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Minimal two-way flow networks with small decay[☆]

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ABSTRACT

Information decay in networks generates two effects. First, it differentiates how well informed different players within the same component are, and therefore how attractive they are to sponsor links to. Second, players may prefer to sponsor links to players they are already connected to. By focusing on small decay we analyze the first effect in isolation. We characterize the set of Nash equilibrium networks in the two-way flow model of network formation with small decay for any increasing benefit function of the players. The results show that small decay is consistent with two well-known stylized facts, namely that (i) many real world networks have high diameters, and (ii) that the diameter of such networks is typically small relative to the population size. We show that even stochastically stable networks may have any diameter when the benefit function is linear or strictly concave. Finally we study implied stability relations. We find that if any non-empty minimal network is stable, then so is the periphery-sponsored star. With strictly convex benefit functions, we find that other stars tend to be stable for a larger range of parameters than larger diameter networks which satisfy our characterization. However, with strictly concave benefit functions the other stars are stable for a smaller range of parameters than the larger diameter networks.

1. Introduction

One of the key roles of socio-professional networks is the exchange of information. Exchanging information allows people to find fitting jobs,[1] recognize business opportunities,[2] and learn about new products.[3] It is therefore no surprise

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1 Classical references are for instance Myers and Shultz (1951) and Granovetter (1974). See also Montgomery (1991).

2 For instance, Shane and Cable (2002) show that social networks facilitate the acquisition of external capital for new ventures. They report that it is information gathering rather than social obligation which drives the result.

3 Coleman et al. (1966) report how the diffusion of new medical drugs is facilitated by the social-professional network of physicians. See also Jackson (2008) for a good textbook on social and economic networks, and many interesting examples.

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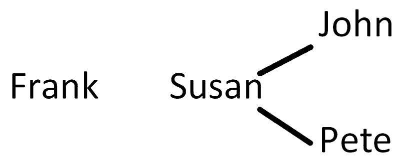

Fig. 1. Example network.

that people invest in their personal network. Networking events are common. People visit conferences;[4] businessmen join trade missions; and many people join network sites such as LinkedIn and Facebook.

An important feature of information networks is that acquired information can be passed on to others. Suppose John and Pete are not directly linked to each other, but are both directly connected to Susan (see Fig. 1). Then Susan may pass on relevant information from Pete to John. In this way information flows through networks. However, for several reasons John may benefit from being directly connected to Pete, rather than indirectly. First, Pete's information may be less timely if it has to pass through Susan before it reaches John. Second, communication is typically noisy. If information is passed on via others, in this case Susan, it accumulates more noise than if Pete provides it to John directly. Lastly, any person who passes on information may forget to pass on some part of it. Such processes are captured by the idea of 'information decay': information that is passed on more often before it reaches him is worth less to an agent. Clearly the presence of information decay affects to whom an agent wants to link.

In particular, information decay has two effects. *First*, ex-ante homogeneous players become heterogeneous by their position in the network. Consider Fig. 1. Suppose Frank wants to connect to John, Susan and Pete. Without decay he does not care to which of these persons he creates a link. Any one of these links would give him the full benefits of being connected to all three. With decay, Frank strictly prefers a link with Susan, as she is in the middle. A link with either John or Pete would result in a higher loss due to decay. Thus a player with many direct links in the network, or a player who is in the middle, may be more attractive to sponsor a link to. *Second*, decay may give the individual player an incentive to sponsor links to players he is already connected to, but indirectly. Consider John in Fig. 1. John may prefer to create a link with Pete even though John and Pete are already connected via Susan. The reason is that the costs of the additional link may be worthwhile because it reduces the information loss due to decay. This second effect only affects the set of equilibrium networks if there is enough decay. In contrast, the first effect exists for all positive levels of decay. Small decay refers to those levels of decay for which the second effect does not affect the results. It follows that it is possible to analyze the first effect of decay in isolation. This is what we do in this paper.

We study the effects of small decay on network architecture in the seminal two-way flow model of Bala and Goyal (2000a). In the two-way flow model agents incur a private cost to link to other agents. The other agent always accepts the link.[5] Once linked, information flows in both directions. We characterize the network architectures which can be supported by (strict) Nash equilibria, and refer to such networks as (strict) Nash networks. We also provide a sufficiency result for general payoff functions. We find that (strict) Nash networks can have high diameters[6] if decay is small. In contrast, earlier literature on decay in the two-way flow model only provides characterizations and examples of strict Nash networks (henceforth SNNs, SNN for singular) with low diameters.

For this reason, we analyze whether high diameter networks can be stochastically stable when we introduce a simple myopic best reply dynamic. Moreover, we study how the returns to information affect which of these networks are SNNs for the widest range of parameters. In both these analyses, we restrict ourselves to benefit functions which are either strictly concave, linear or strictly convex. We show that networks of any diameter can be stochastically stable. Moreover we show that (i) the periphery-sponsored star[7] (PSS) is stable for the widest range of parameters, and (ii) that low diameter networks are relatively more likely under convex benefit functions, while (iii) high diameter networks are relatively more likely if the benefit function is concave.

Our paper is most closely related to Bala and Goyal (2000a) (henceforth BG), Hojman and Szeidl (2008) and Feri (2007). BG introduce the two-way flow model we use. They show that without decay, all non-empty Nash networks are minimally

[4] In the US there were roughly 270,000 conferences with in total about 61 million visitors during 2013, according to a report by PwC (*The economic significance of meetings to the U.S. economy: Interim study update for 2012*). Including other meetings at contracted venues such as business meetings and trade shows, there are 1.8 million events and about 225 million visitors.

[5] In the two-way flow model, only the player who initiates the link (the sponsor) pays the costs of the link. The other player, the link's recipient, incurs no costs for the link. Thus no recipient of a link has a reason to refuse that link. More broadly, given that information flows in two directions, the recipient of a link will always want to accept a link if the costs of accepting a link are sufficiently low. Thus this model captures situations in which the sponsor of a link bears the brunt of the linking costs.

[6] The diameter of a network is the largest distance between any two players in the network. The distance between two connected players is the shortest path of links in the network that connects the two players. See also Section 2.

[7] A periphery-sponsored star is a network in which there is a unique player to whom every other player sponsors a link, and in which no other links are sponsored.

connected,[8] and that all minimal connected networks are Nash networks for some range of parameters. Moreover they show that each SNN is a center-sponsored star (CSS).[9] Although BG do consider decay and note that even with an arbitrarily small level of decay many other types of networks may be SNNs, they only provide a (partial) characterization for non-empty SNNs of diameter three or smaller. Jackson (2008) elegantly extends BG's partial results to a more general decay function (Proposition 11.4). The results are, however, very similar to BG and are focused almost exclusively on stars. In this paper, we characterize all minimal non-empty stable networks. Moreover, our characterization allows for general increasing benefit functions, whereas the results by BG and by Jackson (2008) are derived for linear benefit functions.[10]

Hojman and Szeidl (2008) show that the PSS is the only type of Nash network if (i) there is decay; (ii) there is a maximal distance which information can travel; (iii) the benefit function is concave enough and (iv) the population is large enough. This result is obtained with an elegant, general specification of decay where the only restrictive assumption is key to the result: information cannot travel beyond a certain distance. How do our results compare? We find for small enough decay that if any non-empty network is a SNN, then so is the PSS. This fits with their finding of the PSS being the unique Nash network. However, in contrast to Hojman and Szeidl, we also find that high diameter networks can be (strict) Nash networks. We can understand the difference as follows. First, the final part of our analysis shows that under concave benefit functions for some range of parameters, there may not be any low diameter networks other than the PSS, while there are some high diameter SNNs. Second, note that Hojman and Szeidl rule out high diameter networks by restricting the distance information can travel. Combining this maximum distance with a strong enough concavity of the benefit function can rule out all networks other than the PSS. Thus, by allowing information to always travel one step further, as well as considering a wider set of benefit functions, we obtain a richer set of SNNs. We also differ from Hojman and Szeidl (2008) in the way we impose enough structure on the model to obtain our results. In Hojman and Szeidl, the population is large enough (possibly extremely large), whereas we obtain our results by considering 'small enough decay' (possibly extremely small). Comparing our results with BG and Hojman and Szeidl (2008), we believe that our paper makes an important point: based on the reading of earlier literature the reader may get the impression that even small decay favors low diameter networks such as stars, whereas our results show that this is not the case. In fact, small decay combined with concave benefits favors both the PSS as well as high diameter networks. Moreover, also high diameter networks are stochastically stable. Our paper therefore adds to the literature by allowing for non-large population sizes, more general benefit functions, and information which can always be passed on further.

Feri (2007) is to our knowledge the first who applied stochastic stability to the two-way flow model with decay.[11] For linear benefit functions and constant linking costs, he finds that depending on the linking costs the set of stochastically stable networks either consists of all essential complete networks (there is a link between any pair of players), or it contains at least all star networks, or it consists of the empty network and possibly the PSS. For small decay, his characterization does not exclude high diameter networks. However, it does not show that some high diameter networks can actually be stochastically stable.[12] We do prove that networks of any diameter can be stochastically stable. This result has four requirements, which together are sufficient: first, the population is large enough; second, decay is small enough; third, the benefit function is linear or concave; and fourth, the CSS is a SNN.

From a broader perspective our paper is related to the literature on link reliability. This literature looks at the case where links may occasionally 'fail' in transmitting information. The first paper to consider here is Bala and Goyal (2000b); interesting follow-ups include Haller and Sarangi (2005) and Billand et al. (2006). The case of link reliability is similar to the case of decay. In this literature the focus is on the formation of back-up links. This corresponds most closely to the study of the effects of 'large decay', namely the formation of links to shorten the distance between the sponsor and a set of other players he is already connected to.

From the perspective of the network literature across several disciplines, our paper is related to the literature on small worlds. In networks with the small world property, the network diameter is of an order substantially smaller than that of

[8] A non-empty network is a network with at least one link. In a connected network any pair of players is connected through some path of links. In a minimal network the sponsor and recipient of each link are only connected through that particular link. So in a minimal network every link is necessary for connecting one subset of players to another subset.

[9] A center-sponsored star is a network in which there is a unique player who sponsors a link to every other player, and in which no other links are sponsored.

[10] A series of papers has treated the effect of non-linear benefits of information on network formation. BG's characterization without decay is for a general increasing benefit function. See moreover Vergara Caffarelli (2009) for an example in the one-way flow model; Buechel (2007) for an example with two-sided link formation; Goyal and Joshi (2006) for an example with two-sided link formation and non-linearity of payoffs in the number of own links and the number of links by others; and Bloch and Dutta (2009) for an example with endogenous link strength (and non-linearity of benefits in link strengths).

[11] Stochastic stability has also been applied to alternative network formation models. Cui et al. (2013) apply it to the one-way flow model introduced in BG; Jackson and Watts (2002) study the stochastic stability of networks combining a link formation dynamic based on pairwise stability with a coordination game which is played on the network; Section 5.5 of Kamphorst (2005) applies stochastic stability to the multiple group model. This model is a variant of the two-way flow model with a specific heterogeneity in linking costs.

[12] Feri and Meléndez-Jiménez (2013) endogenizes the decay on any link by making it depend on the result of a coordination game. The results on the stochastic stability of networks by Feri (2007) are robust with respect to this extension. Charoensook (2012) also endogenizes decay. There the level of decay at a player increases convexly in the number of links that player has. Consequently, players want to be connected to players who have few links. Charoensook shows that this can result in Nash networks with multiple non-singleton components (separate groups of connected players). By contrast, in our setting, players would rather link to players with many links.

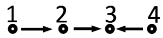

Fig. 2. Example of a four player network g, where $g = \{12, 23, 43\}$.

the population. This is a feature which is shared by many real world networks.[13] For instance, work by Newman (2001) and by Goyal et al. (2004, 2006) on scientific collaboration networks reports larger diameters. Depending on the particular science and time period these diameters range from 14 to 40, with respectively 14845 and 5253 authors in the component. Although we show that the diameters of SNNs and stochastically stable networks can be arbitrarily large, the population of high diameter networks needs to be much larger. In particular, high diameter networks involve population sizes which are substantially larger than the diameter itself.

A second way in which our paper is related to small worlds, is through decay. Barabási and Albert (1999) argue that 'preferential attachment' is key to understanding small world networks. Preferential attachment means that 'new' players are more likely to form links with players who have many links than with players who have few links. The two-way flow model with decay offers a micro-foundation for preferential attachment.[14] Given that you care about the distance to other players, it is typically more attractive to sponsor a link to a player with many links, than to a player with few links. This fit is not perfect. Preferential attachment ignores the rest of the network, while decay does not. Consider for example some player j who has links to each of two well-connected players, including player k. Under decay, it can be better to link to player j rather than to player k, if player k's many links are all to poorly connected players (see e.g. Fig. 6 and Example 1). In contrast, a link to k is more likely under preferential attachment.

The paper is structured as follows. In Section 2 we introduce the network formation game with decay. Section 3 contains the characterization of Nash networks. We prove that stochastically stable networks can have any diameter in Section 4. The role of the benefit function's shape on the set of SNNs is studied in Section 5. Finally, Section 6 concludes.

2. The game

Let $\mathcal{N}$ be the set of players with cardinality n, where $n \geq 4$. Each player decides to which of the other players he will sponsor a link. A link by player i (the *sponsor*) to player j (the *recipient*) is denoted by ij. The set of all links that a player i can possibly sponsor is given by $\mathcal{L}_i \equiv \{ij : j \in \mathcal{N}\backslash\{i\}\}$. $\mathcal{L}$ is defined as the set of all possible links, meaning that $\mathcal{L} \equiv \bigcup_{i \in \mathcal{N}} \mathcal{L}_i = \{ij \in \mathcal{N} \times \mathcal{N} : i \neq j\}$. We typically denote the strategy of player i – the set of links that he sponsors – by g_i. His strategy space $\mathcal{G}_i$ is therefore the power set of $\mathcal{L}_i$, i.e. $\mathcal{G}_i \equiv 2^{\mathcal{L}_i}$. All links together form a network,[15] typically denoted by g, so $g \equiv \bigcup_{i \in \mathcal{N}} g_i$. The strategy space $\mathcal{G}$ is the power set of $\mathcal{L}$, namely $\mathcal{G} \equiv 2^{\mathcal{L}}$. We can depict such a network g as a graph, where the players are the nodes, and each link $ii' \in g$ is represented by an arrow (directed arc) from i to i'. For example, Fig. 2 shows the network $\{12, 23, 43\}$.

