y])(\overline{Z}) \\ &= (d\tilde{\alpha})(X, Y, Z). \end{aligned}$$
□

Corollary 1.2.16. *A fibrewise-symplectic fibration with connected fibres is tame if and only if it is a symplectic fibration.*

Hence, intuitively, while foliations can be interpreted as "fibrations over a non-smooth space", the tame symplectic foliations correspond to "symplectic fibrations over a non-smooth space". Hence, similar to the leading problem mentioned above, we have the following:

Another central problem: which compact (orientable) manifolds admit codimension-one tame symplectic foliations?

This question was studied, among others, in [GMP11], where a cohomological characterisation of a family of such manifolds is given.

In the codimension-one case, we will see in the next chapter (see Proposition 2.1.6) that for dimensional reasons:

Corollary 1.2.17. *A codimension-one symplectic foliation $(M, \mathcal{F}, \omega)$ is tame if and only if the foliated form admits a closed extension $\tilde{\omega} \in \Omega^2(M)$.*

Example 1.2.18 (Cosymplectic structures)**.** Among pairs (α, ω) consisting of a one-form and a two-form on a manifold M satisfying the integrability conditions (1.7); a distinguished class that received special attention in the literature consists of those with the property that both

α as well as ω are closed. Such pairs are called **cosymplectic structures** on M. Hence all manifolds that admit a cosymplectic structure also admit a tame codimension-one symplectic foliation.

However, manifolds that admit cosymplectic structures are very rare. Of course, the total space of any compact symplectic fibration $\pi : M \to S^1$ admits a cosymplectic structure (take any closed extension as in the previous corollary and the pull-back of the one-form $d\phi$). However, from Tischler's theorem (Theorem 1.1.9), one deduces that any manifold that admits a cosymplectic structure must fibre symplectically over S^1 (see Proposition 3.2.13). Cosymplectic structures will be discussed in detail in Chapter 3.

Note that, since on compact manifolds symplectic forms can never be exact, the existence of a tame symplectic foliation on a compact manifold M with $H^2(M) = 0$ implies that all the leaves are non-compact. Therefore, using Novikov's theorem (which states that foliations on simply connected three-manifolds are bound to have compact leaves) we deduce:

Proposition 1.2.19. *An orientable three-manifold M with $\pi_1(M)$ finite (and hence with $H^2(M) = 0$) does not admit a tame codimension-one symplectic foliation. In particular, S^3 does not admit a tame codimension-one symplectic foliation.*

We expect this proposition to be just one in a long list of examples that do not admit tame codimension-one symplectic foliations. However, even for S^5, such a result seems hard to prove. To start with, there is no analogue of Novikov's theorem in higher dimensions and S^5 admits codimension-one foliations with non-compact leaves [Mei12]. In our view, any progress on the last "central problem" should first settle the following:

Question: Does S^5 admit a codimension-one tame symplectic foliation?

1.3 Example: Nilpotent Lie Groups and Nilmanifolds

The aim of this section is to construct left-invariant codimension-one symplectic foliations on five-dimensional nilpotent Lie groups and nilmanifolds (cf. [Kut14]). The purpose of the section is to illustrate the notions of symplectic foliations in some simple, yet interesting type of manifolds and the results from this section will not be used in the rest of the thesis.

Notation. Throughout this section, N denotes a Lie group. The identity element is denoted e and the Lie algebra by $\mathfrak{n}$. A basis of the Lie algebra $\mathfrak{n}$ is denoted by $\{X_1, \dots, X_n\}$, and the basis for the dual differential graded algebra $\mathfrak{n}^*$ by $\{x_1, \dots, x_n\}$. The structural constants for the Lie algebra are defined by $[X_i, X_j] = \sum_k c_{ij}^k X_k$.

Recall that a *nilpotent (Lie) group* N is a (Lie) group for which the lower central series,

$$N \trianglerighteq [N, N] \trianglerighteq [[N, N], N] \trianglerighteq \cdots ,$$

has finite length. For connected and simply connected nilpotent Lie groups, the exponential map is a diffeomorphism (see e.g. [Kna03]), so every simply connected nilpotent Lie group is diffeomorphic to $\mathbb{R}^n$.

The Lie algebra of a nilpotent Lie group N is a *nilpotent Lie algebra* $\mathfrak{n}$, i.e, one for which the lower central series,

$$\mathfrak{n} \trianglerighteq [\mathfrak{n}, \mathfrak{n}] \trianglerighteq [[\mathfrak{n}, \mathfrak{n}], \mathfrak{n}] \trianglerighteq \cdots ,$$

has finite length.

Codimension-one symplectic foliations on Lie groups

Before discussing the case of nilpotent Lie groups, let us deal with arbitrary Lie groups and see how codimension-one symplectic foliations on Lie groups are reflected in their Lie algebras.

Let N be a Lie group (not necessarily nilpotent) and consider the left N-action on N given by left translation. An *invariant* form on N is a form invariant under this action. Since the de Rham differential of an invariant form is also invariant, the invariant forms on N form a differential subcomplex $(\Omega^\bullet_{\text{inv}}(N), d_{dR})$ of the de Rham complex.

On the other hand, recall that there is a differential d on $\wedge^\bullet \mathfrak{n}^*$ coming from the Lie algebra structure on $\mathfrak{n}$ that makes $(\wedge^\bullet \mathfrak{n}^*, d)$ into a differential graded algebra. The differential is defined by its action on the generators as follows:

$$dx_i = \sum_{j<k} c^i_{jk} x_j \cdot x_k = \sum_{j<k} c^i_{jk} x_{jk}, \tag{1.10}$$

where $\cdot$ denotes the wedge product on $\wedge^\bullet \mathfrak{n}^*$ and we use the notation $x_{ij} := x_i \cdot x_j$.

It is well known that the complexes $(\Omega^\bullet_{\text{inv}}(M), d_{dR})$ and $(\wedge^\bullet \mathfrak{n}^*, d)$ are isomorphic (see for instance [Kna03]):

Lemma 1.3.1. *The map* $\text{ev}_e : (\Omega^\bullet_{\text{inv}}(N), d_{dR}) \to (\wedge^\bullet \mathfrak{n}^*, d)$*, that maps an invariant form* α *to its value at* $e \in N$*, is an isomorphism of complexes.*

An *invariant* codimension-one symplectic foliations on a Lie group N is a codimension-one symplectic foliations where both the foliation and the leafwise-symplectic form are invariant under the left-action. We can apply this lemma to the pair of forms defining a codimension-one symplectic foliation (see Remark 1.2.2) to transfer them to the Lie algebra, as follows:

Lemma 1.3.2. *Let* N *be a Lie group of dimension* $2n+1$*. Then*

i. Any pair $(b, a) \in \wedge^2 \mathfrak{n}^* \times \mathfrak{n}^*$ *satisfying the equations*

$$i.\ \ a \cdot da = 0 \qquad ii.\ \ db \cdot a = 0 \qquad iii.\ \ a \cdot b^n \neq 0, \tag{1.11}$$

defines a pair $(\beta, \alpha) \in \Omega^2(N) \times \Omega^1(N)$ *of invariant forms that induces an invariant codimension-one symplectic foliation on* N*, i.e., that satisfies the integrability conditions* (1.7)*.*

ii. Conversely any invariant codimension-one symplectic foliation on N *gives rise to a pair* $(b, a) \in \wedge^2 \mathfrak{n}^* \oplus \mathfrak{n}^*$ *satisfying the previous equations.*

Nilpotent Lie algebras

Here we give some basic results regarding the structure of $\mathfrak{n}^*$, when $\mathfrak{n}$ is a nilpotent Lie algebra.

The nilpotency condition on a Lie algebra $\mathfrak{n}$ has useful consequences for the differential graded structure of $\wedge^\bullet\mathfrak{n}^*$ (recall Equation (1.10)), that we can use to study invariant codimension-one symplectic foliations. One of those is the existence of a basis for $\mathfrak{n}^*$, useful to make computations:

Proposition 1.3.3 ([Mal51])**.** *There is a basis $\{x_1,\dots,x_n\}$ of $\mathfrak{n}^*$ such that for all $j=1,\dots,n$,*

$$dx_j \in \wedge^2\langle x_1,\dots,x_{j-1}\rangle.$$

Proof. Define inductively $\mathfrak{n}^1=[\mathfrak{n},\mathfrak{n}]$ and $\mathfrak{n}^k=[\mathfrak{n}^{k-1},\mathfrak{n}]$ and consider the finite series

$$\mathfrak{n}>\mathfrak{n}^1>\cdots>\mathfrak{n}^i=0,$$

where $i\le n$. Note that $\dim\mathfrak{n}^k<\dim\mathfrak{n}^{k-1}$, $k=1,\dots,i$. Consider the dual series of annihilators

$$\mathfrak{n}^*=\mathrm{Ann}(\mathfrak{n}^i)>\cdots>\mathrm{Ann}(\mathfrak{n}^1)>\mathrm{Ann}(\mathfrak{n})=0$$

This is a filtration of $\mathfrak{n}^*$. Take a basis $\{x_1,\dots,x_n\}$ adapted to the filtration (i.e, such that $\{x_1,\dots,x_{\dim\mathrm{Ann}(\mathfrak{n}^k)}\}\in\mathrm{Ann}(\mathfrak{n}^k)$ for all $k=1,\dots,i$. Note that in this basis, if $x_j\in\mathrm{Ann}(\mathfrak{n}^k)$, then $x_j|_{[\mathfrak{n}^{k-1},\mathfrak{n}]}=0$ and therefore $dx_j(\mathfrak{n}^{k-1},\cdot)=x_j([\mathfrak{n}^{k-1},\cdot])=0$. Thus, $dx_j\in\wedge^2\mathrm{Ann}(\mathfrak{n}^{k-1})$. □

The basis obtained in the previous lemma is called a *Malcev basis* of $\mathfrak{n}^*$. Some geometrical information can be easily derived from the existence of such a basis. For instance,

Corollary 1.3.4. *If $a\in\mathfrak{n}^*$ satisfies $a\cdot da=0$ then $da=0$*

Proof. Let $a=\sum_{i=1}^k\lambda_i x_i$, where $\{x_1,\dots,x_n\}$ is a Malcev basis for $\mathfrak{n}^*$ and $\lambda_k\neq 0$. Note that the only terms in $a\cdot da$ that contain x_k are

$$\lambda_k x_k\cdot\sum_{i=1}^k\lambda_i dx_i=\lambda_k x_k\cdot da,$$

and since $a\cdot da=0$, thus $\lambda_k x_k\cdot da=0$. Moreover, since $\lambda_k\neq 0$ and da does not contain any x_k, it follows that $da=0$. □

Corollary 1.3.5. *Any invariant codimension-one foliation on a nilpotent Lie group N is unimodular.*

Five-dimensional nilpotent Lie groups and nilmanifolds

Here we show that all five-dimensional nilpotent Lie groups admit codimension-one symplectic foliations and we find the ones that admit tame codimension-one symplectic foliations which are invariant.

Theorem 1.3.6. *All five-dimensional nilpotent Lie groups admit left-invariant codimension-one symplectic foliations.*

The proof of the theorem is done by constructing an invariant codimension-one symplectic foliation on each five-dimensional nilpotent Lie group. This can be done by using the classification of nilpotent Lie algebras (see [Gra07]). In dimension five there are only nine non-isomorphic nilpotent Lie algebras:

- $L_{5,1}$: Abelian
- $L_{5,2} : [X_1, X_2] = X_3$
- $L_{5,3} : [X_1, X_2] = X_3, [X_1, X_3] = X_4$
- $L_{5,4} : [X_1, X_2] = X_5, [X_3, X_4] = X_5$
- $L_{5,5} : [X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_2, X_4] = X_5$
- $L_{5,6} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5$
- $L_{5,7} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5$
- $L_{5,8} : [X_1, X_2] = X_4, [X_1, X_3] = X_5$
- $L_{5,9} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5$.

Proof. The table below contains, for every five-dimensional nilpotent Lie algebra $L_{5,j}$, $j = 1, \ldots, 9$, a pair $(b, a) \in \wedge^2\mathfrak{n}^* \times \mathfrak{n}^*$ that satisfies the equations (1.11) and that consequently defines an invariant codimension-one symplectic foliation on the nilpotent Lie group N.

Note that the bases for $\mathfrak{n}^*$ used here are the dual to the bases of $\mathfrak{n}$ used in the previous list where the Lie algebras are classified, and are Malcev bases for the Lie algebras.

Lie Algebra	Differential	a	b	db
$L_{5,1}$	Abelian	x_1	$x_{23} + x_{45}$	0
$L_{5,2}$	$dx_3 = x_{12}$	x_1	$x_{23} + x_{45}$	0
$L_{5,3}$	$dx_3 = x_{12}, dx_4 = x_{13}$	x_5	$x_{14} + x_{23}$	0
$L_{5,4}$	$dx_5 = x_{12} + x_{34}$	x_1	$x_{23} + x_{45}$	x_{124}
$L_{5,5}$	$dx_3 = x_{12}, dx_5 = x_{13} + x_{24}$	x_2	$x_{15} + x_{34}$	0
$L_{5,6}$	$dx_3 = x_{12}, dx_4 = x_{13}, dx_5 = x_{14} + x_{23}$	x_1	$x_{25} - x_{34}$	0
$L_{5,7}$	$dx_3 = x_{12}, dx_4 = x_{13}, dx_5 = x_{14}$	x_1	$x_{25} - x_{34}$	0
$L_{5,8}$	$dx_4 = x_{12}, dx_5 = x_{13}$	x_1	$x_{24} + x_{35}$	0
$L_{5,9}$	$dx_3 = x_{12}, dx_4 = x_{13}, dx_5 = x_{23}$	x_1	$x_{25} + x_{34}$	x_{124}

□

The previous table also shows that in all but two cases, the codimension-one symplectic foliation obtained is tame. The other two-cases where $db \neq 0$ are dealt with in the next proposition.

Proposition 1.3.7. *Let N be a nilpotent Lie group whose Lie algebra is either $L_{5,4}$ or $L_{5,9}$. Then N does not admit invariant tame codimension-one symplectic foliations.*

Proof. We prove in each case that there is no pair $(b, a) \in \wedge^2\mathfrak{n}^* \times \mathfrak{n}^*$ satisfying equations (1.11) for which $db = 0$. We deal with each case separately.

- $L_{5,4}$: Since $a \in \mathfrak{n}^*$ has to satisfy $da = 0$ then we must have $a(X_5) = 0$ (i.e, a does not contain x_5). Consider the expansion $b = \sum_{i<j} \lambda_{ij} x_{ij}$ and its differential

$$db = -\lambda_{15}x_{134} - \lambda_{25}x_{234} - \lambda_{35}x_{312} - \lambda_{45}x_{412}.$$

  If $db = 0$, then $\lambda_{15} = \lambda_{25} = \lambda_{35} = \lambda_{45} = 0$ and therefore $\iota_{X_5} b = 0$. We conclude then that if a and b are closed, they both have X_5 on their kernel, which contradicts the condition $a \cdot b^2 \neq 0$.

- $L_{5,9}$: In this case $a \in \mathfrak{n}^*$ being closed implies that $a(X_3) = a(X_4) = a(X_5) = 0$. Expressing $b = \sum_{i<j} \lambda_{ij} x_{ij}$, its differential becomes

$$db = -\lambda_{15}x_{123} - \lambda_{24}x_{213} + \lambda_{34}x_{124} + \lambda_{35}x_{125} + \lambda_{45}x_{135} - \lambda_{45}x_{423}$$

  The condition $db = 0$ implies $\lambda_{35} = \lambda_{45} = \lambda_{34} = 0$ and $\lambda_{24} = \lambda_{15}$. Therefore, every term of b contains x_1 or x_2, which means that $a \cdot b^2 = 0$ since a is of the form $\lambda_1 x_1 + \lambda_2 x_2$.

  Geometrically, the obstruction here can be seen as follows: any invariant foliation is transverse to the plane generated by X_1 and X_2. On the other hand, for b to be symplectic on the foliation, its kernel would have to be on that plane. But any closed two-form of maximal rank has its kernel transverse to that plane. $\square$

Nilmanifolds

Nilpotent Lie groups give rise to certain homogeneous compact spaces called nilmanifolds:

Definition 1.3.8. *A **nilmanifold** is a compact manifold of the form $M = N/\Gamma$, where N is a simply connected nilpotent Lie group and $\Gamma < N$ is a discrete subgroup.*

To construct nilmanifolds, one first has to know when a given simply connected nilpotent Lie group admits a discrete co-compact subgroup. The answer to this question is due to Malcev [Mal51]. He gave necessary and sufficient conditions in terms of the Lie algebra.

Theorem 1.3.9 (Malcev [Mal51])**.** *A simply connected nilpotent Lie group N admits a discrete co-compact subgroup Γ if and only if there exists a basis for $\mathfrak{n}$ for which the structural constants are rational numbers.*

Therefore, together with the classification of the five-dimensional nilpotent Lie algebras, this theorem proves that all five-dimensional nilpotent Lie algebras can give rise to nilmanifolds. From Theorem 1.3.6 it follows that all these nilmanifolds admit codimension-one symplectic foliations.

Corollary 1.3.10. *All five-dimensional nilmanifolds admit codimension-one symplectic foliations.*

1.4 Symplectic Foliations as Poisson structures

The goal of this section is to discuss symplectic foliations from the point of view of Poisson geometry. In particular, this will give rise to the notion of Poisson cohomology, modular vector field as well as a new interpretation of the variation of leafwise-symplectic forms.

Poisson structures

For a smooth manifold M, the space of multivector fields is the space $\mathfrak{X}^\bullet(M) := \Gamma(\wedge^\bullet TM)$. This space has a natural bracket, called the *Schouten-Nijenhuis bracket*, defined as

$$[f, a] = -\iota_{df}(a), \quad f \in C^\infty(M), \quad a \in \mathfrak{X}^k(M), \quad k \geq 1$$

and on indecomposable multivector fields as

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_k \wedge Y_1 \wedge \cdots \hat{Y}_j \cdots \wedge Y_l$$

and extended by $\mathbb{R}$-linearity. It can also be described as the unique extension of the Lie bracket of vector fields that makes the graded algebra of multivector fields into a Gerstenhaber algebra (see [Vai94]).

Definition 1.4.1. *A **Poisson structure** on a manifold M is a bivector $\pi \in \Gamma(\wedge^2 TM)$ satisfying*

$$[\pi, \pi] = 0.$$

A Poisson structure on a manifold M can be equivalently defined as a Lie bracket on the space of smooth functions, i.e, a bilinear operation

$$\{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying the Leibniz rule

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

and the Jacobi identity

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

The equivalence between the Lie bracket and the bivector definition is given by

$$\{f, g\}_\pi = \pi(df, dg).$$

A **Poisson map** between two Poisson manifolds (M_1, π_1), (M_2, π_2) is a smooth map $\varphi : M_1 \to M_2$ satisfying

$$\{f, g\}_{\pi_2} \circ \varphi = \{f \circ \varphi, g \circ \varphi\}_{\pi_1}, \text{ or equivalently, } \varphi_*(\pi_1) = \pi_2.$$

A **Poisson vector field** on a Poisson manifold (M, π) is a vector field $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X \pi = 0$.

A Poisson structure π associates a vector field $X_f \in \mathfrak{X}(M)$ to any smooth function $f \in C^\infty(M)$ by the formula

$$X_f = -[\pi, f] = \pi^\sharp(df),$$

where

$$\pi^\sharp : T^*M \to TM, \quad \pi^\sharp(\alpha) = \pi(\alpha, \cdot) \tag{1.12}$$

is the vector bundle map induced by π. Equivalently, X_f can be defined as the derivation of $C^\infty(M)$ given by $X_f = \{f, \cdot\}$. The vector field X_f is called the **Hamiltonian vector field** of f.

Note that any bivector $w \in \mathfrak{X}^2(M)$ (not necessarily Poisson) defines a bundle map $w^\sharp : T^*M \to TM$ by the same formula above (Equation (1.12)) and defines 'Hamiltonian' vector fields $X_f^w := w^\sharp(df)$. Moreover, w also induces a bracket on $C^\infty(M)$ defined by $\{f, g\}_w = w(df, dg)$. This bracket satisfies the Leibniz rule but might not satisfy the Jacobi identity. In this case,

Lemma 1.4.2 (See e.g. [Vai94])**.** *The following statements are equivalent:*

- *The bivector $w \in \mathfrak{X}^2(M)$ is Poisson.*
- *The bracket $\{\cdot, \cdot\}_w$ satisfies the Jacobi identity.*
- *$X^w_{\{f,g\}_w} = [X^w_f, X^w_g]$ for all $f, g \in C^\infty(M)$.*

If a bivector π is Poisson, the image of $\pi^\sharp$ is an integrable, possibly singular, distribution in the sense of Sussman [Sus73]. The maximal integral submanifolds of this distribution are immersed submanifolds and each maximal integral submanifold L inherits a symplectic structure ω_L that satisfies:

$$\omega_L(X_f|_L, X_g|_L) = -\pi(df, dg).$$

Since the Hamiltonian vector fields span the tangent space of L, this formula defines uniquely the form $\omega_L \in \Omega^2(L)$. Each (L, ω_L) is called a *symplectic leaf* and the collection of all these leaves is known in the literature as *symplectic foliation* defined by π. Since the foliation might be singular, this is not the definition of symplectic foliation we use in this thesis. However, as we see next, it is a generalisation thereof.

The *rank* of a bivector π at a point $x \in M$ is the dimension of the space $\pi^\sharp_x(T^*_x M)$. If the bivector is Poisson, the rank becomes the dimension of the symplectic leaf passing through x.

Definition 1.4.3. *A **regular Poisson structure** is a Poisson structure π that has the same rank at all the points of M.*

In this case, the singular foliation defined by π becomes a regular foliation in the sense of Definition 1.1.1. We denote this foliation by $\mathcal{F}_\pi$. The collection of symplectic structures on the leaves of $\mathcal{F}_\pi$ form a leafwise-symplectic structure $\eta_\pi \in \Omega^2(\mathcal{F}_\pi)$. Therefore regular Poisson structures induce symplectic foliations on M. Conversely, symplectic foliations also induce Poisson structures: if $(\mathcal{F}, \eta)$ is a symplectic foliation on M, then the following is a Poisson bracket on $C^\infty(M)$:

$$\{f, g\}(x) = -\eta_L(X^L_f, X^L_g)(x), \ \ \forall f, g \in C^\infty(M), \forall\, x \in M,$$

where L is the leaf passing through x and $X_f^L, X_g^L \in \mathfrak{X}(L)$ are the Hamiltonian vector fields on L computed with the symplectic structure η_L.

These assignments are inverses of each other:

Proposition 1.4.4. *The assignment $\pi \mapsto (\mathcal{F}_\pi, \eta_\pi)$ defines a 1-1 correspondence between regular Poisson structures on a manifold M and symplectic foliations on M.*

For the proof of this proposition, see [Vai94]. In virtue of this result, we use interchangeably the words: regular Poisson structure and symplectic foliation.

Weinstein's splitting theorem (see [Wei83]) implies that for every regular Poisson structure π, there are, around every point, a neighbourhood U and coordinates $\{q_1, ..., q_k, p_1, ..., p_k, z_1, ..., z_r\}$ where π can be written as

$$\pi|_U = \sum_{i=1}^{k} \frac{\partial}{\partial q_i} \wedge \frac{\partial}{\partial p_i}.$$

This shows that *regular* Poisson structures do not have local invariants, a phenomenon that also occurs in symplectic geometry and foliation theory.

Poisson cohomology

A Poisson structure π on a manifold M gives rise to a differential d_π on the space of multivector fields $\mathfrak{X}^\bullet(M)$:

$$d_\pi : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet+1}(M), \quad d_\pi(\alpha) = [\pi, \alpha].$$

The graded Jacobi identity satisfied by the Schouten bracket implies that $d_\pi^2 = 0$ and therefore one can talk of the cohomology of this complex.

Definition 1.4.5. *The **Poisson cohomology** of the Poisson manifold (M, π) is the cohomology of the complex $(\mathfrak{X}^\bullet(M), d_\pi)$ and is denoted by $H_\pi^\bullet(M)$.*

This cohomology gives useful information about the Poisson manifold, but it is generally very hard to compute. In low degrees, the Poisson cohomology groups have a geometric interpretation. For instance:

- $H_\pi^0(M) \subset C^\infty(M)$ are all the Casimir functions, i.e, the functions that are constant on the leaves.
- $H_\pi^1(M)$ are the Poisson vector fields modulo the Hamiltonian vector fields.
- $H_\pi^2(M)$ represents the infinitesimal deformations of π modulo the deformations coming from diffeomorphisms (see for instance [Vai94]). This group will be useful in the last chapter, where we study the deformations of a certain class of Poisson structures.

Remark 1.4.6. While we think of elements on $\Omega^k(M)$ as multilinear forms that act on vector fields, elements of $\mathfrak{X}^k(M)$ should be thought of as multilinear forms that act on one-forms. Using the reinterpretation (1.12) of π as a bundle map $\pi^\sharp : T^*M \to TM$, we see that forms on M can be mapped by $\pi^\sharp$ to multi-vector fields on M; this gives rise to a chain map from the de Rham complex of M to the Poisson complex of (M, π). Although the compatibility with the bracket can be checked directly, the best way to understand it is via the interpretations in terms of Lie algebroids (this will be explained in Section 1.5).

In the regular case, since the image of $\pi^\sharp$ is the associated foliation $\mathcal{F}_\pi$, we obtain a chain map

$$\pi^\sharp : (\Omega^\bullet(\mathcal{F}_\pi), d_{\mathcal{F}}) \to (\mathfrak{X}^\bullet(M), d_\pi)$$

and a map in cohomology

$$\pi^\sharp : H^\bullet(\mathcal{F}_\pi) \to H^\bullet_\pi(M). \tag{1.13}$$

In arbitrary codimensions, the two types of cohomologies (and the map $\pi^\sharp$) are part of a spectral sequence converging to Poisson cohomology, involving the foliated cohomologies $H^p(\mathcal{F}_\pi, \Lambda^q\nu^*)$. In the codimension-one case this boils down to a long exact sequence. Here we describe directly this long exact sequence, as the sequence that arises naturally when comparing the Poisson cohomology, the foliated cohomology and the foliated cohomology with coefficients in the normal bundle. An interesting feature of this sequence is that it is intimately related with the variation of foliated forms.

Proposition 1.4.7. *Let (M, π) be a corank-one Poisson manifold and let $(\mathcal{F}, \omega)$ denote its symplectic foliation. Then there is a long exact sequence*

$$\cdots \to H^{k-2}(\mathcal{F}, \nu) \xrightarrow{\mathfrak{d}} H^k(\mathcal{F}) \xrightarrow{\pi^\sharp} H^k_\pi(M) \to H^{k-1}(\mathcal{F}, \nu) \xrightarrow{\mathfrak{d}} H^{k+1}(\mathcal{F}) \to \cdots, \tag{1.14}$$

where the connecting map $\mathfrak{d}$ is given, up to a sign, by the cup-product with the leafwise variation $\mathrm{var}_\omega \in H^2(\mathcal{F}, \nu^*)$ (see Definition 1.2.14).

Note that the cup-product from the statement is the natural pairing in cohomology induced by the wedge-pairing

$$\begin{array}{ccc} \Omega^k(\mathcal{F}, \nu) \times \Omega^l(\mathcal{F}, \nu^*) & \longrightarrow & \Omega^{k+l}(\mathcal{F}) \\ (\alpha_1 \wedge \cdots \wedge \alpha_k \otimes s, \beta_1 \wedge \cdots \wedge \beta_k \otimes \gamma) & \longmapsto & \gamma(s)\alpha_1 \wedge \cdots \wedge \alpha_k \wedge \beta_1 \wedge \cdots \wedge \beta_k, \end{array} \tag{1.15}$$

for all $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_l \in \Omega^1(\mathcal{F})$, $s \in \Gamma(\nu)$, $\gamma \in \Gamma(\nu^*)$.

Proof. We first claim that there is a short exact sequence:

$$0 \to \Gamma(\wedge^k T\mathcal{F}) \longrightarrow \mathfrak{X}^k(M) \xrightarrow{j} \Gamma(\wedge^{k-1} T\mathcal{F} \otimes \nu) \to 0, \tag{1.16}$$

where the first map is the inclusion, while j is given by

$$X_1 \wedge \cdots \wedge X_k \longmapsto \sum_{i=1}^{k} (-1)^{i+1} X_1 \wedge \cdots \hat{X}_i \cdots \wedge X_k \otimes \overline{X}_i, \quad \forall X_1, \ldots, X_k \in \mathfrak{X}(M),$$

where $\overline{X_i} \in \Gamma(\nu)$ denotes, as before, the class of X_i modulo $T\mathcal{F}$. What is perhaps not so clear is the fact that j takes values in the subspace $\Gamma(\wedge^{k-1} T\mathcal{F} \otimes \nu)$ of $\Gamma(\wedge^{k-1} TM \otimes \nu)$. To check this, choose a splitting $\sigma : \nu \to TM$ of the sequence $T\mathcal{F} \to TM \to \nu$. Choose $X \in \sigma(\nu)$

a nowhere-zero vector field on $\sigma(\nu)$. Then, $TM = T\mathcal{F} \oplus \mathbb{R} \cdot X$ and for every $i = 1, \dots, k$, $X_i = V_i + a_i X$ for $V_i \in \Gamma(T\mathcal{F})$ and $a_i \in C^\infty(M)$. Then,

$$\begin{aligned} j(X_1 \wedge \cdots \wedge X_k) &= \sum_{i=1}^{k} (-1)^{i+1} a_i (V_1 + a_1 X) \wedge \cdots \hat{X}_i \cdots \wedge (V_k + a_k X) \otimes \overline{X} \\ &= \sum_{i=1}^{k} \sum_{j \neq i} (-1)^{i+j+1} a_i a_j X \wedge V_1 \wedge \cdots \hat{X}_i \cdots \hat{X}_j \cdots \wedge V_k \otimes X \\ &\qquad\qquad + \sum_{i=1}^{k} (-1)^{i+1} a_i V_1 \wedge \cdots \hat{X}_i \cdots \wedge V_k \otimes \overline{X} \\ &= \sum_{i=1}^{k} (-1)^{i+1} a_i V_1 \wedge \cdots \hat{X}_i \cdots \wedge V_k \otimes \overline{X} \in \Gamma(\wedge^{k-1} T\mathcal{F} \otimes \nu), \end{aligned}$$

where the double sums vanishes because the terms $a_i a_j$ are symmetric in i, j.

Next, the short exact sequence (1.16), combined with the isomorphism $\omega^\sharp : T\mathcal{F} \xrightarrow{\sim} T^*\mathcal{F}$, gives rise to a short exact sequence:

$$0 \to \Omega^k(\mathcal{F}) \to \mathfrak{X}^k(M) \to \Omega^{k-1}(\mathcal{F} \otimes \nu) \to 0. \tag{1.17}$$

It is a routine computation to check that these maps commute with the differentials. This gives rise to the long exact sequence from the statement. It remains to check the connecting homomorphism has the stated form. Let us write this homomorphism explicitly. To do so, let us choose again a splitting $\sigma : \nu \to TM$ of the sequence $0 \to T\mathcal{F} \to TM \to \nu \to 0$ that allows us to write $TM = T\mathcal{F} \oplus \sigma(\nu)$ and $T^*M = \sigma'(\nu^*) \oplus T^*\mathcal{F}$, where σ' is the splitting induced in the dual short exact sequence. Let $\tilde{\omega}$ be an extension of ω whose kernel is $\sigma(\nu)$. Then we obtain, by 'chasing the diagram', that

$$\mathfrak{d}([\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \otimes \overline{N}]) = [d_{\mathcal{F}}(\alpha_1 \wedge \cdots \wedge \alpha_{k-1}) \wedge \tilde{\omega}^\sharp(N)|_{\mathcal{F}}] + [\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge (\wedge^2 \tilde{\omega}^\sharp)(d_\pi N)|_{\mathcal{F}}]$$

for all $\alpha_1, \cdots, \alpha_{k-1} \in \Omega^1(\mathcal{F})$, $N \in \mathfrak{X}(M)$.

On the other hand, we have the identity

$$d(\iota_Y \tilde{\omega})|_{\mathcal{F}} = \iota_Y d\tilde{\omega}|_{\mathcal{F}} + (\wedge^2 \tilde{\omega}^\sharp)(d_\pi Y)|_{\mathcal{F}}, \quad \forall Y \in \mathfrak{X}(M).$$

Using this identity for $Y = N \in \sigma(\nu)$, since $\iota_N \tilde{\omega} = 0$, we get $\iota_N d\tilde{\omega}|_{\mathcal{F}} = -\tilde{\omega}^\sharp(d_\pi N)|_{\mathcal{F}}$. Substituting this into the expression for $\mathfrak{d}$ we get

$$\mathfrak{d}([\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \otimes \overline{N}]) = -[\alpha_1 \wedge \cdots \wedge \alpha_{k-1} \wedge \iota_N(d\tilde{\omega})|_{\mathcal{F}}] = -([\alpha_1 \wedge \cdots \alpha_{k-1} \otimes \overline{N}], d_\nu[\omega]). \quad \square$$

Modular vector fields and the modular class

The first Poisson cohomology group $H^1_\pi(M)$ of any Poisson manifold (M, π) contains a distinguished element, called the modular class of (M, π), which arises as the obstruction for the existence of transverse measures that are invariant under the flows of Hamiltonian vector fields. We recall its definition.

Lemma 1.4.8. *Let (M, π) be an orientable Poisson manifold. Choose a volume form μ on M. Then the equation*

$$\mathcal{L}_{X_f}\mu = \mathcal{L}_{X^\mu_\pi}(f)\mu,$$

defines a vector field X^μ_π on M, where X_f is the Hamiltonian vector field of f. The vector field X^μ_π is Poisson and its Poisson cohomology class does not depend on the choice of the volume form.

This lemma follows directly from the definitions. The vector field X^μ_π from the previous lemma is called the *modular vector field* of the Poisson structure π relative to the volume form μ.

Definition 1.4.9. *The **modular class** of a Poisson manifold (M, π), denoted by*

$$\mathrm{mod}_\pi \in H^1_\pi(M), \tag{1.18}$$

*is the Poisson cohomology class of the vector field X^μ_π from the previous lemma. A Poisson manifold is **unimodular** if its modular class vanishes.*

In the regular case, while the foliated and Poisson cohomologies are related by the map $\pi^\sharp$ (see Equation (1.13)), it is not surprising that the modular class of (M, π) is related to the modular class of the foliation. However, since so far we discussed the modular class of foliations only in the codimension-one case, we consider only that case.

Lemma 1.4.10. *Let (M, π) be a corank-one Poisson manifold. Then $\pi^\sharp : H^1(\mathcal{F}) \to H^1_\pi(M)$ maps $\mathrm{mod}_\mathcal{F}$ to mod_π.*

Proof. Let θ be a one-form defining the symplectic foliation and let $\beta \in \Omega^1(M)$ be any one form satisfying $d\theta = \beta \wedge \theta$. Recall from Definition 1.1.10 that $\beta|_\mathcal{F}$ is $d_\mathcal{F}$ closed and its cohomology class is the modular class of $\mathcal{F}$. We need to see that $\pi^\#(\beta)$ is a representative of the modular class of (M, π).

Let $\omega \in \Omega^2(M)$ be an extension of the leafwise-symplectic form induced by π. Let us take as a volume form the form $\mu = \omega^n \wedge \theta$, where $2n + 1$ is the dimension of M. Note that $\mathcal{L}_{X_f}\omega|_{\ker\theta} = 0$ for any smooth function f and therefore, using that $d\theta = \theta \wedge \beta$,

$$X^\mu_\pi(f)\mu = \mathcal{L}_{X_f}\mu = \omega^n \wedge \mathcal{L}_{X_f}\theta = \omega^n \wedge \iota_{X_f}\, d\theta = -(\iota_{X_f}\beta)\, \omega^n \wedge \theta = -\pi(df, \beta)\mu = \pi^\#(\beta)(f)\mu,$$

which implies that $X^\mu_\pi = \pi^\#(\beta)$. □

Lemma 1.4.11. *For any regular Poisson manifold (M, π),*

$$\pi^\sharp : H^1(\mathcal{F}) \to H^1_\pi(M)$$

is injective.

Note that, in the codimension-one case, this lemma follows directly from the long exact sequence of Proposition 1.4.7.

Proof. Let $\alpha \in \Omega^1(\mathcal{F})$ such that $d_\mathcal{F}\alpha = 0$ and let $\tilde{\alpha}$ be an extension of α. Recall that $\pi^\sharp : T^*M \to TM$ descends to an isomorphism $\pi^\sharp : T^*M/\nu^* \xrightarrow{\sim} T\mathcal{F}$.

If $\pi^\sharp(\tilde{\alpha}) = X_f$ for some $f \in C^\infty(M)$, then $\pi^\sharp(\tilde{\alpha} - df) = 0$, which means that $\alpha - df|_\mathcal{F} = 0$, which in turn implies that the class of α in $H^1(\mathcal{F})$ vanishes. □

Combining the previous two lemmas and using also Lemma 1.1.11, we obtain the following characterizations of unimodularity:

Corollary 1.4.12. *Let (M, π) be a corank-one Poisson manifold and let $\mathcal{F}$ denote its induced foliation. Then the following statements are equivalent:*

i. The Poisson manifold (M, π) is unimodular.

ii. The foliation $\mathcal{F}$ is unimodular.

iii. There exists a closed one-form defining the foliation $\mathcal{F}$.

Note that, combined with the discussion around Tischler's theorem (see Theorem 1.1.9), this implies that any unimodular corank-one Poisson structures on a compact orientable manifold can be perturbed into a fibrewise-symplectic fibration over S^1.

1.5 Intermezzo: Lie Algebroids

Lie algebroids provide a useful framework where various apparently different geometries can be put in the same setup. In particular, they give some more insight into some of the objects that we have already seen. Lie algebroids will become an important tool in the last chapter, in our study of deformations of Poisson structures.

We give a very brief introduction to Lie algebroids and their cohomology. For a more detailed account of the theory of algebroids and groupoids, see for instance [Mac05].

Definition 1.5.1. *A **Lie algebroid** over a manifold M is a vector bundle $A \to M$ endowed with a vector bundle map $\rho : A \to TM$, called the **anchor**, and a Lie bracket $[\cdot,\cdot]$ on the space of sections $\Gamma(A)$, satisfying the Leibniz identity*

$$[\alpha, f\beta] = f[\alpha, \beta] + (\mathcal{L}_{\rho(\alpha)} f)\beta,$$

for every $\alpha, \beta \in \Gamma(A)$, $f \in C^\infty(M)$.

Remark 1.5.2. From the Leibniz and the Jacobi identities it can be seen that the anchor ρ preserves the brackets (see [Mac05]).

Example 1.5.3. A foliation defines a Lie algebroid in a natural way, taking as anchor the inclusion map $\rho : T\mathcal{F} \hookrightarrow TM$ and as bracket the usual bracket of vector fields.

Example 1.5.4. A Poisson structure π on a manifold M defines a Lie algebroid on M, called the *cotangent Lie algebroid*. The vector bundle is the cotangent bundle $T^*M \to M$, the anchor map is $\pi^\sharp : T^*M \to TM$ (see (1.12)) and the bracket is:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - d(\pi(\alpha, \beta)).$$

This bracket can be characterised as the a unique bracket on $\Omega^1(M)$ satisfying the Leibniz identity with respect to $\pi^\sharp$ and such that $[df, dg]_\pi = d\{f, g\}$.

Lie algebroid representations and cohomology

Definition 1.5.5. *Let $A \to M$ be a Lie algebroid over M. An* ***A-connection*** *on a vector bundle $E \to M$ is a bilinear operator*

$$\nabla : \Gamma(A) \otimes \Gamma(E) \longrightarrow \Gamma(E), \quad (\alpha, s) \longmapsto \nabla_\alpha(s),$$

satisfying the following connection-like conditions:

$$\nabla_{f\alpha}(s) = f\nabla_\alpha(s), \quad \nabla_\alpha(fs) = f\nabla_\alpha(s) + \mathcal{L}_{\rho(\alpha)}(f)s, \ \forall f \in C^\infty(M), \alpha \in \Gamma(A), s \in \Gamma(E).$$

An A-connection ∇ is ***flat*** *if $\nabla_{[\alpha,\beta]} = [\nabla_\alpha, \nabla_\beta]$.*

With this notion of A-connection we can now define the notion of a Lie-algebroid representation.

Definition 1.5.6. *A* ***representation*** *of the Lie algebroid $A \to M$ is a vector bundle $E \to M$ together with a flat A-connection ∇.*

Example 1.5.7. Let $A \to M$ be any Lie algebroid and let $E = M \times \mathbb{R}$ be the trivial line bundle over M. We can define an A-connection by the formula $\nabla_\alpha(f) = \mathcal{L}_{\rho(\alpha)}f$. It is flat by Remark 1.5.2 and hence it defines a representation of A on E. This representation is called the *trivial representation* of A.

Example 1.5.8. Let $A \to M$ be a Lie algebroid of rank r over M^n and consider the line bundle $Q_A := \wedge^r A \otimes \wedge^n T^*M$. This line bundle admits a representation ∇ of A given by the equation

$$\nabla_\alpha(\alpha_1 \wedge \cdots \wedge \alpha_r \otimes \mu) = \sum_i (-1)^i [\alpha, \alpha_i] \wedge \alpha_1 \wedge \cdots \hat{\alpha}_i \cdots \wedge \alpha_r \otimes \mu + \alpha_1 \wedge \cdots \wedge \alpha_r \otimes \mathcal{L}_{\rho(\alpha)}\mu,$$

for every $\alpha \in \Gamma(A)$, every r-tuple of elements $\alpha_i \in \Gamma(A)$, $i = 1, \ldots, r$ and every $\mu \in \Gamma(\wedge^n T^*M)$. This representation is used to define the modular class of the Lie algebroid (see the next subsection).

Let $E \to M$ be a representation of a Lie algebroid $A \to M$. Consider the complex

$$\Omega^\bullet(A, E) := \Gamma(\wedge^\bullet A^* \otimes E)$$

and endow it with the differential d_∇ induced by the A-connection ∇ using the usual Koszul formula (see definition of $d^*_{\mathcal{F}}$ in Equation (1.4)). The flatness of ∇ implies that $d^2_\nabla = 0$ and therefore $(\Omega^\bullet(A, E), d_\nabla)$ is a cochain complex.

Definition 1.5.9. *The* ***cohomology of A with coefficients in E****, denoted $H^\bullet(A, E)$, is the cohomology of the complex $(\Omega^\bullet(A, E), d_\nabla)$.*

The ***cohomology of A****, denoted $H^\bullet(A)$, is the cohomology with coefficients in the trivial representation (from Example 1.5.7).*

Example 1.5.10. Let $(M, \mathcal{F})$ be a foliated manifold and $T\mathcal{F} \to M$ its associated Lie algebroid. It follows directly from the definitions that, for the trivial representation of $T\mathcal{F}$, $d_\nabla = d_{\mathcal{F}}$ and thus the cohomology of the Lie algebroid $T\mathcal{F} \to TM$ is the foliated cohomology, $H^\bullet(T\mathcal{F}) = H^\bullet(\mathcal{F})$. Moreover, the Lie algebroid cohomology when taking the $T\mathcal{F}$-representation on ν and ν^* given by the Bott connection coincides with the foliated cohomology with values in ν and ν^*, respectively, i.e, $H^\bullet(T\mathcal{F}, \nu) = H^\bullet(\mathcal{F}, \nu)$, $H^\bullet(T\mathcal{F}, \nu^*) = H^\bullet(\mathcal{F}, \nu^*)$.

Example 1.5.11. Let (M, π) be a Poisson manifold and let $T^*M \to M$ be the induced cotangent Lie algebroid over M (recall Example 1.5.4). If we take the trivial representation of T^*M, then $d_\nabla = d_\pi$ and therefore the cohomology of the cotangent Lie algebroid is isomorphic to the Poisson cohomology, $H^\bullet(T^*M) = H^\bullet_\pi(M)$ (see [ELW99]).

The modular class of a Lie algebroid

Weinstein defined a notion of modular class for a Lie algebroid that generalises the previous notions of modular classes of foliations and Poisson structures [Wei97, ELW99].

Lemma 1.5.12. *Let $A \to M$ be a Lie algebroid of rank r over M^n and consider the representation ∇ on the line bundle Q_A defined in Example 1.5.8.*
i. If Q_A is trivial, take a nowhere-zero section $s \in \Gamma(Q_A)$ and let $\theta_s \in \Gamma(A^*)$ be defined by the equation

$$\nabla_\alpha s = \theta_s(\alpha)s, \qquad \forall \alpha \in \Gamma(A).$$

ii. If the line bundle Q_A is not trivial, consider the trivial line bundle $L = Q_A \otimes Q_A$, extend the A-connection ∇ to L by the Leibniz identity, take a nowhere-zero section $s \in \Gamma(L)$ and let $\theta_s \in \Gamma(A^*)$ be defined by

$$\nabla_\alpha s = 2\,\theta_s(\alpha)s, \qquad \forall \alpha \in \Gamma(A).$$

Then, in both cases, $d_\nabla \theta_s = 0$ and the cohomology class $[\theta_s] \in H^1(A)$ does not depend on the choice of s.

Proof. The flatness of ∇ implies that $d_\nabla \theta_s = 0$. It remains to prove that the class of θ_s does not depend on the choice of the section s: if s' another nowhere-zero section (which would live in $\Gamma(Q_A)$ in case i. and in $\Gamma(Q_A \otimes Q_A)$ in case ii.), then $s' = as$ for some non vanishing function a in $C^\infty(M)$ and

$$\theta_{s'} = \theta_s + d_\nabla \log |a|$$

in case i., or

$$\theta_{s'} = \theta_s + d_\nabla (\tfrac{1}{2} \log |a|)$$

in case ii. □

Definition 1.5.13. *The **modular class** of the Lie algebroid A is the cohomology class $[\theta_s] \in H^1(A)$ from the previous lemma.*

Example 1.5.14. Consider the Lie algebroid $T\mathcal{F} \to TM$ associated with a foliation $\mathcal{F}$ on M. If the foliation is co-oriented of codimension one, then it is easy to see that the modular class of the foliation $\mathcal{F}$ agrees with modular class of the Lie algebroid $T\mathcal{F} \to M$.

Using representations of Lie algebroids, we can give another characterisation of unimodularity of foliations (cf. Corollary 1.4.12).

Lemma 1.5.15. *Let $\mathcal{F}$ be a codimension-one foliation on the manifold M. Then, the foliation $\mathcal{F}$ is unimodular if and only if the Lie algebroid representation of $T\mathcal{F}$ on the normal bundle ν given by the Bott connection is isomorphic to the trivial representation of $T\mathcal{F}$.*

Proof. First recall that forms θ defining $\mathcal{F}$ are in correspondence with trivialisation of ν: first, a one-form defining θ defines a trivialisation ϑ_θ by the formula:

$$\begin{aligned} \vartheta_\theta : \nu &\longrightarrow M \times \mathbb{R} \\ \overline{N} &\longmapsto (x, \theta(N)), \end{aligned}$$

where $N \in T_xM$ is a vector and $\overline{N}$ denotes its class in ν_x. Conversely, any trivialisation ϑ of ν is of the form ϑ_θ for some θ defining $\mathcal{F}$. Using this trivialisation ϑ_θ, we can transfer the $T\mathcal{F}$-representation on ν to a $T\mathcal{F}$-representation on $M \times \mathbb{R}$, as follows:

$$\nabla_X(\theta(N)) = \theta([X,N]) = \mathcal{L}_X(\theta(N)) - d\theta([X,N]),\ \forall X \in \Gamma(T\mathcal{F}),\ N \in \mathfrak{X}(M).$$

Now, it is clear that θ is closed if and only if ϑ_θ is an isomorphism of ν to the trivial $T\mathcal{F}$-representation. □

Example 1.5.16. Take now the cotangent Lie algebroid defined by a Poisson manifold (M, π). Then the modular class of the cotangent Lie algebroid is twice the modular class of the Poisson manifold (see [ELW99]).

