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Identifying a framework for graphing formulas from expert strategies

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ABSTRACT

It is still largely unknown what are effective and efficient strategies for graphing formulas with paper and pencil without the help of graphing tools. We here propose a two-dimensional framework to describe the various strategies for graphing formulas with recognition and heuristics as dimensions. Five experts and three secondary-school math teachers were asked to solve two complex graphing tasks. The results show that the framework can be used to describe formula graphing strategies, and allows for differentiation between individuals. Experts used various strategies when graphing formulas: some focused on their repertoire of formulas they can instantly visualize by graphs; others relied on strong heuristics, such as qualitative reasoning. Our exploratory study is a first step towards further research in this area, with the ultimate aim of improving students' skills in reading and graphing formulas.

1. Introduction

Students often have difficulties with algebra, in particular giving meaning to and grasping the structure of algebraic formulas, and manipulating them (Chazan & Yerushalmy, 2003; Drijvers, Goddijn, & Kindt, 2010; Kieran, 2006; Sfard & Linchevski, 1994). Functions can be represented in several forms, such as algebraic formulas and graphs; the latter are more accessible for students than the former (Janvier, 1987; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993).

A graphical representation gives information on covariation, that is, how the *y*-coordinate (the dependent variable) changes as a result of changes of the *x*-coordinate (the independent variable) (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). A graph shows possible symmetry, intervals of increase or decrease, extreme values, and infinity behavior. In this way, it visualizes the "story" of an algebraic formula. Graphs may help learners to give meaning to algebraic formulas and so make learning algebra easier for them (Eisenberg & Dreyfus, 1994; Kilpatrick & Izsak, 2008; NCTM, 2000; Philipp, Martin, & Richgels, 1993; Yerushalmy & Gafni, 1992).

Graphs are also considered important in problem solving (Polya, 1945; Stylianou & Silver, 2004). In his list of heuristics, Polya (1945) mentions drawing a picture or diagram as one of the first options. Creating and using multiple representations,

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and switching between them, are important tools in problem solving (Janvier, 1987; NCTM, 2000). Stylianou and Silver (2004) and Stylianou (2002, 2010) found how graphs are used to understand the problem situation, to record information, to explore, and to monitor and evaluate results.

For learning about functions, graphing tools such as graphic calculators are recommended (Drijvers & Doorman, 1996; Drijvers, 2002; Hennessy, Fung, & Scanlon, 2001; Kieran & Drijvers, 2006; Philipp et al., 1993; Schwartz & Yerushalmy, 1992; Yerushalmy & Gafni, 1992). With these tools, graphing formulas seems easy. In the past, constructing a graph was itself a goal or the graph itself an end product. To produce one, many algebraic skills (determining domain, zeroes, derivative, etc.) were employed, along with standard methods requiring multiple algebraic manipulations, which were not straightforward for all learners.

Graphing tools now make it possible to study problems that in the past could not be solved or could be solved only with difficulty. In order to use these tools adequately, however, one must know what aspects of graphs to look for (Philipp et al., 1993). According to Stylianou and Silver (2004) novices experience difficulties in the visual explorations of the graphs they have constructed. They concluded that such explorations are restricted to familiar functions. So, in order to make effective and efficient use of technology, learners should know about graphs representing basic functions, and also should have learned to reason about such graphs (Drijvers, 2002; Eisenberg & Dreyfus, 1994; Stylianou & Silver, 2004).

Learners who do graphing with pen and paper may establish the connection between the algebraic and the graphical representations of a function more effectively than learners who only perform computer graphing (Goldenberg, 1988). In this article, graphing to produce a sketch of a graph with its main characteristics without technological help will be called *graphing formulas*.

Despite earlier research on how to learn and how to teach functions, it is still largely unknown what knowledge and skills are necessary to graph formulas effectively and efficiently. In order to learn more about these, we have identified expert strategies in our research. Experts are expected to know and use more effective and efficient strategies than novices (Chi, 2006, 2011). Hence, the focus of this article will be on determining a suitable framework for formula graphing strategies. With the help of this knowledge base, a professional development trajectory for teachers and teaching material for students may eventually be developed.

2. Theory

2.1. Aspects of graphing formulas

Functions are at the core of math education. There are several reasons for students' difficulties with the concept. Functions, like other mathematical concepts, are not directly accessible as physical objects. Access to mathematical concepts can only be gained through representations. To understand mathematical concepts one needs to relate elements of different representations (Janvier, 1987; Kaput, 1998). For functions, these representations are algebraic formulas, graphs, tables, and contexts (Janvier, 1987). These representations have to be combined in order to produce a rich concept image of the function (Thomas, Wilson, Corballis, Lim, & Yoon, 2010; Tall & Vinner, 1981).

The ability to represent concepts, to establish meaningful links between and within representations, and to translate from one representation of a concept to another is at the core of doing and understanding mathematics. Different concepts have been used to refer to this ability: representational flexibility (Nistal, Van Dooren, Clarebout, Elen, & Verschaffel, 2009), representational fluency (Lesh, 1999), representational versatility (Thomas & Hong, 2001). 'Representational versatility' has been defined as the ability to work seamlessly within and between representations and to engage in procedural and conceptual interactions with representations (Thomas et al., 2010). Our research deals with translations between algebraic formulas and graphs, demonstrating representational versatility.

Much research has been done on algebraic and graphical representations and their relations. Students are often found to have difficulties with reading algebraic formulas and the so-called process-object character of a function. For graphing formulas, it is necessary that one can "read" algebraic formulas and deal with the process-object character of a function. These two issues are discussed in the next sections.

2.1.1. Reading algebraic formulas

There are different ways to create meaning for algebraic formulas: from the problem context, from the algebraic structure of the formula, and from its various representations (Kieran, 2006). In order to read an algebraic formula, one has to grasp its structure (Sfard & Linchevski, 1994). In the literature this is called 'symbol sense' (Arcavi, 1994). Symbol sense has several aspects, such as the ability to read through algebraic expressions, to see the expression as a whole rather than a concatenation of letters, and to recognize its global characteristics (Arcavi, 1994). Symbol sense enables people to scan an algebraic expression so as to make rough estimates of the patterns that would emerge in numeric or graphical representations (Arcavi, 1994).

