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Nonstandard Methods in the Theory of Ultrafilters

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Introduction

Ultrafilters are mathematical objects that appear in many fields of mathematics. They are perhaps mostly used in measure theory, model theory and topology. Ultrafilters are therefore often used to connect the theories of those fields. Because ultrafilters are used in so many different ways, there are also many ways to give a motivation behind their definition. One option is to describe them as a way of deciding which subsets of a set are to be called "large", as we will do in the introduction of Chapter 1.

The theory of ultrafilters has expanded vastly since Tarski proved their existence in [1]. A part of that theory will be covered in Chapter 1. We can for example take the image of an ultrafilter under a function, or put a preorder on them. Furthermore, we have a way of defining a topology and an algebraic structure on the space $\beta\mathbb{N}$ of ultrafilters on the natural numbers, that have some interesting properties. Again, the resulting space is also useful in other fields of mathematics, as combinatorial number theory and Ramsey theory.

A prominent application of ultrafilters is the construction of nonstandard analysis, an attempt of Abraham Robinson to revive the pre-Weierstraß analysis with *infinitesimals* in a mathematically rigorous way. In nonstandard analysis, the set of real numbers is extended by adding infinitely small and infinitely large numbers. In Chapter 2, we will construct a limited version of nonstandard analysis, just rich enough that it provides us with infinitely large versions of natural numbers, called *hypernatural numbers*. We will prove the existence of such numbers by using the theory of ultrafilters that is presented in Chapter 1. The reason we will only construct this limited version of nonstandard analysis is that it is strong enough to serve as groundwork for the theory that is described in Chapter 3.

Interestingly, the connection between ultrafilters and nonstandard analysis can also be exploited the other way around. This connection will be studied in Chapter 3. In nonstandard analysis we are concerned with certain extensions of sets called *hyper-extensions*. It turns out that the elements of a hyper-extension ${}^*X$ of X correspond to the ultrafilters on X. While this method of viewing elements of hyper-extensions as ultrafilters is not new, it is still an active field of research nowadays. This has given some interesting results in both the theory and application of ultrafilters. For example, it has lead to the discovery of a short and elegant proof of Hindman's theorem, which will be given at the end of Chapter 3.

In summary, this thesis provides a short introduction to both the theory of ultrafilters and nonstandard analysis, in order to apply the latter to the former. The assumed knowledge is naive set theory, some model theory and a tiny bit of topology and group theory.

Finally, I would like to thank my supervisor dr. Jaap van Oosten for suggesting this interesting topic and pointing me in the right direction at some crucial moments in the process of writing my thesis.

Chapter 1

Ultrafilters

In this chapter we introduce the notion of an ultrafilter, and build a part of the theory of ultrafilters. Most of the theory and proofs of this chapter are based on [4], however, here presented in an extended fashion.

A part of the theory presented here will be used in Chapter 2 to construct an ultrapower model of the hypernatural numbers. More importantly, in Chapter 3 we will prove some interesting properties of the theory of this chapter using the nonstandard methods developed in Chapter 2.

1.1 Filters and Ultrafilters

An *ultrafilter* on a set X can be seen as a way of defining which subsets of X are relatively 'large'. This notion of largeness does not have to be based on cardinality. We could for example call a subset of X large whenever it contains a specific element $x \in X$ (these filters are called *principal ultrafilters*). However, there are some essential properties of largeness that we feel should hold regardless of what notion we choose. Firstly, X itself should be large, and $\emptyset$ should not. A second intuition is that when a subset is large, its complement is not. Third, when two subsets of X are large, we feel that their intersection should be as well. This is motivated by the fact that $A \cap B = A \setminus B^c$ and the intuition that a large set minus a small set is large. The fourth and final intuition is that any subset containing a large set is large. As we will see later, the formal definition of an *ultrafilter* encompasses precisely these four intuitions about largeness.

The intuitions list above may raise some justified objections. For instance, two intuitively large subsets of X might share only one element, making their complement quite small. While this intuition has its flaws, and is by no means the sole motivation behind the definition of an ultrafilter, we still feel that keeping it in mind can be useful in furthering the intuitive understanding of ultrafilters. Especially in Chapter 2, when we use ultrafilters to construct ultrapowers, by seeing them as denoting which subsets of a set are 'large' we gain understanding of their role in the construction.

Definition 1.1.1. A *filter* $\mathcal{F}$ on a set X is a subset of $\mathcal{P}(X)$ with the following properties:

1. $X \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$.

2. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.

3. if $A \in \mathcal{F}$ and $B \subseteq X$ such that $A \subseteq B$, then $B \in \mathcal{F}$. ■

An *ultrafilter* is a maximal filter, i.e. a filter that can not be enlarged.

Definition 1.1.2. A family of sets $\mathcal{F}$ has the *finite intersection property* (FIP) if the intersection of any finite collection of elements of $\mathcal{F}$ is nonempty. That is, if for any $A_1, \ldots, A_n \in \mathcal{F}$ it holds that $A_1 \cap \ldots \cap A_n \neq \emptyset$. ■

note that it follows from the first and second defining properties that any filter has the FIP.

In the presence of the Axiom of Choice we can prove that any filter extends to an ultrafilter. In fact, this holds for any family of sets that has the FIP.

Theorem 1.1.3. *Any family $\mathcal{F}$ that has the FIP can be extended to an ultrafilter on X.*

Proof. We will use Zorn's Lemma. Let P be the poset of all subsets of $\mathcal{P}(X)$ that extend $\mathcal{F}$ and have the FIP, ordered by inclusion. Clearly P is nonempty, since $\mathcal{F} \in P$. If C is a nonempty chain in P, then $\bigcup C$ is an upper bound for C. It holds that $\mathcal{F} \subseteq \bigcup C$ and $\bigcup C$ has the FIP, since for every $A_1, \ldots, A_n \in \bigcup C$, there is $\mathcal{G} \in P$ with the FIP such that $A_1, \ldots, A_n \in \mathcal{G}$. It follows that $\bigcup C \in P$. Hence, by Zorn's Lemma, P has a maximal element $\mathcal{H}$. We claim that $\mathcal{H}$ is an ultrafilter on X. By the maximality of $\mathcal{H}$ we have $X \in \mathcal{H}$. The fact that $\mathcal{H}$ has the FIP implies that $\emptyset \notin \mathcal{H}$. Suppose that $A, B \in \mathcal{H}$ such that $A \cap B \notin \mathcal{H}$, then $\mathcal{H} \cup \{A \cap B\}$ extends $\mathcal{H}$ and has the FIP. This contradicts the maximality of $\mathcal{H}$. Similarly, adding $B \supseteq A$ to $\mathcal{H}$ for $A \in \mathcal{H}$ can not break the FIP. Indeed, if $B \cap A_1 \cap \ldots \cap A_n = \emptyset$, then $A \cap A_1 \cap \ldots \cap A_n = \emptyset$. We see that the maximality of $\mathcal{H}$ implies that $\mathcal{H}$ is a filter, and therefore an ultrafilter. □

We can now prove that the single intuition about largeness that we formulated in the introduction, but that is not included in the definition of a filter, exactly characterizes the ultrafilters.

Proposition 1.1.4. *A filter $\mathcal{F}$ on X is an ultrafilter if and only if for all $A \subseteq X$, either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

Proof. $\Rightarrow$. Let $A \subseteq X$. Clearly it is not the case that $A, A^c \in \mathcal{F}$, since that would imply that $A \cap A^c = \emptyset \in \mathcal{F}$. Now suppose wlog that $A \notin \mathcal{F}$. The family $\mathcal{F} \cup \{A\}$ can not have the FIP, since then (by the previous Theorem) we would be able to extend it to an ultrafilter that strictly extends the ultrafilter $\mathcal{F}$. Therefore, there are $F_1, \ldots, F_n \in \mathcal{F}$ such that $A \cap F_1 \cap \ldots \cap F_n = \emptyset$. We obtain the desired result by noting that $A^c \supseteq F_1 \cap \ldots \cap F_n \in \mathcal{F}$, and thus $A^c \in \mathcal{F}$.

$\Leftarrow$. We extend $\mathcal{F}$ to a ultrafilter $\mathcal{F}'$. If $\mathcal{F}$ is not maximal, then there is a set $A \in \mathcal{F}' \setminus \mathcal{F}$. We then have $A^c \in \mathcal{F} \subset \mathcal{F}'$, so $A \cap A^c = \emptyset \in \mathcal{F}'$, a contradiction. □

The following Lemma follows from the previous proposition.

Lemma 1.1.5. *Let $\mathcal{F}$ be an ultrafilter on a set X. If $A \in \mathcal{F}$ such that $A = A_1 \cup \ldots \cup A_n$ is a union of disjoint sets, then it holds that $A_i \in \mathcal{F}$ for precisely one $i \in 1, \ldots, n$.*

Proof. Clearly, there is at most one $A_i \in \mathcal{F}$, since the intersection of two A_i's is the empty set. To show that there is also at least one $A_i \in \mathcal{F}$, suppose by contradiction that $A_1, \ldots, A_n \notin \mathcal{F}$. Then, by the previous proposition, we would have that $A_1^c, \ldots, A_n^c \in \mathcal{F}$. But then it would also follow that $A_1^c \cap \ldots \cap A_n^c = (A_1 \cup \ldots \cup A_n)^c = A^c \in X$, contradicting the previous proposition. □

We can divide the ultrafilters on X into two categories. The ultrafilters that contain a singleton, i.e. $\{x\} \in \mathcal{F}$ for a $x \in X$, are called *principal*. All ultrafilters that do not have this property fall into the second category and are called *nonprincipal*. It is easily seen that the principal ultrafilter $\mathcal{F}$ on X that contains $\{x\}$ is exactly the set $\mathcal{F} = \{A \subseteq X \mid x \in A\}$. Therefore, the principal ultrafilters are precisely determined by the singleton they contain. The following proposition shows that the principal ultrafilters are precisely those that contain finite sets.

Proposition 1.1.6. *An ultrafilter is principal if and only if it contains a finite set.*

Proof. One direction is trivial. The other direction follows easily from Lemma 1.1.5. Let $\mathcal{F}$ be an ultrafilter and let $A = \{a_1, \ldots, a_n\} \in \mathcal{F}$ be a finite set. We have the disjoint union of singletons

$$\{a_1\} \cup \ldots \cup \{a_n\} \in \mathcal{F}.$$

By Lemma 1.1.5 one of those singletons is in $\mathcal{F}$. □

Since every ultrafilter on X contains X, there can be no nonprincipal ultrafilters on finite sets.

We end this section with a proof that on any infinite set, there is a nonprincipal ultrafilter. Note that this proof is dependent on the Axiom of Choice, since it hinges on Theorem 1.1.3.

Proposition 1.1.7. *On any infinite set there is a nonprincipal ultrafilter.*

Proof. Let X be an infinite set. The set of cofinite subsets of X, i.e. the set $\{A \subseteq X \mid A^c \text{ is finite}\}$, has the FIP. By Theorem 1.1.3, we can extend this set into an ultrafilter containing no finite sets. It follows from the previous proposition that this ultrafilter is nonprincipal. □

1.2 The Rudin-Keisler ordering

In this section we will describe a common ordering on the class of ultrafilters. This ordering first requires the following definition.

Definition 1.2.1. Let $\mathcal{F}$ be an ultrafilter on a set X. Then, for a function $f : X \to Y$, the *image* of $\mathcal{F}$ under f is the ultrafilter on Y given by

$$f(\mathcal{F}) = \{A \subseteq Y \mid f^{-1}(A) \in \mathcal{F}\}.$$ ■

It is easily verified that $f(\mathcal{F})$ is in fact an ultrafilter.

Now we can define the *Rudin-Keisler ordering* on the class of ultrafilters.

Definition 1.2.2. Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on the sets X and Y respectively. Then we say that $\mathcal{G}$ is *Rudin-Keisler below* $\mathcal{F}$, and denote this by $\mathcal{G} \leq_{RK} \mathcal{F}$, if and only if there is a function $f : X \to Y$ such that $f(\mathcal{F}) = \mathcal{G}$. ■

When there is such function f that is a bijection, we say that $\mathcal{F}$ and $\mathcal{G}$ are *isomorphic*. Furthermore, we say that $\mathcal{F}$ and $\mathcal{G}$ are *Rudin-Keisler equivalent* when $\mathcal{G} \leq_{RK} \mathcal{F}$ and $\mathcal{F} \leq_{RK} \mathcal{G}$. In Chapter 3 we will use nonstandard methods to prove that two ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on the natural numbers are Rudin-Keisler equivalent precisely when they are isomorphic.

For the sake of completeness we will show a crucial property of the Rudin-Keisler ordering. Recall that a preorder is a relation that is reflexive and transitive.

Proposition 1.2.3. *$\leq_{RK}$ is a preorder.*

Proof. Reflexivity can be trivially shown by using the identity function.

In order to prove transitivity, we first show a general property of the image of the image of an ultrafilter. Let $\mathcal{F}$ be an ultrafilter on the set X and let $f : X \to Y$ and $g : Y \to Z$ be functions. Then

$$\begin{aligned} g(f(\mathcal{F})) &= \{A \subseteq Z \mid g^{-1}(A) \in f(\mathcal{F})\} \\ &= \{A \subseteq Z \mid g^{-1}(A) \in \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{F}\}\} \\ &= \{A \subseteq Z \mid f^{-1}(g^{-1}(A)) \in \mathcal{F}\} \\ &= \{A \subseteq Z \mid (g \circ f)^{-1}(A) \in \mathcal{F}\} = (g \circ f)(\mathcal{F}). \end{aligned}$$

Now, let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be ultrafilters on the sets X, Y, Z respectively, such that $\mathcal{H} \leq_{RK} \mathcal{G} \leq_{RK} \mathcal{F}$. Then there are functions $f : X \to Y,\ g : Y \to Z$ such that $\mathcal{G} = f(\mathcal{F})$ and $\mathcal{H} = g(\mathcal{G}) = g(f(\mathcal{F})) = (g \circ f)(\mathcal{F})$. The final equality shows that $\mathcal{H} \leq_{RK} \mathcal{F}$. □

1.3 Cartesian and Tensor Products

In this section we will consider ultrafilters on a Cartesian product of sets. Given two ultrafilters on sets X and Y respectively, we can form filters on the Cartesian product $X \times Y$ in two different ways. The most simple way is the following.

Definition 1.3.1. Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on the sets X and Y respectively. Then the *Cartesian product* of $\mathcal{F}$ and $\mathcal{G}$ is the filter on $X \times Y$

$$\mathcal{F} \times \mathcal{G} = \{A \times B \mid A \in \mathcal{F},\ B \in \mathcal{G}\}.$$ ■

The properties of the Cartesian product on sets directly guarantee that the Cartesian product of two (ultra)filters is in fact a filter. We will see below, however, that it is not necessarily the case that the Cartesian product of two ultrafilters is itself an ultrafilter. The second method, defined directly below, does have that property.

Definition 1.3.2. Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on the sets X and Y respectively. Then the *tensor product* $\mathcal{F} \otimes \mathcal{G}$ is the filter on $X \times Y$ defined by setting

$$A \in \mathcal{F} \otimes \mathcal{G} \Leftrightarrow \{x \mid \{y \mid (x, y) \in A\} \in \mathcal{G}\} \in \mathcal{F}.$$

It is the case that $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F} \otimes \mathcal{G}$. Indeed, suppose that $A \times B \in \mathcal{F} \times \mathcal{G}$. Then for all $x \in A$, we have $\{y \mid (x, y) \in A \times B\} = B \in \mathcal{G}$. It follows that $\{x \mid \{y \mid (x, y) \in A \times B\} \in \mathcal{G}\} \supseteq A \in \mathcal{F}$ and thus $A \times B \in \mathcal{F} \otimes \mathcal{G}$.

As said before, it turns out that, unlike the Cartesian product, every tensor product $\mathcal{F} \otimes \mathcal{G}$ is an ultrafilter. Most of the literature leaves the verification of this fact to the reader. Since tensor products can be difficult to understand, we will prove said property in the following lemma.

