Operads

Hopf algebras and coloured Koszul duality

PH.D. THESIS OF
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AT UTRECHT UNIVERSITY

JANUARY 12TH, 2004

TER NAGEDACHTENIS AAN

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(Wiskundige, 1912 – 1968)

Ignoro si creí alguna vez en la Ciudad de los Inmortales:
pienso que me bastó la tarea de buscarla.

Jorge Luis Borges

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Introduction

This chapter intends to give a first intuitive definition of operads, and to give an overview of the results in this thesis. It does not pursue mathematical precision of the statements. The details can be found elsewhere in this thesis.

Operads

Intuitive description

An operad (in the category of topological spaces) consists of a sequence of spaces $P(n)$ for $n \geq 0$ together with a right S_n-action on $P(n)$, an identity element id $\in P(1)$, and continuous composition maps

$$\circ_i : P(n) \times P(m) \longrightarrow P(m+n-1) \qquad (i = 1, \ldots, n)$$

for all n and m. Of course these structures need to satisfy some axioms (eg. May [69], Markl-Shnider-Stasheff [67]). To understand the nature of the axioms, think of $P(n)$ as a space of n-ary operations (i.e. operations with n inputs and one output). The right S_n-action on $P(n)$ corresponds to the permutation of the inputs of n-ary operations. The composition $\circ_i$ constructs the $(m+n-1)$-ary operation from an n-ary and an m-ary operation by using the output of the m-ary operation as the i-th input of the n-ary operation. This is visualised in Figure 1. The identity element in $P(1)$ corresponds to the identity as 1-ary operation. The axioms of an operad are the formal relations that one intuitively obtains from this interpretation. Operads form a category with respect to continuous S_n-equivariant maps that are compatible with the structure maps in the obvious sense.

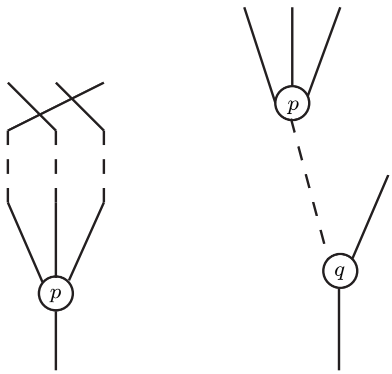


FIGURE 1: A permutation of the inputs applied to $p \in P(3)$ is visualised on the left. Composition $\circ_1$ applied to $q \in P(2)$ and $p \in P(3)$ is visualised at the right. The output of p is used as first input of q.

Examples

Some well known operads are significant both from a historical perspective and in relation to this thesis.

EXAMPLE Let X be a topological space. Define $\mathrm{End}_X(n) = X^{(X^n)}$, the topological space of continuous maps from X^n to X (with the compact-open topology). To make End_X an operad we endow it with the following structure. The S_n-action on $\mathrm{End}_X(n)$ is given by permutation of the inputs, and the composition given by composition of maps. This defines the endomorphism operad End_X of X. For an arbitrary operad P, the operad maps $P \longrightarrow \mathrm{End}_X$ are called P-algebra structures on X. Thus a P-algebra is an interpretation of each $p \in P(n)$ as a continuous map $p : X^n \longrightarrow X$.

EXAMPLE Define the operad S by $S(n) = S_n$, the permutation group on n letters with the action by right multiplication and the discrete topology for $n \geq 1$. For $\sigma \in S_n$ and $\tau \in S_m$ composition $\sigma \circ_i \tau$ is $\hat{\sigma} \circ ((1)^{i-1} \times \tau \times (1)^{n-i})$, where $\hat{\sigma} \in S_{m+n-1}$ is the permutation σ applied to n blocks of letters such that each block has length 1 except for the i-th block which has length m, and $(1) \in S_1$ (indicated in Figure 2). The category of algebras for S is the category of H-spaces. That is, topological spaces X together with an associative continuous multiplication $\mu : X \times X \longrightarrow X$. Throughout a linear version Ass of this operad will play an important role.