A procedure for analyzing the syntactic structure of an expression was formulated by Ernest (1990). A syntactical tree is constructed via an iterative procedure in which the main operator of the expression is identified. The procedure continues until all subexpressions have been given meaning. The decomposition of algebraic expressions into meaningful parts (building blocks) can be considered a heuristic for reading formulas.

Thomas et al. (2010) asked students about their strategies when linking formulas to graphs of linear and quadratic functions. He found that the students tried to imagine the graph and focused on key properties of the functions. Thomas et al.'s findings are consistent with Mason's (Mason, 2002, 2004/2008) argument that both perception of and reasoning about functions are driven by specific properties of the representation (Thomas et al., 2010). Mason (Mason, 2003, 2004/2008) identified levels of attention or awareness, which he used to describe how a person's attention can shift from staring at the whole (for example an algebraic formula) while hardly knowing how to proceed, to discerning details from which objects and sub-objects can be determined in order to recognize relationships, perceive properties, and grasp the essential structure. For this recognition, students need a repertoire of basic functions, and knowledge of the characteristics of the representations of these functions (Eisenberg & Dreyfus, 1994).

In summary, for reading algebraic formulas recognition of basic functions and symbol sense are important, as well as knowledge of procedures such as decomposing formulas into meaningful sub-formulas.

2.1.2. Process and object perspectives

Students often use only the process perspective of a function, because they see a function as a calculation rule in which x and y values are linked. Formulas which differ from the process perspective, for instance $y = x + 3$ and $y = 4 + x - 1$, can belong to the same function. From the object perspective, such formulas are considered identical (Schwartz & Yerushalmy, 1992). The object perspective is needed to perform actions on a function, for instance to transform it (Breidenbach, Dubinsky, Hawks, & Nichols, 1992) and to classify families of functions.

Students have to be able to assume both a process and an object perspective and to switch between the two (Breidenbach et al., 1992; Gray & Tall, 1994; Oehrtman, Carlson, & Thompson, 2008; Sfard, 1991). Moschkovich et al. (1993) formulated a two-dimensional framework, with representations and perspectives as dimensions, and showed that in problem solving different representations and different perspectives are needed. There is a relation between these two dimensions: the algebraic representation of a function makes its process perspective salient, while the graphical representation suppresses the process perspective and thus helps to make a function more entity-like, i.e., an object (Moschkovich et al., 1993; Schwartz & Yerushalmy, 1992).

Three theories consider the complementary aspects of process and object perspectives: APOS, covariational reasoning, and pointwise and global approach in solving problems with formulas and graphs.

The APOS (action, process, object, schema) theory describes how an object perspective is developed through the encapsulation of processes, and how a schema integrating both perspectives is created (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Breidenbach et al., 1992). A well-developed schema can be seen as a cognitive unit, an element of cognitive structure that in its entirety can be the focus of attention at a given time (Barnard & Tall, 1997). Such a cognitive unit can be activated as a single step in a thinking process (Crowley & Tall, 1999). This type of schema makes it possible to instantly switch between process and object perspectives.

Covariational reasoning is the ability to coordinate an image of two varying quantities and to note how they change in relation to each other (Carlson et al., 2002). It is essential to understand major concepts of calculus: functions, limits, derivatives, rates of change, concavity, inflection points, and their real world interpretations (Carlson et al., 2002; Oehrtman et al., 2008). In covariational reasoning, one is able to imagine running through all input-output pairs simultaneously and so to reason about how a function is acting on an entire interval of input values. Such reasoning is not possible operating from a process perspective where each individual computation must be explicitly performed or imagined. Carlson et al. (2002) describe levels of covariational reasoning: from the notion of "y is changing with changes in x", via knowing whether a function increases or decreases, and considerations about the rate of change, to understanding average and instantaneous rates of change, and inflection points.

Even (1998) considers the process and object perspectives in solving problems with formulas and graphs in terms of heuristics. She distinguishes pointwise and global approaches. In the pointwise approach, students plot and read points, whereas in the global approach they focus on the behavior of the function on an interval or in a global way. The global approach is more powerful and gives a better understanding of the relation between formulas and graphs. However, sometimes the pointwise approach is needed to monitor naïve and/or immature interpretations and to construct meaning (Even, 1998).

In summary, in order to graph formulas effectively and efficiently one has to be able to read algebraic formulas and deal with the process and object perspectives. The literature shows that both recognition through schemas and symbol sense are important in this respect. If recognition fails, heuristics are necessary. In our research, we attempted to elucidate which recognition and heuristics are essential for effective and efficient graphing, so as to identify a framework of strategies for graphing formulas. For the necessary background knowledge on how recognition and heuristics are related we consulted the literature on expertise, in which the importance of recognition and heuristics is endorsed.

2.2. Expertise

Experts outperform novices when solving problems in their fields of expertise (Chi, Feltovich & Glaser, 1981; Chi, Glaser, & Rees, 1982; Chi, 2006, 2011). What is the reason for this difference in performance? In order to sketch the main components of expertise and their interrelations, we here provide a short historical review of the changing explanations of expertise during the past forty years (Chi, 2011).

First, experts were believed to have superior search strategies. In expertise research in the 1970s, expertise was often assessed via puzzle-like problems. In order to solve such puzzles participants need general problem-solving strategies rather than much domain-specific knowledge. Within this context, solving a problem is seen as searching for a path in the problem space to connect the problem with the solution. Expertise is defined as the ability to search efficiently and effectively. Therefore, general heuristics have been formulated for mathematical problem solving (Marshall, 1995; Polya, 1945; Schoenfeld, 1985, 1992; Van Streun, 1989). In problem-solving literature, recognition is often mentioned as an all-or-nothing process in the orientation on a problem: either there is recognition or there is not.

