On Structural Completeness of Tabular Superintuitionistic Logics

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As usual, the superintuitionistic (propositional) logics (that is, logics extending intuitionistic logic) are being studied "modulo derivability", meaning such logics are viewed extensionally — they are identified with the set of formulae that are valid (derivable in the corresponding calculus) in this logic. Under this approach, the lattice of all superintuitionistic logics ordered by set-inclusion is dually isomorphic to the lattice of all varieties of pseudo-Boolean algebras. If a logic is defined by a calculus, we introduce a notion of derivability of a formulae from a collection of formulae. The notion of derivability can be generalized to not finitely axiomatizable logics (for instance, as a consequence operator [1]). Sometimes, in such cases the consequences relation can be defined constructively (for instance, it can be defined by a finite matrix [1]). In the present paper, we study precisely these consequence relations: the ones that are defined by finite pseudo-Boolean algebras (regarded as matrices with a unique designated element — the greatest element of the algebra).

1 Modus Rules

We denote propositional variables by p, q, r (perhaps with indexes); we denote propositional formulae by A, B, C, D (perhaps with indexes) constructed in the usual way using propositional variables, the connectives $\&$, $\vee$, $\supset$, $\neg$ and parentheses (,).

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By *modus rule*, we mean a rule of form:

$$\frac{A_1[X,Y,\dots],\dots,A_n[X,Y,\dots]}{A_{n+1}[X,Y,\dots]} \tag{1}$$

that allows to derive the result of a substitution on the formula A_{n+1} from the results of the same substitution on the formulae $A_1,\dots,A_n$.

A *pseudo-Boolean algebra* [2] is a distributive lattice $\mathfrak{A} = \{E; \&, \vee\}$ with relative pseudocomplement $\supset$ and pseudocomplement $\neg$. The greatest element of a pseudo-Boolean algebra is denoted by $\mathbf{1}$ and the smallest — by $\mathbf{0}$.

We say that the rule (1) is *valid in a pseudo-Boolean algebra* $\mathfrak{A}$ if for any list of elements α_1,α_2,… of the algebra $\mathfrak{A}$, $A_{n+1}[\alpha_1,\alpha_2,\dots] = \mathbf{1}$ as long as $A_1[\alpha_1,\alpha_2,\dots] = A_2[\alpha_1,\alpha_2,\dots] = \dots = A_n[\alpha_1,\alpha_2,\dots] = \mathbf{1}$. In other words, the rule (1) is valid in $\mathfrak{A}$ if and only if the following quasi-identity

$$A_1[\alpha_1,\alpha_2,\dots] = \mathbf{1},\dots,A_n[\alpha_1,\alpha_2,\dots] = \mathbf{1} \Rightarrow A_{n+1}[\alpha_1,\alpha_2,\dots] = \mathbf{1}$$

is valid in $\mathfrak{A}$. Modus ponens $\frac{X,(X\supset Y)}{Y}$ is an example of a modus rule that is valid in any pseudo-Boolean algebra.

Let $\mathfrak{A}$ be a pseudo-Boolean algebra. We denote the set of all the formulae valid in $\mathfrak{A}$ (that is, formulae that are equal to $\mathbf{1}$ under any valuation in $\mathfrak{A}$) by $\mathbb{L}(\mathfrak{A})$; we denote the set of all modus rules valid in $\mathfrak{A}$ by $\mathbb{R}(\mathfrak{A})$. It is clear that $\mathbb{L}(\mathfrak{A})$ is closed under applications of rules from $\mathbb{R}(\mathfrak{A})$. $\mathbb{L}(\mathfrak{A})$ is the logic of the algebra $\mathfrak{A}$ (see [3]).

Let R be an arbitrary set of (modus) rules. Denote the set of all pseudo-Boolean algebras in which every rule from R is valid by $\mathbb{K}(\mathsf{R})$. Taking into account the relationship between validity of rules and corresponding quasi-identities, it is clear that $\mathbb{K}(\mathsf{R})$ is a quasivariety, and if $\mathsf{R}_1 \subseteq \mathsf{R}$, then

$$\mathbb{K}(\mathsf{R}_1) \subseteq \mathbb{K}(\mathsf{R}).$$

Let M be a set of formulae and R be a set of rules. We say that a formula A is derivable from M by R (and we write $\mathsf{M} \vdash_{\mathsf{R}} A$), if there is a sequence of formulae $A_1,\dots,A_m$ such that A_m is A and each formula A_i where $1 \le i \le m$ is valid in intuitionistic logic, or belongs to M, or can be derived from the preceding formulae by modus ponens or one of the rules from R. Let $\mathsf{R}(\mathsf{M}) = \{A \mid \mathsf{M} \vdash_{\mathsf{R}} A\}$. Clearly, $\mathsf{R}(\mathsf{M})$ is a superintuitionistic logic.

We call an ordered pair $\langle \mathsf{M}, \mathsf{R}\rangle$ a *modus instance of the logic* $\mathsf{R}(\mathsf{M})$.

We say that a rule (1) is derivable from R by M (we write $\mathsf{R} \vDash_{\mathsf{M}}$ (1)) if $\{A_1,\dots,A_n\} \cup \mathsf{M} \vdash_{\mathsf{R}} A_{n+1}$. If for rule r we have $\mathsf{R} \vDash_{\varnothing} \mathsf{r}$, we say that rule r is *derivable* from R (and we write $\mathsf{R} \vDash \mathsf{r}$). Let $\mathsf{M}(\mathsf{R}) = \{\mathsf{r} \mid \mathsf{R} \vDash_{\mathsf{M}} \mathsf{r}\}$.

We introduce a preorder on the set of all modus instances (of superintuitionistic logics): for modus instances $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ and $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ we let $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \leq \langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ if for each pair of formulae A, B if $A \cup \mathsf{M}_1 \vdash_{\mathsf{R}_1} B$, then $A \cup \mathsf{M}_2 \vdash_{\mathsf{R}_2} B$. Let us note that if $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \leq \langle M_2, \mathsf{R}_2 \rangle$, then for any collection of formulae $A_1, \dots, A_m, B$ if $\{A_1, \dots, A_m\} \cup M_1 \vdash_{\mathsf{R}_1} B$, then $\{A_1, \dots, A_m\} \cup M_2 \vdash_{\mathsf{R}_2} B$.

We say that the modus instances $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ and $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ are *equal* (and we write $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \simeq \langle \mathsf{M}_2, \mathsf{R}_2 \rangle$) if $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \leq \langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ and $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle \leq \langle \mathsf{M}_1, \mathsf{R}_1 \rangle$.

Lemma 1. *Let* $\langle \mathsf{M}, \mathsf{R} \rangle$ *be a modus instance. Then*

$$\langle \mathsf{M}, \mathsf{R} \rangle \simeq \langle \varnothing, \mathsf{M}(\mathsf{R}) \rangle.$$

Proof. Since for every formula C from M the rule

$$\frac{(X \supset X)}{C[X, Y, \dots]}$$

belongs to $\mathsf{M}(\mathsf{R})$, we have $\langle \mathsf{M}, \mathsf{R} \rangle \leq \langle \varnothing, \mathsf{M}(\mathsf{R}) \rangle$.