Remark 1.5.17. Let (M, π) be a regular Poisson manifold and let $\mathcal{F}$ be the foliation defined by π. The maps

$$T^*M \xrightarrow{\pi^\sharp} T\mathcal{F} \overset{i}{\hookrightarrow} TM$$

are Lie algebroid maps, i.e, they preserve the anchors and the brackets. The maps

$$\Omega^\bullet(M) \xrightarrow{i^*} \Omega^\bullet(\mathcal{F}) \xrightarrow{\wedge^\bullet \pi^\sharp} \Gamma(\wedge^k T\mathcal{F})$$

are cochain maps (see [Vai94]). Therefore, we get induced maps

$$H^\bullet_{dR}(M) \xrightarrow{i^*} H^\bullet(\mathcal{F}) \xrightarrow{\pi^\sharp} H^\bullet_\pi(M).$$

1.6 Background material: More on Symplectic Fibrations

In this section we recall some standard material on symplectic fibrations. We follow the presentation from [GLS96], but adopting our terminology from Example 1.2.13.

Lemma 1.6.1. *Any strict symplectic fibration $(M \xrightarrow{\pi} B, \eta)$ is a symplectic fibration (for definitions, see Example 1.2.13).*

Proof. We have to find an extension $\tilde{\eta} \in \Omega^2(M)$ of η such that

$$\iota_{v_1 \wedge v_2} d\tilde{\eta} = 0 \tag{1.19}$$

for all vector fields $v_1, v_2 \in \ker d\pi$. Since $(M \xrightarrow{\pi} B, \eta)$ is a strict symplectic fibration, there is an open cover $\{U_i\}_i$ of B and trivialisations $\phi_i : M|_{U_i} \xrightarrow{\simeq} (U_i \times F_i, \omega_i)$. On each $\pi^{-1}(U_i)$, take the two-form $\eta_i = (\phi_i)^*(\omega_i)$. Take a partition of unity $\{\rho_i\}_i$ on B subordinate to the open cover $\{U_i\}_i$ and consider the two-form $\tilde{\eta} \in \Omega^2(M)$ given by

$$\tilde{\eta} = \sum_i \pi^*(\rho_i)\eta_i.$$

This form extends η and it can be readily seen that it satisfies Equation (1.19). $\square$

The condition for a fibrewise-symplectic fibration of being a symplectic fibration can be rephrased in terms of Ehresmann connections. To see this, let us first discuss the relation between Ehresmann connections and extensions of fibrewise-symplectic forms. Let $(M \xrightarrow{\pi} B, \eta)$ be a fibrewise-symplectic fibration. Denote $\mathrm{Ext}(\eta) \subset \Omega^2(M)$ the two-forms on M that extend η and denote $\mathrm{Ehr}(\pi)$ the set of Ehresmann connections on the fibration. The next lemma describes the relation between the two spaces; its proof is straightforward.

Lemma 1.6.2. *Let $(M \to B, \eta)$ be a fibrewise-symplectic fibration. Then the maps*

$$\mathrm{Ext}(\eta) \longrightarrow \mathrm{Ehr}(\pi), \quad \tilde{\eta} \longmapsto H_{\tilde{\eta}} := \{v \in TM \mid \tilde{\eta}(v, v_1) = 0 \ \ \forall v_1 \in \ker d\pi\}$$

and

$$\mathrm{Ehr}(\pi) \longrightarrow \mathrm{Ext}(\eta), \quad H \longmapsto \tilde{\eta}_H \in \Omega^2(M), \textit{such that } \tilde{\eta}_H|_{\mathcal{F}} = \eta \textit{ and } \iota_v\tilde{\eta}_H = 0 \; \forall v \in H$$

are such that $H_{\tilde{\eta}_H} = H$ for all $H \in \mathrm{Ehr}(\pi)$.

Remark 1.6.3. The previous lemma states that the map $H \to \tilde{\eta}_H$ is injective and that $\tilde{\eta} \to H_{\tilde{\eta}}$ is a left inverse. This last map is, however, not injective: if $\tilde{\eta} \in \mathrm{Ext}(\eta)$ and $\alpha \in \Omega^2(M)$ is any two-form that vanishes when contracted with any vertical vector field, then $H_{\tilde{\eta}+\alpha} = H_{\tilde{\eta}}$.

Using the previous correspondence, one obtains a new interpretation of the symplectic fibration condition (Equation (1.19)).

Lemma 1.6.4 ([GLS96])**.** *Let $(M \xrightarrow{\pi} B, \eta)$ be a fibrewise-symplectic fibration. If $\tilde{\eta}$ is an extension of η that satisfies the equation $\iota_{v_1 \wedge v_2} d\tilde{\eta} = 0$ for all vertical vector fields v_1, v_2, then*

$$\mathcal{L}_v\tilde{\eta}|_{\mathcal{F}} = 0$$

for any vector field $v \in \Gamma(H_{\tilde{\eta}})$.

Definition 1.6.5. *A **symplectic connection** on a fibrewise-symplectic fibration $(M \xrightarrow{\pi} B, \eta)$ is an Ehresmann connection H on $M \to B$ such that $\mathcal{L}_v\tilde{\eta}_H|_{\mathcal{F}} = 0$ for every vector field v in H.*

The name comes from the fact that the horizontal lift of a vector field on B, under a symplectic connection, has a flow that when defined, is a symplectomorphism between the fibres.

Remark 1.6.6. Lemma 1.6.4 says then that if $(M \to B, \eta)$ is a symplectic fibration, there are always symplectic connections. Moreover, the existence of a symplectic connection is equivalent to being a symplectic fibration: if H is a symplectic connection on $(M \to B, \eta)$, then $\tilde{\eta}_H$ is an extension of η that satisfies the condition of a symplectic fibration.

Using a symplectic connection on a symplectic fibration and standard differential-geometric arguments, it is easy to see that, if the fibres are compact, one can "symplectically" trivialise the fibrewise-symplectic fibration.

Proposition 1.6.7. *A symplectic fibration $(M \xrightarrow{\pi} B, \eta)$ with compact fibres is a strict symplectic fibration.*

1.7 Background material: Ehresmann Theorems for Manifolds with Boundary

One of the main goals of this work is to construct codimension-one symplectic foliations on closed manifolds. To do that, the general strategy is to decompose the manifold M into manifolds with boundary M_i, construct codimension-one symplectic foliations on each M_i, and glue the pieces together to obtain a codimension-one symplectic foliation on M.

The foliations on each M_i arise in most cases from a smooth map $f : M_i \to B_i$ where B_i might itself have boundary. Then it becomes important to understand smooth maps between manifolds with boundary.

Maps from manifolds with boundary

For manifolds without boundary, there is a normal form for submersions and the Ehresmann theorem for proper surjective submersions. In order to get equivalent results in the case where the manifolds have boundary, it is necessary to impose extra conditions to control the behaviour of the map near to the boundary.

There are two natural such conditions, depending on whether or not the target manifold has boundary. Let $f : M \to B$ be a submersion and assume M has boundary.

- **Condition i.** $\partial B = \emptyset$, $f|_{\partial M} : \partial M \to B$ is a submersion.
- **Condition ii.** $\partial B \neq \emptyset$, $f^{-1}(\partial B) = \partial M$ and $f|_{\partial M} : \partial M \to \partial B$ is a submersion.

Remark 1.7.1. Condition i. and ii. can be seen as “opposites”:

- Condition i. implies that the fibres of f are transverse to the boundary. Even more, the fibres of f are transverse to ∂M if and only if $f|_{\partial M} : \partial M \to B$ is a submersion. On the other hand, condition ii. implies that the fibres are tangent to ∂M.
- In condition i., the fibres of f are manifolds with boundary (see Corollary 1.7.4) whereas in condition ii., the fibres are manifolds without boundary.

For manifolds with boundary, the existence of collar neighbourhoods around the boundary shows that the manifold around the boundary is entirely determined by its boundary. In a similar spirit, for submersions satisfying either condition i. or ii., the map around the boundary is entirely determined by how it acts on the boundary, as we see next.

Remark 1.7.2 (Collar neighborhoods generated by vector fields)**.** For later references, we recall here that a standard way to produce collar neighborhoods is via flows of vector fields. More precisely, given a manifold M with compact boundary, and a vector field X on M (or just around the boundary) that is transversal to the boundary and points inwards, the flow $\phi^t_X(x)$ is defined for all $t \in [0, \epsilon)$ for some $\epsilon > 0$, and all $x \in \partial M$. This induces a collar neighborhood

$$k_X : \partial M \times [0, \epsilon) \to M, \; k_X(x, t) = \phi^t_X(x).$$

This collar neighborhood puts X in normal form: $k^*_X(X) = \frac{\partial}{\partial t}$.

In the following, we prove normal forms for submersions between manifolds with boundary satisfying condition i. or ii. The arguments are standard geometric-differential arguments (see [Hir97]) and we include all the proofs for completeness.

Condition i.

Proposition 1.7.3. *Let $f : M \to B$ be a surjective submersion where ∂M is compact and condition i. holds. Then, there is a collar neighbourhood U of ∂M diffeomorphic to $\partial M \times [0,\epsilon)$, for $\epsilon > 0$ small enough, where f can be written as*

$$f|_U : \partial M \times [0,\epsilon) \to B, \qquad f(x,t) = f|_{\partial M}(x).$$

Proof. The fibres of the map f are transverse to the boundary. We shall see that we can take a vector field X, around the boundary, tangent to the fibres of f which points inwards. Let us see first how the existence of such a vector field solves the problem. Since the manifold M is compact and the vector field point inwards, we can flow the vector field up to a finite time to get a collar neighbourhood $U \simeq \partial M \times [0,\epsilon)$ where f becomes $f(x,t) = f|_{\partial M}(x)$ since by construction $\mathcal{L}_X f = 0$.

Now let us construct the vector field X by constructing local vector fields and gluing them together using partitions of unity: let $U = \partial M \times [0,1)$ be an arbitrary collar neighbourhood of the boundary and let $p = (z,0)$ be a point in U, $z \in \partial M$. Since f is smooth, it extends to a smooth map from a extended collar neighbourhood $\tilde{U} = \partial M \times (-\delta, 1)$, for some $\delta > 0$ small enough, to B, and the extended map is submersive at $(z,0)$. Therefore, there is an open neighbourhood V_i around $(z,0) \in \tilde{U}$ in which f is submersive. In that open we can find a vector field X_i along the fibres of f, satisfying $df(X) = 0$ and $dt(X) > 0$ (t being the coordinate in $(-\epsilon, 1)$). Take a finite open cover of the boundary $\partial M \times \{0\}$ by those opens where the local vector fields are constructed. Use partitions of unity subordinate to that cover to glue the vector fields together to a vector field X satisfying $df(X) = 0$ and $dt(X) > 0$, since both conditions are preserved by the gluing. □

The previous lemma also gives a collar neighbourhood for the boundary of the fibres of the map f and thus we get the well known result (see [Hir97]):

Corollary 1.7.4. *Let $f : M \to B$ be a surjective submersion between compact manifolds where condition i. holds. Then the fibre $f^{-1}(x)$ at $x \in B$ is a manifold with boundary and its boundary is given by $f^{-1}(x) \cap \partial M$.*

Later on, we encounter submersions to S^1 satisfying condition i. As in the case without boundary, the manifold can be described as a suspension, as follows:

Corollary 1.7.5 (Suspension theorem for manifolds with boundary)**.** *Let $M \xrightarrow{f} S^1$ be a surjective submersion from a compact manifold M satisfying condition i. Then M is the suspension of $N := f^{-1}(1)$ under a diffeomorphism of N, i.e, $M \simeq N \times_{\mathbb{Z}} \mathbb{R}$, with $f([x,t]) = [t] \in S^1$.*

Proof. Take a vector field X that lifts the vector field $\partial/\partial\theta$ on $S^1 = \mathbb{R}/\mathbb{Z}$ that is also tangent to the boundary ∂M. Such a vector field can be readily constructed using the previous lemma. Consider the map

$$\psi : N \times_{\mathbb{Z}} \mathbb{R} \to M, \qquad (x,t) \mapsto \varphi_X^t(x),$$

where the $\mathbb{Z}$ action is given by $n \cdot (x,t) = (\varphi^n(x), t-n)$. The flow of the vector field X exists for all $t \in \mathbb{R}$ since M is compact and X is tangent to the boundary. This map is well defined and it is a smooth injective local diffeomorphism, since X is transverse to the fibres. Since N is compact, ψ is surjective and therefore it is a diffeomorphism. □

Condition i. also allows for a version of the Ehresmann theorem for manifolds with boundary.

Proposition 1.7.6 (Ehresmann's theorem for manifolds with boundary)**.** *Let $M \xrightarrow{f} N$ be a surjective submersion between compact manifolds satisfying condition i. Then M is a locally trivial fibration, i.e, for every point $z \in B$, there is an open $U_z \subset B$ around z and a diffeomorphism $\psi : f^{-1}(U_z) \xrightarrow{\sim} F_z \times U_z$, where $F_z = f^{-1}(z)$ is the fibre above z and $f(x) = \mathrm{pr}_2(\psi(x))$ for all $x \in f^{-1}(U_z)$.*

The proof goes exactly as in the case without boundary, by using an Ehresmann connection tangent to the boundary, which can be seen easily to exist by Proposition 1.7.3, see e.g. [Zou92].

Condition ii.

There is also a normal form for submersions around the boundary when they satisfy condition ii.

Proposition 1.7.7. *Let $f : M \to B$ be a surjective submersion where ∂M is compact and condition ii. holds. Then, there are collar neighbourhoods of the boundaries ∂M and ∂B, diffeomorphic to $\partial M \times [0, \epsilon)$ and $\partial B \times [0, \epsilon)$ for $\epsilon > 0$ small enough, where f can be expressed as*

$$f : \partial M \times [0, \epsilon) \to \partial B \times [0, \epsilon), \qquad f(x, t) = (f|_{\partial M}(x), t).$$

Proof. Choose collar neighbourhoods $U' \simeq \partial M \times [0, 1)$ and $V' \simeq \partial B \times [0, 1)$ and write $f|_{U'}$ becomes $f(x, s) = (\varphi_1(x, s), \varphi_2(x, s))$ for $x \in \partial M$ and $s \in [0, 1)$. We shall see that there is a vector field X such that

$$\mathcal{L}_X \varphi_1 = 0, \qquad X \pitchfork \partial M \times \{0\}, \qquad f_*(X) = \partial/\partial_t.$$

Let us see how the existence of this vector field allows us to finish the proof. Take the flow of the vector field X to define a new collar neighbourhood $U \simeq \partial M \times [0, \epsilon)$ of ∂M (since ∂M is compact, this flow can be taken up to a finite time). On the neighbourhoods U and $V \simeq \partial B \times [0, \epsilon) \subset V'$, the map f becomes $f(x, t) = (\varphi(x), t)$, where $\varphi(x) = \mathrm{pr}_1(f(x, 0))$ and therefore, $\varphi = f|_{\partial M}$.

Now let us construct X. Condition ii. guarantees that $f|_{\partial M}$ is a submersion and therefore $\varphi_1|_{\partial M \times \{0\}}$ is a submersion. Since being a submersion is an open condition and the boundary ∂M is compact, there is a $\delta > 0$ small enough such that $\varphi_1|_{\partial M \times [0,\delta)}$ is a submersion. By rescaling we may assume $\delta = 1$. The map $\varphi_1 : \partial M \times [0, 1) \to \partial B$ is then a submersion and therefore, the fibres of φ_1 are transverse to $\partial M \times \{0\}$. Let $\tilde{X}$ be a vector field on a smaller neighbourhood $W := \partial M \times [0, \delta)$, $\delta > 0$ small enough, which is tangent to the fibres of φ_1 and points inwards (i.e, $dt(\tilde{X}) > 0$). This can be constructed in the same way as the vector field X in the proof of Proposition 1.7.3.

Note that $f_*(\tilde{X}) \neq 0$ since $T_x M = T_x(\partial M) \oplus \mathbb{R} \cdot \tilde{X}_x$ for all $x \in W$ and

$$df|_{T_x M} : T_x M \to T_{f(x)} B \quad \text{and} \quad df|_{T_x(\partial M)} : T_x(\partial M) \to T_{f(x)} \partial B$$

are both surjective. Therefore, for all $x \in W$, $f_*(X_x) = (\varphi_2)_*(X_x) = a(x)\partial_t$ for some smooth function $a > 0$ on W. Consider the vector field $X := (1/a)\tilde{X}$. This vector field satisfies the condition above. $\square$

Chapter 2

Gluing Symplectic Foliations

Many of the most important examples of foliations, starting with Reeb's foliation on S^3 and Lawson's on S^5, are built by breaking the manifold into pieces, endowing each piece with a foliation, and then *gluing* them back. To make sure the gluing in the last step can be performed, the foliation on each piece has to satisfy some type of "tameness" condition at the boundary. One of the standard "tameness" conditions found in the literature is, intuitively, that the foliation can be extended smoothly beyond the boundary, by adding a small cylinder with the trivial foliation.

For symplectic foliations, we will use the same strategy. The gluing procedure in this case will require more care since in addition to gluing the foliations, one also needs to make sure that the leafwise-symplectic structures glue smoothly. *Tameness* conditions similar to the case of foliations will be needed for symplectic foliations. The purpose of this chapter is to discuss these conditions and to show how they make possible the gluing of symplectic foliations.

In the first section we study the possible tameness conditions for leafwise-symplectic forms. Intuitively, this tameness means "no variation of the leafwise-symplectic forms". Although to do the gluing we will only need the case of codimension-one foliations and tameness only around boundary, this section looks at arbitrary codimensions and arbitrary symplectic leaves.

The second section is devoted to the study of tameness (around boundary) and gluing in the more classical setting of foliations. In particular, this section presents some standard material from foliation theory, perhaps from a slightly different point of view.

In the final section we combine the tameness of the leafwise-symplectic forms with the one of the foliations, proving that they can be handled in a compatible way, therefore allowing us to glue symplectic foliations.

The gluing results in the chapter are well known in the literature but we include their proofs here for completeness.

2.1 Tameness Conditions

In this section we discuss the various possible notions of tameness of leafwise-symplectic structures around leaves. Intuitively, tameness around a leaf L means "no variation of the symplectic forms" near L. We will describe three such possibilities, which we call tame, M-tame and strongly tame (around L). In order to give a rough idea of what these notions entail, the following table sketches these conditions.

Tame	Strongly tame	M-tame
• $\exists\, U \supset L$ open nbhd such that $\mathrm{var}_\omega\|_U = 0$	• $\exists\, U \supset L$ open nbhd • $\exists\, \tilde{\omega}_U$ extending $\omega\|_U$ on U such that $d\tilde{\omega}_U = 0$	• $\exists\, U \supset L$ open nbhd • $\exists\, p : U \to L$ retraction such that $\omega\|_U = p^*(\omega_L)\|_{\mathcal{F}}$

Table 2.1: No-variation conditions for a leafwise-symplectic structure ω around a leaf L.

The following diagram explain the relations between the three notions.

$$\begin{tikzcd}
 & \text{Strongly tame} \arrow[dl, Rightarrow, bend right] \arrow[dr, dotted, bend left, "\substack{\text{cod}-1 \\ +\text{ embed}}"] & \\
\text{Tame} \arrow[ur, dotted, bend right, "\text{cod}-1"'] & & \text{M-tame} \arrow[ul, Rightarrow, bend left]
\end{tikzcd}$$

The first notion is just the local version of the tameness that we have already discussed (Definition 1.2.14).

Definition 2.1.1. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M and let L be a leaf. The leafwise-symplectic form ω is called **tame** around L if the restriction of $(\mathcal{F}, \omega)$ to an open neighbourhood U of L is tame (in the sense of Definition 1.2.14). It is called **inv-tame** around L if U can be chosen to be invariant (i.e. a union of leaves).*

To get a feeling about this definition, it is instructive to look at simple symplectic foliations, i.e. at the fibrewise-symplectic fibrations from Example 1.2.11. In this case we have already seen that the (global) tameness condition is equivalent to the fact that the fibration is a symplectic fibration. Furthermore, we have:

Proposition 2.1.2. *Let $\xi = (M \to B, \eta)$ be a fibrewise-symplectic fibration with connected fibres. Then the following are equivalent:*

1) ξ is a symplectic fibration.

2) ξ is tame.

3) ξ is inv-tame around every leaf.

Moreover, if π is proper, then these are equivalent also to:

4) ξ is tame around every leaf.

Proof. The equivalence of 1) and 2) is ensured by Corollary 1.2.16. The implications 2) $\Longrightarrow$ 3) $\Longrightarrow$ 4) are clear. Also, in the proper case, it is clear that 4) implies 3) since any neighborhood of a fibre contains a smaller invariant neighborhood. Hence we are left with proving that

3) implies 2). Let $(M, \mathcal{F}, \eta)$ be a fibrewise-symplectic fibration and assume that η is inv-tame around every leaf. Using Proposition 1.2.15, we can find extensions of η on an invariant neighbourhood around every leaf satisfying Equation (1.19) from Lemma 1.6.1. Therefore, there is an open cover $\{U_i\}_i$ of B and extensions $\eta_i \in \Omega^2(\pi^{-1}(U_i))$ of η satisfying Equation (1.19). Consider a partition of unity $\{\rho_i\}_i$ subordinated to the open cover U_i and consider the two-form $\tilde{\eta} = \sum_i \pi^*(\rho_i)\eta_i$ (compare with proof of Lemma 1.6.1). The two-form $\tilde{\eta} \in \Omega^2(M)$ extends η and satisfies Equation (1.19) and therefore, by Proposition 1.2.15, η is tame. □

Another notion of tameness which is more restrictive (but more appropriate for performing gluings) is inspired by Mitsumatsu's work [Mit11] (therefore the letter M in the definition below).

Definition 2.1.3. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M and let L be an embedded leaf. The leafwise-symplectic form ω is called **M-tame** around L if there exists a tubular neighborhood $U \xrightarrow{p} L$ of L such that*

$$\omega_{L'}|_{L' \cap U} = p^*(\omega_L)|_{L' \cap U}$$

for any other leaf L' of $\mathcal{F}$.

Note that this condition implies in particular that, around L, ω admits an extension which is closed (namely $p^*(\omega_L)$). Therefore, as an intermediate condition between tameness and M-tameness we have:

Definition 2.1.4. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M and let L be an embedded leaf. The leafwise-symplectic form ω is called **strongly-tame** around L if there is an open set $U \supset L$ and an extension $\tilde{\omega}_U \in \Omega^2(U)$ of $\omega|_U$ such that*

$$d\tilde{\omega}_U = 0.$$

Tame symplectic foliations in three-dimensional manifolds are also known as *taut* foliations [ET98] and in higher dimensions they have received the name of two-calibrated foliations [MT09].

Remark 2.1.5. Gotay et al.[GLSW83] show that, in the case of simple foliations, the problem of extending leafwise-closed forms is of topological nature, namely: a closed extension exists if and only if there is a de Rham cohomology class on the total space that restricts to every fibre to the cohomology class defined by the fibrewise-symplectic structure of the fibre. The same result does not hold for non-simple foliations. However, if the leafwise-closed form is required to be non-degenerate on every leaf, it is not known if a similar result holds. For example it is not known if S^5 admits a tame symplectic foliation.

For codimension-one symplectic foliations, the three notion are equivalent:

Proposition 2.1.6. *Let $(\mathcal{F}, \omega)$ be a codimension-one symplectic foliation on M and let L be an embedded leaf. Then the following are equivalent:*

1) ω is tame around L.

2) ω is strongly tame around L.

3) ω is M-tame around L.

Proof. We have to prove that 1) implies 3). If ω is tame around L, there is an extension $\tilde{\omega} \in \Omega^2(U)$ of $\omega|_U$, defined on some open neighborhood U of L, such that $d\tilde{\omega}(X, Y, -) = 0$ for any vector fields X and Y tangent to $\mathcal{F}$ (cf. Proposition 1.2.15). We may assume that $U = M$. Since $\mathcal{F}$ has codimension one, we can write $TM = T\mathcal{F} \oplus C$, where C is a one-dimensional vector bundle over M. A three form in M is defined by its value in $\wedge^3 T\mathcal{F}$ and $\wedge^2 T\mathcal{F} \otimes C$. The three-form $d\tilde{\omega}$ vanishes on $\wedge^3 T\mathcal{F}$ because $d_{\mathcal{F}}\omega = 0$ and on $\wedge^2 T\mathcal{F} \otimes C$ because of the above property of the extension $\tilde{\omega}$. Therefore $d\tilde{\omega} = 0$. This proves that 1) implies 2).

Now we see that 2) implies 3). The kernel of $\tilde{\omega}$ determines a line bundle transverse to $T\mathcal{F}$. Since M is orientable, the foliation $\mathcal{F}$ is co-orientable, and therefore, the line bundle $\ker \tilde{\omega}$ is trivialisable. Let X be a nowhere-zero section of this line bundle. Let $V \subset L \times \mathbb{R}$ be an open around $L \times \{0\}$ in which the flow of X is defined and let $\psi : V \to M$ be given by $\psi(x, t) = \varphi_X^t(x)$, where t denotes the second variable of $V \subset L \times \mathbb{R}$. Note that $d_{(x,0)}\psi(Y, \partial/\partial t) = Y + X$ for any $Y \in \Gamma(TL)$ and therefore, ψ is a local diffeomorphism around $(x, 0)$ for every $x \in L$. We can choose V small enough such that $\psi : V \to M$ is a diffeomorphism onto its image. We prove now that $\psi(V)$ is the tubular neighbourhood of L in which the M-tameness condition holds.

The form $\tilde{\omega}' := \psi^*(\tilde{\omega})$ can be written as $\tilde{\omega}' = dt \wedge \theta_t + \eta_t$ for some one-form θ_t and a two-form η_t that depend in principle on the parameter t.

We have that $i_X\tilde{\omega} = 0$ and therefore $\iota_{\partial/\partial t}\tilde{\omega}' = 0$ which implies that $\theta_t = 0$. Moreover, since $\tilde{\omega}'$ is closed

$$d\tilde{\omega}' = d_L \eta_t + dt \wedge \frac{d}{dt}\eta_t = 0,$$

where d_L denotes the de Rham differential on L. It follows then that $d/dt(\eta_t) = 0$, which finishes the proof. $\square$

2.2 Gluing Codimension-one Foliations

Adding $\partial M \times (-\infty, 0]$ to M

As it was mentioned before, to be able to glue foliations on manifolds with boundary, we have to consider foliations that have a certain "tame behavior" near the boundary. We will consider codimension-one foliations on a manifold M, with the property that (the connected components of) the boundary ∂M is a leaf. Roughly speaking, "tame behavior" near the boundary means that, after enlarging M by adding a cylinder $\partial M \times (-\epsilon, 0]$ foliated by product foliation, the resulting extended foliation is smooth. To do this carefully, we first discuss the process of extending M. First of all, set theoretically, we define

$$M_\infty := \partial M \times (-\infty, 0] \cup_{\partial M} M,$$

where $(x, 0)$ is glued to x, for x in the boundary. This space has no "canonical" smooth structure; instead, it has several smooth structures (diffeomorphic but different), one for each collar neighborhood

$$k : \partial M \times [0, 1) \xrightarrow{\sim} \mathcal{U} \subset M.$$

Indeed, requiring that the inclusion of Int$M$ into M_∞, as well as the obvious extension of k to the infinite cylinder

$$k_\infty : \partial M \times (-\infty, 1) \to M_\infty, \quad k_\infty(y,t) = \begin{cases} (y,t) & \text{if } t \in (-\infty, 0) \\ k(y,t) & \text{if } t \in [0,1) \end{cases} \tag{2.1}$$

to be smooth, one obtains a uniquely defined smooth structure on M_∞, called **the smooth structure induced by $\boldsymbol{k}$**. To avoid confusions, we will denote by $\boldsymbol{M^k_\infty}$ the space M_∞ endowed with this smooth structure. In general, a different k will give rise to a different smooth structure. However, all the resulting manifolds M^k_∞ are diffeomorphic to each other. Since the proof of this statement brings in some constructions that will be useful later on, we explain it here. We will show that each M^k_∞ is actually diffeomorphic to the interior of M. Again, such a diffeomorphism is not "canonical"; to build it we need to fix a smooth diffeomorphism

$$\xi : (0, \infty) \to \mathbb{R}$$

with the property that $\xi(t) = t$ for $t \in [1, \infty)$.

$$\psi_\xi : \ \mathrm{Int}\, M \longrightarrow M^k_\infty, \qquad x \mapsto \begin{cases} x & \text{if } x \in M \setminus \mathcal{U} \\ k_\infty(y, \xi(t)) & \text{if } x = k(y,t). \end{cases} \tag{2.2}$$

The following is immediate:

Lemma 2.2.1. *For any ξ as above, ψ_ξ is a diffeomorphism.*

Tameness of foliations near the boundary

Next, we consider a codimension-one foliation $\mathcal{F}$ on M which is tangent to the boundary, hence with the property that (every connected component of) the boundary is a leaf of $\mathcal{F}$. We then extend $\mathcal{F}$ to a partition $\mathcal{F}_\infty$ on M_∞ so that, in the cylinder part $\partial M \times (-\infty, 0)$, the leaves are the (connected components of the) slices $\partial M \times \{t\}$. We call this **the trivial extension of $\mathcal{F}$**.

Definition 2.2.2. *Let $\mathcal{F}$ be a foliation on a manifold with boundary M. We say that $\mathcal{F}$ is **tame near the boundary** if:*

- *$\mathcal{F}$ is tangent to the boundary, i.e. the connected components of ∂M are leaves of $\mathcal{F}$.*
- *For some collar neighborhood k, $\mathcal{F}_\infty$ is a smooth foliation on M^k_∞.*

Recall that codimension-one co-oriented foliations can be represented by one-forms. Here is a characterization of tameness in terms of such one-forms.

Lemma 2.2.3. *Let $\mathcal{F}$ be a co-orientable codimension-one foliation on a manifold with boundary M and assume that the boundary is compact. Then $\mathcal{F}$ is tame near the boundary if and only if there exists a collar neighbourhood $k : \partial M \times [0,1) \hookrightarrow M$ with the following property: $\mathcal{F}$ can be defined by a one-form α with the property that*

$$k^*(\alpha) = \zeta_t + dt,$$

with $\zeta_t \in \Omega^1(\partial M)$, which is smooth in $t \in [0,1)$, vanishes up to infinite order at $t = 0$.

*A collar neighborhood with this property will be called **adapted to $\mathcal{F}$**.*

Proof. It is clear that any adapted k has the property that $\mathcal{F}_\infty$ is a smooth foliation on M^k_∞. For the converse, assume that $\mathcal{F}$ is tame near the boundary and let $k : \partial M \times [0,1) \hookrightarrow M$ be the collar neighborhood ensured by the tameness. Let $\alpha \in \Omega^1(M)$ be any one-form defining $\mathcal{F}$ and we write it in the collar neighborhood as

$$k^*(\alpha) = \zeta_t + f dt,$$

with $f \in C^\infty(\partial M \times [0,1))$, $\zeta_t \in \Omega^1(\partial M)$. The tameness condition translates into the fact that ζ_t and $f|_{\partial M \times \{0\}} \neq 0$ and therefore, $f \neq 0$ on a smaller neighbourhood $\partial M \times [0,\epsilon)$ around ∂M. Choose any nowhere vanishing function $\tilde{f}$ such that $\tilde{f} \circ k = f$ on this small neighborhood. Replacing α by $\alpha' = \frac{1}{\tilde{f}}\alpha$ (which defines the same foliation) and restricting k to the smaller neighborhood,

$$M \times [0,\epsilon) \subset M \times [0,1) \overset{k}{\hookrightarrow} M,$$

we obtain a collar neighborhood like in the statement, just that it is defined only on $M \times [0,\epsilon)$. To pass to $[0,1)$, compose with

$$M \times [0,\epsilon) \to M \times [0,1), \quad (x,t) \mapsto (x, t\epsilon)$$

and replace α' by $\frac{1}{\epsilon}\alpha'$. □

The previous characterization is sometimes problematic since it appeals to a specific collar neighborhood. For later use, we give the following more intrinsic characterization.

Lemma 2.2.4. *Let $\mathcal{F}$ be a co-orientable codimension-one foliation on a manifold with boundary M and assume that the boundary is compact. Then $\mathcal{F}$ is tame near the boundary if and only if there exists a one-form α defining the foliation such that $d\alpha$ vanishes up to infinite order at ∂M.*

Proof. If $\mathcal{F}$ is tame near the boundary, we just use the α from the previous lemma. Assume now that such an α exists. Since $\mathcal{F} = \ker(\alpha)$ is tangent to the leaf and α is nowhere vanishing, we find a vector field X near the boundary, transversal to the boundary, such that $\alpha(X) = 1$. Consider the induced collar neighborhood $k = k_X$ (cf. Remark 1.7.2). Since $\alpha(X) = 1$, it follows that

$$k^*(\alpha) = \xi_t + dt.$$

Since the condition on α is coordinate invariant, the same property holds for $\xi_t + dt$, i.e.

$$d\alpha = d^M\xi_t - \tfrac{d}{dt}(\xi_t) \wedge dt$$

vanishes to infinite order at $\partial M \times \{0\}$ (where d^M is the de Rham differential in M). In particular, $\frac{d}{dt}\xi_t$ vanishes up to infinite order at $t = 0$. Since $\xi_0 = 0$ (since $\mathcal{F}$ is tangent to the boundary), we have that ξ_t vanishes up to infinite order at $t = 0$, hence we can apply the previous lemma. □

Gluing foliations

Finally, we can discuss the gluing of foliations. As mentioned before, the gluing results are well known and they appear in the literature on foliations.

We assume here that the boundaries are *compact* for simplicity and because this will be the case in the applications. Let $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ two manifolds with boundary with foliations tame near the boundary and let $\varphi : \partial M_1 \xrightarrow{\sim} \partial M_2$ be a diffeomorphism between the boundaries. We consider the glued space

$$M_1 \cup_\varphi M_2 = (M_1 \sqcup M_2)/\sim$$

on which we consider the "foliation" (really a partition at this point) $\mathcal{F}_1 \cup_\phi \mathcal{F}_2$ which is $\mathcal{F}_i$ on M_i, $i \in \{1, 2\}$. As before, one needs to be careful when when describing the smooth structure. The story is very similar to the previous one: First, take two collar neighborhoods, one for each M_i, which we arrange as

$$k_1 : \partial M_1 \times [0, 1) \to M_1, k_2 : \partial M_2 \times (-1, 0] \to M_2$$

(for $(M_2, \mathcal{F}_2)$ we use the parametrization $(-1, 0]$ since it is more intuitive and also avoids unnecessary signs). The corresponding smooth structure is uniquely determined by the conditions that the inclusions $M_i \hookrightarrow M$ are embeddings and the map

$$k : \partial M_1 \times (-1, 1) \to M, \qquad (x, t) \mapsto \begin{cases} k_1(x, t) & \text{if } t > 0 \\ k_2(\varphi(x), t) & \text{if } t < 0, \end{cases} \tag{2.3}$$

is smooth. Again, the resulting smooth structure depends on k_1 and k_2, but its diffeomorphism class does not.

To glue the foliations we use the flexibility that is available: we use collar neighborhoods k_1 and k_2 that are adapted to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, in the sense of Lemma 2.2.3 (with the obvious reparametrisation for $\mathcal{F}_2$): $\mathcal{F}_i$ can be defined by one-forms α_i with the property that

$$k_1^*(\alpha_1) = \xi_t^1 + dt, \;\; k_2^*(\alpha_1) = \xi_t^2 + dt,$$

where ξ_t^1 (with $t \in [0, 1)$) and ξ_t^2 (with $t \in (-1, 0]$) vanish up to infinite order at $t = 0$. Putting α_1 and α_2 together into a one-form α on the resulting $M_1 \cup_\varphi M_2$, one has

$$k^*(\alpha) = \zeta_t + dt,$$

where ζ_t is defined as ζ_t^1 for $t > 0$ and $\varphi^*(\zeta_t^2)$ for $t < 0$. Since both are flat at $t = 0$, ζ_t is a smooth one-form and so is α. The integrability condition can also be checked using the local expression for α. The form α defines thus a foliation on M. This proves that:

Proposition 2.2.5 (Gluing foliations). *Let $(M_1, \mathcal{F}_1)$, $(M_2, \mathcal{F}_2)$ two manifolds with boundary with foliations that are tame near the boundary and let $\varphi : \partial M_1 \xrightarrow{\sim} \partial M_2$ be a diffeomorphism between the boundaries. Then $M_1 \cup_\varphi M_2$ admits a smooth structure such that*

$$\mathcal{F}_1 \cup_\phi \mathcal{F}_2$$

is a smooth foliation with the property that the inclusions $M_i \hookrightarrow M$ are embeddings along which it restricts to $\mathcal{F}_i$, $i \in \{1, 2\}$.

Remark 2.2.6. Another foliation on $M = M_1 \cup_\varphi M_2$ can be defined by using by a form α' such that

$$k^*(\alpha') = \zeta'_t + dt, \tag{2.4}$$

where $\zeta'_t = \zeta_{2t+1}$ for $t < -1/2$, $\zeta'_t = 0$ for $|t| < 1/2$ and $\zeta'_t = \zeta_{2t-1}$ for $t > 1/2$. Geometrically, this foliation can be obtained by the one defined by α just by adding between M_1 and M_2 a collar $\partial M_1 \times (-1/2, 1/2)$ foliated trivially by the leaves $\partial M_1 \times \{t_0\}$. This foliation is useful later on to glue symplectic foliations.

2.3 Gluing Symplectic Foliations

Throughout this section we fix a manifold M with boundary, whose boundary ∂M is *compact*. For simplicity, we also assume that the boundary is connected (in general, as we have already seen, we just consider the connected components of ∂M as leaves).

Definition 2.3.1. *We say that a symplectic foliation $(\mathcal{F}, \omega)$ on M is tame near the boundary if:*

- *The foliation $\mathcal{F}$ is tame near the boundary.*
- *The leafwise-symplectic form ω is tame around the boundary, i.e., on an open neighborhood U of ∂M, ω admits an extension to a closed two-form on U.*

Of course, for the tameness of the symplectic form, we adapt Section 2.1 to the case where the leaf is the boundary. The modified version of Proposition 2.1.6 ensures M-tameness of ω, hence a preferred collar neighborhood on which ω becomes constant. We prove that this collar neighborhood can arranged to also be adapted to $\mathcal{F}$.

Proposition 2.3.2. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M that is tame near the boundary. Denote by ω_∂ the symplectic structure induced on the boundary. Then there exists a collar neighborhood $k : \partial M \times [0, 1) \to M$ such that:*

1. *k is adapted to $\mathcal{F}$ (in the sense of Lemma 2.2.3).*
2. *The pull-back of $(\mathcal{F}, \omega)$ by k is the constant $(\partial M, \omega_\partial) \times [0, 1)$.*

A collar neighborhood with this property will be called ***adapted to*** *$(\mathcal{F}, \omega)$.*

Proof. We have to make sure that the proof of Lemma 2.2.3 (constructing the adapted collar for the foliation) and the proof of Proposition 2.1.6 (part 2) implies 3), which gives the adapted collar for the leafwise-symplectic form) can be carried out at the same time. First fix a defining one-form α for $\mathcal{F}$ as in Lemma 2.2.4. Then we start as in the proof of the proposition: choose $\tilde{\omega}$ closed extension of ω on some open neighborhood around the boundary, open that we may assume to be M. As in the proof of the proposition, the kernel of $\tilde{\omega}$ is trivialisable, hence we find a nowhere-zero section X of the kernel. However, since $\ker(\omega)$ and $\mathcal{F} = \ker(\alpha)$ are complementary inside TM, we may even arrange that $\alpha(X) = 1$. Also, we may also assume that X points inwards (otherwise change it to $-X$ and α to $-\alpha$). Then the desired adapted collar is the one associated to X (cf. Remark 1.7.2)

$$k : \partial M \times [0, \epsilon) \to M, \; k_X(x, t) = \phi_X^t(x).$$

Indeed, first of all, the proof Proposition 2.1.6 continues to apply (we just imposed one extra-condition on X), with the obvious change from $\mathbb{R}$ to $[0, \infty)$; the outcome is that $k^*(\mathcal{F}, \omega)$ is the constant $(\partial M, \omega_{\partial M}) \times [0, \epsilon)$. Secondly, since $\alpha(X) = 1$, we have

$$k^*\alpha = \xi_t + dt$$

and then the argument from the proof of Lemma 2.2.4 implies that ξ_t has the desired vanishing properties, hence k is adapted to $\mathcal{F}$ as well. Strictly speaking we still have to pass from $[0, \epsilon)$ to $[0, 1)$, but that is done like at the end of the proof of Lemma 2.2.3: send t to $t\epsilon$ and replace α by $\frac{1}{\epsilon}\alpha$. □

Corollary 2.3.3. *Let $(\mathcal{F}, \omega)$ be a symplectic foliation on M that is tame near the boundary. Then there exists a collar neighborhood k such that:*

1. *the trivial extension $\mathcal{F}_\infty$ of $\mathcal{F}$ to M^k_∞ is smooth.*
2. *the extension ω_∞ of ω which in the cylinder part $\partial M \times (-\infty, 0]$ is given by $pr_1^*\omega_\partial$ makes $(\mathcal{F}_\infty, \omega_\infty)$ into a smooth symplectic foliation on M^k_∞.*

Finally, we can look at gluings of symplectic foliations. Even though the proof is trivial, the result will be of paramount importance for this work. Hence start with two symplectic foliations $(\mathcal{F}_i, \omega_i)$ defined on manifolds with boundary M_i as above, $i \in \{1, 2\}$, and with a symplectomorphism $\varphi : \partial M_1 \xrightarrow{\sim} \partial M_2$ between the boundaries. While $\mathcal{F}_1 \cup_\varphi \mathcal{F}_2$ (on $M_1 \cup_\varphi M_2$) was already discussed, it is clear that $\omega_1 \cup_\varphi \omega_2$ makes sense as a collection of symplectic forms on the leaves of $\mathcal{F}_1 \cup_\varphi \mathcal{F}_2$.

Theorem 2.3.4 (Gluing symplectic foliations)**.** *Let $(M_i, \mathcal{F}_i, \omega_i)$, $i = 1, 2$ be two codimension-one symplectic foliations on M_i, $i = 1, 2$, tame near the boundary and*

$$\varphi : (\partial M_1, \omega_1|_{\partial M_1}) \xrightarrow{\sim} (\partial M_2, \omega_2|_{\partial M_2})$$

be a symplectomorphism. Then $M_1 \cup_\varphi M_2$ admits a smooth structure such that

$$(\mathcal{F}_1 \cup_\phi \mathcal{F}_2, \omega_1 \cup_\varphi \omega_2)$$

is a smooth codimension-one symplectic foliation with the property that the inclusions $M_i \hookrightarrow M$ are embeddings along which it restricts to $(\mathcal{F}_i, \omega_i)$, $i \in \{1, 2\}$.

Proof. We proceed as in the proof of Proposition 2.2.5, but using collar neighborhoods k_1 and k_2 that are adapted to $(\mathcal{F}_i, \omega_i)$ (as in Proposition 2.3.2). The desired smooth structure is the one associated to (k_1, k_2). Indeed, since the resulting k controls smoothness around $\partial M_1 \times \{0\}$ (given by Equation (2.3)), we just have to make sure that the pull-back of $(\mathcal{F}_1 \cup_\phi \mathcal{F}_2, \omega_1 \cup_\varphi \omega_2)$ by k is smooth. This was already checked for the foliations. For the leafwise-symplectic form, just remark that they are induced by a closed two-form, namely $\mathrm{pr}_1^*(\omega_{\partial M_1})$, where $\mathrm{pr}_1 : \partial M_1 \times (-1, 1) \to M_1$ is the first projection. □

Corollary 2.3.5. *In the previous theorem, if one starts with a diffeomorphism $\varphi : \partial M_1 \xrightarrow{\sim} \partial M_2$ that is not necessarily a symplectomorphism, but satisfies the weaker condition that there is a family of symplectic structures ω_t, $t \in [-1, 1]$ on ∂M_1 joining $\omega_1|_{\partial M_1}$ with $\varphi^*(\omega_2|_{\partial M_2})$, then $M_1 \cup_\varphi M_2$ carries a codimension-one symplectic foliation.*

Proof. Consider the cylinder $\partial M_1 \times [-1,1]$ with the foliation given by $\partial M_1 \times \{t_0\}$ and the leafwise-symplectic structure given by ω' such that $\omega'|_{\partial M_1 \times \{t_0\}} = \omega_{h(t_0)}$, where $h : [-1,1] \to \mathbb{R}$ is a smooth monotone function such that $h(t) = 1$ for $t > 3/4$, $h(t) = -1$ for $t < -3/4$ and $h(0) = 0$. This leafwise-symplectic structure is tame at both boundary leaves and the foliation is tame at the boundary, so the previous theorem can be applied to glue M_1 to boundary $\partial M_1 \times \{-1\}$ using the identity map and M_2 to the side $\partial M_1 \times \{1\}$ using the map φ. The result is then a leafwise-symplectic foliation on the manifold $M_1 \cup_{\mathrm{id}} (\partial M_1 \times [-1,1]) \cup_\varphi M_2 \cong M_1 \cup_\varphi M_2$. $\square$

What is being done in this corollary is, geometrically, endowing the foliation given by the form α' from Remark 2.2.6 and endowing it with a leafwise-symplectic structure, which restricts to $\omega_{h(t)}$ on the leaf $M \times \{t\}$ for some increasing function h.

Remark 2.3.6. These constructions can be rephrased in more algebraic terms, by introducing the notion of (tame Poisson) cobordism between compact symplectic manifolds. Namely, one may say that two compact symplectic manifolds (S_1, ω_1) and (S_2, ω_2) (without boundary) are cobordant, and write $(S_1, \omega_1) \cong (S_2, \omega_2)$, if there is a compact manifold with boundary M, together with a symplectic foliation $\mathcal{F}$ that is tame around the boundary, such that (S_i, ω_i) form the boundary of $(M, \mathcal{F})$. Theorem 2.3.4 implies that cobordism is an equivalence relation. The last corollary implies that, if ω_0 and ω_1 are two symplectic structures on S that are deformation equivalent, then (S, ω_0) is cobordant to (S, ω_1). Several of the constructions that we will present in the next chapters can be re-interpreted in this language. Note however that our interest does not lie that much on the manifolds that are cobordant, but on the cobordism themselves.

Chapter 3

Symplectic Turbulisation

The main goal of this chapter is to develop the *turbulisation procedure*, the main tool used in this thesis to construct tame codimension-one symplectic foliations on manifolds with boundary. Cosymplectic structures are central in the turbulisation procedure and thus we also study them in detail in this chapter.

We will construct codimension-one symplectic foliations on closed manifolds in future chapters, by gluing tame codimension-one symplectic foliations on manifolds with boundary with the gluing theorems from Chapter 2.