Now we come to the (dis)incentives for players to sponsor links. The disincentives arise because sponsoring links is costly. The costs of a link ij are denoted by c, and are incurred completely by the sponsor; the recipient incurs no costs. Let $\mathcal{N}_i^S(g) \subset \mathcal{N}$ be the set of players to whom player i sponsors a link in g, so $\mathcal{N}_i^S(g) \equiv \{j \in \mathcal{N} : ij \in g\}$. Hence the total costs for player i in network g are equal to $|\mathcal{N}_i^S(g)|\, c$.

The benefits of links come from the players' need for information. Each player owns a unit of private information. For simplicity we assume that the information is non-rival in nature. Players benefit from accessing the information of other players. A player can access another player's information if and only if the two players are connected by a path of links. We say that player i observes player j if i and j are connected. On this path, it does not matter by whom the links are sponsored: the benefits of a link flow in both directions, thus 'two-way flow'. To make this precise, we let $\overline{ij} \in g$ denote that $ij \in g$ or $ji \in g$ or both.[16] We say that player i *has* a link with player j, if $\overline{ij} \in g$. A *path of links* ('*path*' for short) is a set of links $\{\overline{i_0 i_1}, \ldots, \overline{i_{k-1} i_k}\} \subseteq g$, where $i_0 = i$ and $i_k = j$. Let $N_i(g)$ denote the set of players to whom player i is connected in network g, including player i himself.

When two players are connected, they exchange their private information. However, information decay is higher the longer the path. We follow BG in our modeling of decay. In this literature only the shortest path between any two players is relevant. The *distance* between two connected players i and j in network g is the length of the shortest path between these two players. We denote this distance by $d_{ij}(g)$.[17] Let $\mathcal{N}_i^d(g) \subset \mathcal{N}$ be the set of players at distance d from player i in network g. So $\mathcal{N}_i^d(g) \equiv \{j \in \mathcal{N} : d_{ij}(g) = d\}$. By definition $\mathcal{N}_i^0(g) = \{i\}$. Every time the information is passed on, a constant

[13] Watts (1999) and Watts and Strogatz (1998) are seminal contributions on small world networks based on a random network approach. The drawback of this approach is that it does not incorporate incentives. Jackson (2008) offers a microeconomic model generating small worlds. Vega Redondo (2007) combines the two approaches.

[14] Note that decay is not necessary to obtain minimal small world networks. See e.g. BG for the two-way flow model without decay (each SNN is a CSS) and also Pin (2011) for strictly pairwise stable networks in a model with two sided costs.

[15] Observe that the strategy profile coincides with the network. In this paper we will refer to any strategy profile as a network. Similarly, we will refer to any (strict) Nash equilibrium as a (strict) Nash network.

[16] In other words $\overline{ij} \in g$ implies that $g \cap \{ij, ji\} \neq \emptyset$.

[17] Note that two-way flow implies $d_{ij} = d_{ji}$ for all $i, j \in N$.

fraction $(1-\delta)$ of the (remaining) information is lost, where $\delta \in (0, 1)$. The total amount of information gathered by player i in network g is then

$$I_i(g) = \sum_{d=0}^{n-1} (\delta^d |\mathcal{N}_i^d(g)|). \tag{1}$$

The benefits derived by player i from network g, $V_i(g)$, are an increasing function of $I_i(g)$, specifically $V_i(g) = f(I_i(g))$ where $f>0$. For several of our results we will focus on specific classes of benefit functions. The first class consists of all increasing functions with constant marginal benefits of information (CMBI), i.e. $f''(I) = 0\ \forall I \geq 0$. The other two classes consist of all increasing functions where the marginal benefits of information are either decreasing (DMBI), thus $f''(I) < 0\ \forall I \geq 0$, or increasing (IMBI), thus $f''(I) > 0\ \forall I \geq 0$.[18]

Observe that decay gives players incentives to sponsor links to players to whom they are already connected with the purpose of reducing the distance to them. In this paper we focus on small decay, so a range of δ close enough to 1, such that this effect plays no role. In particular consider the lowest possible δ_M such that for all $\delta > \delta_M$ there does not exist a player in any network who prefers to sponsor a link to a player he is already connected to. In the appendix, Lemma 4 shows that $\delta_M < 1$ for any population size n, linking costs c and benefit function $f(\cdot)$. Clearly for $\delta > \delta_M$, the results of our characterization cannot be affected by the desire of some player i to add a link to player j if i and j are connected already. Equally clearly, this lower bound δ_M is weakly more restrictive than necessary for our results. We can formulate weaker thresholds for δ to obtain some of our results.[19] However, to keep the exposition clear, we focus only on a single strict definition of small decay, instead of a variety of definitions.

The utility which i obtains in g equals his benefits minus his costs. Formally,

$$U_i(g) = V_i(g) - \left|\mathcal{N}_i^S(g)\right| c. \tag{2}$$

Define g_{-i} as all the links in g excluding the links sponsored by player i. A network g is a Nash network if for each player $i \in \mathcal{N}$ and all $g_{i'} \in \mathcal{G}_i$, $g_{i'} \neq g_i$, we have $U_i(g) \geq U_i(g_{-i} \cup g_{i'})$. Nash network g is a SNN if this inequality is strict for all $i \in \mathcal{N}$. Otherwise g is a weak Nash network. Similarly, in a Nash network every player plays a best reply strategy. Denote by $BR_i^f(g)$ the set of best reply strategies of player i given network g under benefit function f. Formally, for any benefit function f we have

$$BR_i^f(g^*) = \left\{ g_i \in \mathcal{G}_i : U_i(g_{-i}^* \cup g_i) \geq U_i(g_{-i}^* \cup g_{i'}) \text{ for all } g_{i'} \in \mathcal{G}_i \right\}. \tag{3}$$

If in a Nash network the best reply strategy set for each player is singleton then the network is a SNN.

Having described the game, we introduce some additional terminology and notation. Each network g partitions the population into one or more *components* of g. Two players belong to the same component if and only if they are connected. Component x is denoted as $C_x(g)$. A network that contains only one component is called a *connected* network. A network is *minimal* if the deletion of any link in that network will result in an increase of the number of components. A *cycle* is a set of links $\{\overline{j_0 j_1}, \ldots, \overline{j_{k-1} j_k}\}$ such that $j_0 = j_k$, and all players j_1 to j_k are distinct. This implies that a component (or network) is minimal if and only if it contains no cycles. We call a link which is not part of a cycle a *minimal link*, and any link which is part of a cycle a *non-minimal link*. A *non-recipient player* is a player that receives no links, while a *multi-recipient player* is a recipient who receives more than one link. E.g., in the network in Fig. 2 player 3 is a multi-recipient, while players 1 and 4 are both non-recipient players.

Within any connected set of players, an important player is the best-informed player. To formally identify such a best-informed player, for a network g and a set of players M, $M \subseteq \mathcal{N}$, define network g_M as the subset of links of network g of which both the sponsor and the recipient of the link belong to M. Formally, $g_M = \{ij \in g : i, j \in M\}$. Note that g_M may be, but does not have to be, a component.

Definition 1. Let $M \subseteq \mathcal{N}$ be a connected subset of players in network g. Consider $g_M = \{ij \in g : i, j \in M\}$. Then player i, $i \in M$, is a best-informed player of M if $I_i(g_M) \geq I_j(g_M)$ for all $j \in M$.

Remark 1. Consider in network g a player i who is not part of component $C_x(g)$. If player i now wants to sponsor a link to a player $j \in C_x(g)$, then j must be a best-informed player of $C_x(g)$. This is because (i) the additional information i receives from sponsoring a link to j, $j \in C_x(g)$, is $\delta I_j(g)$; (ii) all links cost the same; (iii) i's utility is strictly increasing in information. By the same arguments, if $C_x(g)$ has multiple best-informed players, player i does not care to which one of these players he sponsors a link.

Let $ii' \in g$, then $A_{ii'}(g)$ is the set of players which i observes in g exclusively via link ii'.[20] Setting $M = A_{ii'}(g)$, i gains $\delta I_{i'}(g_M)$ information by sponsoring ii'. In a minimal connected network g, link ii' is said to *point to* player j if $j \in A_{ii'}(g)$ and to *point away* from player j if $j \notin A_{ii'}(g)$. We adopt the same terminology with respect to the relation between links: links $ii' \in g$ and $jj' \in g$ are said to *point to* each other if $i' \in A_{jj'}(g)$ and $j' \in A_{ii'}(g)$.

[18] In Section 5 we discuss which benefit function is plausible in which situation.
[19] This is done in an earlier version of this paper, see De Jaegher and Kamphorst (2010).
[20] So $A_{ii'}(g) \bigcap \mathcal{N}_i(g \backslash \{ii'\}) = \emptyset$ and $A_{ii'}(g) \bigcup \mathcal{N}_i(g \backslash \{ii'\}) = \mathcal{N}_i(g)$.

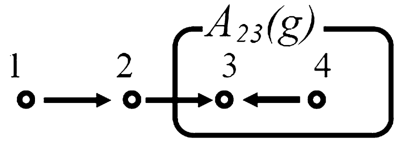

Fig. 3. Illustration of notation ($A_{ij}(g)$).

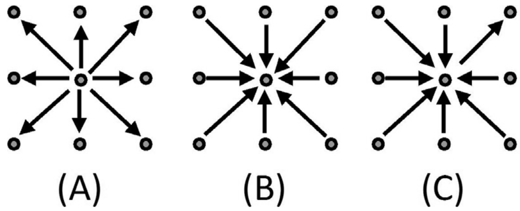

Fig. 4. Example star networks.

Now we define the concepts of end links and end sponsors. These will play an important role in our analysis. We say that a link through which the sponsor observes only the recipient of that link is an *end link*, formally a link $ii' \in g$ is an end link if $A_{ii'}(g) = \{i'\}$. If ii' is an end link then i is an *end sponsor*. To illustrate these concepts, we consider network g in Fig. 3. It consists of one component, and is thus a connected network. In this network, $A_{12}(g) = \{2, 3, 4\}$, and $A_{23}(g) = \{3, 4\}$. Moreover link 23 points to players 3 and 4, and points away from players 1 and 2, while links 12 and 43 point towards each other. Network g contains no end links. Now define $g' = \{21, 23, 43\}$. So g and g' are the same, except that link 12 is replaced by 21. Then 21 is an end link of g', with player 2 being an end sponsor in g'.

We finally define some types of networks that we use throughout the paper. Stars are minimal connected networks where one player, the central player, has a link (either as sponsor or as recipient) with every other player. Networks A–C in Fig. 4 are all stars. The *CSS* is a special case of the star where the central player sponsors each link (network A), while the *PSS* is a special case of the star where the central player receives each link (network B). Of course, a star may also have both center-sponsored and periphery-sponsored links (e.g. network C). BG also defined linked stars. These are diameter three networks, such that any player who has a single link, either receives the link, or sponsors it to the player who has at least two more links than any other player. See networks A and D of Fig. 5 for examples. Note that the player with three links, could not be the recipient of the links with the players above and to the right of him.

More generally, a minimal connected network of this game is a *directed tree*. A *rooted directed tree* is a network with a unique non-recipient player, where such a unique non-recipient player is referred to as a *root*.

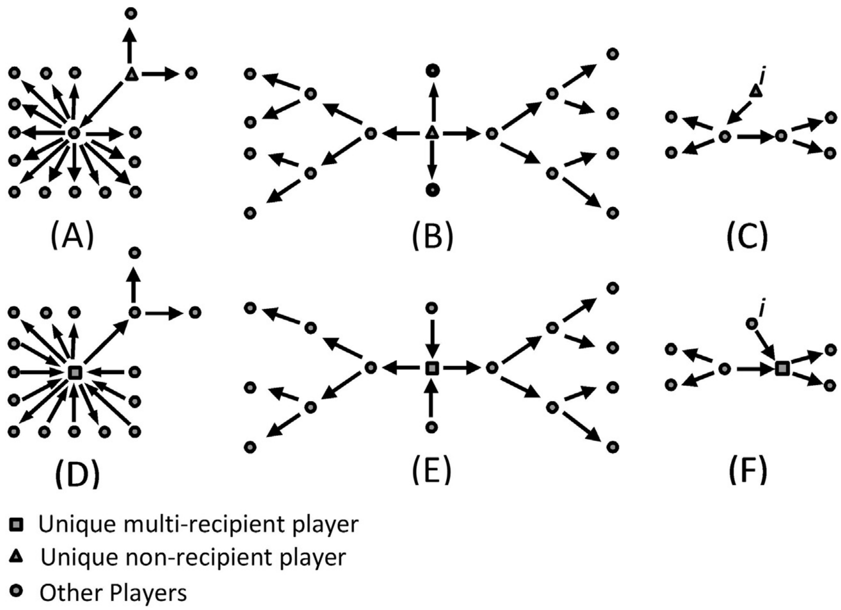

Fig. 5. Examples of non-star networks which satisfy the properties of Proposition 1.

3. Characterization

In this section, we present two of our propositions. The first proposition is a partial characterization of all non-empty minimal Nash networks.[21] As such it applies to any level of decay, not just to small decay. Of course, for larger levels of decay, non-minimal networks may exist as well. After introducing a type of networks which we call 'balanced', the second proposition shows that all balanced minimal networks which satisfy Proposition 1 are SNNs for some range of parameters. At the end of this section we will relate our findings on the diameters of SNNs to earlier findings in the literature. All proofs are relegated to the Appendix.