Subsequently, it was more structured knowledge that was thought to be the decisive factor determining search strategies. Experts do not necessarily have superior general search strategies, but their knowledge is more effectively structured, as cognitive schemas in the long-term memory (Chase & Simon, 1973). A cognitive schema can be seen as a network with (hierarchically) related concepts, procedures, and strategies (Anderson, 1980; Derry, 1996). To a large extent a person's cognitive schemas determine what that person "sees" and recognizes in problem situations (Sweller, Merrienboer, & van Paas, 1998). Thus, when someone is confronted with a problem situation, different levels of recognition are possible, varying from completely recalling the problem situation and the solution to no recognition at all.

In mathematics, when novices see an algebraic formula, as for instance $y = x^2 + 2x$, they may not recognize it as a polynomial function, but see it as just a procedure: $y = \mathrm{x} \cdot \mathrm{x} + 2 \cdot \mathrm{x}$.Experts seeing $y = x^2 + 2x$ will immediately recognize it as a member of the family $y = ax^2 + bx + c$ and will know that the graph will be a parabola with a minimum. In an expert's cognitive schema, activated by the algebraic formula, the formula can be linked to a graph and thus can be instantly visualized graphically, whereas other formulas cannot, if the cognitive schema activated by the formula does not have a link to its graph. In formula graphing the set of functions that can be instantly visualized plays an important role. These functions can be seen as units or building blocks for thinking and reasoning in simple as well as in complex situations.

Recently, it is the representation of the problem that has come to be seen as the dominant factor accounting for expertise: structured knowledge guides representation, which dictates search strategies. Experts and novices focus on different elements of the problem they are confronted with. In physics, experts look for the underlying principles on which a problem is based, whereas novices look at the superficial surface characteristics (Chi et al., 1981). Experts activate schemas that can provide additional information, strategies, and expectations for further elaboration of the problem representation (Chi et al., 1981). In chess research, two main explanations for expertise are given: the ability to access a rich knowledge database through pattern recognition, and the ability to search efficiently through the problem space via a deliberate heuristic search (De Groot, 1965; De Groot, Gobet, & Jongman, 1996; Gobet, 1997, 1998). Berliner and Ebeling (1989) formulated a model in which performance is a function of two variables, i.e., recognition and search. In this model, different combinations of recognition and search may give the same level of performance, as there is a trade-off between recognition and search. Berliner and Ebeling (1989) concluded that the degree of recognition determines the problem space and, as a consequence, the heuristic search.

In mathematics, when experts have to graph a function such as $y = x^2 + 2x$ their algebraic knowledge will tell them that $y = (x + 1)^2 - 1$ and $y = x(x + 2)$ are equivalent formulas. So, the expert can use all these algebraic representations to acquire detailed information about the graph. Novices faced with the formula $y = x^2 + 2x$ lack the ability to switch to an alternative problem representation. Thus, in problem solving levels of recognition depend on knowledge. Experts' structured knowledge facilitates high levels of recognition, which gives them superior problem representation and efficient heuristic search options. In Section 2.3, we will describe different levels of recognition in graphing formulas, and show how in graphing formulas recognition and heuristic search may be related.

2.3. Towards a two-dimensional framework

Research on expertise has indicated that recognition guides heuristic searching, i.e., the level of recognition determines the heuristic search. This interplay can result in a two-dimensional framework by which to describe strategies in graphing formulas. Recognition and heuristic search will form the two dimensions of our framework. Below, we will first give an example to illustrate how different levels of recognition allow different heuristic searches. Then we will formulate levels of recognition and of efficient heuristic searching in graphing formulas.

Example: When we have to graph a complex formula, such as $\mathrm{f}(x) = x^2\sqrt{8-x} + 2x$, it is difficult to immediately visualize its graph. A heuristic search is needed. A first step can be to consider the domain: $< \leftarrow, 8]$. It is possible to decompose the function into two subformulas, $y = x^2\sqrt{8-x}$ and $y = 2x$ (Ernest, 1990). While the second subformula can probably be visualized instantly, the first, $y = x^2\sqrt{8-x}$ probably cannot. Via qualitative reasoning about its infinity behavior at $-\infty$ (if $x \to -\infty$ then $y \to \infty$) and its zeroes 0 and 8, the graph can be sketched (see Fig. 1). Then the graph for $\mathrm{f}(x) = x^2\sqrt{8-x} + 2x$ can be constructed via qualitative reasoning and/or by making a table (Fig. 2).

Another strategy is to reason qualitatively about domain and zeroes of the function $\mathrm{f}(x) = x^2\sqrt{8-x} + 2x$, and then to calculate the derivative in order to find the extreme values. Novices may not recognize the structure of the formula and will probably be limited to using a heuristic such as "make a table". This example shows how recognition determines the problem representation and hence the heuristic search: recognition guides heuristic search. In Section 2.3.1 different levels of recognition and of heuristic search are identified.

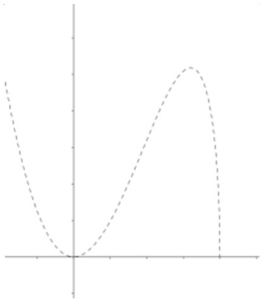


Fig. 1. Graph of $y = x^2\sqrt{8-x}$

2.3.1. Levels of recognition and heuristic search

In this section on the different levels of recognition and heuristics used in our framework we will first focus on recognition, and then on heuristic searching in graphing formulas. The literature on reading formulas and expertise mentioned in Sections 2.1 and 2.2, and the above example, suggest that different levels of recognition are possible: from complete recognition (the graph is instantly known), to decomposing a formula into known subgraphs, to no recognition of the graph. On the basis of the literature, our own reflections, and several pilot interviews before our research, we propose the following six levels of recognition:

Level A: the graph is immediately recognized. For instance, graphs of $y = -2x + 4$ and $y = x^2$ are instantly visualized.

Level B: the equation is recognized as a member of a family of which the possible graphs are known. Only a brief analysis is needed to graph the formula. For instance, $y = 4 \times 0.75^x + 3$ is recognized as belonging to the family of decreasing exponential functions; $y = -x^4 + 6x^2$ is described as a polynomial function of grade 4, so its graph has an M or Λ form because of the negative main coefficient.