This chapter is divided in two parts: In the first part we recall the notion of presymplectic structures and look at them as structures that are naturally induced on hypersurfaces of symplectic manifolds. Then we discuss cosymplectic structures as a particular case of presymplectic structures and study the geometry of cosymplectic manifolds. In the second part, we develop the turbulisation procedure: first in the simple case of $M \times S^1$ and later in the more general case of manifolds which fibre over S^1.

Conventions

- In this chapter, manifolds are assumed to be oriented with a given orientation. Symplectic manifolds are assumed to have the orientation induced by the symplectic form.
- If N is an manifold with a given orientation, $\overline{N}$ denotes the manifold with the opposite orientation.
- Boundaries are endowed with the orientation inherited from the ambient manifold by choosing a vector field pointing inwards, i.e, a basis $\{X_1, \dots, X_k\}$ of the tangent space of the boundary is oriented if the basis $\{X_1, \dots, X_k, Y\}$ is oriented on the ambient manifold, where the vector Y points inwards.

3.1 Presymplectic Structures

Definition 3.1.1. *Let N^{2n+1} be a smooth manifold. A* ***presymplectic structure*** *on N is a closed two-form of maximal rank on N.*

There are different notions of presymplectic structures in the literature: some authors call a presymplectic structure any closed two-form [SKT00], some others require the two-form to be of constant rank, [Got82], while others require the rank to be constant and maximal [HW09].

Definition 3.1.2. *Let η be a presymplectic structure on N. A one-form α is called* ***admissible*** *for η if $\eta^n \wedge \alpha \neq 0$ and is called* ***+admissible*** *for η if $\eta^n \wedge \alpha > 0$ for the given orientation of N.*

Lemma 3.1.3. *Let (N, η) be a presymplectic manifold. Then*

i. There exist admissible and +admissible forms for η.

ii. If α is an admissible form, then so is $f\alpha + \iota_X \eta$ for all $f \in C^\infty(N)$ nowhere zero, $X \in \mathfrak{X}(N)$.

iii. If α and β are two admissible forms for η, there are $f \in C^\infty(N)$ nowhere zero, $X \in \mathfrak{X}(N)$ such that $\beta = f\alpha + \iota_X \eta$.

The proof is simple linear algebra.

Definition 3.1.4. *Let N^{2n+1} be a smooth oriented manifold. An* ***almost cosymplectic structure*** *on N is a pair of forms (η, α), where $\eta \in \Omega^2(N)$ is a presymplectic structure and $\alpha \in \Omega^1(N)$ is an admissible one-form.*

Remark 3.1.5. Note that the condition for being a presymplectic structure is a C^0-open condition in the space of closed two-forms, i.e, if η is presymplectic on a compact manifold N and κ is any closed two-form on N then $\eta + \epsilon\kappa$ is presymplectic for ϵ small enough. Also, admissibility is an open condition on the C^0-topology of one-forms. If α is admissible for η, then any α' close enough to α is admissible for η.

Contact forms

We are mainly interested in two types of presymplectic structures: closed extensions of the leafwise-symplectic structures on tame codimension-one symplectic foliations and differentials of contact forms.

Here we recall the notion of a contact structure. They are structures in odd dimensional manifolds which are closely related to both symplectic structures and codimension-one symplectic foliations.

Definition 3.1.6. *A* ***contact structure*** *on a manifold M^{2n+1} is a codimension-one distribution ξ on M that can be given, locally, by the kernel of a one-form α satisfying $\alpha \wedge d\alpha^n \neq 0$. The contact structure ξ is* ***co-orientable*** *if TM/ξ is orientable.*

If the contact structure is co-orientable, there is a global one-form α on M defining the distribution ξ such that $\alpha \wedge d\alpha^n \neq 0$, or equivalently, $d\alpha|_{\ker \alpha}$ is non-degenerate. A one-form defining a contact structure is called a **contact form**. A contact form α is called **positive** if $\alpha \wedge d\alpha^n$ is positive with respect to the orientation already given.

Starting with a symplectic manifold, one can obtain contact manifolds as certain hypersurfaces on the symplectic manifold (see Lemma 3.1.11). Conversely, starting with a contact manifold, one can find a symplectic manifold by crossing with $\mathbb{R}$:

Definition 3.1.7. *Let α be a contact form on the manifold M. The **symplectisation** of (M, α) is the symplectic manifold $(M \times \mathbb{R}, d(e^t\alpha))$, where t is the coordinate in $\mathbb{R}$.*

Symplectic manifolds with boundary

An important source of presymplectic structures is hypersurfaces in symplectic manifolds.

Lemma 3.1.8. *Let $N \subset M$ be a hypersurface in a symplectic manifold (M, ω). Then:*

i. $\eta := \omega|_N$ is a presymplectic structure on N.

ii. For any vector field $X \in \mathfrak{X}(M)$ transverse to N, $(\iota_X\omega)|_N$ is admissible for η. Conversely, if N is embedded, then any admissible form of η is of the form $(\iota_X\omega)|_N$ for some vector field $X \in \mathfrak{X}(M)$ transverse to N.

Proof. This proof is also simple symplectic linear algebra. Let $\dim M = 2n$. Note that if $X \in \mathfrak{X}(M)$ is transverse to N, then $(\iota_X\omega^n)|_N$ is a volume form on N. Since $(\iota_X\omega)^n|_N = n\iota_X\omega \wedge \omega^{n-1}|_N \neq 0$, then $\omega|_N$ is a presymplectic structure and that $(\iota_X\omega)|_N$ is admissible for every $X \in \mathfrak{X}(M)$ transverse to N.
The fact that any admissible form of $\omega|_N$ is of the form $\alpha_X := (\iota_X\omega)|_N$ for some $X \in \mathfrak{X}(M)$ transverse to N follows directly from Lemma 3.1.3, part iii: since $(\iota_Z\omega)|_N$ is admissible for η for any $Z \in \mathfrak{X}(M)$ transverse to N, if α is any admissible one-form for η, then there is a nowhere-zero function $f \in C^\infty(N)$ and a vector field $Y \in \mathfrak{X}(N)$ such that $\alpha = f(\iota_Z\omega)|_N + \iota_Y(\omega|_N)$. Let $\tilde{f} \in C^\infty(M)$, $\tilde{Y} \in \mathfrak{X}(M)$ be extensions of f and Y, respectively (they exist because N is embedded). Then

$$\alpha = (\iota_{\tilde{f}Z+\tilde{Y}}\omega)|_N,$$

and the vector field $\tilde{f}Z + \tilde{Y}$ is transverse to N. □

Definition 3.1.9. *Let $N \subset M$ be a hypersurface in a symplectic manifold (M, ω). Then $\omega|_N$ is called the presymplectic structure **induced** on N by ω .*

We will be interested mainly in the case where the hypersurface is the boundary of manifold with boundary and there are two types of boundaries that will be relevant.

Definition 3.1.10. *Let (M, ω) be a symplectic manifold with boundary. Then ∂M is of:*

- ***cosymplectic type*** *if there exists a vector field $X \pitchfork \partial M$ such that $\mathcal{L}_X\omega|_{\partial M} = 0$.*
- ***contact type*** *if there exists a vector field $X \pitchfork \partial M$ such that $\mathcal{L}_X\omega|_{\partial M} = \omega|_{\partial M}$.*

We also say that the form ω is of cosymplectic (contact) type at the boundary.

Note that the vector fields on the definition only need to be defined near the boundary. In the case of boundaries of contact type, the vector field X is usually called *Liouville*. It is well known that those vector fields induce contact structures.

Lemma 3.1.11. *If (M^{2n}, ω) is a symplectic manifold with boundary of contact type, then $\iota_X \omega|_{\partial M}$ is a contact form on ∂M.*

Remark 3.1.12. In a symplectic manifold (M, ω) with boundary of contact type, it makes a big difference from the point of view of contact geometry whether the vector field X can be chosen pointing inwards or outwards and if it can be chosen in one way, it usually can't be chosen the other. The boundary is then called *concave* or *convex*, respectively. In the cosymplectic case, however, the difference is less important since the vector field can be chosen to point inwards or outwards, each choice giving a different orientation to the boundary.

The cosymplectic-type condition is discussed in detail in the next section.

Normal form around the boundary

Since presymplectic manifolds appear as boundaries of symplectic manifolds, it is useful to understand symplectic structures around boundaries. The following result can be seen as a particular case of Gotay [Got82], who proved that if (M, ω) is a symplectic manifold, the symplectic structure on a neighbourhood of certain submanifolds S (hypersurfaces included) only depends on $\omega|_S$. The next result follows directly using the classical Moser argument.

Proposition 3.1.13. *Let (M, ω) be a symplectic manifold with compact boundary. Let $\eta := \omega|_{\partial M} \in \Omega^2(\partial M)$ be the presymplectic structure induced by ω and let $\theta \in \Omega^1(\partial M)$ be any +admissible one-form for η. Then there exists a neighbourhood U of the boundary and a diffeomorphism $\varphi : \partial M \times [0, c) \to U$ such that*

$$\varphi^*(\omega|_U) = \eta - d(r\theta),$$

for some $c \in [0, 1)$, where r denotes the coordinate on $[0, c)$.

Remark 3.1.14. Rescaling the +admissible form θ by a positive constant we can always take $U \simeq \partial M \times [0, 1)$

Remark 3.1.15 (Boundary of contact type)**.** Let (M, ω) is a symplectic manifold with boundary of contact type. Then, by Lemma 3.1.11, the one-form $\theta := \iota_X \omega|_{\partial M}$ is a contact form on ∂M, where X is the vector field in the definition of contact type. If X is chosen pointing inwards, then from the same lemma it follows that θ is +admissible for $\omega|_{\partial M}$. Therefore, from Proposition 3.1.13, it follows that there is a neighbourhood $U \simeq [0, c) \times \partial M$ of the boundary where $\omega|_U = d((1 - r)\theta)$. We can rescale the transverse coordinate to obtain a neighbourhood U', where we can write $\omega|_U = d(e^t\theta)$. Therefore, there is a neighbourhood U' of ∂M where $(U, \omega|_U)$ is isomorphic to the "one-side" symplectisation of $(\partial M, \theta)$.

3.2 Cosymplectic Structures

Cosymplectic structures were defined by Libermann in the late 50s [Lib59] and manifolds admitting such structures have been studied by a number of people, for example [FM74, Li08, FV11, CMNY13]. These structures and their geometry, despite their simplicity, are of paramount importance in this thesis. In this section we discuss the notion, the geometry behind it and a generalisation of it. The results in this section are well known (appear in some of the previous papers) and simple to prove.

Definition 3.2.1. *Let M be a $2n+1$ dimensional manifold.*

- *A **cosymplectic form** on M is a presymplectic form η that admits an admissible one-form that is closed.*
- *A **cosymplectic structure** on M is a pair of closed forms $(\eta, \theta) \in \Omega^2(M) \times \Omega^1(M)$, where η is a cosymplectic form and θ is a closed admissible one-form for η. A cosymplectic structure (η, θ) is called **positive** if $\eta^n \wedge \theta > 0$.*

Symplectic manifolds with boundary of cosymplectic type provide an important source of examples of cosymplectic structures.

Lemma 3.2.2. *Let (M, ω) be a symplectic manifold with boundary of cosymplectic type. Then every vector field X transverse to the boundary for which $\mathcal{L}_X\omega|_{\partial M} = 0$ defines a cosymplectic structure $(\omega|_{\partial M}, \iota_X\omega|_{\partial M})$ on the boundary.*

The previous lemma follows directly from the definitions and the cosymplectic structure it defines is called the cosymplectic structure **induced** by (ω, X).

In this work, presymplectic manifolds (N, η) are often the boundary of a symplectic manifold with boundary (M, ω), i.e, $\eta = \omega|_{\partial M}$. In this case the cosymplectic condition for a presymplectic structure can be characterised in different ways.

Lemma 3.2.3. *Let (M^{2n}, ω) be a symplectic manifold with boundary. The following conditions are equivalent:*

i. The form $\omega|_{\partial M}$ is cosymplectic on ∂M.

ii. There is a vector field X on M transverse to ∂M for which $\mathcal{L}_X\omega|_{\partial M} = 0$.

iii. There is a vector field X on a neighbourhood of U of ∂M, which is transverse to ∂M and such that $\mathcal{L}_X\omega|_U = 0$.

iv. There is a neighbourhood $U \simeq \partial M \times [0,1) \subset M$ of ∂M where ω can be written as

$$\omega = \omega|_{\partial M} + \theta \wedge dr,$$

where θ is a closed one-form admissible for $\omega|_{\partial M}$.

The proof of the lemma is straightforward by using Definition 3.1.10, Proposition 3.1.13 and Lemma 3.2.2.

Geometry of cosymplectic structures

The following lemma follows immediately from the definitions.

Lemma 3.2.4. *Let (M, η, θ) be a cosymplectic manifold, then $\mathcal{F} = \ker\theta$ is a codimension-one foliation on M and $\eta|_{\mathcal{F}}$ defines a leafwise-symplectic structure on it. This symplectic foliation (Poisson structure) on M is called the symplectic foliation (Poisson structure) **induced** by (η, θ).*

This symplectic foliation is rather special:

Proposition 3.2.5 ([GMP11]). *Let (M, π) be a corank-one Poisson manifold and $(\mathcal{F}, \omega)$ be its symplectic foliation. The following statements are equivalent:*

i. There is a cosymplectic structure (η, θ) inducing π.

ii. There is a Poisson vector field transverse to $\mathcal{F}$.

iii. The foliation $\mathcal{F}$ is unimodular and the leafwise-symplectic structure ω is tame.

Proof. The equivalence *i.* $\Leftrightarrow$ *iii.* follows from the definitions. The implication *i.* $\Leftrightarrow$ *ii.* appears in [GMP11]. Here we sketch the argument: assume (η, θ) is a cosymplectic structure inducing π. Consider the vector field X defined as the unique vector field satisfying $\iota_X \eta = 0$, $\theta(X) = 1$. Then X is transverse to the foliation and since $\mathcal{L}_X \theta = \mathcal{L}_X \eta = 0$, then $\mathcal{L}_X \pi = 0$. Conversely, suppose there is a Poisson vector field X transverse to $\mathcal{F}$. Let θ be a one-form defining the foliation such that $\theta(X) = 1$ and let η be an extension of the leafwise-symplectic structure ω such that $\iota_X \eta = 0$. Then, the pair (η, θ) is a cosymplectic structure, and by the definition of η and θ, it induces π. $\square$

Remark 3.2.6. Given a tame symplectic foliation that is unimodular, there is no canonical cosymplectic structure inducing it because of the freedom in choosing a transverse direction to the leaves. This issue is addressed in the next section (Lemma 3.2.19).

The previous proposition shows that manifolds with cosymplectic structures come with "nice" symplectic foliations. The price paid for this "niceness" is the scarceness of examples, due to the restrictiveness of the cosymplectic condition: there are some trivial, yet severe topological obstructions to the existence of cosymplectic structures on closed manifolds.

Lemma 3.2.7. *Let (M^{2n+1}, η, θ) be a closed cosymplectic manifold, then there are cohomology classes $\alpha \in H^1(M)$ and $\beta \in H^2(M)$ such that $\alpha^i \wedge \beta^k \neq 0$, $i = 0, 1$, $k = 1, \ldots n$.*

Proof. Since $\eta^n \wedge \theta$ is a volume form and M is closed, the cohomology classes $[\eta] \in H^1(M)$ and $[\theta] \in H^2(M)$ satisfy that $[\eta]^n \wedge [\theta] \neq 0$ in $H^{2n+1}(M)$. Therefore, for each $k = 1, \ldots, n$, $i = 0, 1$, the cohomology class $[\eta]^k \wedge [\theta]^i$ does not vanish in $H^{2k+i}(M)$. $\square$

Definition 3.2.8. *Let (M, η, θ) be a cosymplectic manifold. The **symplectisation** of this cosymplectic manifold is the symplectic manifold $(M \times S^1, \eta + \theta \wedge d\varphi)$.*

Proper cosymplectic structures and symplectic fibrations

Cosymplectic manifolds appear also as manifolds which admit a fibration over S^1 and a closed two-form that is non-degenerate on the fibres.

Definition 3.2.9. *A cosymplectic structure (η, θ) on a manifold M is called **proper** if there is a submersion $M \xrightarrow{f} S^1$ for which $\theta = f^*(d\varphi)$, where φ is the angle variable on S^1. The function f is called a **defining function** for θ.*

Proper cosymplectic structures correspond to fibrations in the following sense:

Lemma 3.2.10. *If (M, η, θ) is a proper cosymplectic manifold and $f : M \to S^1$ is a defining function for θ, then the fibration $(f : M \to S^1, \eta|_{\ker\theta})$ is a symplectic fibration. Conversely, if $(f : M \to S^1, \eta)$ is a symplectic fibration, then there is an extension $\tilde{\eta}$ of η such that $(\tilde{\eta}, f^*(d\varphi))$ is a proper cosymplectic structure on M.*

The previous lemma appears in [GMP11] and follows directly from the definitions. Proper cosymplectic structures are related to suspensions:

Lemma 3.2.11. *Let (S, ω) be a symplectic manifold and φ a symplectomorphism of S. Then the suspension $S \times_{\mathbb{Z}} \mathbb{R}$ admits a proper cosymplectic structure.*

Proof. Recall that the $\mathbb{Z}$-action is given by $n \cdot (x, t) = (\varphi^n(x), t - n)$. Let $\mathrm{pr}_1, \mathrm{pr}_2$ denote the first and second projection on $S \times \mathbb{R}$. Then the pair $(\mathrm{pr}_1^*(\omega), \mathrm{pr}_2^*(dt))$ is a cosymplectic structure on $S \times \mathbb{R}$ which is invariant under the $\mathbb{Z}$-action and therefore, it descends to a cosymplectic structure (η, θ) on $S \times_{\mathbb{Z}} \mathbb{R}$. The defining function for θ is the second projection onto S^1. □

Definition 3.2.12. *Let (S, ω) be a symplectic manifold and let φ be a symplectomorphism of S. The cosymplectic structure **defined** by the suspension $S \times_{\mathbb{Z}} \mathbb{R}$ is the proper cosymplectic structure defined in the proof of the previous lemma.*

From Tischler's theorem (Theorem 1.1.9), it follows that every cosymplectic structure is "close" to a proper cosymplectic structures.

Proposition 3.2.13. *For every cosymplectic structure (η, θ) on a compact manifold M, there is a constant $c > 0$ such that $(\eta, c\theta)$ is arbitrarily close to a proper cosymplectic structure (η, θ').*

Proof. Tischler ([Tis70]) proves that there is a constant $c > 0$ and a closed one-form θ', arbitrarily closed to $c\theta$, coming from a submersion to S^1, i.e, $\theta' = f^*(d\varphi)$ for some submersion $f : M \to S^1$. Since the condition $c\eta^n \wedge \theta > 0$ is an open condition (see Remark 3.1.5), choosing θ' close enough to $c\theta$ the condition $\eta^n \wedge \theta' > 0$ also holds and the pair (η, θ') is a cosymplectic structure. □

Remark 3.2.14. If (M, ω) is a symplectic manifold with boundary of cosymplectic type, then by the previous proposition, there is a closed one-form $\theta = f^*(d\varphi)$ on ∂M coming from a submersion $f : \partial M \to S^1$ which admissible for the cosymplectic form $\omega|_{\partial M}$. Therefore, by Proposition 3.1.13, there is a vector field Z around the boundary, transverse with the boundary such that (ω, Z) induces the proper cosymplectic structure $(\omega|_{\partial M}, \theta)$ on the boundary.

Flat S^1-bundles

In the previous part we saw how symplectic fibrations over S^1 induce (proper) cosymplectic structures on the total space (Lemma 3.2.10). We can also obtain cosymplectic structures by considering S^1-bundles.

Lemma 3.2.15. *Let $f : M \to B$ be an S^1-bundle over a symplectic manifold (B, ω_B). If the bundle $M \to B$ admits a horizontal distribution defined by a closed one-form θ, then the pair $(f^*(\omega_B), \theta)$ is a cosymplectic structure on M.*

Proof. Since both forms of the pair are clearly closed, it remains to check that $f^*(\omega_B)^n \wedge \theta \neq 0$, where $\dim M = 2n+1$. Since the horizontal distribution $H \subset TM$ is transverse to the fibres, $d_x f : H_x \xrightarrow{\simeq} T_{f(x)}B$ is an isomorphism for all $x \in M$. Therefore, $f^*(\omega_B)$ is symplectic on $\ker \theta$, which implies that $f^*(\omega_B)^n \wedge \theta \neq 0$. □

Remark 3.2.16. Consider a fibration $f : M \to B$ as in the previous lemma. The one-form θ defining the horizontal distribution can be deformed slightly to one with rational periods, from which it follows that M is a k-cover of $B \times S^1$. Pulling back to M the foliation on $B \times S^1$ whose leaves are $B \times \{e^{i\varphi}\}$ one gets a codimension-one foliation on M whose leaves are also horizontal with respect to f. One could have also used Tischer's results from Proposition 3.2.13 to obtain a map from M to S^1 and consider the foliation by the fibres of this map. It is an interesting question and a matter of further research to see how precisely these foliations relate to the original foliation on M.

Poisson cohomology for cosymplectic structures

The Poisson cohomology of the Poisson structures induced by cosymplectic structures can be expressed in terms of the foliated cohomology. The following result appears in a less conceptual way in [FIL96].

Theorem 3.2.17. *Let (M, π) be a corank-one Poisson structure for which there is a cosymplectic structure (η, θ) inducing π. Then, the Poisson cohomology of (M, π) is given by*

$$H^\bullet_\pi(M) \simeq H^\bullet(\mathcal{F}) \oplus H^{\bullet-1}(\mathcal{F}), \tag{3.1}$$

where $\mathcal{F}$ is the symplectic foliation of π.

Proof. Consider the long exact sequence in cohomology from Proposition 1.4.7:

$$\cdots \to H^{k-2}(\mathcal{F}, \nu) \xrightarrow{\mathfrak{d}} H^k(\mathcal{F}) \to H^k_\pi(M) \to H^{k-1}(\mathcal{F}, \nu) \xrightarrow{\mathfrak{d}} H^{k+1}(\mathcal{F}) \to \cdots . \tag{3.2}$$

The foliation $\mathcal{F}$ is defined by a closed one-form and then, by Corollary 1.4.12 and Lemma 1.5.15, it is unimodular and the $T\mathcal{F}$-representation on ν is trivial. Therefore, $H^k(\mathcal{F}, \nu) = H^k(\mathcal{F})$.

Moreover, since ω is a tame leafwise-symplectic structure, $d_\nu[\omega] = 0$ and therefore $\mathfrak{d} = 0$. Then, the long exact sequence in cohomology splits into short exact sequences

$$0 \to H^k(\mathcal{F}) \to H^k_\pi(M) \to H^{k-1}(\mathcal{F}) \to 0, \tag{3.3}$$

which finishes the proof. □

Remark 3.2.18. If the Poisson structure π is induced by a symplectic fibration $M \to S^1$ on M, i.e, the cosymplectic structure inducing π is proper, then $H^0_\pi(M) = C^\infty(S^1)$, showing that the Poisson cohomology even in this simple case is infinite dimensional.

Generalised cosymplectic structures

There is a subtle duality between two-forms and bivectors. In the non-degenerate case, there is a one-to-one correspondence between non-degenerate Poisson bivectors and symplectic forms. When the rank is not maximal, the duality is less straightforward: recall from Remark 1.2.2 that a corank-one Poisson bivector π is induced by a pair of forms (η, θ). There are however, several such pairs that induce the Poisson bivector π. This ambiguity comes from the freedom in choosing the transverse direction to the leaves: there is a 1-1 correspondence between pairs of forms satisfying equations (1.7) and pairs (π, X) where π is a Poisson bivector of corank one and X is a vector field transverse to the foliation $\mathcal{F}_\pi$. A refinement of this correspondence (cf. Proposition 3.2.5) is:

Lemma 3.2.19. *Cosymplectic structures on a manifold M are in 1-1 correspondence with pairs $(\pi, X) \in \mathfrak{X}^2(M) \times \mathfrak{X}(M)$, where π is a corank-one Poisson structure and X is a Poisson vector field transverse to the leaves of π.*

This lemma is a particular case of a more general result (Lemma 3.2.29), for *k-cosymplectic structures* (a type of generalised cosymplectic structure) and their dual Poisson structures. The purpose of this section is to discuss these generalised structures. We define them first pointwise and later impose integrability conditions.

Definition 3.2.20. *An* ***almost k-cosymplectic structure*** *on M^n is a tuple $(\omega, \theta_1, \dots, \theta_k)$, where ω is a two-form of constant rank $2l$ and $\theta_1, \dots, \theta_k$ are one-forms, $k = n - 2l$, such that*

$$\omega^l \wedge \theta_1 \wedge \cdots \wedge \theta_k \neq 0.$$

Definition 3.2.21. *An* ***almost k-Poisson structure*** *on M^n is a tuple $(\pi, X_1, \dots, X_k)$, where π is a bivector of constant rank $2l$ and $X_1, \dots, X_k$ are vector fields, $k = n - 2l$, such that*

$$\pi^l \wedge X_1 \wedge \cdots \wedge X_k \neq 0.$$

Using simple linear algebra, it can be easily seen that

Lemma 3.2.22. *There is a $1-1$ correspondence between almost k-cosymplectic structures and almost k-Poisson structures.*

Let us now introduce the integrability conditions.

Definition 3.2.23. *A* ***k-cosymplectic structure*** *is an almost k-cosymplectic structure $(\omega, \theta_1, \dots, \theta_k)$ for which all the forms are closed.*

Definition 3.2.24. *A* ***k-Poisson structure*** *is an almost k-Poisson structure $(\pi, X_1, \dots, X_k)$ such that π is a Poisson bivector and $X_1 \dots, X_k$ are pairwise commuting Poisson vector fields.*

Example 3.2.25. A 0-cosymplectic structure is a symplectic structure. A 0-Poisson structure is a non-degenerate Poisson bivector.

Example 3.2.26. A 1-cosymplectic structure is a cosymplectic structure.

Not surprisingly, the previous lemma can be stated without the word ‘almost’. First, let us consider the corresponding notion of Hamiltonian vector fields.

Lemma 3.2.27. *Let $(M, \omega, \theta_1, \ldots, \theta_k)$ be an almost k-cosymplectic structure on M, and $(M, \pi, X_1, \ldots, X_k)$ its corresponding almost k-Poisson structure (see Lemma 3.2.22). Then, the vector field $X_f := \pi^\sharp(df)$ associated to any function $f \in C^\infty(M)$ satisfies the equations*

$$\iota_{X_f}\omega = df - \sum_{i=1}^{k} X_i(f)\theta_i, \qquad\qquad \iota_{X_f}\theta_i = 0 \ \ \forall i.$$

Moreover, the previous equations define uniquely the vector field X_f.

Proof. It is clear that $\iota_{\pi^\sharp(df)}\theta_i = 0$ for all i. It remains to check that

$$\iota_{\pi^\sharp(df)}\omega = df - \sum_{i=1}^{k} X_i(f)\theta_i,$$

which, since $TM = \cap_i \ker(\theta_i) \oplus \ker\omega$, can be checked easily by contracting with vectors $X_1, \ldots, X_k \in \ker\omega$ and vectors $Z \in \cap_i \ker\theta_i$. The second assertion follows from the previous decomposition of TM and the fact that $\omega^\sharp$ is injective on $\cap_i \ker\theta_i$. $\square$

Remark 3.2.28. Using the vector fields X_f, called the *Hamiltonian* vector fields, an "almost Poisson" bracket induces a bracket on $C^\infty(M)$ by

$$\{f, g\} = \pi(df, dg) = -\omega(X_f, X_g).$$

This bracket satisfies the Jacobi identity (and hence is a genuine Poisson bracket) if and only if $[\pi, \pi] = 0$.

Now we can state the main correspondence

Lemma 3.2.29. *There is a $1-1$ correspondence between k-cosymplectic structures and k-Poisson structures on M.*

Proof. Let $(M, \omega, \theta_1, \ldots, \theta_k)$ be an almost k-cosymplectic structure on M and let $(M, \pi, X_1, \ldots, X_k)$ be the almost k-Poisson structure corresponding to it according to Lemma 3.2.22. Consider the form

$$\Omega = \omega + \sum_{i=1}^{k} \theta_i \wedge dt_i \in \Omega^2(M \times \mathbb{R}^k)$$

is non-degenerate, where t_i is the i-th coordinate in $\mathbb{R}^k$. Then, Ω^{-1} is the bivector

$$\Omega^{-1} = \Pi = \pi + \sum_{i=1}^{k} \partial_{t_i} \wedge X_i \in \mathfrak{X}^2(M \times \mathbb{R}^k).$$

Then, Ω is symplectic if and only if Π is Poisson. Thus, since Ω is closed if and only if $(M, \omega, \theta_1, \ldots, \theta_k)$ is a k-cosymplectic structure and Π is Poisson if and only if $(M, \pi, X_1, \ldots, X_k)$ is k-Poisson, we get the desired 1-1 correspondence. $\square$

Cosymplectic manifolds with boundary.

Recall from Lemma 3.2.2 that symplectic structures (0-cosymplectic) on manifolds with boundary induce, under some conditions, cosymplectic structures (1-cosymplectic) on the boundary. This holds in general for k-cosymplectic structures.

Lemma 3.2.30. *Let $(M, \omega, \theta_1, \ldots, \theta_k)$ be a k-cosymplectic manifold with boundary. Assume that $\omega^l|_{\partial M} = 0$ and that there exists a vector field X on M transverse to ∂M such that $\mathcal{L}_X\omega|_{\partial M} = 0$. Then $(\omega|_{\partial M}, \iota_X\omega|_{\partial M}, \theta_1|_{\partial M}, \ldots, \theta_k|_{\partial M})$ is a (k+1)-cosymplectic structure on ∂M.*

Proof. First, note that since ω is closed, the condition $(d\iota_X\omega)|_{\partial M} = 0$ is equivalent to the condition $\mathcal{L}_X\omega|_{\partial M} = 0$. Moreover, since X is transverse to the boundary,

$$\iota_X(\omega^l \wedge \theta_1 \wedge \cdots \wedge \theta_k)|_{\partial M} = l\,(\omega^{l-1} \wedge \iota_X\omega \wedge \theta_1 \wedge \cdots \wedge \theta_k)|_{\partial M} \neq 0. \qquad \square$$

3.3 Turbulisation: Case $M \times S^1$

Let C be a compact manifold with boundary endowed with a foliation coming from a fibration

$$f : C \to S^1.$$

This foliation might not be tame at the boundary and therefore, it might not be easily glued to another foliation on another manifold C' (see Chapter 2). However, Lawson [Law71] devised a method to change the foliation given by the fibres of f into a foliation that is tame at the boundary (recall Example 1.1.14), provided f satisfied condition i. from Section 1.7. This procedure is called *turbulisation*.

In the rest of this chapter we generalise this procedure to change symplectic foliations into symplectic foliations tame at the boundary. More precisely: if the manifold C admits a a codimension-one symplectic foliation where the foliation is given by the fibres of f and the fibrewise-symplectic structure is "good enough" (namely, if it has a cosymplectic-type behaviour at the boundary), we can modify this codimension-one symplectic foliation to make it tame at the boundary. This procedure will be called *symplectic turbulisation* and the symplectic foliation that is obtained with it coincides with the original symplectic foliation outside a neighbourhood of the boundary.

In this section, we discuss the symplectic turbulisation when $C = M \times S^1$, where M is a compact manifold with boundary, the leaves of the foliation are $M \times \{x\}$, for $x \in S^1$, and the leafwise-symplectic structure is induced by a symplectic structure ω on M. In this case, the "goodness" required for the fibrewise-symplectic structure will be that the symplectic structure ω is of cosymplectic type at the boundary.

Discussing the case of $C = M \times S^1$ first will help us understand better the more general case when the manifold C has a non-trivial fibration over S^1, which will be discussed in the next section.

As we will see shortly, we need a symplectic structure of cosymplectic type to apply the turbulisation procedure. Unfortunately, these type of symplectic structures do not always exist and can be sometimes elusive in dimensions higher than two. There is an obstruction for the existence of these structures similar to the obstruction for closed symplectic manifolds. Recall that if (M, ω) is a closed symplectic manifold, then $H^2(M) \neq \{0\}$; but this does not hold if M has boundary. However,

Lemma 3.3.1. *If (M, ω) is a symplectic manifold with compact boundary of cosymplectic type and $\dim M > 2$, then ω defines a non-zero cohomology class in $H^2(M)$.*

Proof. Let X be the vector field from the definition of boundary of cosymplectic type and recall from Lemma 3.2.2 that the pair $(\iota_X \omega|_{\partial M}, \omega|_{\partial M})$ defines a cosymplectic structure on ∂M. Since $\dim M > 2$, $\omega|_{\partial M} \neq 0$. If the cohomology class of ω were zero then so would the cohomology class of $\omega|_{\partial M}$, but that can not happen (Lemma 3.2.7). □

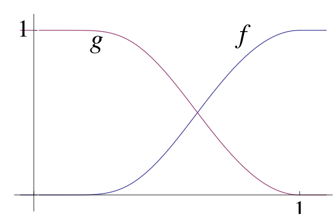
Remark 3.3.2. The condition $\dim M > 2$ is necessary since it is straightforward to see that if M is a two-dimensional manifold with boundary, then any symplectic structure on M is of cosymplectic type at the boundary.

Remark 3.3.3. The obstruction given in the previous lemma is not the only obstruction for the existence of this type of symplectic structures. In Theorem 5.3.8, we give an example of a four-manifold with boundary, with "enough" second cohomology so that the previous lemma does not apply, but yet for which there is not a symplectic structure of cosymplectic type at the boundary.

In the constructions we frequently use pairs of bump functions satisfying the following.

Definition 3.3.4 (Good pair)**.** *The functions $f, g : \mathbb{R} \to \mathbb{R}$ are said to form a **good pair** (f, g) if they are monotone on $(0, 1)$ and*

$$f(x) = \begin{cases} 0 & \textit{if } x \leq 0 \\ 1 & \textit{if } x \geq 1 \end{cases} \qquad g(x) = \begin{cases} 1 & \textit{if } x \leq 0 \\ 0 & \textit{if } x \geq 1 \end{cases}$$



<u>Convention</u>. We always assume, unless otherwise specified, that $f(x) > 0$, $g(x) > 0$ for $x \in (0, 1)$ and that f and g are strictly monotone on $(0, 1)$. However, sometimes, we will need that the pair of functions be constant if $x \leq \epsilon$ for ϵ is small enough. We always state explicitly in those cases what is precisely needed from the good pair.

Foliation

Proposition 3.3.5. *Let M be a manifold with compact boundary. Then $M \times S^1$ admits a foliation tame at the boundary.*

Proof. Consider a collar neighbourhood U of ∂M and a fixed trivialisation

$$k : \partial M \times [0, 1) \xrightarrow{\sim} U \subset M.$$

Denote the second coordinate by r and consider a good pair of functions (f,g). Consider the one-form on $M \times S^1$ defined by

$$\alpha = \begin{cases} d\varphi & \text{on } (M \setminus U) \times S^1 \\ f(r)d\varphi + g(r)dr & \text{on } U \times S^1, \end{cases} \tag{3.4}$$

where φ is the angle coordinate on S^1. Since α is a nowhere-zero form and $\alpha \wedge d\alpha = 0$, this form defines a foliation on $M \times S^1$. Since $d\alpha = f'(r)dr \wedge d\varphi$, by Lemma 2.2.4 it follows that the foliation is tame. □

We want to understand now the leaves of the foliation obtained in the previous proposition

Lemma 3.3.6. *Let $\mathcal{F}$ be the foliation on $M \times S^1$ obtained in the previous lemma. Then*

i. The connected components of $\partial M \times S^1$ are leaves.

ii. The foliation in $\mathrm{Int}M \times S^1$ is diffeomorphic to the trivial product foliation.

Proof. Part i. follows from the fact that $\ker \alpha_x = T_x(\partial M \times S^1)$ for all $x \in \partial M \times S^1$. To see part ii., consider the function $\tilde{h} : (0,\infty) \to \mathbb{R}$ defined by the initial value problem

$$\frac{d}{dr}\tilde{h}(r) = g(r)/f(r), \qquad h(1) = 0.$$

and consider the function $h : \mathrm{Int}M \to \mathbb{R}$, given by

$$h(x) = \begin{cases} \tilde{h}(r) & \text{if } x = (z,r) \in \mathrm{Int}(U) \simeq \partial M \times (0,1) \\ 0 & \text{if } x \in M \setminus U, \end{cases} \tag{3.5}$$

In the interior $\mathrm{Int}\, U \times S^1$, α can be written as $\alpha = f(d\varphi + dh)$. Since $f > 0$ in $\mathrm{Int}\, U$, the form $d\varphi + dh$ defines the same foliation as α. The diffeomorphism of $\mathrm{Int}M \times S^1$

$$\mathrm{Int}M \times S^1 \xrightarrow{\sim} \mathrm{Int}M \times S^1, \quad (x,[\varphi]) \mapsto (x,[\varphi - h(x)])$$

pulls back the form $\alpha/f = d\varphi + dh$ to $d\varphi$ and therefore, it maps the leaves $\mathrm{Int}M \times \{\varphi\}$ of the product foliation on $\mathrm{Int}M \times S^1$ to the leaves of $\mathcal{F}$.

The leaf of $\mathcal{F}$ which is the image of $\mathrm{Int}M \times \{\varphi_0\}$ under this map is denoted by L_{φ_0}. □

Remark 3.3.7. We can describe the leaves of the foliation $\mathcal{F}$ defined in Lemma 3.3.5 in different ways: consider the map $h + t : U \times \mathbb{R} \to \mathbb{R}$, $(x,t) \mapsto h(x) + t$, where h is defined in the proof of the previous lemma. The fibres of this map, which are also the graphs of the functions $\{-h + t_0 : U \to \mathbb{R}\}_{t_0 \in \mathbb{R}}$, define a foliation on $U \times \mathbb{R}$ that descends to the foliation $\mathcal{F}|_{U \times S^1}$ on $U \times S^1 = U \times \mathbb{R}/\mathbb{Z}$.

Intuitively, on $U \times S^1$, the leaves turn around in the "φ direction" asymptotically (since $-h(x) \xrightarrow{x \to \partial M} \infty$) when approaching ∂M as to make the foliation tame at the boundary. Replacing $d\varphi$ by α is therefore called *turbulising* (see Figure 3.1).

Remark 3.3.8 (Opposite turbulisation)**.** Note that, in the definition of α, we could have used $-g(r)dr$ instead of $g(r)dr$. The form obtained in that way would also define a foliation tame at the boundary. It is denoted by α_-. The previous analysis is still valid for the foliation defined for α_- if in the definition of h, $g(r)/f(r)$ is replaced by $-g(r)/f(r)$. Geometrically, the difference is that the leaves defined by α_- turn in the direction of "$-\varphi$" (see Figure 3.1).

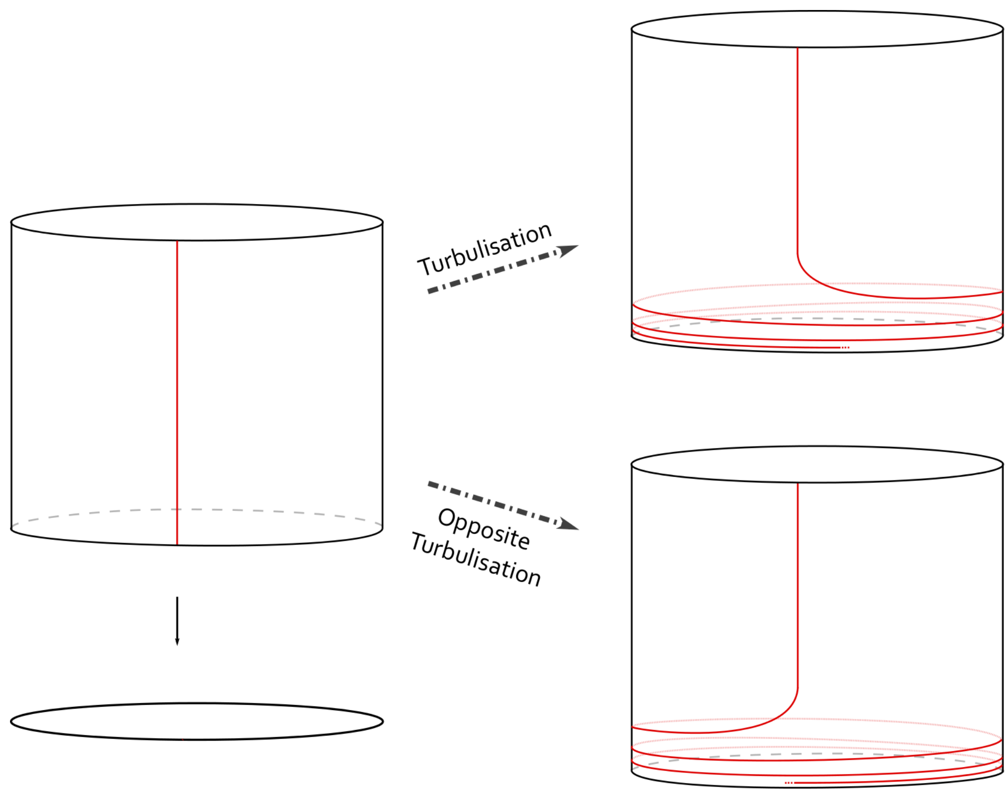


Figure 3.1: Turbulisation procedure.

Leafwise-symplectic structure

Proposition 3.3.9. *Let (M^{2n}, ω) be a symplectic manifold with boundary of cosymplectic type. Then the foliation on the manifold $M \times S^1$ obtained in Lemma 3.3.5 admits a leafwise-symplectic structure that is tame at the boundary in the sense of Definition 2.1.4. The symplectic structure at the boundary leaf is given by*

$$\omega|_{\partial M \times S^1} = \eta - \theta \wedge d\varphi,$$

where (η, θ) is an induced cosymplectic structure on ∂M.

Proof. Since ω is of cosymplectic type at the boundary, there is a collar neighbourhood of the boundary

$$k : \partial M \times [0, 1) \xrightarrow{\sim} U \subset M,$$

where ω can be written as $\omega|_U = \eta + \theta \wedge dr$ and (η, θ) is a cosymplectic structure on ∂M (Propositon 3.1.13). Consider the two-form $\Omega \in \Omega^2(M \times S^1)$ defined as:

$$\Omega = \begin{cases} \omega & \text{on } (M \setminus U) \times S^1 \\ \eta - b(r)\theta \wedge d\varphi + a(r)\theta \wedge dr & \text{on } U \times S^1, \end{cases}$$

where (a, b) is a good pair of functions. Here we abuse notation and denote by ω the pullback

$\mathrm{pr}_1^*(\omega)$ by the first projection $\mathrm{pr}_1 : M \times S^1 \to M$. Note that

$$d\Omega \wedge \alpha = 0, \qquad \Omega^n \wedge \alpha = \begin{cases} \omega^n \wedge d\varphi & \text{on } (M \setminus U) \times S^1 \\ (n-1)(af+bg)\eta^{n-1} \wedge \theta \wedge d\varphi \wedge dr & \text{on } U \times S^1, \end{cases}$$

where (f, g) is the good pair in the proof of Proposition 3.3.5. Therefore, ensuring $af + bg > 0$ on $[0, 1]$, we get that Ω defines a leafwise-symplectic form on $M \times S^1$ with the foliation given in the previous lemma. Note that it is tame at the boundary if we take the functions (a, b) such that $a(r) = 0$ and $b(r) = 1$ for $r < \epsilon$ for some small $\epsilon > 0$. □

We want to understand the symplectic leaves obtained in the previous lemma. To do so, let us first recall the manifold M^k_∞ obtained by gluing $\partial M \times (-\infty, 0]$ to M defined in Section 2.2. To define M^k_∞, we use a collar $k : \partial M \times [0, 1) \xrightarrow{\sim} U \subset M$. Take a collar such that $k^*(\omega) = \eta + \theta \wedge dr$, as before.

Recall the diffeomorphism $\psi_\xi : \mathrm{Int}M \to M^k_\infty$ given by Equation (2.2), which depends on the choice of a diffeomorphism $\xi : (0, \infty) \to \mathbb{R}$ satisfying $\xi(t) = t$ for $t \in [1, \infty)$. Since the functions a, b, f, g used in the proof of the previous lemma satisfy $a(t) = f(t) = 1$, $b(t) = g(t) = 0$ for $t \geq 1$, then we can choose ξ as a solution to the initial value problem:

$$\frac{d\xi}{dt} = \frac{bg}{f} + a, \qquad \xi(1) = 1. \tag{3.6}$$

M_∞ has a natural symplectic structure given by

$$\omega_\infty = \begin{cases} \omega & \text{on } M \\ \eta + \theta \wedge dt & \text{on } \partial M \times (-\infty, 0]. \end{cases}$$

To check the smoothness of this form, we can use the collar k_∞ from Equation (2.1):

$$k^*_\infty(\omega_\infty) = \eta + \theta \wedge dt \in \Omega^2(\partial M \times (-\infty, 1)),$$

and therefore ω_∞ is smooth.

Recall the notation $\mathcal{F} = \{L_\varphi\}_{\varphi \in S^1}$ from the proof of Lemma 3.3.6.

Lemma 3.3.10. *The leaf* $(L_{\varphi_0}, \Omega|_{L_{\varphi_0}})$ *is symplectomorphic to* $(M^k_\infty, \omega_\infty)$.

Proof. Consider the map $\psi_\xi : \mathrm{Int}M \to M^k_\infty$ described above (equation 2.2) and the map

$$\phi : \mathrm{Int}M \to L_{\varphi_0} \subset \mathrm{Int}M \times S^1, \quad x \mapsto \begin{cases} (x, [\varphi_0]) & \text{if } x \in \mathrm{Int}M \setminus U \\ (z, r, [\varphi_0 - h(r)]) & \text{if } x = (z, r) \in U \simeq \partial M \times (0, 1), \end{cases}$$

where h is defined by Equation (3.5). Then,

$$\psi^*_\xi(\omega_\infty) = \begin{cases} \omega & \text{on } \mathrm{Int}M \setminus U \\ \eta + \theta \wedge d\xi(r) & \text{on } U \end{cases}$$

$$\phi^*(\Omega|_{L_{\varphi_0}}) = \begin{cases} \omega & \text{on } \mathrm{Int}M \setminus U \\ \eta + b(r)\theta \wedge dh(r) + a(r)\theta \wedge dr & \text{on } U \end{cases}$$

Then, using the differential equation that ξ satisfies (see Equation (3.6)), we get that the map $\phi \circ \psi_\xi^{-1} : M^k_\infty \to L_{\varphi_0}$ is a symplectomorphism. □

Remark 3.3.11 (Opposite turbulisation)**.** Had we used the form α_- instead of α, then we should have defined the leafwise-symplectic form differently, with $+b(r)\theta \wedge d\varphi$ instead of $-b(r)\theta \wedge d\varphi$. The two-form obtained is denoted Ω_- and it satisfies that $\Omega_-^n \wedge \alpha_- \neq 0$.

The previous analysis of the symplectic leaves remains intact (the minus signs coming from the new definition of h cancel). The main change is then that the form Ω_- gives a symplectic form on $\partial M \times S^1$ defining the opposite orientation than that of Ω, or sticking to the convention that symplectic manifolds inherit an orientation from the symplectic form, Ω_- is a symplectic form on $\overline{\partial M} \times S^1$. The choice between (Ω, α) and (Ω_-, α_-) reflects the fact that we can choose the cosymplectic structure induced on the boundary of M by using an inwards or outwards pointing vector field, which changes the sign of the one-form of the cosymplectic structure. Equivalently, we can see that these two choices come from the two orientations of S^1.