Proposition 1. *Let g be a non-empty minimal Nash network of the two-way flow model with decay. Then*

1. *g has one of the following two configurations:*
   (a) *g is a rooted directed tree with all links pointing away from its root: the unique non-recipient player. Each best-informed player in g is either the root player, or receives a link from the root player;*
   (b) *g is a directed tree with a unique multi-recipient player. Any link not received by this player points away from him. Moreover, this player is the unique best-informed player in g;*
2. *In both configurations of part 1, for any path of links in g pointing in the same direction, say* $\{j_0j_1, j_1j_2, \ldots, j_{k-1}j_k\}$, *we have* $I_{j_1}(g) > I_{j_2}(g) >, \ldots, > I_{j_{k-1}}(g) > I_{j_k}(g)$.
3. *In both configurations of part 1, for* $ij \in g$ *if* $\left|A_{ij}(g)\right| = 2$, *then g is not a SNN; if* $\left|A_{ij}(g)\right| \geq 3$, *then there exist* $\overline{jk}, \overline{jk'} \in g$ *such that* $k \neq k'$;
4. *for all* $\delta \in (\delta_M, 1)$, *any non-empty Nash network of the two-way flow model with decay satisfies Parts 1–3.*

Before we discuss the results, we briefly sketch the proof. The connectedness of the Nash network follows from standard arguments. To understand the rest of Part 1, observe that if, in a minimal Nash network *g*, player *i* observes *j* through a path containing links ii' and jj', then $i' = j'$. To see why $i' \neq j'$ is impossible in a Nash network, note that apparently i' is a best-informed player of his component in $g\backslash \left\{ii'\right\}$. Moreover, due to decay i' receives more information from the link ii' than player j' does. Consequently, in *g* player i' is strictly better informed than j'. However, by the same arguments j' is strictly better informed than i', which cannot be. From this observation, it follows that we can have at most one player who receives multiple links, and that all other links point outward. Moreover, if no player receives multiple links, by minimal connectedness $(n-1)$ players receive exactly one link.[22] Also Part 2 follows from the fact that any player sponsors a link to a best-informed player among the accessed players, combined with the fact that there is decay. Part 3 follows from the observation that if *j* has at most one other link, say $\overline{jk} \in g$, then in network $g\backslash \left\{ij\right\}$ player *k* is at least as well informed as *j*. Thus *i* would (weakly) prefer to link to *k* rather than to *j*. Finally, Part 4 follows by continuity of payoffs in decay. As players strictly prefer not to sponsor non-minimal links when there is no decay, they also prefer not to sponsor such links if decay is small enough.

Note that the first effect of decay, which is the one we study here, creates a discontinuity around $\delta = 1$. For $\delta = 1$, BG show that any minimal connected network is a Nash network. We find that the set of Nash networks is substantially smaller if there is any decay. The reason is that decay generates preference relations about whom to sponsor a link to. Moreover, BG show that for $\delta = 1$ the CSS is the only SNN. Other minimal networks cannot be strict Nash equilibria, because then some sponsor can replace a link by some other link and still receive the same payoff. In contrast, any positive amount of decay typically creates strict preference relations over possible recipients, thereby allowing other non-empty minimal networks to be SNNs as well. Thus, decay substantially reduces the set of Nash networks, while it substantially enlarges the set of SNNs.[23] Examples of non-star networks which satisfy the conditions are presented in Fig. 5. Networks A–C satisfy Part 1a, while networks D–F satisfy Part 1b. Moreover, networks A and D are *linked stars*, while networks B and E have diameter 6. Finally sponsor *i* in networks F and G is indifferent between these two networks, which differ only in the link *i* sponsors. Therefore they cannot be SNNs, although they can be weak Nash networks.

Note that Proposition 1 is not a full characterization. It does not contain any sufficient conditions which guarantee that a network satisfying these conditions is indeed a SNN for some range of parameters. The main obstacle here is that the identity of the best-informed player in a particular component – so the preferred recipient for any link to that component – may depend on δ. We illustrate this in the following example.

Example 1. Consider network *g* in Fig. 6. There the component on the right contains both players *j* and *k*. If *i* wants to link to that component, to whom should he link? In other words, who is the best-informed player of that component? Calculating the information gathered by these two players we find that $I_j(g) = 1 + 2\delta + 8\delta^2$ while $I_k(g) = 1 + 5\delta + \delta^2 + 4\delta^3$. We see that for

[21] We thank two referees. Due to their questions Proposition 1 now covers all Nash networks, rather than only SNNs.

[22] Proposition 11.4 part 3 in Jackson (2008) states that for $f(I_i) = I_i$ and $\delta < c$ other networks than the PSS can be stable, for instance the empty network. Part 1 of our Proposition 1 shows that, in this case, all other non-empty Nash networks are non-minimal. The reason is that by the way in which our links are oriented, all other minimal networks contain an end link which, by $\delta < c$, is not worth its costs.

[23] A taste of the richness of the set of SNNs under decay has been presented in BG. They point out that for particular ranges of parameters both stars and linked stars are SNNs.

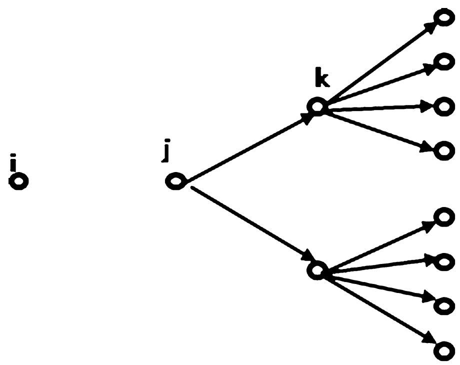

Fig. 6. Who is the best-informed player?

$\delta \in (0, (3/4))$, player k is a best-informed player while player j is not. In contrast, for $\delta \in ((3/4), 1)$ player j is the best-informed player of that component. So for $\delta > (3/4)$, player i would prefer link ij to ik, whereas for $\delta < (3/4)$ his preference would be reversed.

Clearly, finding the best-informed player, as a function of the level of decay, in components with a larger diameter than in the example, requires the solution of higher-degree polynomials. These higher-degree polynomials may have multiple roots, such that the link between the level of decay, and the identity of the best-informed player, is not straightforward. Nonetheless we can state a sufficiency result for δ close enough to 1. In that case, if a player is positioned in the middle of a component, then this middle player is the best-informed player of that component. Before we state the sufficiency result, we will first define what we mean by a player in the middle of a set of players in a given network. Typically, we will let this set of players be the players of a component or the set of players accessed through a link, *e.g.* $A_{ij}(g)$. Then we formulate the balancing condition. Finally we state our sufficiency result saying that all networks which satisfy Proposition 1 and the balancing condition are SNNs for some range of parameters.

Definition 2. Consider a minimal connected subset of players M, $M \subset N$. We say that player j is in the middle of set M in network g if for each neighbor k of j in network g the following holds: in g more than half of the players in M (including k and j) are closer to j than to k.

For example, in Fig. 6 player j is in the middle of the big component and player k is not.[24]

Definition 3. A minimal network g satisfies the balancing condition if for any $ij \in g$ we have that j is in the middle of $A_{ij}(g)$. In that case, we say that the network g is balanced.

For instance, in Fig. 5 networks A, B, D and E are balanced. The following lemma prepares for our sufficiency result. It says that for small enough decay the optimal recipient in a component is the middle player, if present. Note that this is equivalent to saying that for small enough decay, the middle player of any component is also the best-informed player of that component.

Lemma 1. *Consider the case of small decay. There exists a $\delta_B < 1$ such that the following property holds for any minimal Nash network g. Consider any $ij \in g$ such that $A_{ij}(g)$ has a middle player, say k. Then $j = k$ for all $\delta \in [\delta_B, 1)$.*

Note that δ_B is not a restriction which is based on our small decay assumption. In particular the small decay assumption is based on the incentives to sponsor non-minimal links. Both benefit function $f(\cdot)$ and link cost c affect δ_M, while neither affects δ_B. As a result, in some cases we have $\delta_B > \delta_M$, while in other cases $\delta_B < \delta_M$.

Lemma 1 shows that for small enough decay, we know that in a balanced network each link is sponsored to the unique optimal recipient, namely the middle player of the accessed part of the network. This allows us to derive the remaining results in this paper. The consequence is that our remaining results are about SNNs only. A weak Nash network would require some player to be indifferent about which link to sponsor. Yet, in a balanced network with small enough decay each link is sponsored to the unique optimal recipient. By focusing on balanced networks we exclude all weak Nash networks, such as networks C and F of Fig. 5, from the analysis.[25]

[24] Not all connected subsets of a network have a middle player. If two players have exactly half the population closest to them, the definition says that neither player is the middle player. Consider Networks C and F of Fig. 5 without the link sponsored by player i. The resulting non-singleton components have no middle player.

[25] In Appendix H, we separately explore the plausibility of weak Nash networks. Based on the myopic best reply dynamic used in BG, we show there that networks C and F in Fig. 5 can be stochastically stable, while some other weak Nash networks cannot. We thank an anonymous referee for asking about this.

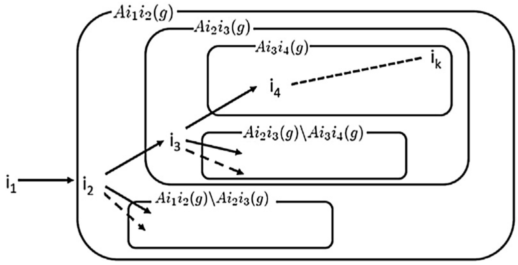

Fig. 7. Why would population need to increase exponentially in diameter? The dashed line refers to a connecting path. Dashed arrows indicate the potential presence of more links to a particular subset of players.

We are now ready for stating the second main proposition of the paper: the sufficiency result. It states that all balanced, minimal networks satisfying Proposition 1 are in fact SNNs in some cases. This implies that all such network types are relevant, albeit for a possibly small range of parameters.

Proposition 2. *Consider any balanced, minimal network g that satisfies the properties of Proposition 1. Then network g is a SNN for a range of δ and c.*

The intuition is simple. By minimality of g we have that for low enough c no player prefers to delete a link if there is no decay. By continuity the same is true if decay is small enough. Given c, we know that if decay is small enough (i) no player prefers to add a link to a connected network and (ii) the optimal recipient of any minimal link is the middle player of the accessed group of players. As g is connected and balanced both conditions are satisfied. Thus given some low c, no player is willing to delete, add or replace links in g if decay is small enough.

We end this section with a discussion of the diameter of SNNs. Earlier results in the literature of this model with decay (see BG and Hojman and Szeidl, 2008), only identified low diameter networks as SNNs, namely stars (diameter 2) and 'linked stars' (diameter 3). In contrast, neither Proposition 1 nor Proposition 2 implies that SNNs should have low diameters. In fact, Proposition 2 shows that networks of any diameter can be SNNs, provided that the population is large enough and the parameters are chosen from the right range. More strongly, as balanced SNNs can have any diameter, and as this already requires δ large enough (both for minimality and balancedness), especially the smaller amounts of decay can result in high diameter networks. It is rather for larger amounts of decay that all high diameter networks will fail to be SNNs.[26][27]

Despite the potentially arbitrarily high diameters of SNNs, we want to stress that the diameter of SNNs will be small compared to the total population size for higher diameter networks. The population size needed to construct a SNN of diameter d tends to become larger in d at an increasing rate. To see why more and more nodes are needed to increase the diameter of a network,[28] consider some minimal SNN g with diameter $d > 4$ and denote by player i_0 either the unique non-recipient player or the unique multi-recipient player (as appropriate). Let $(\overline{i_0 i_1}, i_1 i_2, \ldots, i_{k-1} i_k)$ be a path in g from player i_0 to a player who is furthest away from him, denoted by i_k. By the characterization, any player at distance two or greater from $i_0$receives a link. Fig. 7 depicts this path from i_1 to i_k. Note that i_1 is only willing to sponsor the link to i_2, rather than for instance a link to i_3, if the information gathered through other branch-offs at i_2 is large enough: $A_{i_1 i_2}(g)\backslash A_{i_2 i_3}(g)$ should be large enough compared to $A_{i_2 i_3}(g)$. The same applies to the link $i_2 i_3$. For i_2 to prefer a link to i_3 rather than to i_4, $A_{i_2 i_3}(g)\backslash A_{i_3 i_4}(g)$ should be large enough compared to $A_{i_3 i_4}(g)$, etc. The longer the path between i_0 and i_k is, the more branch-offs there are, all of which should give access to enough information. Moreover the additional players in later branch-offs, say $A_{i_2 i_3}(g)\backslash A_{i_3 i_4}(g)$ also increase the minimally needed amount of information in all earlier branch-offs, in this case $A_{i_1 i_2}(g)\backslash A_{i_2 i_3}(g)$. Thus, the minimal population size tends to become larger as a function of the diameter of SNNs, at an increasing rate. For this reason, although the diameter can be arbitrarily large, it will tend to be small relative to the population size.

4. Stochastic stability of high diameter networks

In the light of earlier research,[29] the stability of high diameter networks is novel. In this section we study whether perhaps high diameter networks, despite being SNNs, are less plausible. To do so we follow Feri (2007) in looking at the stochastic stability of networks.[30] In particular we want to know whether only low diameter networks, say stars and linked stars, survive this refinement. That is not what we find. We prove for DMBI and CMBI that stochastically stable networks may have any diameter. Thus our finding of high diameter stable networks when there is small decay is robust from an evolutionary perspective.

[26] We thank Francis Bloch for pointing this out.
[27] Recall also that δ_M is defined for a given n. Typically, δ_M is higher if n is larger. Also in this respect will higher diameter SNNs require higher δ.
[28] The formal derivation of the minimal number of players needed for a balanced SNN of diameter d is provided in Appendix I.
[29] See e.g. BG and Hojman and Szeidl (2008).
[30] We thank an anonymous referee for urging us in this direction.

To consider the stochastic stability[31] of networks, we first need to specify the underlying dynamics. We adopt the standard myopic best reply dynamic process which was also employed in BG. Denote by g^t the network in period t, $t \in \mathbb{N}$, and by g_i^t the strategy of player i in period t. At the start of each period, each player updates his strategy with probability μ, $\mu \in (0, 1)$, independently of which other players update their strategies. If the player updates his strategy, he chooses a best reply to the previous network, g^{t-1}, thus $g_i^t \in BR_i^f(g^{t-1})$. If the player has multiple best replies, each best reply is chosen with some positive probability. With probability $(1-\mu)$ the player cannot change his strategy, i.e. $g_i^t = g_i^{t-1}$. In time, the dynamics will converge to a recurrent class. A *recurrent class* is a minimal set of networks X, $X \subseteq \mathcal{G}$, with the property that if $g^t \in X$ then $g^{t'} \in X$ for all $t' > t$. Clearly, each SNN forms a singleton recurrent class of this dynamic process. We are interested in the set of *stochastically stable* recurrent classes, meaning recurrent classes which are robust with respect to very scarce mutations in the players' strategies. In particular, we perturb the dynamics above by having each updating player select a strategy at random with some positive probability ε: a *mutation*. In case of a mutation, each strategy in the player's strategy set is chosen with positive probability. With the remaining probability $(1-\varepsilon)$ the player selects a best reply as before. These perturbed dynamics constitute an aperiodic and irreducible Markov chain. Therefore the long run probability distribution μ_ε is unique and time invariant. We say that a recurrent class, and any network it contains, is stochastically stable if the networks in that class occur a positive fraction of the time, when ε approaches zero. Formally, g is stochastically stable if $\hat{\mu}(g) > 0$ where $\hat{\mu} = \lim_{\varepsilon \to 0} \mu_\varepsilon$. Our aim is to establish whether stochastic stability rules out high diameter networks.