Lemma 1.3.3. *For any two ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on the sets X and Y respectively, the tensor product $\mathcal{F} \otimes \mathcal{G}$ is an ultrafilter on $X \times Y$.*

Proof. First we will go trough the definition step-by-step to show that $\mathcal{F} \otimes \mathcal{G}$ is a filter.

1. Note that $\{x \mid \{y \mid (x, y) \in \emptyset\} \in \mathcal{G}\} = \{x \mid \emptyset \in \mathcal{G}\} = \emptyset \notin \mathcal{F}$ and thus $\emptyset \notin \mathcal{F} \otimes \mathcal{G}$.
   Furthermore, for any $x \in X$ we have $\{y \mid (x, y) \in X \times Y\} = Y \in \mathcal{G}$. This means that $\{x \mid \{y \mid (x, y) \in X \times Y\} \in \mathcal{G}\} = X \in \mathcal{F}$, hence $X \times Y \in \mathcal{F} \otimes \mathcal{G}$.

2. Let $A, B \in \mathcal{F} \otimes \mathcal{G}$. Note that for any ultrafilter $\mathcal{U}$ and sets V, W we have $V, W \in \mathcal{U} \Leftrightarrow V \cap W \in \mathcal{U}$. We have the following chain of equivalences:

$$\begin{aligned}
&A \cap B \in \mathcal{F} \otimes \mathcal{G} \\
&\Leftrightarrow \{x \mid \{y \mid (x, y) \in A \cap B\} \in \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x, y) \in A\} \cap \{y \mid (x, y) \in B\} \in \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x, y) \in A\}, \{y \mid (x, y) \in B\} \in \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x, y) \in A\} \in \mathcal{G}\} \cap \{x \mid \{y \mid (x, y) \in B\} \in \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x, y) \in A\} \in \mathcal{G}\}, \{x \mid \{y \mid (x, y) \in B\} \in \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow A, B \in \mathcal{F} \otimes \mathcal{G}.
\end{aligned}$$

   Following the chain backwards gives the required result.

3. Let $A \in \mathcal{F} \otimes \mathcal{G}$ and $B \subseteq X \times Y$ such that $A \subseteq B$. Then for all $x \in X$, we have that $\{y \mid (x, y) \in A\} \subseteq \{y \mid (x, y) \in B\}$. From the fact that $\mathcal{G}$

is an ultrafilter it follows that

$$\{x \mid \{y \mid (x,y) \in A\} \in \mathcal{G}\} \subseteq \{x \mid \{y \mid (x,y) \in B\} \in \mathcal{G}\}.$$

We conclude, by the fact that $\mathcal{F}$ is an ultrafilter, that whenever we have $\{x \mid \{y \mid (x,y) \in A\} \in \mathcal{G}\} \in \mathcal{F}$, we have $\{x \mid \{y \mid (x,y) \in B\} \in \mathcal{G}\} \in \mathcal{F}$.

To show that $\mathcal{F} \otimes \mathcal{G}$ is an ultrafilter, we use Proposition 1.1.4 in both directions. Let $A \subseteq X \times Y$, the following chain of equivalences shows that either A or A^c is in $\mathcal{F} \otimes \mathcal{G}$.

$$\begin{aligned}
A \in \mathcal{F} \otimes \mathcal{G} &\Leftrightarrow \{x \mid \{y \mid (x,y) \in A\} \in \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x,y) \in A\}^c \notin \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x,y) \in A^c\} \notin \mathcal{G}\} \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x,y) \in A^c\} \in \mathcal{G}\}^c \in \mathcal{F} \\
&\Leftrightarrow \{x \mid \{y \mid (x,y) \in A^c\} \in \mathcal{G}\} \notin \mathcal{F} \Leftrightarrow A^c \notin \mathcal{F} \otimes \mathcal{G}.
\end{aligned}$$

□

As promised, the following corollary of the previous lemma shows that the Cartesian product of two ultrafilters need not be itself an ultrafilter.

Corollary 1.3.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be nonprincipal ultrafilters on the set $\mathbb{N}$. Then the Cartesian product $\mathcal{F} \times \mathcal{G}$ is* ***not*** *an ultrafilter.*

Proof. Consider the set $\Delta^+ = \{(n,m) \mid n < m\}$. By the nonprincipality of $\mathcal{G}$, we have for all $n \in \mathbb{N}$, that $\{m \mid (n,m) \in \Delta^+\} = [n+1, \infty] \in \mathcal{G}$. That means that $\{n \mid \{m \mid (n,m) \in \Delta^+\} \in \mathcal{G}\} = \mathbb{N} \in \mathcal{F}$ and thus $\Delta^+ \in \mathcal{F} \otimes \mathcal{G}$. However, $\Delta^+ \notin \mathcal{F} \times \mathcal{G}$, since Δ^+ can not be the Cartesian product of two sets. To see this, note that $(0,1), (1,2) \in \Delta^+$, but not $(1,1) \in \Delta^+$. We obtain the result by recalling that $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F} \otimes \mathcal{G}$ and thus $\mathcal{F} \times \mathcal{G}$ is not maximal. □

With this we conclude this section and chapter. The reader might feel left with some questions. For instance, one might wonder if the tensor product is commutative, i.e. if $\mathcal{F} \otimes \mathcal{G} = \mathcal{G} \otimes \mathcal{F}$ for all ultrafilters $\mathcal{F}, \mathcal{G}$.

The reason that these questions are left unanswered is that the purpose of this chapter is to present a basic version of the theory of ultrafilters, such that we can prove some facts *about* this theory using nonstandard methods. This will be done in Chapter 3, where also the question about the commutativity of $\otimes$ will be answered.

In the next chapter we will use ultrafilters to construct the *hypernatural numbers*.

Chapter 2

The Hypernatural Numbers

In 1961 Abraham Robinson presented the theory he named *nonstandard analysis*. His goal was to construct a method of doing analysis that is both intuitive and mathematically rigorous. Before this new approach, there were two methods of analysis.

The oldest method, that was used by Newton and Leibniz, is characterized by using the notion of infinitesimal small change, denoted by dx. While this method is quite intuitive, it lacks a rigorous basis.

The other method, invented by Weierstraß, is based on the ϵ-δ definition of a limit. This method *has* a rigorous foundation, but is far less easy to use. Results that were already 'known' by Newtons and Leibniz' method can be very hard to proof using Weierstraß's method.

Robinsons solution is to extend the real numbers to a set containing numbers that are larger than all standard real numbers. He calls these numbers *infinite* numbers. The resulting set is called the set of *hyperreals*.

The hyperreals are constructed in such a way, that they behave the same way as the standard real numbers. This property is called *the transfer principle* and will be defined in Section 2. Since, for example, the standard reals form a field under ordinary addition and multiplication, the hyperreals do as well.

From this particular fact, it follows that the hyperreals do not only contain infinitely large numbers, but also their inverses, i.e. infinitely small numbers. These infinitely small numbers are the key elements in performing analysis using the hyperreals.

In this chapter we will be concerned not with the set of hyperreals, but with one of its subsets, called the *hypernatural numbers*. Analogously, this set contains the standard natural numbers, together with *infinite* numbers that are larger than every standard natural number.

There are several different methods of constructing the hypernatural numbers. We choose the method using ultrapowers, which are defined in the first section. In the second section we will use these ultrapowers to construct the hypernatural numbers. Finally, in Section 3, we will give a general definition of a *model of the hypernatural numbers* and show an important property such models may have.

Throughout this chapter we will use the model theoretic notions as defined in Chapter 1 of [9]. Consequently, the exact definitions used here of languages of first order logic and of their interpretations can be found in said text.

2.1 Ultraproducts

The *ultraproduct construction* is a way of forming a structure from a certain family of structures in the same language. It has many applications in model theory. Perhaps the two most important are its use in the direct proof of the compactness theorem (see, for instance, [3, Corollary 4.1.11]) and in the construction of nonstandard models. The latter is, of course, our purpose in this chapter.

We will first define the notion of an *ultraproduct* on sets, after which we will define it on structures.

Definition 2.1.1. Let $\{A_i\}_{i\in I}$ be a family of sets and let $\mathcal{F}$ be an ultrafilter on I. Then the *ultraproduct* $\prod_{\mathcal{F}} A_i$ is the quotient set

$$\prod_{\mathcal{F}} A_i = \prod_{i\in I} A_i/{\sim},$$

where $\sim$ is the equivalence relation given by

$$\vec{a} \sim \vec{b} \Leftrightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{F},$$

for all $\vec{a}, \vec{b} \in \prod_{i\in I} A_i$. ■

The special case in which all A_i are equal, i.e. in which there is an A such that $A = A_i$ for all $i \in I$, is also called an *ultrapower*. This will also be denoted $\prod_{\mathcal{F}} A$ or $A^{|I|}/\mathcal{F}$.

Recall the intuition behind ultrafilters as determining which subsets are 'large'. We have seen that this intuition is not entirely correct, but in the context of ultrafilters it can be quite illuminating. Two elements of the

Cartesian product are equal up-to-$\sim$ if and only if their indices are equal in a 'large' part of the individual sets. It is for this reason that we will only consider ultraproducts under nonprincipal ultrafilters. For, suppose we use a principal ultrafilter, i.e. an ultrafilter $\mathcal{F}$ with $\{x\} \in \mathcal{F}$. Then the ultraproduct is equal to A_x.

For the sake of completeness, we will show that $\sim$ is, in fact, an equivalence relation.

Lemma 2.1.2. $\sim$ *is an equivalence relation.*

Proof. We will go through the properties of an equivalence relation one-by-one.

1. $\vec{a} \sim \vec{a} \Leftrightarrow \{i \in I \mid a_i = a_i\} = I \in \mathcal{F}$. This holds by the first property of the definition of a filter.

2. $\vec{a} \sim \vec{b} \Leftrightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{F} \Leftrightarrow \{i \in I \mid b_i = a_i\} \in \mathcal{F} \Leftrightarrow \vec{b} \sim \vec{a}$.

3. Using both the second and third properties of the definition of a filter, we obtain:

$$\begin{aligned}\vec{a} \sim \vec{b} \text{ and } \vec{b} \sim \vec{c} &\Rightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{F} \text{ and } \{i \in I \mid b_i = c_i\} \in \mathcal{F} \\ &\Rightarrow (\{i \in I \mid a_i = b_i\} \cap \{i \in I \mid b_i = c_i\}) \in \mathcal{F} \\ &\Rightarrow \{i \in I \mid a_i = b_i = c_i\} \in \mathcal{F} \\ &\Rightarrow \{i \in I \mid a_i = c_i\} \in \mathcal{F} \Rightarrow \vec{a} \sim \vec{c}.\end{aligned}$$

□

We will denote the equivalence class of an element $\vec{a}$ under $\sim$ by $[\vec{a}]_{\mathcal{F}}$.

In the following definition we extend the definition of ultrapowers to structures. Recall that for a language $\mathcal{L}$, an $\mathcal{L}$-structure is a non-empty set, together with interpretations of the constants, function symbols and relation symbols.

Definition 2.1.3. Let $\{\mathcal{M}_i \mid i \in I\}$ be a family of $\mathcal{L}$-structures and let $\mathcal{F}$ be an ultrafilter on I. Then the *ultraproduct* is the set $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{M}_i$, together with the following interpretations.

- For each constant c of $\mathcal{L}$:

$$c^{\mathcal{M}} = [\vec{a}]_{\mathcal{F}} :\Leftrightarrow \{i \in I \mid c^{\mathcal{M}_i} = a_i\} \in \mathcal{F}.$$

- For each n-place function symbol f of $\mathcal{L}$:

$$f^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) = [\vec{b}]_{\mathcal{F}} :\Leftrightarrow \{i \in I \mid f^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = b_i\} \in \mathcal{F}.$$

- For each n-place relation symbol R of $\mathcal{L}$:

$$([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) \in R^{\mathcal{M}} :\Leftrightarrow \{i \in I \mid (a_i^1, \ldots, a_i^n) \in R^{\mathcal{M}_i}\} \in \mathcal{F}.$$

Note that we write a_i^1 for the ith element of $\vec{a}_1$. ■

Again for the sake of completeness, we will show that the ultraproduct of structures is well-defined. That is, that for every ultrafilter $\mathcal{F}$, each family of $\mathcal{L}$-structures has a unique ultraproduct.

Lemma 2.1.4. *Let $\{\mathcal{M}_i\}_{i\in I}$ be a family of $\mathcal{L}$-structures and let $\mathcal{F}$ be an ultrafilter on I. Then there is a unique $\mathcal{L}$-structure $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{M}_i$.*

Proof. We have already seen that $\mathcal{F}$ induces an equivalence relation, so the quotient set is well-defined. What is left is to show that the interpretations are well-defined. We need to show that for each interpretation there *exists* an equivalence class in $\mathcal{M}$ which works, that it works *independent* of the choice of representative and that this equivalence class is *unique*. We will do this one-by-one.

1. *Constants.*
   (a) *Existence.* It holds for $[(c^{\mathcal{M}_1}, c^{\mathcal{M}_2}, \ldots)]_{\mathcal{F}} \in \mathcal{M}$.
   (b) *Independence.* Pick any $\vec{a} \in \prod_{i\in I} \mathcal{M}_i$ such that its equivalence class $[\vec{a}]_{\mathcal{F}} = [(c^{\mathcal{M}_1}, c^{\mathcal{M}_2}, \ldots)]$. Then, by the equivalence relation, we have $\{i \in I \mid c^{\mathcal{M}_i} = a_i\} \in \mathcal{F}$.
   (c) *Uniqueness.* Suppose there are $[\vec{a}]_{\mathcal{F}}, [\vec{b}\,]_{\mathcal{F}} \in \mathcal{M}$, such that we have $\{i \in I \mid c^{\mathcal{M}_i} = a_i\}, \{i \in I \mid c^{\mathcal{M}_i} = b_i\} \in \mathcal{F}$. Then, by closure under intersection, we have $\{i \in I \mid c^{\mathcal{M}_i} = a_i = b_i\} \in \mathcal{F}$. It follows by closure under extension that $\{i \in I \mid a_i = b_i\} \in \mathcal{F}$. We conclude that $[\vec{a}]_{\mathcal{F}} = [\vec{b}\,]_{\mathcal{F}}$.
2. *Function symbols.*
   (a) *Existence.* Let $[\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}} \in \mathcal{M}$. Then it holds for

$$[\vec{b}\,]_{\mathcal{F}} = [(f^{\mathcal{M}_1}(a_1^1, \ldots, a_n^1), f^{\mathcal{M}_1}(a_1^2, \ldots, a_n^2), \ldots)]_{\mathcal{F}} \in \mathcal{M}.$$

   (b) *Independence.* Let a_i and b as above and pick any $\vec{u}_i$ and $\vec{v}$ such that $[\vec{v}\,]_{\mathcal{F}} = [\vec{b}\,]_{\mathcal{F}}$ and $[\vec{u}_i]_{\mathcal{F}} = [\vec{a}_i]_{\mathcal{F}}$ for all $i = 1, \ldots, n$. Then, by the equivalence relation, we have

$$\{i \in I \mid f^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = v_i\} \in \mathcal{F}.$$

And thus, by closure under intersection, we have

$$(\{i \in I \mid f^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = v_i\} \cap \bigcap_{1 \leq k \leq n} \{i \in I \mid a_i^k = u_i^k\}) \in \mathcal{F}.$$

By closure under extensions, we obtain

$$\{i \in I \mid f^{\mathcal{M}_i}(u_i^1, \ldots, u_i^n) = v_i\} \in \mathcal{F}.$$

(c) *Uniqueness.* Suppose that for $[\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}$, $[\vec{b}]_{\mathcal{F}}$ and $[\vec{c}]_{\mathcal{F}}$, we have

$$\{i \in I \mid f^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = b_i\} \in \mathcal{F} \text{ and}$$
$$\{i \in I \mid f^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = c_i\} \in \mathcal{F}.$$

Then, by the same considerations as with constants, we obtain that $[\vec{b}]_{\mathcal{F}} = [\vec{c}]_{\mathcal{F}}$.