On the other hand, since $\mathsf{R} \vDash_{\mathsf{M}} \mathsf{r}$ holds for every r from $\mathsf{M}(\mathsf{R})$, if $A \vdash_{\mathsf{M}(\mathsf{R})} B$ then $\{A\} \cup \mathsf{M} \vdash_{\mathsf{R}} B$, that is, $\langle \varnothing, \mathsf{M}(\mathsf{R}) \rangle \leq \langle \mathsf{M}, \mathsf{R} \rangle$. □

Corollary 1. *Let* $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ *and* $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ *be modus instances. Then* $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \leq \langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ *if and only if* $\mathsf{M}_1(\mathsf{R}_1) \subseteq \mathsf{M}_2(\mathsf{R}_2)$.

Proof. Indeed, if $\mathsf{M}_1(\mathsf{R}_1) \subseteq \mathsf{M}_2(\mathsf{R}_2)$ then

$$\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \simeq \langle \varnothing, \mathsf{M}_1(\mathsf{R}_1) \rangle \leq \langle \varnothing, \mathsf{M}_2(\mathsf{R}_2) \rangle \simeq \langle \mathsf{M}_2, \mathsf{R}_2 \rangle. \quad \square$$

If $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \leq \langle \mathsf{M}_2, \mathsf{R}_2 \rangle$, then $\langle \varnothing, \mathsf{M}_1(\mathsf{R}_1) \rangle \leq \langle \varnothing, \mathsf{M}_2(\mathsf{R}_2) \rangle$. Suppose (1) is in $\mathsf{M}_1(\mathsf{R}_1)$. Then $A_1, \dots, A_n \vdash_{\mathsf{M}_1(\mathsf{R}_1)} A_{n+1}$ and, due to $\langle \varnothing, \mathsf{M}_1(\mathsf{R}_1) \rangle \leq \langle \varnothing, \mathsf{M}_2(\mathsf{R}_2) \rangle$, we have $A_1, \dots, A_n \vdash_{\mathsf{M}_2(\mathsf{R}_2)} A_{n+1}$. Thus, the rule (1) is in $\mathsf{M}_2(\mathsf{R}_2)$, that is, $\mathsf{M}_1(\mathsf{R}_1) \subseteq \mathsf{M}_2(R_2)$.

A set of all modus instances forms a lattice:

$$\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \cup \langle \mathsf{M}_2, \mathsf{R}_2 \rangle = \langle \mathsf{M}_1 \cup \mathsf{M}_2, \mathsf{R}_1 \cup \mathsf{R}_2 \rangle;$$
$$\langle \mathsf{M}_1, \mathsf{R}_1 \rangle \cap \langle \mathsf{M}_2, \mathsf{R}_2 \rangle = \langle \mathsf{R}_1(\mathsf{M}_1) \cap \mathsf{R}_2(\mathsf{M}_2), \mathsf{M}_1(\mathsf{R}_1) \cap \mathsf{M}_2(\mathsf{R}_2) \rangle.$$

A modus instance of the inconsistent logic is the greatest element of this lattice, and the modus instance $\langle I, \text{modus ponens} \rangle$, where I is intuitionistic logic, is the least element of this lattice.

Denote by $\mathcal{L}_{\mathsf{M}}$ the lattice of all modus instances of superintuitionistic logics, and denote by $\mathcal{L}_{\mathsf{K}}$ the lattice of all quasivarieties of pseudo-Boolean algebras.

Theorem 1. *The lattices $\mathcal{L}_\mathsf{M}$ and $\mathcal{L}_\mathsf{K}$ are dually isomorphic.*

Proof. Let $\langle \mathsf{M}, \mathsf{R}\rangle$ be modus instance and κ be a mapping $\mathcal{L}_\mathsf{M}$ to $\mathcal{L}_\mathsf{K}$ such that

$$\kappa : \langle \mathsf{M}, \mathsf{R}\rangle \mapsto \mathbb{K}(\mathsf{M}(\mathsf{R})).$$

Let us verify that κ is a dual isomorphism of $\langle \mathsf{M}, \mathsf{R}\rangle$ onto $\mathbb{K}(\mathsf{M}(\mathsf{R}))$. Let $\mathsf{K} \in \mathcal{L}_\mathsf{K}$. Consider a mapping $\mu :\mapsto \langle \varnothing, \bigcap_{\mathfrak{A}\in\mathsf{K}} \mathbb{R}(\mathfrak{A})\rangle$. Using Lemma 1, it is not hard to check that $\mu = \kappa^{-1}$, thus, κ is a 1-1-mapping of $\mathcal{L}_\mathsf{M}$ onto $\mathcal{L}_\mathsf{K}$. Let $\langle \mathsf{M}_1, \mathsf{R}_1\rangle$ and $\langle \mathsf{M}_2, \mathsf{R}_2\rangle$ be modus instances and $\langle \mathsf{M}_1, \mathsf{R}_1\rangle \le \langle \mathsf{M}_2, \mathsf{R}_2\rangle$. Then, by Corollary 1, $\mathsf{M}_1(\mathsf{R}_1) \subseteq \mathsf{M}_2(\mathsf{R}_2)$ which entails $\mathbb{K}(\mathsf{M}_2(\mathsf{R}_2)) \subseteq \mathbb{K}(\mathsf{M}_1(\mathsf{R}_1))$, that is, $\kappa(\langle \mathsf{M}_2, \mathsf{R}_2\rangle) \subseteq \kappa(\langle \mathsf{M}_1, \mathsf{R}_1\rangle)$. Thus, κ is a dual isomorphism of $\mathcal{L}_\mathsf{M}$ onto $\mathcal{L}_\mathsf{K}$. $\square$

A.V. Kuznetsov brought to author's attention the fact that the lattices $\mathcal{L}_\mathsf{M}$ and $\mathcal{L}_\mathsf{K}$ are dually isomorphic.

2 Defining modus instances by pseudo-Boolean algebras

Let Q be a set of pseudo-Boolean algebras and let $\langle \mathsf{M}, \mathsf{R}\rangle$ be a modus instance. We say that $\langle \mathsf{M}, \mathsf{R}\rangle$ is *defined by a set of pseudo-Boolean algebras* Q if $\mathsf{M}(\mathsf{R})$ consists of all rules that are valid in each algebra of Q. In other words, Q defines $\langle \mathsf{M}, \mathsf{R}\rangle$ if and only if $\mathsf{M}(\mathsf{R}) = \bigcap_{\mathfrak{A}\in\mathsf{Q}} \mathbb{R}(\mathfrak{A})$. It is clear that a set Q defines an instance $\langle \mathsf{M}, \mathsf{R}\rangle$ if and only if the set Q generates [4] the quasivariety $\mathbb{K}(\mathsf{M}(\mathsf{R}))$.