Level C: the formula is split into subformulas that can be instantly visualized. For instance, $y = x + 4/x$ is decomposed into $y = x$ and $y = 4/x$, or $y = 2x\sqrt{x+6}$ into $y = 2x$ and $y = \sqrt{x+6}$

Level D: characteristic aspects of the graph are recognized but the rest of the graph is not. For instance, the graph of $y = x + 4/x$ is described as 'having a slanted asymptote $y = x$ and a vertical asymptote $x = 0$,' but no other features of the graph are described.

Level E: the graph is not even partly recognized, but the participant is able to use the algebraic formula for deliberate exploration of the graph. For instance, the formula $y = x^2/(x^2 + 2)$ is analyzed by qualitative reasoning: domain is, zero at $x = 0$; the graph is symmetrical; if $x \to \infty$ then $y \to 1$ (infinity behavior), so $y = 1$ is a horizontal asymptote; the graph increases for positive x values.

Level F: the graph is not recognized, and neither are any features of the algebraic formula. At this level one is restricted to using a standard repertoire to find characteristics of the graph, i.e., domain, zeroes, and extreme values via the derivative, or making a table with random x values.

These levels of recognition can be linked to Mason's levels of attention (Mason, 2003) and Thomas and Hong's model of interaction with representations (Thomas & Hong, 2001). The latter gives a hierarchy of observations of a representation: from surface observation, via noting properties, to actions on the representation in order to obtain further information or understanding of a concept (Thomas & Hong, 2001). Our levels of recognition can be linked to Mason's levels of attention as follows: gazing at the whole (an algebraic formula) while hardly being aware how to proceed can be related to level F, discerning details to level E, recognizing relationships between different details, perceiving properties, and seeing the essential structure to levels C, B, or A.

Expertise research shows that experts use more efficient heuristics than novices do. These efficient heuristics are called 'strong'. The stronger the heuristics, the more characteristics of the problem situation and domain-specific knowledge will be used, which results in faster problem solving. When graphing formulas, these stronger heuristics will result in more

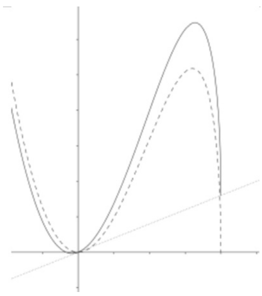


Fig. 2. Construction of the graph $f(x) = x^2\sqrt{8-x} + 2x$

Table 1
Two-dimensional framework.

		Heuristic search (strong → weak)				
Levels of recognition (high → low)	A	A1. Graph is instantly recognized as a whole				
	B	B1. Recognition of family (with characteristics); possible graphs are known	B2. Search for 'parameters' of the graph	B3. Investigate the family characteristics, for instance via zeroes, derivative		
	C	C1. Split formula in subformulas, graphs of subformulas being known	C2. Compose the graphs by qualitative reasoning	C3. Compose the graphs by making a table		
	D	D1. Characteristic aspect of graph is recognized; rest of graph is unknown				
	E	E1. Graph is not recognized, algebraic formula is starting point for strategic exploration	E2. Qualitative reasoning for instance about domain, or vertical asymptote, or symmetry, or infinity behavior, or increase/decrease	E3. Algebraic manipulation	E4. Strategic search, for instance for zeroes or extreme values (via derivative)	E5. Calculate strategically chosen point(s)
	F	F1. No recognition at all	F2. Standard repertoire of research	F3. Make table with random x values		

information about the whole graph. These descriptions are in line with the global versus the pointwise approach formulated by Even (1998). We will give two examples of strong vs. weak heuristics.

Someone having to graph $y = \ln(x - 4)$ can recognize this function as a logarithmic function with a graph that increases and has a vertical asymptote. A strong heuristic is to sketch the standard function $y = \ln(x)$ and use a translation, because the focus is on the whole graph. A weak heuristic is to find the zero and vertical asymptote and calculate some points, because in that case the person only looks locally and tries to construct the graph with this local information.

When someone has to graph a formula and does not even partly recognize the graph, making a table is considered a weaker heuristic than qualitative reasoning about infinity behavior or about symmetry, because of the difference in local and global information about the graph.

Thus, regarding graphing formulas we have identified six levels of recognition (from direct recall to no recognition at all), and strong and weak heuristics. From the expertise literature, we have learned that recognition guides heuristic search. This interplay between recognition and heuristics results in a two-dimensional framework: For every level of recognition, we can formulate strong and weak heuristics, and so construct a two-dimensional framework (Table 1).

2.4. Research questions

We wanted to check if the differences between the strategies for graphing formulas of experts, teachers, and learners could be accommodated within the framework of Table 1. We also wanted to check how often a person would use more than one strategy, so that the description of sequences of strategies may result in a "path" in the framework. At this stage of our research, having established the framework, we focused first on strategies of experts. Experts are expected to have more knowledge, and thus to recognize more and to use stronger heuristics. Therefore we expected their paths in the framework to contain fewer steps compared to the non-experts, and to be situated predominately on the upper side (more recognition), and/or on the left side (strong heuristics) of the framework in comparison with the paths of non-experts.

This raised the following questions:

Does the framework describe strategies in graphing formulas appropriately and discriminatively?

Which strategies do experts use in formula-graphing tasks?

3. Method

This study can be characterized as an exploratory study, allowing for a portrayal of the richness of the situation in which knowledge elements and heuristics have to be established.

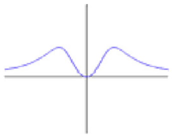
Before we started the experimental part of our research, we discussed formula-graphing strategies in interviews with three well-known researchers in mathematical education. These interviews provided us with an inside view of the levels of recognition and heuristics used in graphing formulas such as $y = x + 4/x$ and $y = \sqrt{1 + x^2}$.