Below we show that stochastically stable networks may have any diameter (the proofs are in Appendix E). To do this we use two different sufficient conditions for a recurrent class to be stochastically stable. First, any recurrent class which can be reached from *any* other recurrent class by a single mutation (followed by unperturbed dynamics) is stochastically stable.[32] We show that this applies to the CSS, provided that the CSS is a SNN. Second, any recurrent class which can be reached from some *stochastically stable* recurrent class through a single mutation (again, followed by unperturbed dynamics) is itself stochastically stable.[33] We apply this second condition iteratively to construct a path from the CSS to stochastically stable SNNs of arbitrarily high diameters. For this procedure we need to verify for each network to which we let the unperturbed dynamics converge, that it is indeed a SNN. The following lemma allows us to do so. It says that if the benefit function features CMBI or DMBI any balanced network satisfying Proposition 1 is a SNN, provided the CSS is a SNN too. Let $\mathcal{G}^*$, $\mathcal{G}^* \subset \mathcal{G}$, denote the set of all non-empty balanced networks which satisfy Proposition 1.

Lemma 2. *Let $f(\cdot)$ have CMBI or DMBI. Moreover let $\max\{\delta_M, \delta_B\} < \delta < 1$ and let the CSS be a SNN, so $f(1+(n-1)\delta) - f(1+(n-2)\delta) > c$. Then any network $g \in \mathcal{G}^*$ is a SNN.*

Our proposition states that stochastically stable networks may have any diameter d, $d \geq 2$.[34] The proof is by construction. After each mutation we will let the network converge in one step to a connected balanced rooted directed tree. Lemma 11 (Appendix E) shows that such a network belongs to $\mathcal{G}^*$. An illustration of the first four iterations of the construction procedure is given in Fig. 8. In particular, we start the construction procedure at a CSS which is a SNN and thus stochastically stable. Let i_c be its central player. Now, each iteration of the procedure consists of two steps. In the first step, a mutation occurs such that the unique non-recipient player, i_c, deletes his links to two players, say j and k, who have symmetric positions in the network. In the second step there is no mutation. Instead, one player is allowed to update his strategy. We let this be a player who receives a link from i_c. This player sponsors a link to both j and k. In this way the network stays balanced while it also satisfies the properties of Proposition 1. By iteration, we can construct ever larger networks. As each SNN on this path is attained through a single mutation from a stochastically stable SNN, each SNN on this path is stochastically stable.

Proposition 3. *Let $f(\cdot)$ have CMBI or DMBI. Moreover let $\max\{\delta_M, \delta_B\} < \delta < 1$ and let the CSS be a SNN, so $f(1+(n-1)\delta) - f(1+(n-2)\delta) > c$. For any diameter $d \geq 2$, there exists a stochastically stable network with diameter d provided the population is large enough.*

Evidently, high diameter networks are not ruled out by stochastic stability. Thus stochastic stability does not suggest that such networks are implausible. The range of decay for which Feri (2007) finds that stochastically stable networks must be stars involves large decay. In particular, he obtains this result when decay is such that for all networks of diameter 3 or more, some player will prefer to sponsor a non-minimal link, while decay is still small enough such that no player has corresponding incentives in stars. Here too we see that especially small decay allows for high diameter networks.

Our proof does not extend to the case of IMBI. The reason is that under IMBI the CSS may be a SNN while larger diameter networks in $\mathcal{G}^*$ are not. Even if some high diameter network $g \in \mathcal{G}^*$ would be a SNN, there is no guarantee that any other network $g' \in \mathcal{G}^*$ is a SNN too. Consequently, when we let the dynamics converge to some constructed $g' \in \mathcal{G}^*$ it is unclear whether g' is stochastically stable. For instance, under IMBI g^0 in Fig. 8 can be a SNN even when g^1 is not. In such a case,

[31] see e.g. Young (1993, 1998) for more extensive discussions of stochastic stability.

[32] Let Z be a recurrent class, and let there be x recurrent classes in total. In terms of Freidlin and Wentzell (1984) and Young (1993), this condition ensures that the 'stochastic potential' of Z is equal to $x-1$, which is the minimal possible. By Theorem 2 of Young (1993) it follows that Z is stochastically stable.

[33] Let Z and Z' be recurrent classes. If Z can be reached from Z' after a single mutation followed by unperturbed dynamics, then the stochastic potential of Z cannot be larger than that of Z'. Consequently, if Z' is stochastically stable then so is Z (see also Samuelson, 1997; and Feri, 2007).

[34] Note that in our proofs we will let only one player update at a time. Even though potentially many or all players could update simultaneously, inertia makes it possible that only one player updates. We use this possibility. As a consequence our proofs can be applied to the dynamics of Feri (2007) as well.

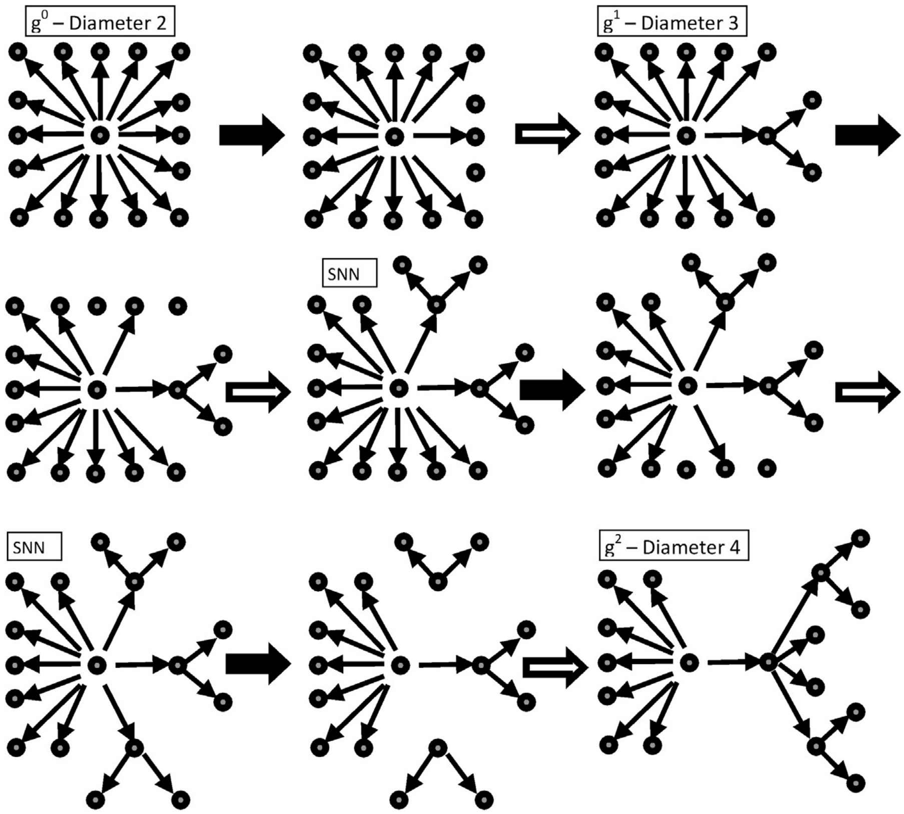

Fig. 8. Example dynamics of CSS g^0 to diameter 4 network g^2. The big arrows indicate transitions from one network to the next. Each transition involves a single player updating. Filled arrows indicate that the updating player suffers from a mutation. Open arrows indicate a normal best reply response by the updating player.

our argument fails. In that sense, Lemma 2 plays a crucial role in our proof. Under DMBI the strict Nash stability of the CSS implies stability of the higher diameter networks, but not vice versa. Under IMBI this implied stability relation does not hold. In the following section we explore how the shape of the benefit function affects such implied stability relations between the PSS, other stars, and other networks in $\mathcal{G}^*$.

5. The role of the benefit function's shape

In this section we extend the result from Lemma 2 to analyze how the shape of the benefit function affects implied stability relations between the PSS, other stars, and other networks in $\mathcal{G}^*$.[35] We find that the shape matters in this respect. Under IMBI the opposite result is obtained: stability of larger diameter networks in $\mathcal{G}^*$ implies stability of different types of stars but not vice versa. We also find that the PSS plays a special role: if any non-empty minimal network is a SNN then the PSS is a SNN too. This last result fits well with earlier literature, see e.g. Hojman and Szeidl (2008).

The effect of the benefit function's shape on the implied stability of the SNNs begs the question: which benefit function is more plausible? The answer depends on the nature of information. If some information helps to understand other information, then IMBI is relatively likely.[36] On the other hand, if information consists of independent signals, for instance about the user value of a particular product, information may be characterized by DMBI. In particular, the expected marginal effect of the first signal on the quality of your decision may well be larger than that of the 10th or 100th signal.[37]

We now turn the main proposition of this section. We denote the set of PSS by $\mathcal{G}^{PSS}$. The proof is in Appendix F.

Proposition 4. *Let $f(\cdot)$ have CMBI, DMBI or IMBI. Moreover let* $\max\left\{\delta_M, \delta_B\right\} < \delta < 1$.

1. *Suppose any minimal non-empty network g is a SNN. Then any $g' \in \mathcal{G}^{PSS}$ is a SNN.*

[35] Recall that $\mathcal{G}^*$ is the set of networks which are balanced and satisfy Proposition 1

[36] Cohen and Levinthal (1989) argue that this typically true for R&D.

[37] Bikhchandani and Mamer (2013) provide sufficient conditions for i.i.d. signals to have DMBI. Moreover Moscarini and Smith (2002) show that, in the case of i.i.d. signals, the marginal benefit of another signal is eventually decreasing.

2. *For CMBI, suppose any* $g \in \mathcal{G}^* \backslash \mathcal{G}^{PSS}$ *is a SNN. Then any* $g' \in \mathcal{G}^*$ *is a SNN.*
3. *For DMBI, suppose any star* $g \in \mathcal{G}^* \backslash \mathcal{G}^{PSS}$ *is a SNN. Then any* $g' \in \mathcal{G}^*$ *is a SNN. Moreover, for some range of parameters any non-star* $g \in \mathcal{G}^*$ *is a SNN while no star* $g' \in \mathcal{G}^* \backslash \mathcal{G}^{PSS}$ *is a SNN.*
4. *For IMBI, suppose any non-star* $g \in \mathcal{G}^*$ *is a SNN and that player i sponsors k end links in g,* $k \geq 1$. *Then any star with at most k end links is a SNN. Moreover, there exists a range of parameters such that any star with at most k end links is a SNN while g is not.*

These results suggest that higher diameter networks (or rather, non-star networks) are relatively implausible under IMBI.[38] By the same notion, however, they appear relatively plausible under DMBI. This result is explained by the following intuition. There are two forces that give players sufficient incentives to sponsor their links.[39] *First*, players may sponsor links because they can access a lot of information by sponsoring a link to a well-informed player. This effect is equally at work for CMBI, DMBI and IMBI, and is strongest in the PSS, explaining why overall the PSS is a SNN for the widest range of parameters under small decay. *Second*, players may sponsor a link because, given the amount of information gained, the benefit of additional information is large enough. In any SNN other than the PSS, there are end sponsors who sponsor an end link. All end sponsors receive the minimal amount of information possible through a minimal link. If there is DMBI, additional information is valued most by players who know little. Thus end sponsors will be more willing to sponsor end links if they have little information to start with, so when distances between players tend to be large. This is the case in high diameter networks. If instead there is IMBI, additional information is worth more to players who are already well-informed. Thus end sponsors are especially willing to sponsor end links if they are well-informed to start with. This is more likely in low diameter networks. Concluding, we can explain the relative stability of the PSS by the large amount of information gained by the sponsor of each link. Similarly, the rest of the results of Proposition 4 can be explained by whether δ information is more valuable to a well-informed player, or rather to a poorly informed player.

Thus we have an intuition that under DMBI, high diameter networks may be SNNs on top of PSS networks. The intuition is not specific to comparing stars and other networks. To illustrate this, our final proposition shows that under DMBI if any linked star is a SNN then so are some networks of diameter 4 or larger. The reverse does not hold. The proof is in Appendix G. A noteworthy implication of the last proposition is that for a range of parameters the set of SNNs can contain a diameter gap: all SNNs are either PSS (diameter 2), or they have a minimal diameter of at least 4.[40]

Proposition 5. *Let* $n \geq 7$, *and let* $f(.)$ *feature DMBI. Moreover, let* δ *be such that* $\delta > \max\left\{\delta_M, \delta_B\right\}$. *If any SNN g,* $g \notin \mathcal{G}^{PSS}$, *is a star or linked star, then there exists a network* $g' \in \mathcal{G}^*$ *of diameter 4 which is also a SNN. The opposite does not apply: for some range of parameters there exists a SNN g′ of diameter 4 or higher, while no SNN* $g \in G^* \setminus G^{PSS}$ *is a star or linked star.*

6. Conclusion

The purpose of this paper has been to characterize the Nash networks for the standard network formation model by BG with information decay. We have focused on minimal networks, which are relevant in three ways. First, for small decay, minimal networks are the only Nash networks. Second, for larger decay, minimal Nash networks may co-exist with non-minimal Nash networks. For such cases, our results provide necessary conditions on what stable minimal networks look like. Third, focusing on minimal networks allows us to focus on one of the two effects of decay, namely that nodes become heterogenous. Admittedly, our focus on minimal networks leaves out the important class of non-mimimal networks, as non-minimal networks are only excluded for a specific range of decay. Yet, the characterization of SNNs that are not minimal requires techniques others than those used in the current paper. This deserves separate analysis, which will be complementary to our work.

Generally speaking, our paper has three contributions. First, it provides a characterization of the stable networks in the BG two-way flow setting under small decay. Although we cannot give a precise characterization of the range of parameters for which any characterized network is stable, we follow the literature's solution in providing a sufficiency result of the following type: we show that any network satisfying the characterization and one additional property, the balancing condition, is a SNN for a range of parameters.