3. *Relation symbols.* In this case it is not necessary to show existence and uniqueness. There may be no elements in $R^{\mathcal{M}}$, or there may be multiple. We *do* need to show that the property of 'being an element of $R^{\mathcal{M}}$' for any equivalence class in $\mathcal{M}$, is independent of the representative we use to verify it.

   (a) *Independence.* Let $\vec{a}_1, \ldots \vec{a}_n, \vec{b}_1, \ldots, \vec{b}_n \in \prod_{i \in I} \mathcal{M}_i$ such that $[\vec{a}_i]_{\mathcal{F}} = [\vec{b}_i]_{\mathcal{F}}$ for all $i = 1, \ldots, n$. Then, again by the closure of ultrafilters under intersection and extension, we have

$$\begin{aligned} [\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}} \in R^{\mathcal{M}} &\Leftrightarrow \{i \in I \mid (a_i^1, \ldots, a_i^n) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \\ &\Leftrightarrow \{i \in I \mid (b_i^1, \ldots, b_i^n) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \\ &\Leftrightarrow [\vec{b}_1]_{\mathcal{F}}, \ldots, [\vec{b}_n]_{\mathcal{F}} \in R^{\mathcal{M}} \end{aligned}$$

□

We have seen that in ultraproducts of sets, the elements of the Cartesian product are equal precisely when they are equal according to a 'large' part of the individual sets. The amazing thing about ultraproducts is that this also holds for ultraproducts of $\mathcal{L}$-structures and first-order $\mathcal{L}$-sentences. That is, an $\mathcal{L}$-sentence is true in the ultraproduct if and only if it is true in a 'large' part of the individual $\mathcal{L}$-structures. This is what makes ultraproducts so useful in model theory, as it can be used for instance in proving the compactness theorem. The statement was first proved by Polish mathematician

Jerzy Łoś and we will prove it here as well. Since the proof is quite extensive, we will build towards the theorem with the following propositions.

First, we will show that it holds for the interpretation of terms. That is, that terms in an ultraproduct have a certain interpretation, precisely when they have that interpretation in a 'large' part of the individual structures.

Proposition 2.1.5. *Let $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{M}_i$ be the ultraproduct of a collection of $\mathcal{L}$-structures $\{\mathcal{M}_i\}_{i\in I}$. Then for any $\mathcal{L}$-term t with n variables, it holds that*

$$t^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) = [\vec{b}]_{\mathcal{F}} \Leftrightarrow \{i \in I \mid t^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = b_i\} \in \mathcal{F},$$

for all $[\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}} \in \mathcal{M}$.

Proof. We will prove this by induction on the complexity of closed $\mathcal{L}$-terms. As induction base we note that if t is a constant, then the thesis holds by definition.

For the induction step, suppose that $t(x_1, \ldots, x_n)$ is of the form

$$f(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n)),$$

such that the thesis holds for $t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n)$. Then, by the definition of the interpretation of terms, the statement

$$t^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) = [\vec{b}]_{\mathcal{F}} \tag{2.1}$$

is equivalent to

$$f^{\mathcal{M}}(t_1^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}), \ldots, t_m^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}})) = [\vec{b}]_{\mathcal{F}}. \tag{2.2}$$

For each $k = 1, \ldots, m$, let

$$\vec{u}_k := (t_k^{\mathcal{M}_1}(a_1^1, \ldots, a_1^n),\ t_k^{\mathcal{M}_2}(a_2^1, \ldots, a_2^n), \ldots).$$

It follows that

$$\{i \in I \mid t_k^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = u_i^k\} = I \in \mathcal{F}.$$

Again, we write u_i^k for the ith element of $\vec{u}_k$. By the induction hypothesis, we obtain that $t_k^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) = [u_k]_{\mathcal{F}}$ for all $k = 1, \ldots, m$. Therefore, the following statement is equivalent to 2.2.

$$f^{\mathcal{M}}([\vec{u}_1]_{\mathcal{F}}, \ldots, [\vec{u}_m]_{\mathcal{F}}) = [\vec{b}]_{\mathcal{F}}. \tag{2.3}$$

By the definition of the interpretation of function symbols in ultraproducts, this is equivalent to

$$\{i \in I \mid f^{\mathcal{M}_i}(u_i^1, \ldots, u_i^m) = b_i\} \in \mathcal{F}. \tag{2.4}$$

Using the definition of u_k, this is equivalent to

$$\{i \in I \mid f^{\mathcal{M}_i}(t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n), \ldots, t_m^{\mathcal{M}_i}(a_1^1, \ldots, a_1^n)) = b_i\} \in \mathcal{F}. \tag{2.5}$$

Again using the definition of the interpretation of terms, but now in the opposite direction, we obtain a final equivalent statement:

$$\{i \in I \mid t^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = b_i\} \in \mathcal{F}. \tag{2.6}$$

The equivalence (2.1) $\Leftrightarrow$ (2.6) is our required equivalence. □

Now that we have shown this property of terms, it is quite easy to show that Łoś's Theorem holds in the special case of atomic formulas. Formally,

Proposition 2.1.6. *Let $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{M}_i$ be the ultraproduct of a collection of $\mathcal{L}$-structures $\{\mathcal{M}_i\}_{i \in I}$. Then for every atomic $\mathcal{L}$-formula φ with n free variables, it holds that*

$$\mathcal{M} \models \varphi([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F},$$

for all $[\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}} \in \mathcal{M}$.

Proof. We separate two cases. In each we use the definition of the interpretation of atomic formulas.

Case 1. $\varphi(x_1, \ldots, x_n)$ is of the form $t_1(x_1, \ldots, x_n) = t_2(x_1, \ldots, x_n)$ for terms t_1, t_2 with n variables. Then, by Proposition 2.1.5, we have

$$\begin{aligned}
&\mathcal{M} \models \varphi([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) \\
&\quad \Leftrightarrow \mathcal{M} \models t_1([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) = t_1([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) \\
&\quad \Leftrightarrow t_1^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) = t_2^{\mathcal{M}}([\vec{a}_1]_{\mathcal{F}}, \ldots, [\vec{a}_n]_{\mathcal{F}}) \\
&\quad \Leftrightarrow \{i \in I \mid t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = t_2^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F},
\end{aligned}$$

as desired.

Case 2. $\varphi(x_1, \ldots, x_n)$ is of the form $R(t_1(x_1, \ldots, x_n), \ldots, t_m(x_1, \ldots, x_n))$ for some relation symbol R of the language $\mathcal{L}$. Then, again by Proposition 2.1.5, we have

$$\begin{aligned}
&\mathcal{M} \models \varphi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\quad \Leftrightarrow \mathcal{M} \models R(t_1([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}), \ldots, t_m([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F})) \\
&\quad\quad \Leftrightarrow (t_1^\mathcal{M}([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}), \ldots, t_m^\mathcal{M}([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F})) \in R^\mathcal{M}
\end{aligned}$$

Now let u_k be as in the proof of Proposition 2.1.5 for all $k = 1, \ldots, m$. Then we continue the chain of equivalences as follows

$$\begin{aligned}
&\Leftrightarrow ([\vec{u}_1]_\mathcal{F}, \ldots, [\vec{u}_m]_\mathcal{F}) \in \mathcal{R}^\mathcal{M} \\
&\quad \Leftrightarrow \{i \in I \mid (u_i^1, \ldots, u_i^m) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \\
&\quad\quad \Leftrightarrow \{i \in I \mid (t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n), \ldots, t_m^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)) \in R^{\mathcal{M}_i}\} \in \mathcal{F} \\
&\quad\quad\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F},
\end{aligned}$$

and we obtain our required equivalence. □

Now that we have found it to hold for the atomic formulas, we can use that as induction base in proving that it holds for *all* formulas.

Theorem 2.1.7 (Łoś's Theorem)**.** *Let $\mathcal{M} = \prod_\mathcal{F} \mathcal{M}_i$ be the ultraproduct of a collection of $\mathcal{L}$-structures $\{\mathcal{M}_i\}_{i \in I}$. Then for every $\mathcal{L}$-formula φ with n free variables, it holds that*

$$\mathcal{M} \models \varphi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F},$$

for all $[\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F} \in \mathcal{M}$.

Proof. We will prove this by induction on the complexity of $\mathcal{L}$-formulas. Proposition 2.1.6 is our induction base. Since the connectives $\{\neg, \wedge, \exists\}$ are sufficient to write all $\mathcal{L}$-sentences (up to logical equivalence), we only have to perform the induction step for those connectives.

Suppose that $\varphi(x_1, \ldots, x_n)$ is of the form $\neg\psi(x_1, \ldots, x_n)$, such that the thesis holds for ψ. Then we obtain the required result as follows.

$$\begin{aligned}
\mathcal{M} \models \varphi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) &\Leftrightarrow \mathcal{M} \not\models \psi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \psi(a_i^1, \ldots, a_i^n)\} \notin \mathcal{F} \\
&\Leftrightarrow \{i \in I \mid \mathcal{M}_i \not\models \psi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F},
\end{aligned}$$

where in the third equivalence, we use Proposition 1.1.4. It is worth noting that this is the first time in this section that we have used the fact that $\mathcal{F}$ is an ultrafilter, and not merely a filter.

Suppose that $\varphi(x_1, \ldots, x_n)$ is of the form $\psi(x_1, \ldots, x_n) \wedge \phi(x_1, \ldots, x_n)$, such that the thesis holds for ϕ and ψ. Then we have

$$\begin{aligned}
&\mathcal{M} \models \varphi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\quad \Leftrightarrow \mathcal{M} \models (\psi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \wedge \phi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F})) \\
&\quad \Leftrightarrow \mathcal{M} \models \psi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \text{ and } \mathcal{M} \models \phi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \psi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\qquad\qquad \text{and } \{i \in I \mid \mathcal{M}_i \models \phi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \psi(a_i^1, \ldots, a_i^n) \text{ and } \mathcal{M}_i \models \phi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models (\psi(a_i^1, \ldots, a_i^n) \wedge \mathcal{M}_i \models \phi(a_i^1, \ldots, a_i^n))\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n)\} \in \mathcal{F},
\end{aligned}$$

as desired. Note that we used the fact that ultrafilters are closed under intersection and extension. Combining these properties gives us the property that for all A, B,

$$A \cap B \in \mathcal{F} \Leftrightarrow A \in \mathcal{F} \text{ and } B \in \mathcal{F}.$$

Finally, suppose that $\varphi(x_1, \ldots, x_n)$ is of the form $\exists x \psi(x, x_1, \ldots, x_n)$, such that the thesis holds for ψ. We then have

$$\begin{aligned}
&\mathcal{M} \models \varphi([\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\quad \Leftrightarrow \mathcal{M} \models \exists x \psi(x, [\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\quad \Leftrightarrow \text{ there exists an } [\vec{a}_0]_\mathcal{F} \in \mathcal{M} \text{ s.t. } \mathcal{M} \models \psi([\vec{a}_0]_\mathcal{F}, [\vec{a}_1]_\mathcal{F}, \ldots, [\vec{a}_n]_\mathcal{F}) \\
&\quad \Leftrightarrow \text{ there exists an } \vec{a}_0 \in \prod_{i \in I} \mathcal{M}_i \text{ s.t. } \{i \in I \mid \mathcal{M}_i \models \psi(a_i^0, a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \text{ there exists an } a_i \in \mathcal{M}_i \text{ s.t. } \mathcal{M}_i \models \psi(a_i^0, a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \exists x \psi(x, a_i^1, \ldots, a_i^n)\} \in \mathcal{F} \\
&\quad \Leftrightarrow \{i \in I \mid \mathcal{M}_i \models \varphi(a_i^1, \ldots, a_i^n\} \in \mathcal{F},
\end{aligned}$$

where the third equivalence holds due to the induction hypothesis and the fourth due to the fact that we can pick $(a_1^0, a_2^0, \ldots) := \vec{a}_0$ or $\vec{a}_0 := (a_1^0, a_2^0, \ldots)$, depending on which direction we take in the bi-implication. This completes the induction. □

Recall that an ultrapower is an ultraproduct $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{M}_i$, such that each $\mathcal{M}_i$ is equal. That is, such that there is an $\mathcal{N}$ for which $\mathcal{M}_i = \mathcal{F}$ for all $i \in I$. Such ultrafilter is denoted $\prod_{\mathcal{F}} \mathcal{N}$. We have the following natural way of assigning to each $a \in \mathcal{N}$ an element $[\vec{a}]_{\mathcal{F}} \in \mathcal{M}$.

Definition 2.1.8. Let $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{N}$ be an ultrapower. The *natural embedding* of $\mathcal{N}$ into $\mathcal{F}$ is the function $d : \mathcal{N} \to \mathcal{M}$ such that

$$d(a) = [(a, a, a, \ldots)]_{\mathcal{F}},$$

for all $a \in \mathcal{N}$. ■

We obtain the following corollary of Łoś's Theorem.

Corollary 2.1.9. *Let $\mathcal{M} = \prod_{\mathcal{F}} \mathcal{N}$ be an ultrapower. Then the natural embedding $d : \mathcal{N} \to \mathcal{M}$ is an elementary embedding.*

Proof. Let φ be a formula with n free variables and let $a_1, \ldots, a_n \in \mathcal{N}$. Then, by Łoś's Theorem, we have the following chain of equivalences.

$$\begin{aligned} \mathcal{M} \models \varphi(d(a_1), \ldots, d(a_n)) &\Leftrightarrow \{i \in I \mid \mathcal{N} \models (a_1, \ldots, a_n)\} \in \mathcal{F} \\ &\Leftrightarrow \mathcal{N} \models \varphi(a_1, \ldots, a_n). \end{aligned}$$

The fact that d also preserves interpretations follows trivially from the definitions, making it an elementary embedding. □

2.2 Ultrapower Construction of the Hypernatural Numbers

In this section we will consider a specific ultraproduct construction. Precisely, we will consider the family consisting of $\mathbb{N}$ to the power $\mathbb{N}$. The resulting ultraproduct will be an instance of a *model of the hypernatural numbers*, of which a formal definition will be given in the next section.

The ultrapower construction of nonstandard models is primarily used for nonstandard analysis. Recall from the introduction of this chapter that nonstandard analysis requires the set of *hyperreals*. Therefore, in the literature we mostly see the ultrapower $\mathbb{R}^{\mathbb{N}}$. We, however, only need the hypernatural numbers so for us the ultrapower $\mathbb{N}^{\mathbb{N}}$ is sufficient.

We consider the language of first order logic $\mathcal{L}_{\mathbb{N}}$ with the following symbols (the idea for using this language is from [8]).

- For all $n \in \mathbb{N}$ a constant symbol c_n.

- For all $n > 0$ and every function $f : \mathbb{N}^n \to \mathbb{N}$ an n-ary function symbol F_f.
- For all $n > 0$ and every subset $X \subseteq \mathbb{N}^n$ an n-ary relation symbol P_X.

Furthermore, we consider the $\mathcal{L}_{\mathbb{N}}$-structure $\mathcal{N} = \mathbb{N}$, together with the following, natural interpretations.

- For constant c_n, we let $c_n^{\mathcal{N}} = n$.
- For every function symbol F_f, we let $F_f^{\mathcal{N}} = f$.
- For every relation symbol P_X, we let $P_X^{\mathcal{N}} = X$.

Let $\mathcal{F}$ be a fixed ultrafilter on $\mathbb{N}$, which, as we have seen in Chapter 1, exists. We define $\mathfrak{N} := \prod_{\mathcal{F}} \mathcal{N}$. That is, $\mathfrak{N}$ is the $\mathcal{L}_{\mathbb{N}}$-structure that is the ultrapower of $\mathcal{N}$ to the power $\mathbb{N}$. Since we are working with a fixed ultrafilter $\mathcal{F}$, we will omit that specification in the notation of the equivalence classes, i.e. we will write $[\vec{a}]$ instead of $[\vec{a}]_{\mathcal{F}}$.