Theorem 2. *Let a set of pseudo-Boolean algebras* Q *define a modus instance* $\langle \mathsf{M}, \mathsf{R}\rangle$*. Then* $\langle \mathsf{M}, \mathsf{R}\rangle$ *is defined by a single pseudo-Boolean algebra* $\mathfrak{A}$*, which is a Cartesian product of algebras from* Q.

Proof. Let $\mathsf{r} \in \mathsf{M}(\mathsf{R})$. Since rule r is valid in every algebra from Q, this rule is valid in a Cartesian product of algebras from Q, that is, this rule is valid in $\mathfrak{A}$ (see the properties of quasivarieties in [4]).

Let $\mathsf{r} \notin \mathsf{M}(\mathsf{R})$ and let r be rule (1). Then (1) is invalid in some algebra $\mathfrak{B} \in \mathsf{Q}$. Let $\alpha_1, \alpha_2, \dots \in \mathfrak{B}$ and $A_1[\alpha_1, \alpha_2, \dots] = \dots = A_n[\alpha_1, \alpha_2, \dots] = \mathbf{1}$, but $A_{n+1}[\alpha_1, \alpha_2, \dots] \neq \mathbf{1}$. We can assume that the rule (1) is valid in the 2-element Boolean algebra Z_2 (otherwise (1) is invalid in $\mathfrak{A}$, for Z_2 is embedded in every non-trivial pseudo-Boolean algebra). Let $\phi : \mathfrak{B} \to \mathsf{Z}_2$ be a homomorphism of $\mathfrak{B}$ onto Z_2 and $\beta_1 = \phi(\alpha_1), \beta_2 = \phi(\alpha_2), \dots$. Then $A_1[\beta_1, \beta_2, \dots] = \dots = A_n[\beta_1, \beta_2, \dots] = \mathbf{1}$. Let us consider elements $\overline{\alpha}_1, \overline{\alpha}_2, \dots$ from $\mathfrak{A}$, such that the projection of $\overline{\alpha}_i$ in $\mathfrak{B}$ is α_i and the projection of $\overline{\alpha}_i$ in any other algebra from Q is β_i, where $i = 1, 2, \ldots$. Then $A_1[\overline{\alpha}_1, \overline{\alpha}_2, \ldots] = \cdots = A_n[\overline{\alpha}_1, \overline{\alpha}_2, \ldots] = \mathbf{1}$, while $A_{n+1}[\overline{\alpha}_1, \overline{\alpha}_2, \ldots] \neq \mathbf{1}$, because the projection of $A_{n+1}[\overline{\alpha}_1, \overline{\alpha}_2, \ldots]$ is the element $A_{n+1}[\alpha_1, \alpha_2, \ldots]$ and it is distinct from $\mathbf{1}$. □

Theorem 3. *Every modus instance is defined by some pseudo-Boolean algebra.*

Proof. Let $\langle \mathsf{M}, \mathsf{R} \rangle$ be a modus instance. Let us verify that the set $\mathbb{K}(\mathsf{M}(\mathsf{R}))$ defines $\langle \mathsf{M}, \mathsf{R} \rangle$. Indeed, by definition, if $\mathsf{r} \in \mathsf{M}(\mathsf{R})$, then r is valid in every algebra from $\mathbb{K}(\mathsf{M}(\mathsf{R}))$. Let $\mathsf{r} \notin \mathsf{M}(\mathsf{R})$ and $\mathsf{R}_1 = \mathsf{M}(\mathsf{R}) \cup \{\mathsf{r}\}$. Then $\langle \mathsf{M}, \mathsf{R} \rangle \simeq \langle \varnothing, \mathsf{M}(\mathsf{R}) \rangle \leq \langle \varnothing, \mathsf{R}_1 \rangle$ and $\langle \varnothing, \mathsf{M}(\mathsf{R}) \rangle \not\simeq \langle \varnothing, \mathsf{R}_1 \rangle$. Since κ is a dual isomorphism, $\kappa(\langle \varnothing, \mathsf{R}_1 \rangle)) \subseteq \kappa(\langle \varnothing, \mathsf{M}(\mathsf{R}) \rangle)$, we have $\mathbb{K}(\mathsf{R}_1) \subseteq \mathbb{K}(\mathsf{M}(\mathsf{R}))$ and the inclusion is proper. Let $\mathfrak{A} \in \mathbb{K}(\mathsf{M}(\mathsf{R}))$ and $\mathfrak{A} \notin \mathbb{K}(\mathsf{R}_1)$. Then r is invalid in $\mathfrak{A}$. So, it has been proven that the set $\mathbb{K}(\mathsf{M}(\mathsf{R}))$ defines the modus instance $\langle \mathsf{M}, \mathsf{R} \rangle$, and we can apply Theorem 2 and complete the proof. □

Remark. Theorem 3 follows also from the results from [5].

Theorem 4. *Let an algebra $\mathfrak{A}$ define a modus instance $\mathcal{L}_1 = \langle M_1, \mathsf{R}_1 \rangle$ and an algebra $\mathfrak{B}$ define a modus instance $\mathcal{L}_1 = \langle M_2, \mathsf{R}_2 \rangle$. Then algebra $\mathfrak{A} \times \mathfrak{B}$ defines the modus instance $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$.*

Proof. Let us verify that the set $\{\mathfrak{A}, \mathfrak{B}\}$ defines $\mathcal{L}$. Since $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{L}_2$ and κ is a dual isomorphism (see the proof on the Theorem 1), $\kappa\mathcal{L} = \kappa\mathcal{L}_1 \cup' \kappa\mathcal{L}_2$. $\kappa\mathcal{L}_1$ is a quasivariety defined by $\mathfrak{A}$, $\kappa\mathcal{L}_2$ is a quasivariety defined by $\mathfrak{B}$. Therefore $\kappa\mathcal{L}$ is generated by the algebras $\mathfrak{A}$ and $\mathfrak{B}$, that is, $\mathcal{L}$ is generated by $\{\mathfrak{A}, \mathfrak{B}\}$. An application of Theorem 2 completes the proof. □

3 Kuznetsov's theorem about finitely generated pseudo-Boolean algebras

In [6], A.V. Kuznetsov announced a result the proof of which was not published. Below, we offer a proof of this result significantly different from the proof communicated to the author by Kuznetsov.