In later chapters, when applying the previous lemmas, the choice of (Ω, α) is referred to as *turbulisation*, while the choice of (Ω_-, α_-) is referred to as *opposite turbulisation*.

Remark 3.3.12. Geometrically, the turbulisation procedure states that when having a symplectic manifold of cosymplectic type at the boundary (M, ω), then one can modify the product symplectic foliation on $M \times S^1$ to make it trivial and tame at the boundary by turning the leaves, which are initially transverse to the boundary, around the boundary $\partial M \times S^1$, hence the name turbulisation.

The Poisson bivector

Here we write explicitly the Poisson bivector obtained in Lemma 3.3.9. This can be found as the unique bivector π_T such that $\pi_T^\sharp(\alpha) = 0$ and such that the map $\pi_T^\sharp \circ \Omega^\sharp|_{T\mathcal{F}} : T\mathcal{F} \to T\mathcal{F}$ is the identity.

In this case, we can guess that $\pi_T|_{(M\setminus U)\times S^1} = \omega^{-1}$ and $\pi_T|_{U\times S^1} = \pi_0 + \sigma_1 X_0 \wedge \partial_r + \sigma_2 X_0 \wedge \partial_\varphi$ for some functions σ_1 and σ_2. Then we should find the functions σ_1 and σ_2 that make the conditions on π_T hold.

To check these conditions, it is convenient to have an explicit form for $T\mathcal{F}$.

$$T\mathcal{F} = \ker \alpha = \begin{cases} TM & \text{on } (M \setminus U) \times S^1 \\ T\partial M \oplus \langle f\partial_r - g\partial_\varphi \rangle & \text{on } U \times S^1. \end{cases}$$

After solving for σ_1 and σ_2, we find:

$$\pi_T = \begin{cases} \omega^{-1} & \text{on } (M \setminus U) \times S^1 \\ \pi_0 + \dfrac{1}{af + bg} X_0 \wedge (-f\partial_r + g\partial_\varphi) & \text{on } U \times S^1, \end{cases} \tag{3.7}$$

where (π_0, X_0) is the 1-Poisson structure on ∂M corresponding to the cosymplectic structure (η, θ) induced by ω in the boundary.

When performing the opposite turbulisation, we need to change $-g\partial_\varphi$ to $g\partial_\varphi$ in $T\mathcal{F}$ to obtain $\ker \alpha_-$ and $+g\partial_\varphi$ to $-g\partial_\varphi$ in the expression for π_T.

Example 3.3.13 (Codimension-one symplectic foliation on S^3)**.** The Reeb foliation on S^3 can be obtained using Lemma 3.3.5 and it can be endowed with a leafwise-symplectic structure using Lemma 3.3.9.

Let D^2_+, D^2_- be two copies of $D^2 = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ and let S^1_1, S^1_2 be two copies of $S^1 = \partial D^2$. Denote by $\varphi_\pm$ the S^1-coordinates on $\partial D^2_\pm$ and $\varsigma_{1,2}$ the S^1-coordinates on $S^1_{1,2}$. We can write

$$S^3 = D^2_+ \times S^1_1 \cup_{T^2} D^2_- \times S^1_2,$$

where the gluing diffeomorphism joins ∂D^2_+ with S^1_2 and ∂D^2_- with S^1_1. The standard area form ω_{D^2} on D^2 is of cosymplectic type at the boundary, so let us apply the turbulisation procedure to $D^2_+ \times S^1_1$ and the opposite turbulisation procedure to $D^2_- \times S^1_2$. The symplectic structures on the leaves are then $-d\varphi_+ \wedge d\varsigma_1$ and $d\varphi_- \wedge d\varsigma_2$. The gluing diffeomorphism maps these symplectic structures to each other and therefore, there using Theorem 2.3.4 these two symplectic foliations glue together to a codimension-one symplectic foliation on S^3.

3.4 Turbulisation: General Case

The results from the previous section can be generalised to manifolds which fibre over S^1. Before doing that, we introduce a class of cosymplectic structures on manifolds with boundary that will allow us to make this generalisation.

Cosymplectic manifolds with boundary of s-type

To generalise the previous construction we use cosymplectic structures on manifolds with boundary. We need, however, to impose some regularity conditions on the behavious of these cosymplectic structures around the boundary.

Definition 3.4.1. *Let (C, η, θ) be a cosymplectic manifold with boundary. The boundary ∂C is said to be of* ***s-type*** *if there exists a non-vanishing vector field V in a collar neighbourhood U of the boundary, such that:*

- *V is transverse to the boundary,*
- *$\mathcal{L}_V \theta|_U = 0$,*
- *$\mathcal{L}_V \eta|_U = 0$ and $\eta|_{\partial C}$ is nowhere symplectic.*

We also say in this case that the cosymplectic structure is of s-type at the boundary.

Remark 3.4.2. In the previous definition, the fact that $\eta|_{\partial C}$ is nowhere symplectic implies that the leaves defined by θ are transverse to the boundary, since η is non-degenerate on the leaves defined by θ. The converse is not true but we can characterise this condition in terms of the 1-Poisson structure defined by (η, θ):

Lemma 3.4.3. *Let (C, η, θ) be a cosymplectic manifold with boundary of s-type. Let (π, X) be the 1-Poisson structure on M corresponding to (η, θ). Then $\eta|_{\partial C}$ is nowhere symplectic if and only if X is tangent to the boundary.*

Proof. Let $U \simeq \partial C \times [0, 1)$ be an arbitrary collar neighbourhood of the boundary. In this collar, we can write

$$\eta = \beta_t \wedge dt + \gamma_t, \qquad \theta = \alpha_t + f_t dt, \qquad X = Z_t + g_t \partial_t,$$

for some forms $\beta_t, \alpha_t \in \Omega^1(\partial C)$, $\gamma_t \in \Omega^2(\partial C)$, functions $f_t, g_t \in C^\infty(\partial C \times [0,1))$ and some vector field $Z_t \in \mathfrak{X}(\partial C)$, depending smoothly on t. Recall that the vector field X is defined by the equations $\iota_X \eta = 0$ and $\iota_X \theta = 1$. These equations, when evaluated at the boundary give

$$\iota_{Z_0}\gamma_0 = g_0\beta_0, \qquad \iota_{Z_0}\beta_0 = 0, \qquad \iota_{Z_0}\alpha_0 + g_0 f_0 = 1.$$

We need to prove that $\gamma_0^n = 0$ if and only if $g_0 = 0$. Note that if $g_0 = 0$, then $Z_0 \in \mathfrak{X}(C)$ is nowhere zero and lies in the kernel of γ_0, so γ_0 can not have maximal rank and $\gamma_0^n = 0$. Conversely, if $\gamma_0^n = 0$, then $\eta^n \wedge \theta = (n-1)\gamma_0^{n-1} \wedge \beta_0 \wedge dt \wedge \alpha_0 \neq 0$. However

$$0 = \iota_{Z_0}(\gamma_0^n \wedge \alpha_0) = ng_0\beta_0 \wedge \gamma_0^{n-1} \wedge \alpha_0.$$

SInce, $\beta_0 \wedge \gamma_0^{n-1} \wedge \alpha_0$ is a volume form on ∂C, it follows that $g_0 = 0$. □

The definition of boundary of s-type requires the existence of a vector field transverse to the boundary that leaves the cosymplectic structure invariant. There is some freedom in choosing this vector field.

Lemma 3.4.4. *Let (C, η, θ) be a cosymplectic structure of s-type at the boundary. Then there is a vector field W, around the boundary, transverse to the boundary such that $\iota_W \theta = \mathcal{L}_W \eta = 0$.*

Proof. Let V be the vector field in the definition of s-type and let (π, X) be the 1-Poisson structure corresponding to (η, θ). Since $\mathcal{L}_V \theta = 0$, then $\theta(V)$ is locally constant. By the previous lemma, we know that X is tangent to ∂C and therefore, $W = V - \theta(V)X$ is transverse to the boundary. Then, since $\iota_X \theta = 1$, $\iota_W \theta = 0$ and since $\iota_X \eta = 0$, $\mathcal{L}_W \eta = \mathcal{L}_V \eta = 0$. □

Remark 3.4.5. Let (C, η, θ) be a cosymplectic manifold with boundary of s-type. By the previous lemma, we can choose the vector field on the s-type condition to be tangent to the leaves defined by θ. Therefore, the s-type condition ensures that η restricts to the leaves defined by θ to a symplectic structure of cosymplectic type at the boundary.

The vector field in the definition of boundary of s-type allows us to create a collar neighbourhood where the cosymplectic structure can be put in a standard form:

Lemma 3.4.6. *Let (C, η, θ) be a cosymplectic manifold with boundary of s-type. Then, there is a collar neighbourhood $U \simeq \partial C \times [0,1)$ and there is a 2-cosymplectic structure (γ, β, α) on ∂C such that*

$$\theta|_U = p^*(\alpha), \qquad \eta|_U = p^*(\beta) \wedge dt + p^*(\gamma), \qquad V|_U = \partial_t,$$

where $p : U \to \partial M$ is the projection.

Proof. Let V be the vector field from Lemma 3.4.4. We may assume that V points inwards. Let $U \simeq \partial C \times [0, \epsilon)$ be the tubular neighbourhood around the boundary defined by the flow of V. We can rescale the second coordinate and assume $U \simeq \partial C \times [0,1)$. On this neighbourhood U we can write,

$$\theta|_U = \alpha_t + adt \qquad \eta|_U = \beta_t \wedge dt + \gamma_t,$$

for some forms $\alpha_t, \beta_t \in \Omega^1(\partial C)$, $\gamma_t \in \Omega^2(\partial C)$, depending smoothly on t and a function $a \in C^\infty(U)$. Since $\theta|_U(\partial_t) = 0$, then $a = 0$. Moreover, since θ is closed, $\alpha_t = \alpha_0$. Furthermore, $\mathcal{L}_{\partial_t}\eta = d(\iota_{\partial_t}\eta) = -d\beta_t = 0$, which implies that $\beta_t = \beta_0$ and that $d^B\beta_0 = 0$, where d^B

denotes the de Rham differential on the boundary ∂M. Since η and β_0 are closed, then $d\gamma_t = 0$, which implies that $\gamma_t = \gamma_0$ and $d^B\gamma_0 = 0$.

Finally, since $\eta|_{\partial C} = \gamma_0$ is nowhere symplectic, then $\gamma_0^n = 0$, where $\dim C = 2n+1$. Thus, from $\eta^n \wedge \theta \neq 0$, it follows that $\gamma_0^{n-1} \wedge \beta_0 \wedge \alpha_0 \wedge dt \neq 0$. Therefore, we get that $(\gamma_0, \beta_0, \alpha_0)$ is a 2-cosymplectic structure on ∂M. □

We can obtain cosymplectic structures of s-type at the boundary from S^1-bundles over some symplectic manifolds (recall Lemma 3.2.15).

Proposition 3.4.7. *Let $f : C \to B$ be an S^1-bundle over a symplectic manifold with boundary of cosymplectic type (B, ω_B). Suppose $f^{-1}(\partial B) = \partial C$. Then, if the bundle $C \to B$ admits a horizontal distribution defined by a closed one-form θ, then the pair $(f^*(\omega_B), \theta)$ is a cosymplectic structure of s-type at the boundary.*

Moreover, if X is a vector field around ∂B for which $\mathcal{L}_X\omega_B = 0$, then, the horizontal lift $\tilde{X}$ of X, satisfies $\mathcal{L}_{\tilde{X}} f^*(\omega_B) = 0$ and $\theta(\tilde{X}) = 0$.

Proof. Let $\eta = f^*(\omega_B)$. From Lemma 3.2.15, we know that (η, θ) is a cosymplectic structure on C. It remains to check that it is of s-type at the boundary.

Since ω_B is of cosymplectic type at the boundary, there is a vector field X around ∂B such that $\mathcal{L}_X\omega_B = 0$. Consider the horizontal vector field $\tilde{X}$ on a neighbourhood of ∂C that projects to X. Therefore, $\mathcal{L}_{\tilde{X}}\eta = 0$. This, together with the fact that $\theta(\tilde{X}) = 0$, proves that (η, θ) is a cosymplectic structure on C of s-type at the boundary. □

General symplectic turbulisation

Note that if (M, ω) is a symplectic manifold with boundary of cosymplectic type, then $(M \times S^1, \omega, d\varphi)$ is a cosymplectic manifold with boundary of s-type. Therefore, the following theorem is a generalisation of the results from the previous section.

Theorem 3.4.8 (Symplectic Turbulisation)**.** *Let (C^{2n+1}, η, θ) be a cosymplectic manifold with boundary of s-type. Then there is a Poisson structure π on C of corank one that is tame at the boundary. The symplectic structure on the boundary is*

$$\pi|_{\partial C}^{-1} = \eta|_{\partial C} + \iota_V\eta|_{\partial C} \wedge \theta|_{\partial C},$$

where V is a vector field in the definition of boundary of s-type satisfying the conditions of Lemma 3.4.4.

Proof. First, we use Lemma 3.4.6 to find a collar neighbourhood $U \simeq \partial C \times [0, 1)$ where the cosymplectic structure can be written in U as

$$\theta|_U = \alpha \qquad \eta|_U = \beta \wedge dt + \gamma,$$

where the triple (γ, β, α) is a 2-cosymplectic structure on ∂M and we omit the p^* to simplify the notation.

Consider the pair:

$$\theta' = \begin{cases} \theta & \text{on } C \setminus U \\ f(t)\alpha - g(t)dt & \text{on } U, \end{cases} \qquad \eta' = \begin{cases} \eta & \text{on } C \setminus U \\ \gamma + \beta \wedge (a(t)dt + b(t)\alpha) & \text{on } U, \end{cases} \tag{3.8}$$

where (a,b), (f,g) are good pairs of functions satisfying $af+bg>0$. Then, since $\theta'\wedge d\theta'=0$, then θ' defines a foliation. Moreover,

$$
\begin{aligned}
d\eta'\wedge\theta' &= \begin{cases} d\eta\wedge\theta & \text{on } C\setminus U\\ db\wedge\beta\wedge\alpha\wedge(f(t)\alpha-g(t)dt) & \text{on } U,\end{cases}\\
\eta'^n\wedge\theta' &= \begin{cases} \eta^n\wedge\theta & \text{on } C\setminus U\\ -(n-1)(af+bg)\gamma^{n-1}\wedge\beta\wedge\alpha\wedge dt & \text{on } U,\end{cases}
\end{aligned}
\tag{3.9}
$$

then $d\eta'\wedge\theta'=0$ and $\eta'^n\wedge\theta'\neq 0$, and therefore, the pair (η',θ') defines a codimension-one symplectic foliation. The foliation defined by θ' is tame at the boundary and if the functions a,b,f,g are chosen such that $a(t)=f(t)=0$, $b(t)=g(t)=1$ for $t<\epsilon$ for some small ϵ, then the leafwise-symplectic structure is tame at the boundary. □

Note that the s-type condition is a condition only around the boundary. Therefore, the previous proof shows something slightly stronger than the statement: the fact that the forms (η,θ) are both closed far away from the boundary is not actually needed.

Remark 3.4.9. The s-type condition from Definition 3.4.1 can be understood from the turbulisation procedure, as the condition that allows us to change the original cosymplectic structure around the boundary $(\alpha,\beta\wedge dt+\gamma)$ to another cosymplectic structure $(dt,\gamma+\beta\wedge\alpha)$ (determined by the original structure and the vector field V) which gives a symplectic foliation tame at the boundary.

Theorem 3.4.10 (Symplectic turbulisation II)**.** *Let (C,π) be a corank-one Poisson manifold with boundary such that there exist:*

- *a collar neighbourhood U of ∂C and*
- *a pair of forms (η,θ) inducing π,*

such that $(\eta|_U,\theta|_U)$ is a cosymplectic structure of s-type at the boundary on U. Then there is a Poisson structure π on C of corank one that is tame at the boundary. The symplectic structure on the boundary is

$$\pi|_{\partial C}^{-1}=\eta|_{\partial C}+\iota_V\eta|_{\partial C}\wedge\theta|_{\partial C},$$

where V is a vector field in the definition of boundary of s-type.

Remark 3.4.11 (Opposite turbulisation in the general case)**.** Similarly as in the case of $M\times S^1$, when turbulising, one can choose the direction towards which the leaves (which are initially transverse to the boundary) are going to "rotate". In this case, change α for $-\alpha$ in the expressions for η' and θ' (see Equation (3.8)). This defines a new Poisson structure which induces a symplectic structure on the boundary leaf with the opposite orientation as the symplectic structure induced by π.

Tameness and unimodularity

First, note that the codimension-one symplectic foliation obtained in Theorem 3.4.8 is not globally tame in general, however it is tame if the compact leaves are removed, as we now see.

For that, let (C, η, θ) be a cosymplectic manifold with boundary of s-type and let π' be the Poisson structure on C obtained in Theorem 3.4.10. Recall the good pair (f, g) and the collar U of the boundary used in the construction (equation (3.8)) and consider $\bar{f} : C \to \mathbb{R}$ a function on C defined as: $\bar{f}(x) = f(t)$ for $x = (z, t) \in \partial C \times [0, 1) \simeq U$, and $\bar{f}|_{C \setminus U} = 1$. Consider the open set

$$C_0 := \{x \in C \mid \bar{f}(x) > 0\} = C \setminus (\partial C \times [0, \epsilon]) \subset C,$$

which is diffeomorphic to the interior of C. The symplectic foliations induced on C_0 by θ and θ' have no compact leaves. In this setting,

Lemma 3.4.12. *The symplectic foliation on C obtained in Theorem 3.4.8 is unimodular and tame in C_0.*

Proof. Consider the one-form θ' and two-form η' given in Equation (3.8), that define the foliation and the leafwise-symplectic structure. To check unimodularity, note that $(1/\bar{f})\theta'$ is a one-form that defines the same foliation as θ' and $(1/\bar{f})\theta'$ is closed. To check tameness, consider the good pair (a, b) from the proof of Theorem 3.4.10 (see Equation (3.8)) as functions of C in the same way it was done for f. Consider the two-form

$$\tilde{\eta} = \eta' - \frac{b}{\bar{f}}\beta \wedge \theta' = \begin{cases} \eta & \text{on } C_0 \setminus U \\ \gamma + (a + (1 - \bar{f})/\bar{f})\beta \wedge dt & \text{on } C_0 \cap U \end{cases}$$

This form defines the same leafwise-symplectic structure as η since $\pi'^{\sharp}(\theta') = 0$ and is $\tilde{\eta}$ is closed. $\square$

The following corollaries of Theorem 3.4.8 are used in later chapters.

Corollary 3.4.13. *For any cosymplectic manifold (M, η, θ) without boundary, the manifold $M \times D^2$ admits a codimension-one symplectic foliation tame at the boundary, with the symplectic structure on the boundary leaf given by*

$$\eta + d\varphi \wedge \theta.$$

Proof. It suffices to check that $M \times D^2$ can be endowed with a cosymplectic structure where the boundary is of s-type. Let $\omega_{D^2} = r dr \wedge d\varphi$ be the standard area form on D^2 in polar coordiantes and let V be the vector field around the boundary of D^2 given by $V = (1/r)\partial_r$. It satisfies that $\iota_V \omega_{D^2} = d\varphi$. Extend the vector field V trivially to the product $M \times D^2$ to get a vector field around the boundary $M \times \partial D^2$. We can use this vector field to check that $(\eta + \omega_{D^2}, \theta)$ on $M \times D^2$ is a cosymplectic structure of s-type at the boundary. $\square$

Several manifolds appear as suspensions and therefore it is useful to have a version of Theorem 3.4.10 for suspensions.

Corollary 3.4.14 (Symplectic turbulisation for suspensions)**.** *Let (M, ω) be a symplectic manifold with boundary of cosymplectic type and let $\psi \in \mathrm{Symp}(M, \omega)$ be a symplectomorphism. If M admits a vector field X on a neighbourhood U of the boundary satisfying:*

- *X is transverse to the boundary,*
- *X is invariant under ψ, i.e, $d\psi(X|_U) = X|_{\psi(U)}$ and*
- *X preserves ω, i.e $\mathcal{L}_X\omega = 0$,*

then, the suspension $M \times_{\mathbb{Z}} \mathbb{R}$ of M under the diffeomorphism ψ admits a codimension-one symplectic foliation tame at the boundary.

Proof. Recall the cosymplectic structure (η, θ) defined by the suspension (Definition 3.2.12), where $\theta := \mathrm{pr}_2^*(d\varphi)$, where the map $\mathrm{pr}_2 : M \times_{\mathbb{Z}} \mathbb{R} \to S^1 = \mathbb{R}/\mathbb{Z}$ is the one induced by the projection.

Let us first see that we can construct a neighbourhood V of ∂M which is invariant under ψ. Consider the collar neighbourhood $\chi : \partial M \times [0, \epsilon) \to M$, $\chi(x,t) = \varphi_X^t(x)$, where φ_X^t denotes the flow of X at time t, such that $\chi(\partial M \times [0,\epsilon)) \subset U \cap \psi(U)$. Then, since $\psi(\varphi_X^t(x)) = \varphi_X^t(\psi(x))$, $V = \chi(M \times [0, \epsilon))$ is a neighbourhood of ∂M invariant under ψ.

Let us extend X on V to a vector field $\tilde{X}$ around $\partial(M \times_{\mathbb{Z}} \mathbb{R})$ that is tangent to the fibres of pr_2 and therefore it satisfies $\theta(\tilde{X}) = 0$. Moreover, since $\mathcal{L}_X\omega = 0$, then $\mathcal{L}_{\tilde{X}}\eta = 0$. Therefore, the cosymplectic structure (η, θ) is of s-type at the boundary. □

Remark 3.4.15 (Proper cosymplectic manifolds with boundary of s-type)**.** Let (C, η, θ) be a cosymplectic manifold with boundary of s-type such that the cosymplectic structure is proper. In this case, Theorem 3.4.8 follows from the previous corollary. To see this, let $f : C \to S^1$ be a defining function for θ. Consider the vector field X of the 1-Poisson structure (π, X) associated with (η, θ), defined by the equations:

$$\iota_X\eta = 0, \qquad \iota_X\theta = 1.$$

By Lemma 3.4.3, X is tangent to the boundary and since, $\mathcal{L}_X\theta = 0$, its flow maps fibres of f to fibres of f. Therefore, its flow at times 1 (considering $S^1 = \mathbb{R}/\mathbb{Z} = \{e^{2\pi i z} | z \in [0,1]\}$) defines a monodromy φ that allows us to express $f : C \to S^1$ as a suspension $C \simeq f^{-1}(1) \times_{\mathbb{Z}} \mathbb{R}$. Now, consider the vector field W from the Lemma 3.4.4. It is transverse to the boundary and it satisfies that $\iota_W\theta = \mathcal{L}_W\eta = 0$. Note that $\theta([X,W]) = -d\theta(X,W) = 0$ and, $\forall A \in \mathfrak{X}(C)$

$$0 = \mathcal{L}_W(\eta(X,A)) = (\mathcal{L}_W\eta)(X,A) + \eta([W,X],A) + \eta(X,[W,A]) = \eta([W,X],A),$$

and therefore, $\iota_{[X,W]}\omega = 0$. Thus, $[X,W] = 0$ and the flow of X leaves W invariant. Finally, since $\mathcal{L}_X\eta = 0$, the flow of X preserves the symplectic structure $\eta|_{f^{-1}(1)}$. Therefore, we are in the setting of the previous corollary.

Chapter 4

Log-symplectic Structures and Symplectic Foliations

We saw in Chapter 1 that all closed orientable five-manifolds of the form $N^2 \times M^3$ admit a codimension-one symplectic foliation (Remark 1.2.5). A natural next step in finding codimension-one symplectic foliations on manifolds of the form $N^1 \times M^4$. Since we are interested in closed manifolds, then $N^1 = S^1$. We deal with this question in the coming two chapters.

In this chapter we introduce log-symplectic structures, a special type of Poisson structure that is helpful when constructing codimension-one symplectic foliations: starting from a log-symplectic structure on a manifold M we can easily construct a codimension-one symplectic foliation on $M \times S^1$. This method is called *regularisation*. Log-symplectic structures have a lot in common with symplectic structures of cosymplectic type and the regularisation procedure is closely related to the turbulisation procedure discussed in the previous chapter.

The regularisation method for log-symplectic structures allows us to construct symplectic foliations on manifolds of the form $M \times S^1$ where M admits a log-symplectic structure. Since there are manifolds that admit a log-symplectic structure but do not admit any symplectic structure, in this way we can create new non-trivial examples of symplectic foliations. Unfortunately, some interesting examples (e.g. S^4) are not symplectic and do not admit log-symplectic structures either. We will develop different methods to deal with these examples in the next chapter.

In this chapter we first recall the basics of log-symplectic structures, discussing their geometry and proving a normal form theorem for log-symplectic structures around the set of singularities. Then we discuss the regularisation method and its relation with the turbulisation procedure. In the last part, we discuss the obstructions to the existence of log-symplectic structures.

4.1 Geometry and Normal Form

Definition 4.1.1 (Log-symplectic structure)**.** *A Poisson structure π on a manifold M^{2n} is a log-symplectic structure if the map*

$$\wedge^n\pi : M \longrightarrow \bigwedge^{2n} TM, \qquad x \longmapsto \wedge^n\pi(x)$$

is transverse to the zero section.

These structures have had several names since their introduction in the late 90s. They were initially studied in the framework of deformation quantization in [NT96], where they are called *b-symplectic* structures. Later, their complex analogue was considered in [Got02], where they were first given the name log symplectic. In the context of Poisson geometry, this class of Poisson structures was introduced on two-dimensional surfaces in [Rad02] (under the name of *topologically stable Poisson structures*) where a complete classification was obtained. In higher dimensions a systematic investigation of the geometric properties of log symplectic structures appeared in [GMP14]. Their integrations by symplectic groupoids were studied in [GL14].

The set $Z := (\wedge^n\pi)^{-1}(0)$ is called the *singular set* of π. This singular set is the preimage of a transverse map and therefore, when non empty, it is a codimension-one submanifold of M. The transversality condition also ensures that the rank of π is $2n$ at points outside the singular set Z and $2n-2$ at points in Z. When $Z = \emptyset$, the Poisson structure is non-degenerate and its inverse defines a symplectic structure.

The Hamiltonian diffeomorphisms have to fix the singular locus since they preserve the Poisson structure and thus the (singular) foliation. Therefore, the Poisson bivector is tangent to the singular locus and the singular locus is then a Poisson submanifold of M.

The symplectic foliation induced by a log-symplectic structure consists of the connected components of $M \setminus Z$ and the codimension-one foliation induced on Z. This codimension-one foliation on Z is fairly special.

Modular class and cosymplectic structure on Z

Let (M,π) be an orientable log-symplectic manifold. Let μ be a volume form on M and denote by $t := \langle \pi^n, \mu\rangle$. The volume form μ defines a trivialisation of the bundle $\wedge^{2n}TM \xrightarrow{\sim} M \times \mathbb{R}$ by $a \mapsto (x, \langle a, \mu\rangle)$, where $a \in \wedge^{2n}T_xM$. Therefore, the log-symplectic condition implies that t is a submersion along Z. Thus, there is a vector field Y transverse to Z which projects under t to the vector field $\partial/\partial r$ in $\mathbb{R}$. Taking the flow of Y, we construct a neighbourhood U and a diffeomorphism $U \xrightarrow{\sim} Z \times (-\delta, \delta)$, $x \mapsto (p(x), t(x))$, where $p : U \to Z$ is a retraction. In this neighbourhood the Poisson bivector can be written as

$$\pi = V_t \wedge t\frac{\partial}{\partial t} + w_t,$$

with $V_t(dt) = 0$, $w_t(dt,\cdot) = 0$ and therefore, $\pi_Z = w_0$ and $X_Z = V_0$ are tangent to Z. The vector field V_t is the modular vector field corresponding to μ, since

$$tV_t(f) = \{f,t\} = L_{H_f}\langle \pi^n, \mu\rangle = \langle \pi^n, L_{H_f}\mu\rangle = \langle \pi^n, L_X(f)\mu\rangle = tX(f).$$

The equation $[\pi, \pi] = 0$ implies that $[\pi_Z, \pi_Z] = 0$ and $[\pi_Z, X_Z] = 0$, i.e, π_Z is Poisson and X_Z is a Poisson vector field. Since $\partial/\partial t \wedge V_t \wedge w_t^{n-1} = \mu^{-1} \neq 0$ for all $t \in (-\delta, \delta)$, we have that π_Z is regular of corank one and that X_Z is a transverse Poisson vector field.

By the Lemma 3.2.19, the pair (π_Z, X_Z) induces a cosymplectic structure (η, θ) on Z. The pair (π_Z, X_Z) determines π in a neighbourhood of the singular locus, as shown in the next lemma, which appeared in [GMP14].

Proposition 4.1.2 (Normal form for log-symplectic structures, see [GMP14])**.** *Let π be log-symplectic structure on an orientable manifold M, with compact singular locus $Z \neq \emptyset$ and let X be a representative of the modular vector field. Then*

1. *$\pi_Z := \pi|_Z$ is a regular corank-one Poisson structure on Z which has a transverse Poisson vector field given by $X_Z := X|_Z$. Moreover, there is a tubular neighbourhood $U \simeq Z \times (-1, 1)$, in which Z corresponds to $t = 0$, such that*

$$\pi|_U = X_Z \wedge t\partial_t + \pi_Z.$$

2. *Let (η, θ) be the cosymplectic structure corresponding to the 1-Poisson structure (π_Z, X_Z) (Lemma 3.2.19). Then, denoting $\omega := \pi|_{M\setminus Z}^{-1}$, we have that*

$$\omega|_{U\setminus Z} = d(\log |t|) \wedge \theta + \eta.$$

Proof. As before, let μ be a volume form on M and denote by $t := \langle \pi^n, \mu\rangle$ and let $U \simeq Z\times(-\delta, \delta)$ be a tubular neighbourhood of the singular locus, where the second coordinate is given by t and where the bivector can be written as

$$\pi = V_t \wedge t\frac{\partial}{\partial t} + w_t.$$

It was observed before that $\pi_Z := w_0$ is a Poisson structure on Z and that $X_Z := V_0$ is a nowhere-zero Poisson vector field on (Z, π_Z). Therefore, the bivector

$$\pi_0 = X_Z \wedge t\frac{\partial}{\partial t} + \pi_Z$$

is Poisson. We see that π and π_0 are related by an exact gauge transformation, and then we use the Moser argument to produce a diffeomorphism that maps π to π_0. Note that the inverses of π and π_0 on $U \setminus Z$ are given by

$$\omega := \pi|_{U\setminus Z}^{-1} = d\log |t| \wedge \theta_t + \eta_t, \qquad \omega_0 := \pi_0|_{U\setminus Z}^{-1} = d\log |t| \wedge \theta + \eta,$$

where θ_t and η_t are determined by $\theta_t(\partial/\partial t) = 0$, $\theta_t(V_t) = 1$, $w_t^\sharp(\theta_t) = 0$, $\eta_t^\sharp(\partial/\partial t) = 0$, $\eta_t^\sharp(V_t) = 0$ and $\eta_t^\sharp \circ w_t^\sharp(\xi) = \xi$, for all ξ with $\xi(V_t) = 0$. Clearly $\theta_0 = \theta$, $\eta_0 = \eta$ and $(n-1)dt \wedge \theta_t \wedge \eta_t^{n-1} = \mu \neq 0$. Also, since π_0 is Poisson, we have that ω_0 is closed, hence also θ and η are closed.

Observe that the difference $\Omega := \omega - \omega_0$ is a closed two-form that extends smoothly to Z. Since its pullback to Z vanishes, by the relative Poincaré lemma (see e.g. [Sil00]), we can write $\Omega = d\zeta$ for some one-form ζ vanishing on Z. Consider the family of two-forms $\omega_\epsilon := \omega_0+\epsilon\Omega$, for

$\epsilon \in [0,1]$. The two-forms ω_ϵ are non degenerate on $U' \setminus Z$, where U' is a small neighbourhood of Z and their inverses $\pi_\epsilon := \omega_\epsilon^{-1}$ define Poisson tensors on U'. The usual computation from the Moser lemma implies that the flow of the time dependent vector field $Y_\epsilon := -\pi_\epsilon^\sharp(\zeta)$ pulls ω_ϵ back to ω_0; hence it pushes forward π_0 to π_ϵ. Since Y_ϵ vanishes on Z, its flow is defined up to $\epsilon = 1$ on an open around Z. Finally, since the normal form is invariant under rescaling, we may assume the normal form holds on $Z \times (-1, 1)$. □

4.2 Regularisation

We discuss now the relation between log-symplectic structures on a manifold M and codimension-one symplectic foliations on $M \times S^1$.

In this section we deal with log-symplectic structures in manifolds with boundary, so we want to fix the following:

Definition 4.2.1. *A log-symplectic structure on a manifold with boundary $(M, \partial M)$ is a log-symplectic structure on M whose singular locus is the boundary ∂M.*

Log-symplectic structures can be used to construct codimension-one symplectic foliations.

Theorem 4.2.2 (Regularisation of log-symplectic structures)**.** *Let (M^{2n}, π) be a log-symplectic manifold and let Z be the singular locus of π. Let X be a Poisson vector field that is transverse to the symplectic foliation defined on Z. Then $\pi_R = \pi + X \wedge \partial_\varphi$ is a regular corank-one Poisson structure on $M \times S^1$.*

Proof. Since X is a Poisson vector field, the bivector $\pi_R = \pi + X \wedge \partial_\varphi$ is a Poisson structure. To see that it is regular, let us compute its rank. Note that $\pi_R^n = \pi^n + (n-1)\pi^{n-1} \wedge X \wedge \partial_\varphi$ and we get two cases: around $Z \times S^1$ the second term does not vanish, and in $M \setminus Z \times S^1$, the first one does not vanish. Since the terms can not cancel each other, the rank of π_R is $2n$. □

Remark 4.2.3. A vector field X as in the hypothesis of the theorem always exist; take for instance a representative of the modular vector class of (M, π).

Definition 4.2.4. *The regular Poisson structure π_R on $M \times S^1$ from Theorem 4.2.2 is called the **regularisation** of π with respect to the vector field X.*

Lemma 4.2.5. *If (M, π) is a log-symplectic manifold, then the Poisson structure π_R with respect to a representative X of the modular class of π is tame around $Z \times S^1$, where Z is the singular hypersurface.*

Proof. Let U be the open neighbourhood around Z where the normal form (Proposition 4.1.2) holds. Let $X_Z = X|_Z$ and (θ, η) be the cosymplectic structure induced on Z by (π, X_Z). The two-form

$$\omega = (dt - \theta) \wedge d\varphi + \eta$$

on $U \times S^1$ is closed and it satisfies

- $\pi_R^\sharp(\omega^\sharp(\pi_R^\sharp(dt))) = -tX_Z = \pi_R^\sharp(dt)$,
- $\pi_R^\sharp(\omega^\sharp(\pi_R^\sharp(\theta))) = t\partial_t + \partial_\varphi = \pi_R^\sharp(\theta)$,

- $\pi_R^\sharp(\omega^\sharp(\pi_R^\sharp(d\varphi))) = -X_Z = \pi_R^\sharp(d\varphi)$,

And therefore, $\pi_R^\sharp \circ \omega^\sharp|_{\mathrm{im}\pi_R^\sharp} = \mathrm{id}|_{\mathrm{im}\pi_R^\sharp}$. Then, ω extends the leafwise-symplectic structure on $U \times S^1$. Thus π_R is tame in $U \times S^1$. □

Log-symplectic structures vs symplectic structures of cosymplectic type

Lemma 4.2.6. *A manifold with boundary $(M, \partial M)$ admits a log-symplectic structure if and only if it admits a symplectic structure of cosymplectic type at the boundary.*

Proof. Let π be a log-symplectic structure on (M, ∂_M). By the normal form for log-symplectic structures (Proposition 4.1.2), there is a tubular neighbourhood $U \simeq \partial M \times [0,1)$ for which

$$\pi|_{U\setminus\partial M}^{-1} = \eta + dt/t \wedge \theta,$$

where (η, θ) is a cosymplectic structure on ∂M induced by π and t is the coordinate in [0,1). Take a positive function $\xi : [0,1] \longrightarrow \mathbb{R}$ that is equal to $1/t$ around $t = 1$ and equal to 1 around $t = 0$. Let ω_U be the two-form:

$$\omega_U = \eta + \xi(t) dt \wedge \theta.$$

This two-form can be extended by $\pi|_{M\setminus\partial M}^{-1}$ outside U to a symplectic form ω that, by its local form around ∂M, is of cosymplectic type at the boundary.

In a similar manner, for the reverse implication, consider ω a symplectic structure on $(M, \partial M)$ of cosymplectic type at ∂M. Then, there is a tubular neighbourhood $U \simeq \partial M \times [0,1)$ for which

$$\omega|_U = \eta + dt \wedge \theta,$$

where θ, η is the cosymplectic structure on ∂M induced by ω. Let $\psi : (0,1) \to \mathbb{R}$ be a smooth function equals to $1/t$ around 0 and 1 around 1. Consider the two-form on $U \setminus \partial M$

$$\omega_U' = \eta + \psi(t) dt \wedge \theta.$$

This form can be extended to a form on $M \setminus \partial M$. Its inverse defines then a log-symplectic Poisson structure on M with singular locus ∂M, inducing on M the same cosymplectic structure as ω. □

This interplay between log-symplectic structures and symplectic structures of cosymplectic type at the boundary becomes more subtle when one tries to do surgery.

Lemma 4.2.7 (Surgery, [FMTM13])**.** *Let $(M_1, \partial M_1, \omega_1)$ and $(M_2, \partial M_2, \omega_2)$ be two symplectic manifolds with boundary of cosymplectic type. If the cosymplectic manifolds ∂M_1 and $\overline{\partial M_2}$ are diffeomorphic, then $M_1 \sqcup_\partial M_2$ admits a symplectic structure. If the cosymplectic manifolds ∂M_1 and ∂M_2 are diffeomorphic then $M_1 \sqcup_\partial M_2$ admits a log-symplectic structure.*

Regularisation vs turbulisation

Given a symplectic manifold with boundary of cosymplectic type (M, ω), we can apply the turbulisation procedure (Proposition 3.3.9) to obtain a codimension-one symplectic foliation on $M \times S^1$ that corresponds to the bivector π_T from Equation (3.7). We can also obtain a log-symplectic structure π by using Lemma 4.2.6 and then apply the regularisation procedure to obtain a codimension-one symplectic foliation on $M \times S^1$ given by the bivector π_R from Theorem 4.2.2. For both procedures there are several choices involved. We would like to compare the result obtained by answering the following question:

Are there choices when constructing π_T and π_R such that $(M \times S^1, \pi_T)$ is isomorphic to $(M \times S^1, \pi_R)$?

Let us start with a symplectic manifold with boundary of cosymplectic type (M, ω). In a neighbourhood $U \simeq \partial M \times [0, 1)$, ω can be written as $\omega|_U = \eta + \theta \wedge dt$, where (η, θ) is the cosymplectic structure induced on ∂M. Its inverse is

$$\pi|_U = \omega|_U^{-1} = \pi_0 - X_0 \wedge \partial_t,$$

where (π_0, X_0) is the 1-Poisson structure corresponding to (η, θ).

Turbulisation. To obtain the Poisson bivector π_T that the turbulisation procedure gives, we have to choose two good pairs of functions (f, g) and (a, b) (see Equation (3.7)). Let us choose them such that $af + bg = 1$. In this case, π_T reads,

$$\pi_T|_{U \times S^1} = \pi_0 - fX_0 \wedge \partial_t + gX_0 \wedge \partial_\varphi.$$

on $U \times S^1$, and it extends as ω^{-1} to the rest of M.

Regularisation. To obtain π_R, we first need to apply the previous lemma to get log-symplectic structure $\pi_{\log}$ from π. This involves choosing a function $\psi : [0, 1] \to \mathbb{R}$ that is equal to $\psi(t) = t$ around $t = 0$ and $\psi(t) = 1$ around $t = 1$. To apply the regularisation procedure, we also need to choose a vector field X that is Poisson and transverse to the leaves on Z. Let j be a decreasing function that takes the value 1 around $t = 0$ and 0 around $t = 1$. It is easy to check that the vector field $j(t)X_0$ on U extends to M and is Poisson. The Poisson bivector obtained by regularisation on the bivector π_l is

$$\pi_T = \pi_0 - \psi X_0 \wedge \partial_t + jX_0 \wedge \partial_\varphi.$$

on $U \times S^1$ and it extends also as ω^{-1} to the rest of M.

Geometrically, the procedures of turbulising or regularising are qualitatively the same. Now we can see that they are also quantitatively the same: we claim that π_T and π_R are diffeomorphic in the interior of M (we assume $f > 0$ for $t > 0$). For that, consider a diffeomorphism ϕ that maps $f\partial_t|_{U \setminus \partial M}$ to $\psi\partial_t|_{U \setminus \partial M}$, and is the identity outside U and choose the function j so that $\phi_*(g\partial_t) = j\partial_t$. This diffeomorphism maps $\pi_T|_{\mathrm{Int}(M)}$ to $\pi_R|_{\mathrm{Int}(M)}$.

Remark 4.2.8. We could have chosen $-X$ to write π_R in Theorem 4.2.2. The choice of the sign of X is related, using the comparison between turbulisation and regularisation, to whether we use turbulisation or opposite turbulisation (Remark 3.3.11).

4.3 Obstructions

In this section we discuss the problem of existence of log-symplectic structures on compact manifolds. This question becomes important when trying to construct codimension-one symplectic foliations using Theorem 4.2.2.

The following lemma states that a compact log-symplectic manifold has a class in the second cohomology group whose powers, except maybe for the top, are nontrivial. This result gives cohomological obstructions for the existence of log-symplectic structures similar to those in symplectic geometry. We prove it here using the normal form of log-symplectic structures (Proposition 4.1.2), although the proof can be also be done without the normal form result, as in [MOT14b]. Compare also with Lemma 3.3.1.

Theorem 4.3.1. *Let (M^{2n},π) be a compact log-symplectic manifold. Then there exists a class $c \in H^2(M)$ such that $c^{n-1} \in H^{2n-2}(M)$ is nonzero.*

Proof. Assume first that M is orientable. In this case, from Proposition 4.1.2, there is a tubular neighbourhood $U \simeq Z \times (-1,1)$ where the inverse of π outside the singular locus can be written as

$$\pi|^{-1}_{M\setminus Z} = d(\log|t|)\wedge\alpha + \beta,$$

where (α,β) is a cosymplectic structure on Z induced by π. Note that $\alpha\wedge\beta^{n-1}$ is a volume form and therefore since Z is compact, the cohomology class of $[\beta]^{n-1} \in H^{2(n-1)}(Z)$ can not vanish. Here we construct a cohomology class $c \in H^2(M)$ whose pullback to Z is $[\beta]$ and therefore which satisfies the conclusion of the theorem. Consider the function $h : (-1,1) \to \mathbb{R}$ such that $h(t) = -1$ for $|t| < 1/3$ and $h(t) = \log|t|$ for $|t| > 2/3$. Then, the form

$$\omega' = d(h(t))\wedge\alpha + \beta$$

agrees with π^{-1} outside a tubular neighbourhood of Z and can be extended to Z, defining a closed two-form on M. Note that the pullback of the form $\omega' \in \Omega^2(M)$ is β and therefore, $c = [\omega'] \in H^2(M)$.

If M is not orientable, consider $p : \widetilde{M} \to M$ the orientable double cover, and let $\gamma : \widetilde{M} \xrightarrow{\simeq} \widetilde{M}$ be the corresponding deck transformation. We first construct a tubular neighbourhood $(\widetilde{r},t) : \widetilde{U} \xrightarrow{\simeq} \widetilde{Z}\times(-\delta,\delta)$ of the singular locus $\widetilde{Z} := p^{-1}(Z)$ of $\widetilde{\pi} := p^*(\pi)$, with $\widetilde{U} = p^{-1}(U)$, and such that the action of γ corresponds to $\gamma(z,t) = (\gamma(z),-t)$, for $(z,t) \in \widetilde{Z}\times(-\delta,\delta)$. The map $\widetilde{r} : \widetilde{U} \to \widetilde{Z}$ can be constructed by lifting a retraction $r : U \to Z$. Consider a volume form μ_0, and denote by f the smooth function satisfying $\gamma^*(\mu_0) = -e^f\mu_0$. Then the volume form $\mu := e^{f/2}\mu_0$ satisfies $\gamma^*(\mu) = -\mu$. Thus, by shrinking U, we can use $t := \langle\widetilde{\pi}^n,\mu\rangle$ to construct the desired tubular neighbourhood. As before, on $\widetilde{Z}\times(-\delta,\delta)$ we can write $p^*(\omega|_{U\setminus Z}) = \alpha_t \wedge dt/t + \beta_t$. Invariance under γ implies that $(\gamma|_{\widetilde{Z}})^*(\alpha_t) = \alpha_{-t}$ and $(\gamma|_{\widetilde{Z}})^*(\beta_t) = \beta_{-t}$. In particular α_0 and β_0 are invariant. Thus, choosing the function $\chi(t)$ from the construction from the orientable case to satisfy $\chi(t) = \chi(-t)$, we obtain an invariant closed two-form ω' on $\widetilde{M}$ that satisfies $[\omega']^{n-1} \neq 0$. Invariance implies that $\omega' = p^*(\omega'')$ for a closed two-form ω'' on M; hence $c := [\omega'']$ satisfies the conclusion. $\square$

Observe that for $Z \neq \emptyset$ the proof of the theorem uses only the compactness of Z and not that of M.

Remark 4.3.2. This theorem rules out the possibility of using the regularisation method to construct codimension-one symplectic foliations on manifolds like $S^4 \times S^1$ by constructing a log-symplectic structure on S^4.

Despite the obstruction from the previous theorem being similar to the one there is in symplectic geometry, there are manifolds which are not symplectic but still admit a log-symplectic structure, such as $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ for $m > 1$, $n > 0$ (see [Tau94] for the non-symplectic part and [Cav13] for the existence of a log-symplectic structure).

There is an additional obstruction for the existence of log-symplectic structures that have non-empty singular locus, which reduces further the applicability of the regularisation method.

Theorem 4.3.3 ([Cav13]). *If a compact orientable manifold M admits a log-symplectic structure with nonempty singular locus then there is a nontrivial class $b \in H^2(M)$ such that $b^2 = 0$.*

On the other hand, there are some interesting examples of log-symplectic structures that can be used to construct codimension-one symplectic foliations.

Theorem 4.3.4 ([Cav13]). *If M is a simply connected compact four-manifold, then $M\#(S^2 \times S^2)$ and $M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ admit log-symplectic structures.*

Using Theorem 4.2.2, we immediately get the following existence result.

Corollary 4.3.5. *If M is a simply connected compact four-manifold, then*

$$(M\#(S^2 \times S^2)) \times S^1, \qquad (M\#\mathbb{C}P^2\#\overline{\mathbb{C}P^2}) \times S^1$$

admit a codimension-one symplectic foliation.

Chapter 5

Lefschetz-type Fibrations and $M^4 \times S^1$

As mentioned in the introduction, the goal of this chapter is to prove results of the form

Theorem 5.0.6. *If M is a closed oriented four-manifold that admits either a genuine fibration or an achiral Lefschetz fibration onto a two-dimensional manifold, then $M \times S^1$ admits a codimension-one symplectic foliation.*

Using this theorem, we can produce codimension-one symplectic foliations on a family of five-dimensional manifolds of the form $M \times S^1$, including $S^4 \times S^1$, where it was not possible to apply the methods of previous chapters (see Remark 4.3.2).