We denote the set of elements of $\mathfrak{N}$ by ${}^*\mathbb{N}$. The set ${}^*\mathbb{N}$ consists of equivalence classes of sequences of natural numbers. For example, we have $[(4, 4, 4, \ldots)] \in {}^*\mathbb{N}$ and $[(0, 1, 2, \ldots)]$ in ${}^*\mathbb{N}$. By Corollary 2.1.9 the natural embedding $d : \mathbb{N} \to {}^*\mathbb{N}$ is an elementary embedding. When we restrict the codomain of d to its image, i.e. to the set $\{[(n, n, n, \ldots)] \mid n \in \mathbb{N}\}$ we obtain an isomorphism between $\mathbb{N}$ and a subset of ${}^*\mathbb{N}$. This allows us to talk about corresponding elements in $\mathbb{N}$ and ${}^*\mathbb{N}$ interchangeably. We will exploit this fact by renaming the elements $[(n, n, n, \ldots)]$ to just n. As a result, the set $\mathbb{N}$ will from now on be a subset of ${}^*\mathbb{N}$.

The elements in ${}^*\mathbb{N} \cap \mathbb{N}$ are called *finite* numbers. An elementary result is given by the following proof of the existence of *infinite* numbers.

Proposition 2.2.1. *There exists an infinite number $\alpha \in {}^*\mathbb{N}$*

Proof. Let $\vec{a} = (1, 2, 3, \ldots)$ and $\alpha = [\vec{a}] \in {}^*\mathbb{N}$. Then for all $n \in \mathbb{N}$ it holds that $\{i \in \mathbb{N} \mid a_i = n\} = \{n\}$. Since $\mathcal{F}$ is an nonprincipal ultrafilter, it follows that $[\vec{a}] \neq [(n, n, n, \ldots)]$, i.e. $\alpha \neq n$. Therefore α is not equal to any finite number, and thus α is infinite. □

Now Łoś's Theorem guarantees us that $\mathbb{N}$ and ${}^*\mathbb{N}$ behave in the same way. Take for for example the fact that $\mathbb{N}$ is an ordered semigroup. This is axiomatized by the following $\mathcal{L}_{\mathbb{N}}$-sentences.

- $\forall a \forall b \forall c((a + b) + c = a + (b + c))$;

- $\forall a \forall b \forall c((a \leq b) \rightarrow (a+c) \leq (b+c))$;
- $\forall a \forall b \forall c((a \leq b) \rightarrow (c+a) \leq (c+b))$.

Note that we write the $\mathcal{L}_{\mathbb{N}}$-sentences in a simplified (readable) form. The first two axioms are formally written as

- $\forall a \forall b \forall c(F_+(F_+(a,b),c) = F_+(a, F_+(b,c)))$;
- $\forall a \forall b \forall c(P_{\leqslant}(a,b) \rightarrow P_{\leqslant}(F_+(a,c), F_+(b,c)))$,

where $+ : \mathbb{N}^2 \rightarrow \mathbb{N}$ is the function of ordinary addition on $\mathbb{N}$ and $\leqslant = \{(n,m) \mid n \leq m\} \subseteq \mathbb{N}^2$ and $\leq$ is the ordinary order of $\mathbb{N}$. Since these sentences hold for $\mathcal{N}$ they, by Łoś's Theorem, also hold for $\mathfrak{N}$.

In particular, the natural numbers are a *natural* ordered semigroup. That means, among other things, that they are well-ordered by $\leq$. That is, any non-empty subset of $\mathbb{N}$ has a least element. One may think that, by Łoś's Theorem, this also applies to the hypernatural numbers. However, consider the following axiomatization of well-orderedness.

- $(\forall X \subseteq \mathbb{N})(X \neq \emptyset \rightarrow (\exists x \in X)(\forall y \in X)(x \leq y))$

We can not apply Łoś's Theorem to this axiom, since it can not be written in the language $\mathcal{L}_{\mathbb{N}}$. Indeed, this sentence quantifies over *sets* and is therefore second order, while $\mathcal{L}_{\mathbb{N}}$ is a language in first order logic, quantifying over *elements* of $\mathbb{N}$.

An important note is that an expression in the form $(\exists x \in X)$, i.e. a bounded quantifier, *can* be we written in the language $\mathcal{L}_{\mathbb{N}}$. Suppose, for instance that we would want to express the sentence "all x in X are also in Y". Naively we would write $(\forall x \in X)(x \in Y)$, but formally we have the $\mathcal{L}_{\mathbb{N}}$ sentence $\forall x(P_X(x) \rightarrow P_Y(x))$.

We will distinguish between *informally written* $\mathcal{L}_{\mathbb{N}}$-formulas, where functions and relations will be written with their usual symbols, and *formally written* $\mathcal{L}_{\mathbb{N}}$-formulas, which are strictly in the language $\mathcal{L}_{\mathbb{N}}$. Informally written formulas are in a sense hybrids between the syntactic formulas and their semantic interpretations. The syntactical expression $P_X(x)$ is interpreted as $x \in X$. Informally it is also written as $x \in X$. The informally written formulas only have a rigorous basis insofar as they can also be written formally.

We see that we can not derive the well-orderedness of ${}^*\mathbb{N}$ from the well-orderedness of $\mathbb{N}$ by using Łoś's Theorem. We can then ask ourselves: is there another way to the derive the well-orderedness of ${}^*\mathbb{N}$? The answer is no, since ${}^*\mathbb{N}$ is *not* well-ordered by $\leq$ (and first order logic is sound). A proof of this result will be given later in this section.

First we will elaborate further on the connection between the natural and the hypernatural numbers. We have already seen that each number $n \in \mathbb{N}$ has a counterpart in ${}^*\mathbb{N}$. This counterpart is not so special, since we have defined it to be equal to n.

In line with the tradition of nonstandard analysis, we call the elements in $\mathbb{N}$ *standard*. All other numbers, i.e. those in ${}^*\mathbb{N} \setminus \mathbb{N}$ are *nonstandard*. For numbers the property of being standard coincides precisely with the property of being finite.

However, the standard numbers are not the only standard objects under consideration. The language $\mathcal{L}_{\mathbb{N}}$ also provides us with relation and function symbols that allow us to express statements about functions $f : \mathbb{N}^n \to \mathbb{N}$ and sets $X \subseteq \mathbb{N}^n$. Later in this section we will see under which condition such functions and sets are called standard.

It is a convention in nonstandard analysis to talk about certain maps, called *star-maps*, that map every standard object to its counterpart object in ${}^*\mathbb{N}$, called its *hyper-extension*. We will now construct such a star-map ourselves.

On the domain of numbers our star-map coincides with the natural embedding of $\mathbb{N}$ into ${}^*\mathbb{N}$. We write ${}^*n$ for the star-map applied to n. It follows that ${}^*n = n$ for all $n \in \mathbb{N}$ (recall that we renamed the equivalence classes corresponding to natural numbers). This demonstrates that a hyper-extension need not be nonstandard.

On the domain of functions and sets we define the star-map as follows.

- For all $f : \mathbb{N}^k \to \mathbb{N}$, we let

$$
{}^*f([\vec{a}_1], \ldots, [\vec{a}_k]) = [\vec{b}\,] \Leftrightarrow \{n \in \mathbb{N} \mid f(a^1_n, \ldots, a^k_n) = b_n\} \in \mathcal{F}.
$$

- For all $X \subseteq \mathbb{N}^k$, we let

$$
([\vec{a}_1], \ldots, [\vec{a}_k]) \in {}^*X \Leftrightarrow \{n \in \mathbb{N} \mid (a^1_n, \ldots, a^k_n) \in X\} \in \mathcal{F}.
$$

As in the previous section, we write a^k_n for the nth elements of $\vec{a}_k$.

Showing that the star-map is well-defined for these objects is analogous to the way it was done in Lemma 2.1.4.

Notice that the standard numbers are precisely those that are the hyper-extension of some $n \in \mathbb{N}$. The *standard* functions and *standard* sets are defined analogously. For instance, a function $f' : ({}^*\mathbb{N})^n \to {}^*\mathbb{N}$ is called *standard* if there is some $f : \mathbb{N} \to \mathbb{N}$, such that $f' = {}^*f$.

Next to numbers, sets, and functions, there is one more class of objects in the domain of the star-map. These objects will be considered later in this section. First let us prove two properties of the hyper-extensions of sets.

Proposition 2.2.2. *Let* $X_1, \ldots, X_k \subseteq \mathbb{N}$. *Then*

$$^*(X_1 \times \ldots \times X_k) = {}^*X_1 \times \ldots \times {}^*X_k.$$

In particular, it follows that $({}^*\mathbb{N})^k = {}^*(\mathbb{N}^k)$.

Proof. We have the following chain of equivalences.

$$
\begin{aligned}
&([\vec{a}_1], \ldots, [\vec{a}_k]) \in {}^*(X_1 \times \ldots \times X_k) \\
&\quad \Leftrightarrow \{n \in \mathbb{N} \mid (a_n^1, \ldots, a_n^k) \in X_1 \times \ldots \times X_k\} \in \mathcal{F} \\
&\quad\quad \Leftrightarrow \{n \in \mathbb{N} \mid a_n^1 \in X_1\} \cap \ldots \cap \{n \in \mathbb{N} \mid a_n^k \in X_k\} \in \mathcal{F} \\
&\quad\quad\quad \Leftrightarrow \{n \in \mathbb{N} \mid a_n^1 \in X_1\}, \ldots, \{n \in \mathbb{N} \mid a_n^k \in X_1\} \in \mathcal{F} \\
&\quad\quad\quad\quad \Leftrightarrow [\vec{a}_1] \in {}^*X_1, \ldots, [\vec{a}_k] \in {}^*X_k \Leftrightarrow ([\vec{a}_1], \ldots, [\vec{a}_k]) \in {}^*X_1 \times \ldots \times {}^*X_k.
\end{aligned}
$$

For $X_i = \mathbb{N}$ for $i = 1, \ldots, k$ we obtain the particular case. □

Proposition 2.2.3. *Let* $X \subseteq \mathbb{N}^k$. *Then* $X = {}^*X$ *if and only if* X *is finite.*

Proof. Let $X_i := \{x_i \mid \vec{x} \in X\}$ for all $i \in 1, \ldots, k$. It follows that that $X \subseteq X_1 \times \ldots \times X_k$. We will prove both sides of the bi-implication separately.

$\Rightarrow$. Suppose that $X = {}^*X$ and suppose, by contradiction, that X is infinite. Then there is an i such that $1 \leq i \leq k$ for which X_i is infinite.

We can order the elements of X_i by the ordinary ordering $\leq$ such that we have $x_0 \leq x_1 \leq x_2 \leq \ldots$ for $x_0, x_1, x_2, \ldots \in X_i$.

By similar reasoning as in the proof of Proposition 2.2.1, we obtain that there is an $\alpha := [(x_0, x_1, x_2, \ldots)] \in {}^*X_i \setminus X_i$. Pick $(a_1, \ldots, a_k) \in X$. It follows $(a_1, \ldots, a_{i-1}, \alpha, a_{i+1}, \ldots, a_k) \in {}^*X \setminus X$, a contradiction.

$\Leftarrow$. Suppose that X is finite. Then X_i is finite for all all $1 \leq i \leq k$. We will show that $X_i = {}^*X_i$ for arbitrary i, and thus that $X = {}^*X$.

Let $[\vec{a}] \in {}^*X_i$. Then we have $\{n \in \mathbb{N} \mid a_n \in X_i\} \in \mathcal{F}$. We denote the elements in X_i by $x_0, \ldots, x_j$ (order them by, say, the ordinary ordering $\leq$). It follows that

$$\{n \in \mathbb{N} \mid a_n \in X_i\} = \{n \in \mathbb{N} \mid a_n = x_1\} \cup \ldots \cup \{n \in \mathbb{N} \mid a_n = x_j\}.$$

Note that the set that is the left part of the equation is in $\mathcal{F}$, and the right part is a union of disjoint sets. By Lemma 1.1.5, we obtain that the set $\{n \in \mathbb{N} \mid a_n = x\} \in \mathcal{F}$ for some $x \in X_i$. We conclude that $[\vec{a}] = x \in X_i$. Therefore $X_i = {}^*X_i$ and we are done. □

It follows from the previous proposition that standard sets do not need to contain standard numbers. In fact, any infinite standard set contains nonstandard numbers.

The final class of objects in the domain of the star-map consists of a subset of $\mathcal{L}_{\mathbb{N}}$-sentences. Precisely, those $\mathcal{L}_{\mathbb{N}}$-sentences that contain only *bounded quantifiers*. That means that every quantifier is of the form $(\exists x \in X)$ or $(\forall x \in X)$ for some variable x and set X. As we have seen, this is formally written as $\exists x(P_X(x) \to \ldots)$ and $\forall x(P_X(x) \to \ldots)$.

The star-map is applied to the *informal* versions of $\mathcal{L}_{\mathbb{N}}$-sentences. Let φ be an informally written $\mathcal{L}_{\mathbb{N}}$ sentence with bounded quantifiers. Then we let the hyper-extension ${}^*\varphi$ of φ be the same sentence, where each number, function, and set is replaced by its hyper-extension.

For example, the sentence

$$\varphi : (\forall x \in X)(x \in Y \to f(x) = a)$$

becomes under the star-map

$${}^*\varphi : (\forall x \in {}^*X)(x \in {}^*Y \to {}^*f(x) = {}^*a).$$

note that when we write φ formally, we obtain the following sentence.

$$\varphi : \forall x(P_X \to (P_Y \to P_Y(F_f(x) = a))).$$

We aim to show that the informally written hyper-extension ${}^*\varphi$ of every informally written $\mathcal{L}_{\mathbb{N}}$-sentence φ are the equal when written formally. For then, we can use Łoś's Theorem to show that φ is true in $\mathbb{N}$, precisely when ${}^*\varphi$ is true in ${}^*\mathbb{N}$. In the tradition of nonstandard analysis this fact is called *transfer* and it is proved in the following theorem.

Theorem 2.2.4 (The Transfer Principle). *Let φ be an informally written $\mathcal{L}_{\mathbb{N}}$-sentence with bounded quantifiers. Then it holds that*

$$\mathbb{N} \models \varphi \iff {}^*\mathbb{N} \models {}^*\varphi.$$

Proof. The first thing to show is that the informally written φ and ${}^*\varphi$ are (syntactically) equal when formally written. The cases of numbers, functions, and sets will be handled separately. We will only show that it holds for an example in each case as that could convince the reader of the truth of this theorem. A fully rigorous proof would involve a more formal definition of informally written sentences, and yet another induction on the complexity of $\mathcal{L}_{\mathbb{N}}$-formulas.

Numbers. We will demonstrate this by looking at the formally written sentence $\varphi : a = a$. We have

$$\begin{aligned} \mathbb{N} &\models \varphi \Leftrightarrow a = a && \text{and} \\ {}^*\mathbb{N} &\models \varphi \Leftrightarrow [(a, a, a, \ldots)] = [(a, a, a, \ldots)] \Leftrightarrow {}^*a = {}^*a. \end{aligned}$$

Thus φ is written informally in $\mathbb{N}$ as $a = a$ and in ${}^*\mathbb{N}$ as ${}^*a = {}^*a$. This indeed corresponds with taking the hyper-extension of φ.

Functions. Let formally written $\varphi : F_f(a) = b$. This implies that

$$\begin{aligned} \mathbb{N} &\models \varphi \Leftrightarrow f(a) = b && \text{and} \\ {}^*\mathbb{N} &\models \varphi \Leftrightarrow [(f(a), f(a), f(a), \ldots)] = [(b, b, b, \ldots)] \Leftrightarrow {}^*f(a) = {}^*b. \end{aligned}$$

Again, we see that writing φ informally in $\mathbb{N}$ and ${}^*\mathbb{N}$ corresponds to taking the hyper-extension ${}^*\varphi$.

Sets. Let $\varphi : P_X(a)$ when formally written. It follows that

$$\begin{aligned} \mathbb{N} &\models \varphi \Leftrightarrow a \in X && \text{and} \\ {}^*\mathbb{N} &\models \varphi \Leftrightarrow \{n \in \mathbb{N} \mid a \in X\} \in \mathcal{F} \Leftrightarrow [(a, a, a, \ldots)] \in X \Leftrightarrow {}^*a \in X, \end{aligned}$$

as required.