Our characterization implies that networks of any diameter can be SNNs. The question arises whether high diameter networks are plausible. The answer to this is our second contribution. Under DMBI and CMBI we show that networks of any diameter can be stochastically stable, whenever the CSS is a SNN. Thus also from an evolutionary perspective high diameter networks occur a positive fraction of the time in the long run.

The third contribution is that we consider implied stability relations. Comparing the PSS, other stars and non-star networks satisfying our characterization we find that the stability of any such network implies that the PSS is a SNN too, but not vice versa. Roughly speaking, this is because no single link gives access to as much information as a link in the PSS. When we consider such relations between the other networks, we see the importance of the shape of the benefit function. The

[38] By Proposition 1 any minimal non-empty Nash networks has end links, unless it is a PSS.

[39] Recall that by $\max\left\{\delta_M, \delta_B\right\} < \delta < 1$, no player wants to add or replace a link in any $g \in \mathcal{G}^*$.

[40] Examples of larger diameter gaps can be obtained from the authors.

willingness to sponsor a link, given the amount of information gained, depends on how much information the player would have without the link. Under DMBI, players have a higher need to acquire some amount of information the less information they have. This tends to favor non-stars, in the sense that if any star is stable, then so is any other network (satisfying our conditions), but not the other way around. Under IMBI, players value links more the better informed they are. This favors star networks. Under CMBI the stability of any network other than the PSS implies the stability of all networks considered.

As usual, the results of this paper are derived from a model that is based on many assumptions. Some are relatively minor. For instance, typically links cannot be formed without consent of the receiver of the link. However, player consent is unlikely to be a problem if links can only increase the payoffs of the receiving player. This is true in a setting of two-way flow, when the costs of accepting links are small enough. Similarly, the function describing information decay in distance is not only standard, but also quite specific. Nonetheless, by continuity our results should also hold for any decay function which is strictly decreasing in distance yet remains positive. Note in this regard that a binding limit to the distance information can travel, as in Hojman and Szeidl (2008), would correspond to large decay.[41]

A less innocuous assumption is that in our model the costs of the links are constant. However, the intuitions for a model variant with non-linear costs seem relatively straightforward. Suppose that the benefit function is CMBI while linking costs are non-linear. For convex linking costs, the CSS may not be a SNN, whereas all other balanced networks that meet our necessary conditions are, because the central player in the CSS finds the costs of the last links sponsored too high. For concave linking costs, the CSS may be a SNN whereas no other balanced networks that meet our necessary conditions are (except the PSS). The reason is that only a player who sponsors many links finds it rewarding to sponsor all of his links. This suggests at first sight that there is a one-to-one relation between convex costs and decreasing marginal benefits of information on the one hand; and concave costs and increasing marginal benefits of information on the other hand. Yet, the effect of the shape of the linking cost function is not so much related to the diameter of SNNs, but to how many links players sponsor. The latter effect is not specific to the presence of information decay. This is because a player's costs depend only on the number of links he sponsors, while his benefits depend on the architecture of the whole network.

Of course, our objective to study one of the effects in isolation, small decay, is not without costs. First, numerically the upper bound on the level of decay may be very low, especially for larger population sizes. In addition, for most of our result there is an additional upper bound from the balancing condition, which may or may not impose an even tighter upper bound. Second, empirical studies typically find non-minimal networks. This limits the relevance of the original goal of the paper, which is to understand decay better rather than explaining real world networks directly. Nonetheless, our findings are consistent with two stylized facts from the empirical literature: (i) networks may have high diameters, which are (ii) small relative to the population size. In addition, the intuitions for the relationship between the set of balanced Nash networks and the shape of the benefit function do not depend on either minimality or balancedness of those networks. Instead, they depend on the desirability of obtaining some additional information. In principle, these intuitions could carry over to non-minimal links and networks. Whether this is indeed the case remains to be verified by future empirical and theoretical work.[42]

Appendix

The appendix starts with preliminary lemmata. The first two results are standard in the literature, namely first that for small enough decay, all SNNs will be minimal and, second, that all non-empty networks are connected. The last preliminary lemmata are about the optimal recipient to link to, and about link orientation. These results are instrumental in the results of Section 3. After that we have separate sections of the appendix for respectively Proposition 1, Lemma 1, 2, Section 4, and Propositions 4 and 5. We end with two separate sections on the stochastic stability of weak Nash networks (Appendix H), and on the maximal diameter of networks (Appendix I).

Appendix A. Preliminary Lemmata

A.1. Minimality

Lemma 3. *In the absence of decay, no player in any network g prefers to sponsor non-minimal links.*

Proof. Any non-minimal link yields a marginal benefit of zero. As linking is costly, a player sponsoring a non-minimal link is better off when deleting any one of his non-minimal links. □

Now we show that for small enough decay, no player wishes to sponsor a non-minimal link.

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[41] We thank a referee for pointing this out.

[42] One possible starting point is offered by the literature on interfirm networks. This literature documents how different industries have different network characteristics, including a different average path length. Moreover; this literature identifies some of the factors that explain differences in network structure. These include the degree of separability of innovation activities (Rosenkopf and Schilling, 2007), the degree of interdependency between firms (Gulati and Gargiulo, 1999), the degree to which expertise is dispersed (Powell et al., 1996), or the degree of technological similarity between firms (Gomes-Casseres et al., 2006). In terms of our analysis, all of these characteristics are aspects of the shape of the benefits of information.

Lemma 4. *For any $c > 0$, $n \geq 4$, and $f(I)$, there exists $\delta_M < 1$ such that for all $\delta > \delta_M$ no player in any possible network wishes to sponsor a non-minimal link.*

Proof. Note that the benefit function is continuous in δ. By this continuity it follows from Lemma 3 that $\delta_M < 1$. □

A.2. Connectedness

The following two lemmata together show that all non-empty networks are connected.[43]

Lemma 5. *Let network g be a non-empty Nash network. Then g has no singleton component.*

Proof. Suppose that g does have a singleton component. Then there exists a player, say j, who is isolated. Moreover, by minimality and non-emptiness of g, there exists some player i who receives no links, but does sponsor a set of links himself. Note that $I_i(g_{-i}) = I_j(g_{-i}) = I_j(g)$. Now let player j consider strategy $g_{j'}$ in which he sponsors a link to every recipient of links from player i. So $g_{j'} = (ji' \in \mathcal{L} : i' \in \mathcal{N}_i^S(g))$. This costs him the same as g_i costs player i. But the benefits to player j are strictly larger because he accesses the same players at the same distance as player i does and in addition he will be connected to player i. Hence if g_i is a best reply for player i to network g, then $g_j = \{\emptyset\}$ cannot be a best reply to player j. Since g is a Nash network by assumption, this forms a contradiction. □

Lemma 6. *Let network g be a non-empty Nash network. Then g is connected.*

Proof. Suppose not. Then by Lemma 5 there is a Nash network g which contains multiple non-singleton components. W.l.o.g. we have $ii', jj' \in g$ such that i and i' belong to one component, say $C_1(g)$, and j and j' to another, say $C_2(g)$. Because g is a Nash network, it is a best reply for player i to sponsor a link to i' while not sponsoring a link to any player in $C_2(g)$. So player i' has access to at least as much information in $g \setminus \{ii'\}$ than j' in g. Hence we obtain that $I_{i'}(g) > I_{i'}(g \setminus \{ii'\}) \geq I_{j'}(g) > I_{j'}(g \setminus \{jj'\})$. Because g is Nash, we also have that $I_{i'}(g) < I_{j'}(g \setminus \{jj'\})$, which gives us a contradiction. Hence any Nash network has only one component and is therefore connected. □

It follows that *all non-empty* Nash networks are connected. Put otherwise, a Nash network is either connected, or it is the empty network. Note that the minimal benefit which an isolated player would derive from sponsoring a link is $f(1 + \delta) - f(1)$. Thus Lemma 6 implies the following corollary.

Corollary 1. *Let $c < f(1 + \delta) - f(1)$. Then any Nash network is connected.*

Before starting our actual characterization, we have thus shown that for small decay all non-empty SNNs must be minimal connected networks.

A.3. Optimal link recipients and link orientation

The following two Lemmata are instrumental in proving the main characterization results.

Lemma 7. *Let g be a minimal connected Nash network. For all $ij \in g$, it must be the case that, among the players in set $M = A_{ij}(g)$, j is a best-informed player in network g_M, and that j is the unique best-informed player in g_M if g is a SNN. Moreover, among the players in set M, player j is the unique best-informed player in g.*

Proof. Consider any minimal network g, and any sponsor of a link, say player i with link $ij \in g$. Denote $A_{ij}(g)$ by M. Then the amount of information gained by i through $ij \in g$ equals $\delta I_j(g_M)$, namely the amount of information which j gathers in the network without link ij, discounted one time. If i would replace ij by the link ij', where $j' \in M$, then i would receive $\delta I_{j'}(g_M)$ instead. As each link costs the same i strictly prefers ij to ij' if and only if $I_j(g_M) > I_{j'}(g_M)$. Noting that network g is a Nash network, $ij \in g$ thus implies that $I_j(g_M) \geq I_{j'}(g_M)$ for any $j' \in M$. This inequality is a strict inequality if the preference is strict, thus if g is a SNN. Moreover, j is also the best-informed player of M in network g. The reason is that the link ij increases the information advantage of player j over the other players in M, since the information which travels from i to j will decay at least one more time before it reaches any other player in M. This concludes the proof. □

Lemma 7 allows us to derive a result central to the characterization, namely that links pointing to one another must be sponsored to the same player.

Lemma 8. *Let g be a minimal connected Nash network. If g contains links ii' and jj' which point towards each other, then $i' = j'$.*

Proof. We prove this by contradiction. Suppose not, so $j \in A_{ii'}(g)$ and $i \in A_{jj'}(g)$, while $i' \neq j'$. Note that by minimality there is one path connecting i and j, and this path goes via players i' and j'. By Lemma 7, among the nodes in the set $A_{ii'}(g)$, i' is the node that receives strictly the most information from network g, and in the set $A_{jj'}(g)$, j' is the node that receives strictly the most information from network g. But as $j' \in A_{ii'}(g)$ and $i' \in A_{jj'}(g)$, Lemma 7 also implies that both $I_{i'}(g) > I_{j'}(g)$ and $I_{j'}(g) > I_{i'}(g)$: a contradiction. □

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[43] This result is a variant of BG, Proposition 5.1, which applied to linear benefit functions only.

This implies that if no end-link is sponsored in a minimal connected SNN, then all links are received by the same player. As this insight proves useful in Section 4, we state it in the following Corollary.

Corollary 2. *Let g be a minimal connected Nash network. Then g is either a PSS, or g contains at least one end link.*

Appendix B. Proof of Proposition 1

We prove the different parts of the proposition in turn. Before we prove Parts 1a and 1b, we first state a lemma showing that only two types of non-empty minimal Nash networks are possible.

Lemma 9. *Let network g be a non-empty minimal Nash network. Then g contains either a unique non-recipient player and no multi-recipient players, or a unique multi-recipient player and multiple non-recipient players.*

Proof. First note that by Lemma 6, g is connected. By minimality, g has precisely $(n-1)$ directed links. It follows that g contains at least one non-recipient player. If there is exactly one non-recipient player, there are thus $(n-1)$ players who receive one link. Consequently there cannot be any multi-recipient players. If instead there are multiple non-recipient players, then there must exist at least one multi-recipient player. It remains to be shown that SNNs cannot have multiple multi-recipient players. Suppose that players i and j are both multi-recipient players. Because g is minimally connected, players i and j are connected by at most one path. This means that both players receive at least one link, say $i'i$ for i and $j'j$ for j, which is not part of the path connecting them. These links point toward each other which implies by Lemma 8 that $i=j$, a contradiction. □

Therefore we have two cases for a non-empty minimal Nash network g. Either g has a unique non-recipient player, or it has a unique multi-recipient player. We will now consider each case in turn, and prove respectively each part of the proposition.

Proof for Proposition 1. **Part 1a:** Let i be the unique non-recipient player. We now show that all links point away from him. Suppose not. Then w.l.o.g. there exists a player j who is the closest player to i who sponsors a link which points towards i, say link jj'. Clearly $j' \neq i$. By construction j' is a multi-recipient player: he receives a link connecting him to i, and he receives another link from j pointing towards i. This violates Lemma 9. Thus all links point away from i. Consequently i is the root of a directed tree.

Finally, we show by construction that the set of best-informed players in g is a subset of i and all the recipients of a link by i. Suppose not. Then there exists $ii' \in g$ and $j \in A_{ii'}(g)$ such that $I_j(g) \geq I_k(g)$ for all $k \in \mathcal{N}$. However, by Lemma 7 we have $I_{i'}(g) > I_j(g)$. The statement follows.

Part 1b: let i be the unique multi-recipient player. First we show that for all $jj' \in g$, either jj' points away from i, or $j'=i$. Suppose not. As i receives multiple links, by construction there exists $i'i \in g$ such that $i'i$ and jj' point towards each other, while $j' \neq i$. This violates Lemma 8.

Now we prove that i is the unique best-informed player of g. As i is a multi-recipient player, there exist two players j and j' such that g comprises ji and $j'i$. Let $M = A_{ji}(g)$ and $M' = A_{j'i}(g)$. From Lemma 7 it follows that for all $k \in M$ we have $I_i(g_M) \geq I_k(g_M)$ and thus $I_i(g) > I_k(g)$. Similarly $I_i(g) > I_{k'}(g)$ for all $k' \in M'$. By connectedness of g we have $M \cup M' = \mathcal{N}$. Thus i is the unique best-informed player of g.

Part 2: suppose not. Then along this path there exists a link $j_\ell j_{\ell+1}$, where $\ell \in \{1, \ldots, k-1\}$, such that $I_{j_\ell}(g) \leq I_{j_{\ell+1}}(g)$. Letting $M = A_{j_{(\ell-1)}j_\ell}(g)$this implies that $I_{j_\ell}(g_M) < I_{j_{\ell+1}}(g_M)$, as j_ℓ receives more information than $j_{\ell+1}$ via link $j_{(\ell-1)}j_\ell$. Consequently player $j_{(\ell-1)}$ would strictly prefer to sponsor a link to $j_{(\ell+1)}$ rather than to j_ℓ, and g would not be a Nash network: a contradiction.