EXAMPLE Define the operad D_m of little m-disks as follows. The space $D_m(n)$ is the

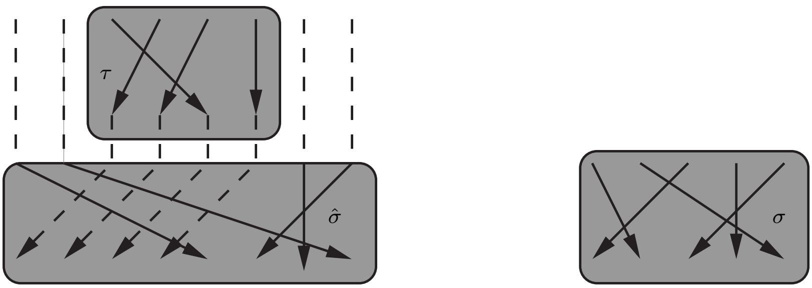


FIGURE 2: Visualisation of $\sigma \circ_3 \tau$. The passage from σ to $\hat{\sigma}$ amounts to replacing the third arrow in σ by parallel arrows, one for each arrow in τ.

space of ordered n-tuples of inclusions of the unit disk $D^m = \{x \in \mathbb{R}^m \mid |x| \leq 1\}$ into D^m that preserve the directions of the coordinate axes and have disjoint images. Permutation of the inclusions give the symmetric group action, and composition of the inclusions defines the operadic composition, as described in Figure 3. The operad D_2 was first used in the study of double loop spaces by May [69]. The starting point of this work is the observation that every double loop space $Y = \Omega^2 X$ has the structure of a D_2-algebra since $y \in Y$ can be interpreted as a map $y : D^2 \longrightarrow X$ that is constant and equal to the base point on the boundary $S^1 \subset D^2$.

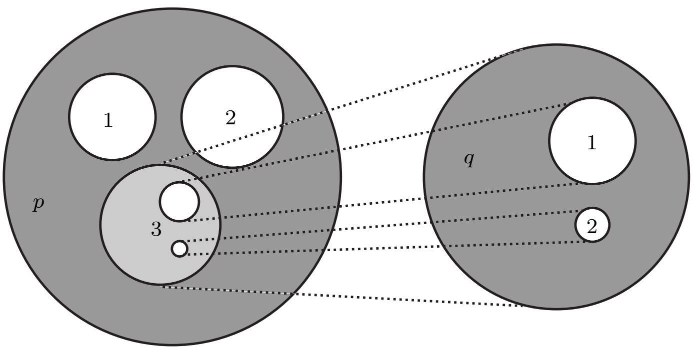


FIGURE 3: Composition $p \circ_3 q$ in the little disks operad D_2 is given by shrinking the disk q and identifying it with the interior of the third little disk in p.

Applications

Algebraic topology

The first applications of operads were in algebraic topology. For example, operads were used to classify the homotopy types of H-spaces (Stasheff [78]) and infinite loop spaces (Boardman-Vogt [3], May [69]). In most applications of this type one starts from spaces with a certain algebraic structure, and one replaces the algebraic theory by a weaker one which is equivalent to the initial algebraic theory modulo homotopy equivalences.
The key development in this field was the definition of homotopy invariant algebraic theories in terms of operads. The conditions necessary to assure that the weakened algebraic theory leads to an equivalent structure (modulo homotopy equivalences) are easiest to understand from the operadic point of view. This approach amounts to doing homotopy theory for operads themselves, and studying the interplay of homotopy theory on the level of operads and homotopy theory at the level of spaces with algebraic structure.

Homological algebra

The operad axioms can directly be translated to any symmetric monoidal category $(\mathcal{C}, \otimes, I)$. Above we described the situation for $(\mathcal{C}, \otimes, I) = (\mathsf{Top}, \times, *)$, the category of topological spaces with the direct product as monoidal structure. Other examples of symmetric monoidal categories are the category $(R\text{-}\mathsf{Mod}, \otimes, R)$ of modules over a commutative ring R and the usual tensor product $\otimes$, or the category $(k\text{-}\mathsf{dgVect}, \otimes, k)$ of differential graded vector spaces over a field k with the usual tensor product.