The construction in this chapter follows the general scheme we have used before: the manifold M is split into compact pieces $M = \cup_i X_i$ intersecting only at the boundary ∂X_i. Each piece X_i is endowed with a symplectic structure of cosymplectic type at the boundary and tame corank-one Poisson structures in each piece $X_i \times S^1$ and gluing them all together.

The new feature in this chapter is the use of fibrations to construct the symplectic structures of cosymplectic type at the boundary on the pieces X_i, from which the codimension-one symplectic foliations on $X_i \times S^1$ arise.

Conventions. As in the previous chapters, all manifolds are supposed to be oriented.

Preliminaries

As mentioned above, we will construct symplectic structures on manifolds which admit some type of fibrations. These fibrations will satisfy condition ii. from Section 1.7. To be able to use these symplectic structures to construct codimension-one symplectic foliations, we need to make sure that the cosymplectic structures are of cosymplectic type. This is ensured in this chapter by a compatibility of the symplectic structures with the fibrations:

Lemma 5.0.7. *Let $M \xrightarrow{f} B$ be a surjective submersion from a compact manifold with boundary M to a surface B such that $f^{-1}(\partial B) = \partial M$. Let ω be a symplectic form on M such that $\omega|_{f^{-1}(b)}$ is symplectic for every $b \in \partial B$. Then ω is of cosymplectic type at the boundary and $f^*(d\varphi)$ is a closed admissible one-form for $\omega|_{\partial M}$, where φ is the angle coordinate in ∂B.*

The proof is this lemma follows directly from simple linear algebra.

Remark 5.0.8. In the same setting as in the previous Lemma, using the symplectic structure ω on M we can construct a tame codimension-one symplectic foliation on $M \times S^1$. The symplectic structure on the boundary leaf of $M \times S^1$ can be explicitly described as follows: $\partial M \xrightarrow{f} \partial B$ is a surjective submersion, and since ∂B is a disjoint union of circles. The symplectic form on the boundary leaf $\partial M \times S^1$ can be described as

$$\omega = \eta|_{\partial M} + f^*(d\varphi) \wedge d\psi,$$

where φ is the angle in the S^1s on the base ($\partial B \simeq \sqcup S^1$) and ψ is the angle variable on the S^1 in the product $M \times S^1$.

Thurston's trick

To apply Lemma 5.0.7, we need a symplectic form on M which is symplectic on the fibres. In this chapter, we will often find a closed two-forms η which is symplectic on the fibres but not necessarily on M. This form might degenerate along the horizontal directions of the fibration. Thurston [Thu76b] proved that if the total space is compact and the base is symplectic, one can modify η to make it symplectic by adding the missing horizontal components.

Lemma 5.0.9 (Thurston's trick, [Thu76b]). *Let $f : M \to B$ be a fibration of a compact manifold M over a symplectic manifold (B, ω_B) such that there is a closed two-form ω on M that restricts to the fibres symplectically. The form $\omega + Kf^*(\omega_B)$ is symplectic for $K >> 0$.*

Gluing

It will be useful to have a version of the gluing Theorem 2.3.4 in terms of the cosymplectic structures that induce the symplectic structures on the boundary leaves:

Corollary 5.0.10 (Of Theorem 2.3.4). *Let M_i, $i = 1, 2$, be compact symplectic four-manifolds with boundary and $\psi : \partial_0 M_2 \xrightarrow{\sim} \partial_0 M_1$ be a diffeomorphism between components of the boundaries $\partial_0 M_i \subset \partial M_i$. Let $(M_i \times S^1, \mathcal{F}_i, \vartheta_i)$ be codimension-one symplectic foliations tame at the boundary. Suppose that there are cosymplectic structures (η_i, θ_i) on $\partial_0 M_i$ such that:*

- $\vartheta_i|_{\partial_0 M_i \times S^1} = \eta_i + d\varphi \wedge \theta_i$,
- *there is a path of cosymplectic structures (η_t, θ_t), $t \in [0, 1]$, on $\partial_0 M_2$, joining (η_2, θ_2) to $(\psi^*(\eta_1), \psi^*(\theta_1))$.*

Then, the symplectic foliations $(\mathcal{F}_1, \vartheta_1)$ and $(\mathcal{F}_2, \vartheta_2)$ can be glued to a codimension-one symplectic foliation on $(M_1 \cup_\psi M_2) \times S^1$.

If $\partial(M_1 \cup_\psi M_2) \neq \emptyset$, then the symplectic foliation obtained is tame at the boundary.

Proof. The path $\omega_t = \eta_t + d\varphi \wedge \theta_t$ is a path of symplectic structures joining $\vartheta_1|_{\partial_0 M_1 \times S^1}$ and $\psi^*(\vartheta_2|_{\partial_0 M_2 \times S^1})$. Using Lemma 2.3.4 we obtain a codimension-one symplectic foliation on $(M_1 \cup_\psi M_2) \times S^1$. Since we glued at the boundary $\partial_0 M_i \times S^1$, $i = 1, 2$ and the other boundaries were left untouched, then the resulting symplectic foliation is tame at the remaining boundaries. $\square$

Lemma 5.0.11. *Let M be a closed three-manifold. Then the cosymplectic structures on M which have the same closed one-form and induce the same orientation on M form a convex set.*

Proof. Let (η, θ) be a cosymplectic structure on M. Note that if (η', θ) is another cosymplectic structure, then for $t \in [0,1]$, $(\eta + t\eta', \theta)$ is a cosymplectic structure on M. $\square$

From the previous two results, it follows immediately that:

Corollary 5.0.12. *Let M_i, $i = 1, 2$, be compact symplectic four-manifolds with boundary and $\psi : \partial_0 M_2 \xrightarrow{\sim} \partial_0 M_1$ be a diffeomorphism between components of the boundaries $\partial_0 M_i \subset \partial M_i$. Let $(M_i \times S^1, \mathcal{F}_i, \vartheta_i)$ be codimension-one symplectic foliations tame at the boundary. Suppose that there are cosymplectic structures (η_i, θ_i) on $\partial_0 M_i$ such that*

- $\vartheta_i|_{\partial_0 M_i \times S^1} = \eta_i + d\varphi \wedge \theta_i$,
- $\theta_2 = \psi^*(\theta_1)$,
- (η_2, θ_2) *and* $(\psi^*(\eta_1), \theta_2)$ *define the same orientation on* $\partial_0 M_2$.

Then, the symplectic foliations $(\mathcal{F}_1, \vartheta_1)$ and $(\mathcal{F}_2, \vartheta_2)$ can be glued to a codimension-one symplectic foliation on $(M_1 \cup_\psi M_2) \times S^1$. In case $\partial(M_1 \cup_\psi M_2) \neq \emptyset$, then the symplectic foliation obtained is tame at the boundary.

Remark 5.0.13. If $f : N \to S^1$ is a fibration and $\eta_1, \eta_2 \in \Omega^2(\partial N)$ are two cosymplectic forms which are non-degenerate on the fibres, then from Lemma 5.0.7, we know that $\theta := f^*(d\varphi)$ is an admissible one-form for η_1 and η_2. Then, the cosymplectic structures (η_1, θ) and (η_2, θ) define the same orientation on N if and only if η_1 and η_2 define the same orientation on the fibres of f.

In this chapter, we will need to apply the previous lemmas when the codimension-one symplectic foliations on $M_1 \times S^1$ and $M_2 \times S^2$ come from certain symplectic structures of cosymplectic type on M_1 and M_2. We state now a gluing lemma in terms of these symplectic structures.

Lemma 5.0.14 (Gluing lemma)**.** *Let $f : M \to B$ be a smooth surjective map from a compact connected four-manifold M onto an oriented surface B such that*

- *The set of singular points $S := \mathrm{Sing}(f) \subset M$ is a finite union of points and circles and $f|_S$ is injective.*
- *f has connected fibres.*

Suppose there is a decomposition $B = \cup_{k=1}^{l} B_k$ into compact surfaces with boundary intersecting only at the common boundaries and symplectic structures $\omega_k \in \Omega^2(f^{-1}(B_k))$ such that

- *$f(S) \cap \partial B_k = \emptyset$ for all k*
- *$\omega_k|_{f^{-1}(x)}$ is symplectic for every regular value $x \in B_k$.*

Then, $M \times S^1$ admits a codimension-one symplectic foliation.

Proof. Let $M_k := f^{-1}(B_k)$. Consider the fibration $f|_{M\setminus S} : M \setminus S \to B$. Since M and B are orientable, then the vector bundle $\ker(df|_{M\setminus S}) \to M \setminus S$ is orientable. Fix an orientation for this bundle. This fixes an orientation on the regular fibres of f and on the regular part of the singular fibres.

For each k, $M_k \setminus S$ is connected, and therefore, the vector bundle $\ker(df|_{M_k\setminus S}) \to M_k \setminus S$ has only two orientations. We may assume that $\omega_k|_{\partial M_k}$ induces the given orientation on the this vector bundle. Since if it induces the opposite orientation, we may take $-\omega_k$.

By Lemma 5.0.7, ω_k is of cosymplectic type at the boundary and the one-form $f^*(d\varphi)$ is admissible for $\omega_k|_{\partial M_k}$, where φ is the angle coordinate on ∂B_k. Therefore, each ω_k induces a codimension-one symplectic foliation on $M_k \times S^1$. We want to glue all these symplectic foliations using Corollary 5.0.12 to obtain a codimension-one symplectic foliation on $(\cup_k M_k) \times S^1 = M \times S^1$. Note that in this case, all the gluing diffeomorphisms are the identity.

Note first that for each l, ω_l induces the given orientation on the regular fibres of ∂M_l and therefore, by Remark 5.0.13, for all pairs $k \neq j$ for which M_j and M_k are glued along a common boundary, say $\partial_1 M_k = \partial_1 M_j$, $(\omega_k|_{\partial_1 M_k}, f^*(d\varphi))$ and $(\omega_j|_{\partial_1 M_j}, f^*(d\varphi))$ are two cosymplectic structures on $\partial_1 M_k$ inducing the same orientation, where φ is a volume form on $\partial_1 B_k$.

Let us fix volume forms $d\varphi_k$ on ∂B_k for all k such that if B_j and B_k have a common boundary $\partial_0 B_j = \partial_0 B_k$, then $d\varphi_k|_{\partial_0 B_k} = d\varphi_j|_{\partial_0 B_j}$. Construct symplectic foliations on $M_k \times S^1$ for all k, which are tame at the boundary, where the symplectic structures on the boundary $\partial_1 M_k$ is induced by the cosymplectic structure $(\omega_k|_{\partial M_k}, f^*(d\varphi_k))$. Since the cosymplectic structures define the same orientations on the common boundaries, we can use Corollary 5.0.12 to glue these codimension-one symplectic foliations to obtain a codimension-one symplectic foliation on $M \times S^1 = (\cup_k M_k) \times S^1$. $\square$

Remark 5.0.15. For the conclusion of the previous lemma to hold we do not need the orientability of M and B but only the orientability of the bundle $\ker d(f|_{M\setminus S}) \to B$.

Remark 5.0.16. Note that in the statement of the previous lemma there is no mention of orientations. The orientations issues that arise are solved in the proof. The reason they can be solved is that using the fibration, we can choose the orientations consistently on each one of the pieces of M in a way that the pieces can be glued without any problem.

5.1 Fibrations

The aim of this section is to prove the following proposition:

Theorem 5.1.1. *Let M^4 be a closed four-manifold. If M admits a surjective submersion onto an orientable surface B, then $M \times S^1$ admits a codimension-one symplectic foliation.*

Remark 5.1.2. Under the assumptions of the previous theorem, Thurston [Thu76b] proved that if the fibre is not T^2, then M is symplectic and hence the proposition is automatic. However, there are interesting cases in which M is not symplectic, e.g., $M = S^3 \times S^1$ admits the Hopf fibration $h : S^3 \times S^1 \to S^2$ but it is not symplectic. Nonetheless, a codimension-one symplectic foliation on $M \times S^1 = S^3 \times T^2$ can be easily constructed by taking the codimension-one symplectic foliation of S^3 from Example 3.3.13 and crossing it with T^2 with the standard symplectic structure.

The proof of Theorem 5.1.1 is done by reducing to the case when the surface B has non-empty boundary or equivalently, $H^2(B) = 0$.

Proposition 5.1.3. *Let $M \xrightarrow{\pi} B$ be a surjective submersion of a compact orientable four-manifold M onto an orientable surface B for which $H^2(B) = 0$, such that $\partial M = \pi^{-1}(\partial B)$. Then M admits a symplectic structure which restricts symplectically to the fibres.*

Theorem 5.1.1 follows directly from the previous lemma:

Proof of Theorem 5.1.1 assuming Proposition 5.1.3. Take a closed disk $D^2 \subset B$ and split the surface B as $B = D^2 \cup (B \setminus \mathrm{Int} D^2)$. Then apply Proposition 5.1.3 to obtain symplectic structures on $\pi^{-1}(D^2)$ and $M \setminus \pi^{-1}(D^2)$ which are symplectic on the fibres. Use finally Lemma 5.0.14 to obtain a codimension-one symplectic foliation on $M \times S^1$. □

Proof of Proposition 5.1.3. Note that B is a two-disk with one-handles attached. Decomposing M as the union of a closed tubular neighbourhood of the fibre and the closure of its complement and using the Mayer Vietoris sequence in homology for this decomposition, it follows that the homology class of the fibre is non-zero in the homology of the space. Take then a two-form $\eta \in \Omega^2(M)$ whose integral over the fibre is non-zero. This two-form can be modified into a two-form which is fibrewise symplectic using standard differential-geometric modifications (see details in the proof of Proposition 5.2.7). Finally, using Thurston's trick, this form can be made symplectic. □

5.2 Lefschetz Fibrations

The goal of this section is to prove Theorem 5.2.5, that states that if a manifold M admits a Lefschetz fibration, then $M \times S^1$ admits a codimension-one symplectic foliation.

First let us recall the notion of Lefschetz singularity and Lefschetz fibration.

Definition 5.2.1 ((Achiral) Lefschetz singularities)**.** *A smooth map $f : M \to N$ from a four-dimensional oriented manifold M, possibly with boundary, into a two-dimensional oriented manifold N is said to have a* ***achiral Lefschetz singularity*** *at $p \in \mathrm{Int}\, M$ if there is a chart U around p and V around $f(p)$ such that one can find complex coordinates (z_1, z_2) on U, centered at p, and z on V, centered at $f(p)$, such that $z = f(z_1, z_2) = z_1^2 + z_2^2$.*
These complex coordinates do not have to define the aready-given orientation on M and N. If they do, the singularity is called a ***Lefschetz singularity****.*

Definition 5.2.2 ((Achiral) Lefschetz fibration)**.** *An* ***(achiral) Lefschetz fibration*** *is a smooth map $f : M \to N$ from a four-dimensional oriented manifold M into a two-dimensional oriented manifold N that has (achiral) Lefschetz singularities at a finite set $\{x_1, \ldots, x_k\}$, but is everywhere else submersive. If M and N have boundary, we require $\{x_1, \ldots, x_k\} \subset \mathrm{Int}(M)$ and $\pi^{-1}(\partial N) = \partial M$.*

Remark 5.2.3. If $\pi : M \to \Sigma$ is an achiral Lefschetz fibration on M, by perturbing π slightly if necessary, we can ensure that π is injective on the set of singular points (see [GS99]).

Remark 5.2.4. Note first that an achiral Lefschetz fibration with only one achiral Lefschetz singularity can be made into a Lefschetz fibration by adjusting the orientation of the base.

The main theorem proved in this section is:

Theorem 5.2.5. *Let M be a compact, oriented four-manifold, with or without boundary. If M admits an (achiral) Lefschetz fibration, then $M \times S^1$ admits a corank-one Poisson structure. If M has boundary, then the corank-one Poisson structure is tame at the boundary.*

If M admits a Lefschetz fibration whose fibre is non-trivial in homology, this result follows directly from Gompf's Theorem:

Theorem 5.2.6 (Gompf [GS99])**.** *Let M be a compact oriented 4-manifold which admits a Lefschetz fibration $f : M \to B$. If the homology class of the fibres of f are non-zero in the homology of M, then M admits a symplectic structure symplectic on the fibres.*

However, even the fibre is zero in homology, we can easily reduce the proof Theorem 5.2.5 to the following proposition, which can be seen as a particular case of Gompf's theorem.

Proposition 5.2.7. *Let $M \xrightarrow{\pi} D^2$ be a Lefschetz fibration of a compact four dimensional manifold M onto the disk D^2 with only one Lefschetz singularity at $p \in \pi^{-1}(0)$. Then, M admits a symplectic structure ω that restricts to the regular fibres symplectically.*

Let us now see how Theorem 5.2.5 follows from the previous proposition.

Proof of Theorem 5.2.5, assuming Proposition 5.2.7. Let $\pi : M \to \Sigma$ be an achiral Lefschetz fibration on M. If the set $C \subset M$ of achiral Lefschetz singularities is empty, we are in the setting of Theorem 5.1.1 and the result follows automatically. We may assume then that $C \neq \emptyset$. Choose a collection of closed neighbourhoods $U_x \subset \Sigma$, with smooth boundaries, around points in $\pi(C)$ such that $U_x \cap U_{x'} = \emptyset$ for $x \neq x'$. Let $V_x = \pi^{-1}(U_x)$ and decompose M as:

$$M = \cup_{x \in C} V_x \cup \big(M \setminus \cup_{x \in C} \mathrm{Int} V_x\big).$$

For every $x \in \pi(C)$, $\pi|_{V_x} : V_x \to U_x$ is a Lefschetz fibration with only one singular point. We can use Proposition 5.2.7 to obtain a symplectic structure $\omega_x \in \Omega^2(V_x)$ that is non-degenerate on the regular fibres and use Proposition 5.1.3 to obtain a symplectic structure ω on $M \setminus \cup_{x \in C} \mathrm{Int} V_x$ which also restricts symplectically to the fibres. Using the Gluing Lemma 5.0.14, we obtain a codimension-one symplectic foliation on $M \times S^1$, tame at the boundary. $\square$

The proof of Proposition 5.2.7 is given by Gompf [GS99] as a step of the proof of Theorem 5.2.6. His proof does not contain many details and therefore we give a detailed proof of Proposition 5.2.7 in the Appendix at the end of this chapter.

The ideas of Gompf's proof were used by Cavalcanti [Cav13] to produce log-symplectic structures on manifolds with achiral Lefschetz fibrations in which the fibre is homologically non-trivial.

Example 5.2.8 (Codimension-one symplectic foliation on $S^4 \times S^1$)**.** Using Theorem 5.2.5 and the fact that S^4 admits an achiral Lefschetz fibration with T^2-fibres over S^2 with two singular points (see [GS99], Example 8.4.7), we get that $S^4 \times S^1$ admits a codimension-one symplectic foliation.

Example 5.2.9. Etnyre and Fuller [EF06] proved the following result

Theorem 5.2.10 ([EF06]). *Let X be a smooth, closed, oriented four-manifold. Then there exists a framed circle in X such that the manifold obtained by surgery along that circle admits an achiral Lefschetz fibration with base S^2.*

Combining this with Theorem 5.2.5, we conclude that for every smooth, closed, oriented four-manifold X, there is a framed circle in X such that, if $\tilde{X}$ denotes the manifold obtained from surgery along that circle, $\tilde{X} \times S^1$ admits a codimension-one symplectic foliation.

Remark 5.2.11. In [GS99], it is shown that some manifolds (for example the connected sum of $S^1 \times S^3$ with itself n-times, for $n > 1$) do not admit achiral Lefschetz fibrations, something that shows the limitations of Theorem 5.2.5 and shows that a stronger result is necessary. In the next section we try prove such a result.

5.3 Broken Lefschetz fibrations: a failed attempt

Broken Lefschetz fibrations are a type of fibration that generalises Lefschetz fibrations. They are surjective submersions where the set of singularities is allowed to have Lefschetz singularities (not achiral) and some other type of singularities, called round singularities. Every compact four-manifold admits a broken Lefschetz fibration onto S^2 and not every four-manifold admits an achiral Lefschetz fibration [BK15]. Therefore, if we could generalise Theorem 5.2.5 to have broken Lefschetz fibrations instead of achiral Lefschetz fibrations, we would prove that for all compact four-manifolds M, the manifold $M \times S^1$ admits a codimension-one symplectic foliation.

In this section we see that there are certain problems with such a generalisation and we see that a theorem analogous to Theorem 5.2.5 for broken Lefschetz fibrations can not be proven along the same lines as it was done for Lefschetz fibrations. However, while proving that this method does not work we will need some results which are interesting on its own: we will prove that if W is the three-manifold with boundary giving the basic cobordism between T^2 and S^2 (see Example 5.3.4), then $W \times S^1$ does not admit a symplectic structure of cosymplectic type (Theorem 5.3.8).

Geometry of round singularities

Definition 5.3.1 (Round Singularities). *A smooth map $\pi : M \to N$ from a four-dimensional oriented manifold M into a two-dimensional oriented manifold N is said to have a* ***round singularity*** *along an embedded one-manifold $Z \in \operatorname{Int} M$ if around every point $z \in Z$, there are coordinates (t, x, y, z) with t a local coordinate on Z, in which π can be expressed as $\pi(t, x, y, z) = (t, z^2 - x^2 - y^2)$.*

These singularities, together with the Lefschetz singularities are important because any generic map from a closed four-manifold M to S^2 can be homotoped to one that has only these type of singularities ([BK15]).

Definition 5.3.2. *A* ***broken Lefschetz fibration*** *(BLF) on a four-manifold M is a smooth map $\pi : M \to S^2$ that is a submersion outside a finite number of points $C \subset \operatorname{Int} M$ and a finite number of circles $Z \subset \operatorname{Int} M$, where it has Lefschetz singularities and round singularities, respectively.*

Conventions. Similarly to the Lefschetz fibration case, we may assume that the BLFs are injective when restricted to the singular set $C \cup Z$. Throughout this section,

$$I := [-1, 1]$$

and

$$A := \{x \in \mathbb{R}^2, |x| \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right]\}$$

denotes the annulus.

Lemma 5.3.3. *Let $M \xrightarrow{\pi} A$ be a broken Lefschetz fibration of a compact four-dimensional manifold M over the annulus A, with only round singularities along a circle $\Gamma \simeq S^1$ such that $\pi(\Gamma) = \{|x| = 1\}$ and $\pi^{-1}(\partial A) = \partial M$. Then M is the suspension of a three-dimensional manifold N with boundary. Moreover, N admits a real valued Morse function $g : N \to I$ with only one singular point in the interior, of index 2, such that $g^{-1}(\partial I) = \partial N$.*

Proof. Consider the map $h : M \to S^1$, $x \mapsto \pi(x)/|\pi(x)|$, which can be seen as the composition of π with pr_r, the radial projection. Note that this map is a submersion of M onto S^1: outside the circle of singularities Γ, h is a composition of submersions so it is a submersion. The map π has rank 1 on the points $x \in \Gamma$, so to check that h is submersive at points $x \in \Gamma$ we need to check that the one-dimensional image of $d\pi(x)$ is transverse to the radial vector field on A (the latter being the kernel of $d\mathrm{pr}_r$). To see that, consider the normal form of π around x: there is an open U around x with coordinates (t, x, y, z) and an open V with coordinates (t, r) around $\pi(x)$ such that π can be written as

$$\pi|_U : U \to V, \qquad (t, x, y, z) \mapsto (t, z^2 - x^2 - y^2),$$

where $(t, 0, 0, 0) = U \cap \Gamma$ and $(t, 0) = V \cap \pi(\Gamma)$. The image of $d\pi$ at the points in Γ is then $\partial/\partial t$, which is tangent to $\pi(\Gamma)$ and therefore transverse to the radial vector field on A.

Using the submersion h, M can be expressed as the suspension of the fibre $N := h^{-1}\{1\}$ under a return map, i.e, $M \simeq N \times_{\mathbb{Z}} \mathbb{R}$.

The function $\pi|_N : N \to [\frac{1}{2}, \frac{3}{2}]$ has only one singular point at $N \cap \Gamma =: \{p\}$. It remains to show that $\pi|_N$ is a Morse function on N and that p has index 2. Let us take the coordinate neighbourhoods of $U \subset M$ of p and $V \subset A$ of $\pi(p)$ described above. In the coordinates (t, r) on V (where $\pi(p) = (0, 0)$),

$$B_\delta = \{(0, r) \in V,\ r \in [-\delta, \delta]\} \subset A$$

is transverse to $\pi(\Gamma)$ for δ small enough and therefore it is diffeomorphic to a neighbourhood of $\pi(p)$ in $\pi(N)$. Therefore, there are coordinates around $p \in N$ and around $0 \in \pi(N) \simeq I$ such that $\pi|_N$ has the form $\pi|_N(x, y, z) = z^2 - x^2 - y^2$. The function $g := 2(\pi|_N - 1)$ satisfies then the conclusions of the lemma. ∎

The previous lemma allows us to translate a problem around a neighbourhood of the round singularities into a problem of a three-dimensional manifold with boundary with a Morse function.

It is useful to have in mind the following example of such a three-dimensional manifold.

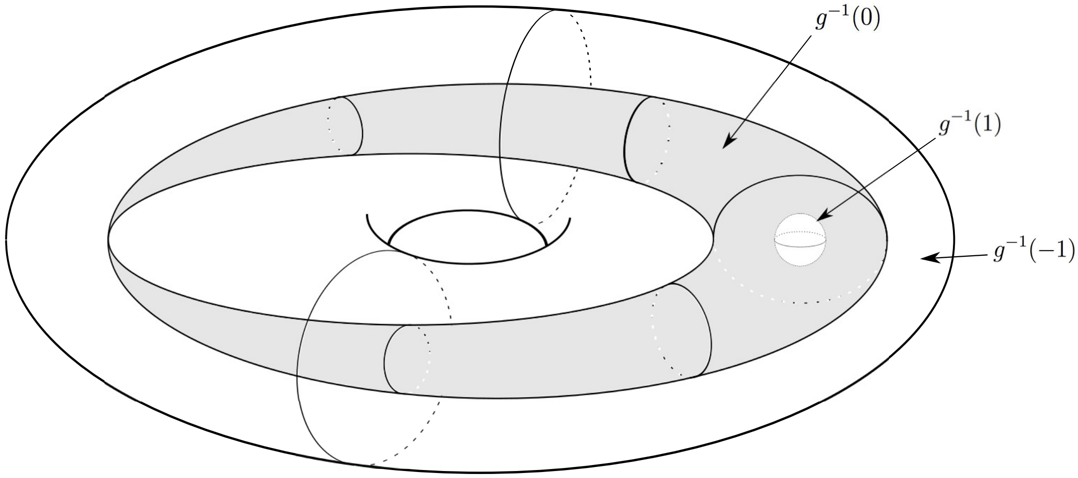


Figure 5.1: Cobordism W between S^2 and T^2

Example 5.3.4. Let $W := T^2 \times S^1 \setminus D^3$ be the three-manifold with boundary obtained from a solid torus by removing a small ball. This manifold gives a cobordism between the two boundaries $\partial_+ W = S^2$ and $\partial_- W = T^2$. Let $g : W \to I$ be a Morse function on W that has only one critical point $p \in g^{-1}(0)$ in the interior, whose singular fibre is a pinched torus (pinched at p) and the regular fibres inside the pinched torus are spheres S^2 "parallel" to the boundary $\partial_+ W$ and outside are tori "parallel" to the boundary $\partial_- W$, as shown in Figure 5.1 below.

We would like now to prove a result analogue to Proposition 5.2.7 for round singularities, i.e, if $f : M \to A$ is a surjective map whose set of singularities is a circle of round singularities, we would like to find a symplectic structure on M of cosymplectic type around the boundary. If we try to follow the proof of Proposition 5.2.7 given here (see Appendix), we need to prove results analogous to Lemma 5.4.1 and Corollary 5.4.2 in the case of a three-manifold N with a Morse function with index two instead of a four-manifold with a Lefschetz fibration over the disk. These results can be proven and their proofs are done in the same way, changing the disk by the interval. This shows that Lefschetz singularities can be seen as four-dimensional analogues of Morse singularities of index two in three-manifolds.

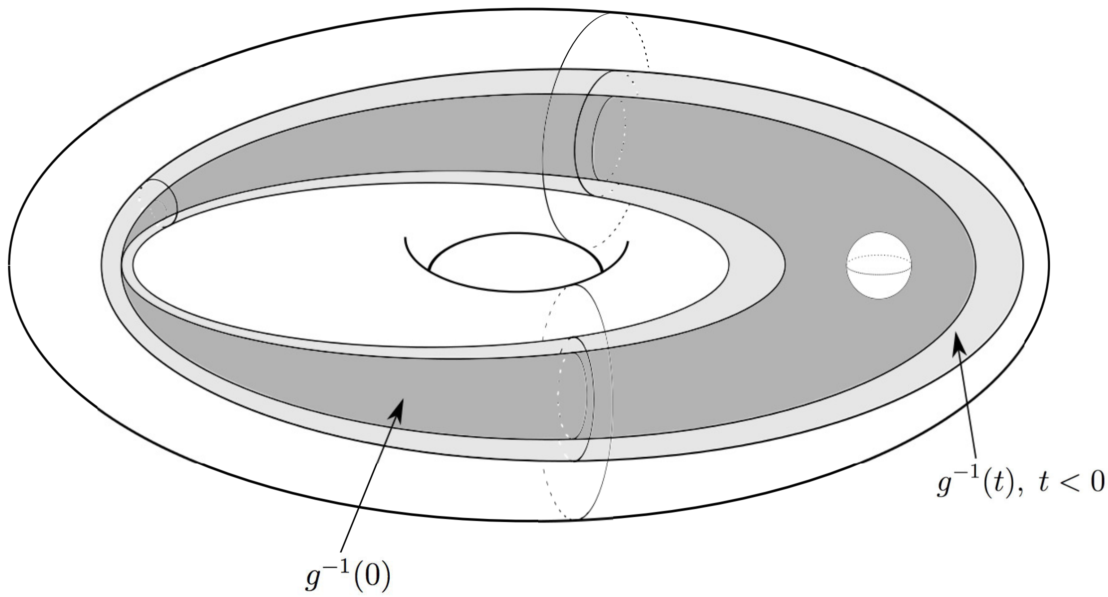
Lemma 5.3.5. *Let N be a compact orientable manifold with boundary and $g : N \to I$ be a Morse function with only one singularity $p \in \mathrm{Int}(M)$ for which $g^{-1}(\partial I) = \partial N$. Let F_0 denote the singular fibre of g. Then for every open neighbourhood $A \subset F_0$ around p there is an interval I_0 around $0 \in I$, an open neighbourhood $O \subset N$ of p and a diffeomorphism*

$$\psi : (F_0 \setminus F_0 \cap O) \times I_0 \xrightarrow{\sim} g^{-1}(I_0) \setminus O,$$

such that $g(x) = \mathrm{pr}_2(\psi(x))$ for every $x \in N$ and such that $F_0 \cap O \subset A$.

Corollary 5.3.6. *Let N be a compact orientable manifold with boundary and $g : N \to I$ a Morse function with only one singularity $p \in \mathrm{Int} N$ and such that $g^{-1}(\partial I) = \partial N$. Let F_0 denote the singular fibre. For every open neighbourhood $U \subset N$ of p, there is a map $j : N \to N$ such that*

Singular fibre and regular fibres $g^{-1}(t)$, $t < 0$



Singular fibre and regular fibres $g^{-1}(t)$, $t > 0$

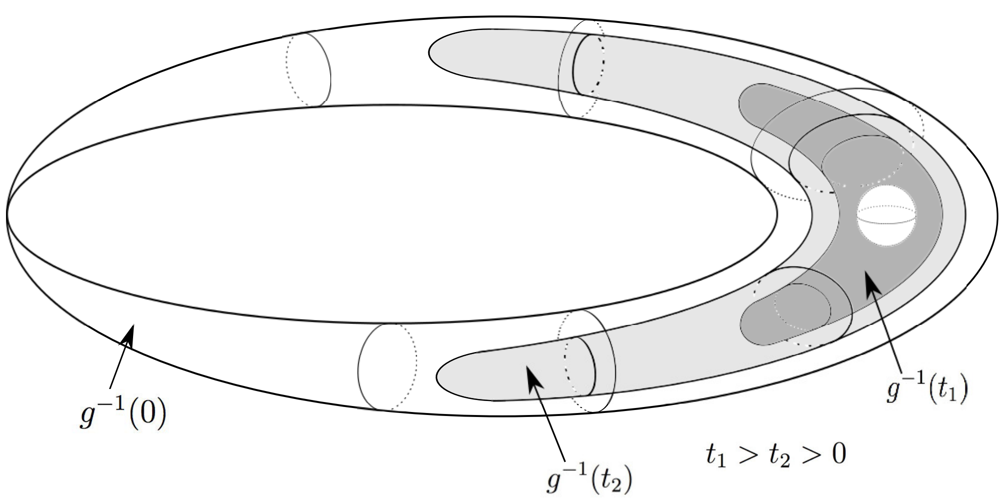


Figure 5.2: Fibres of g

- *j is the identity on F_0 and on a small neighbourhood V of p,*
- $\mathrm{im}(j) \subset U \cup F_0$.

Despite having these two results, an analogue version of Proposition 5.2.7 does not hold, as we prove in the next subsection.

Broken Lefschetz fibration on S^4

S^4 admits a broken Lefschetz fibration that can be easily understood using standard decompositions of the spheres

Remark 5.3.7 (Decomposition of the spheres)**.** There are natural decompositions for the sphere S^n: for each $k = 1, \dots, n$, there is the decomposition

$$S^n \simeq (D^k \times S^{n-k}) \cup_{S^{k-1}\times S^{n-k}} (S^{k-1} \times D^{n-k+1}).$$

To see how these decompositions arise, look at S^n as a subset of $\mathbb{R}^{n+1}$,

$$S^n = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} |\ x_1^2 + \cdots + x_{n+1}^2 = 1\ \} \subset \mathbb{R}^{n+1},$$

and consider the closed subsets

$$\begin{aligned} C &= \{(x_1, \dots, x_{n+1}) \in S^n |\ \ x_1^2 + \cdots + x_k^2 \leq 1/2\,\}, \\ D &= \{(x_1, \dots, x_{n+1}) \in S^n |\ x_{k+1}^2 + \cdots + x_n^2 \leq 1/2\,\}. \end{aligned}$$

It is easy to see that $C \simeq D^k \times S^{n-k}$ and $D \simeq S^{k-1} \times D^{n-k+1}$ and that $S^n = C \cup_{\partial C} D$, where the gluing occurs along the set

$$\partial C = \partial D = \{x_1^2 + \cdots + x_k^2 = \tfrac{1}{2},\ x_{k+1} + \cdots + x_n^2 = \tfrac{1}{2}\} \simeq S^{k-1} \times S^{n-k}.$$

If we apply this for S^4 for $k = 3$ we get:

$$S^4 \simeq (D^3 \times S^1) \cup_{S^2\times S^1} (S^2 \times D^2).$$

We can further decompose $D^3 \times S^1 = (W \times S^1) \cup_{T^3} (D^2 \times T^2)$, where W is the three manifold from Example 5.3.4: since $D^3 \cup \{\infty\} = S^3 \simeq (D^2 \times S^1) \cup_{T^2} (S^1 \times D^2)$, we can see W as the three-dimensional manifold obtained by removing a obtained by removing the standard solid torus $D^2 \times S^1$ from D^3.

Then,

$$S^4 = (D^2 \times T^2) \cup_{T^3} (W \times S^1) \cup_{S^2\times S^1} (S^2 \times D^2).$$

Consider the maps $\mathrm{pr}_1 : D^2 \times S^2 \to D^2$, $\mathrm{pr}_1 : D^2 \times T^2 \to D^2$ and $g \times \mathrm{id} : W \times S^1 \to A$, where g is the Morse function described in Example 5.3.4. These maps can be glued to a map from S^4 to $D^2 \cup_{S^1} A \cup_{S^1} D^2 \simeq S^2$,

$$\pi : S^4 \longrightarrow S^2$$

which is a broken Lefschetz fibration with a connected set of round singularities and without Lefschetz singularities.

If we try to find a codimension-one symplectic foliation on $S^4 \times S^1$ using this broken Lefschetz fibration of S^4, we would need to find symplectic structures of cosymplectic type on each one of the three pieces that form S^4. The pieces $D^2 \times S^2$ and $T^2 \times D^2$ clearly admit such structures compatible with the fibrations to D^2. However, for $W \times S^1$ is more complicated.

$W \times S^1$ does admit a symplectic structure. To see this, recall that the higher-dimensional solid torus $D^n \times S^1$ can be embedded in $\mathbb{R}^{n+1}$ in the same way the solid torus $D^2 \times S^1$ can be embedded in $\mathbb{R}^3$. Thus, since $W \times S^1 \subset D^3 \times S^1$, W can be embedded in $\mathbb{R}^4$ and the usual symplectic structure on $\mathbb{R}^4$ defines a symplectic structure on $W \times S^1$. This symplectic structure, however, is not of cosymplectic type, and can not be made so:

Theorem 5.3.8. *$W \times S^1$ does not admit a symplectic structure of cosymplectic type at the boundary.*

A more general result was proven in [GZ14]. Here we give a different, more down-to-earth proof of a particular case of Geiges' result.

Proof. Let $M := W \times S^1$ and suppose ω is a symplectic structure on M of cosymplectic type at the boundary. Let

$$N_1 = T^2 \times S^1 \subset \partial M, \qquad N_2 = S^2 \times S^1 \subset \partial M$$

be the two components of the boundary of M. For $i = 1, 2$, the symplectic structure ω induces cosymplectic structures (η_i, θ_i) on N_i, i.e, there are collar neighbourhoods $U_i \simeq N_i \times [0, 1)$, where $\omega|_{U_i} = \eta_i + \theta_i \wedge dt$, with t the coordinate in $[0, 1)$. Note that from Remark 3.2.14, we may assume the cosymplectic structures induced at the boundary are proper, i.e, that $\theta_i = f_i^*(d\varphi)$ for some fibrations $f_i : N_i \to S^1$, $i = 1, 2$. We may assume also that the fibrations f_i have connected fibres. Otherwise, we can pass to new fibrations $\tilde{f}_i : N_i \to S^1$ with connected fibres and $(\tilde{f}_i)^*(d\varphi) = k f^*(d\varphi)$ for some $k \in \mathbb{Z}_{>0}$.

The idea of the proof is to find some almost complex structure on M, "compatible" with the cosymplectic structures at the boundary and use Chern classes to find a contradiction.

Compatible almost complex structure. For $i = 1, 2$, consider the following decomposition of the tangent space of M at the boundary N_i: for $x \in N_i$,

$$T_x M = K_i \oplus \mathbb{R} \cdot X_i \oplus \mathbb{R} \cdot \partial_t,$$

where $K_i := \ker \theta_i$, $X_i \in \mathfrak{X}(N_i)$ is the Reeb-like vector field defined by $\iota_{X_i}\theta_i = 1$ and $\iota_{X_i}\eta_i = 0$. Note that η_i is a symplectic structure on the vector bundle $K_i \to N_i$.

We claim there exists one almost complex structure J on M such that, for $i = 1, 2$, $J|_{N_i}$ restricts to an almost complex structure on the vector bundle $K_i \to N_i$. To see this, consider a metric g on M such that around each boundary N_i, $i = 1, 2$, the decomposition above of T_M becomes orthogonal. Take the almost complex structure J compatible with ω constructed using g (see [Sil00], page 84), i.e, take

$$J = (AA^*)^{-1/2} A, \quad \text{where } A : TM \to TM \text{ is defined by } \quad \omega(U, V) = g(AU, V).$$

The linear map A is skewsymmetric with respect to g. The almost complex structure J satisfies

$$\omega(U, JV) = g((AA^*)^{1/2} U, V).$$

We want to see that $J|_{N_i}$ preserves the subspace K_i. Note that for every $V \in K_i$,

$$\omega(V, X_i) = 0 = g(AV, X_i), \quad \text{and} \quad \omega(V, \partial_t) = 0 = g(AV, \partial_t),$$

which means that $A(K_i)$ is orthogonal to $K_i^\perp = \mathbb{R} \cdot \partial_t \oplus \mathbb{R} \cdot X_i$ and $A(K_i^\perp)$ is orthogonal to K_i. This implies that A preserves the subspaces K_i and $K_i^\perp$ and therefore, by the definition of J we get that $J|_{N_i}(K_i) = K_i$, and $J|_{N_i}(K_i^\perp) = K_i^\perp$, for $i = 1, 2$. Therefore, we can decompose $TM|_{N_i} = K_i \oplus K_i^\perp$ as a direct sum of the complex vector bundle K_i and the trivial complex line bundle $K_i^\perp$.

Let J_i be $J|_{N_i}$ restricted to K_i. Let $c \in H^2(M)$ be the Chern class associated with the complex vector bundle (TM, J) and $c_i \in H^2(N_i)$ be the first Chern class of the complex vector bundles (K_i, J_i), $i = 1, 2$. We will use these Chern classes to reach a contradiction, as follows:

For $i = 1, 2$, let $r_i : H^2(M) \to H^2(N_i)$ be the maps induced in cohomology by the restriction of forms. Since $TM|_{N_i} = K_i \oplus K_i^\perp$ and $K_i^\perp$ is trivial, then $r_1(c) = c_1$ and $r_2(c) = c_2$.

We will see in what follows that r_1 is injective, that $c_1 = 0$ and that $c_2 \neq 0$. This leads to a contradiction because since r_1 is injective and $c_1 = 0$, then $c = 0$; however, since $c_2 \neq 0$, then $c \neq 0$. This will finish the proof.

Injectivity of r_1. Now we see that $r_1 : H^2(M) \to H^2(N_1)$ is injective. To prove this, recall that W can be described as a three-ball minus a solid torus, $W = D^3 \setminus (D^2 \times S^1)$. From this we have the decomposition

$$D^3 \times S^1 = W \times S^1 \cup D^2 \times T^2.$$

One part of the Mayer-Vietoris sequence for the de Rham cohomology for this decomposition reads:

$$0 \longrightarrow H^2(M) \oplus H^2(D^2 \times T^2) \xrightarrow{p} H^2(T^2 \times S^1) \longrightarrow 0.$$

Therefore p is an isomorphism. Since $p(x, 0) = r_1(x)$ for all $x \in H^2(M)$, it follows that r_1 is injective.

Computation of c_1 and c_2. Let us now compute c_1 and c_2. Consider the fibration $f_i : N_i \to S^1$. Let F_i be the fibres of f_i, $i = 1, 2$. To see what the fibres are, we can use the long exact sequence in homotopy groups for a fibration.

$$1 \longrightarrow \pi_1(F_i) \longrightarrow \pi_1(N_i) \longrightarrow \pi_1(S^1) \longrightarrow 1.$$

Using that $\pi_1(N_1) = \pi_1(T^2 \times S^1) = \mathbb{Z}^3$ and that $\pi_1(N_2) = \pi_1(S^2 \times S^1) = \mathbb{Z}$, it follows that $\pi_1(F_1) \simeq \mathbb{Z}^2$ and $\pi_1(F_2) = 1$. Therefore, since F_1 and F_2 are compact surfaces without boundary, it follows that $F_1 \simeq T^2$ and $F_2 \simeq S^2$.

Consider the fibration $f_2 : S^2 \times S^1 \to S^1$. This fibration is isomorphic to the suspension $S^2 \times_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{pr}_2} S^1$, for some diffeomorphism $\varphi \in \mathrm{Diff}^+(S^2)$. Since $\pi_0(\mathrm{Diff}^+(S^2)) = 1$ (See [EE69]), then φ is isotopic to the identity. Since isotopic diffeomorphisms φ yield isomorphic fibrations, there is an diffeomorphism ϕ that makes the following diagram commute:

$$\begin{tikzcd}
S^2 \times S^1 \arrow[rr, "\phi"] \arrow[dr, "f_2"'] & & S^2 \times S^1 \arrow[dl, "\mathrm{pr}_2"] \\
& S^1 &
\end{tikzcd}$$

This implies that $K_2 = \ker\theta_2 = (\mathrm{pr}_1 \circ \phi)^*(TS^2)$, where $\mathrm{pr}_1 : S^2 \times S^1 \to S^2$. The first Chern class $c(K_2, J_2)$ coincides with the Euler class $e(K_2)$ and the latter, by functoriality of the Euler class, is $e(K_2) = (\mathrm{pr}_1 \circ \phi)^*(e(S^2))$. Therefore, since $(\mathrm{pr}_1 \circ \phi)^*$ is injective, and $e(S^2) \neq 0$, then $c_2 = c(K_2) \neq 0$.

Consider now the fibration $f_1 : T^2 \times S^1 \to S^1$. Since the fibre is T^2, the fibration is isomorphic to the suspension $N_{1,\varphi} := T^2 \times_{\mathbb{Z}} \mathbb{R}$ for some diffeomorphism $\varphi \in \mathrm{Diff}^+(T^2)$. Recall that the map $\pi_0(\mathrm{Diff}(T^2)) \to SL_2(\mathbb{Z})$ that assigns to a isotopy class of diffeomorphisms φ its induced map in $H_1(T^2) = \mathbb{Z}^2$ is a bijection ([EE69]). Therefore, we may assume then that φ is a matrix $A \in SL_2(\mathbb{Z})$. Recall the long exact sequence in homology for a suspension (see [Hat01]):

$$\cdots \longrightarrow H_2(T^2) \longrightarrow H_1(T^2) \xrightarrow{A-\mathrm{Id}} H_1(T^2) \longrightarrow H_1(N_{1,A}) \longrightarrow H_0(F) \longrightarrow 1$$

Since $N_{1,A} = T^2 \times S^1$, then $H_1(N_{1,A}) = \mathbb{Z}^3$. Moreover, since $H_0(F) = \mathbb{Z}$, and $H_1(T^2) = \mathbb{Z}^2$, then it follows that the map $H_1(T^2) \to H_1(N_{1,A})$ should be surjective and therefore that the map $A - \mathrm{Id}$ has to be zero. Therefore, $A = \mathrm{Id}$ and the monodromy of the suspension $N_{1,A}$ can be chosen to be trivial. Thus, we get an isomorphism of fibrations ψ similar to the one we got for f_2:

$$\begin{tikzcd} T^2 \times S^1 \arrow[rr, "\psi"] \arrow[dr, "f_1"'] & & T^2 \times S^1 \arrow[dl, "\mathrm{pr}_2"] \\ & S^1. & \end{tikzcd}$$

In this case, similarly to what we did for f_1, we can conclude that

$$c_1 = c(K_1, J_1) = e(K_1) = (\mathrm{pr}_1 \circ \psi)^*(e(T^2)) = 0$$

since T^2 is parallelizable. Therefore $c_1 = e(T^2) = 0$. □

5.4 Appendix

Proof of Proposition 5.2.7

In this appendix we give the detailed proof of Proposition 5.2.7. The proof of this Proposition appears in [GS99] without many details.

Lemma 5.4.1. *Let $M \xrightarrow{\pi} D^2$ be a Lefschetz fibration of a compact four dimensional manifold M onto the disk D^2 with only one Lefschetz singularity at $p \in \pi^{-1}(0)$. Let $F_0 = \pi^{-1}(0)$ be the singular fibre. Then, for every open neighbourhood $A \subset F_0$ of p, there is a disk $D_0^2 \subset D^2$ around 0, an open $O \subset M$ around p, and a diffeomorphism*

$$\psi : F_0 \setminus (F_0 \cap O) \times D_0^2 \xrightarrow{\sim} \pi^{-1}(D_0^2) \setminus O$$

such that $\pi(x) = \mathrm{pr}_2(\psi^{-1}(x))$ and $O \cap F_0 \subset A$.