Part 3: Let $M = A_{ij}(g)$. By construction $\left|M\right| \geq 2$. Thus $\exists k \in M$ such that $\overline{jk} \in g$. If $\left|M\right| = 2$, then $I_j(g_M) = I_k(g_M) = 1 + \delta$. By Lemma 7 g is not a SNN. Now let $\left|M\right| > 2$, while for all $k' \in M \setminus \left\{k\right\}$ we have $\overline{jk'} \notin g$. Then k connects j to all other players in M. Thus k is strictly better informed than j in g_M. By Lemma 7 we obtain a contradiction.

Part 4: this follows from combining Lemma 4 with Part 1. □

Appendix C. Proof of Lemma 1

Proof. It is sufficient to prove that this property holds for any arbitrarily chosen link ij in any arbitrarily chosen minimal network g. Define $M = A_{ij}(g)$. First, for small enough decay, player i will not prefer to sponsor two links to members of M. Thus we verify that i does not want to replace ij by a single other link, ik where $k \in M$. To do this, denote by $\mathcal{N}_i^d(g,k)$, $\mathcal{N}_i^d(g,k) \subseteq M$, the set of players in M who are at distance d from player i in g, and who i observes via k, where obviously $k \in M$. We denote an arbitrary member of $\mathcal{N}_i^d(g,j)$ by j^d. So j^d is some player in M at distance d from i. Finally note that $j^1 = j$.

Consider $\mathcal{N}_i^2(g, j)$, so players in M who are at distance 2 from i in g (thus $\overline{jj^2} \in g$). We now derive the condition under which player i does not want to replace $ij^1(=ij)$ by ij^2, where $j^2 \in \mathcal{N}_i^2(g, j)$. The information gathered by player i from M by link ij^1 equals

$$\sum_{d=1} \delta^d \left|\mathcal{N}_i^d(g, j^1)\right|. \tag{4}$$

In comparison, the information gathered by player i from M if he replaces ij by $ij^2$is equal to

$$\sum_{d=1} \left\{\delta^{d+1}\left[\left|\mathcal{N}_i^d(g, j^1)\right| - \left|\mathcal{N}_i^d(g, j^2)\right|\right] + \delta^{d-1}\left|\mathcal{N}_i^d(g, j^2)\right|\right\}. \tag{5}$$

This formula follows from observing that all players in set $\mathcal{N}_i^d(g, j^2)$ are one step closer than before by switching the link from $ij^1(=ij)$ to ij^2, while the remainder of the players in M, namely $\left[\left|\mathcal{N}_i^d(g, j^1)\right| - \left|\mathcal{N}_i^d(g, j^2)\right|\right]$, are now one step further away. So i strictly prefers to keep his link ij, rather than replacing it with a link to j^2 if the cost of getting one step further away from some agents ($\left[\left|\mathcal{N}_i^d(g, j^1)\right| - \left|\mathcal{N}_i^d(g, j^2)\right|\right]$), outweighs the benefits of getting one step closer to some other agents ($\left|\mathcal{N}_i^d(g, j^2)\right|$). This is the case if and only if

$$\sum_{d=1} \left\{\left[\delta^d - \delta^{d+1}\right]\left|\mathcal{N}_i^d(g, j^1)\right| + \left[\delta^{d+1} - \delta^{d-1}\right]\left|\mathcal{N}_i^d(g, j^2)\right|\right\} > 0 \tag{6}$$

Define as $\delta_B^{ij} < 1$ the largest root of the polynomial on the left-hand side of (6) that is smaller than 1. Another root of (6) is $\delta = 1$, reflecting the fact that in the absence of decay, it does not matter where the player connects. If the derivative with respect to δ of the left-hand side of (6) is negative at $\delta = 1$, then a range of large δs exists such that the player prefers to connect to j^1 rather than to any j^2. Taking the derivative of (6) with respect to δ and putting $\delta = 1$, the condition becomes

$$-\sum_{d=1} \left|\mathcal{N}_i^d(g, j^1)\right| + 2\sum_{d=1} \left|\mathcal{N}_i^d(g, j^2)\right| < 0 \tag{7}$$

or

$$\sum_{d=1} \left|\mathcal{N}_i^d(g, j^1)\right| > 2\sum_{d=1} \left|\mathcal{N}_i^d(g, j^2)\right| \tag{8}$$

and thus

$$\sum_{d=1} \left|\mathcal{N}_i^d(g, j^1)\right| - \sum_{d=1} \left|\mathcal{N}_i^d(g, j^2)\right| > \sum_{d=1} \left|\mathcal{N}_i^d(g, j^2)\right| \tag{9}$$

Note that condition (9) is satisfied by definition for any $j^2 \in \mathcal{N}_i^d(j)$, if player j is the middle player of g_M. This proves that for large enough δ, player i will strictly regret replacing his link ij by a link ij^2. However, we have only derived a condition assuring that no player wants to reconnect to a player at distance 2 in the accessed component. To see that this condition also implies that he does not want to switch his link ij to a player in M who is at distance 3 or more, consider a path $\left\{\overline{j^1j^2}, \overline{j^2j^3}, \ldots, \overline{j^pj^{p+1}}, \overline{j^{p+1}j^{p+2}}, \ldots, \overline{j^{z-1}j^z}\right\}$ in g_M, where player j^z has no other links than $\overline{j^{z-1}j^z}$. Now suppose i needs to replace ij by a link to either j^p or to j^{p+1}. Which recipient would he prefer? Let $g' = \left\{ij^p\right\} \cup g \backslash \left\{ij\right\}$. By the given condition, for high δ, player i prefers connecting to j^p rather than to j^{p+1} if $\sum_{d=1} \left|\mathcal{N}_i^d(g', j^p)\right| > 2\sum_{d=1} \left|\mathcal{N}_i^d(g', j^{p+1})\right|$. Note now that $\sum_{d=1} \left|\mathcal{N}_i^d(g', j^p)\right| = \sum_{d=1} \left|\mathcal{N}_i^d(g, j^1)\right|$, whereas $\sum_{d=1} \left|\mathcal{N}_i^d(g', j^{p+1})\right| < \sum_{d=1} \left|\mathcal{N}_i^d(g, j^2)\right|$. From condition (6) it follows that player i prefers linking to any j^p along the path rather than to j^{p+1}. Thus if j is the middle player in g_M, then there exists some δ_B such that for all $\delta \in [\delta_B, 1)$ we have that i strictly prefers ij to ik for all $k \in M$, $k \neq j$. □

Appendix D. Proof of Proposition 2

Proof. We prove this by construction. First we will argue that for $\delta = 1$, there exists a range of c such that no player wants to add a non-minimal link or delete a minimal link. Then by continuity the same applies for δ close enough to 1. Moreover, for δ sufficiently close to 1, the balancing condition is satisfied as well.

Let $\delta = 1$. As g is connected, no player in g is willing to sponsor an additional link for any $c > 0$. Now consider the incentives to delete links. Let $b_i^{\min}(g)$ be the minimal average benefit which i receives over any subset of links, $s_i \subseteq g_i$, he sponsors. So

$$b_i^{\min}(g) = \min\left\{\frac{1}{|s_i|}(f(I_i(g)) - f(I_i(g \backslash s_i))) : s_i \subseteq g_i\right\}. \tag{10}$$

As g is minimal and $f(\cdot)$ is increasing, we have that $b_i^{\min}(g) > 0$. Clearly, no player has an incentive to delete any subset of his links if $c < b^{\min}(g)$, where $b^{\min}(g) = \min \left\{ b_i^{\min}(g) \right\}_{i \in \mathcal{N}}$.

Note now that $f(I_i(g)) - f(I_i(g'))$ is continuous in δ, as $I_i(g)$ is continuous in δ and $f(I_i(g))$ is continuous in $I_i(g)$. It follows that there exists some $\bar{\delta}(g) < 1$ such that for all $\delta > \bar{\delta}(g)$ all players in g strictly prefer to keep all links they sponsor, while they prefer not to add any links. By setting $\delta > \max \left\{ \bar{\delta}(g), \delta_B \right\}$ the balancing condition is satisfied, i.e. there is also no player who wants to replace any subset of the links he sponsors. The result follows. □

Appendix E. Proofs for Section 4

Proof for Lemma 2. By $f'(\cdot) \leq 0$ and the minimality of all networks in $\mathcal{G}^*$ it follows that no sponsor in any $g \in \mathcal{G}^*$ can gain less by sponsoring a link than $f(1+(n-1)\delta) - f(1+(n-2)\delta)$. More strongly, under DMBI, so $f'(\cdot) < 0$, any sponsor in any non-star network gains strictly more from any single link. Consequently, no sponsor in any network $g \in \mathcal{G}^*$ will prefer to delete his link if the central player in a star strictly prefers to sponsor a link. Moreover, by $\delta > \delta_M$ no player wants to add links. By $\delta > \delta_B$ no sponsor of a link wants to change the recipient of a link he sponsors. Therefore if the CSS is SNN so is any $g \in \mathcal{G}^*$. □

Before the proof of the two main propositions we present two other lemmata. The first lemma shows that Lemma 1 in Feri (2007) applies to DMBI benefit functions as well. The second lemma shows that any balanced directed rooted tree is a member of $\mathcal{G}^*$.

Lemma 10. *Let $f(\cdot)$ have CMBI or DMBI. Moreover let $\max \left\{ \delta_M, \delta_B \right\} < \delta < 1$ and let the CSS be a SNN, so $f(1+(n-1)\delta) - f(1+(n-2)\delta) > c$. Then any CSS is stochastically stable.*

Proof. Consider any network g. Let i_c update and, by mutation, sponsor a link to each other player. Now let all players but i_c update. They will find it optimal to delete all of their links. The resulting CSS is a SNN, and thus a recurrent class. □

Lemma 11. *Let $\delta > \delta_B$. If network g is a connected balanced directed rooted tree, then $g \in \mathcal{G}^*$.*

Proof. As g is a rooted directed tree, it satisfies part 1a of Proposition 1. As the network is balanced, part 2 of Proposition 1 is satisfied as well. Moreover, by $\delta > \delta_B$, each link ij in a balanced network g is sponsored to the best-informed player of $A_{ij}(g)$. Thus part 3 is also satisfied. Part 4 is not related to the architecture of the network. Finally, as g is balanced, $g \in \mathcal{G}^*$. □

Proof for Proposition 3. The proof is by construction. We start with the CSS g^0 with central player i_c. By Lemma 10, any CSS which is a SNN is stochastically stable. The structure of the construction procedure is as follows. Each step consists of two periods, in the first period a single mutation takes place, while in the second period there is convergence to a SNN through a single updating player. In particular, in the first period i_c deletes by mutation two links, creating two new, identical components. If these components are not singletons, then each of these components is a rooted directed tree, where the root player is also the middle player. In the second period, a player i who receives a link from i_c (so $i_c i \in g$) sponsors a link to each of these two components. Specifically, by $\delta > \delta_B$, i sponsors a link to the root players of each of these components. This has two consequences. First, the network becomes again a connected rooted directed tree with i_c as the root player. Second, as each sponsor other than i_c always adds two links to the middle players of two symmetric components, the network is also balanced. Thus after each second step, by Lemma 11, the network has converged to a network in $\mathcal{G}^*$. By Lemma 2 this network is a SNN. As each network is reached from a stochastically stable network (g^0) by a chain of single updates interspersed with convergence to a SNN, each such SNN is stochastically stable too.

The two-period procedure described above will be the building block for our proof. We now show how to use this procedure to get from g^0 to a stochastically stable diameter 3 network g^1, and then to a stochastically stable diameter 4 network g^2. Finally we show how to use this procedure to get from network g^k (with diameter $k+2$) to g^{k+1} (with diameter $k+3$, provided n is so large that i_c still sponsors an end link). Note that any SNNs on the path between g^k and g^{k+1} will have a diameter no smaller than g^{k+1}. The construction procedure leading to g^1 and g^2 respectively is illustrated in Fig. 8.

Consider g^0. Let i_c delete two links to peripheral players other than player i^*, $i^* \neq i_c$. Then let player i^* update. He will sponsor two links, one to each of the isolated players. Call the resulting network g^1. As discussed above $g^1 \in \mathcal{G}^*$ and thus both SNN and stochastically stable. Note that $A_{i_c j}(g^1) = \left\{ j \right\}$ unless $j = i^*$.

Now iterate the procedure used to obtain g^1 twice, without involving any player in $A_{i_c i^*}(g^1)$. Let i^1 and $i^{1'}$ be the players who are symmetric to i^* in the resulting network. Now let i_c delete the links to i^1 and $i^{1'}$. Finally, let i^* update. He will add two links, namely to i^1 and $i^{1'}$. As discussed above, $g^2 \in \mathcal{G}^*$. Note that to obtain g^2 from g^1 it is necessary to apply the two-period procedure three times, whereas g^1 was obtained after a single application from g^0. Moreover note that again $A_{i_c j}(g^2) = \left\{ j \right\}$, unless $j = i^*$. In other words, i^* is the only sponsor in g^2 who receives a link from i_c.

By iteration we can obtain any diameter. Consider g^k, where $k \geq 1$. Let i^* be the unique sponsor in g^k who receives a link from i_c. Suppose s^k applications of the two-period procedure were needed to obtain g^k from g^{k-1}. Then in $2s^k + 1$ applications of the two-period procedure g^{k+1} can be obtained. First apply twice more the procedure to obtain g^k ($2s^k$ repetitions each), so that players i^k and $i^{k'}$ become symmetric to i^* in the resulting network, say g^{k+}. Note that the furthest player from i_c in $A_{i_c i^*}(g^k)$ was distance $k+1$ from i_c. So also the furthest players from i_c in $A_{i_c i^k}(g^{k+})$ and $A_{i_c i^{k'}}(g^{k+})$ are distance $k+1$ away from him. Let j be one of these furthest players from i_c in $A_{i_c i^k}(g^{k+})$. Finally let (first) i_c delete his links to i^k and $i^{k'}$; and (second) let i^* add links to i^k and $i^{k'}$. Call the resulting network g^{k+1}. Note that in g^{k+1} there is only one sponsor who receives a link from i_c, namely i^*. By the arguments above $g^{k+1} \in \mathcal{G}^*$ and is therefore SNN and stochastically stable. Finally note that distance of i_c to j has now increased by one. Similarly, all other players in $A_{i_c i^k}(g^{k+})$ and $A_{i_c i^{k'}}(g^{k+})$ are now one further from i_c. Thus if i_c sponsors an end link in g^{k+1}, the diameter of g^{k+1} is now $k+3$. As k was chosen arbitrarily, the proof follows. □

Appendix F. Proof to Proposition 4

Proof. Observe that by $\delta > \delta_M$ no player prefers to sponsor a non-minimal link. By connectedness (all $g \in \mathcal{G}^*$ are connected) this implies that no player prefers to sponsor an additional link. Similarly, as all $g \in \mathcal{G}^*$ are balanced, $\delta > \delta_B$ implies that for all $ij \in g$, j is the best-informed player of g_M, where $M = A_{ij}(g)$. Thus no player i with $ij \in g$ prefers to replace ij by a link ik, $k \in A_{ij}(g)$. Therefore if any $g \in \mathcal{G}^*$ is not a SNN, it is because some player i wants to delete one or more links. Based on this observation, we now prove parts 1 to 4 of Proposition 4.