Applications of operads related to homotopy theory are not restricted to the topological category. Homotopy theory for dg vector spaces is the example most relevant to this thesis. In the category of dg vector spaces chain homotopies are a suitable replacements for homotopy equivalence. One can consider dg vector spaces with a certain algebraic structure and try to characterize these objects modulo homotopies. The relevance of this algebraic analogon of the topological case is illustrated by Quillen's work on rational homotopy theory [73] (cf. also Bousfield-Gugenheim [4]), and by Kontsevich' result on deformation-quantisation of Poisson manifolds [42] (cf. Tamarkin [80] for a more operadic proof in Euclidian space). Quillen shows rational homotopy theory of spaces can be reduced to algebra (the study of quasi isomorphism classes of dg coalgebras). Kontsevich shows that every Poisson structure on a manifold is induced by a deformation of its algebra of functions (applying the ideas sketched above to dg Lie algebras).

Recall that operads describe algebraic theories. Just as in the topological case we can replace a given operad by another one describing a weaker algebraic theory which has better invariance properties, but leads to the same characterisation modulo homotopies. This trick applied to dg Lie algebras is one of the essential ingredients of Kontsevich' work mentioned above.
One of the central tools to construct operads with nice invariance properties is Koszul duality as developed in Ginzburg-Kapranov [29]. The great advantage of Koszul duality is that it leads to particularly concise invariant theories. Examples of algebraic structures which fit in this framework of Koszul duality are A_∞-algebras (a weak version of associative algebras), L_∞-algebras (a weak version of Lie algebras), and G_∞-algebras (a weak version of Gerstenhaber algebras). In Chapters 3 and 4 the reader will encounter some applications of Koszul duality in the context of coloured operads.

Hopf algebras

Apart from these applications based on homotopy theory, there are more combinatorial applications of operads in algebra. Some of these exploit the fact that operads give a structured way to study algebraic theories. For example Loday et. al. [59], and Livernet [55]) which study non-antisymmetric versions of Lie algebras such as Leibniz (or Loday) algebras, dialgebras, etc. motivated by non-commutative geometry.
Milnor-Moore [70] establishes an equivalence between Lie algerbas and cocommutative Hopf algebras (over a field of characteristic 0). Non-antisymmetric versions of Lie algebras led to generalisations of this result to classes of non-commutative Hopf algebras (cf. Ronco [74], Chapoton [11]).
This work is connected to the study of combinatorial Hopf algebras of quasi symmetric functions through the Hopf algebras of planar binary trees (Loday-Ronco [60]) and the Hopf algebra of permutations (Malvenuto-Reutenhauer [63]).

A different relation between operads and Hopf algebras is due to Moerdijk [71] and uses Hopf operads to construct the Connes-Kreimer Hopf algebra of trees (cf. Chapter 2) as just one of a large family of Hopf algebra structures. A third approach is developed in this thesis (an example is treated in joint work with Moerdijk [82]).

Other applications

To stress the lopsidedness of this introduction, a short list of applications of operads to different fields now follows. Huang [35] shows an equivalence between vertex algebras and algebras for the partial operad of conformal disks. Hinich-Vaintrob

[32] uses operads to prove a conjecture on the Kirilov-Duflo isomorphism for algebras of chord diagrams related to Vassiliev knot invariants. Kontsevich [43] proves the Formality Theorem and generalisations thereof by operadic means. Kimura-Stasheff-Voronov [40] shows the BRST complex in cohomological quantum field theory has a homotopy Gerstenhaber algebra structure. Kontsevich-Manin [44, 45] show that formal solutions of the WDVV equations are in 1-1 correspondence with algebras for the operad of moduli spaces of algebraic curves. It goes without saying that this short list contains just some of the many recent applications of operads.

The remainder of this Chapter gives an overview of the main results in this thesis.

Part I: Operads and Hopf algebras

The first Part of this thesis is dedicated to applications of operads to Hopf algebras. As indicated above, there are several relations between operads and Hopf algebras. We consider two of these. Chapter 1 provides a new tool in understanding the Hopf algebras in renormalisation. Chapter 2 gives a detailed study of some aspects of a construction of Moerdijk [71].

Operads and renormalisation

Kreimer [48] shows Hopf algebras play an important role in renormalistion of quantum field theories. More precisely, the counterterms in the renormalisation process can be constructed using the antipode of the Hopf algebra. This was mathematically further developed and applied to other theories by Connes-Kreimer [15, 16, 17], Brouder-Frabetti [7], Krajewski-Wulkenhaar [47], and others.