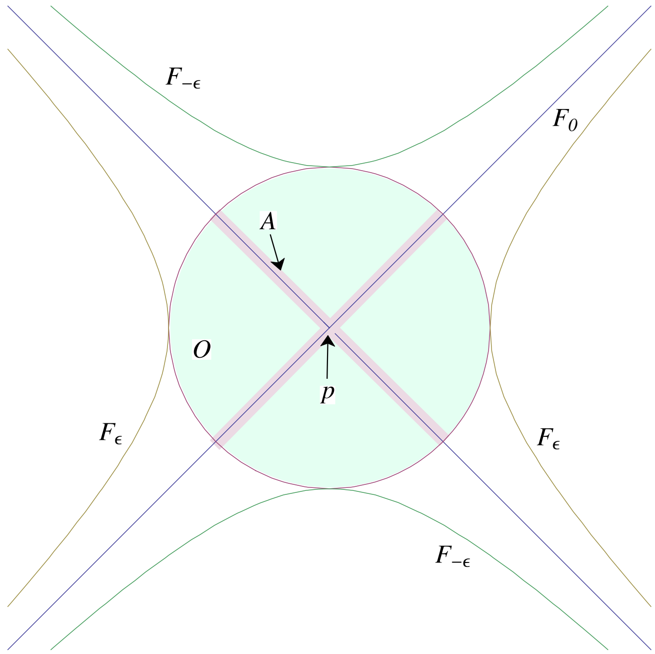


Figure 5.3: Lower dimensional analogue of Proposition 5.4.1. The picture shows three fibres: $F_0, F_\epsilon, F_{-\epsilon}$, of the map $\pi : \mathbb{R}^2 \to \mathbb{R}$, $\pi(x, y) = x^2 - y^2$ around the singular point $p = (0, 0)$. In this case O is the shaded disk and $A = O \cap F_0$.

Proof. Let U be an open neighbourhood around p and D_0^2 be a closed neighbourhood of $0 \in D^2$ where there are coordinates $(z_1, z_2) \in \mathbb{C}^2$ on U and $z \in \mathbb{C}$ on D_0^2 in which π can be written as $\pi : U \to \mathbb{C}$, $\pi(z_1, z_2) = z_1^2 + z_2^2$. In real coordinates, (x_1, y_1, x_2, y_2) on U and (x, y) on D_0^2, the map π can be written as $\pi(x_1, y_1, x_2, y_2) = (x_1^2 - y_1^2 + x_2^2 - y_2^2, 2x_1y_1 + 2x_2y_2)$. Since we want to prove a local statement, we may assume that $D_0^2 = D^2$ and $\pi^{-1}(D_0^2) = M$.

Consider the following vector fields on D^2

$$X = \partial_x := \partial/\partial x, \qquad Y = \partial_y := \partial/\partial y.$$

We claim that there exist an open neighbourhood $V \subset U$ of p and two vector fields $\tilde{X}$ and $\tilde{Y}$ on M satisfying:

- Outside V, $\tilde{X}$ and $\tilde{Y}$ project to X and Y.

- The flow lines of $\tilde{X}$ and $\tilde{Y}$ that pass through $F_0 \setminus (V \cap F_0)$ never enter V.

Let us see how this claim allows us to finish the proof: take the map

$$\psi : F_0 \setminus (F_0 \cap V) \times \mathbb{R}^2 \to M, \qquad (q, t_1, t_2) \mapsto \varphi_{\tilde{X}}^{t_1} \circ \varphi_{\tilde{Y}}^{t_2}(q).$$

defined on a neighbourhood around $0 \in \mathbb{R}^2$ where the flows of $\tilde{X}$ and $\tilde{Y}$ exist. Since $\tilde{X}$ and $\tilde{Y}$ are projectable, $\pi(\psi(q, t_1, t_2)) = (t_1, t_2)$. Furthermore, since $\tilde{X}$ and $\tilde{Y}$ are transverse to the fibres and linearly independent, it is a local diffeomorphism around $F_0 \setminus (F_0 \cap V) \times \{0\}$. Therefore, there is a neighbourhood D_0^2 around $0 \in \mathbb{R}^2$ where the map $\psi|_{F_0 \setminus (F_0 \cap V) \times D_0^2}$ is a diffeomorphism. Since the image is compact, its complement is open, let us call it O. The map ψ is then a diffeomorphism onto its image $\pi^{-1}(D_0^2) \setminus O$. Since the map is the identity on $F_0 \setminus (F_0 \cap V) \times \{0\}$, it follows that $O \cap F_0 \subset V \cap F_0 \subset A$.

Construction of $V, \tilde{X}, \tilde{Y}$. Let us construct $\tilde{X}$, $\tilde{Y}$ and V. We may assume

$$U = \{x \in M, |x| < 2\epsilon\}$$

for some ϵ small enough, for a metric on M that restricts to the standard metric on U (seen as a subset of $\mathbb{R}^4$). Consider the vector fields

$$\begin{aligned} X_U &= \mathrm{Re}(\bar{z}_1\partial_{z_1} + \bar{z}_2\partial_{z_2}) = x_1\partial_{x_1} - y_1\partial_{y_1} + x_2\partial_{x_2} - y_2\partial_{y_2}, \\ Y_U &= \mathrm{Im}(\bar{z}_1\partial_{z_1} + \bar{z}_2\partial_{z_2}) = x_1\partial_{y_1} + y_1\partial_{x_1} + x_2\partial_{y_2} + y_2\partial_{x_2}. \end{aligned}$$

Note that $\pi_*(X|_U) = 2(x_1^2 + y_1^2 + x_2^2 + y_2^2)\partial_x$ and $\pi_*(Y|_U) = 2(x_1^2 + y_1^2 + x_2^2 + y_2^2)\partial_y$.

Denote $V_t := \{x \in U, |x| < t\epsilon\}$, $V := V_1$. Construct vector fields $\overline{X}$, $\overline{Y}$ on M that coincide with X_U and Y_U on $V_{5\epsilon/4}$ and that project to X and Y outside $V_{7\epsilon/4}$ using partitions of unity. Then, take positive functions g, h such that $\tilde{X} = g\overline{X}$ and $\tilde{Y} = h\overline{Y}$ project to X and Y outside $V_{\epsilon/2}$, respectively.

We now check the second condition. Consider now V as a subset of $\mathbb{R}^4$. The inner product of the vector fields $\tilde{X}|_V$ and $\tilde{Y}|_V$ with the radial vector field yield

$$\tilde{X} \cdot (x_1, y_1, x_2, y_2) = g(x)\mathrm{pr}_1(\pi(x)), \qquad \tilde{Y} \cdot (x_1, y_1, x_2, y_2) = h(x)\mathrm{pr}_2(\pi(x)),$$

in particular $\tilde{X}, \tilde{Y}$ are tangent to the sphere $\partial V \cap F_0$ at the singular fibre where $\pi(x) = 0$. Using this inner product we prove now that the flow lines of the vector field $\tilde{X}$ that intersect $F_0 \setminus V \cap F_0$ never enter V. The argument for $\tilde{Y}$ is completely analogous. Let $D^2_{x>0} = \{(x, y) \in D^2 / x > 0\}$. Since $\pi_*(\tilde{X})$ is a positive multiple of ∂_x and the the inner product of $\tilde{X}$ with the radial vector field is positive on $\pi^{-1}(D^2_{x>0})$, if a flow line passing through $F_0 \setminus V \cap F_0$ were to enter the ball V at a positive time, then it would do it on a point $p \in \pi^{-1}(D^2_{x>0})$ and therefore the inner product of $\tilde{X}_p$ with the radial vector field would have to be negative. If it enters V in a negative time, a similar reasoning applies changing $D^2_{x>0}$ by $D^2_{x<0}$. □

Corollary 5.4.2. *Let $M \xrightarrow{\pi} D^2$ be a Lefschetz fibration of a compact four dimensional manifold M onto the disk D^2 with only one Lefschetz singularity at $p \in \pi^{-1}(0)$. Let $F_0 = \pi^{-1}(0)$ denote the singular fibre. Then for every open neighbourhood U of p, there is a smooth map $j : M \to M$ satisfying*

- *j is the identity on F_0 and on a small neighbourhood V of p.*
- *The image of j is contained in $F_0 \cup U$.*

Proof. The map j is constructed as the composition of two smooth maps, $j_1 : M \to M$ whose image is contained on $\pi^{-1}(\pi(U))$ and $j_2 : \mathrm{im}(j_1) \to M$ such that $j_2(\mathrm{im}(j_1)) \subset F_0 \cup U$.

Construction of j_1. Since $\pi(U)$ is open, it contains an open ball $D_\delta = \{|x| < \delta\} \subset D^2$ for some δ small enough.

Consider the vector field $Y = h(r)\partial_r$ on D^2, expressed in polar coordinates (r, θ), where $h(r) = 0$ for $r < \delta/2$ and $h(r) = -1$ for $r > \delta$ and let $\tilde{Y}$ be a lift of this vector field to M. Let j_1 be the flow of the vector field at time 1. This flow exists and maps M into $M_\delta := \pi^{-1}(D_\delta)$.

Construction of j_2. Let us now construct the map j_2. Apply first the previous lemma for a non-empty open neighbourhood A of p, strictly contained in $U \cap F_0$, and let D_0^2, O and ψ be the disk around $0 \in D^2$, the open neighbourhood of p and the isomorphism obtained, respectively. First, note that $O \cap U$ is an open neighbourhood of p and therefore, since π is a submersion everywhere away from p, $\pi(O \cap U)$ contains a small disk $D_\delta \subset D_0^2$ for δ small enough.

Let W be an open set in F_0 such that $F_0 \cap O \subsetneq W \subsetneq F_0 \cap U$. It can be assumed that $\psi^{-1}(W \times D_\delta) \subset U$ by choosing δ small enough. Consider polar coordinates (r, θ) on D^2 and take the map

$$j_2' : (F_0 \setminus (F_0 \cap O)) \times D^2 \to (F_0 \setminus (F_0 \cap O)) \times D^2, \qquad j_2'(x, r, \theta) = (x, rg(x), \theta),$$

where $g : F_0 \setminus (F_0 \cap O) \to \mathbb{R}$ equals to $g(x) = 1$ around O and $g(x) = 0$ around $F_0 \setminus W$. Conjugating j_2' by ψ we get a function $j_2 : M_\delta \setminus O \to M$ that extends to a function $j : M_\delta \to M$ by defining it to be the identity on O. Let

$$V = (\psi^{-1}\{(W \setminus W \cap O) \times I\} \cup O) \cap U \subset U.$$

Note that j_2 maps $\pi^{-1}(D_\delta)$ into $F_0 \cup V$.

Finally, the function $j := j_2 \circ j_1$ satisfies the conditions of the theorem. □

We have now all the ingredients to prove Lemma 5.2.7.

Proof of 5.2.7. This proof is made in three steps. First we construct a two-form ω_V in M that is non-degenerate on the union of F_0 with a small neighbourhood of the singular point p. Using this form we construct another two-form ω which is closed and symplectic along the fibres on the regular points. Finally, we apply a slight modification of Thurston trick to ω to obtain a symplectic form on M, which is symplectic on the fibres.

Construction of ω_V. Let F_0 denote the singular fibre and U be an open ball around x such that on U we have the normal form for the Lefschetz singularity, i.e, we have orientation-preserving complex coordinates (z_1, z_2) on U such that $\pi|_U : U \to D^2$ can be written as $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Consider the symplectic structure $\omega_U = dx_1 \wedge dy_1 + dx_2 \wedge dy_2 \in \Omega^2(U)$. For every regular point q, ω_U is non-degenerate on $\ker d_q\pi$ since

$$\omega_U \wedge d\pi_1 \wedge d\pi_2 = (x_1^2 + y_1^2 + x_2^2 + y_2^2) dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2,$$

where π_1 and π_2 denote the coordinates of π in $D^2 \subset \mathbb{R}^2$.

Apply Corollary 5.4.2 to an open subset $U' \subsetneq U$ around p and let j and V be the map and the open neighbourhood of p obtained, i.e, j is a contraction which is the identity on $V \cup F_0$ and $\mathrm{im}(j) \subset U' \cup F_0$.

Take a two-form $\omega_V \in \Omega^2(M)$ that restricts to $\omega_U|_V$ on V and is non-degenerate on $F_0 \setminus \{p\}$ (we put it in italics because that is what we will use of ω_V). This form can be constructed as follows: take an open V' such that $\overline{V} \subsetneq V' \subsetneq U$ and take a two-form ξ on $M \setminus V$ such that its restriction to $F_0 \setminus F_0 \cap V$ is an area form defining the same orientation on $F_0 \cap (U \setminus V)$ as $\omega_U|_{F_0 \cap U}$. Consider now the form

$$\omega_V = \rho_1 \omega_U + \rho_2 \xi,$$

where $\rho_1, \rho_2 \in C^\infty(M)$ are bump functions satisfying $\rho_1 + \rho_2 = 1$, ρ_1 is supported on U and takes the value 1 on V' while ρ_2 is supported on $M \setminus V'$ and takes the value 1 on $M \setminus U$. The two-form ω_V is positive on $F_0 \setminus \{p\}$ and agrees with ω_U on V.

<u>Construction of ω</u>. Consider the two-form $\omega = j^*(\omega_V) \in \Omega^2(M)$. This form is closed. Indeed, closedness can be checked on the two opens V' and $M \setminus \overline{V}$. On V', $d\omega|_{V'} = j^*(d\omega_U|_{V'}) = 0$. On $M \setminus \overline{V}$, note that $j|_{M \setminus \overline{V}}$ becomes a projection onto F_0, i.e., $j : M \setminus \overline{V} \to F_0 \setminus (F_0 \cap \overline{V})$ and therefore

$$d\omega|_{M \setminus \overline{V}} = dj^*(\omega_V|_{F_0 \setminus (F_0 \cap \overline{V})}) = j^*(d\omega_V|_{F_0 \setminus (F_0 \cap \overline{V})}) = 0,$$

since $\omega_V|_{F_0 \setminus (F_0 \cap V)}$ is a two-form on a two-dimensional manifold.

Since ω is an area form when restricted to $F_0 \setminus \{p\}$ and $\omega|_{\ker d_q \pi}$ is non-degenerate for points q close to p, there is a small disk D_ϵ, for small ϵ such that $\omega|_{\ker d_q \pi}$ is non-degenerate for every $q \in \pi^{-1}(D_\epsilon)$, $q \neq p$. Since $M \simeq \pi^{-1}(D_\epsilon)$ under a diffeomorphism that maps fibres to fibres and leaves a small neighbourhood $\pi^{-1}(D_{\epsilon/2})$ fixed, we may assume $M = \pi^{-1}(D_\epsilon)$.

<u>Adapted Thurston trick</u>. Now, take $\omega' = \omega + K\pi^*(\omega_D)$, where ω_D is a symplectic structure on D^2 such that $\omega \wedge \pi^*(\omega_D) > 0$ on $M \setminus \{p\}$. Note that since $(d\pi)_p = 0$, $\pi^*(\omega_D)_p = 0$ and therefore for $K > 0$ big enough ω' is symplectic everywhere and its restriction to the regular fibres is symplectic. Using Lemma 5.0.7, we finish the proof. $\square$

Chapter 6

Open Book Decompositions, Symplectic Cobordisms and S^5

In this chapter we apply the turbulisation techniques from Chapter 3 to fibrations with S^1 fibres that are not necessarily products. S^5 is one of the first interesting examples of such manifolds.

First, we apply the turbulisation techniques of Chapter 3 to construct symplectic foliations on some open book decompositions. The applicability of theorems 3.4.8 and 3.4.14 - and therefore, of the resulting constructions for open books - relies on the possibility of finding a symplectic structure of cosymplectic type at the boundary. However, the symplectic structures that appear naturally are frequently not of this type. Sometimes they can be modified to be so. In this chapter we deal with one way of doing this modification, called symplectic cobordisms.

Finally, we use an open book decomposition of S^5, the Turbulisation Theorem and the symplectic cobordisms to construct a codimension-one symplectic foliation on S^5. Such a structure on S^5 was found first by Mitsumatsu [Mit11] and the construction we make here, inspired by his, frames his methods in a more general context.

Conventions.

We follow the same conventions regarding orientations as we did in Chapter 2. We further assume that contact manifolds with a given contact form are endowed with the orientation given by the contact form.

6.1 Open Book Decompositions

The goal of this section is to apply the turbulisation theorem to construct codimension-one symplectic foliations on manifolds that admit certain types of open book decompositions.

First we recall the basic notions and results in open books.

Definition 6.1.1. *An* ***open book decomposition*** *(OBD) of a manifold M consists of:*

i. A codimension-two submanifold B with trivial normal bundle.

ii. A submersion $f : M \setminus B \to S^1$ such that there is a tubular neighbourhood $\nu_B \simeq B \times D^2$

of B where the following diagram commutes:

$$\begin{tikzcd} \nu_B \setminus B \arrow[r, "\sim"] \arrow[d, "f"'] & B \times (D^2 \setminus \{0\}) \arrow[dl, "\mathrm{pr}_2/|\mathrm{pr}_2|"] \\ S^1 & \end{tikzcd}$$

Note that $P := \overline{f^{-1}(1)}$ is a manifold with boundary because of the normal form of f around B. P is called the *page* of the open book and the submanifold $B := \partial P = f^{-1}(0)$ is called the *binding* of the open book.

Definition 6.1.2. *An **abstract open book** is a pair (P, ϕ), where*

i. P is a compact manifold with boundary B,

*ii. $\phi \in \mathrm{Diff}(P)$ is a diffeomorphism of P that is the identity in a neighbourhood of ∂P. The diffeomorphism ϕ is called the **monodromy** of the abstract open book.*

An **isomorphism** between abstract open books (P_1, φ_1) and (P_2, φ_2) is a diffeomorphism $\psi : P_1 \to P_2$ such that $\varphi_1 = \psi^{-1} \circ \varphi_2 \circ \psi$.

Lemma 6.1.3. *An abstract open book decomposition (P, ϕ) gives rise to a closed manifold $M_{(P,\phi)}$ with an open book decomposition for which the page is diffeomorphic to P.*

Proof. Let $N := P \times_{\mathbb{Z}} \mathbb{R}$ be the suspension of P formed using ϕ. Since ϕ is the identity near the boundary of P, $\partial N = \partial P \times S^1$ and we can glue a copy of $\partial P \times D^2$ to N using the identity map on the boundary:

$$M_{(P,\phi)} := (P \times_{\mathbb{Z}} \mathbb{R}) \cup_{\partial P \times S^1} (\partial P \times D^2).$$

$M_{(P,\phi)}$ is a closed manifold. The map $\mathrm{pr}_2 : P \times_{\mathbb{Z}} \mathbb{R} \to S^1$ induced by the second projection extends to a map $\pi : M_{(P,\phi)} \setminus (\partial P \times \{0\}) \to S^1$ by taking the angular coordinate in $\partial P \times D^2$. Therefore, $(\partial P \times \{0\}, \pi)$ is an open book decomposition of $M_{(P,\phi)}$. The page $P \cup_{\partial P} \partial P \times [0,1]$ is clearly diffeomorphic to P. □

Remark 6.1.4. It is clear that if $(P_1, \phi_1) \simeq (P_2, \phi_2)$ are diffeomorphic abstract open books, then $M_{(P_1,\phi_1)} \simeq M_{(P_2,\phi_2)}$.

The following lemma shows the inverse construction: how to construct an abstract open book from an open book decomposition.

Lemma 6.1.5. *Let (B, π) an OBD on a closed manifold M, then there is an abstract open book (P, ϕ) such that $M_{(P,\phi)} \simeq M$, where P is the page of (B, π).*

Proof. Take $\nu_B \simeq \mathrm{Int}(B \times D^2)$ a small open tubular neighbourhood of B where the map π becomes the angle coordinate in D^2. Consider the map $\pi : M \setminus \nu_B \to S^1$. It satisfies condition i. from section 1.7 and therefore, by Proposition 1.7.6, we can write $M \setminus \nu_B \simeq (P \setminus (P \cap \nu_B)) \times_{\mathbb{Z}} \mathbb{R}$ for some monodromy ϕ' of $P \setminus (P \cap \nu_B)$. By the choice of ν_B, the monodromy can be chosen to be the identity near the boundary. We can extend ϕ by the identity on $P \cap \nu_B$ to obtain a diffeomorphism of P such that (P, ϕ) an abstract open book. By construction, it is clear that $M_{(P,\phi)} \simeq M$. □

Let (B, π) be an open book decomposition on M. The abstract open books (P, ϕ) constructed in the previous lemma are called abstract open books *corresponding* or *associated* to (B, π).

Remark 6.1.6. In the literature, an open book decomposition on a manifold M is also defined as an isomorphism between M and $M_{(P,\phi)}$ for some abstract open book (P, ϕ).

Remark 6.1.7 (Changing the page)**.** Let (B, π) be an open book decomposition. Let $P = \overline{\pi^{-1}(1)}$ be the page and (P, ϕ) an associated abstract open book. Let ν'_B be a tubular neighbourhood of B where the normal form of π holds, and let ν_B a smaller tubular neighbourhood properly contained on ν'_B. Let $P_0 := P \setminus (P \cap \nu_B)$, $\phi_0 := \phi|_{P_0}$. Then $(P_0, \phi_0) \simeq (P, \phi)$. To see this, note first that

$$P := P_0 \cup_{\partial P_0 \times \{1\}} \partial P_0 \times [0, 1],$$

where the gluing is done with a collar $U = \partial P_0 \times [1, 2) \subset P_0$ where ϕ_0 is the identity. Then there is a diffeomorphism $\psi : P_0 \xrightarrow{\sim} P$, that is the identity outside U and maps $\partial M \times [1, 2] \subset P_0$ to $\partial M \times [0, 2] \subset P$ such that $\psi^{-1} \circ \phi \circ \psi = \phi_0$.

Example 6.1.8 (Open book decomposition of S^5)**.** We describe in this example an open book decomposition of S^5 that we will use later to construct a codimension-one symplectic foliation on S^5. This open book comes from the Milnor fibration from [Mil68]. This open book is closely related to the Hopf fibration of S^5, something that we will exploit later in our constructions.

Consider S^5 as a submanifold of $\mathbb{C}^3$:

$$S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \,|\, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\} \subset \mathbb{C}^3.$$

Then $S^1 \subset \mathbb{C}$ acts on $\mathbb{C}^3$ by multiplication and this action restricts to a free S^1-action on S^5, called the *Hopf action.* The infinitesimal generators of the actions are the vector fields

$$H := \left.\frac{d}{dt}\right|_{t=0} e^{it}(x_i, y_i) \in \mathfrak{X}(S^5),$$

for $(x_i, y_i)_{i=1,2,3} \in S^5$. The quotient map

$$h : S^5 \longrightarrow S^5/S^1 \simeq \mathbb{C}P^2 \tag{6.1}$$

is called the *Hopf map* and it endows S^5 with a principal S^1-bundle structure over $\mathbb{C}P^2$.

Proposition 6.1.9. *There is a open book decomposition (B, f) of S^5 with the following properties:*

i. The binding B is invariant under the Hopf action and $h(B) \simeq T^2$.

ii. The fibres of the map $f : S^5 \setminus B \to S^1$ are transverse to the orbits of the Hopf action.

Proof. Consider $S^5 \subset \mathbb{C}^3$ as written above and take the polynomial

$$\tilde{p} : \mathbb{C}^3 \to \mathbb{C}, \quad z \mapsto z_1^3 + z_2^3 + z_3^3.$$

Let $p := \tilde{p}|_{S^5}$. We see now that 0 is a regular value for p: note that $\tilde{p}$ is submersive for all $x \in \mathbb{C}^3 \setminus \{0\}$. Let $x \in p^{-1}(0) \subset S^5$ and recall that $T_x\mathbb{C}^3 = T_xS^5 \oplus \mathbb{R} \cdot x$ and $(d_x\tilde{p})(x) = 3\tilde{p}(x) = 0$. Therefore, for all $x \in p^{-1}(0)$, $d_x\tilde{p}|_{T_xS^5}$ is surjective, i.e, p is submersive at $p^{-1}(0)$ and

$$B := p^{-1}(0) \subset S^5$$

is a codimension-two submanifold. Since $dp|_{TS^5|_B} : TS^5|_B \to B \times T_0\mathbb{C}$ is a vector bundle map with kernel $\ker d(p|_B)$, then $\nu(B) = TS^5/\ker d(p|_B) \simeq B \times T_0\mathbb{C}$, i.e, B has a trivial normal bundle. For $\epsilon > 0$ small enough,

$$\tau_\epsilon(B) = \{z \in S^5, |p(z)| < \epsilon\} \simeq B \times \operatorname{Int}(D^2_\epsilon)$$

is a tubular neighbourhood of B. Indeed, there is a retraction

$$\pi_B : \tau_\epsilon(B) \to B \tag{6.2}$$

such that

$$\begin{aligned} \varphi : \tau_\epsilon(B) &\xrightarrow{\sim} \; B \times D^2_\epsilon \\ z &\longmapsto (\pi_B(z), p(z)) \end{aligned} \tag{6.3}$$

is an isomorphism. Now define

$$f : S^5 \setminus B \to S^1, \quad x \mapsto p(x)/|p(x)|. \tag{6.4}$$

Note that $df(H) = 3\partial_\theta$, where ∂_θ is the angular vector field on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$. Therefore, f is a submersion and the fibres of f are transverse to the orbits of the Hopf action.

The pair (B, f) is then an open book decomposition of S^5. The fibration $f : S^5 \setminus B \to S^1$ is called the *Milnor fibration* on S^5 (see [Mil68]).
The binding B is clearly invariant under the Hopf action. The quotient of B by this action can be described using the degree formula for the genus of a smooth algebraic curve in $\mathbb{C}P^2$ (see e.g. [GJ94]). The formula says the genus of a complex curve in $\mathbb{C}P^2$ that is given by the zeros of an irreducible homogeneous polynomial of degree d is given by

$$g = \frac{(d-1)(d-2)}{2}.$$

Therefore, $h(B)$ is a curve of genus one in $\mathbb{C}P^2$, hence diffeomorphic to T^2. □

6.2 Codimension-one Symplectic Foliations from Open Books

Open books are an important tool in contact geometry to prove the existence of contact structures. They were first introduced in contact topology by Thurston and Winkelnkemper in dimension 3 [TW75, Win73] and later studied extensively by Giroux (see [TEG96] for a review). In this section we show that they open bookds can also be used when constructing codimension-one symplectic foliations.

In contact geometry, there is an important existence result using open books. Before stating it, recall that a contact form α is said to be **supported** by the open book decomposition (B, π) if α is positive and $d\alpha$ is symplectic on the pages and it restricts to the binding as a positive contact form.

Theorem 6.2.1 (see e.g. [Gei08])**.** *Let M be a closed manifold that admits an abstract open book decomposition (P, ϕ) satisfying the following conditions:*

a) P admits an exact symplectic structure $\omega = d\beta$ and the vector field defined by $\iota_Y \omega = \beta$ is transverse to ∂P and points outwards.

b) The map ϕ is a symplectomorphism of (P, ω).

Then M admits a contact structure supported by the open book decomposition (B, f) corresponding to (P, ϕ).

For codimension-one symplectic foliations we have an analogous existence result:

Theorem 6.2.2 (Open book decomposition)**.** *Let M be a closed manifold that admits an abstract open book decomposition (P, ϕ) satisfying the following conditions:*

a) P admits a symplectic structure ω of cosymplectic type at the boundary.

b) The map ϕ is a symplectomorphism of (P, ω).

Then M admits a codimension-one symplectic foliation.

Proof. Let (η, θ) be the cosymplectic structure induced on B by ω. By Corollary 3.4.13, the manifold $B \times D^2$ admits a codimension-one symplectic foliation, where the boundary leaf $B \times S^1$ has the symplectic structure given by $\eta + \theta \wedge d\varphi$.

The complement $C := M \setminus \operatorname{Int}(B \times D^2)$ fibres over S^1 with a monodromy ϕ which is trivial in a neighbourhood of the boundary. Since ϕ is the identity around the boundary, we can use Corollary 3.4.14 and Theorem 3.4.8 to get a codimension-one symplectic foliation on $P \times_{\mathbb{Z}} \mathbb{R}$ for which the boundary leaf $B \times S^1$ has the symplectic structure $\eta + \theta \wedge d\varphi$. Using Theorem 2.3.4 we can glue these two pieces using the identity map to obtain a codimension-one foliation on M. $\square$

Remark 6.2.3. In the last section of this chapter we construct a codimension-one symplectic foliation on S^5. However, despite using a decomposition of S^5 obtained with the open book decomposition from Example 6.1.8, we will not use the previous theorem to construct the symplectic foliation.

6.3 Symplectic Cobordisms

Definition 6.3.1. *Let η_1, η_2 be two presymplectic structures on compact manifolds N_1 and N_2. Then η_1 is **symp-cobordant** to η_2 if there is a compact connected symplectic manifold (M, ω) with boundary $\partial M = N_1 \sqcup \overline{N}_2$ such that ω induces η_1 on $N_1 \subset M$ and η_2 on $\overline{N}_2 \subset M$. It is denoted by $\eta_1 \prec \eta_2$.*
*If $N_1 = N_2$ and $M = N_1 \times [0, 1]$ we say that η_1 is **elementary symp-cobordant** to η_2 and it is denoted by $\eta_1 \prec_e \eta_2$.*

We use the notation $\eta_1 \prec^M \eta_2$ to indicate that the manifold M gives the cobordism between N_1 and N_2.

Example 6.3.2. Any symplectic structure on $N \times I$ gives rise to an elementary symplectic cobordism. A symplectic structure on $N \times I$ is of the form $\eta_t + \alpha_t \wedge dt$, where (η_t, α_t) is an almost cosymplectic structure on N for each t. This symplectic structure gives that $\eta_0 \prec_e \eta_1$ where η_0, η_1 are presymplectic structures on N.

Definition 6.3.3. *A presymplectic structure η on the closed manifold N is said to be cobordant to the empty set, denoted by $\eta \prec \emptyset$, if there is a compact manifold M such that $\partial M = N$ and ω induces η on N. Similarly, the empty set is cobordant to the presymplectic manifold (N, η), denoted $\emptyset \prec \eta$, if there is a compact manifold M such that $\partial M = \overline{N}$ and ω induces η on $\overline{N}$.*

Example 6.3.4. Any symplectic manifold with boundary (M, ω) gives rise to a cobordism $\omega|_{\partial M} \prec^M \emptyset$.

In order to understand symplectic cobordisms, it is important to be able to glue symplectic manifolds along the boundary (cf. Lemma 4.2.7).

Lemma 6.3.5 (Gluing symplectic manifolds)**.** *Let (M_i, ω_i), $i = 1, 2$ be symplectic manifolds with compact boundaries and let $\varphi : \partial M_1 \xrightarrow{\simeq} \partial M_2$ be an orientation reversing diffeomorphism between the boundaries. Suppose that $\omega_1|_{\partial M_1} = \varphi^*(\omega_2|_{\partial M_2}) = \eta \in \Omega^2(\partial M_1)$. Then the manifold $M = M_1 \cup_\varphi M_2$ admits a symplectic structure on M which restricts to ω_i on M_i, $i = 1, 2$.*

Proof. Let $N := \partial M_1$ and let us identify ∂M_2 with $\overline{N}$ using φ. Let $\alpha \in \Omega^1(N)$ be a +admissible one-form for η. We can use Proposition 3.1.13 to get a collar neighbourhood $U \simeq N \times [0, c_1)$ in M_1, for some $c_1 > 0$, where

$$\omega_1|_U = \eta - d(r\alpha).$$

The form $-\alpha$ is +admissible for η on $\overline{N}$. Therefore, there is a collar neighbourhood $V \simeq N \times [0, c_2)$ in M_2, for some $c_2 > 0$, in which

$$\omega_2|_V = \eta + d(r\alpha).$$

Changing $r \mapsto -r$ we get a new collar $V' \simeq N \times (-c_2, 0]$ where we can write $\omega_2|_{V'} = \eta - d(r\alpha)$. Therefore, if we glue M_1 and M_2 using the collars U and V' and we can define a symplectic structure equals to ω_1 on M_1 and ω_2 on M_2. This symplectic form is smooth because $\omega_1|_U$ and $\omega_2|_{V'}$ have the same formula. □

Remark 6.3.6. Symp-cobordism is not an equivalence relation: it is in general not reflexive nor symmetric in the space of presymplectic structures.

However, if the presymplectic structure η in N is a cosymplectic form, then $\eta \prec_e \eta$, symp-cobordism given by $(N \times [0,1], \omega = \eta + \theta \wedge dt)$, where θ is a closed admissible one-form.

Lemma 6.3.7. *Symp-cobordism is a transitive relation.*

Proof. Let η_i be a presymplectic structure on the manifold N_i, $i = 1, 2, 3$ and assume that $\eta_1 \prec \eta_2$ and $\eta_2 \prec \eta_3$. This means that there are symplectic manifolds (M_a, ω_a) and (M_b, ω_b) with $\partial M_a = N_1 \sqcup \overline{N}_2$, $\partial M_b = N_2 \sqcup \overline{N}_3$ such that ω_a induces η_2 on $\overline{N}_2 \subset M_a$ and ω_b induces η_2 on $N_2 \subset M_b$. Using Lemma 6.3.5, M_a and M_b can be glued along N_2 to obtain $M := M_a \cup_{N_2} M_b$ and a symplectic structure ω that agrees with ω_a on M_a and ω_b on M_b. Therefore, M gives the symp-cobordism between η_1 and η_3. $\square$

Symplectic cobordisms allow us to create new examples of symplectic manifolds with boundary from existing ones. The elementary cobordisms allow us, in addition, to fix the diffeomorphism type of the manifold.

Lemma 6.3.8. *Let (M, ω) be a compact symplectic manifold and let η be the presymplectic structure induced by ω in ∂M. If $\eta' \prec_e \eta$, then there is another symplectic structure on M that induces the presymplectic form η' in the boundary.*

Proof. By assumption $\eta \prec^M \emptyset$ and $\eta' \prec_e \eta$. Using the transitivity of the relation (Lemma 6.3.7) we get that $\eta' \prec \emptyset$. This last symp-cobordism is given by gluing M with a cylinder $\partial M \times [0,1]$ along ∂M and $\partial M \times \{0\}$, which is diffeomorphic to M. Thus $\eta' \prec^M \emptyset$. $\square$

Some symplectic cobordisms

Here we give some simple examples of symplectic cobordisms that will be used later.

Lemma 6.3.9. *Let M be a compact manifold, $\eta \prec^M \emptyset$ and $\kappa \in \Omega^2(M)$ be a closed form. Then $\eta + \epsilon\kappa|_{\partial M} \prec^M \emptyset$ for ϵ small enough.*

Proof. Since $\eta \prec^M \emptyset$, there exists a symplectic structure ω on M such that $\omega|_{\partial M} = \eta$. Let $\kappa \in \Omega^2(M)$ be a closed form. Since $\omega + \epsilon\kappa$ is symplectic and defines the same orientation as ω for ϵ small enough, we have that $\omega|_{\partial M} + \epsilon\kappa|_{\partial M} = \eta + \epsilon\kappa|_{\partial M} \prec^M \emptyset$. $\square$

A non-trivial elementary symp-cobordism can be extracted from the normal form for symplectic manifolds around the boundary.

Lemma 6.3.10. *Let η be a presymplectic structure on a closed manifold N and α any +admissible form for η. Then $\eta \prec_e \eta + \epsilon d\alpha$ for $\epsilon > 0$ small enough.*

Proof. Let $M = N \times [0,1]$ and let α an +admissible form for η. The form $\omega' = d(r\alpha) + \eta \in \Omega^2(M)$ is closed in M and is non-degenerate in $TM|_{N\times\{0\}}$ and therefore there exists $\epsilon > 0$ small enough such that ω' is symplectic on $N \times [0, \epsilon]$. Equivalently, $\omega = d(\epsilon r\alpha) + \eta$ is symplectic in $N \times [0,1]$. Similarly to Example 6.3.2, (M, ω) gives the elementary symp-cobordism $\eta \prec_e \eta + \epsilon d\alpha$. $\square$

Lemma 6.3.11. *Let ξ_0 and ξ_1 be isotopic contact forms in a compact manifold N. Then there exists a constant $C > 0$ big enough such that $d\xi_0 \prec_e c d\xi_1$ for every $c \geq C$.*

Proof. Let $\dim N = 2n+1$ and let ξ_t be the isotopy between ξ_0 and ξ_1 and $\eta_t = \frac{d}{dt}\xi_t$. For each t, $\eta_t \wedge (d\xi_t)^n = f_t\, \xi_t \wedge (d\xi_t)^n$ for some function f_t depending smoothly on t, where d denotes the differential in N. Let $a \in \mathbb{R}$ be such that $f_t > -a$ for all t. Take $b > a$ and let $c = e^b$, $C = e^a$. Consider now $\Omega := d^{\mathrm{tot}}(e^{bt}\xi_t) \in \Omega^2(N \times I)$, where d^{tot} is the differential in $N \times I$. The form Ω is symplectic, since

$$\Omega = be^{bt}dt \wedge \xi_t + e^{bt}dt \wedge \eta_t + e^{bt}d\xi_t = e^{bt}dt \wedge (b\xi_t + \eta_t) + e^{bt}d\xi_t,$$

and therefore

$$(e^{-bt}\Omega)^{n+1} = n\, dt \wedge (b\xi_t + \eta_t) \wedge (d\xi_t)^n = n(b + f_t)dt \wedge \xi_t \wedge (\xi_t)^n,$$

which never vanishes because $b + f_t > b - a > 0$. The form Ω gives then the cobordism between $d\xi_0$ and $e^b d\xi_1$. □

Lemma 6.3.12. *Let η be a cosymplectic form on an oriented three-manifold N. Let θ be a closed +admissible one-form for η and let $\xi \in \Omega^1(N)$ be such that $\theta \wedge (\eta + d\xi) > 0$. Then, $\eta \prec_e d\xi + \eta$.*

Proof. Consider the closed two-form $\omega \in \Omega^2(N \times [0,1])$ given by

$$\omega = \lambda dr \wedge \theta + \eta + d(r\xi),$$

This form gives the desired cobordism, provided it is symplectic. Let us see that it is non-degenerate. Its top power is

$$\tfrac{1}{2}\omega^2 = \lambda dr \wedge \theta \wedge (\eta + rd\xi) + \eta \wedge dr \wedge \xi + rdr \wedge \xi \wedge d\xi.$$

Using, Lemma 5.0.11, since $\theta \wedge \eta > 0$ and $\theta \wedge (\eta + d\xi) > 0$, it follows that $\theta \wedge (\eta + rd\xi) > 0$ for $r \in [0,1]$. Therefore, by choosing λ big enough, we can ensure that the first term dominates the sum and therefore, that $\omega^2 > 0$. □

We summarize now the elementary symp-cobordisms found.

Proposition 6.3.13. *Let η be a presymplectic structure on a closed manifold N.*

1. *If $\eta \prec^M \emptyset$ and $\kappa \in \Omega^2_{cl}(M)$, then $\eta + \epsilon\kappa \prec^M \emptyset$ for ϵ small enough.*

2. *$\eta \prec_e \eta + \epsilon d\alpha$ for any α +admissible for η, given ϵ small enough.*

3. *If ξ_0, ξ_1 are isotopic contact forms, then $d\xi_0 \prec_e cd\xi_1$ for some constant $c \neq 0$.*

4. *If $\dim N = 3$ and η is a cosymplectic form, θ is a closed +admissible one-form, ξ is any one-form satisfying $\theta \wedge (\eta + d\xi) > 0$, then $\eta \prec_e \eta + d\xi$.*

Towards structures of cosymplectic type

For applications, we will be interested in modifying symplectic structures on manifolds with boundary to make them of cosymplectic type at the boundary.

Note that if η is a presymplectic structure, a necessary condition to have the cobordism $\eta \prec^M \emptyset$ is that η be in the image of the restriction form $\Omega^2(M) \to \Omega^2(\partial M)$. In the next lemma we prove that this is only required at a cohomological level.

Lemma 6.3.14. *Let $N \subset M$ be an embedded submanifold. If $\eta \in \Omega^k(N)$ is a closed form whose cohomology class lies in the image of the restriction map $i^* : H^k(M) \to H^k(N)$, then there is a closed form $\kappa \in \Omega^k(M)$ such that $\kappa|_N = \eta$.*

Proof. Let $\eta \in \Omega^k(N)$ be closed. By hypothesis there is $\kappa' \in \Omega^k(M)$ such that $i^*[\kappa'] = [\eta]$ and therefore $i^*\kappa' - \eta = d\xi$ for some $\xi \in \Omega^{k-1}(N)$. Let $E \to N$ be a tubular neighborhood of N and let $x \mapsto |x|$ be a metric on E. Let $h : E \to \mathbb{R}$ be defined as 1 when $|x| < 1$ and 0 when $|x| > 2$. The form $d(h\xi) \in \Omega^k(E)$ can be extended by zero to a smooth form on M and the form $\kappa = \kappa' - d(h\xi) \in \Omega^k(M)$ is closed and satisfies that $i^*\kappa = i^*\kappa' - d\xi = \eta$. □

The next proposition will allow us to change symplectic structures of contact type at the boundary to symplectic structures of cosymplectic type at the boundary.

Proposition 6.3.15. *Let M^4 be a compact four-manifold with boundary and let $N := \partial M$ denote its boundary. Let $\xi \in \Omega^1(N)$ be such that $d\xi \prec^M \emptyset$ and $\eta \in \Omega^2(N)$ be a cosymplectic form for which:*

- *$[\eta] \in H^2(N)$ is in the image of the restriction map $H^2(M) \to H^2(N)$.*
- *There is a closed +admissible one-form θ for η such that $\theta \wedge d\xi \geq 0$.*

Then, $\eta \prec^M \emptyset$.

Proof. Let ω be the symplectic form on M such that $\omega|_N = d\xi$. Since $[\eta]$ is the image of the map $H^2(M) \to H^2(N)$, then by Lemma 6.3.14, there exists a two-form $\kappa \in \Omega^2(M)$ such that $\kappa|_N = \eta$. Then, since M is compact, $\omega + \epsilon\kappa$ is symplectic for ϵ small enough and $(\omega + \epsilon\kappa)|_N = d\xi + \epsilon\eta$. Therefore,

$$d\xi + \epsilon\eta \prec^M \emptyset.$$

On the other hand, since $\theta \wedge d\xi \geq 0$, then $\theta \wedge (\epsilon\eta + d\xi) > 0$ and we can apply Lemma 6.3.12 to the form $\epsilon\eta$ and we obtain $\epsilon\eta \prec_e d\xi + \epsilon\eta$. Therefore,

$$\epsilon\eta \prec_e d\xi + \epsilon\eta \prec^M \emptyset \quad \Longrightarrow \quad \epsilon\eta \prec^M \emptyset \quad \Longrightarrow \quad \eta \prec^M \emptyset.$$ □

6.4 Application: S^5

We saw in Chapter 1, Example 3.3.13, that S^3 admits a codimension-one symplectic foliation. The next natural question is then if S^5 admits a codimension-one symplectic foliation. Mitsumatsu (see [Mit11]) gave a positive answer to this question by constructing explicitly a leafwise-symplectic structure on the foliation that Lawson (see [Law71]) constructed on S^5. He proved the following theorem:

Theorem 6.4.1 ([Mit11])**.** *There exists a codimension-one symplectic foliation on S^5.*

He actually proved a slightly more general result, but for the purposes of the present work, we will be only interested on this part of his theorem. In this section we use the turbulisation theorems and the symplectic cobordisms to reprove this result. These methods were developed when trying to understand Mitsumatsu's proof and are therefore inspired by his construction.

To prove Theorem 6.4.1, we will use the decomposition arising from the open book of S^5 described in Example 6.1.8. We describe first this decomposition:

Outside and inside components

Recall now the open book decomposition from Example 6.1.8. The closure of the tubular neighbourhood $\tau_\epsilon(B)$ from Equation (6.3) is called the **inside component** and the complement of its interior,

$$C_\epsilon := S^5 \setminus \tau_\epsilon(B)$$

is called the **outside component**. Both components are invariant under the Hopf action and the quotient of C_ϵ under the S^1-action is $C_\epsilon/S^1 \simeq \mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$. Consider the map $f_\epsilon := f|_{C_\epsilon} : C_\epsilon \to S^1$ and let

$$P_\epsilon := f_\epsilon^{-1}(1) \subset C_\epsilon.$$

We prove Theorem 6.4.1 by constructing a codimension-one symplectic foliation on the inside and outside component and gluing them together.

To construct the codimension-one symplectic foliation on the outside component, we use the geometry of $\mathbb{C}P^2$ to construct a suitable symplectic structure on $\mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$, that we will then pullback to C_ϵ to construct the codimension-one symplectic foliation. This symplectic structure will arise as a modification to the Fubini-Study form on $\mathbb{C}P^2$ that comes from the standard contact structure on S^5 which is adapted to the open book of S^5 just described. Then we will use the geometry of the binding, specially the fact that the binding is a cosymplectic manifold, to modify this symplectic structure.

The codimension-one symplectic foliation on the inside component will be constructed easily using the one we get in the outside component.

Standard contact structure on S^5 and coframe on B

In this part we first recall the contact structure on S^5 that induces the Fubini-Study form on $\mathbb{C}P^2$. Second, we construct a coframe in the binding that is invariant under the Hopf action, that will allow us to modify this symplectic structure.

Let us now state and prove some well-known properties of the standard contact structure on S^5, with the purpose of setting up some notation.

Lemma 6.4.2. *The standard contact structure ξ on S^5 satisfies:*

i. ξ is invariant under the Hopf action.

ii. $d\xi = h^*(\omega_{FS})$, where $\omega_{FS} \in \Omega^2(\mathbb{C}P^2)$ is the Fubini-Study form in $\mathbb{C}P^2$.

iii. $\xi_B := \xi|_B$ *is a contact form on* B.

iv. $d\xi$ *restricts to a symplectic structure on the fibres of* f_ϵ.

Proof. i. Let

$$\tilde{\xi} = \sum_{i=1}^{3}(x_i dy_i - y_i dx_i) = \frac{i}{2}\sum_{i=1}^{3}(z_i d\bar{z}_i - \bar{z}_i dz_i) \in \Omega^1(\mathbb{C}^3)$$

be a primitive of the usual symplectic structure in $\mathbb{C}^3$. The one-form $\xi := \xi|_{S^5}$ is the usual contact structure on S^5. Its explicit formula shows that it is invariant.

ii. - iii. It is known that $d\xi = h^*(\omega_{FS})$, where $\omega_{FS} \in \Omega^2(\mathbb{C}P^2)$ is the Fubini-Study form in $\mathbb{C}P^2$. This symplectic structure, together with the usual complex structure of $\mathbb{C}P^2$, endows $\mathbb{C}P^2$ with a Kähler structure. Since $h(B)$ is a complex submanifold of $\mathbb{C}P^2$, $\omega_{FS}|_{h(B)}$ is symplectic in $h(B)$. Thus, $d\xi_B = h^*(\omega_{FS}|_{h(B)}) = f\theta_1 \wedge \theta_2$ for some coframe $\theta_1, \theta_2 \in \Omega^1(h(B))$ and some $f > 0$. Thus $\xi_B \wedge d\xi_B = f\xi_B \wedge \theta_1 \wedge \theta_2 \neq 0$.

iv. To check the last item, note that

$$\ker d\xi = \mathbb{R} \cdot \left(\sum_{i=1}^{3} y_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial y_i}\right) = \mathbb{R} \cdot \left(\left.\frac{d}{dt}\right|_{t=0} e^{it}(x_i, y_i)\right)$$

is the vector field generating Hopf action, which is transverse to the fibres of f everywhere (see Proposition 6.1.9, part ii.). Therefore, $d\xi|_{P_\epsilon}$ defines a symplectic structure on the fibres of f_ϵ. $\square$

Remark 6.4.3. We may take the orientations of S^5 and B such that ξ and ξ_B become positive contact forms.