3.1. Tasks

To elicit the participants' strategies we developed two tasks. Because we wanted to move participants out of their recognition zone, these tasks contained complex formulas and graphs. To challenge the participants' adaptive expertise we presented them with a new situation, such as task B below, which required them to work from graph to formula, which is the reverse of the usual order. Such a reverse task calls on the same thinking processes, because participants are expected to graph formulas when they elaborate and test potential solutions.

During both tasks, participants were allowed to use only paper and pen. The available time was 10 min for each problem. Participants were asked to think aloud; this was not expected to disturb their thinking process, and should give reliable information about their problem-solving activities (Ericsson, 2006).

Tasks: 20 min (10 min per task)
In the first task, we ask you to graph a formula. In the second task, we ask you to find a formula that fits a given graph. We are interested in your strategies. Please voice your thoughts and think aloud while solving these tasks.
Task A. Graph the formula $y = 2x\sqrt{8-x} - 2x$
Task B. Find a formula that fits the graph.



3.2. Participants

To make sure we covered the full range of possible strategies, we invited highly regarded mathematical experts along with a range of teachers to participate. Five experts were selected, from different backgrounds: three mathematicians who teach first-year students at university, one author of a mathematical textbook, and one teacher educator. All of them had a Master's degree or Ph.D. in mathematics and had more than ten years of experience in teaching. In their education and in their work they often have to graph formulas, and therefore we considered them experts in graphing formulas. We assigned the letters P, Q, R, S, and T to our five experts.

We would expect the experts' paths in the framework to be located predominantly on the upper and left sides of the table. In order to provide a contrast with the experts we invited three math teachers to solve the same tasks. Teachers are not novices, but we do not assume all teachers to have the same level of expertise in graphing formulas. We would expect teachers' paths to be situated more on the lower and right sides of the framework. The teachers, labeled U, V, and W had 30, 6, and 2 years of teaching experience, respectively.

3.3. Coding for task A

All participants' performances were videotaped and transcribed, and we used the framework to analyze the results. The transcriptions of the think-aloud protocols were cut into sections to allow encoding according to our framework. Unspoken actions and observations of the experimenter were indicated by [. . ..]. The recognition levels mentioned in Section 2.3.1 were used to start the encoding. Coding was done according to the following instructions:

- If a participant immediately recognizes the graph $y = 2x$ as a straight line or $y = \sqrt{x}$ as a half horizontal parabola, encode A1 (the graph is instantly recognized)
- If a participant sees $y = \sqrt{8-x}$ as a member of the family of square root functions and uses transformations on $y = \sqrt{x}$ encode B1–B2; however, if the participant does not use transformations but instead determines starting point (8,0) and calculates one or more points to determine whether the graph is to the right or to the left, encode B1–B3
- If a participant sees $y = 2x\sqrt{8-x}$ and decomposes this formula into $y = 2x$ and $y = \sqrt{8-x}$ (C1), graphs both formulas immediately (A1), and then multiplies the graphs by qualitative reasoning (C2), encode C1-A1-C2. If a participant multiplies these graphs by multiplying several y values, encode C1-A1-C3
- If a participant sees $y = 2x\sqrt{8-x}$ and describes the graph around (8,0) in terms of 'from (8,0) it starts to the left with a vertical tangent' (D1), and then makes a table with several well-chosen x values (E5), encode D1 → E5
- If a participant sees $y = 2x\sqrt{8-x}$, starts factorizing and calculates zeroes, domain, and extreme values, encode E1-E3-E4-E2-E4

All fragments in the protocols were coded by two coders working independently. The few differences were discussed and agreement was reached in all cases. The encodings were displayed in the framework. If the same encoding appeared several times in a row, this was noted as only one point in the relevant segment of the framework. For every participant

Table 2
Derived framework for task B: from graph to formula.

		Heuristic search (strong → weak)		
Levels of recognition(high → low)	A	A1. Formula is instantly recognized		
	B	B1. Recognition of family of formulas ('is something like. . .')	B2. Searching for 'parameters' of the formula (e.g., translation or via zeroes)	
	C	C1. Graph is decomposed into subgraphs	C2. Finding formulas for subgraphs and composing the formula	
	D	D1. 'Parts of the graph give parts of formula' (part of formula is recognized)	D2. Adjusting a formula to characteristics via qualitative reasoning	
	E	E1. Mentioning algebraic formulas ('can it be something like this?') without direct link to graph	E2. Checking a formula via qualitative reasoning	E3. Checking a formula by general methods (zeroes, extreme values, table)
	F	F1. No recognition at all		

this resulted in a path inside the framework. In addition, the time needed to solve the problem and the extent to which a participant was successful was indicated.

3.4. Task B: From graph to formula

From the two-dimensional framework, we derived a special framework to analyze the performances on task B. We used the same recognition levels; the heuristics on every recognition level in the two-dimensional framework were "translated" into heuristics for this new task. Table 2. For this derived framework similar encoding instruction were formulated.

4. Results

4.1. Results: graphing a formula

We first present the results of task A, graphing the formula $y = 2x\sqrt{8-x} - 2x$. For all participants we encoded the protocols based on the framework. These encodings can be tabulated as in the example below. We illustrate this process for expert *P* (Table 3); the results for all participants are given in Table 4.

For every participant the data are displayed as a path in the framework. Starting point "***S***" and end point "***E***" are indicated, as well as the time (in minutes) needed, and whether the graph was correct, partially correct, or not correct (Fig. 3).

Fig. 3 shows that in task A all experts found a correct graph, although T made a mistake when calculating the extreme value of the function. Experts solved this task in about 5 min on average. Four out of five experts started at recognition

Table 3
Expert P's results on task A.