Chapter 1 gives a construction of a commutative graded connected Hopf algebra from a 1-reduced cooperad (cf. Theorem 1.2.5), and proves that many of the Hopf algebras that occur in renormalisation are of this type. The interesting point in the construction is that we can describe structure induced by the Hopf algebra explicitly in terms of the operad dual to the cooperad. In particular the Lie algebra of primitive elements of the dual and the group of characters admit such a description (Theorems 1.2.9 and 1.2.12). The situation can somewhat informally be summarised in the following Theorem.

THEOREM *Given a 1-reduced cooperad, the symmetric algebra on the total space of invariants enjoys a natural commutative, graded connected Hopf algebra structure. The Lie algebra of primitives of the dual and the group of characters allow an explicit description in terms of the dual operad.*

This translates to the examples from quantum field theory as follows. Introducing the operad makes checking the Hopf algebra structure easier since the operad is a much smaller object, and the renormalisation group and its Lie algebra can be written directly in terms of the underlying operad and do not need separate investigation. Most of the examples in Chapter 1 are related to well known operads (cf. Section 1.3). The most important exception is the Connes-Kreimer Hopf algebra of graphs [16] which led to a new operad of graphs (cf. Section 1.4). Moreover, the Wick rotation formula suggests how to make the 'cofree' symmetric coalgebra on a vector space with a symmetric bilinear form an algebra for this operad.

Hopf operads and trees

Partly motivated by the developments in renormalisation sketched above, Hopf algebras generated by combinatorial objects attracted a lot of attention over the previous decennium (eg. Loday-Ronco [60] describes a Hopf algebra of planar binary trees, Malvenuto-Reutenauer [63] a Hopf algebra of permutation, and Connes-Kreimer [15] a Hopf algebra of rooted trees). In most cases these Hopf algebras occur one at a time. A construction by Moerdijk [71] gives whole families of Hopf algebras of rooted trees and of planar rooted trees. The construction is based on operad extensions of Hopf operads P, such that the initial algebra of the extended operad still possesses a family of Hopf P-algebra structures.

Chapter 2 considers Moerdijk's construction for a more general class of extensions (cf. Theorem 2.2.7), and studies special cases in more detail. The Chapter obtains Hopf algebras generated by different combinatorial objects: rooted trees with coloured edges. For these Hopf algebras we obtain a closed formula for the coproduct. The Lie algebra of primitive elements of the dual Hopf algebra is examined.

THEOREM *For $n \geq 1$ there is a family of Hopf algebra structures on the free symmetric algebra generated by rooted trees with n-coloured edges, parametrised by sequences $(q_{11}, \ldots, q_{1n}, q_{21} \ldots q_{2n}) \in k^{2n}$. The coproduct is given on a tree t by the formula*

$$\Delta(t) = \sum_{s \subset t} \prod_j q_{1j}^{\sum_{v \in s} p_j(v,s,t)} \cdot \prod_j q_{2j}^{\sum_{v \in s^c} p_j(v,s^c,t)} \cdot s \otimes s^c,$$

where the sum is over subforests $s \subset t$, and where for v a vertex in the subforest s we denote *by $p_k(v, s, t)$ the number of edges of colour k in the path in t from v to the root of t that have their lower vertex in s^c. For forests t we define $p_k(v, s, t)$ as $p_k(v, s \cap t', t')$, where t' is the connected component of t containing v.*

In addition, the Chapter studies versions for planar trees (Corollary 2.3.6), and planar n-ary trees (Corollary 2.3.8). These modifications amount to a change of the Hopf operad P in the construction. Moreover, it is shown how some well known Hopf algebras of trees can be constructed using this approach (cf. Section 2.4).

Part II: Coloured Koszul duality

The second Part of this thesis is based on the application of operads in homological algebra, and in particular Koszul duality. The operads to which we apply this theory are the operad describing algebroids (i.e. Lie-Rinehart algebras) and the operad describing non-symmetric pseudo operads. There is a complication, these are coloured operads, not operads. To solve this problem, Koszul duality is generalised to coloured operads (Corollary 3.3.8).