Lemma 6.4.4 (Coframes)**.** *Let* $M := \mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$*. There are:*

(a) A one-form $\alpha_M \in \Omega^1(M)$ *such that* $d\alpha_M = \omega_{FS}|_M$.

(b) A coframe on ∂M *of closed one-forms* $\alpha_B, \vartheta_1, \vartheta_2$.

Proof. (a) Note that P_ϵ inherits a $\mathbb{Z}_3$-action from the S^1-action on C_ϵ. Moreover, the inclusion $P_\epsilon \hookrightarrow C_\epsilon$ descends to a diffeomorphism $P_\epsilon/\mathbb{Z}_3 \simeq C_\epsilon/S^1 \simeq M$.

The form $\xi_{P_\epsilon} := \xi|_{P_\epsilon}$ is $\mathbb{Z}_3$-invariant and therefore, it descends to a form

$$\alpha_M \in \Omega^1(M).$$

Since $d\xi_{P_\epsilon}$ is symplectic on P_ϵ and $\mathbb{Z}_3$-invariant, then $d\alpha_M$ is also symplectic. Since $d\xi = h^*(\omega_{FS})$, it follows that $d\alpha_M = \omega_{FS}|_M$.

(b) The boundary of P_ϵ, $\partial P_\epsilon = p^{-1}\{\epsilon\}$, is identified with B using the tubular neighbourhood $\tau_\epsilon(B)$ (see Equation (6.3)). The trivialisation φ of this tubular neighbourhood can be chosen to be S^1-equivariant by choosing the retraction π_B (recall Equation (6.2)) to be defined by an S^1-invariant metric. Therefore, the isomorphism between B and ∂P_ϵ given by π_B from Equation (6.3) can be chosen to be $\mathbb{Z}_3$-equivariant (note that ∂P_ϵ inherits a $\mathbb{Z}_3$-action and not an S^1-action like B).

Since the retraction $\pi_B : \partial C_\epsilon \to B$ is S^1-equivariant, it descends to a map $r : \partial M \to h(B)$. We have then the following commutative diagram:

$$\begin{tikzcd} P_\epsilon \arrow[d, "/\mathbb{Z}_3"'] & \partial P_\epsilon \arrow[l, hook'] \arrow[r, "\pi_B|_{\partial P_\epsilon}", "\sim"'] \arrow[d, "/\mathbb{Z}_3"'] & B \arrow[d, "h_B"] \\ M & \partial M \arrow[l, hook'] \arrow[r, "r"] & h(B) \end{tikzcd}$$

where $h_B := h|_B$. Let $\theta_1, \theta_2 \in \Omega^1(h(B))$ be a co-frame for the torus $h(B)$ and let

$$\vartheta_1 := r^*(\theta_1), \qquad \vartheta_2 := r^*(\theta_2)$$

be two closed one-forms on ∂M.

Recall the form $\xi_B := \xi|_B$ defined in the previous lemma. The form $\pi_B^*(\xi_B)|_{\partial P_\epsilon} \in \Omega^1(\partial P_\epsilon)$ is $\mathbb{Z}_3$-invariant and therefore it descends to a form

$$\alpha_B \in \Omega^1(\partial M).$$

Note that if $\zeta : \partial P_\epsilon \to \partial M$ denotes the quotient map by the $\mathbb{Z}_3$-action, then

$$\zeta^*(\alpha_B \wedge \vartheta_1 \wedge \vartheta_2) = (\pi_B|_{\partial P_\epsilon})^*(\xi_B \wedge h_B^*(\theta_1) \wedge h_B^*(\theta_2)).$$

Since $\xi_B \wedge h_B^*(\theta_1) \wedge h_B^*(\theta_2)$ is a volume form on B and π_B^* is injective, then $\alpha_B \wedge \vartheta_1 \wedge \vartheta_2 \neq 0$. □

Remark 6.4.5. In the previous lemma, we may choose the frame θ_1, θ_2 such that $\alpha_B \wedge \vartheta_1 \wedge \vartheta_2 > 0$. The forms α_B and $\alpha_M|_{\partial M}$ are related and their relation will be given in the proof of the next Lemma.

Now we use the coframe from the previous Lemma to reduce the proof of Theorem 6.4.1 to the problem of the existence of a suitable symplectic structure on M.

Let us fix the trivialisation of $\partial C_\epsilon \xrightarrow{\sim} B \times S^1$, $x \mapsto (\pi_B(x), f_\epsilon(x))$, where π_B is given by Equation (6.2) and let us also identify ∂M with $B/\mathbb{Z}_3$ as in the previous proof.

Lemma 6.4.6. *The manifold $M := \mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$ admits a symplectic structure ω of cosymplectic type at the boundary such that $\omega|_{\partial M} = \vartheta_2 \wedge \alpha_B$.*

Proof of Theorem 6.4.1 assuming Lemma 6.4.6. Let us recall the two fibrations

$$f_\epsilon : C_\epsilon \to S^1, \qquad h_\epsilon := h|_{C_\epsilon} : C_\epsilon \to \mathbb{C}P^2 \setminus h(\tau_\epsilon(B)),$$

with typical fibres P_ϵ and S^1 which are everywhere transverse. First we prove that the pair $(\eta, \theta) := (h_\epsilon^*(\omega), f_\epsilon^*(d\varphi))$ is a cosymplectic structure on C_ϵ of s-type at the boundary, where ω is the symplectic structure on $\mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$ obtained from Lemma 6.4.6. The map $h_\epsilon : C_\epsilon \to \mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$ satisfies the hypothesis of Proposition 3.4.7: the transversality of the fibres of the fibrations above implies that the kernel of the closed one-form θ defines a distribution on C_ϵ which is horizontal. Therefore, by Proposition 3.4.7, (η, θ) is a cosymplectic structure on C_ϵ of s-type at the boundary.

Recall that we are identifying ∂C_ϵ with $B \times S^1$ using the map $x \mapsto (\pi_B(x), f_\epsilon(x))$ and $\partial M \simeq B/\mathbb{Z}_3$. Under this identification, the map $h_\epsilon|_{\partial C_\epsilon} : \partial C_\epsilon \to \partial M$ becomes $h_\epsilon|_{\partial C_\epsilon} : B \times S^1 \to B/\mathbb{Z}_3$, $h_\epsilon|_{\partial C_\epsilon}(x, e^{i\theta}) = [xe^{-i\theta/3}]_3$, where $[\cdot]_3$ denotes the equivalence class under the $\mathbb{Z}_3$ action given by multiplication by the third roots of unity. We want to compute the pullback of the forms $\alpha_B, \vartheta_1, \vartheta_2 \in \Omega^1(B/\mathbb{Z}_3)$ under h_ϵ. To compute this pullback, consider the commutative diagram

$$\begin{tikzcd} B \times S^1 \arrow[r, "q"] \arrow[d, "3:1"'] & B \arrow[d, "\zeta"] \\ B \times S^1 \arrow[r, "h_\epsilon|_{\partial C_\epsilon}"] & B/\mathbb{Z}_3 \end{tikzcd}$$

where $q(x, e^{i\theta}) = xe^{-i\theta}$, the vertical map $3:1$ is given by $(x, e^{i\theta}) \mapsto (x, e^{i3\theta})$, and the map ζ is the quotient by $\mathbb{Z}_3$. We claim that for every one-form $\gamma \in \Omega^1(B)$ which is invariant under the S^1-action on B, $q^*(\gamma) = \gamma - (\iota_V \gamma) d\varphi$, where V is the vector field generating the S^1-action on B. To see this, note that

$$\begin{aligned} q^*(\gamma)(X_i) &= \gamma(q_* X_i) = \gamma(X_i), \quad i = 1,2,3 \\ q^*(\gamma)(\partial_\varphi) &= \gamma(q_* \partial_\varphi) = -\iota_V \gamma \end{aligned}$$

where X_i, $i = 1,2,3$ are a frame of vector fields on B which are invariant under the action of S^1.

Let us abuse notation and denote $h_B^*(\theta_i)$ by $\theta_i \in \Omega^1(B)$, $i = 1,2$.

Now, since $\zeta^*(\alpha_B) = \xi_B$ and $\zeta^*(\vartheta_i) = \theta_i$ and $\iota_V \theta_i = 0$, $i = 1,2$, $\iota_V \xi_B = 1$, we get that $(h_\epsilon|_{\partial C_\epsilon})^*(\vartheta_i) = \theta_i$, $i = 1,2$, and $(h_\epsilon|_{\partial C_\epsilon})^*(\alpha_B) = \xi_B + \frac{1}{3} d\varphi$.

Therefore, since $\omega|_{\partial M} = \vartheta_2 \wedge \alpha_B$, then $\eta|_{B \times S^1} = \theta_2 \wedge \xi_B + \frac{1}{3}\theta_2 \wedge d\varphi$. Moreover, since ϑ_1 is a closed +admissible form for $\omega|_{\partial M}$, then there is a vector field X around ∂M such that $\iota_X \omega|_{\partial M} = \vartheta_1$ and therefore, using Proposition 3.4.7, we find a vector field V satisfying the conditions of Lemma 3.4.4 for which $\iota_V \eta|_{B \times S^1} = \theta_1$. Therefore, applying Theorem 3.4.8 we get a codimension-one symplectic foliation on C_ϵ for which the symplectic structure at the boundary $B \times S^1$ is

$$\theta_2 \wedge \xi_B + (\tfrac{1}{3}\theta_2 + \theta_1) \wedge d\varphi.$$

On the other side, consider the cosymplectic structure $(\theta_2 \wedge \xi_B, \frac{1}{3}\theta_2 + \theta_1)$ on B and use Corollary 3.4.13 to construct a codimension-one symplectic foliation on $B \times D^2$ such that the symplectic structure at the boundary coincides with the one above. Using the Gluing Theorem 2.3.4 we can glue these two symplectic foliations to obtain a codimension-one symplectic foliation on S^5. □

Symplectic structure on $\mathbb{C}P^2 \setminus h(\tau_\epsilon(B))$

Proof of Lemma 6.4.6. In this proof we modify the symplectic structure $d\alpha_M \in \Omega^2(M)$ obtained in Lemma 6.4.4 to make it of cosymplectic type at the boundary. This modification is made in two steps. These steps appear already implicitly in Mitsumatsu's construction, although not in the form of cobordisms.

Step 1 $d\alpha_B \prec^M \emptyset$

Here we see that there is an isotopy of contact forms joining α_B and $\alpha_M|_{\partial M}$. Using this isotopy, since $d\alpha_M|_{\partial M} \prec^M \emptyset$, by Lemma 6.3.11 there is a constant $\lambda > 0$ such that $\lambda\, d\alpha_B \prec^M \emptyset$ and therefore $d\alpha_B \prec^M \emptyset$.

Let us construct this isotopy. Recall the identification of the tubular neighbourhood $\tau_\epsilon(B)$ with $B \times D^2$ using the trivialisation (6.3). Using this trivialisation, the identification of $B = B \times \{0\}$ with $\partial P_\epsilon = B \times \{\epsilon\}$ becomes the projection on the second component. Moreover, using this trivialisation, we have $\xi_B = i_0^*(\xi)$ and $\xi|_{\partial P_\epsilon} = i_\epsilon^*(\xi)$, where $i_t : B \times \{t\} \to B \times D^2$, $t \in [0,\epsilon]$ is the inclusion. Therefore, $\xi|_{\partial P_\epsilon}$ is a one-form on $\Omega^1(\partial P_\epsilon)$ arbitrarily close to ξ_B for ϵ close enough to 0. Since $\xi_B \in \Omega^1(\partial P_\epsilon)$ is contact, $\xi|_{\partial P_\epsilon}$ is also contact and the path $\xi_t = i_t^*(\xi)$, $t \in [0,\epsilon]$ is a path of $\mathbb{Z}_3$-invariant contact forms on ∂P_ϵ connecting ξ_B and $\xi|_{\partial P_\epsilon}$. It descends to the quotient to a path α_t of contact forms joining α_B and $\alpha_M|_{\partial M}$.

Step 2 $\vartheta_2 \wedge \alpha_B \prec^M \emptyset$.

This follows by applying Proposition 6.3.15 for $\xi = \alpha_B$, $\eta = \vartheta_2 \wedge \alpha_B$ and $\theta = \vartheta_1$. Note that θ is +admissible for η. To apply this proposition we need to check two conditions:

- $\vartheta_2 \wedge \alpha_B$ is in the image of the map $H^2(M) \to H^2(\partial M)$. This can be done by standard Mayer-Vietoris argument (Lemma 5.3 in [Mit11]). For completeness, we prove this in the lemma below.
- $\theta \wedge d\xi \geq 0$. This holds because, by the definition of the coframe $\vartheta_1, \vartheta_2, \alpha_B$ (Lemma 6.4.4), it follows that $\vartheta_1 \wedge d\alpha_B = 0$.

Therefore, we obtain a symplectic structure ω on M such that $\omega|_{\partial M} = \vartheta_2 \wedge \alpha_B$ and thus the result follows. This symplectic structure is of cosymplectic type since ϑ_1 is a closed admissible one-form for $\vartheta_2 \wedge \alpha_B$. $\square$

Lemma 6.4.7. *The pullback map $H^2(M) \to H^2(\partial M)$ is surjective.*

Proof. The Mayer-Vietoris sequence for cohomology of $\mathbb{C}P^2$ using the decomposition $\mathbb{C}P^2 = (\mathbb{C}P^2 \setminus h(B)) \cup h(\tau_\epsilon(B))$ and the fact that $H^3(\mathbb{C}P^2) = \{0\}$, reads

$$\cdots \to H^2(\mathbb{C}P^2) \to H^2(\mathbb{C}P^2 \setminus h(B)) \oplus H^2(h(\tau_\epsilon(B))) \to H^2(h(\tau_\epsilon(B) \setminus B)) \to 0 \tag{6.5}$$

Note that the last term in the sequence can be replaced by $H^2(\partial M)$ since $h(\tau_\epsilon(B) \setminus B)$ is homotopically equivalent to ∂M with the equivalence given by the inclusion of ∂M into $h(\tau_\epsilon(B))$. In a similar way we can replace $\mathbb{C}P^2 \setminus h(B)$ for M.

We now prove that the restriction $i^* : H^2(h(\tau_\epsilon(B))) \to H^2(\partial M)$ is zero and thus from the Mayer-Vietoris sequence, the map $H^2(M) \to H^2(\partial M)$ is surjective.
Consider the commutative diagrams

$$\begin{tikzcd} \partial M \arrow[r, "i"] \arrow[d, "r"'] & h(\tau_\epsilon(B)) \arrow[dl, "\pi_{h(\tau_\epsilon(B))}"] \\ h(B) & \end{tikzcd} \qquad \begin{tikzcd} H^2(\partial M) & H^2((h(\tau_\epsilon(B))) \arrow[l, "i^*"'] \\ & H^2(h(B)) \arrow[ul, "r^*"] \arrow[u, "\pi^*_{h(\tau_\epsilon(B))}"'] \end{tikzcd}$$

Since $h_B : B \to h(B)$ is a non trivial S^1 principal bundle over $h(B)$, $\partial M = B/\mathbb{Z}_3$ is also a non trivial principal S^1 bundle over $h(B)$. The Euler class e of $r : \partial M \to h(B)$ (seen as an S^1 bundle) determines a non trivial element in $H^2(h(B))$. Since $H^2(h(B))$ is one dimensional, the map r^* is determined by $r^*(e)$. Note that $h_B^*(e) = d\xi_B$, which is the curvature of the bundle $B \to h(B)$. Consider the following diagram:

$$\begin{tikzcd} B \arrow[r, "/\mathbb{Z}_3"] \arrow[dr, "h_B"'] & \partial M \arrow[d, "r"] \\ & h(B) \end{tikzcd}$$

Since ξ_B is invariant, it descends to a form $\bar{\xi}_B$ on ∂M and therefore, $r^*(e) = d\bar{\xi}_B$, which means that in cohomology, $r^* = 0$. On the other hand, $\pi^*_{h(\tau_\epsilon(B))}$ is an isomorphism, since $\pi_{h(\tau_\epsilon(B))}$ is a retraction from $h(\tau_\epsilon(B))$ to $h(B)$. Since the diagram in the right commutes, we get that $i^* = 0$. □

Remark 6.4.8 (Leaves of the foliation)**.** The leaves of the foliation which are in the interior component are diffeomophic to $T^2 \times D^2$. The compact leaf separating the two components is diffeomorphic to $B \times S^1$ and the leaves in the outside component are three-covers of $\mathbb{C}P^2$ minus a symplectic torus.

Remark 6.4.9 (Unimodularity and tameness of the foliation)**.** The foliation has compact leaves and $H^2(S^5) = 0$. Therefore, the codimension-one symplectic foliation is not tame. It is however tame both in the inside and outside component since they are obtained by cosymplectic structures of s-type at the boundary using Theorem 3.4.8.

Remark 6.4.10 (Boothby-Wang contact manifolds)**.** This construction of S^5 can be framed in a general setting of Boothby-Wang contact manifolds.
Let (S, ω_S) be a compact symplectic manifold with ω_S of integral class. Let $L \to S$ be the complex line bundle with Chern class $c_1(L) = [\omega_S]$ and M the S^1-bundle associated to L. Recall that a connection form θ on M such that $d\theta = \pi^*(\omega_S)$ is a contact form on M, called a *Boothby-Wang contact form* (see [Gei08]). To construct codimension-one symplectic foliation on M, let us first obtain an open book decomposition on M and then apply Theorem 6.2.2. In the case of S^5, $S = \mathbb{C}P^2$, $\omega_S = \omega_{FS}$ is the Fubiny-Study form, $M = S^5$ and $\theta = \xi$.

Recall that Donaldson ([Don96]) proves that, for $k \in \mathbb{Z}$ big enough, the Poincare dual of the class $k[\omega_S] \in H^2(S)$ can be represented by a codimension-two symplectic submanifold $\tau \subset S$. He constructs τ as the zeros of a section s of the vector bundle $L^{\otimes k} \to S$, $\tau = s^{-1}(0) \subset S$.

Let $\pi : M \to S$ denote the projection and $B = \pi^{-1}(\tau)$. We have the following diagram:

$$\begin{tikzcd} M \arrow[r, "\pi"] & S \\ B \arrow[u, hook, "i"] \arrow[r, "\pi|_B"] & \tau \arrow[u, hook, "i"'] \end{tikzcd}$$

The line bundle $\pi^*(L^{\otimes k}) \to M$ is trivial and therefore, the pullback section

$$\tilde{s} := \pi^*(s) : M \to \pi^*(L^{\otimes k}) = M \times \mathbb{C}$$

defines a map $\tilde{s} : M \to \mathbb{C}$ that vanishes on B. Therefore, we get a map $f : M \setminus B \to S^1$, $x \mapsto \tilde{s}(x)/|\tilde{s}(x)|$. Note also that the normal bundle of B is $\pi^*(L)|_B \to B$ which is trivial. Moreover, the pair (B, f) defines an open book decomposition on M (see [CDK14]). In the case S^5, this open book decomposition coincides with the one given in Example 6.1.8.

We want to use Theorem 6.2.2 to construct a codimension-one foliation on M. Let ν_B be a tubular neighbourhood of B and $\nu_\tau := \pi(\nu_B)$. Then, $M \setminus \nu_B$ projects to $S \setminus \nu_\tau$, and ω_S defines a symplectic structure on $S \setminus \tau$. To apply Theorem 6.2.2 we need a cosymplectic structure on $M \setminus \nu_B$ satisfying some conditions. The pair $(\pi^*(\omega_S), f^*(d\varphi))$ defines a cosymplectic structure on M, but it does not satisfy the condition of the Theorem: to start with, $\pi^*(\omega_S) = d\theta$ so the form $\pi^*(\omega_S)$ does not define a symplectic structure of cosymplectic type at the boundary when restricted to the fibres of f. If one could obtain a symplectic structure ω_C of cosymplectic type on $S \setminus \nu_\tau$, then $(\pi^*(\omega_C), f^*(d\varphi))$ has a good chance of satisfying the conditions of the theorem. Then, the main obstacle in constructing a codimension-one foliation on M by applying Theorem 6.2.2 is obtaining a suitable symplectic structure on $S \setminus \nu_\tau$. In the case of S^5, this is achieved by modifying ω_S using symplectic cobordisms.

Remark 6.4.11 (Symplectic vector bundles on S^5)**.** As an aside remark, we classify the symplectic vector bundles on S^5 and we see that the one defined by the symplectic foliation on S^5 from Theorem 6.4.1 is the unique symplectic vector bundle of rank 4, up to isomorphism, which is a subbundle of TS^5.

Lemma 6.4.12. *There are two isomorphism classes of symplectic vector bundles of rank four on S^5 and only one of them can be seen as a subbundle of TS^5.*

Proof. Let $\mathrm{Symp}_4(S^5)$ denote the isomorphism classes of symplectic four-bundles over S^5. These isomorphism classes are classified by the homotopy classes of maps $S^4 \to \mathrm{Sp}(4)$ (see [BT95]). Since $\mathrm{Sp}(4)$ is homotopically equivalent to its maximal compact subgroup $SU(2)$, these are the same as homotopy classes of maps $S^4 \to SU(2)$. The latter reflects the fact that a vector bundle is symplectic if and only if it admits a complex structure.

The homotopy classes of maps $S^4 \to SU(2) \simeq S^3$ can be classified using the homotopy groups. Indeed,

$$[S^4, X] \simeq \pi_4(X, x)/\pi_1(X, x),$$

where the action of $\pi_1(X, x)$ on $\pi_4(X, x)$ is given as follows: the space $\Omega_x X$, of loops with base point x, acts on itself by conjugation, and this action it induces an action of $\pi_1(X, x)$ on $\pi_{q-1}(\Omega_x X, x) \simeq \pi_q(X, x)$ for every q (see [BT95] for more details). Therefore,

$$\mathrm{Symp}_4(S^5) \simeq \pi_4(S^3)/\pi_1(S^3) \simeq \mathbb{Z}_2/\{0\},$$

see [Hat01]. Therefore, there are two isomorphism classes of symplectic vector bundles over S^5.

We would like to see which one of those vector bundles can be seen as subbundles of TS^5. Since S^5 is simply connected, all line bundles over S^5 are trivial. We deduce then that a vector bundle E of rank 4 over S^5 can be seen as a subbundle of its tangent bundle if and only if, when we add a trivial bundle to it, we get a bundle isomorphic to TS^5, which is not trivial (see e.g., [BT95]). This means that the trivial four-bundle over S^5 can not be seen as a subbundle of TS^5 and therefore, there is only one isomorphism class of symplectic subbundles of TS^5 of rank 4. □

Chapter 7

Deformations of Log-symplectic Structures

In this chapter we deviate from constructing symplectic foliations and study the deformations of log-symplectic structures. These are a particular type of Poisson structures, close to being non-degenerate, that were used in Chapter 3 to construct symplectic foliations.

Log-symplectic structures are interesting by themselves from the point of view of Poisson geometry. They have an interesting non-trivial geometry and yet are simple enough to enable us to make a complete description of their deformations.

In the first section, we present the language of b-geometry, the natural formalism to study log-symplectic structures. In the next section we compute the Poisson cohomology of log-symplectic structures and use this computation to motivate the section that follows, that deals with deformations of log-symplectic structures. There, we describe the space of Poisson structures close to a log-symplectic structure, up to diffeomorphisms close to the identity.

Except Lemma 7.2.3 and the last subsection **??**, everything that is included in this chapter appears in [MOT14a].

7.1 B-geometry

A very effective tool to deal with log-symplectic structures is using an adaptation of the b-geometry of manifolds with boundary developed by Melrose in [Mel93] (see also [GMP14]).

We consider pairs (M, Z), where M is a smooth manifold (without boundary) and Z is a distinguished codimension-one closed embedded submanifold of M. Such a pair is called a *b-manifold*. A *b-map* is a smooth map $\varphi : (M_1, Z_1) \to (M_2, Z_2)$ between b-manifolds that is transverse to Z_2 and satisfies $\varphi^{-1}(Z_2) = Z_1$.

Associated to a pair (M, Z) there is a natural *b-tangent bundle* denoted by $T_Z M \to M$, whose sections are vector fields on M that are tangent to Z (the Serre-Swan Theorem ensures that this vector bundle is well defined) . Let $E \to M$ be a tubular neighbourhood of Z in M, and denote by ξ the corresponding Euler vector field on E; i.e. the flow of ξ is fibrewise multiplication with e^t. Regarding ξ as a section of $T_Z E$, we see that ξ is nowhere vanishing, and moreover, that

$$\xi_Z := \xi|_Z \in \Gamma(T_Z M|_Z) \tag{7.1}$$

is independent of the tubular neighbourhood.

There is also a notion of *b-de-Rham cohomology*, denoted by $H^\bullet_Z(M)$, which is computed on the complex of *b-differential forms* $\Omega^\bullet_Z(M) := \Gamma(\wedge^\bullet T^*_Z M)$ (i.e. on the space of multi-linear forms on $T_Z M$). The differential is determined by the fact that the restriction map $\Omega^\bullet_Z(M) \to \Omega^\bullet(M\backslash Z)$ is a chain map. It is well-known that this complex fits in a (canonical) short exact sequence of complexes:

$$0 \longrightarrow \Omega^\bullet(M) \longrightarrow \Omega^\bullet_Z(M) \xrightarrow{\iota_{\xi_Z}} \Omega^{\bullet-1}(Z) \longrightarrow 0, \tag{7.2}$$

where ξ_Z is defined by (7.1). This sequence splits. To see this, fix E a tubular neighbourhood of Z, and consider a function $\lambda : M\backslash Z \to (0,\infty)$ satisfying that $\lambda(x) := |x|$, for $x \in E$ with $|x| \leq 1/2$, and $\lambda \equiv 1$ on $M\backslash\{x \in E : |x| < 1\}$. We call such a function *a distance function adapted to* E. The one-form $d\log(\lambda)$ extends to a closed one-form on $T_Z M$ which is supported in E. A splitting of (7.2) commuting with the differentials is given by the map

$$\sigma : \Omega^{\bullet-1}(Z) \longrightarrow \Omega^\bullet_Z(M), \quad \omega \mapsto d\log(\lambda) \wedge p^*(\omega), \tag{7.3}$$

where $p : E \to Z$ is the projection. This implies the Mazzeo-Melrose theorem [Mel93], i.e. we have the following decomposition for b-cohomology:

$$H^\bullet_Z(M) = H^\bullet(M) \oplus \sigma(H^{\bullet-1}(Z)) \simeq H^\bullet(M) \oplus H^{\bullet-1}(Z). \tag{7.4}$$

Note that the image of $1 \in H^0(Z)$ under σ is

$$\sigma(1) = [d\log(\lambda)].$$

Now, if λ' is another distance function (associated to a second tubular neighbourhood), it is easy to see that there is a smooth function $f \in C^\infty(M)$ such that $\lambda' = e^f\lambda$. Hence,

$$d\log(\lambda') = d\log(\lambda) + df,$$

and therefore, the class $[d\log(\lambda)] \in H^1_Z(M)$ is independent of the choice of λ.

This example motivates the results below. The case when M is orientable of Corollary 7.1.2 appeared in [Sco13] in the more general setting of b^k-forms. Here orientability is not assumed.

Lemma 7.1.1. *The elements in $\sigma(H^{k-1}(Z)) \subset H^k_Z(M)$ are characterized by the following property: if $\omega \in \Omega^k_Z(M)$ is a closed b-form, then $[\omega] \in H^k_Z(M)$ belongs to $\sigma(H^{k-1}(Z))$ if and only if for every b-map $i : (N, W) \to (M, Z)$ from a k-dimensional compact oriented b-manifold (N, W), the following holds:*

$$\int_N i^*(\omega) = 0,$$

where the integral is the regularized volume defined in [MT04].

Proof of Lemma 7.1.1. Recall from [MT04] the definition of the *regularized volume*: namely, let ω be a top b-form on a compact oriented b-manifold (N, W). Its regularized volume is defined by the limit:

$$\mathrm{Vol}_N(\omega) := \lim_{\epsilon\to 0} \int_{\lambda\geq\epsilon} \omega,$$

where $\lambda : N\backslash W \to (0,\infty)$ is a distance function. This limit exists for any λ, and does not dependent on the choice of λ.

First we show that the map $\sigma : \Omega^{k-1}(Z) \to \Omega^k_Z(M)$ maps closed forms to forms that satisfy the condition from the lemma. For this, consider a b-map $i : (N,W) \to (M,Z)$ from a compact oriented manifold N of dimension k, and consider a closed $k-1$-form μ on Z. Since i is a b-map, note that $\lambda \circ i$ is a distance function for (N,W); thus the volume can be computed by:

$$\mathrm{Vol}_N(i^*(\sigma(\mu))) := \lim_{\epsilon\to 0}\int_{N_\epsilon} i^*\Big(d\big(\log(\lambda)p^*(\mu)\big)\Big) = \lim_{\epsilon\to 0}\int_{N_\epsilon} d\big(\log(\lambda\circ i)(p\circ i)^*(\mu)\big),$$

where $N_\epsilon := \{x \in N : \lambda(i(x)) \geq \epsilon\}$. For ϵ small enough, we will prove that:

$$\int_{N_\epsilon} d\big(\log(\lambda\circ i)(p\circ i)^*(\mu)\big) = 0. \tag{7.5}$$

For small enough ϵ, N_ϵ is a manifold with boundary $W_\epsilon := \{x \in N : \lambda(i(x)) = \epsilon\}$, and by Stokes' theorem, the integral in (7.5) equals $\log(\epsilon)\int_{W_\epsilon}(p\circ i)^*(\mu)$. Note that W_ϵ, with the opposite orientation, is also the boundary of the manifold $E_\epsilon := \{x \in N : \lambda(i(x)) \leq \epsilon\}$. Therefore, again using Stokes' theorem we obtain

$$\int_{W_\epsilon}(p\circ i)^*(\mu) = -\int_{E_\epsilon} d(p\circ i)^*(\mu) = -\int_{E_\epsilon}(p\circ i)^*(d\mu) = 0,$$

which proves that (7.5) holds.

Next, we claim that for any b-map $i : (N,W) \to (M,Z)$ from a k-dimensional, compact oriented b-manifold (N,W), and any exact k-b-form $d\eta$, we have that $\mathrm{Vol}_N(i^*(d\eta)) = 0$. Write $\eta = \alpha + \sigma(\beta)$, for $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^{k-2}(Z)$. Then

$$\mathrm{Vol}_N(i^*(\omega)) = \mathrm{Vol}_N(di^*(\alpha)) + \mathrm{Vol}_N(\sigma(d\beta)).$$

The first term vanishes, since it is the integral of an exact de Rham form, and the second vanishes by the above. Thus, we conclude that the b-map $i : (N,W) \to (M,Z)$ induces a map

$$\mathrm{Vol}_N : H^k_Z(M) \longrightarrow \mathbb{R},$$

which vanishes on $\sigma(H^{k-1}(Z))$.

Let $C^k_Z(M) \subset H^k_Z(M)$ be the space of b-cohomology classes $[\omega] \in H^k_Z(M)$ satisfying $\mathrm{Vol}_N(i^*(\omega)) = 0$ for all b-maps $i : (N,W) \to (M,Z)$ where (N,W) is a compact oriented k-dimensional b-manifold. By the above, we have that $\sigma(H^{k-1}(Z)) \subset C^k_Z(M)$. Now, $H_k(M,\mathbb{R})$ is generated by smooth oriented submanifolds $i : N \to M$, which can be assumed to be transverse to Z, hence yielding b-maps $i : (N, i^{-1}(Z)) \to (M,Z)$. This implies that $C^k_Z(M)\cap H^k(M) = \{0\}$. Thus, using the decomposition (7.4), this shows that $C^k_Z(M) = \sigma(H^{k-1}(Z))$. □

Corollary 7.1.2 ((see also [Sco13])). *The decomposition (7.4) is functorial in the b-category.*

Proof. Let $\varphi : (M_1,Z_1) \to (M_2,Z_2)$ be a b-map. We have to show that the induced map

$$\varphi^* : H^\bullet_{Z_2}(M_2) \longrightarrow H^\bullet_{Z_1}(M_1)$$

maps $\sigma(H^{\bullet-1}(Z_2))$ into $\sigma(H^{\bullet-1}(Z_1))$, where σ is the map from (7.3). Let $[\omega] \in \sigma(H^{k-1}(Z_2))$ be represented by a closed b-forms $\omega \in \Omega^k_{Z_2}(M_2)$. Let $i : (N, W) \to (M_1, Z_1)$ be a b-map from a compact oriented k-dimensional b-manifolds (N, W). Then $\varphi \circ i : (N, W) \to (M_2, Z_2)$ is also a b-map; therefore, by the lemma,

$$\int_N i^*(\varphi^*(\omega)) = \int_N (\varphi \circ i)^*(\omega) = 0.$$

Again, the lemma implies that $[\varphi^*(\omega)] = \varphi^*[\omega] \in \sigma(H^{k-1}(Z_2))$. This finishes the proof. □

Log-symplectic structures

The framework of b-geometry allows us to regard log-symplectic structures as "symplectic" structures on the b-tangent bundle $T_Z M$. To see this, note first that a log-symplectic structure on M with singular locus Z is the same as a non-degenerate section $\pi \in \Gamma(\wedge^2 T_Z M)$ satisfying $[\pi, \pi] = 0$. This is equivalent to having a non-degenerate b-two-form $\omega \in \Gamma(\wedge^2 T_Z^* M)$ that is closed, $d\omega = 0$; the two being related by $\pi^{-1} = \omega$. Using a tubular neighbourhood of Z in M and a distance function λ adapted to E, we can decompose

$$\omega = \alpha + d\log(\lambda) \wedge p^*(\theta), \tag{7.6}$$

where α is a closed two-form on M and θ is a closed one-form on Z. Note that the image of any closed b-one-form under the isomorphism $\pi^\sharp : T_Z^* M \xrightarrow{\sim} T_Z M$ is a Poisson b-vector field. In particular, $X := -\pi^\sharp(d\log(\lambda))$ represents the modular class of π (see [GMP14]) and $V := X|_Z$ is a Poisson vector field on (Z, π_Z) transverse to the symplectic leaves. As mentioned before, the pair (π_Z, V) corresponds to the cosymplectic structure (η, θ) on Z, where $\eta := \alpha|_Z$. Note that θ does not depend on the tubular neighbourhood and on λ, since it is the image of ω under the map ι_{ξ_Z} from (7.2). However, η does depend, but in a rather mild fashion, as it changes only by exact two-forms of the type $d(f\theta)$, with $f \in C^\infty(Z)$. Likewise, V is determined up to Hamiltonian vector fields.

Non-degenerate b-forms

In the proof of Theorem 7.3.1 we use a version of the Moser trick for closed non-degenerate b-two-forms. As in the classical case, this stability result holds for the following type of forms:

Definition 7.1.3. *A b-form $\zeta \in \Omega^k_Z(M)$ of degree k on (M, Z) is called* non-degenerate *if the following map is surjective:*

$$T_Z M \longrightarrow \bigwedge^{k-1} T_Z^* M, \quad V \mapsto \iota_V \zeta.$$

Comparing dimensions, we see that non-degenerate b-forms can exist only in degrees $1, 2$ and $\dim(M)$. Correspondingly, closed non-degenerate b-forms give rise to the following geometric objects:

(1) Let ϑ be a closed non-degenerate b-one-form. Non-degeneracy is equivalent to ϑ being nowhere vanishing. Now $\iota_{\xi_Z}\vartheta$ is a closed 0-form on Z, thus on each connected component

Z_i of Z it is a constant $c_i \in \mathbb{R}$. Geometrically, ϑ encodes a codimension one foliation $\mathcal{F}_\vartheta$ on M which on $M \backslash Z$ is given by the kernel of ϑ, it is transverse to the components Z_i for which $c_i = 0$, and it contains the components Z_i with $c_i \neq 0$ as leaves. To see that this defines indeed a smooth foliation, let $E_i \to M$ be tubular neighbourhoods of Z_i, such that $E_i \cap E_j = \emptyset$, for $i \neq j$, and let λ_i be adapted distance functions. We decompose ϑ as the locally finite sum

$$\vartheta = \theta + \sum_i c_i d \log(\lambda_i),$$

where θ is a closed one-form. Around a component with $c_i = 0$, we have that $\vartheta = \theta$, and non-degeneracy implies that $\theta|_{TZ_i} : TZ_i \to \mathbb{R}$ is nowhere vanishing; hence the foliation extends to Z_i by the kernel of θ, and moreover, it is transverse to Z_i. If Z_i is a component with $c_i \neq 0$, then around Z_i we can write $\vartheta = \theta + c_i d \log(\lambda)$. On small open neighbourhoods U_i around points in Z_i, we can write $\lambda_i = |t|$, where t is a coordinate function with $Z_i \cap U_i = \{t = 0\}$. Then the kernel of ϑ is also the kernel of the one-form

$$t/c_i \vartheta = dt + t/c_i \theta,$$

which shows that the foliation extends smoothly to Z_i, with Z_i as a leaf.

(2) As discussed in the previous section, closed non-degenerate b-two-forms ω are the same as log-symplectic structure $\pi (= \omega^{-1})$ with singular locus Z.

(3) Let μ be a b-top-form on (M^m, Z). Then $w := \mu^{-1} \in \Gamma(\wedge^m TM)$ is a multi-vector field of top degree on M, which intersects the zero-section of $\wedge^m TM$ transversally at Z. These structures are called generic *Nambu structures* of top degree and were studied in [MT04]. In Corollary 7.1.5 below, we show that the b-geometric Moser argument implies the main result from *loc.cit.*

We give now the b-geometric Moser lemma for non-degenerate b-forms. The proof is the same as in the case for symplectic b-two-forms, which appeared first in [NT96].

Lemma 7.1.4 ((see [NT96])). *Let $\zeta \in \Omega^k_Z(M)$ be a closed non-degenerate b-k-form on a compact b-manifold (M, Z), where $k \in \{1, 2, \dim(M)\}$. If $\zeta' \in \Omega^k_Z(M)$ is a closed b-k-form, such that $(1-t)\zeta + t\zeta'$ is non-degenerate for all $t \in [0, 1]$, and*

$$[\zeta] = [\zeta'] \in H^k_Z(M),$$

then there exists a b-diffeomorphism $\varphi : (M, Z) \xrightarrow{\sim} (M, Z)$, such that $\varphi^*(\zeta') = \zeta$.

As a consequence of this result, we obtain the following:

Corollary 7.1.5. *Let $\mu, \mu' \in \Omega^{top}_Z(M)$ be nowhere vanishing b-top-forms on a compact b-manifold (M, Z). If $[\mu] = [\mu'] \in H^{top}_Z(M)$, then there exists a b-diffeomorphism φ of (M, Z) such that $\varphi^*(\mu') = \mu$.*

Proof. Since $\wedge^{\mathrm{top}} T^*_Z M$ is a rank-one bundle, we can write $\mu' = f\mu$, for some nowhere vanishing function $f \in C^\infty(M)$. Let us show that $f > 0$. If $Z = \emptyset$, it follows that M is orientable, therefore the claim follows by integrating μ and μ' on M with respect to the same orientation.

If $Z \neq \emptyset$, we have that $\iota_{\xi_Z}\mu$ and $\iota_{\xi_Z}\mu'$ are volume forms on Z in the same cohomology class, thus we can apply the same argument as before to conclude that $f|_Z > 0$. Hence, $f > 0$ everywhere. This implies that $(1-t)\mu + t\mu' = ((1-t)+tf)\mu$ is nowhere vanishing for all $t \in [0,1]$ and the result follows from the Moser Lemma. □

Using the decomposition from Corollary 7.1.2

$$H_Z^{\mathrm{top}}(M) \simeq H^{\mathrm{top}}(M) \oplus H^{\mathrm{top}}(Z),$$

we see that Corollary 1 extends the classification of generic Nambu structures of top degree from [MT04] to the case of non-orientable manifolds M. In the orientable case, fixing orientations on M and on the components of Z, we have that a generic Nambu structure w of top degree with singular locus Z is determined, up to orientation preserving diffeomorphisms, by the regularized volume of $\mu := w^{-1}$ and by the volumes of the components Z_i of Z computed with the aid of $\iota_{\xi_Z}\omega|_{Z_i}$. In the non-oriented case, it is determined entirely by the volumes of the components.

7.2 Poisson Cohomology of Log-symplectic Manifolds

The cohomology of log-symplectic manifolds can be computed explicitly with the help of b-cohomology.

Proposition 7.2.1. *Let (M,π) be a log-symplectic manifold and Z its singular locus. The Poisson cohomology of (M,π) is isomorphic to the b-cohomology of (M,Z):*

$$H_\pi^\bullet(M) \simeq H_Z^\bullet(M).$$

The proof uses the space $\mathfrak{X}_Z^\bullet(M)$ of multi-vector fields on M tangent to Z, which can also be identified with the space of b-multi-vector fields on (M,Z), i.e. $\mathfrak{X}_Z^\bullet(M) = \Gamma(\wedge^\bullet T_Z M)$. Note that, since Z is a Poisson submanifold, $(\mathfrak{X}_Z^\bullet(M), d_\pi)$ is a subcomplex of $(\mathfrak{X}^\bullet(M), d_\pi)$. The resulting cohomology, denoted by $H_{\pi,Z}^\bullet(M)$, is called the "b-Poisson-cohomology" in [GMP14]. In *loc.cit.* it is shown that the log-symplectic structure π gives an isomorphism of complexes

$$\wedge^\bullet \pi^\sharp : (\Omega_Z^\bullet(M), d) \xrightarrow{\sim} (\mathfrak{X}_Z^\bullet(M), d_\pi),$$

and therefore $H_Z^\bullet(M) \simeq H_{\pi,Z}^\bullet(M)$. Hence the following lemma implies Proposition 7.2.1.

Lemma 7.2.2. *The inclusion $\mathfrak{X}_Z^\bullet(M) \subset \mathfrak{X}^\bullet(M)$ induces an isomorphism in cohomology; i.e. the Poisson cohomology is isomorphic to the b-Poisson cohomology:*

$$H_\pi^\bullet(M) \simeq H_{\pi,Z}^\bullet(M).$$

Proof. We will construct linear maps $h : \mathfrak{X}^\bullet(M) \to \mathfrak{X}^{\bullet-1}(M)$ that satisfy

$$\zeta(w) := w + d_\pi \circ h(w) + h \circ d_\pi(w) \in \mathfrak{X}_Z^\bullet(M), \text{ for all } w \in \mathfrak{X}^\bullet(M). \tag{7.7}$$

This implies the conclusion, since ζ induces a map in cohomology $\zeta : H_\pi^\bullet(M) \to H_{\pi,Z}^\bullet(M)$, which is the inverse of the map induced by the inclusion.

Let $E \to M$ be a tubular neighbourhood of Z in M, with projection $p : E \to Z$. Let χ be a smooth function supported in E, such that $\chi = 1$ in a neighbourhood of Z. Denote $\omega := \pi^{-1} \in \Omega^2_Z(M)$ and consider the canonical one-form $\theta \in \Omega^1(Z)$ defining the foliation on Z. Now, define the operators h by

$$h : \mathfrak{X}^\bullet(M) \longrightarrow \mathfrak{X}^{\bullet-1}(M), \quad h(w) := \iota_{p^*(\theta)}\left(\chi w\right).$$

It suffices to check (7.7) locally around Z, where we have that $\chi = 1$. First, note that, since $p^*(\theta)$ is closed, a Poisson version of the Cartan formula holds:

$$\iota_{p^*(\theta)} \circ d_\pi + d_\pi \circ \iota_{p^*(\theta)} = L_{\pi^\sharp(p^*(\theta))}.$$

Thus, for (7.7) it suffices to check that $w + L_{\pi^\sharp(p^*(\theta))}w \in \mathfrak{X}^\bullet_Z(M)$. Since π is tangent to Z, $\xi := \pi^\sharp(p^*(\theta))$ is a b-vector field. Recall that $\theta = \omega^\sharp|_Z(\xi_Z)$, where ξ_Z is the canonical section of $T_Z M|_Z$. Since

$$\omega^\sharp(\xi)|_Z = p^*(\theta)|_Z = \theta,$$

and ω is invertible, we have that $\xi|_Z = \xi_Z$. Now, an easy local computation shows that every b-vector field ξ extending ξ_Z satisfies $w + [\xi, w] \in \mathfrak{X}^\bullet_Z(M)$. □

Analyzing the proof of Lemma 7.2.2, we see that the quotient complex $A^\bullet := \mathfrak{X}^\bullet(M)/\mathfrak{X}^\bullet_Z(M)$ is acyclic. In fact, this complex is isomorphic to the complex computing the Poisson cohomology of the Poisson manifold (Z, π_Z) with coefficients in the normal bundle ν_Z of Z, endowed with the canonical representation. Thus, we conclude that $H^\bullet_{\pi_Z}(Z; \nu_Z) = 0$. This is surprising, since the cohomology with trivial coefficients $H^\bullet_{\pi_Z}(Z)$ never vanishes, and moreover, this cohomology is infinite dimensional when the cosymplectic structure is proper (see section 3.2).

Since Z is a Poisson submanifold, π induces a Lie algebroid structure T^*Z over Z. Moreover, we have the short exact sequence

$$0 \longrightarrow \nu^*_Z \longrightarrow T^*M|_Z \longrightarrow T^*Z \longrightarrow 0,$$

where ν^*_Z is the conormal bundle of Z. Choosing a splitting of the sequence $\sigma : T^*Z \longrightarrow T^*M|_Z$, we get a representation of T^*Z on ν^*_Z given by

$$\nabla'_a t^* = [\sigma(a), t^*] = \mathcal{L}_{\pi^\sharp(\sigma(\alpha))} t^*,$$

with $t^* \in \Gamma(\nu^*_Z)$, $a \in \Gamma(T^*Z)$ and the bracket is the bracket induced by π. From now on we regard $T^*Z \subset T^*M|_Z$ via the splitting σ and we write $\sigma(a) = a$ for any $a \in \Gamma(T^*Z)$.
The dual connection induces a representation of T^*Z on ν_Z. This dual connection is given by

$$(\nabla_a X)(t^*) = \mathcal{L}_{\pi^\sharp(a)}(t^*(X)) - X(\mathcal{L}_{\pi^\sharp(a)} t^*) = (\mathcal{L}_{\pi^\sharp(a)} X)(t^*),$$

$a \in \Gamma(T^*Z)$, $X \in \Gamma(\nu_Z)$, $t^* \in \Gamma(\nu^*_Z)$, which means that $\nabla_a X = \mathcal{L}_{\pi^\sharp(a)} X \mod TZ$.
This connection induces a differential d_∇ in the complex $\Omega^\bullet(T^*Z, \nu_Z)$. The cohomology hereby computed is the Lie algebroid T^*Z with coefficients in the normal bundle with the representation ∇. This cohomology is denoted by $H^\bullet(T^*Z, \nu_Z)$ and we will prove that it is isomorphic to the cohomology of the complex $A^\bullet(M)$. Combined with the previous lemma, we see that the Lie algebroid cohomology with coefficients in the normal with the representation given by ∇ vanishes, whereas the Lie algebroid cohomology of T^*Z with coefficients in the normal using the trivial representation is, in general, infinite dimensional (see Lemma 3.2.17).