We have (roughly) shown that the star-map is a map between informally written $\mathcal{L}_{\mathbb{N}}$ sentences in $\mathbb{N}$ and ${}^*\mathbb{N}$ that *preserves the underlying formally written sentence.* Therefore, this theorem is nothing more than the special case of Łoś's Theorem for sentences with bounded quantifiers, and is thus true. □

Transfer can be seen as a tool for intuitively exploiting the connection between $\mathbb{N}$ and ${}^*\mathbb{N}$ that was already established by Łoś's Theorem. By way of demonstration of *transfer*, we will show that the infinite numbers are, in fact, larger than the finite numbers.

Proposition 2.2.5. *Let $\alpha \in {}^*\mathbb{N}$ be infinite and $n \in {}^*\mathbb{N}$ be finite. Then it holds that $\alpha > n$.*

Proof. Suppose, by contradiction, that $\alpha \leq n$. For every $m \in \mathbb{N}$ such that $m \leq n$ it holds that $m \in A_n := \{1, \ldots, n\}$. Therefore, the following sentence is true in $\mathbb{N}$.

$$\varphi : (\forall x \in \mathbb{N})((x \leq n) \rightarrow x \in A_n)$$

By *transfer*, the following sentence is true in ${}^*\mathbb{N}$.

$${}^*\varphi : (\forall x \in {}^*\mathbb{N})((x \leq {}^*n) \rightarrow x \in {}^*A_n)$$

We have $\alpha \leq n = {}^*n$, but by Proposition 2.2.3 we have $\alpha \notin A_n = {}^*A_n$, a contradiction with φ. □

As promised, the following proposition shows that the hypernatural numbers are *not* well-ordered by $\leq$.

Proposition 2.2.6. *The set* ${}^*\mathbb{N}\setminus\mathbb{N}$ *of infinite numbers has no least element.*

Proof. Suppose, by contradiction, there is a least element $\alpha \in {}^*\mathbb{N}\setminus\mathbb{N}$. Then, by the previous Proposition, $\alpha > n$ for all $n \in \mathbb{N}$.

Now consider the element $\alpha - 1$. Suppose $\alpha - 1 = k$ for some $k \in \mathbb{N}$. Then $\alpha = k + 1 \in \mathbb{N}$, a contradiction.

We conclude that $\alpha - 1 \in {}^*\mathbb{N} \setminus \mathbb{N}$ and thus that α is *not* the least element. □

2.3 Models of the Hypernatural Numbers

The construction in the previous section is an instance of a *model of the hypernatural numbers*. In this section we will give a (semi-)formal definition of such models and show how two models might differ.

Definition 2.3.1. A *model of the hypernatural numbers* is a triple $\langle *, \mathbb{N}, {}^*\mathbb{N}\rangle$ such that

(HN1) $\mathbb{N} \subset {}^*\mathbb{N}$;

(HN2) ${}^*\mathbb{N} \setminus \mathbb{N} \neq \emptyset$;

(HN3) The star-map $*$ maps each number $n \in \mathbb{N}$, function $f : \mathbb{N}^n \to \mathbb{N}$, set $X \subseteq \mathbb{N}^n$ to a their hyper-extensions ${}^*n \in {}^*\mathbb{N}$, ${}^*f : ({}^*\mathbb{N})^n \to {}^*\mathbb{N}$ and $X \subseteq ({}^*\mathbb{N})^n$;

(HN4) For all $n \in \mathbb{N}$, it holds that ${}^*n = n$;

(HN5) The star-map $*$ also maps sentences φ about $\mathbb{N}$ to their hyper-extensions by replacing every number, function and set by their hyper-extensions. This mapping satisfies **The Transfer Principle**:

$$\mathbb{N} \models \varphi \;\Leftrightarrow\; {}^*\mathbb{N} \models {}^*\varphi.$$ ■

Clearly the ultrapower construction of the previous section provides a model of the hypernatural numbers. (HN1) follows from Corollary 2.1.9 and the renaming of the equivalence classes. (HN2) is proved in Proposition 2.2.1. (HN3) and (HN4) follow directly from the construction of the star-map. (HN5) is proved in Theorem 2.2.4.

Their are many methods of constructing a model of the hypernatural numbers. See [12] for a detailed exposition on some of these methods. The properties (HN1) - (HN5) are shared by all models. Note that Proposition 2.2.5 and Proposition 2.2.6 are proven independently of the ultrapower construction and thus hold for all models of the hypernatural numbers.

There are, however, ways in which two models of the hypernatural numbers may differ. Even when using only the ultrapower construction to construct two models of the hypernatural numbers, they can still differ. This difference is caused by the ultrafilter that is used. We have already seen an example of this: when a principal ultrafilter is used, the resulting ultrapower is not even a model of hypernatural numbers at all, as it does not satisfy (HN1).

There is one particular property in which models of models of the hypernatural numbers may differ that we want to highlight in this section, as it will be used in the next chapter. This difference can be caused by using different methods of construction of the hypernatural numbers as well as by using different nonprincipal ultrafilters in the ultraproduct construction.

Definition 2.3.2. Let κ be an infinite cardinal. A model of the hypernatural numbers is a κ*-enlargement* if for any collection $\mathcal{G}$ of subsets of $\mathbb{N}$ such that $|\mathcal{G}| < \kappa$ and $\mathcal{G}$ has the FIP, it holds that

$$\bigcap\{{}^*X \mid X \in \mathcal{G}\} \neq \emptyset. \qquad \blacksquare$$

We call a model of the hypernatural numbers an *enlargement* if it is a κ-*enlargement* for some $\kappa > \aleph_0$.

It turns out that in models of the hypernatural numbers the property (HN2) is equivalent to being an enlargement. That is, (HN2) could be replaced by the definition of κ-enlargement for some $\kappa > \aleph_0$. This is proved in the following Theorem (the general idea of this proof is from [11]).

Theorem 2.3.3. *Let* $\langle *, \mathbb{N}, {}^*\mathbb{N}\rangle$ *satisfy (HN1,3,4,5). Then it is an enlargement if and only if it satisfies (HN2).*

Proof. $\Rightarrow$. Suppose $\langle *, \mathbb{N}, {}^*\mathbb{N}\rangle$ is an enlargement. Define the family $\mathcal{G}$ by $\mathcal{G} = \{\mathbb{N} \setminus \{n\} \mid n \in \mathbb{N}\}$. Since $|\mathcal{G}| = \aleph_0$, the enlargement property implies that the intersection $\bigcap\{{}^*(\mathbb{N} \setminus \{n\}) \mid n \in \mathbb{N}\} \neq \emptyset$. For any $n \in \mathbb{N}$, by *transfer*, we have ${}^*(\mathbb{N} \setminus \{n\}) = {}^*\mathbb{N} \setminus \{{}^*n\} = {}^*\mathbb{N} \setminus \{n\}$. Therefore, it holds that ${}^*\mathbb{N} \setminus \mathbb{N} = \bigcap\{{}^*(\mathbb{N} \setminus \{n\}) \mid n \in \mathbb{N}\} \neq \emptyset$. It follows that axiom (NS2) holds.

$\Leftarrow$. Suppose that $\langle *, \mathbb{N}, {}^*\mathbb{N}\rangle$ satisfies (HN2). Let $\mathcal{G} = \{X_n \mid n \in \mathbb{N}\}$ be a countable family of subsets of $\mathbb{N}$ that has the FIP. We will show that $\bigcap\{{}^*X \mid X \in \mathcal{G}\} \neq \emptyset$.

If $X_n = \mathbb{N}$ for all $n = \mathbb{N}$, then $\bigcap\{^*X \mid X \in \mathcal{G}\} = {}^*\mathbb{N} \neq \emptyset$. Otherwise, suppose wlog that $0 \notin X_0$.

We define a function $f : \mathbb{N} \to \mathbb{N}$ that maps each $n \in \mathbb{N}$ to an arbitrary element in $X_0 \cap \ldots \cap X_n$ if that set is non-empty, and to 0 otherwise. Since $\mathcal{G}$ has the FIP, we have $f(n) \neq 0$ for all $n \in \mathbb{N}$.

By *transfer* of the sentence $\varphi : (\forall x \in \mathbb{N})(f(x) \neq 0)$, we obtain that ${}^*\varphi : (\forall x \in {}^*\mathbb{N})({}^*f(x) \neq 0)$ holds. By (NS2) there is an $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ and for that α we thus have ${}^*f(\alpha) \neq 0$.

We claim that ${}^*f(\alpha) \in \bigcap\{^*X_n \mid n \in \mathbb{N}\}$, or equivalently that ${}^*\alpha \in X_n$ for all $n \in \mathbb{N}$. To see this, pick n arbitrarily. Then by transfer on the true sentence

$$\varphi : (\forall x \in \mathbb{N})((x \geq n \wedge f(x) = 0) \to f(x) \in X_n)$$

we obtain

$${}^*\varphi : (\forall x \in {}^*\mathbb{N})((x \geq n \wedge {}^*f(x) = 0) \to f(x) \in X_n).$$

By Proposition 2.2.5, we have that $\alpha \geq n$ for all $n \in \mathbb{N}$. It follows that ${}^*f(\alpha) \in {}^*X_n$ for all $n \in \mathbb{N}$, as required. □

We see that enlargement guarantees the richness of models of hypernatural numbers: κ-enlargement for some $\kappa > \aleph_0$ is even a necessary condition for the existence of infinite numbers.

In particular in the ultrapower construction of the hypernatural numbers, the extent to which the resulting model is an enlargement depends on the ultrafilter that is used. Details about this fall beyond the scope of this thesis, but can be found in [3, p. 300].

In the next chapter we will work in a fixed model of the hypernatural numbers. We will show a way to assign to each hypernatural number an ultrafilter on the natural numbers. Exploiting this connection, we can use what we know about the hypernatural numbers to prove new facts about ultrafilters on $\mathbb{N}$, of which the general theory was presented in Chapter 1.

Chapter 3

Hypernatural Numbers as Ultrafilters

In this chapter we build and explore the connection between ultrafilters on $\mathbb{N}$ and the hypernatural numbers. Once the connection is made, we can use the tools of nonstandard analysis, of which *transfer* is a particularly useful one, to prove some interesting properties of ultrafilters. This chapter owes its name, as well as most of its content to [10]. In all sections, except for Section 2, we will assume that we are working in a model of the hypernatural numbers that is an $\mathfrak{c}^+$-enlargement, where $\mathfrak{c}$ is the cardinality of the continuum and $\mathfrak{c}^+$ is its successor cardinal.

3.1 The u-Equivalence

The following definition describes a way to associate a (not necessarily unique) ultrafilter to each hypernatural number. It is the fundamental building block in our nonstandard theory of ultrafilters on the natural numbers.

Definition 3.1.1. Let $\alpha \in {}^*\mathbb{N}$. We define the *ultrafilter generated by* α as the family

$$\mathfrak{U}_\alpha = \{X \subseteq \mathbb{N} \mid \alpha \in {}^*X\}.$$ ■

That $\mathfrak{U}_\alpha$ is in fact an ultrafilter follows from *transfer*. Indeed, it holds trivially that $\alpha \in {}^*\mathbb{N}$ and $\alpha \notin {}^*\emptyset = \emptyset$. Suppose that $X_1, X_2 \in \mathfrak{U}_\alpha$. Then, by *transfer*, it holds that ${}^*(X_1 \cap X_2) = {}^*X_1 \cap {}^*X_2 \ni \alpha$, and thus $X_1 \cap X_2 \in \mathfrak{U}_\alpha$. Furthermore, suppose that $X_1 \in \mathfrak{U}_\alpha$ and $X_2 \supseteq X_1$. Then, by *transfer*, it holds that ${}^*X_2 \supset {}^*X_1$. Hence, we have $\alpha \in {}^*X_2$, and thus $X_2 \in \mathfrak{U}_\alpha$. Finally, let $X \subseteq \mathbb{N}$ and consider X^c. By *transfer* we obtain that α is in either ${}^*X$ or ${}^*X^c$, as required.

Note that $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$ if and only if for all $X \subseteq \mathbb{N}$ we have the equivalence $\alpha \in {}^*X \Leftrightarrow \beta \in {}^*X$. That is, α and β can not be separated by any hyper-extension.

A first characterization of ultrafilters on $\mathbb{N}$ in terms of hypernatural numbers is given by the following Proposition.

Proposition 3.1.2. *The ultrafilter $\mathfrak{U}_\alpha$ is (non)principal if and only if α is (non)standard.*

Proof. $\mathfrak{U}_\alpha$ is principal if and only if there is an $n \in \mathbb{N}$ such that $\{n\} \in \mathfrak{U}_\alpha$. The following chain of equivalences proves the proposition.

$$\{n\} \in \mathfrak{U}_\alpha \Leftrightarrow \alpha \in {}^*\{n\} = \{n\} \Leftrightarrow \alpha = n. \qquad \square$$

Definition 3.1.3. For any $\alpha, \beta \in {}^*\mathbb{N}$, we say that α and β are *u-equivalent* if they generate the same ultrafilter, i.e. if $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$. We denote the u-equivalence of α and β by $\alpha \underset{u}{\sim} \beta$. For any $\alpha \in {}^*\mathbb{N}$, the u-equivalence class of α is denoted by $u(\alpha) = \{\beta \mid \alpha \underset{u}{\sim} \beta\}$. ■

The following proposition gives us another way to characterize the u-equivalence class of an $\alpha \in {}^*\mathbb{N}$.

Proposition 3.1.4. *Let $\alpha \in {}^*\mathbb{N}$. Then $u(\alpha) = \bigcap\{{}^*X \mid X \in \mathfrak{U}_\alpha\}$.*

Proof. Let $\beta \in u(\alpha)$. Since $\alpha \underset{u}{\sim} \beta$, we have $\mathfrak{U}_\alpha = \mathfrak{U}_\beta$. This implies that α and β can not be separated by any hyper-extension. It follows that

$$\beta \in \bigcap\{{}^*X \mid \beta \in {}^*X\} = \bigcap\{{}^*X \mid \alpha \in {}^*X\} = \bigcap\{{}^*X \mid X \in \mathfrak{U}_\alpha\}.$$

Therefore, we have the inclusion $u(\alpha) \subseteq \bigcap\{{}^*X \mid X \in \mathfrak{U}_\alpha\}$. The converse inclusion is obtained from following the same steps backwards. $\square$

When working in a certain model of the hypernatural numbers that is a $\mathfrak{c}^+$-enlargement, it holds that each ultrafilter is generated by some $\alpha \in {}^*\mathbb{N}$. To illustrate this fact, let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$, then $\mathcal{F}$ has the FIP. Since $\mathcal{F} \subseteq \mathcal{P}(\mathbb{N})$, we have that $|\mathcal{F}| \leq \mathfrak{c} < \mathfrak{c}^+$. The set $\{{}^*X \mid X \in \mathcal{F}\}$ has the same cardinality and, by *transfer*, also has the FIP. By $\mathfrak{c}^+$-enlargement, there is an $\alpha \in \bigcap\{{}^*X \mid X \in \mathcal{F}\}$. It follows that $\mathcal{F} \subseteq \mathfrak{U}_\alpha$, and thus (since $\mathfrak{U}_\alpha$ is a filter and $\mathcal{F}$ is an ultrafilter) we conclude that $\mathcal{F} = \mathfrak{U}_\alpha$.

It follows that there is a 1-1 correspondence between the ultrafilters on $\mathbb{N}$, the set of which is denoted $\beta\mathbb{N}$, and the u-equivalence classes of the hypernatural numbers, i.e. the quotient space ${}^*\mathbb{N}/\underset{u}{\sim}$.

We will often exploit the correspondence that is stated above by picking, for some ultrafilter $\mathcal{F}$, the hypernatural number α that it is generated by. That is, by picking $\alpha \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\alpha$.

A first property we will prove about the u-equivalence, is that it is preserved under standard functions.