Algebroids and Koszul duality

Lie algebroids (eg MacKenzie [62], Cannas da Silva-Weinstein [9]) are geometric structures generalising at the same time Lie algebras, tangent bundles, and Poisson manifolds. Lie algebroids are relevant to physics (Cattaneo-Felder [10], Landsman [52]) and provide the infinitessimal structure of smooth groupoids (MacKenzie [62], Crainic-Fernandez [19]). The algebraic conterpart of this structure is an algebroid or Lie-Rinehart algebra (eg. Huebschmann [36]).

Chapter 3 develops Koszul duality for coloured operads and applies it to algebroids. It shows that the 2-coloured operad of algebroids is a self-dual Koszul coloured operad (Theorem 3.5.3). The explicit form of the homology complex of an algebroid (R, A) can be given in terms of a dg Lie algebra structure on the bar complex $B_*(R, A)$ of the commutative algebra R with coefficients in the Lie algebra A (Theorem 3.5.2). Here I only state part of the result.

THEOREM *Given a commutative algebra R and an R-module A, a Lie algebra structure on*

A and an A-module structure on R define an algebroid (Lie-Rinehart algebra) iff the bracket

$$\begin{aligned}[(f_1,\dots,f_n,X),&(g_1,\dots,g_m,Y)] = \\ &(\mathrm{Sh}(f_1,\dots,f_n;g_1,\dots,g_m),[X,Y]) \\ &\pm\sum_i(\mathrm{Sh}(f_1,\dots,f_n;g_1,\dots,g_{i-1}),X(g_i),\dots,g_m),Y) \\ &\mp\sum_i(\mathrm{Sh}(f_1,\dots,f_{i-1};g_1,\dots,g_m),Y(f_i),\dots,f_n,X)\end{aligned}$$

on the bar complex $B_*(R,A)$ of R with coefficients in A defines a dg Lie algebra structure, where Sh *is the shuffle product. The homology complex of the algebroid contains the Chevalley-Eilenberg complex of this dg Lie algebra as a direct summand.*

As an intermediate step we study Koszul duality for operad algebras together with a module (Theorems 3.4.2 and 3.4.6). For well known operads this shows how complexes from classical homological algebra occur in this framework (Theorem 3.4.8). This extends results of Ginzburg-Kapranov [29]. The relation between Koszul duality for algebroids and Poisson algebras is explained in section 3.6.

The results in Chapter 3 induce a definition of a strongly homotopy algebroid, and a deformation theory for algebroids (cf. Chapter 4). However, these notions are a bit too naive since they do not consider the geometric origin of the objects concerned: in all geometric examples the deformation theory is trivial. For this reason the subject is not treated here. Restricting to a suitable class of deformations is expected to provide the correct notion of deformation.

Strongly homotopy operads

Sometimes the axioms for operads and their morphisms are too strict. I give three examples.

(i). The little disks D_2 form an operad, but the homotopy equivalent spaces of configurations of ordered points in the unit disk do not unless we use the Fulton-McPherson compactification (eg. Voronov [83]).

(ii). The application $V \longmapsto \mathrm{End}_V$ from vector spaces to operads is not functorial. In general it is hard to construct maps between endomorphism operads. This is a pitty, since these maps could be used to study homotopy invariance results for algebras.

(iii). There need not exist a quasi isomorphism of operads between quasi isomophic operads.

Chapter 4 develops a weaker notion, strongly homotopy operads, and shows to what extent this solves the points mentioned above. The first and third point can be solved completely.

THEOREM *The singular $\mathbb{Q}$-chains on the configuration spaces of distinct points in the unit disk in $\mathbb{R}^2$ form a strongly homotopy operad, homotopy equivalent to the $\mathbb{Q}$-chains on the little disks operad D_2.*

This solves the first point. Regarding the third point, two operads are quasi isomorphic iff there exists a quasi isomorphism of strongly homotopy operads between them (Theorem 4.2.11). The second point is a bit more subtle and is addressed in Theorem 4.3.3 and leads to homotopy invariance results for strongly homotopy P-algebras. The crucial ingredient for these results is that the $\mathbb{N}$-coloured operad for non-Σ pseudo operads is a self dual Koszul operad, and that the homology complex of a non-Σ pseudo operad is its bar complex (Theorem 4.2.3).