A pseudo-Boolean algebra in which the order is linear is called a *chain algebra.*

Lemma 2. *Let $\mathfrak{B}$ be a finite chain pseudo-Boolean algebra embedded in a homomorphic image of some algebra $\mathfrak{A}$. Then $\mathfrak{B}$ is also embedded in $\mathfrak{A}$.*

Proof. Let $\mathfrak{B}$ be embedded in $\mathfrak{A}/\nabla$, where ∇ is a filter [2] of the algebra $\mathfrak{A}$, and let $\mathfrak{B}'$ be a subalgebra of $\mathfrak{A}/\nabla$ that is isomorphic to $\mathfrak{B}$. The algebra $\mathfrak{B}'$ consists of conjugated classes relative to the filter ∇. In each of these conjugated classes, we take elements $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ (where n is a cardinality of $\mathfrak{B}$), one element per class, in such a way that $\alpha_0 = \mathbf{0}$, $\alpha_{n_1} = \mathbf{1}$ and if $i \leq j$, then $\alpha_i \supset \alpha_j \in \nabla$ $(i, j = 1, 2, \ldots, n-2)$. Let

$$\alpha = \underset{i<j}{\&}(\alpha_i \supset \alpha_j) \ \& \ \underset{j<i}{\&}((\alpha_i \supset \alpha_j) \supset \alpha_j).$$

Due to $\mathfrak{B}'$ being a chain algebra, if $i < j$ then the element $((\alpha_i \supset \alpha_j) \supset \alpha_j)$ is in ∇ $(i, j \in \{0, \ldots, n-1\})$, that is, $\alpha \in \nabla$. Let $\beta_0 = \mathbf{0}$, $\beta_i = (\alpha \supset \alpha_i)$ for $i = 1, \ldots, n-1$. Let us check that the elements $\beta_0, \beta_1, \ldots, \beta_{n-1}$ form a chain subalgebra of $\mathfrak{A}$. Indeed, if $0 \leq i \leq j \leq n-1$, then $\beta_i \supset \beta_j \mathbf{1}$, because $((\alpha \supset \alpha_i) \supset (\alpha \supset \alpha_j)) = (\alpha \supset (\alpha_i \supset \alpha_j)) = \mathbf{1}$. Let us check that if $0 \leq i \leq j \leq n-1$, then $(\beta_j \supset \beta_i) = \beta_i$ holds. It is clear that $\beta_i \leq (\beta_j \supset \beta_i)$. On the other hand,

$$\begin{aligned}
&((\beta_j \supset \beta_i) \supset \beta_i) = (((\alpha \supset \alpha_j) \supset (\alpha \supset \alpha_i)) \supset (\alpha \supset \alpha_i)) = \\
&((\alpha \supset (\alpha_j \supset \alpha_i)) \supset (\alpha \supset \alpha_i)) = \alpha \supset (((\alpha_j \supset \alpha_i) \supset \alpha_i)) = \mathbf{1},
\end{aligned} \tag{1}$$

i.e. $(\beta_j \supset \beta_i) \leq \beta_i$. Hence, $(\beta_j \supset \beta_i) = \beta_i$ for all $0 \leq i < j \leq n-1$. In case of $0 < i \leq n-1$, we get $\neg\beta_i = (\beta_i \supset \mathbf{0}) = (\beta_i \supset \beta_0) = \beta_0 = \mathbf{0}$. So, $\beta_0 \leq \beta_1 \leq \cdots \leq \beta_{n-1}$ and elements $\beta_0, \beta_1, \ldots, \beta_{n-1}$ form a chain subalgebra which is obviously isomorphic to $\mathfrak{B}$. □

Theorem 5 (Kuznetsov's Theorem). *Let $\mathfrak{A}$ be a finitely generated pseudo-Boolean algebra, the cardinalities of all its chain subalgebras is bounded by a number n. Then $\mathfrak{A}$ is finite.*

Proof. By induction on n we will prove that if $\mathfrak{A}$ is m-generated pseudo-Boolean algebra and all chain subalgebras of $\mathfrak{A}$ have cardinality $\leq n$, then

$$|\mathfrak{A}| \leq k_m(n),$$

where $k_m(n)$ is defined in the following way:

$$\begin{aligned}
&k_m(2) = 2^{2^n}, \\
&k_m(i+1) = (k_m(i)+1)^{s(i)(k_m(i)+1)^m},
\end{aligned} \tag{2}$$

where $s_m(i)$ is a number of all distinct modulo isomorphism m-generated pseudo-Boolean algebras which chain subalgebras have cardinalities $\leq i$.

If an m-generated pseudo-Boolean algebra $\mathfrak{A}$ has chain subalgebras only of cardinality ≤ 2, then $\mathfrak{A}$ is a Boolean algebra and $|\mathfrak{A}| \leq 2^{2^n}$ (see [4]).

Let every m-generated algebra, the cardinalities of all chain subalgebras of which are bounded by n, have cardinality $\leq k_m(n)$; and let there also be only $\leq s_m(n)$ distinct modulo isomorphism such algebras.

Let $\mathfrak{A}$ be a m-generated algebra, all chain subalgebras of which have cardinalities $\leq n+1$. Let us consider two cases: 1) $\mathfrak{A}$ is subdirectly irreducible; 2) $\mathfrak{A}$ is subdirectly reducible.

1) If $\mathfrak{A}$ is subdirectly irreducible, then it has a pre-top element [2], that is, the unique element covered by $\mathbf{1}$.

Let us denote by ω the pre-top element of $\mathfrak{A}$ and let $\nabla = \{\mathbf{1}, \omega\}$. ∇ is a filter of $\mathfrak{A}$. Let $\mathfrak{B} = \mathfrak{A}/\nabla$. All chain subalgebras of $\mathfrak{B}$ have cardinalities at most n. Since $\mathfrak{B}$ is a homomorphic image of an m-generated algebra, $\mathfrak{B}$ is an m-generated algebra. Therefore, we can apply to $\mathfrak{B}$ the induction assumption, i.e. we can assume $|\mathfrak{B}| \leq k_m(n)$. It is not hard to see that $|\mathfrak{A}| = |\mathfrak{B}| + 1$, so $|\mathfrak{A}| \leq k_m(n) + 1$ and modulo isomorphism there are at most $s_m(n)$ such algebras.

2) Let $\mathfrak{A}$ be reducible to a subdirect product of subdirectly irreducible algebras. It is known [4] that the subdirect factors are homomorphic images of the algebra $\mathfrak{A}$ and all are m-generated. By Lemma 2, homomorphic images of $\mathfrak{A}$ cannot contain chain subalgebras of cardinality $> n+1$. Hence, the cardinalities of subdirect factors are bounded by $k_m(n) + 1$, and the number of such factors does not exceed s_m. Using the formula 1 from [4, p. 358], we get

$$|\mathfrak{A}| \leq (k_m(n) + 1)^{s_m(n)(k_m(n)+1)^m}. \qquad \square$$

4 Generalization of Troelstra–Kuznetsov's Theorem

A modus instance $\langle \mathsf{M}, \mathsf{R} \rangle$ is called *tabular* if it can be defined by a finite pseudo-Boolean algebra.

A modus instance $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ is an *immediate predecessor* of a modus instance $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ if in the lattice $\mathcal{L}_{\mathsf{M}}$ the modus instance $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$ covers the modus instance $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$.