Proposition 3.1.5. *Let $\alpha, \beta \in {}^*\mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ be a function. If $\alpha \underset{u}{\sim} \beta$, then ${}^*f(\alpha) \underset{u}{\sim} {}^*f(\beta)$.*

Proof. First note that *transfer* of the true sentence

$$\varphi : (\forall n \in \mathbb{N})(f(n) \in A \leftrightarrow n \in \{n \in \mathbb{N} \mid f(n) \in A\})$$

gives us

$${}^*\varphi : (\forall n \in {}^*\mathbb{N})({}^*f(n) \in {}^*A \leftrightarrow n \in {}^*\{n \in \mathbb{N} \mid f(n) \in A\}).$$

Now let $\alpha \underset{u}{\sim} \beta$ and let f be a function. We will show that ${}^*f(\alpha)$ and ${}^*f(\beta)$ can not be separated by any hyper-extension. For every $A \subseteq \mathbb{N}$, this is shown by the following chain of equivalences:

$$\begin{aligned} {}^*f(\alpha) \in {}^*A &\Leftrightarrow \alpha \in {}^*\{n \in \mathbb{N} \mid f(n) \in A\} \\ &\Leftrightarrow \beta \in {}^*\{n \in \mathbb{N} \mid f(n) \in A\} \Leftrightarrow {}^*f(\beta) \in {}^*A. \end{aligned}$$

We find that ${}^*f(\alpha)$ and ${}^*f(\beta)$ can not be separated by any hyper-extension, and are thus u-equivalent. □

The following Lemma states a well-known combinatorial property of the natural numbers. We will formulate it here, since it will be useful for developing our nonstandard theory of ultrafilters. For a proof we refer the reader to [10]. Although this proof uses ultrafilters, it makes no use of nonstandard methods.

Lemma 3.1.6 ([10], 2.5)**.** *Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n) \neq n$ for all n. Then there exists a 3-coloring $\chi : \mathbb{N} \to \{1, 2, 3\}$ such that $\chi(n) \neq \chi(f(n))$ for all n.*

Using the previous Lemma, we can prove an important property of the u-equivalence.

Theorem 3.1.7. *Let $\alpha \in {}^*\mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that ${}^*f(\alpha) \underset{u}{\sim} \alpha$. Then ${}^*f(\alpha) = \alpha$.*

Proof. Suppose, with the aim of contradiction, that ${}^*f(\alpha) \neq \alpha$. Now let $A = \{n \in \mathbb{N} \mid f(n) \neq n\}$. By *transfer* of the obviously true sentence $(\forall n \in \mathbb{N})(n \in A \leftrightarrow f(n) \neq n)$, we obtain that $\alpha \in {}^*A$. Now let the function $g : \mathbb{N} \to \mathbb{N}$ be such that $g|_A = f$ and $g(n) \neq n$ for all $n \in \mathbb{N}$ and let $B = \{n \in \mathbb{N} \mid f(n) = g(n)\}$. Since $A \subseteq B$, by *transfer* it holds that ${}^*A \subseteq {}^*B$. Therefore, $\alpha \in {}^*A$ implies $\alpha \in {}^*B$. Using *transfer* on the sentence $(\forall n \in \mathbb{N})(n \in B \leftrightarrow f(n) = g(n))$, we find that ${}^*f(\alpha) = {}^*g(\alpha)$.

Now we can apply the previous Lemma on the function g to obtain a 3-coloring $\chi : \mathbb{N} \to \{1, 2, 3\}$ such that $\chi(n) \neq \chi(g(n))$ for all n. Using *transfer* on the distinctive property of χ, we find ${}^*\chi({}^*f(\alpha)) = {}^*\chi({}^*g(\alpha)) \neq {}^*\chi(\alpha)$. Now let X be the set of natural numbers with the same color as α, i.e. $X = \{n \in \mathbb{N} \mid \chi(n) = i\}$, where $i = {}^*\chi(\alpha)$. By *transfer*, $\alpha \in {}^*X$ but ${}^*f(\alpha) \notin {}^*X$, and thus α and ${}^*f(\alpha)$ can be separated by a hyper-extension. This means that ${}^*f(\alpha) \not\underset{u}{\sim} \alpha$, a contradiction. □

The following propositions are other important properties of the u-equivalence.

Proposition 3.1.8. *Let $\alpha \in {}^*A$ for a set $A \subseteq \mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f|_A$ is injective. Then there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that ${}^*f(\alpha) = {}^*\varphi(\alpha)$.*

Proof. If $\alpha \in \mathbb{N}$, then we can define:

$$\varphi(x) = \begin{cases} f(\alpha) & \text{if } x = \alpha \\ \alpha & \text{if } x = f(\alpha) \\ x & \text{else.} \end{cases}$$

If $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $\alpha \in {}^*A \neq A$. By the converse of Proposition 2.2.3, A is infinite. This means we can partition A into disjoint infinite sets B and C such that $A = B \cup C$. Wlog, assume that $\alpha \in {}^*B$. Since $f|_B$ is injective, it is a bijection when we restrict its codomain to its image $f(B)$. Because the complements of both B and $f(B)$ are (countably) infinite, we can extend this bijection to the totality of the natural numbers. □

Proposition 3.1.9. *Let $\alpha, \beta \in {}^*\mathbb{N}$ and $f, g : \mathbb{N} \to \mathbb{N}$ functions such that ${}^*f(\alpha) \underset{u}{\sim} \beta$ and ${}^*g(\beta) \underset{u}{\sim} \alpha$. Then there exists a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that ${}^*\varphi(\alpha) \underset{u}{\sim} \beta$.*

Proof. By Proposition 3.1.5, ${}^*g({}^*f(a)) \underset{u}{\sim} {}^*g(\beta) \underset{u}{\sim} \alpha$. From Theorem 3.1.7 it follows that ${}^*g({}^*f(\alpha)) = \alpha$. Now let $A = \{n \in \mathbb{N} \mid g(f(n)) = n\}$. By *transfer* we have that $\alpha \in {}^*A$. Furthermore, it is the case that $f|_A$ is injective. Indeed, for all $n, m \in A$ it holds that if $f(n) = f(m)$, that then $n = g(f(n)) = g(f(m)) = m$. By the previous Proposition there exists a bijection φ such that ${}^*f(\alpha) = {}^*\varphi(\alpha)$ and by the hypothesis ${}^*\varphi(\alpha) \underset{u}{\sim} \beta$. □

We will now use the above results to prove two fundamental properties of ultrafilters on the natural numbers. Note that there are also proofs available for the following properties that do not make use of the u-equivalence. The reason we prove these properties again, using the u-equivalence, is to further show its connection to the theory of ultrafilters. In order to do this, we first need the following proposition about the image of an ultrafilter generated by a hypernatural number.

Proposition 3.1.10. *Let $\alpha \in {}^*\mathbb{N}$. Then for every function $f : \mathbb{N} \to \mathbb{N}$, the image of the ultrafilter generated by α under the function f is the ultrafilter generated by ${}^*f(\alpha)$. That is, $f(\mathfrak{U}_\alpha) = \mathfrak{U}_{{}^*f(\alpha)}$.*

Proof. We will show that both ultrafilters contain precisely the same subsets of $\mathbb{N}$. For every $A \subseteq \mathbb{N}$, this is shown by the following chain of equivalences.

$$\begin{aligned} A \in f(\mathfrak{U}_\alpha) &\Leftrightarrow f^{-1}(A) \in \mathfrak{U}_\alpha \\ &\Leftrightarrow \alpha \in {}^*(f^{-1}(A)) \\ &\Leftrightarrow {}^*f(\alpha) \in {}^*A \Leftrightarrow A \in \mathfrak{U}_{{}^*f(\alpha)}, \end{aligned}$$

where the third equivalence holds due to *transfer*. □

Theorem 3.1.11. *Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ a function such that $f(\mathcal{F}) = \mathcal{F}$. Then $\{n \in \mathbb{N} \mid f(n) = n\} \in \mathcal{F}$.*

Proof. Since every ultrafilter on $\mathbb{N}$ is generated by some hypernatural number, we can pick $\alpha \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\alpha$. By the previous Proposition, it holds that $\mathfrak{U}_\alpha = \mathcal{F} = f(\mathcal{F}) = f(\mathfrak{U}_\alpha) = \mathfrak{U}_{{}^*f(\alpha)}$. This means that $a \underset{u}{\sim} {}^*f(\alpha)$ and thus by Theorem 3.1.7, we have that ${}^*f(\alpha) = \alpha$. By *transfer*, it holds that $\alpha \in {}^*\{n \in \mathbb{N} \mid f(n) = n\}$ and hence $\{n \in \mathbb{N} \mid f(n) = n\} \in \mathfrak{U}_\alpha = \mathcal{F}$. □

As announced in Chapter 1, we can now prove the following fact about the Rudin-Keisler ordering.

Proposition 3.1.12. *Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on $\mathbb{N}$. Then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic if and only if $\mathcal{G} \leq_{RK} \mathcal{F}$ and $\mathcal{F} \leq_{RK} \mathcal{G}$.*

Proof. $\Rightarrow$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be a bijection such that $\varphi(\mathcal{F}) = \mathcal{G}$. Then, as shown in the proof of Proposition 1.2.3, we have that

$$\varphi^{-1}(\mathcal{G}) = \varphi^{-1}(\varphi(\mathcal{F})) = (\varphi^{-1} \circ \varphi)(\mathcal{F}) = \mathrm{id}_{\mathbb{N}}(\mathcal{F}) = \mathcal{F}.$$

$\Leftarrow$. Pick $\alpha, \beta \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\alpha$ and $\mathcal{G} = \mathfrak{U}_\beta$. Then there are functions $f, g : \mathbb{N} \to \mathbb{N}$ such that (using Proposition 3.1.10) we have the equations

$$\mathfrak{U}_{{}^*f(\alpha)} = f(\mathfrak{U}_\alpha) = f(\mathcal{F}) = \mathcal{G} = \mathfrak{U}_\beta \qquad \text{and}$$
$$\mathfrak{U}_{{}^*g(\beta)} = g(\mathfrak{U}_\beta) = g(\mathcal{G}) = \mathcal{F} = \mathfrak{U}_\alpha.$$

It follows that ${}^*f(\alpha) \underset{u}{\sim} \beta$ and ${}^*g(\beta) \underset{u}{\sim} \alpha$. Applying Proposition 3.1.9, we obtain a bijection $\varphi : \mathbb{N} \to \mathbb{N}$ such that ${}^*\varphi(\alpha) \underset{u}{\sim} \beta$. We conclude that $\varphi(\mathcal{F}) = \varphi(\mathfrak{U}_\alpha) = \mathfrak{U}_{{}^*\varphi(\alpha)} = \mathfrak{U}_\beta = \mathcal{G}$. □

3.2 The S-Topology

Abraham Robinson, the inventor of nonstandard analysis, introduced the notion of S-topology which we will study in this section. The "S" stands for "standard", since the S-topology on a hyper-extension ${}^*X$ (a standard element) is the topology generated by the basis consisting of hyper-extensions (standard elements) of subsets of X. Precisely,

Definition 3.2.1. Let $X \subseteq \mathbb{N}$. The *S-topology* on ${}^*X$ is the topology generated by the basis $\mathcal{B} = \{{}^*A \mid A \subseteq X\}$. ■

For every ${}^*A \in \mathcal{B}$ it holds by *transfer* that ${}^*X \setminus {}^*A = {}^*(X \setminus A)$ and thus that ${}^*A$ is closed. This means that every S-topology has a basis of clopen sets.

Again, we will use our nonstandard theory to prove two interesting properties about this topology. We will not assume any κ-enlargement in this section, since these properties hold regardless. Both results show a connection between the S-topology on ${}^*\mathbb{N}$ and the *ultrafilter map*, that is the map $\mathfrak{U} : {}^*\mathbb{N} \to \beta\mathbb{N}$ that sends each hypernatural number to the ultrafilter that it generates.

Proposition 3.2.2. *Let τ be the S-topology on ${}^*\mathbb{N}$ and let $\mathfrak{U} : {}^*\mathbb{N} \to \beta\mathbb{N}$ be the ultrafilter map. Then*

1. *τ is compact if and only if $\mathfrak{U}$ is surjective;*
2. *τ is Hausdorff if and only if $\mathfrak{U}$ is injective.*

Proof. Since both topological properties have many equivalent definitions, we will specify which of those we will use.

(1). For compactness, we will use the definition that a space is compact if and only if any collection of closed sets with the FIP has non-empty intersection.

$\Rightarrow$. Let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. Then the set $\mathcal{C} = \{{}^*X \mid X \in \mathcal{F}\}$ contains only (basic) closed sets and, by *transfer*, has the FIP. By compactness, there is an $\alpha \in \bigcap \mathcal{C}$. Therefore, we have that $\mathcal{F} \subseteq \{X \subseteq \mathbb{N} \mid \alpha \in {}^*X\} = \mathfrak{U}_\alpha$. By the maximality of $\mathcal{F}$, the previous inclusion is an equality.

$\Leftarrow$. Let $\mathcal{C}$ be a family of closed sets. Recall that our topology has a basis of clopen sets. Therefore, any closed set can be written as an intersection of basic sets. This means that, wlog, we can take $\mathcal{C}$ to be a family of basis elements, i.e. $\mathcal{C} = \{{}^*A_i \mid i \in I\}$. Suppose that $\mathcal{C}$ has the FIP. Then $\mathcal{C}' = \{A_i \mid i \in I\}$ contains subsets of $\mathbb{N}$ and, by *transfer*, has the FIP, and thus we can use Theorem 1.1.3 to extend $\mathcal{C}'$ to an ultrafilter $\mathcal{F}$ on $\mathbb{N}$. By the hypothesis, there is an $\alpha \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\alpha$. It follows that for every ${}^*A_i \in \mathcal{C}$, we have $A_i \in \mathcal{F} = \mathfrak{U}_\alpha$ and thus $\alpha \in {}^*A_i$. We conclude that $\alpha \in \bigcap \mathcal{C} \neq \emptyset$.

(2). For Hausdorffness, we will use the definition that a space is Hausdorff if and only if any two distinct elements can be separated by basic open sets.

In particular, τ is Hausdorff if and only if for all $\alpha, \beta \in {}^*\mathbb{N}$ with $\alpha \neq \beta$ there are ${}^*A, {}^*B \in \mathcal{B}$ such that $\alpha \in {}^*A$, $\beta \in {}^*B$ and ${}^*A \cap {}^*B = \emptyset$. This means precisely that there are $A \in \mathfrak{U}_\alpha$ and $B \in \mathfrak{U}_\beta$ such that $A \cap B = \emptyset$ and that holds if and only if $\mathfrak{U}_\alpha \neq \mathfrak{U}_\beta$. □

In the previous section we saw that under the assumption of $\mathfrak{c}^+$-enlargement, the ultrafilter map is surjective. As a corollary of the above proposition we obtain that in that case the S-topology on ${}^*\mathbb{N}$ is compact.

There is more to say about the Hausdorffness of the S-topology on an arbitrary hyper-extension ${}^*X$. For example, under the ultrapower model of the hypernatural numbers using an ultrafilter $\mathcal{F}$ on $\mathbb{N}$, the S-topology on ${}^*\mathbb{N}$ is Hausdorff if and only if $\mathcal{F}$ satisfies certain properties. In that case we the ultrafilter $\mathcal{F}$ is called *Hausdorff*. This, however, does not fall within the scope of this text, so again we refer the reader to [10].

3.3 Ultrafilters Generated by Pairs

We will now extend the u-equivalence to pairs. As in the first section of this chapter, we will assume to be working in a model of the hypernatural numbers that is a $\mathfrak{c}^+$-enlargement. Under this extension, every pair of hypernatural numbers generates an ultrafilter on $\mathbb{N} \times \mathbb{N}$. Formally,

Definition 3.3.1. Let $(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$. We define the *ultrafilter generated by* (α, β) as the family

$$\mathfrak{U}_{(\alpha,\beta)} = \{X \subseteq \mathbb{N} \times \mathbb{N} \mid (\alpha, \beta) \in {}^*X\}. \quad \blacksquare$$

By the same reasoning as with the regular u-equivalence, we have that $\mathfrak{c}^+$-enlargement implies that every ultrafilter on $\mathbb{N} \times \mathbb{N}$ is generated by some pair of hypernatural numbers. The definition of the u-equivalence relation $\underset{u}{\sim}$ on ${}^*\mathbb{N} \times {}^*\mathbb{N}$ is also analogous. Namely, we set $(\alpha, \beta) \underset{u}{\sim} (\alpha', \beta')$ if and only if $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_{(\alpha',\beta')}$.