Fragments	encoding
'Yes, then I write down the formula; I always start like this' 'yes, I establish the domain: $x \leq 8$'	E1/E2
'Anyway the graph goes through (0,0)' [starts sketch]	E4
'If I take 8 then the graph is on -16' [draws point (8,-16)]	E5
'That will be vertical there' [draws vertical part of graph in (8,-16)]	D1
'What is happening if I look at minus infinity' [draws part of graph down left]	E2
[Factorizes the formula and write $2x(\sqrt{8-x}-1)$]	E3
[Calculates the derivative; makes a mistake with the chain rule; calculation gives an equation which has no solutions]	E4
'This is a hassle; there has to be a simpler way' 'i'll start once more; I have $\sqrt{8-x}$' [draws graph of $y=\sqrt{8-x}$]	C1/A1
'-1' [graph of $y=\sqrt{8-x}-1$]	B2
'Multiplies this by $2x$'	C2
'That means that the graph here is still a little lower' [points around $x=8$]	C2
'And if I multiply by $2x$, anyway on this side it is still positive; and this will be negative; [points near $x=8$]	C2
'And here it is 0' [points to the zero of $y=\sqrt{8-x}-1$]	C2
'The turning point should be here' [points near $x=4$]	C2
'And here [points left of y axis] I multiply by something negative and the graph will go like this'	C2
[Sketches graph left of y axis] 'and the graph will go down very quickly' 'it will result in this graph' 'and the zero will be at the moment that $\sqrt{8-x}$ equals 1; so $x=7$'	
[Looks at the calculations of the derivative; but cannot find the mistake]	
'Well it has to be something like this; I made a mistake in the calculation of the derivative'	

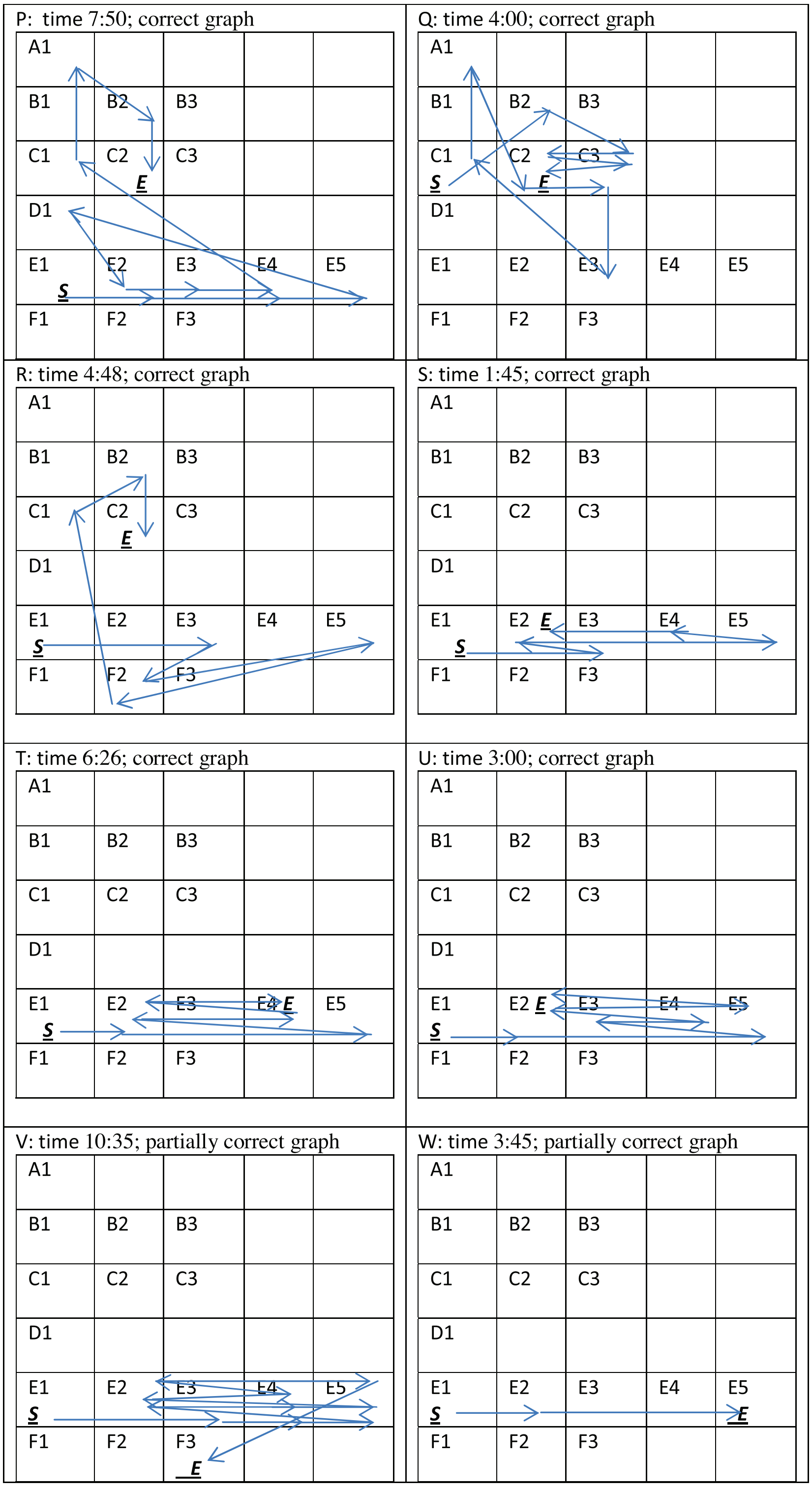


Fig. 3. Results on task A in the two-dimensional framework

Table 4
Results on task A.

Experts														
P	E1/E2	E4	E5	D1	E2	E3	E4	C1/A1	B2	C2	C2	C2	C2	C2	C2
Q	C1/A1	C2	C3	E3	C1/B2	C3	C3	C2	C2	C2	C2	C3	C2	C2	
R	E1/E3	F2	F2	E5	F2	C1/B2	C2								
S	E1/E3	E2	E5	E4	E2										
T	E1/E2	E5	E5	E2	E4	E2	E4								
Teachers															
U	E1/E2	E5	E3	E4	E2	E5	E2	E2							
V	E1/E3	E4	E5	E5	E2	E4	E5	E2	E4	E4	E2	E5	F3		
W	E1/E2	E5	E5												

level E: the algebraic formula is the starting point for exploration. Only expert Q started by decomposing the formula into sub-formulas, and later considered a different decomposition.