The results above are closely related to the deformation theory for operads and their algebras. The cotangent complex [64] (related to deformations of operads themselves) can be constructed from a strongly homotopy operad. This exhibits a natural L_∞-algebra structure on the cotangent complex (Theorems 4.4.4, and 4.5.12). Moreover, one recovers a L_∞-algebra governing algebra deformations described in Kontsevich-Soibelman [46] (Theorem 4.5.16). This allows us to compare the two L_∞-algebras. The result can be summarised as follows.

THEOREM *There exists a functorial construction of an L_∞-algebra from a strongly homotopy operad. As an application, one finds a natural L_∞-algebra structure on Markl's cotangent complex of an operad [64]. This complex is embedded as an L_∞-algebra in the L_∞-algebra governing the deformation of operad algebras as described in Kontsevich-Soibelman [46].*

CHAPTER **0**

Preliminaries

This chapter discusses some classical algebraic structures, and introduces operads. The definition of collections and operads is not completely standard, though it is obviously equivalent to the usual one.

0.1 Assumptions, conventions, and notations

The reader is assumed to be acquainted with the language of Category Theory. I use categorical language since it leads to concise and clear statements. The most common reference to this is Mac Lane [54]. The reader is also assumed to have some background in Homological Algebra, as offered by eg. Weibel [84], or Hilton-Stammbach [30].

Throughout this thesis, the natural numbers $\mathbb{N}$ are meant to include 0. The symbol k (if not used as an index) will denote a field of characteristic 0. A vector space will be a vector space over k. The category of vector spaces and linear maps will be denoted k-Vect. The category Ch(k-Vect) of (unbounded) complexes of vector spaces (or differential graded (dg) vector spaces) will be denoted k-dgVect. If V is a dg vector space, and $v \in V$ is an homogeneous element, then its degree will be denoted by $|v|$. I use the cohomological convention: the differential d of the dg vector space V is a map of degree $+1$. Let $V^n = \{v \in V || v| = n\}$ be the space of homogeneous elements of degree n. Then $V[m]$ is the dg vector space with $(V[m])^n = V^{n-m}$. Later on I might be a bit sloppy and leave out the 'dg' where it is clear from the context that we work with differentially graded objects. Let V and W be (dg) vector spaces. Denote by $\tau : V \otimes W \longrightarrow W \otimes V$ the symmetry (or twist) of the tensor product. In the dg case this involves the natural signs $\tau : v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v$ on homogeneous elements. I use the Koszul convention $f \otimes g(x \otimes y) := (-1)^{|g||x|} f(x) \otimes g(y)$ for homogeneous maps $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ applied to homogeneous elements $x \in V$ and $y \in W$. In particular in combination with the shift $[m]$ this allows us to reduce the number of sign we have to write. If G is a finite group we will denote its group algebra kG. The product is linearly extended from the product on G. With the coproduct that is the linear extension of the diagonal $G \longrightarrow G \times G$ and the antipode given by the linear extension of the inverse, kG is a Hopf algebra. The example that occurs most in this thesis is the case $G = S_n$, the permutation group on n letters.

0.2 Classical concepts

The concepts introduced below are standard. The aim is to fix notation. Details can be found in text books (eg. Weibel [84], Loday [56], and Sweedler [79]).

§1 Algebras, coalgebras, Hopf algebras

0.2.1 DEFINITION An *(associative) algebra* is a vector space $A \in k\text{-}\mathsf{Vect}$ together with an associative multiplication $\mu : A \otimes A \longrightarrow A$. A *differentially graded (dg) algebra* is a dg vector space $A \in k\text{-}\mathsf{dgVect}$ together with an associative dg multiplication $\mu : A \otimes A \longrightarrow A$. That is, the (dg) linear map satisfies

$$\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu) : A^{\otimes 3} \longrightarrow A.$$

An algebra is *unital* if there exists a map $\eta : k \longrightarrow A$ such that

$$\mu \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = \mu \circ (\eta \otimes \mathrm{id}) : A \longrightarrow A.$$