In [7], A.S. Troelstra stated a theorem about superintuitionistic logics, but the proof was based on an incorrect statement (see [8, 9]). The proof of this theorem was obtained by Kuznetsov [9]. Below we prove this theorem for modus instances.

Theorem 6 (Generalization of Troelstra–Kuznetsov theorem). *Let a modus instance $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ be an immediate predecessor of a tabular modus instance $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$. Then the modus instance $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ is tabular.*

Proof. Let the finite pseudo-Boolean algebra $\mathfrak{A}$ define the modus instance $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$. Let us show that there is a finite pseudo-Boolean algebra $\mathfrak{B}$ such that all rules from $\mathsf{M}_1(\mathsf{R}_1)$ are valid in $\mathfrak{B}$, but at least one rule from $\mathsf{M}_2(\mathsf{R}_2)$ is invalid in $\mathfrak{B}$. Indeed, since $\mathsf{M}_1(\mathsf{R}_1) \subseteq \mathsf{M}_2(\mathsf{R}_2)$, we have $\mathbb{K}(\mathsf{M}_2(\mathsf{R}_2)) \subseteq \mathbb{K}(\mathsf{M}_1(\mathsf{R}_1))$ and the inclusion is proper. Due to $\mathbb{K}(\mathsf{M}_2(\mathsf{R}_2))$ being generated by finite algebra $\mathfrak{A}$, all subdirectly irreducible algebras from $\mathbb{K}(M_2(\mathsf{R}_2))$ have cardinalities bounded by $m = |\mathfrak{A}|$ (because the formula E_m from [10] is valid in all algebras of $\mathbb{K}(M_2(\mathsf{R}_2))$). Suppose that there are no finite algebras in $\mathbb{K}(\mathsf{M}_1(\mathsf{R}_1)) \setminus \mathbb{K}(\mathsf{M}_2(\mathsf{R}_2))$ and let $\mathfrak{B}$ be a finitely generated algebra from $\mathbb{K}(\mathsf{M}_1(\mathsf{R}_1)) \setminus \mathbb{K}(\mathsf{M}_2(\mathsf{R}_2))$. Then, by our assumption, $\mathfrak{B}$ is infinite and, by virtue of Theorem 5, the cardinalities of chain subalgebras of $\mathfrak{B}$ are not bounded, that is, among chain subalgebras of $\mathfrak{B}$ there are algebras of cardinality $> m$. These algebras cannot be in $\mathbb{K}(M_2(R_2))$, so they are in $\mathbb{K}(\mathsf{M}_1(\mathsf{R}_1)) \setminus \mathbb{K}(M_2(R_2))$. The obtained contradiction shows that $\mathbb{K}(\mathsf{M}_1(\mathsf{R}_1)) \setminus \mathbb{K}(M_2(R_2))$ contains at least one finite algebra. Let K be the quasivariety generated by the algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then $\mathbb{K}(M_2(R_2)) \subseteq \mathsf{K} \subseteq \mathbb{K}(M_1(R_1)$ and the inclusion $\mathbb{K}(M_2(R_2)) \subseteq \mathsf{K}$ is proper. Because the modus instance $\langle \mathsf{M}_1, \mathsf{R}_1 \rangle$ **immediately** preceding $\langle \mathsf{M}_2, \mathsf{R}_2 \rangle$, we have $\mathsf{K} = \mathbb{K}(\mathsf{M}_2(\mathsf{R}_2))$, that is $\{\mathfrak{A}, \mathfrak{B}\}$ defines $\mathbb{K}(\mathsf{M}_1(\mathsf{R}_1))$. An application of Theorem 2 completes the proof.

5 Finitely Presented pseudo-Boolean Algebras

Let A be a formula and $p_1, \dots, p_n$ be a list containing all propositional variables occurring in A, $\mathfrak{A}$ be a pseudo-Boolean algebra and ϕ be a mapping of $p_1, \dots, p_n$ to $\mathfrak{A}$.

We say that a formula A with a set of generating symbols $p_1, \dots, p_n$ *defines* the algebra $\mathfrak{A}$ by ϕ, if

1) the elements $\phi(p_1), \dots, \phi(p_n)$ generate $\mathfrak{A}$;

2) $A[\phi(p_1), \dots, \phi(p_n)] = \mathbf{1}$;

3) for any formula $B(p_1, \dots, p_n)$ such that $B[\phi(p_1)], \dots, \phi(p_n) = \mathbf{1}$, the formula $A \supset B$ is valid in intuitionistic logic.

From now on, we assume that all variables $p_1, \ldots, p_n$ occur in A (if p_i does not occur in A, we can replace A with A', where $A' = A \& (p_i \supset p_i)))$. So, we will say that A defines $\mathfrak{A}$ by ϕ and we will write $\mathfrak{A} = \mathfrak{A}(A, \phi)$.

We say that a *formula A defines an algebra* if there exists such a map ϕ that $\mathfrak{A} = \mathfrak{A}(A, \phi)$.

Let $\mathfrak{A}$ be a finite subdirectly irreducible pseudo-Boolean algebra and ω be its pre-top element. Let $\mathfrak{A} = \mathfrak{A}(A, \phi)$ and $B(p_1, \ldots, p_n)$ be a formula such that $B[\phi(p_1), \ldots, \phi(p_n)] = \omega$. The rule

$$\frac{A[X_1, \ldots, X_n]}{B[X_1, \ldots, X_n]} \tag{2}$$

is called a *quasi-characteristic rule of algebra* $\mathfrak{A}$ (comp. with the notion of characteristic formula [11])

Theorem 7. *Let* r *be a quasi-characteristic rule of a pseudo-Boolean algebra* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be a pseudo-Boolean algebra. The following conditions are equivalent:*

1) r *is refuted in* $\mathfrak{B}$

2) $\mathfrak{A}$ *is isomorphically embedded in* $\mathfrak{B}$.

Proof. 2) ⇒ 1). Suppose $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{B}$. Then r is refuted in the subalgebra of $\mathfrak{B}$ that is isomorphic to $\mathfrak{A}$, that is r is invalid in $\mathfrak{B}$.

1) ⇒ 2). Suppose (2) is a quasi-characteristic rule of algebra $\mathfrak{A}(A, \phi)$ and rule (2) is refuted in $\mathfrak{B}$. Then for some $\beta_1, \ldots, \beta_n \in \mathfrak{B}$ the following holds

$$A[\beta_1, \ldots, \beta_n] = \mathbf{1} \text{ and } B[\beta_1, \ldots, \beta_n] \neq \mathbf{1}.$$

Let $\mathfrak{B}'$ be a subalgebra of $\mathfrak{B}$ generated by elements $\beta_1, \ldots, \beta_n$. By Theorem 1 [4, p. 276], there is a homomorphism of algebra $\mathfrak{A}$ onto $\mathfrak{B}'$ such that $\phi(\psi(p_i)) = \beta_i$ for $i = 1, \ldots, n$. Since

$$\begin{aligned} \psi(\omega) &= \psi(B[\phi(p_1), \ldots, \phi(p_n)]) = B[\psi(\phi(p_1)), \ldots, \psi(\phi(p_n))] \\ &= B[\beta_1, \ldots, \beta_n] \neq \mathbf{1} \end{aligned} \tag{3}$$

and, since, the pre-top element is in every non-trivial filter, ψ is an isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$. And this proves Theorem 7 (comp. theorem 7 with results from [12]). □

A pseudo-Boolean algebra $\mathfrak{A}$ is called *finitely presented* if $\mathfrak{A} = \mathfrak{A}(A)$ for some formula A.