Lemma 7.2.3. *The complex $A^\bullet(M)$ is isomorphic to the complex $\Omega^{\bullet-1}(T^*Z,\nu_Z)$ computing the Lie algebroid cohomology.*

Proof. Consider the map $l:\mathfrak{X}^k(M)\longrightarrow\Gamma(\wedge^kTM|_Z)/\Gamma(\wedge^kTZ)$ given by the composition of the restriction map to the submanifold Z and the quotient map. l is surjective and its kernel is precisely $\mathfrak{X}_Z(M)$ and therefore, we get a chain map isomorphism between $A^k(M)$ and $\Gamma(\wedge^kTM|_Z)/\Gamma(\wedge^kTZ)$. Moreover, note that the splitting $\sigma:T^*Z\longrightarrow T^*M|_Z$ induces isomorphisms $T^*M|_Z\simeq T^*Z\oplus\nu_Z^*$ and $TM|_Z\simeq TZ\oplus\nu_Z$ and therefore an isomorphism

$$\Gamma(\wedge^kTM|_Z)/\Gamma(\wedge^kTZ)\simeq\Gamma(\wedge^{k-1}TZ\otimes\nu_Z),$$

where we now regard $\nu_Z\subset TM|_Z$ and $T^*Z\subset T^*M|_Z$.
All in all, we get an isomorphism $\varphi:A^k(M)\xrightarrow{\sim}\Gamma(\wedge^{k-1}TZ\otimes\nu_Z)$. We have to prove that φ commutes with the differentials that we have in both complexes: in $A^\bullet(M)$ we have the differential d_A coming from d_π and in $\Gamma(\wedge^\bullet TZ\otimes\nu_Z)=\Omega^\bullet(T^*Z,\nu_Z)$ we have the differential d_B induced by ∇.
Both complexes are generated by their 0 and 1 degree elements so we just have to check that $\varphi\circ d_A=d_B\circ\varphi$ for elements in $A^0(M)$ and $A^1(M)$.
Let $X\in\Gamma(\nu_Z)$, $a\in\Gamma(T^*Z)$, $\tilde{X}$ be an extension of X to a vector field on M. Then,

$$d_B(X)(a)=\nabla_aX=\mathcal{L}_{\pi^\sharp(a)}X\mod TZ.$$

On the other hand,

$$\begin{aligned}(\varphi\circ d_A\circ\varphi^{-1}(X))(a)&=\iota_a[\pi,\tilde{X}]\mod TZ=-(\mathcal{L}_{\tilde{X}}\pi)^\sharp(a)\mod TZ\\&=-\mathcal{L}_{\tilde{X}}(\pi^\sharp(a))-\pi^\sharp(\mathcal{L}_{\tilde{X}}a)\mod TZ=-\mathcal{L}_{\tilde{X}}(\pi^\sharp(a))\mod TZ\\&=\mathcal{L}_{\pi^\sharp(a)}\tilde{X}\mod TZ=\mathcal{L}_{\pi^\sharp(a)}X\mod TZ\end{aligned}$$

Now, consider $Z\in\Gamma(TZ),X\in\Gamma(\nu_Z)$, $a_1,a_2\in\Gamma(T^*Z)$

$$\begin{aligned}d_B(Z\otimes X)(a_1,a_2)&=\nabla_{a_1}(a_2(Z)X)-\nabla_{a_2}(a_1(Z)X)-Z([a_1,a_2])X\mod TZ\\&=(\mathcal{L}_{\pi^\sharp(a_1)}(a_2(Z))-\mathcal{L}_{\pi^\sharp(a_2)}(a_1(Z)))X+a_2(Z)\mathcal{L}_{\pi^\sharp(a_1)}X-a_1(Z)\mathcal{L}_{\pi^\sharp(a_2)}X\\&\qquad-(\mathcal{L}_{\pi^\sharp(a_1)}a_2(Z)-\mathcal{L}_{\pi^\sharp(a_2)}a_1(Z)-Z(\pi(a_1,a_2)))X\mod TZ\\&=a_2(\mathcal{L}_{\pi^\sharp(a_1)}Z)X-a_1(\mathcal{L}_{\pi^\sharp(a_2)}Z)X+a_2(Z)\mathcal{L}_{\pi^\sharp(a_1)}X-a_1(Z)\mathcal{L}_{\pi^\sharp(a_2)}X\\&\qquad+Z(\pi(a_1,a_2))X\mod TZ\end{aligned}$$

On the other hand

$$\begin{aligned}(\varphi\circ d_A\circ\varphi^{-1})(Z\otimes X))(a_1,a_2)&=-\iota_{a_1\wedge a_2}(d_\pi Z\wedge X-Z\wedge d_\pi X)\mod TZ\\&=-(\iota_{a_1\wedge a_2}d_\pi Z)X+a_2(Z)\iota_{a_1}d_\pi X-a_1(Z)\iota_{a_2}d_\pi X\mod TZ\\&=-(\mathcal{L}_Z\pi)(a_1,a_2)X+a_2(Z)\mathcal{L}_{\pi^\sharp(a_1)}X-a_1(Z)\mathcal{L}_{\pi^\sharp(a_2)}X\mod TZ\\&=-Z(\pi(a_1,a_2))X+\pi(\mathcal{L}_Za_1,a_2)+\pi(a_1,\mathcal{L}_Za_2)\\&\qquad+a_2(Z)\mathcal{L}_{\pi^\sharp(a_1)}X-a_1(Z)\mathcal{L}_{\pi^\sharp(a_2)}X\mod TZ\\&=-Z(\pi(a_1,a_2))X-(\mathcal{L}_Za_1)(\pi^\sharp(a_2))X+(\mathcal{L}_Za_2)(\pi^\sharp(a_1))X\\&\qquad+a_2(Z)\mathcal{L}_{\pi^\sharp(a_1)}X-a_1(Z)\mathcal{L}_{\pi^\sharp(a_2)}X\mod TZ\\&=Z(\pi(a_1,a_2))X-a_1(\mathcal{L}_{\pi^\sharp(a_2)}Z)+a_2(\mathcal{L}_{\pi^\sharp(a_1)}Z)\\&\qquad+a_2(Z)\mathcal{L}_{\pi^\sharp(a_1)}X-a_1(Z)\mathcal{L}_{\pi^\sharp(a_2)}X\mod TZ\end{aligned}$$

This proves that d_B and $\varphi \circ d_A \circ \varphi^{-1}$ are equal on the elements of degree 0 and 1 and since both are derivations, they are the same in all $\Omega^\bullet(T^*Z, \nu_Z)$. □

7.3 Deformations of Log-symplectic Structures

In this section we study the deformations of log-symplectic structures in the space of Poisson structures.

There are only two ways to deform a log-symplectic structure π on a compact manifold M. These are described as follows:

i. The first type of deformation is the *gauge transformation* by a (small enough) closed two-form $\varpi \in \Omega^2(M)$. This is a general operation in Poisson geometry, which transforms π by adding the restriction of ϖ to the symplectic form on each leaf of π.

ii. The second type of deformation is specific to log-symplectic structures and it transforms π locally around Z. For a (small enough) closed one-form $\gamma \in \Omega^1(Z)$, the transformed Poisson structure is also log-symplectic with singular locus Z, but with foliation on Z given by the kernel of $\theta + \gamma$.

These transformations can be described algebraically in terms of their inverses on $M \backslash Z$; the result of transforming π by the pair $(\varpi, \gamma) \in \Omega^2(M) \times \Omega^1(Z)$ is given by

$$\left(\pi_\gamma^\varpi\right)^{-1} := \pi^{-1} + \varpi + d\log(\lambda) \wedge p^*(\gamma). \tag{7.8}$$

The cosymplectic structure on Z induced by π_γ^ϖ is the pair $(\eta + \varpi|_Z, \theta + \gamma)$.

In fact, these two types of deformations cover all Poisson structures near a log-symplectic structure.

Theorem 7.3.1. *Let (M, π, Z) be a compact log-symplectic manifold. Consider $\varpi_1, \dots, \varpi_l$ closed two-forms on M and $\gamma_1, \dots, \gamma_k$ closed one-forms on Z such that their cohomology classes form a basis of $H^2(M)$ and of $H^1(Z)$ respectively. For $\epsilon \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^k$, denote by $\varpi_\epsilon := \sum_{i=1}^l \epsilon_i \varpi_i$ and $\gamma_\delta := \sum_{i=1}^k \delta_i \gamma_i$. Then,*

(a) for all small enough $\epsilon \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^k$, we have that $\pi_{\gamma_\delta}^{\varpi_\epsilon}$ (given by (7.8)) is a log-symplectic structure;

(b) there is a C^1-open neighbourhood $\mathcal{U} \subset \Gamma(\wedge^2 TM)$ around π, such that every Poisson structure $\pi' \in \mathcal{U}$ is isomorphic to $\pi_{\gamma_\delta}^{\varpi_\epsilon}$ for some vectors $\epsilon \in \mathbb{R}^l$, $\delta \in \mathbb{R}^k$;

(c) there is a C^1-neighbourhood $\mathcal{D} \subset \mathrm{Diff}(M)$ around id_M such that for $\varphi \in \mathcal{D}$, the equality $\varphi_*(\pi_{\gamma_\delta}^{\varpi_\epsilon}) = \pi_{\gamma_{\delta'}}^{\varpi_{\epsilon'}}$ implies $\epsilon = \epsilon'$ and $\delta = \delta'$.

Theorem 7.3.1 states that the cohomology classes $([\alpha], [\theta])$ from (7.6) give a parametrization of the space of Poisson structures near π up to small diffeomorphisms. In terms of b-cohomology, this says that Poisson structures near a log-symplectic manifold (M, π, Z) are parametrised by the second b-cohomology group $H_Z^2(M)$. This is in perfect analogy with the situation in symplectic geometry, where deformations of symplectic forms are classified by the second de

Rham cohomology group. More specifically, we can reformulate the main result in the b-language. If M is compact and $\mu \in \Omega^2_Z(M)$ is a C^0-small closed b-two-form, then $\omega + \mu$ is still non-degenerate, therefore $\pi^\mu := (\omega+\mu)^{-1}$ is a small deformation of π. This remark implies part (a) of Theorem 7.3.1. Part (b) says that every Poisson structure close to π is isomorphic to one of this form, and moreover, whenever $[\mu] = [\mu'] \in H^2_Z(M)$, we have that π^μ and $\pi^{\mu'}$ are isomorphic. Conversely, part (c) says that there is an open neighbourhood $\mathcal{D}$ around id_M in the space of diffeomorphisms of M, such that if $[\mu] \neq [\mu']$, then π^μ and $\pi^{\mu'}$ are not related by an element in $\mathcal{D}$. In other words, Poisson structures around π are parametrised up to small diffeomorphisms by an open neighbourhood in $H^2_Z(M)$ around $[\omega]$; or equivalently, by an open neighbourhood in $H^2(M) \oplus H^1(Z)$ around the pair $([\alpha],[\theta])$. Moreover, by Corollary 7.1.2, the pair $([\alpha],[\theta])$ is canonically associated to π.

As mentioned in chapter 1, the second Poisson cohomology group $H^2_\pi(M)$ has the heuristic interpretation of being the "tangent space" at π of the moduli-space of all Poisson structures on M. From the theorem and the previous section, we show that for log-symplectic structures this interpretation is in fact accurate. By Theorem 7.3.1, deformations of a compact log-symplectic manifold (M,π) with singular locus Z are parametrised by $H^2_Z(M) \simeq H^2_\pi(M)$.

For M a compact oriented surface, Theorem 7.3.1 follows from the classification in [Rad02]: writing Z as a union of circles $Z_1,\dots,Z_k$ and decomposing $[\theta] = \sum_i [\theta_i]$, with $[\theta_i] \in H^1(Z_i)$, the classes $[\theta_i]$ correspond to the periods of the modular vector field along the curves Z_i and $[\alpha]$ corresponds to the *generalized Liouville volume* of (M,π).

Recall that a log-symplectic structure (M,π,Z) is called *proper* if the cosymplectic induced on Z is proper, or equivalently, if the foliation on Z is given by a submersion to S^1. Now this is equivalent to the cohomology class $[\theta]$ being a real multiple of an integral class (see [Tis70]). Since such one-forms are dense in the space of all closed one-forms, Theorem 7.3.1 implies

Corollary 7.3.2. *The proper log-symplectic structures form a dense set in the space of all log-symplectic structures.*

This result also appeared also in [Cav13].

Stability of the singular locus

In this section we begin the proof of Theorem 7.3.1. We prove that a Poisson structure C^1-close to a log-symplectic structure π is also log symplectic and moreover, its singular locus is diffeomorphic to that of π.

This is related to the problem of stability of Poisson submanifolds, which studies persistence of Poisson submanifolds under deformations of the Poisson bivector, and was treated in the case of symplectic leaves in [CF10]. Heuristically, the infinitesimal condition for stability of the Poisson submanifold N of (M,π) is that the quotient complex $\mathfrak{X}^\bullet(M)/\mathfrak{X}^\bullet_N(M)$ is acyclic in degree two, where $\mathfrak{X}^\bullet_N(M)$ denotes as before the space of multi-vector fields tangent to N. In the case of N being a compact symplectic leaf, it is proved in [CF10] that the condition

$$H^2\left(\mathfrak{X}^\bullet(M)/\mathfrak{X}^\bullet_N(M)\right) = 0$$

ensures the stability of N and moreover that the space of Poisson submanifolds nearby N is parametrised by $H^1\left(\mathfrak{X}^\bullet(M)/\mathfrak{X}^\bullet_N(M)\right)$.

For a log-symplectic structure (M, π), we saw in the previous section that the complex $\mathfrak{X}^\bullet(M)/\mathfrak{X}^\bullet_Z(M)$ is acyclic in all degrees. This observation is the infinitesimal counterpart of the next Lemma.

Remark on the topologies.

We use the C^1-topologies on $\mathrm{Diff}(M)$ and on $\Gamma(\wedge^2 TM)$, and the C^0-topology on $\Gamma(\wedge^2 T_Z M)$. Similarly, we endow the space $\mathrm{Poiss}(M)$, of all Poisson structures on M, with the induced C^1-topology, and the space $\mathrm{Log}(M, Z) \subset \Gamma(\wedge^2 T_Z M)$, of all log-symplectic structures on M with singular locus Z, with the induced C^0-topology.

The reader should be warned that the C^0-topology on $\Gamma(\wedge^2 T_Z M)$ is not the subspace C^0-topology induced from $\Gamma(\wedge^2 TM)$. To see this, let U be an open neighbourhood around a point in Z with coordinates $(t, x) \in \mathbb{R} \times \mathbb{R}^{2n-1}$ such that $Z \cap U = \{t = 0\}$. A bivector of the form $f(t,x)\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x_i}$ is in $\Gamma(\wedge^2 T_Z M)$ if and only if $f(t, x) = tg(t, x)$ for some smooth function $g(t, x)$. Now the C^0-norm on $\Gamma(\wedge^2 TM)$ computes the supremum of $f(t, x)$, whereas the C^0-norm on $\Gamma(\wedge^2 T_Z M)$ computes the supremum of $g(t, x)$. This makes the following lemma more subtle than one would expect.

Lemma 7.3.3. *Given a compact log-symplectic manifold (M, π) with singular locus Z, any C^1-close Poisson structure is log symplectic and has singular locus diffeomorphic to Z. More precisely, there is a C^1-open $\mathcal{V} \subset \mathrm{Poiss}(M)$ around π and there is a map $\Phi : \mathcal{V} \to \mathrm{Diff}(M)$ satisfying $\Phi(\pi) = \mathrm{id}_M$ and $\Phi(\pi')_*(\pi') \in \mathrm{Log}(M, Z)$ for every $\pi' \in \mathcal{V}$. Moreover, the map*

$$\Psi : \mathcal{V} \longrightarrow \mathrm{Log}(M, Z), \qquad \Psi(\pi') := \Phi(\pi')_*(\pi')$$

is continuous for the C^1-topology on $\mathcal{V}$ and the C^0-topology on $\mathrm{Log}(M, Z)$.

Proof. First we assume that M is orientable. We prove this case in two steps: first we construct Φ and $\mathcal{V}$ and second we prove that Φ has the desired continuity property.

Step 1. Construction of Φ and $\mathcal{V}$.

Since M is orientable, the normal bundle to Z is trivial, and we can find a tubular neighbourhood $E \to M$, $E \simeq \mathbb{R} \times Z$. Denote by t the coordinate on $\mathbb{R}$. Now, $\mu := \frac{1}{t} \wedge^n \pi|_E$ is a nowhere vanishing section of $\wedge^{2n} TE$, and for every smooth bivector w, we have that $\wedge^n w|_E = h_w(t, x)\mu$, for some smooth function h_w on E. Moreover, the assignment $w \mapsto h_w|_{[-2,2]\times Z}$ is continuous with respect to the C^1-topologies, and $h_\pi = t$.

Consider the C^1-open neighbourhood $\mathcal{D} \subset C^\infty([-2, 2] \times Z)$ around the t consisting of functions h such that $\partial h(t, x)/\partial t > 0$ and $|h(t, x) - t| < 1$. Then, for any $h \in \mathcal{D}$, we have that the function

$$\varphi_h : [-2, 2] \times Z \longrightarrow \mathbb{R} \times Z, \quad (t, x) \mapsto (h(t, x), x),$$

is a diffeomorphism onto its image and that $\varphi_h\left((-2, 2) \times Z\right) \supset [-1, 1] \times Z$. Moreover, the assignment $\varphi_h \mapsto \varphi_h^{-1}|_{[-1,1]\times Z}$ is continuous with respect to the C^1-topologies.

Let $\mathcal{V}'$ be the C^1-open neighbourhood in $\Gamma(\wedge^2 TM)$ consisting of elements w that satisfy

$$\left(\wedge^n w\right)^{-1}(0) \subset (-2, 2) \times Z \quad \text{and} \quad h_w|_{[-2,2]\times Z} \in \mathcal{D}.$$

Clearly, $\pi \in \mathcal{V}'$. Write $\varphi_{h_w}^{-1}(t,x) = (g_w(t,x), x)$ and $g_w(x) := g_w(0,x)$. We have that

$$(\wedge^n w)^{-1}(0) = Z_w := \{(g_w(x), x) : x \in Z\},$$

and since $\partial h_w/\partial t \neq 0$ on $[-2,2] \times Z$, $\wedge^n w$ is transverse to the zero-section. Thus, if $\pi' \in \mathcal{V}'$ is Poisson, then π' is log symplectic with singular locus $Z_{\pi'}$.

We fix a smooth compactly supported function $\chi : \mathbb{R} \to [0,1]$, such that $\chi(t) = 1$ for $t \in [-2,2]$ and $|\chi'(t)| < 1/2$ for $t \in \mathbb{R}$. Consider the following diffeomorphism:

$$\Phi(w) : M \xrightarrow{\sim} M,\ \Phi(w)|_{M\backslash E} = \mathrm{id}_{M\backslash E},\ \Phi(w)|_E(t,x) := (t - \chi(t)g_w(x), x).$$

The conditions $|\chi'(t)| < 1/2$ and $|g_w(x)| \leq 2$ imply that $\mathrm{pr}_1 \circ \partial\Phi(w)/\partial t > 0$; and hence $\Phi(w)$ is indeed a diffeomorphism. The fact that $\chi(t) = 1$ on $[-2,2]$ implies that $\Phi(w)$ maps Z_w onto Z. Let $\mathcal{V} := \mathcal{V}' \cap \mathrm{Poiss}(M)$.

Step 2. Continuity of Ψ.

Clearly, the assignment $w \mapsto \Phi(w)$ is continuous for the C^1-topologies, and therefore the assignment $w \mapsto \Phi(w)_*(w)$ is continuous for the C^1-topology on $\mathcal{V}'$ and the C^0-topology on $\Gamma(\wedge^2 TM)$. For a Poisson structure $\pi' \in \mathcal{V}'$, we have that $\Phi(\pi')_*(\pi')$ belongs to $\mathrm{Log}(M,Z)$. Now, here comes the more subtle point of the proof: the fact that the C^0-topology on $\mathrm{Log}(M,Z)$ is not the subspace topology induced from $\Gamma(\wedge^2 TM)$ does not allow us to conclude yet the proof. Also, for an arbitrary $w \in \mathcal{V}'$, $\Phi(w)_*(w)$ is not an element of $\Gamma(\wedge^2 T_Z M)$; therefore we have to restrict Φ to $\mathcal{V} = \mathcal{V}' \cap \mathrm{Poiss}(M)$.

Consider a finite open cover of Z by coordinate charts $\{U_l \simeq \mathbb{R}^{2n-1}\}_l$ such that the closed balls $\{\overline{B}_l\}_l$ of radius 1 still cover Z. In one of these charts U_l with coordinates (x_i), an element $\sigma \in \Gamma(\wedge^2 T_Z M)$ can be written as

$$\sigma = \sum_i a_i^l(t,x) t \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x_i} + \sum_{i,j} b_{ij}^l(t,x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The C^0-norm of $\sigma|_{[-2,2]\times Z}$ (as an element in $\Gamma(\wedge^2 T_Z M)$) is the supremum of the functions a_i^l and b_{ij}^l on $[-2,2] \times \overline{B}_l$. Thus to compute the C^0-norm of σ on M, one needs to measure its C^0-norm as an element in $\Gamma(\wedge^2 TM)$ and the supremum norm of the coefficients a_i^l in a small neighbourhood of Z. In our case, we know that the map $\mathcal{V} \ni \pi' \mapsto \Phi(\pi')_*(\pi') \in \Gamma(\wedge^2 TM)$ is continuous for the C^0-topology on the second space, thus it suffices to check that the corresponding functions "a_i^l" also vary continuously.

Consider $\pi' \in \mathcal{V}$, and denote $g := g_{\pi'}$. Denote the local expression of π' on U_l by

$$\pi' := \sum_i A_i^l(t,x) \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j} B_{ij}^l(t,x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}.$$

The fact that π' is tangent to Z_g is written in coordinates as follows:

$$\sum_i A_i^l(g(x),x)\left(\frac{\partial}{\partial x_i} + \frac{\partial g}{\partial x_i}(x)\frac{\partial}{\partial t}\right) + \sum_{i,j} B_{ij}^l(g(x),x)\frac{\partial g}{\partial x_i}(x)\frac{\partial}{\partial x_j} = 0. \tag{7.9}$$

Now, $\Phi(\pi')|_{[-2,2]\times Z}$ is of the form $(t,x)\mapsto(t-g(x),x)$, and therefore

$$\Phi(\pi')_*(\pi')|_{[-1,1]\times Z}=\frac{1}{2}\sum_{i,j}B^l_{ij}(t+g(x),x)\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}+ \\ +\sum_i\Big(A^l_i(t+g(x),x)-\sum_j B^l_{ij}(t+g(x),x)\frac{\partial g}{\partial x_j}(x)\Big)\frac{\partial}{\partial t}\wedge\frac{\partial}{\partial x_i}.$$

Equation (7.9) implies that the coefficient of $\frac{\partial}{\partial t}\wedge\frac{\partial}{\partial x_i}$ vanishes at $t=0$ and therefore we get that the coefficient of $t\frac{\partial}{\partial t}\wedge\frac{\partial}{\partial x_i}$ is equal to

$$a^l_i(t,x):=\frac{1}{t}\Big(A^l_i(t+g(x),x)-\sum_j B^l_{ij}(t+g(x),x)\frac{\partial g}{\partial x_j}(x)\Big)= \\ =\int_0^1\Big(\frac{\partial A^l_i}{\partial t}(st+g(x),x)-\sum_j\frac{\partial B^l_{ij}}{\partial t}(st+g(x),x)\frac{\partial g}{\partial x_j}(x)\Big)ds.$$

We see now that this explicit formula, which holds on $[-1,1]\times Z$, implies that these coefficients depend continuously on π', i.e., the map $\pi'\mapsto(a^l_i)$ is continuous with respect to the C^1-topology on $\mathcal{V}$ and the C^0-topology on $C^\infty([-1,1]\times Z)$. Indeed, let π'' be a Poisson structure C^1-close to π'. Denote the coefficients of π'' on U_l by $\tilde{A}^l_i$, $\tilde{B}^l_{ij}$. Similarly, for π'' we get a function $\tilde{g}$ taking values in $[-2,2]$ which, by the previous argument, is C^1-close to g. We have then

$$\tilde{A}^l_i=A^l_i+\alpha^l_i,\quad \tilde{B}^l_{ij}=B^l_{ij}+\beta^l_{ij},\quad \tilde{g}=g+\delta,$$

for some functions α^l_i, β^l_{ij}, δ which are C^1-small. We need to compute the C^0-norm of $\tilde{a}^l_i-a^l_i$ on $[-1,1]\times\overline{B}_l$, for which we use the formula

$$(\tilde{a}^l_i-a^l_i)(t,x)=\int_0^1\Big(\int_0^1\frac{\partial^2A^l_i}{\partial t^2}\Big(\delta(x)s'+st+g(x),x\Big)\delta(x)ds'+\frac{\partial\alpha^l_i}{\partial t}(st+\tilde{g}(x),x)\Big)ds- \\ -\sum_j\Big(\frac{\partial g}{\partial x_j}+\frac{\partial\delta}{\partial x_j}\Big)\int_0^1\Big(\int_0^1\frac{\partial^2B^l_{ij}}{\partial t^2}\big(\delta(x)s'+st+g(x),x\big)\delta(x)ds'+\frac{\partial\beta^l_{ij}}{\partial t}(st+\tilde{g}(x),x)\Big)ds$$

Note that the C^1-norms of α^l_i and β^l_{ij} on $[-3,3]\times\overline{B}_l$ are bounded by a multiple of $||\pi''-\pi'||_{C^1}$. Now, since $st+\tilde{g}(x)\in[-3,3]$ for $s,t\in[0,1]$ and $x\in Z$, there is a constant $C>0$ such that

$$||\tilde{a}^l_i-a^l_i||_{C^0}\leq C\big(||\delta||_{C^0}+||\pi''-\pi'||_{C^1}\big).$$

This implies the desired continuity and finishes the proof in the case when M is orientable.

Step 3. The non-orientable case.

If M is not orientable, consider the orientable double cover $p:\widetilde{M}\to M$, and let $\gamma:\widetilde{M}\xrightarrow{\sim}\widetilde{M}$ be the corresponding deck transformation. Then $\widetilde{\pi}:=p^*(\pi)$ is a γ-invariant log-symplectic structure on $\widetilde{M}$ with singular locus $\widetilde{Z}:=p^{-1}(Z)$. Consider a γ-invariant tubular neighbourhood of $\widetilde{E}\to\widetilde{M}$, which admits a trivialization $\widetilde{E}\xrightarrow{\sim}\mathbb{R}\times\widetilde{Z}$ on which γ acts by $\gamma(t,x):=(-t,\gamma(x))$ (for more details on this construction, see [MOT14b]). Let $\widetilde{\mathcal{V}}$ be the C^1-neighbourhood of $\widetilde{\pi}$

constructed in the first part, and let $\widetilde{\Phi} : \widetilde{\mathcal{V}} \to \mathrm{Diff}(\widetilde{M})$ be the corresponding map. Let $\mathcal{V}$ be the set of Poisson structures π' such that $p^*(\pi') \in \widetilde{\mathcal{V}}$. Clearly, $\mathcal{V}$ is a C^1-neighbourhood of π. Note that the construction of $\widetilde{\Phi}$ is such that $\widetilde{\Phi}(p^*(\pi'))$ is γ-equivariant: we have that $\gamma^*(\mu) = -\mu$, thus $h_{p^*(\pi')} \circ \gamma(t,x) = -h_{p^*(\pi')}(t,x)$, thus also $g_{p^*(\pi')} \circ \gamma(x) = -g_{p^*(\pi')}(x)$; and therefore, by choosing $\chi(t)$ to be an even function, we obtain that $\widetilde{\Phi}(p^*(\pi'))$ is γ-equivariant. This implies that $\widetilde{\Phi}(p^*(\pi'))$ covers a diffeomorphism which we denote by $\Phi(\pi') \in \mathrm{Diff}(M)$ and which satisfies

$$\widetilde{\Phi}(p^*(\pi'))_*(p^*(\pi')) = p^*\left(\Phi(\pi')_*(\pi')\right).$$

Since $\widetilde{\Phi}(p^*(\pi'))_*(p^*(\pi')) \in \mathrm{Log}(\widetilde{M}, \widetilde{Z})$, it follows that $\Phi(\pi')_*(\pi') \in \mathrm{Log}(M,Z)$. Moreover, since the pull-back maps $p^* : \mathcal{V} \to \widetilde{\mathcal{V}}$ and $p^* : \mathrm{Log}(M,Z) \to \mathrm{Log}(\widetilde{M},\widetilde{Z})$ are embeddings for the C^1-topology and for the C^0-topology, respectively, it follows that the map $\pi' \mapsto \Phi(\pi')_*(\pi')$ is continuous. This concludes the proof. $\square$

Proof of Theorem 7.3.1

In this section we prove part (b) and (c) of Theorem 7.3.1. By the result in the previous section (Lemma 7.3.3), all small deformations can be represented by nearby log-symplectic structures which have the same singular locus. In this setup, the b-version of the Moser argument (Lemma 7.1.4) is used to prove that log-symplectic structures are determined up to diffeomorphism by their b-cohomology. This concludes the proof of part (b) of Theorem 7.3.1. Finally, part (c) is proved by showing that small diffeomorphisms do not change the b-cohomology class.

Fix (M^{2n}, π) a compact log-symplectic manifold with singular locus Z. Let $\omega := \pi^{-1} \in \Omega^2_Z(M)$ denote the inverse of π. Consider, as in the statement of Theorem 7.3.1, $\varpi_1, \ldots, \varpi_l$ closed two-forms on M and $\gamma_1, \ldots, \gamma_k$ closed one-forms on Z such that their cohomology classes form a basis of $H^2(M)$ and $H^1(Z)$ respectively. For $\epsilon \in \mathbb{R}^l$ and $\delta \in \mathbb{R}^k$, denote by $\varpi_\epsilon := \sum_{i=1}^l \epsilon_i \varpi_i$ and $\gamma_\delta := \sum_{i=1}^k \delta_i \gamma_i$. Fix a tubular neighbourhood E of Z, with projection $p : E \to Z$, and let λ be a distance function for E. Denote by

$$\omega_{\epsilon,\delta} := \omega + \varpi_\epsilon + d\log(\lambda) \wedge p^*(\gamma_\delta).$$

The Mazzeo-Melrose decomposition implies that the map $\mathbb{R}^{l+k} \to H^2_Z(M)$, $(\epsilon, \delta) \mapsto [\omega_{\epsilon,\delta}]$ is a linear isomorphism.

Next, we construct a convex neighbourhood consisting of non-degenerate b-forms, which is used in the proof of part (b).

Lemma 7.3.4. *There is a C^0-open neighbourhood $\mathcal{W}$ around ω in the space of all closed b-two-forms $\Omega^2_{Z,closed}(M)$, such that every $\omega' \in \mathcal{W}$ is non-degenerate, and moreover, if $(\epsilon,\delta) \in \mathbb{R}^{l+k}$ is such that $[\omega'] = [\omega_{\epsilon,\delta}]$, then the entire path $(1-t)\omega' + t\omega_{\epsilon,\delta}$ for $t \in [0,1]$ is contained in $\mathcal{W}$.*

Proof. Fix a metric on $T_Z M$, and consider the following map on $\Omega^2_Z(M)$:

$$\omega' \in \Omega^2_Z(M),\ \omega' \mapsto ||\omega'|| := \sup\{|\pi^\sharp \circ \omega'^\sharp(V) - V| : V \in T_Z M, |V| = 1\}.$$

Clearly, $||\omega|| = 0$, thus the following is a C^0-neighbourhood of ω:

$$\widetilde{\mathcal{W}} := \{\omega' \in \Omega^2_Z(M) : d\omega' = 0,\ ||\omega'|| < 1\} \subset \Omega^2_{Z,closed}(M).$$

Note that $\widetilde{\mathcal{W}}$ is a convex neighbourhood consisting entirely of log-symplectic structures on (M, Z). Let $U \subset \mathbb{R}^{l+k}$ be the space consisting of pairs (ϵ, δ) such that $\omega_{\epsilon,\delta} \in \widetilde{\mathcal{W}}$. Clearly, U is a convex open neighbourhood of 0. Next, note that taking the cohomology class of a closed b-form $\omega' \mapsto [\omega']$ is continuous for the C^0-topology on the space of all closed b-forms. To see this, observe that the decomposition form Section 7.1

$$\Omega^{\bullet}_{Z,closed}(M) \xrightarrow{\sim} \Omega^{\bullet}_{closed}(M) \oplus \Omega^{\bullet-1}_{closed}(Z)$$

is C^0-continuous, and that taking de Rham cohomology is also C^0-continuous, because de Rham cohomology can be detected by integrating along compact submanifolds, and integration is C^0-continuous. This shows that the following is C^0-open.

$$\mathcal{W} := \{\omega' \in \widetilde{\mathcal{W}} : [\omega'] = [\omega_{\epsilon,\delta}] \text{ for some } (\epsilon, \delta) \in U\}.$$

Clearly, $\mathcal{W}$ has the required properties. □

Proof of Theorem 7.3.1, part (b). Consider the C^0-neighbourhood $\mathcal{W}$ from the previous lemma. Consider also the C^1-open neighbourhood $\mathcal{V} \subset \mathrm{Poiss}(M)$, the map $\Phi : \mathcal{V} \to \mathrm{Diff}(M)$, and the map $\Psi : \mathcal{V} \to \mathrm{Log}(M, Z)$, $\Psi(\pi') := \Phi(\pi')_*(\pi')$ from Lemma 7.3.3. The inversion map $\mathrm{Log}(M, Z) \to \Omega^2_Z(M)$ is continuous for the C^0-topologies, and by the continuity of Ψ, the following is a C^1-open neighbourhood around π:

$$\mathcal{U} := \{\pi' \in \mathcal{V} : \Psi(\pi')^{-1} \in \mathcal{W}\} \subset \mathrm{Poiss}(M).$$

Let $\pi' \in \mathcal{U}$. Then π' is diffeomorphic to the log-symplectic structure $(\omega')^{-1} := \Psi(\pi') \in \mathrm{Log}(M, Z)$ via the map $\Phi(\pi')$. Let $(\epsilon, \delta) \in \mathbb{R}^{l+k}$ be such that $[\omega'] = [\omega_{\epsilon,\delta}]$. Since $\omega' \in \mathcal{W}$, the Moser argument from Lemma 7.1.4 implies that ω' is diffeomorphic to $\omega_{\epsilon,\delta}$. □

To prove part (c), first we show the existence of path connected neighbourhoods of id_M in the space of diffeomorphisms that fix Z.

Lemma 7.3.5. *Let (M, Z) be a compact b-manifold. Then there exists a C^1-open neighbourhood $\mathcal{D}$ around id_M in $\mathrm{Diff}(M)$, such that, for every $\varphi \in \mathcal{D}$ that leaves Z invariant, there is a smooth isotopy φ_t, for $t \in [0, 1]$, with $\varphi_0 = \mathrm{id}_M$ and $\varphi_1 = \varphi$, consisting of diffeomorphisms that send Z to Z.*

Proof. First, recall the standard construction of a C^1-neighbourhood of id_M in $\mathrm{Diff}(M)$. Let g be a metric on M, with exponential map $\exp : TM \to M$. There is a C^1-open neighbourhood $\mathcal{A} \subset \Gamma(TM)$ around the zero-section, such that for every $V \in \mathcal{A}$, the induced map

$$\exp_*(V) \in C^\infty(M, M), \quad \exp_*(V)(x) := \exp(V_x)$$

is a diffeomorphism. Moreover, $\exp_*$ is a homeomorphism for the C^1-topologies onto a C^1-open neighbourhood $\mathcal{D} \subset \mathrm{Diff}(M)$ around id_M.

Consider a metric for which Z is geodesically closed. We claim that for such a metric, if $\mathcal{A}$ is small enough then the set $\mathcal{D}$ has the required property. Let $\varphi \in \mathcal{D}$ be such that φ sends Z to Z, and write $\varphi = \exp_*(V)$, for some $V \in \mathcal{A}$. We consider the isotopy $\varphi_t := \exp_*(tV)$ for $t \in [0, 1]$. For $x \in Z$, the curve $\varphi_t(x) = \exp(tV_x)$ is a geodesic that starts and ends in Z. For small enough $\mathcal{A}$, we may assume that V_x is shorter than the injectivity radius of exp. Since Z is geodesically closed, this implies that $V_x \in T_xZ$, and so $\varphi_t(x) \in Z$ for all t. □

Proof of Theorem 7.3.1, part (c). Consider the open neighbourhood $\mathcal{D}$ described in the previous lemma. Let $\pi_0, \pi_1 \in \mathrm{Log}(M, Z)$ be two log-symplectic structures such that there is some $\varphi \in \mathcal{D}$ for which $\varphi_*(\pi_0) = \pi_1$. Denote by $\omega_0 := \pi_0^{-1}$ and $\omega_1 := \pi_1^{-1}$. We claim that $[\omega_0] = [\omega_1]$. Since φ sends Z to Z, there is an isotopy φ_t with the properties from the lemma, i.e. φ_t is a b-diffeomorphism for all $t \in [0, 1]$. Denote the generating time-dependent b-vector field by X_t,

$$X_t(x) = \frac{d\varphi_t}{dt}\left(\varphi_t^{-1}(x)\right).$$

We apply now the "reverse Moser trick". Since $\varphi_1^*(\omega_1) = \omega_0$, we have that

$$\omega_0 - \omega_1 = \int_0^1 \frac{d}{dt}\left(\varphi_t^*(\omega_1)\right)dt = \int_0^1 \varphi_t^*(L_{X_t}\omega_1)dt = d\int_0^1 \varphi_t^*(\iota_{X_t}\omega_1)dt.$$

Hence $[\omega_0] = [\omega_1] \in H^2_Z(M)$, and this proves part (c) of Theorem 7.3.1. □

Examples

In this section we present some simple examples to illustrate some phenomena that appear in deforming log-symplectic structures.

Example 7.3.6. Consider the unit sphere S^2 embedded in $\mathbb{R}^3 = \{(x, y, z)\}$. We use the angle $\theta := \tan^{-1}(y/x) \in S^1$ and $z \in (-1, 1)$ as coordinates on S^2 minus the two poles. The bivector $\partial_\theta \wedge \partial_z$ extends to a non-degenerate Poisson structure on S^2. Consider the log-symplectic structure

$$\pi := z\partial_\theta \wedge \partial_z.$$

The singular locus of π is the equator $S^1 := \{z = 0\} \cap S^2$, and the induced one-form on S^1 is $d\theta|_{S^1}$. The corresponding b-two-form is

$$\omega := \frac{dz}{z} \wedge d\theta.$$

The generators of the b-cohomology group $H^2_{S^1}(S^2)$ are $[\omega]$ and $[dz \wedge d\theta]$. Under the canonical decomposition $H^2_{S^1}(S^2) \simeq H^2(S^2) \oplus H^1(S^1)$, $[\omega]$ corresponds to the generator of $H^1(S^1)$ and $[dz \wedge d\theta]$ to the generator of $H^2(S^2)$. Every Poisson structure C^1-near π is isomorphic to one of the form

$$\pi_{\epsilon,\delta} := (\omega + \epsilon dz \wedge d\theta + \delta\omega)^{-1} = \frac{z}{1 + \epsilon z + \delta}\partial_\theta \wedge \partial_z.$$

Of course, this example fits in the classification of log-symplectic structures on compact orientable surfaces from [Rad02].

Example 7.3.7. The log-symplectic structure $\pi = z\partial_\theta \wedge \partial_z$ on S^2 is invariant under the symmetry $\gamma(x, y, z) := (-x, -y, -z)$. Let $p : S^2 \to \mathbb{P}^2 := S^2/\{\mathrm{id}, \gamma\}$ be the projection onto the real projective plane. Invariance of π implies that $p_*(\pi)$ is a log-symplectic structure on $\mathbb{P}^2$ with singular locus $p(S^1) = S^1/\{\mathrm{id}, \gamma\} \simeq S^1$. Since $H^2(\mathbb{P}^2) = 0$, we have that the b-cohomology is concentrated in $H^1(p(S^1)) \simeq \mathbb{R}$. The corresponding 1-parameter family of deformations is

$$p_*(\pi_{0,\delta}) = 1/(1 + \delta)p_*(\pi).$$

Example 7.3.8. In the symplectic world, the analogous statement to Theorem 7.3.1 allows us to find a C^0-neighbourhood of the symplectic structure in which symplectic structures are classified by $H^2(M)$. In the log-symplectic case, the neighbourhood in $\mathrm{Poiss}(M)$ has to be C^1, merely because transversality is a C^1 condition. Actually, for the proof of Lemma 7.3.3, one needs C^1-closeness only in a small neighbourhood around the singular locus, while outside one can consider C^0-open neighbourhoods.

To illustrate this phenomenon, consider again the log-symplectic structure $\pi = z\partial_\theta \wedge \partial_z$ on S^2. We look at families of Poisson structures of the form

$$\pi_\varepsilon := h_\varepsilon(z) z\partial_\theta \wedge \partial_z,$$

for small $\varepsilon > 0$, where $h_\varepsilon : \mathbb{R} \to [-1, 1]$ are functions such that $h_\varepsilon(z) = 1$ for $|z| \geq \varepsilon$. Note that the C^0-distance between π_ε and π is less than 2ε.

First, consider h_ε such that it vanishes on $[-\epsilon/2, \epsilon/2]$. We see that π can be C^0-approximated by Poisson structures that are not log symplectic.

Consider now h_ε such that it vanishes linearly at $\pm\varepsilon/2$, and only at these points. The resulting structures are log symplectic with singular locus the three circles $z = -\varepsilon/2$, $z = 0$, $z = \varepsilon/2$. This shows that π can be C^0-approximated by log-symplectic structures whose singular locus is not diffeomorphic to that of π.

Example 7.3.9. Consider $S^2 \times T^2$ endowed with the product log-symplectic structure

$$\pi := z\partial_\theta \wedge \partial_z + \partial_{\theta_2} \wedge \partial_{\theta_1},$$

where θ_1, θ_2 are the coordinates on the two-torus $T^2 = S^1 \times S^1$. The singular locus of π is the three-torus $S^1 \times T^2$, where the first S^1 denotes the equator in S^2. The induced cosymplectic structure on $S^1 \times T^2$ is $(d\theta_1 \wedge d\theta_2, d\theta)$. The b-two-form is

$$\omega = d\log|z| \wedge d\theta + d\theta_1 \wedge d\theta_2.$$

The second b-cohomology $H^2_{S^1\times T^2}(S^2 \times T^2)$ is spanned by

$$[dz \wedge d\theta],\ [d\theta_1 \wedge d\theta_2],\ [d\log|z| \wedge d\theta],\ [d\log|z| \wedge d\theta_1],\ [d\log|z| \wedge d\theta_2],$$

where the first two generators correspond to $H^2(S^2 \times T^2)$ and the last three to $H^1(S^1 \times T^2)$. By Theorem 7.3.1, we conclude that any Poisson structure C^1-close to π is diffeomorphic to a log-symplectic structure on $(S^2 \times T^2, S^1 \times T^2)$ with b-two-form

$$\omega' = \epsilon_1 dz \wedge d\theta + (1 + \epsilon_2)d\theta_1 \wedge d\theta_2 + d\log|z| \wedge ((1 + \delta_1)d\theta + \delta_2 d\theta_1 + \delta_3 d\theta_2),$$

for some $\epsilon_1, \epsilon_2, \delta_1, \delta_2, \delta_3 \in \mathbb{R}$ small enough. The cosymplectic structure is

$$((1 + \epsilon_2)d\theta_1 \wedge d\theta_2, (1 + \delta_1)d\theta + \delta_2 d\theta_1 + \delta_3 d\theta_2).$$

In particular, the foliation is a "generalized Kronecker foliation", i.e. its pullback to $\mathbb{R}^3$ via the projection $\mathbb{R}^3 \to S^1 \times T^2$ is a foliation by the parallel affine planes

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 : (1 + \delta_1)x_1 + \delta_2 x_2 + \delta_3 x_3 = C\}_{C\in\mathbb{R}}.$$

This is quite remarkable, since the foliation $\ker(d\theta)$ can be C^∞-approximated by foliations which behave very differently, e.g. the family of foliations $\mathcal{F}_\epsilon$ given by the kernel of $d\theta + \epsilon\chi(\theta)d\theta_1$, where χ is a function such that $\chi(\theta) = \theta$ for small $|\theta|$.

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Samenvatting

In dit proefschrift worden er twee hoofdonderwerpen behandeld: *symplectische foliaties van codimensie één* en *log-symplectische structuren*.

Een *foliatie van codimensie één* van een variëteit M is een verdeling van M in deelvariëteiten van één dimensie lager, waar de elementen van de verdeling ordelijk naast elkaar liggen. Ordelijk betekent, heuristisch gezien, dat de verdeling lokaal lijkt op de verdeling van rechte lijnen in een vlak. Elke samenhangende deelvariëteit van de verdeling heet een *blad*. Een *symplectische* foliatie van codimensie één is een foliatie van codimensie één waar elke blad over een symplectische structuur beschikt en de symplectische structuren op een gladde manier variëren van blad tot blad.

Hier richten we ons op de vraag: welke compacte variëteiten laten symplectische foliaties van codimensie één toe? We ontwikkelen een methode waarmee symplectische foliaties van codimensie één kunnen worden gebouwd. Deze methode heet *symplectische omwenteling* ("symplectic turbulisation" in het Engels) en is een generalisatie van de *omwentelingsmethode* die al bestond om foliaties van codimensie één te bouwen.

In hoofdstukken 1 en 2 geven we een uitgebreide inleiding waar we de belangrijkste begrippen in herinnering brengen die voor het proefschrift nodig zijn. Vervolgens leggen we in hoofdstuk 3 de symplectische omwentelingsmethode uit en in hoofdstuk 5 en 6 passen we de methode toe om symplectische foliaties te bouwen in sommige vijfdimensionale variëteiten van de vorm $M \times S^1$ (hoofdstuk 5) en variëteiten die bepaalde openboekdecomposities toelaten (hoofdstuk 6).

In hoofdstukken 4 en 7 worden log-symplectische structuren bestudeerd. Ze zijn Poissonstructuren (Poissonstructuren kunnen worden gezien als een generalisatie van symplectische foliaties) die simpel maar interessant genoeg zijn om een grondige studie toe te laten. In hoofdstuk 4 worden de basisdefinities en de grondeigenschappen omtrent log-symplectische structuren bestudeerd. Deze structuren zijn ook gerelateerd aan de symplectische omwentelingsmethode en de precieze relatie wordt ook in hoofdstuk 4 behandeld. In hoofdstuk 7 bestuderen we de vervormingen van deze structuren, een probleem die interessant is vanuit het standpunt van de Poissonmeetkunde.

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Curriculum Vitae

Boris Osorno Torres was born in Barbosa, Antioquia, Colombia, on November 22, 1987. After obtaining his high school degree in November 2003, he did his bachelor in physics engineering between 2004 and 2009 at the National University of Colombia, Medellín. In September 2009, he moved to the Netherlands to take part in the Master Class program "Arithmetic geometry and non commutative geometry", organised at Utrecht University by the Mathematical Research Institute. From September 2010 until January 2011, he participated in the first half of the masterclass "Moduli spaces", organised also by the Mathematical Research institute at Utrecht University. On February 2011, he started his PhD at Utrecht University under the supervision of prof. dr. Marius Crainic.