We begin our theory of the u-equivalence of pairs with the following lemma.

Lemma 3.3.2. *For any $\alpha, \beta \in {}^*\mathbb{N}$, we have that $\mathfrak{U}_\alpha \times \mathfrak{U}_\beta \subseteq \mathfrak{U}_{(\alpha,\beta)}$.*

Proof. Let $A \times B \in \mathfrak{U}_\alpha \times \mathfrak{U}_\beta$. By definition we have $\alpha \in {}^*A$ and $\beta \in {}^*B$. Using *transfer*, we obtain $(\alpha, \beta) \in {}^*A \times {}^*B = {}^*(A \times B)$. We conclude that $A \times B \in \mathfrak{U}_{(\alpha,\beta)}$. $\square$

The following theorem gives a characterization of those pairs of hypernatural numbers $(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ that generate an ultrafilter that can be written as a tensor product. It turns out that, in that case, the pair (α, β) always generates the same ultrafilter as the tensor product of the ultrafilter they generate individually. Such a pair of hypernatural numbers is called a *tensor pair.*

Theorem 3.3.3. *Let $\alpha, \beta \in {}^*\mathbb{N}$. Then (α, β) generates a tensor product if and only if $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$.*

Proof. One direction is trivial. For the other direction, suppose that (α, β) generates a tensor product, say $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\gamma \otimes \mathfrak{U}_\delta$. By the previous Proposition, we have $\mathfrak{U}_{(\alpha,\beta)} \supseteq \mathfrak{U}_\alpha \times \mathfrak{U}_\beta$ and, as we have seen in Chapter 1, $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\gamma \otimes \mathfrak{U}_\delta \supseteq \mathfrak{U}_\gamma \times \mathfrak{U}_\delta$. Suppose, by contradiction, that $\alpha \underset{u}{\not\sim} \gamma$. Then there is a hyper-extension that separates α and γ, i.e. there is an $X \subseteq \mathbb{N}$, such that $a \in {}^*X$, but not $\gamma \in {}^*X$. Pick arbitrary $Y \in \mathfrak{U}_\beta$ and $Z \in \mathfrak{U}_\delta$, then we have disjoint sets $X \times Y \in \mathfrak{U}_\alpha \times \mathfrak{U}_\beta$ and $X^c \times Z \in \mathfrak{U}_\gamma \times \mathfrak{U}_\delta$, which both belong to $\mathfrak{U}_{(\alpha,\beta)}$, a contradiction. We conclude that $\alpha \underset{u}{\sim} \gamma$, i.e. $\mathfrak{U}_\alpha = \mathfrak{U}_\gamma$. Analogously it can be shown that $\mathfrak{U}_\beta = \mathfrak{U}_\delta$, and thus that $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\gamma \otimes \mathfrak{U}_\delta = \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$, as required. $\square$

3.4 Algebra on $\beta\mathbb{N}$

A well-known known topology on the space of ultrafilters $\beta\mathbb{N}$ is the one generated by the basis $\mathcal{B} = \{\mathcal{O}_A \mid A \subseteq \mathbb{N}\}$, where

$$\mathcal{O}_A = \{\mathcal{F} \in \beta\mathbb{N} \mid A \in \mathcal{F}\}$$

for all $A \subseteq \mathbb{N}$. Note that, like the S-topology, this topology is also generated by a basis of clopen sets.

Under this topology, the space $\beta\mathbb{N}$ is what is called the *Stone-Čech compactification* of $\mathbb{N}$. This implies that $\beta\mathbb{N}$ is compact, Hausdorff and that $\mathbb{N}$ is dense in $\beta\mathbb{N}$. Further explanation of said compactification falls beyond the scope of this thesis, but we will show that the topological space $\beta\mathbb{N}$ has at least two of the implied properties.

Theorem 3.4.1. *Let τ be the topology on $\beta\mathbb{N}$ that is generated by $\mathcal{B}$. Then*

1. *τ is compact;*
2. *τ is Hausdorff;*

Proof. We will use the same definitions of compactness and Hausdorffness as in the proof of Proposition 3.2.2.

(1). Suppose that $\mathcal{C}$ is a collection of closed sets that has the FIP. Like in the proof of Proposition 3.2.2, we can wlog take $\mathcal{C}$ to be a family of basic sets, i.e. $\mathcal{C} = \{\mathcal{O}_{A_i} \mid A_i \subseteq \mathbb{N} \text{ and } i \in I\}$.

For any finite number of sets $\mathcal{O}_{A_1}, \ldots, \mathcal{O}_{A_n} \in \mathcal{C}$, we have that their intersection is non-empty. This means that there is an ultrafilter $\mathcal{F}$, such that $A_1, \ldots, A_n \in \mathcal{F}$. By the definition of a filter this implies that $A_1, \ldots, A_n$ have non-empty intersection.

From the reasoning above we can conclude that the set $\{A_i \mid i \in I\}$ has the FIP. By Theorem 1.1.3, we can extend said set to an ultrafilter $\mathcal{G}$ on $\mathbb{N}$. We conclude that $\mathcal{G} \in \bigcap \mathcal{C} \neq \emptyset$.

(2). Let $\mathcal{F}, \mathcal{G} \in \beta\mathbb{N}$, such that $\mathcal{F} \neq \mathcal{G}$. Then there is an $A \subseteq \mathbb{N}$ such that A is contained in one ultrafilter, but not in the other.

Say wlog that $A \in \mathcal{F}$. By Proposition 1.1.4, we have $A^c \in \mathcal{G}$. It follows that $\mathcal{F} \in \mathcal{O}_A$ and $\mathcal{G} \in \mathcal{O}_{A^c}$. By that same proposition we obtain that $\mathcal{O}_A \cap \mathcal{O}_{A^c} = \emptyset$, as required. □

We define yet another binary operation on two ultrafilters.

Definition 3.4.2. Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on the sets X and Y respectively. Then the *pseudo-sum* $\mathcal{F} \oplus \mathcal{G}$ is the image of the tensor product $\mathcal{F} \otimes \mathcal{G}$ under the sum function $S(x, y) = x + y$. ■

We will consider this operation on the space $\beta\mathbb{N}$ with the above topology. The operation owes its name to the fact that it is similar to ordinary addition. In fact, it extends the sum on the natural numbers. To see this, let $\mathcal{F}(n)$ and $\mathcal{F}(m)$ be principal ultrafilters on $\mathbb{N}$, generated by the natural numbers n and m respectively. Then we have

$$\begin{aligned} A \in \mathcal{F}(n) \oplus \mathcal{F}(m) &\Leftrightarrow A \in S(\mathcal{F}(n) \otimes \mathcal{F}(m)) \\ &\Leftrightarrow S^{-1}(A) \in \mathcal{F}(n) \otimes \mathcal{F}(m) \\ &\Leftrightarrow \{(a,b) \in \mathbb{N} \times \mathbb{N} \ \mid a+b \in A\} \in \mathcal{F}(n) \otimes \mathcal{F}(m) \\ &\Leftrightarrow \{a \mid \{b \mid a+b \in A\} \in \mathcal{F}(m)\} \in \mathcal{F}(n) \\ &\Leftrightarrow n \in \{a \mid \{b \mid a+b \in A\} \in \mathcal{F}(m)\} \\ &\Leftrightarrow n \in \{a \mid m \in \{b \mid a+b \in A\}\} \\ &\Leftrightarrow n+m \in A \Leftrightarrow A \in \mathcal{F}(n+m), \end{aligned}$$

where $\mathcal{F}(x)$ is the ultrafilter generated by x.

The following result is a generalization of this idea.

Proposition 3.4.3. *Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ be ultrafilters on $\mathbb{N}$. Then $\mathcal{F} \oplus \mathcal{G} = \mathcal{H}$ if and only if there exists a tensor pair (α, β) such that $\mathfrak{U}_\alpha = \mathcal{F}$, $\mathfrak{U}_\beta = \mathcal{G}$ and $\mathfrak{U}_{\alpha+\beta} = \mathcal{H}$. In consequence:*

$$\{\mathcal{F} \oplus \mathcal{G} \mid \mathcal{F}, \mathcal{G} \in \beta\mathbb{N}\} = \{\mathfrak{U}_{\alpha+\beta} \mid (\alpha, \beta) \textit{ is a tensor pair}\}.$$

Proof. $\Rightarrow$. Pick $\xi, \eta \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\xi$ and $\mathcal{G} = \mathfrak{U}_\eta$ and also pick $(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ such that (α, β) generates the ultrafilter that is the tensor product of $\mathfrak{U}_\xi$ and $\mathfrak{U}_\eta$, i.e. $\mathfrak{U}_{(\alpha,\beta)} = \mathfrak{U}_\xi \otimes \mathfrak{U}_\eta$. By Theorem 3.3.3, we have that $\mathfrak{U}_\alpha = \mathfrak{U}_\xi$ and $\mathfrak{U}_\beta = \mathfrak{U}_\eta$. We obtain the required result by noting that

$$\begin{aligned} \mathcal{H} = S(\mathcal{F} \otimes \mathcal{G}) = S(\mathfrak{U}_\xi \otimes \mathfrak{U}_\eta) &= S(\mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta) \\ &= S(\mathfrak{U}_{(\alpha,\beta)}) = \{A \subseteq \mathbb{N} \mid S^{-1}(A) \in \mathfrak{U}_{(\alpha,\beta)}\} \\ &= \{A \subseteq \mathbb{N} \mid (\alpha, \beta) \in {}^*\{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n+m \in A\}\}. \end{aligned}$$

By *transfer*, the last set is equal to $\{A \subseteq \mathbb{N} \mid \alpha + \beta \in {}^*A\} = \mathfrak{U}_{\alpha+\beta}$, as required.

$\Leftarrow$. Let (α, β) be such a tensor pair. Then $\mathcal{W} = \mathcal{F} \oplus \mathcal{G} = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta = S(\mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta) = S(\mathfrak{U}_{(\alpha,\beta)})$. As above, we have that the last set is equal to $\mathfrak{U}_{\alpha+\beta}$, as required.

In consequence, we have $A \in \{\mathcal{F} \oplus \mathcal{G} \mid \mathcal{F}, \mathcal{G} \in \beta\mathbb{N}\}$ if and only if there exists a tensor pair $(\alpha, \beta) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ such that $A \in \mathfrak{U}_{\alpha+\beta}$. □

In order to prove some more properties about the pseudo-sum operation, we will first consider the following notion that was introduced by M. Beiglböck in [2].

Definition 3.4.4. Let $A \subseteq \mathbb{N}$ and let $\mathcal{F}$ be an ultrafilter on $\mathbb{N}$. We define the *ultrafilter-shift* of A by $\mathcal{F}$ as the set

$$A - \mathcal{F} = \{n \in \mathbb{N} \mid A - n \in \mathcal{F}\},$$

with $A-n$ the leftward shift of A by n, i.e. $A-n = \{m \in \mathbb{N} \mid n+m \in A\}$. ■

The following definition corresponds to taking the ultrafilter-shift of a subset of $\mathbb{N}$ by an ultrafilter generated by a hypernatural number. Unlike in [10], we directly define it as such, instead of using a different, equivalent definition.

Definition 3.4.5. Let $A \subseteq \mathbb{N}$ and let $\gamma \in {}^*\mathbb{N}$. The *hyper-shift* A_γ of A by γ is the ultrafilter-shift of A by $\mathfrak{U}_\gamma$. That is,

$$A_\gamma = A - \mathfrak{U}_\gamma = \{n \in \mathbb{N} \mid A - n \in \mathfrak{U}_\gamma\}.$$ ■

The following proposition demonstrates the connection between hyper-shifts and the pseudo-sum operation.

Proposition 3.4.6. *Let $\alpha, \beta, \gamma \in {}^*\mathbb{N}$. Then:*

1. *For every $A \subseteq \mathbb{N}$, $A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow A_\beta \in \mathfrak{U}_\alpha$;*
2. *For every $n \in \mathbb{N}$, $(A - n)_\beta = A_\beta - n$;*
3. *For every $n \in \mathbb{N}$, $A - n \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow n \in (A_\beta)_\alpha$;*
4. *$\mathfrak{U}_\gamma = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow A_\gamma = (A_\beta)_\alpha$ for every $A \subseteq \mathbb{N}$.*

Proof. We first show a general property about hyper-shifts. Namely, that for any $A \subseteq \mathbb{N}$ and $\gamma \in {}^*\mathbb{N}$, we have that the hyper-shift $A_\gamma = ({}^*A - \gamma) \cap \mathbb{N}$. This holds by the fact that for every $n \in \mathbb{N}$ we have the following chain of equivalences:

$$n \in A_\gamma \Leftrightarrow A - n \in \mathfrak{U}_\gamma \Leftrightarrow \gamma \in {}^*(A - n) \Leftrightarrow \gamma \in {}^*A - n \Leftrightarrow n \in {}^*A - \gamma,$$

where the second to last equivalence holds due to *transfer*.

(1) follows from yet another chain of equivalence classes that holds for any $A \subseteq \mathbb{N}$:

$$\begin{aligned} A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta &\Leftrightarrow A \in S(\mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta) \\ &\Leftrightarrow S^{-1}(A) \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \\ &\Leftrightarrow \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid n+m \in A\} \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta \\ &\Leftrightarrow \{a \mid \{b \mid a+b \in A\} \in \mathfrak{U}_\beta\} \in \mathfrak{U}_\alpha \\ &\Leftrightarrow \{a \mid A-a \in \mathfrak{U}_\beta\} \in \mathfrak{U}_\alpha \\ &\Leftrightarrow \{a \mid a \in A_\beta\} \in \mathfrak{U}_\alpha \Leftrightarrow A_\beta \in \mathfrak{U}_\alpha. \end{aligned}$$

For (2), we have for all $k \in \mathbb{N}$ that $k \in (A-n)_\beta \Leftrightarrow k \in {}^*(A-n) - \beta \Leftrightarrow k \in ({}^*A - n) - \beta \Leftrightarrow k + n \in {}^*A - \beta \Leftrightarrow k+n \in A_\beta \Leftrightarrow k \in A_\beta - n$.

To prove (3), we use both of the above properties and obtain $A - n \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow (A-n)_\beta \in \mathfrak{U}_\alpha \Leftrightarrow A_\beta - n \in \mathfrak{U}_\alpha \Leftrightarrow \alpha \in {}^*(A_\beta - n) \Leftrightarrow \alpha \in {}^*(A_\beta) - n \Leftrightarrow n \in {}^*(A_\beta) - \alpha \Leftrightarrow n \in (A_\beta)_\alpha$.

For (4), we will cover both sides of the bi-implication separately.

$\Rightarrow$. Suppose that $\mathfrak{U}_\gamma = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta$ and let $A \in \mathbb{N}$, then the result holds because of the fact that for every $n \in \mathbb{N}$, we have $n \in A_\gamma \Leftrightarrow n \in {}^*A - \gamma \Leftrightarrow \gamma \in {}^*A - n \Leftrightarrow \gamma \in {}^*(A-n) \Leftrightarrow A - n \in \mathfrak{U}_\gamma = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \Leftrightarrow n \in (A_\beta)_\alpha$, where the last equivalence holds due to (3).