Experts S and T started by exploring the algebraic formula and used qualitative reasoning. Both strategies can give fast results. Some experts (P and R) took an in-between position: they started by exploring the algebraic formula but later switched to decomposition. The protocols show their considerations, in which they state that they do not make sufficient progress and will start to look for alternative strategies: 'this is a hassle, there has to be a simpler way' (from the protocol of expert P). Only experts P, Q, and R used their repertoire of formulas that can be instantly visualized in this task.

All teachers started on recognition level E, which shows that they did not use their repertoire of formulas that can be instantly visualized. Two teachers were less successful in completing this task: their graphs were only partially correct. In general, the teachers' paths are located in the lower and right sides of the framework.

4.2. Results on task B: From graph to formula

In order to analyze the performances on task B (finding a formula that fits a given graph), we encoded the fragments of the thinking-aloud protocols. These encodings are tabulated below. First, we illustrate this process for expert S (in Table 5); we give the results for all participants in Table 6.

In task B, four out of five experts found a correct formula: $y = x^2e^{-x^2}$ (experts P, Q, R) and $y = x^2/(x^2+1)^2$ (expert S). Halfway through, expert T (total time 6:26 min) hit on a correct formula ($y = x^2e^{-|x|}$) but did not recognize this as correct and proceeded without finding another correct formula. Teacher U found $y = x^2/(x^4+1)$ in 4:40 min. The other two teachers had

Table 5
S's results on task B.

Fragments	encoding
'You need an asymptote and at $x = 0$ the graph is at zero; that [means] the graph is symmetrical, so I do something with x^2'	D1
'First I look for something that starts at 0 for positive values, increases and then will decrease to 0; it is hard' [pause] 'Around here is a maximum' 'You can do something with x^2; $(x-1)^2$ has a minimum; the 1/... has a maximum '[Writes down $1/(x-1)^2$]	D2
'But then it will go in this direction and at $x = 0$ is not 0'	E2
'Thus if I multiply by x [writes down $y = x/(x-1)^2$] 'then at $x = 0$ it will be 0and if x goes to infinity it will still go to 0'	D2
'This seems correct; I do x^2; so $y = x^2/(x-1)^2$	D2
'No that is not correct'	E2
íThen instead of a minus I make + because otherwise I will get an asymptote [writes $y = x^2/(x+1)^2$]í	D2
íLet's see; again this is not correct; I want a maximum at $x = 1$ not at $x = 0$; If I look at $x/(x+1)$; $x + 1/x$?, [pause], this is not correctí	E2
'Oh, this one might perhaps correct after all: $y = x^2/(x+1)^2$; seems OK; fine function'	E2

Table 6
Results for task B.

Experts																
P	C1	C2	D2	E2												
Q	D1	C1	C2	E2												
R	B1	E2	D2	E2	C1/2	D2	E2									
S	D1	D2	E2	D2	D2	E2	D2	E2	E2							
T	B1	E2	D2	E2	D2	E2	C1/2	D2	E2	D2	D2	E2	E3	D2	D2	E2
Teachers																
U	D1	D2	E3	E2	D2	E2										
V	D1	D2	E2	D2	E2	D2										
W	D1	D2	E2	D2	E2											

Table 7
Comparison of strategies on tasks A and B.

Expert	Task A strategy	Time	Task B strategy	Time
P	$\frac{1}{2}-\frac{1}{2}$	7:50	1 – 0	3:43
Q	1 – 0	4:00	1 – 0	1:40
R	$\frac{1}{2}-\frac{1}{2}$	4:48	$\frac{1}{2}-\frac{1}{2}$	2:32
S	0 – 1	1:45	0 – 1	3:32
T	0 – 1	6:26 (P)	$\frac{1}{2}-\frac{1}{2}$	6:26 (X)
U	0 – 1	3:00	0 – 1	2:36
V	0 – 1	10:35 (P)	0 – 1	6:22 (P)
W	0 – 1	3:45 (P)	0 – 1	1:14 (X)

1 – 0 indicates that the recognition strategy was used.
0 – 1 indicates that the recognition strategy was not used, only analysis of the algebraic formula.
$\frac{1}{2}-\frac{1}{2}$ indicates that the participant started with an analysis of the algebraic formula and later switched to recognition strategies.
"X": the problem was not solved correctly.
"P": the solution was only partially correct.

more difficulties: V found $y=|x|/e^{|x|}$ in 6:20 min and W presented $y=(\frac{1}{2})^{x^2}$ within 1:15 min as a formula, which would fit the graph.

So, overall, most experts used encodings at level C, in which the graph is decomposed into subgraphs (expert R), or decomposed by describing subgraphs (experts P, Q, T). The strategies used most frequently, and used by every participant, were those of adjusting a formula to characteristics via qualitative reasoning, followed by checking a formula via qualitative reasoning.

4.3. Comparing choices of strategies

In both tasks, it was possible to use recognition strategies (levels A, B, C) or do an analysis of the algebraic formula (levels D, E, F). Fig. 3 and Table 6 show whether a participant used a recognition strategy or not. In this way, strategies on the two tasks can be compared (Table 7).

Most participants were consistent in their choice of strategy on both tasks; they used either recognition strategies or analysis of algebraic formulas (see Table 7). Both strategies can be successful and can give fast results.

The videotapes show that all participants started without any hesitations. They appear to recognize the type of task and do not seem to consider any general problem-solving heuristics.

5. Conclusions and discussion

Learners have difficulties reading algebraic formulas and the underlying process-object duality of functions. In graphing a formula, they have to read the formula and to use both a process and an object perspective. It was largely unknown what are effective and efficient strategies for graphing formulas. The purpose of our research was to identify a framework of strategies involving graphing formulas and to describe experts' strategies in graphing formulas.