Theorem 8. *Every finitely presented pseudo-Boolean algebra is finitely approximated.*

Proof. Let $\mathfrak{A} = \mathfrak{A}(A, \phi)$ and $\beta \in \mathfrak{A}$. Suppose $\beta \neq \mathbf{1}$ and $A = A(p_1, \ldots, p_n)$. Due to elements $\phi(p_1), \ldots, \phi(p_n)$ generating $\mathfrak{A}$, for some formula B, we have $\beta = B[\phi(p_1), \ldots, \phi(p_n)]$. Since $B[\phi(p_1), \ldots, \phi(p_n)] \neq \mathbf{1}$, formula $A \supset B$ is invalid in the intuitionistic logic. Hence, there is a finite pseudo-Boolean algebra $\mathfrak{B}$ and elements $\beta_1, \ldots, \beta_n$ from $\mathfrak{A}$, such that $A[\beta_1, \ldots, \beta_n] = \mathbf{1}$ and $B[\beta_1, \ldots, \beta_n] \neq \mathbf{1}$. It is clear that $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, so $\mathfrak{A}$ is finitely approximated.□

Corollary 2. *Every subdirectly irreducible finitely approximated pseudo-Boolean algebra is finite.*

6 Modus Complete Table Modus Instances

Let $\langle \mathsf{M}, \mathsf{R} \rangle$ be a modus instance and let r be a modus rule. The rule r is said to be *admissible* in $\langle \mathsf{M}, \mathsf{R} \rangle$ (comp. [13]), if logic $\mathsf{R}(\mathsf{M})$ is closed relative to applications of r.

A rule r is said to be *admissible in a superintuitionistic logic* L, if L is closed relative to applications of r.

A rule r is called *derivable in a modus instance* $\langle \mathsf{M}, \mathsf{R} \rangle$, if $\mathsf{r} \in \mathsf{M}(\mathsf{R})$.

A rule r is called *derivable in a superintuitionistic logic* L if r is derivable in $\langle \mathsf{L}, \text{modus ponens} \rangle$.

A modus instance is called *modus complete*, if each admissible in it modus rule is derivable (comp. with the notion of structural completeness [14]).

A superintuitionistic logic is called *modus complete*, if each admissible in it modus rule is derivable.

We will call a pseudo-Boolean algebra $\mathfrak{A}$ *modus poor*, if validity on $\mathfrak{A}$ of the rule (1) yields validity in $\mathfrak{A}$ of the formula $((A_1 \& \ldots \& A_n) \supset A_{n+1})$. In other words, an algebra $\mathfrak{A}$ is modus poor, if it defines the modus instance $\langle \mathbb{L}(\mathfrak{A}), \text{modus ponens} \rangle$.

Theorem 9. *Let $\mathfrak{A}$ be a finite pseudo-Boolean algebra. The following conditions are equivalent:*

- *$\mathfrak{A}$ is modus poor;*
- *every subdirectly irreducible homomorphic image of $\mathfrak{A}$ is isomorphically embedded in $\mathfrak{A}$.*

Proof. 1) ⇒ 2). Let $\mathfrak{A}'$ be a subdirectly irreducible homomorphic image of $\mathfrak{A}$ and let (2) be a characteristic rule of $\mathfrak{A}'$. Since the formula $A \supset B$ is invalid in $\mathfrak{A}'$, it is invalid in $\mathfrak{A}$ too. Since $\mathfrak{A}$ is modus poor, the rule (2) is invalid in $\mathfrak{A}$. By virtue of Theorem 7, $\mathfrak{A}'$ is isomorphically embedded in $\mathfrak{A}$.

2) ⇒ 1). By contradiction: suppose the rule (1) is valid in $\mathfrak{A}$ while the formula $((A_1 \& \ldots \& A_n) \supset A_{n+1})$ is invalid in $\mathfrak{A}$, that is for some $\alpha_1, \ldots, \alpha_m \in \mathfrak{A}$ we have

$$((A_1[\alpha_1, \ldots, \alpha_m] \& \ldots \& A_n[\alpha_1, \ldots, \alpha_m]) \supset A_{n+1}[\alpha_1, \ldots, \alpha_m] \neq \mathbf{1}.$$

Let $\alpha = ((A_1[\alpha_1, \ldots, \alpha_m] \& \ldots \& A_n[\alpha_1, \ldots, \alpha_m])$. Then

$$\alpha \supset A_{n+1}[\alpha_1, \ldots, \alpha_m] \neq \mathbf{1},$$

that is, there is a filter ∇ of algebra $\mathfrak{A}$ such that

$$\alpha \in \nabla \text{ and } A_{n+1}[\alpha_1, \ldots, \alpha_m] \notin \nabla.$$

Let ∇' be a maximal filter such that

$$\alpha \in \nabla' \text{ and } A_{n+1}[\alpha_1, \ldots, \alpha_m] \notin \nabla'.$$

Then $\mathfrak{A}/\nabla'$ is subdirectly irreducible (an image of element $A_{n+1}[\alpha_1, \ldots, \alpha_m]$ under the canonical map is a pre-top element). Let ψ be the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}'$. Then

$$\psi(\alpha) = A_1[\psi(\alpha_1), \ldots, \psi(\alpha_m)] \& \ldots \& A_n[\psi(\alpha_1), \ldots, \psi(\alpha_m)] = \mathbf{1}$$

but

$$A_{n+1}[\psi(\alpha_1), \ldots, \psi(\alpha_m)] \neq \mathbf{1},$$

which means that the rule (1) is invalid in $\mathfrak{A}/\nabla'$. Since, due to 2), algebra $\mathfrak{A}/\nabla'$ is isomorphically embedded in $\mathfrak{A}$, the rule (1) is invalid in $\mathfrak{A}$, and the latter contradicts the assumption. □

Theorem 10. *There is an algorithm that, given a finite pseudo-Bollean algebra $\mathfrak{A}$, recognizes whether $\mathfrak{A}$ is modus poor.*

Proof. The desired algorithm consists of enumerating all subdirectly irreducible homomorphic images of $\mathfrak{A}$ and verifying of whether they are isomorphically embedded in $\mathfrak{A}$. Due to $\mathfrak{A}$ being finite, the process is effective. □

Theorem 11. *Let L be a superintuitionistic logic and let $\mathfrak{F}(\mathsf{L})$ be its Lindenbaum algebra. Then the algebra $\mathfrak{F}(\mathsf{L})$ defines the modus complete instance of logic L.*

Proof. Let R be a set of all admissible in L rules. Then $\langle \mathsf{L}, \mathsf{R} \rangle$ is a modus complete instance. Let us verify that algebra $\mathfrak{F}(\mathsf{L})$ defines the instance $\langle \mathsf{L}, \mathsf{R} \rangle$. Note that every derivable in $\langle \mathsf{L}, \mathsf{R} \rangle$ rule is admissible in L, that is, $\mathsf{L}(\mathsf{R}) = \mathsf{R}$. Let us check that a rule r is valid in $\mathfrak{F}(\mathsf{L})$ if and only if it is in R, i.e. it is admissible in L.