$\Leftarrow$. Suppose that for every $A \subseteq \mathbb{N}$, we have $A_\gamma = (A_\beta)_\alpha$ and assume by contradiction that there is an $A \subseteq \mathbb{N}$, such that $A \in \mathfrak{U}_\gamma$, but $A \notin \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta$. Then by (3), it holds that $0 \in (A_\beta)_\alpha$. By definition of the u-equivalence, however, we have that $\gamma \notin {}^*A$ and thus $0 \notin A_\gamma$, a contradiction. □

We can now show that $(\beta\mathbb{N}, \oplus)$, that is the topological space $\beta\mathbb{N}$ equipped with the pseudo-sum operation, is a *topological left semigroup*, i.e. that it has the following two properties:

Theorem 3.4.7. *Let $(\beta\mathbb{N}, \oplus)$ be the Stone-Čech compactification of $\mathbb{N}$ together with the pseudo-sum operation. Then:*

1. *The operation $\oplus$ is associative, i.e. for all $\mathcal{F}, \mathcal{G}, \mathcal{H} \in \beta\mathbb{N}$, it holds that $(\mathcal{F} \oplus \mathcal{G}) \oplus \mathcal{H} = \mathcal{F} \oplus (\mathcal{G} \oplus \mathcal{H})$.*

2. *For any $\mathcal{G} \in \beta\mathbb{N}$, the map $\oplus_\mathcal{G} : \mathcal{F} \mapsto \mathcal{F} \oplus \mathcal{G}$ is continuous.*

Proof. (1). Pick $\alpha \in {}^*\mathbb{N}$ such that it holds that $\mathcal{F} = \mathfrak{U}_\alpha$ and, using Proposition 3.4.3, pick a tensor pair $(\beta, \gamma) \in {}^*\mathbb{N} \times {}^*\mathbb{N}$ such that $\mathcal{G} = \mathfrak{U}_\beta$, $\mathcal{H} = \mathfrak{U}_\gamma$

and $\mathcal{G} \oplus \mathcal{H} = \mathfrak{U}_\beta \oplus \mathfrak{U}_\gamma = \mathfrak{U}_{\beta+\gamma}$. Then for all $A \subseteq \mathbb{N}$:

$$\begin{aligned} A \in (\mathcal{F} \oplus \mathcal{G}) \oplus \mathcal{H} &\Leftrightarrow A \in (\mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta) \oplus \mathfrak{U}_\gamma \\ &\Leftrightarrow A_\gamma \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \\ &\Leftrightarrow (A_\gamma)_\beta \in \mathfrak{U}_\alpha \\ &\Leftrightarrow A_{\beta+\gamma} \in \mathfrak{U}_\alpha \\ &\Leftrightarrow A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_{\beta+\gamma} \Leftrightarrow A \in \mathcal{F} \oplus (\mathcal{G} \oplus \mathcal{H}). \end{aligned}$$

(2). Recall that a function on a topological space is continuous if the preimage of any base element is open. We aim to show that for any $\mathcal{G} \in \beta\mathbb{N}$, and any $\mathcal{O}_A \in \mathcal{B}$, we have that $\oplus_\beta^{-1}(\mathcal{O}_A)$ is open. This holds by the fact that for any $\mathcal{F} \in \beta\mathbb{N}$ we can pick $\alpha, \beta \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\alpha$ and $\mathcal{G} = \mathfrak{U}_\beta$ to obtain the following chain of equivalences:

$$\begin{aligned} \mathfrak{U}_\alpha \in \oplus_\mathcal{G}^{-1}(\mathcal{O}_A) &\Leftrightarrow \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \in \mathcal{O}_A \\ &\Leftrightarrow A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta \\ &\Leftrightarrow A_\beta \in \mathfrak{U}_\alpha \Leftrightarrow \mathfrak{U}_\alpha \in \mathcal{O}_{A_\beta}, \end{aligned}$$

as required. □

While the pseudo-sum operation is transitive, it is not commutative. In fact, as Di Nasso puts it in [10], it "*fails badly to be commutative*". This is because the only elements of $\beta\mathbb{N}$ that commute with all other elements (in other words, the elements of the *center* of $(\beta\mathbb{N}, \oplus)$) are the principal ultrafilters. The following theorem shows this using hyper-shifts.

Theorem 3.4.8. *Let $\mathcal{F} \in \beta\mathbb{N}$. Then there is a $\mathcal{G} \in \beta\mathbb{N}$ such that we have $\mathcal{F} \oplus \mathcal{G} \neq \mathcal{G} \oplus \mathcal{F}$, if and only if $\mathcal{F}$ is nonprincipal.*

Proof. $\Rightarrow$. By contradiction, suppose that $\mathcal{F}$ is the principal ultrafilter generated by $n \in \mathbb{N}$ and let $\beta \in {}^*\mathbb{N}$ such that $\mathcal{G} = \mathfrak{U}_\beta$. Then for all $A \subseteq \mathbb{N}$, it holds that

$$\begin{aligned} A \in \mathcal{F} \oplus \mathcal{G} &\Leftrightarrow A \in \mathfrak{U}_n \oplus \mathfrak{U}_\beta \Leftrightarrow A_\beta \in \mathfrak{U}_n \Leftrightarrow n \in {}^*(A_\beta) \\ &\Leftrightarrow n \in A_\beta \Leftrightarrow n \in {}^*A - \beta \Leftrightarrow \beta \in {}^*A - n \Leftrightarrow \beta \in {}^*A_n \\ &\Leftrightarrow A_n \in \mathfrak{U}_\beta \Leftrightarrow A \in \mathfrak{U}_\beta \oplus \mathfrak{U}_n \Leftrightarrow A \in \mathcal{G} \oplus \mathcal{F}. \end{aligned}$$

This contradicts the defining property of $\mathcal{G}$.

$\Leftarrow$. Let $\mathcal{F}$ be a nonprincipal ultrafilter and pick $\gamma \in {}^*\mathbb{N}$ such that $\mathcal{F} = \mathfrak{U}_\gamma$. By Proposition 3.1.2, we have that γ is infinite. By *transfer*, we can divide

${}^*\mathbb{N}$ into disjoint subsets $[\alpha^2, (\alpha+1)^2)$ for all $\alpha \in {}^*\mathbb{N}$. Now pick the unique $\nu \in {}^*\mathbb{N}$ such that $\gamma \in [\nu^2, (\nu+1)^2)$. From the fact that γ is infinite, it follows that ν must be infinite.

When ν is even, we let the set A be the union of precisely those intervals $[n, (n+1)^2)$ where $n \in \mathbb{N}$ is even, i.e.

$$A = \bigcup_{\text{n even}} [n^2, (n+1)^2).$$

When ν is odd, we define A in the similar way, but for all odd $n \in \mathbb{N}$. By *transfer*, the hyper-extension ${}^*A$ of A consists of all intervals $[\alpha^2, (\alpha+1)^2)$ for $\alpha \in {}^*\mathbb{N}$ even/odd. Now we distinguish two cases.

Case 1. $(v+1)^2 - \gamma$ is infinite. Let $\beta = (v+1)^2$. Since the difference between γ and $(v+1)^2$ is infinite, we have $\gamma + n \in [\nu^2, (\nu+1)^2) \subseteq {}^*A$ for all $n \in \mathbb{N}$. Therefore it holds for all $n \in \mathbb{N}$ that $\gamma + n \in {}^*A$ and thus $n \in ({}^*A - \gamma) \cap \mathbb{N} = A_\gamma$. That means that $A_\gamma = \mathbb{N}$. Since the interval $[(v+1)^2, (v+2)^2) \cap {}^*A = \emptyset$ is infinite, we have for all $n \in \mathbb{N}$, that $\beta + n \notin {}^*A$ and thus $n \notin ({}^*A - \beta) \cap \mathbb{N} = A_\beta$. That means that $A_\beta = \emptyset$. By definition, we have $A_\gamma \in \mathfrak{U}_\beta$, but $A_\beta \notin \mathfrak{U}_\gamma$ and thus $A \in \mathfrak{U}_\beta \oplus \mathfrak{U}_\gamma$, but $A \notin \mathfrak{U}_\gamma \oplus \mathfrak{U}_\beta$, as required.

Case 2. $(v+1)^2 - \gamma$ is finite. Let $\beta = \nu^2$. Then it holds that the interval $(\gamma, (v+1)^2) \subseteq {}^*A$ is finite, while the interval $[(\nu+1)^2, (\nu+2)^2) \cap {}^*A = \emptyset$ is infinite. This means that the set of $n \in \mathbb{N}$, for which $\gamma + n \in {}^*A$ is finite. As above, it follows that A_γ is finite. However, for all $n \in \mathbb{N}$ it holds that $\beta + n \in {}^*A$, and thus $A_\beta = \mathbb{N}$. Since β is infinite, we that $\mathfrak{U}_\beta$ is nonprincipal, and thus $A_\gamma \notin \mathfrak{U}_\beta$. By definition, it holds that $A_\beta \in \mathfrak{U}_\gamma$. We now conclude that $A \notin \mathfrak{U}_\beta \oplus \mathfrak{U}_\gamma$, but $A \in \mathfrak{U}_\gamma \oplus \mathfrak{U}_\beta$, as required. □

The following corollary, that was announced in Chapter 1, follows immediately from the previous theorem and the definition of the pseudo-sum operation.

Corollary 3.4.9. *The tensor product $\otimes$ is not commutative.*

Particularly important in applications are those ultrafilters that are *idempotent* under the pseudo-sum operation. That is, those $\mathcal{F} \in \beta\mathbb{N}$, for which $\mathcal{F} = \mathcal{F} \oplus \mathcal{F}$.

Since we take $\mathbb{N}$ to include 0, there is precisely one principal idempotent ultrafilter. In most literature regarding idempotent ultrafilters $\mathbb{N}$ is defined *not* to include 0, precisely because of this issue. Indeed, it can be convenient to let idempotence imply non-principality. We solve this issue by referring to the principal ultrafilter generated by 0 as the only *trivially idempotent* ultrafilter.[1]

The following theorem gives some nonstandard characterizations of the idempotent ultrafilters.

Theorem 3.4.10. *Let $\alpha \in {}^*\mathbb{N}$. Then the following properties are equivalent.*

1. *$\mathfrak{U}_\alpha$ is idempotent;*
2. *For every $A \subseteq \mathbb{N}$, it holds that $A_\alpha = (A_\alpha)_\alpha$;*
3. *For every $A \subseteq \mathbb{N}$ such that $\alpha \in {}^*A$, it holds that $\alpha \in {}^*(A_\alpha)$;*
4. *For every $A \subseteq \mathbb{N}$ such that $\alpha \in {}^*A$, there exists $B \subseteq A \cap B_\alpha$ such that $\alpha \in {}^*B$.*

Proof. (1) $\Leftrightarrow$ (2). This follows directly from (4) of Proposition 3.4.6.

(1) $\Leftrightarrow$ (3). By (1) of Proposition 3.4.6, we have as particular case the equivalence $\alpha \in {}^*(A_\alpha) \Leftrightarrow A \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$. Therefore, (3) states that it holds that $\alpha \in \mathfrak{U}_\alpha \Rightarrow \alpha \in \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$, i.e. $\mathfrak{U}_\alpha \subseteq \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$. By the maximality of ultrafilters this is equivalent to $\mathfrak{U}_\alpha = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\alpha$.

(2), (3) $\Rightarrow$ (4). This holds for $B = A \cap A_\alpha$. First we will show that the required inclusion holds for B. Trivially, we have that $B \subseteq A$. By (2), we have that

$$B \subseteq A_\alpha = A_\alpha \cap A_\alpha = A_\alpha \cap (A_\alpha)_\alpha = (A \cap A_\alpha)_\alpha = B_\alpha.$$

By (3) and *transfer*, we find that ${}^*B = {}^*(A \cap A_\alpha) = {}^*A \cap {}^*A_\alpha \ni \alpha$.

(4) $\Rightarrow$ (3). Let $\alpha \in {}^*A$ and B as in the hypothesis. We wish to show that also $B \subseteq A_\alpha$. Indeed,

$$B \subseteq B_\alpha \subseteq (A \cap B_\alpha)_\alpha = A_\alpha \cap (B_\alpha)_\alpha \subseteq A_\alpha,$$

where the second inclusion is due to the defining property of B.

By closure under extensions it follows that, since $B \in \mathfrak{U}_\alpha$, we also have $A_\alpha \in \mathfrak{U}_\alpha$. We conclude that $\alpha \in {}^*(A_\alpha)$, as required. □

It is natural to wonder whether non-trivially idempotent ultrafilters actually exist. We have given some nonstandard characterizations, but are there actually infinite hypernatural numbers that satisfy these properties?

[1] In [10], Di Nasso seems to ignore this issue, or at least he does not address it. He clearly takes $\mathbb{N}$ to include 0, as he uses that fact in, for instance, the proof of [10, 8.5]. Contradictory, in the proof of [10, 8.3], he assumes that any idempotent ultrafilter is generated by an *infinite* hypernatural number and hence is nonprincipal.

The answer to the above question is: yes, these numbers exist. It follows from the general fact that any compact Hausdorff topological left semigroup has idempotent elements, which was first proved in [5].

Our proofs that $(\beta\mathbb{N}, \oplus)$ is a topological left semigroup hold regardless of whether we include 0 in the natural numbers. Therefore, Ellis' theorem implies that $(\beta\mathbb{N}, \oplus)$ contains a non-trivially idempotent.

It would be nice if we could give an alternative prove of the existence of hypernatural numbers generating idempotent ultrafilters. However, such a proof seems harder to find than one might expect. In fact, Di Nasso poses the formulation of such an alternative proof as an open question in [10].

We conclude this chapter with an application of idempotent ultrafilters and their nonstandard characterization. Specifically, we we will give a non-standard ultrafilter proof of Hindman's theorem, a combinatorial theorem first proved by N. Hindman in [7].

This theorem has an interesting history. The original proof was so complicated that Hindman says in [6]: *"If the reader has a graduate student that she wants to punish, she should make him read and understand that original proof."*

Later, a far more easy proof using ultrafilters was presented by F. Galvin and S. Glazer (see [6]). Now, using our connecting between ultrafilters and hypernatural numbers, we can formulate an analogous, nonstandard proof.

Theorem 3.4.11 (Hindman's Theorem)**.** *Let $\{C_1, \ldots, C_n\}$ be a finite family of disjoint sets, such that $C_1 \cup \ldots \cup C_n = \mathbb{N}$. Then there is an infinite set $X \subseteq \mathbb{N}$, such that the sum of elements of any finite set $F \subseteq X$, belongs to the same set C_i.*

Proof. By the existence of idempotent ultrafilters and the surjectivity of the ultrafilter map, we can pick an $\alpha \in {}^*\mathbb{N}$ such that $\mathfrak{U}_\alpha$ is *non-trivially* idempotent. Recall that this implies that α is infinite. Furthermore, by *transfer*, we find that there is a unique C_i such that $\alpha \in {}^*C_i$.

By (4) of Theorem 3.4.10, there is a $B \subseteq \mathbb{N}$, such that $B \subseteq C_i \cap B_\alpha$ and $\alpha \in B$. Note that if $x \in B$, then $x \in B_\alpha$. This implies that $B - x \in \mathfrak{U}_\alpha$, i.e. $\alpha \in {}^*(B - x)$.

Now pick some $x_1 \in B$. Trivially we have that $\alpha \in {}^*B$ and by the above considerations, we find that $\alpha \in {}^*(B - x_1)$. This means that we have the following true sentence.

$$^*\varphi : (\exists \alpha \in {}^*\mathbb{N})(\alpha \in {}^*B \wedge \alpha \in {}^*(B - x_1) \wedge \alpha > x_1),$$

which under *transfer* becomes

$$\varphi : (\exists n \in \mathbb{N})(n \in B \wedge n \in (B - x_1) \wedge n > x_1).$$

Denote the element that satisfies the conditions of the above existential quantifier by x_2. We then have $x_1, x_2, x_1 + x_2 \in B$ with $x_1 < x_2$.

We still have $\alpha \in {}^*(B - x_1)$ and now, by the definition of B, we obtain $\alpha \in {}^*(B - x_2)$ and $\alpha \in {}^*(B - (x_1 + x_2)) = {}^*((B - x_1) - x_2)$. By the same reasoning as above, there is an $x_3 > x_2 > x_1$, such that it holds that $x_3 \in B \cap (B - x_1) \cap (B - x_2) \cap (B - x_1 - x_2)$. It follows that the elements $x_1, x_2, x_3, x_1 + x_2, x_1 + x_3, x_2 + x_3, x_1 + x_2 + x_3 \in B$.

Repeating this process, we eventually end up with a set $X = \{x_1, x_2, \ldots\}$, such that $x_1 < x_2 < \ldots$ and each finite sum of elements of X is in B. We finish the proof by recalling that any element in B is also in C_i. $\square$

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