Part 1. We first show that if a PSS is not a SNN, then any non-recipient player in a minimal non-empty network g (weakly) prefers to delete all links. Suppose that a PSS is not a SNN. Then, by $\delta > \delta_M$, we have for any $l \geq 1$

$$f(1+\delta+(n-2)\delta^2) - c > f(1+l\delta+(n-1-l)\delta^2) - lc \tag{11}$$

As the central player is clearly the optimal recipient, it follows that the PSS is not a SNN because

$$f(1) \geq f(1+\delta+(n-2)\delta^2) - c \tag{12}$$

Consequently we obtain

$$f(1) > f(1+l\delta+(n-1-l)\delta^2) - lc. \tag{13}$$

Note that in any minimal network no non-recipient sponsoring l links can gain more information than $(1+l\delta+(n-1-l)\delta^2)$, namely l players at distance 1 and the rest at distance 2. Moreover, every minimal component in any non-empty network g contains at least one non-recipient. By connectedness of the component, this non-recipient sponsors at least one link. It follows that if a PSS network is not a SNN, then neither is any non-empty minimal network g.

Part 2. Under CMBI, the willingness to sponsor a link depends solely on how much additional information is obtained through that link. The sponsors who gain the least amount of information through a single link are the end sponsors. Under CMBI the network architecture does not matter for how the end sponsor values the information. Hence either all end sponsors in all $g \in \mathcal{G}^*$ prefer to delete their links, or all strictly prefer to keep their links. The result follows.

Part 3. This follows straightforwardly from the proof of Lemma 2 for DMBI.

Part 4. Consider any non star network $g \in \mathcal{G}^*$. W.l.o.g. let i be the end sponsor who sponsors the most end links in g. Let k be the number of end links sponsored by i in g. As g is not a star, $k < n-1$. Consider now star network g' with central player i_c, where i_c sponsors k links. The stability of the PSS (see Part 1) implies that in g' no peripheral sponsor prefers to delete a link. Thus if g' is not SNN if and only if i_c prefers to delete a number of links. As all links sponsored by i_c are end links, IMBI implies that if i_c prefers to delete any number of links, he also prefers to delete all k links. The amount of information gained through the k end links for i in g is as for i_c in g'. In addition i_c is better informed than i, formally $I_{i_c}(g') > I_i(g)$. Consequently, if i_c weakly prefers to delete any number of his links in g', then player i in g strictly prefers to delete at least the same number of links in g'. The proof follows. □

Appendix G. Proof of Proposition 5

Proof. By $\delta > \max\{\delta_M, \delta_B\}$, if a network $g \in \mathcal{G}^*$ is not a SNN, then some sponsor in g prefers to delete one or more links. By Part 3 of Proposition 4 we know that the stability of any star with a center-sponsored link implies the stability of all higher diameter networks in $\mathcal{G}^*$, but not the other way around. Thus it remains for us to show that the stability of diameter 3 networks implies that some $g \in \mathcal{G}^*$ with diameter 4 or larger is stable, but not vice versa.

For $n \geq 7$, we can construct a diameter 4 network which is a SNN when any diameter 3 network is a SNN, but not the other way around. Take a PSS network and delete the links sponsored by two distinct peripheral players, j and j'. Consider two distinct other peripheral players, i and i' and let them form the links ij and $i'j'$. This network, say g^4, has diameter 4, and meets all the conditions of Propositions 1 and 2. For $\delta > \delta_M$, no player wants to add a link in this network. The added benefit of link ij in this network (and thus of $i'j'$) is $f(1+2\delta+(n-4)\delta^2+\delta^3) - f(1+\delta+(n-4)\delta^2+\delta^3)$. Under DMBI, the added benefit of each other link in g^4 is higher. Thus, such a network is a SNN if

$$f(1+2\delta+(n-4)\delta^2+\delta^3) - f(1+\delta+(n-4)\delta^2+\delta^3) > c \tag{14}$$

We now construct a network of diameter 3, g^3, where the smallest benefit to any player of sponsoring a link is as high as possible among diameter 3 networks. In all diameter 3 networks we can uniquely identify two adjacent players, say i_c and $i_{c'}$ such that for any other player j we have that either $\overline{i_c j} \in g$ or $\overline{i_{c'} j} \in g$. Of the remaining $(n-2)$ players, let k players, with $1 \leq k \leq (n-3)$, have a link (as sponsor or recipient) with i_c and $(n-k-2)$ have a link with $i_c{}'$. At least one of i_c and $i_c{}'$ must

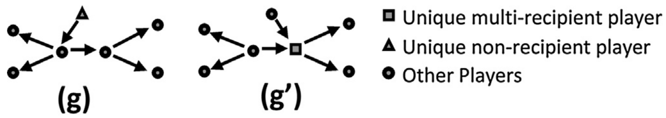

Fig. 9. Two weak Nash networks.

sponsor all links other than the link $\overline{i_c i_{c'}}$. Suppose not. Then there exist i, $i' \in N$ such that $ii_c \in g$ and $i'i_c{}' \in g$. These two links point towards each other, violating Lemma 8. W.l.o.g. let $i_c j \in g^3$. Now we derive the strongest possible incentives for i_c to sponsor $i_c j$ in g^3, to compare this with the stability condition for g^4, (14). By DMBI, i_c has the strongest incentives to acquire this additional δ information the less information i_c has without the end link. Thus g^3 has $k = 1$, while the remaining $n - 3$ peripheral players have a link with $i_c{}'$. Consequently g^3 is a SNN only if

$$f(1 + 2\delta + (n-3)\delta^2) - f(1 + \delta + (n-3)\delta^2) > c \tag{15}$$

Comparing (14) and (15) we see that in both cases difference in information is δ. However the amount of information without the link is lower in g^4 than for i_c in g^3. Therefore (15) implies (14) under DMBI, while for some range of parameters (14) is satisfied while (15) is not. This concludes the proof. □

Appendix H. Stochastic stability of weak Nash networks

We first prove that for CMBI and DMBI benefit functions, stochastically stable weak Nash networks can exist. For this we use networks C and F from Fig. 7 in the paper. We depict them again in Fig. 9 for reference, where network C is g and network F is g'. Second we show that not all weak Nash networks can be stochastically stable.

Fig. 10 depicts the steps of the proof. The proof follows the same lines as that for Proposition 3 in the main text.

Proof. The proof is by construction. As we need to prove existence of stochastically stable weak Nash networks, an example suffices. Let $n = 7$, let $f'(I_i) \le 0$ and let $c < f(1 + 6\delta) - f(1 + 5\delta)$, and $\delta > \max\{\delta_M, \delta_B\}$. For these parameters, networks g and g' in Fig. 9 are weak Nash networks, and any CSS is a SNN. We start with the CSS g^0 and central player 2. By Lemma 10 in the paper, any CSS which is a SNN is stochastically stable. Let player 2 suffer from a mutation and delete links to 6 and 7, which results in network g^1. Now let only player 3 update. His best reply is to sponsor links 36 and 37. Thus we obtain g^2. As this is a SNN (by Lemma 2 from the paper) obtained after a single mutation, g^2 is stochastically stable. Let 2 suffer from a mutation again, and delete his link to player 1, which gives us g^3. Finally, let 1 update. One of his two best responses is to sponsor link 12. Thus we obtain $g = g^4$ after a single mutation from a stochastically stable state (g^2). The last thing to do is to prove that g belongs to a recurrent class. Note that in both g and g' from Fig. 9 player 1 is the only player who has multiple best replies. Depending on the best reply he chooses, the resulting network can either be g or g', but no other network. Thus there is a recurrent class X, where $X = \{g, g'\}$. This concludes the proof. □

We now show by example that there may exist weak Nash networks which cannot be stochastically stable. Fig. 11 is useful for that regards. Note that g^I is basically g^3 from Fig. 10 with link 41 added. In contrast to earlier proofs, this proof relies on several agents updating simultaneously as in BG. It therefore does not extend to the dynamics specified in Feri (2007).

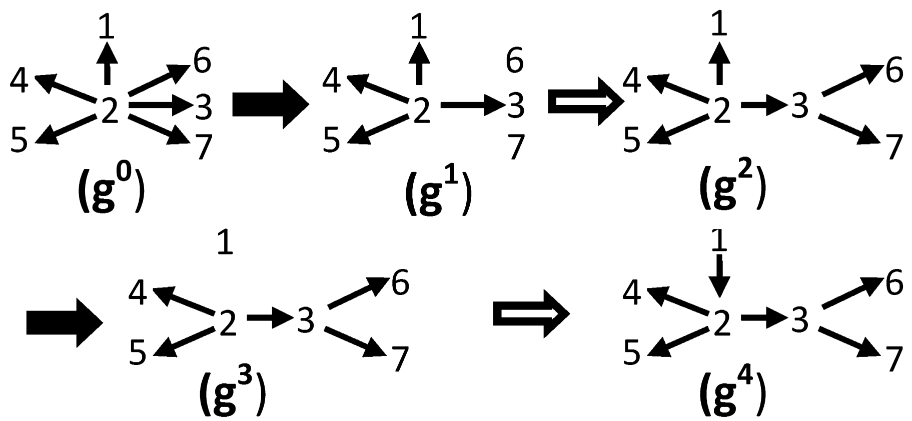

Fig. 10. Example dynamics of CSS g^0 to network g^4, which is network g from Fig. 9. The big arrows indicate transitions from one network to the next. Each transition involves a single player updating. Filled arrows indicate that the updating player suffers from a mutation. Open arrows indicate a normal best reply response by the updating player.

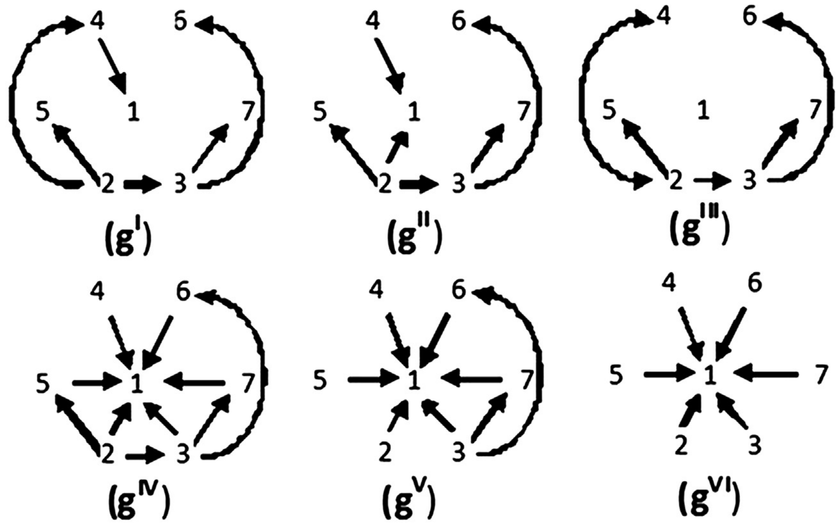

Fig. 11. Convergence to PSS.

Proof. First we note that g^I is a weak Nash networks under DMBI and CMBI, provided that there is small decay and that the CSS is a SNN. Now we show that g^I is not stochastically stable. We do this by allowing convergence to the PSS, g^{VI}. As the PSS is a SNN, it is a recurrent class by itself. It follows that g^I is not in a recurrent class of the dynamics. Consequently, g^I is not stochastically stable.

Consider g^I. Every player plays a best reply. Only player 2 has an alternative best reply, namely $\{21, 23, 25\}$. In that case, he replaces 24 by 21. Let player 2 update and select his other best reply. We obtain g^{II}. Network g^{II} is not a Nash network. By decay, player 4 strictly prefers link 42 to link 41. Let players 4 and 2 update simultaneously, and let player 2 switch back to g_2^I. Then we obtain g^{III}, such that $g^{III} \supset \{24, 42\}$. Now let all players except player 1 update. As the CSS is a SNN and by non-increasing marginal benefits of information, all players prefer to sponsor a link to the isolated player 1. Moreover, 2 and 4 delete their links to each other, as those links are non-minimal. We obtain g^{IV}. Now let player 2 update. Under small decay, he wants to delete two of his three links. Player 2 will sponsor link 21. Sponsoring anyone other link instead would put some players at distance 3, whereas 21 keeps all other players within distance 2. Thus we obtain g^V. Finally, let player 3 update. For the same reasons he sponsors only link 31 and deletes links 36 and 37. Thus we obtain network g^{VI} which is a PSS. This concludes the proof. □

Appendix I. Maximal diameter of balanced networks

In this appendix we derive the maximal diameter which a balanced minimal connected SNN can have as a function of n, if decay is small enough. To do so we need the following technical lemma. The intuition for this lemma is provided in the main paper (see Fig. 7 and nearby intuition). Moreover, the recipient of an end link is called an *end recipient.*

Lemma 12. *Consider a balanced network g. Let link $ij \in g$ be such that it gives i access to an end recipient at distance d from player i. There are two cases. In Case 1, i gets access to the end recipient through a link sponsored by j. In Case 2, i gets access to the end recipient through a link received by j.*

1. *If i gets access to the end recipient at distance d through a link jk, then in the component g_M with $M = A_{ij}(g)$ player i has at least $\sum_{l=1}^{x+1} 2^l$ players at distance $(d-x)$ or larger from him, where $0 \le x \le (d-2)$.*
2. *If i gets access to the end recipient at distance d through a link kj (where $kj \neq ij$), then in the component g_M with $M = A_{ij}(g)$, for $d \ge 3$, (a) player i has at least $\sum_{l=1}^{x+1} 2^l$ players at distance $(d-x)$ or larger from him, where $0 \le x \le (d-4)$ (with $d \ge 4$); (b) player i has at least $1 + \sum_{l=1}^{d-3} 2^l$ players at distance 3 or larger in component g_M; and (c) player i has at least $4 + 2\sum_{l=1}^{d-3} 2^l$ players at distance 2 or larger in component g_M.*

Proof. We first prove Part 1 by induction. By Lemma 8 in the main paper, in g_M with $M' = A_{jk}(g)$, all links point away from k. Let m be an end sponsor in M' who is furthest from k. Thus, m receives a link from some player l. By Part 3 of Proposition 1 in the main paper, m should sponsor at least two links.