Suppose the rule (1) is admissible in L. Let $B_1, B_2, \dots$ be formulae such that $A(B_1, \dots), A_2(B_1, \dots), \dots, A_n(B_1, \dots) \in \mathsf{L}$, that is, if we regard the formulae $B_1, B_2, \dots$ as elements of $\mathfrak{F}(\mathsf{L})$, then

$$A_1[B_1, \dots] = A_2[B_1, \dots] = \dots = A_n[B_1, \dots] = \mathbf{1}.$$

Due to admissibility of the rule (1) in L, we have $A_{n+1}(B_1, \dots) \in \mathsf{L}$, that is,

$$A_{n+1}[B_1, \dots] = \mathbf{1}.$$

Hence, the rule (1) is valid in $\mathfrak{F}(\mathsf{L})$.

Now, suppose that rule (1) is valid in $\mathfrak{F}(\mathsf{L})$. Using the arguments analogous to above, one can conclude that the rule (1) is admissible in L. $\square$

By $\mathfrak{F}_m(\mathsf{L})$ we denote the subalgebra of $\mathfrak{F}(\mathsf{L})$ consisting of all formulae not containing variables distinct from $p_1, p_2, \dots, p_m$.

Theorem 12. *Let L be the logic of a finite pseudo-Boolean algebra $\mathfrak{A}$ and let the elements $\alpha_1, \dots, \alpha_m$ generate $\mathfrak{A}$. Then the algebras $\mathfrak{F}(\mathsf{L})$ and $\mathfrak{F}_n(\mathsf{L})$, where $n \geq m$, define the same modus instance.*

Proof. Since $\mathfrak{F}_n(\mathsf{L})$ is a subalgebra of $\mathfrak{F}(\mathsf{L})$, we have $\mathbb{R}(\mathfrak{F}(\mathsf{L})) \subseteq \mathbb{R}(\mathfrak{F}_n(\mathsf{L}))$.

Now, let us verify $\mathbb{L}(\mathfrak{F}_n(\mathsf{L})) \subseteq \mathbb{L}(\mathfrak{F}(\mathsf{L}))$. Indeed, if a formula A is invalid in $\mathfrak{F}(\mathsf{L})$, then $A \notin \mathsf{L}$ and, hence, A is invalid in $\mathfrak{A}$. Let $\beta_1, \dots, \beta_s \in \mathfrak{A}$ and $A[\beta_1, \dots, \beta_s] \neq \mathbf{1}$. Due to $\mathfrak{A}$ being generated by the elements $\alpha_1, \dots, \alpha_m$, there are formulae $B_1(p_1, \dots, p_m), \dots, B_s(p_1, \dots, p_m)$ such that $\beta_i = B_i[\alpha_i, \dots, \alpha_m]$ for $i = 1, \dots, s$. Therefore,

$$A[B_1[\alpha_1, \dots, \alpha_m], \dots, B_s[\alpha_1, \dots, \alpha_m]] \neq \mathbf{1}. \qquad \square$$

In the latter formula, let us replace every occurrence of α_i with $p_i, i = 1, \dots, m$. We obtain a formula $A'(p_1, \dots, p_m)$ which is invalid in L (otherwise it would be valid in $\mathfrak{A}$, which is not true). Due to $m \leq n$, the formula A' is invalid in $\mathfrak{F}_n(\mathsf{L})$, and, hence, the formula A is invalid in $\mathfrak{F}_n(\mathsf{L})$. Thus, the algebra $\mathfrak{F}_n(\mathsf{L})$ defines a modus instance of logic L. By Theorem 11, it cannot be greater than the modus instance defined by $\mathfrak{F}(\mathsf{L})$. We can apply Corollary 1 and conclude that $\mathbb{R}(\mathfrak{F}_n(\mathsf{L})) \subseteq \mathbb{R}(\mathfrak{F}(\mathsf{L}))$, i.e. $\mathbb{R}(\mathfrak{F}_n(\mathsf{L})) = \mathbb{R}(\mathfrak{F}(\mathsf{L}))$ and the algebras $\mathfrak{F}_n(\mathsf{L})$ and $\mathfrak{F}(\mathsf{L})$ define the same modus instance.

Corollary 3. *A superintuitionistic logic* L *is modus complete if and only if the algebra* $\mathfrak{F}(\mathsf{L})$ *defines the modus instance* $\langle \mathsf{L}$, *modus ponens*$\rangle$.

Lemma 3. *Let* $\mathfrak{A}$ *be a pseudo-Boolean algebra.* $\mathfrak{A}$ *is modus poor if and only if* $\mathfrak{A}$ *defines the modus instance* $\langle \mathbb{L}(\mathfrak{A})$, *modus ponens*$\rangle$

Proof. Suppose $\mathfrak{A}$ defines the modus instance $\langle \mathbb{L}(\mathfrak{A}), \text{modus ponens}\rangle$ and the rule (1) is valid in $\mathfrak{A}$. Then the formula A_{n+1} is derivable from the formulae $A_1, \ldots, A_n$ and $\mathbb{L}(\mathfrak{A})$ by modus ponens. By the deduction theorem, $((A_1 \& \ldots A_n) \supset A_{n+1}) \in \mathbb{L}(\mathfrak{A})$, that is, formula $((A_1 \& \ldots A_n) \supset A_{n+1})$ is valid in $\mathfrak{A}$.

Suppose $\mathfrak{A}$ is modus poor and it defines a modus instance $\langle \mathbb{L}(\mathfrak{A}), \mathsf{R}\rangle$. Due to $\mathfrak{A}$ being modus poor, if the rule (1) is in R, then $((A_1 \& \ldots A_n) \supset A_{n+1}) \in \mathbb{L}(\mathfrak{A})$. So, $\mathfrak{A}$ defines the modus instance $\langle \mathbb{L}(\mathfrak{A}), \text{modus ponens}\rangle$.