5.1. Conclusions

The first aim of our research was to identify a framework with which we could describe formula-graphing strategies appropriately and discriminatively. We consider the framework *appropriate* if all strategies used by participants can be encoded in it. The results show that all statements from the protocols of all eight participants could actually be encoded within the two-dimensional framework. Therefore, we conclude that strategies used by the participants in our tasks can be described appropriately within the two-dimensional framework. The framework is discriminative if different strategies used by participants result in different paths in the framework. Fig. 3 shows the differences and similarities in strategies, which also appear in the protocols. The videotapes and the protocols show that Q is more straightforward in his strategy choices, works faster and in a more straightforward way, and uses his domain knowledge and skills more efficiently. The different paths of experts P and Q in the framework, resulting from differences in strategy according to the protocols, are clearly seen: Q is faster (4:00 versus 7:50 min), Q's path contains fewer steps, and Q's steps are situated more at the upper side of the framework (indicating more recognition), and more on the left side (indicating stronger heuristics). Therefore, we conclude that the framework is also discriminative.

The second aim of our research was to describe experts' strategies in graphing-formula tasks. The experts in our research used a range of strategies in graphing formulas. Qualitative reasoning and recognizing and using formulas that can be instantly visualized by a graph seem to be the main strategies used by the experts in our research. Table 3 and Table 7 show that some experts focused on recognition and used their repertoire of formulas that can be instantly visualized by a graph. Other experts focused on analysis of algebraic formulas and used their strong heuristics, such as qualitative reasoning about characteristics as, for instance, domain, infinity behavior, and symmetry.

Expertise in graphing formulas does not involve calculations of derivatives. All our experts seemed to hesitate to start such calculations. In addition, when two of them did, they made mistakes.

We formulated task B in order to establish whether expertise can be used functionally. In this task, participants were forced to use their repertoire of formulas that can be instantly visualized. Results from Table 6 show that recognition was used to formulate hypotheses about the formula, and that formula graphing was used to test these hypotheses (E2, E3 in the derived framework). This shows that for task B the same thinking processes were needed as for graphing formulas. Since all fragments from the protocols could actually be encoded within the derived framework, the results for task B are in accordance with the two-dimensional framework.

Although in graphing formulas (task A) some of our experts did not use their repertoire of formulas that can be instantly visualized by a graph, when they were forced to, as they were by task B, most experts showed that they do have a large repertoire and were able to use that repertoire.

Not all our teachers solved these tasks adequately. As expected, teachers' paths are situated on the lower side of the framework, because they did not use high levels of recognition (levels A, B, or C). The variation in the teachers' performances (correct graph/formula, time needed) is large. The performance of the most experienced teacher (U) closely resembles that of expert S. Teacher U can be considered an expert in this domain of graphing formulas. Teacher V produced only partially correct solutions and needed many steps in his solving process, as well as a lot of time. Teacher W worked very fast, used weak heuristics, and produced inaccurate solutions.

5.2. Discussion

In literature, several aspects are mentioned which are important for graphing formulas (see Section 2) but there was still no framework to describe strategies involving graphing formulas. In our framework knowledge about expertise, about recognition, and about heuristic search for graphing formulas is integrated. From expertise literature, it is found that recognition and heuristic search are two components of expertise. To investigate these components we used theory about reading formulas and the process and object perspectives.

For recognition a repertoire of basic functions (Eisenberg & Dreyfus, 1994), symbol sense (Arcavi, 1994), decomposition of algebraic expressions into smaller expressions (Ernest, 1990), and the classification of function-families are considered important aspects. Different levels of awareness have been formulated by Mason (2003). In this research, these aspects have been combined into a scale of recognition, from complete recognition to no recognition at all.

For heuristic search, we use aspects of process and object perspectives. The process perspective in which a function is seen as a calculation rule in which x and y values are linked, gives often only local information about the graph and is called a pointwise approach by Even (1993, whereas the global approach (Even, 1993) or covariational reasoning (Carlson et al., 2002; Oehrtman et al., 2008) gives information about a function's behavior on an interval or in a global way. We consider heuristics that result in information about intervals as strong and heuristics that result in local information as weak. In this way, heuristics are ordered into a scale of heuristics, form strong to weak.

We found that the framework did cover the strategies used by the participants. However, not all strategies in the framework were found in the participants' protocols (see Fig. 3). An explanation might be the limited number of experts and teachers that could be included in the study. This limitation is due to the labor-intensive method for strategy assessment. Further research, with a larger group of experts and teachers, may provide more information about the strategies used in graphing formulas. In addition, students should be included so that the lower range of the framework, i.e., levels E and F, may be explored further.

Another aspect that could have influenced the strategies we found can be the choice of the formula and the graph used in tasks A and B. For instance, recognition level D ("characteristic aspect of graph is recognized; rest of graph is unknown") was found only once. In the earlier pilot interviews this level was used by experts in the case of rational functions such as $y = x + 4/x$ and $y = 4/(x^2 - 4)$. Future research, involving other functions, such as these rational functions, can provide information on whether alternative strategies not mentioned in the framework are used regularly.

Reflecting on the results of task A, we were surprised by the form of the paths in the framework (Fig. 3). Although we expected the experts' paths in the framework to be located predominantly in the upper and left range of the framework, the paths of the experts S and T were situated in the lower range of the framework and the paths of experts P and R started in the lower range of the framework. Task A seems to have triggered our experts' strong heuristics and not their repertoire of formulas that can be instantly visualized by a graph. When a task has been practiced many times, a program of self-instruction can be developed. The familiar task triggers a set of personal metacognitive instructions, evoking specific activities (Veenman, Van Hout-Wolters, & Afflerbach, 2006). Although this can help in problem solving, routine can also be a risk. For chess, Saariluoma (1992) found that strong players tend to choose stereotyped solutions and sometimes miss non-typical, shorter solutions. Perhaps this can explain why some of our experts did not use their repertoire of formulas that can be instantly visualized by a graph.

In this research, a framework with hierarchies of recognition and heuristic search in graphing formulas has been defined. This framework can be used to assess expertise in graphing formulas. In this way, teachers' and students' current strategies can be compared with expert strategies. In addition, the framework might be used to indicate a development trajectory for teaching efficient strategies in graphing formulas. Further research is necessary to elucidate if, and how, the framework can indeed be helpful in learning and teaching to graph formulas.

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