Let an end sponsor m receive a link from a player l sponsoring exactly two end links. Then, if player l is himself a recipient, by the balancing condition (see Lemma 1 in the main paper), player l's links should point to at least 6 players at distance 1 or 2 from him, and at least 2 players at distance 2 from him.

Let player l's links point to exactly 6 players, at distance 1 or 2 from him, and let player l himself receive a link from player h. Then, if player h is himself a recipient, his links should point to at least 14 players at distance 1, 2 or 3 from him, at least 6 players at distance 2 or 3 from him, and at least 2 players at distance 3 from him. And so forth.

The proof for 2(a) is similar. Define $M' = \mathcal{N}_k\left(g\backslash\left\{kj\right\}\right)$. By Lemma 8 in the main paper, all links in $g_{M'}$ point away from k. The rest is the same as for Part 1 above.

We next prove 2(b). This follows simply from the fact that there must be at least $\sum_{l=1}^{d-3} 2^l$ players at distance 4 or larger plus at least one player at distance 3.

In order to prove 2(c), note first that given the above, there is at least one player k connected to player j who gives access to at least $1+\sum_{l=1}^{d-3} 2^l$ players. Consequently, $\left|A_{kj}(g)\right| \geq 2+\sum_{l=1}^{d-3} 2^l$. By the balancing condition, $\left|A_{ij}(g)\right| \geq 4+2\sum_{l=1}^{d-3} 2^l$. □

We now derive the maximal diameter networks that can be achieved for a particular range of n. In order to do this, we derive the minimal number of players n that are needed to achieve a network of given diameter D. For this n, and for a range of populations just above it, this is then also the maximal diameter network that can be achieved.

The focus in the proposition below is on a planner who aims to maximize the size of a balanced SNN network. In the proof below we use 'player' and 'nodes' interchangeably, depending whether we want to emphasize an action or decision by the player or not. Finally, define 'candidate networks' as balanced networks which satisfy the parts of Proposition 1 in the main text.

Proposition 6. *Consider any odd x, with $x \geq 3$. For n such that $1+2^{(x-1)/2}+2\sum_{l=1}^{(x-1)/2} 2^{l-1} \leq n < 4+3\sum_{l=1}^{(x-1)/2} 2^{l-1}$ (where $(4+3\sum_{l=1}^{(x-1)/2} 2^{l-1})-(1+2^{(x-1)/2}+2\sum_{l=1}^{(x-1)/2} 2^{l-1}) = 2$), the maximal diameter d over all candidate networks is equal to x. For n such that $4+3\sum_{l=1}^{(x-1)/2} 2^{l-1} \leq n < 1+2^{(x+1)/2}+2\sum_{l=1}^{(x+1)/2} 2^{l-1}$ (where $(1+2^{(x+1)/2}+2\sum_{l=1}^{(x+1)/2} 2^{l-1})-(4+3\sum_{l=1}^{(x-1)/2} 2^{l-1}) >> 2$), the maximal diameter d over all candidate networks is $(x+1)$.*

Proof. It suffices to derive the minimal number of nodes $n_1(d_1)$ needed to achieve a network with any even diameter d_1, and the minimal number of nodes $n_2(d_2)$ needed to achieve a network with any odd diameter d_2. Once we have these results, it immediately follows that for any n with $n_1(d_1) < n < n_2(d_2)$, the maximal achievable diameter is d_1, and for any n with $n_2(d_2) < n < n_1(d_1)$, the maximal achievable diameter is d_2. The reason that any additional player i can be linked to the network by link ii_v, where i_v is the best informed player in the network. Note that this will not expand the diameter, as the best informed player has at least 2 links. We will first derive the minimal number of nodes for even diameter networks. Then we will derive the minimal number of nodes for odd diameter networks.

Minimal number of nodes for an even diameter. We now show that one needs at least $n = 4+3\sum_{l=1}^{d_1/2-1} 2^{l-1}$ nodes in order to have networks of diameter d_1 with $d_1 \geq 4$.

We *first* show that the minimal number of nodes needed to achieve a network with a *multi*-recipient player of even diameter $d_1 \geq 4$ is the given n. By the definition of a multi-recipient player, at least two of the nodes at distance 1 from such a multi-recipient player must sponsor links towards him (though the multi-recipient player may sponsor links himself as well). W.l.o.g. let j be the multi-recipient player and let players i and k sponsor a link to j. Then in order to put an end recipient at any distance s from i using a minimal number of nodes, it should be that i accesses this end recipient through link kj rather than through some link jk'. This follows from Lemma 12 and the fact that $4+2\sum_{l=1}^{s-3} 2^l < \sum_{l=1}^{s-1} 2^l$. Thus, in order to construct with a minimal number of nodes a multi-recipient network where two end recipients l and m are at distance d_1 from each other, we can limit ourselves to networks with links ij and kj, where i (respectively k) accesses end recipient l (m) through k (i).

Consider in particular a multi-recipient network with diameter d_1. From Lemma 12 it follows that no end recipient is further than distance $d_1/2$ of the multi-recipient player. Otherwise the number of additional nodes needed on one side of j is larger than the number of nodes which can be taken away at the other side of j. We now derive the minimal number of nodes necessary to construct such a network. By Lemma 12, sponsor i (respectively k) of the multi-recipient player must give sponsor k (i) access to at least $1+\sum_{l=1}^{d_1/2-1} 2^{l-1}$ nodes (where i (k) is included himself). This means that sponsor k (respectively i) has this same number of nodes at distance 2 or more in the component that includes i (k) and the links sponsored by i (k). Applying the balancing condition now, it follows that k (i) should have at least this same number of links at distance 2 or more in components to which multi-recipient player j gives access, but to which i (k) does not give access. It follows that $n = 4+3\sum_{l=1}^{d_1/2-1} 2^{l-1}$ is the minimal number of nodes necessary to construct a multi-recipient network with diameter d_1. To show that a candidate network with such a number of nodes indeed exists, consider a two-sponsor network (a network in which each sponsor sponsors exactly 2 links) where exactly three players sponsor a link to the multi-recipient player. There are exactly $3 \times 2^{l-1}$ players at distance l from the multi-recipient player, with $2 \leq l \leq d_1/2-1$, whose links point away from the multi-recipient player. Such a network indeed meets Part 3 of Proposition 1 in the main paper, and meets the balancing condition. An example is the network in Fig. 12 for $n = 25$ and $d_1 = 8$.

Second, we show that with the given $n = 4+3\sum_{l=1}^{d_1/2-1} 2^{l-1}$, there is no non-recipient candidate network which has a higher diameter than d_1. In order to show this, we show that, for even d_1, the minimal number of nodes needed to achieve a diameter-d_1 network with a non-recipient player is always at least as high as the minimal number of nodes needed to achieve a diameter-d_1 network with a multi-recipient player. Consider a non-recipient network with diameter d_1 with two end recipients at distance d_1 from one another, and each at distance $d_1/2$ from the non-recipient player. In such a network, the non-recipient player sponsors at least two links. By Lemma 12, the non-recipient player has at least $2\sum_{l=1}^{d_1/2-1} 2^l$

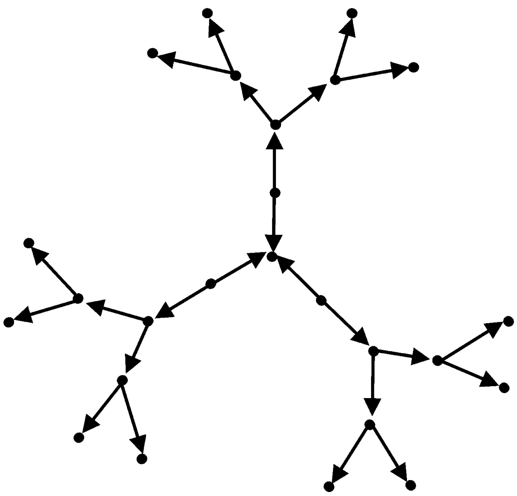

Fig. 12. Maximal-diameter network for $n = 25$.

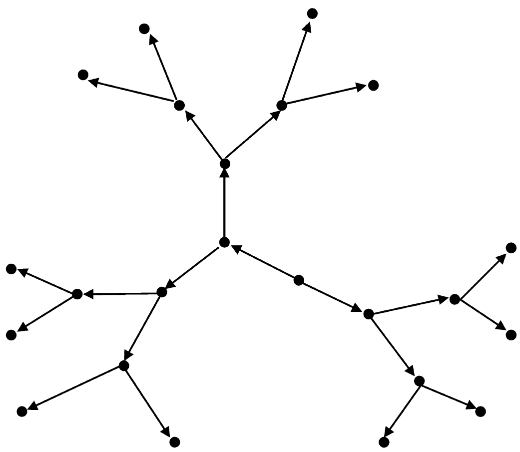

Fig. 13. Maximal diameter network for $n = 23$.

nodes at distance 2 or more from him. Together with the minimum of two players at distance 1, this means a minimum of $2+2\sum_{l=1}^{d_1/2-1}2^l = 2\sum_{l=1}^{d_1/2}2^{l-1}$ nodes at distance 1 or more. Thus, to construct such a network, we need a minimum of $n = 1+2\sum_{l=1}^{d_1/2}2^{l-1}$ nodes. To show that a network using this number of nodes indeed exists, consider a symmetric network where each sponsor sponsors exactly 2 links. This is a candidate network. In this network there are exactly 2 times 2^{l-1} nodes at distance l from the non-recipient player, with $1 \le l \le d_1/2$. By Lemma 12, decreasing the distance between the non-recipient player and an end recipient at one side of the non-recipient player in order to increase the distance between the non-recipient player and an end recipient on the other side of the non-recipient player is only possible with the use of more nodes. It follows that $n = 1+2\sum_{l=1}^{d_1/2}2^{l-1}$ is the minimal number of nodes with which we can construct a non-recipient candidate network with diameter d_1.

Finally, note that for $d_1 \ge 4$, $1+2\sum_{l=1}^{d_1/2}2^{l-1} \ge 4+3\sum_{l=1}^{d_1/2-1}2^{l-1}$ (where equality is obtained only for $d_1 = 4$). It follows that to construct a network with even diameter $d_1 \ge 4$, one needs at least $4+3\sum_{l=1}^{d_1/2-1}2^{l-1}$ nodes.

Odd diameter and minimal number of nodes. Next, we show that in order to construct a candidate that has odd diameter d_2 with $d_2 \ge 3$, one needs at least $n = 1+2^{(d_2-1)/2}+2\sum_{l=1}^{(d_2-1)/2}2^{l-1}$ nodes. Consider first a non-recipient candidate network with diameter d_2 and with two end recipients at distance d_2 from each other. By the arguments above, one of these two end recipients is at distance $(d_2-1)/2$ of the non-recipient player, while the other end recipient at distance $(d_2+1)/2$. In such a network, the non-recipient player has at least two players at distance 1. By Lemma 12, the non-recipient player has at least $\sum_{l=1}^{(d_2-1)/2-1}2^l + \sum_{l=1}^{(d_2+1)/2-1}2^l$ at distance 2 or more from him. Together with the minimum of two nodes at distance 1, this means a minimum of $\sum_{l=1}^{(d_2-1)/2}2^{l-1} + \sum_{l=1}^{(d_2+1)/2}2^{l-1}$ nodes at distance 1 or more. Thus, in order to construct this network, we need at least $n = 1+2^{(d_2-1)/2}+2\sum_{l=1}^{(d_2-1)/2}2^{l-1}$ nodes. To show that a network using this number of nodes indeed exists, consider a network where each sponsor sponsors exactly two links. In each of these components, there are exactly 2 times 2^{l-1} nodes at distance l from the non-recipient player, with $1 \le l \le (d_2-1)/2$. In one of the two components, there are additionally exactly $2^{(d_2-1)/2}$ players at distance $(d_2-1)/2$ from the non-recipient player. An example is the network in Fig. 13 for $n=23$ and $d_2=7$. Note that such a network is a candidate network.

Second, we show that in order to achieve a network with odd diameter d_2 in a candidate network with a multi-recipient player, more nodes are needed than in the specified network with a non-recipient player. By the analysis above, in order to achieve a network with even diameter (d_2-1), the minimal number of nodes is achieved in e.g. the two-sponsor network with a multi-recipient player, an example of which is given in Fig. 12. Note now that applying the balancing condition to the sponsors of the multi-recipient player, in order to increase the diameter of such a network to d_2, one needs to add at least the same number of nodes as are needed to increase the diameter to (d_2+1). As $4+3\sum_{l=1}^{(d_2+1)/2-1}2^{l-1}-1-2^{(d_2-1)/2}-2\sum_{l=1}^{(d_2-1)/2}2^{l-1}=2$, this means that more nodes are needed than to achieve diameter d_2 for the non-recipient candidate network.

Other numbers of nodes. Note finally that for n such that $1+2^{(x-1)/2}+2\sum_{l=1}^{(x-1)/2}2^{l-1}\leq n<4+3\sum_{l=1}^{(x-1)/2}2^{l-1}$, one can achieve diameter $d=x$ e.g. by letting the non-recipient player sponsor additional links. For n such that $4+3\sum_{l=1}^{(x-1)/2}2^{l-1}\leq n<1+2^{(x+1)/2}+2\sum_{l=1}^{(x+1)/2}2^{l-1}$, one can achieve diameter $d=(x+1)$ e.g. by letting each additional node sponsor one link, namely to the multi-recipient player. □

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