Corollary 4. *A logic* L *is modus complete if and only if the algebra* $\mathfrak{F}(\mathsf{L})$ *is modus poor.*

Theorem 13. *Let a finite pseudo-Boolean algebra* $\mathfrak{A}$ *define a logic* L. *There exists an algorithm that by algebra* $\mathfrak{A}$ *is checking whether logic* L *is modus complete.*

Proof. The algorithm: using $\mathfrak{A}$, construct the algebra $\mathfrak{F}_n(\mathsf{L})$, where n is at least the number of generators of $\mathfrak{A}$, and check whether $\mathfrak{F}_n(\mathsf{L})$ is modus poor. □

Example 1. *Let* L *be a logic of a finite chain pseudo-Boolean algebra* $\mathfrak{A}$ *and* $n = |\mathfrak{A}|$. *Then* L *is modus complete (comp. [15]). Indeed, all subdirectly irreducible homomorphic images of algebra* $\mathfrak{F}_n(\mathsf{L})$ *are chain algebras and, by Lemma 2, they are isomorphically embedded in* $\mathfrak{F}_n(\mathsf{L})$. *By Theorem 9,* $\mathfrak{F}_n$ *is modus poor. By Theorem 12, logic* L *is modus complete.*

Example 2.

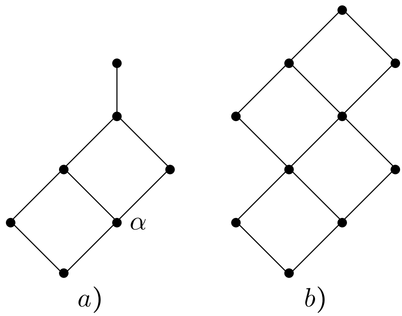

Fig. 1.

The diagram of the pseudo-Boolean algebra Z_7 *is depicted at Fig.1 a), and the diagram of the pseudo-Boolean algebra* Z_{10}*, which is isomorphic to* $\mathfrak{F}_1(\mathsf{L}(\mathsf{Z}_7))$*, is depicted at Fig.1 b). Note, that the algebra* Z_7 *is generated by the element* α *and is a homomorphic image of the algebra* Z_{10}*, while* Z_7 *is not isomorphically embedded in* Z_{10}*. Thus, the logic* $\mathbb{L}(\mathsf{Z}_7)$ *is not modus complete. In particular, the following quasi-characteristic rule of the algebra* Z_7 *is admissible but not derivable in this logic:*

$$\frac{((\neg\neg X \supset X) \supset (X \vee \neg X))}{(\neg X \vee \neg\neg X)}.$$

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References

[1] J. Łoś and R. Suszko. "Remarks on sentential logics". In: *Nederlandse Akademie van Wetenschappen. Proceedings. Series A. Indagationes Mathematicae* 20 (1958), pp. 177–183. ISSN: 0023-3358. Zbl: 0092.24802.

[2] H. Rasiowa and R. Sikorski. *The mathematics of metamathematics.* Vol. 41. Monografie Matematyczne. Państwowe Wydawnictwo Naukowe, 1963. Zbl: 0122.24311.

[3] A. V. Kuznecov and V. J. Gerčiu. “The superintuitionistic logics and finitary approximability”. In: *Doklady Akademii Nauk SSSR* 195 (1970), pp. 1029–1032. ISSN: 0002-3264. MR: 0295891. A. Kuznetsov and V. Gerchiu. “Superintuitionistic logics and finite approximability”. In: *Soviet Mathematics Doklady* 11.6 (1970), pp. 1614–1619. ISSN: 0197-6788. Zbl: 0219.02014.

[4] A. I. Mal’cev. *Algebraic Systems.* Vol. 192. Die Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 1973. DOI: 10.1007/978-3-642-65374-2. Zbl: 0266.08001. Trans. of A. I. Mal’cev. *Algebraic systems.* Posthumous edition, edited by D. Smirnov and M. Taĭclin. Springer-Verlag, New York-Heidelberg, 1973. MR: 0349384.

[5] R. Wójcicki. “Matrix approach in methodology of sentential calculi”. In: *Studia Logica* 32.1 (1973), pp. 7–37. DOI: 10.1007/BF02123806.

[6] A. V. Kuznetsov. “On finitely generated pseudo-Boolean algebras and finitely approximable varieties”. In: *Proceedings of the 12th USSR Algebraic Colloquium.* (in Russian). Sverdlovsk, 1973, p. 281.

[7] A. S. Troelstra. “On intermediate propositional logics”. In: *Indagationes Mathematicae* 27 (1965), pp. 141–152.

[8] V. A. Jankov. “The construction of a sequence of strongly independent superintuitionistic propositional calculi”. In: *Doklady Akademii Nauk SSSR* 181 (1968), pp. 33–34. ISSN: 0002-3264. MR: 0232661. V. Jankov. “Constructing a sequence of strongly independent superintuitionistic propositional calculi”. In: *Soviet Mathematics Doklady* 9 (1968), pp. 801–807. Zbl: 0198.31803.

[9] A. V. Kuznetsov. “Some properties of the lattice of varieties of pseudo-Boolean algebras”. In: *XI All-Union Algebraic Colloquium. Abstracts.* in Russian. 1971, pp. 255–256.

[10] C. G. McKay. “On finite logics”. In: *Nederlandse Akademie van Wetenschappen. Proceedings. Series A. Indagationes Mathematicae* 29.3 (1967), pp. 363–365. ISSN: 0023-3358. MR: 0215689. Zbl: 0153.00702.

[11] V. A. Jankov. “Conjunctively irresolvable formulae in propositional calculi”. In: *Izv. Akad. Nauk SSSR Ser. Mat.* 33 (1969), pp. 18–38. V. Jankov. “Conjunctively Indecomposable Formulas in Propositional Calculus”. In: *Mathematics of the USSR-Izvestiya* 3.1 (1969), pp. 17–35. DOI: 10.1070/IM1969v003n01ABEH000744.

[12] A. I. Budkin and V. A. Gorbunov. “On the theory of quasivarieties of algebraic systems”. In: *Algebra i Logika* 14.2 (1975), pp. 123–142, 240. A. Budkin and V. Gorbunov. “Quasivarieties of algebraic systems”. In: *Algebra and Logic* 14.2 (1975), pp. 73–84. DOI: 10.1007/BF01668420.

[13] Г. Е. Минц. “Производность Допустимых Правил”. In: *Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta Imeni V.A. Steklova* 32 (1972), pp. 85–89. ISSN: 0373-2703. MR: 0344076. Zbl: 0358.02031. G. E. Mints. “Derivability of admissible rules”. In: *Journal of Mathematical Sciences* 6.4 (1976), pp. 417–421. DOI: 10.1007/BF01084082. Zbl: 0375.02014.

[14] W. A. Pogorzelski. “Concerning the notion of completeness of invariant propositional calculi”. In: *Studia Logica* 33.1 (1974). DOI: 10.1007/BF02120868.

[15] W. Dzik and A. Wroński. “Structural Completeness of Gödel’s and Dummett’s Propositional Calculi”. In: *Studia Logica* 32 (1973), pp. 69–75. DOI: 10.1007/BF02123815.