If unital algebras are used this will be stated explicitly, the convention is to use the non-unital species. If A is an algebra we denote by A^{op} the algebra A with the opposite multiplication $\mu^{\mathrm{op}} = \mu \circ \tau$ where τ is the symmetry of the tensor product, and denote by A^+ the unital associative algebra $A \oplus k$, where multiplication on A^+ is determined by multiplication on A and the unit $u : k \longrightarrow A \oplus k$ that is the inclusion of the second summand.
An (dg) algebra is called *(graded) commutative* if the multiplication is invariant under the symmetry in the sense that $\mu = \mu \circ \tau$. The forgetful functor from algebras to (dg) vector spaces has a left adjoint $\bar{T}$. If V is a vector space, $\bar{T}V = \bigoplus_{n \geq 1} V^{\otimes n}$ where μ is

defined by concatenation of tensor factors. That is, $\mu(v_1 \otimes \ldots \otimes v_k, v_{k+1} \otimes \ldots \otimes v_n) = v_1 \otimes \ldots \otimes v_n$. The algebra $\bar{T}V$ is called the *free algebra* on V. Similarly, the forgetful functor from (dg) commutative algebras to (dg) vector spaces has a left adjoint $\bar{S}$. If V is a (dg) vector space, $\bar{S}V = \bigoplus_{n\geq 1}(V^{\otimes n})_{S_n}$, where the subscript S_n indicates that we that coinvariants with respect to the action of S_n which permutes the tensor factors. The commutative algebra is the quotient of $\bar{T}V$ by the commutator ideal. The commutative algebra $\bar{S}V$ is called the *free commutative algebra* on V. There exist unital versions of both constructions. The *free unital associative algebra* is given by $TV = \bigoplus_{n\geq 0} V^{\otimes n}$, the *free unital commutative algebra* by $SV = \bigoplus_{n\geq 0}(V^{\otimes n})_{S_n}$. Similarly, $\bar{S}$ (resp. S) is left adjoint to the forgetful functor from (unital) commutative algebras to vector spaces.

Let A be an algebra. A *left A-module* is a vector space M together with an algebra map $\lambda : A \longrightarrow \mathrm{Hom}_k(M, M)$, where $\mathrm{Hom}_k(M, M) = k\text{-}\mathsf{Vect}(M, M)$ (the vector space of k-linear maps from M to M) is an algebra with composition as product. A *right A-module* is a vector space M together with an algebra map $\rho : A^{\mathrm{op}} \longrightarrow \mathrm{Hom}_k(M, M)$. An *$A$-bimodule* is a vector space M with a left and a right module structure such that $\lambda(a)\rho(b) = \rho(b)\lambda(a)$ for $a, b \in A$. One usually writes maps λ and ρ as

$$\lambda : A \otimes M \longrightarrow M, \quad \text{and} \quad \rho : M \otimes A \longrightarrow M.$$

0.2.2 DEFINITION A *(dg) coalgebra* can be defined as an (dg) algebra C in the opposite category $(k\text{-}\mathsf{dgVect})^{\mathrm{op}}$. Explicitly, it is a (dg) vector space together with a cocommutative comultiplication $\Delta : C \longrightarrow C \otimes C$. That is,

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta : C \longrightarrow C^{\otimes 3}.$$

For the coproduct we use *Sweedler's notation* $\Delta(x) = \sum_{(x)} x' \otimes x''$, reminiscent of the form of the coproduct when written in terms of a basis. Coassociativity states $\sum\sum(x')' \otimes (x')'' \otimes x'' = \sum\sum x' \otimes (x'')' \otimes (x'')''$, which allows us to write unambiguously $\sum x' \otimes x'' \otimes x'''$ for this expression. A coalgebra is *counital* if there exists a map $\varepsilon : C \longrightarrow k$ such that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta : C \longrightarrow C.$$

A coalgebra is *cocommutative* if Δ is invariant under the symmetry as $\Delta = \tau \circ \Delta$. Let V be a vector space. Inspired by the algebra case, one defines a functor $\bar{T}' : k\text{-}\mathsf{dgVect} \longrightarrow \mathsf{Coalg}$ by $\bar{T}'V = \bigoplus_{n\geq 1} V^{\otimes n}$. The comultiplication on $\bar{T}'V$ is $\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n-1}(v_1 \otimes \ldots \otimes v_k) \otimes (v_{k+1} \otimes \ldots \otimes v_n)$, where each of the bracketed factors is interpreted as an element of $\bar{T}'V$. This coalgebra is called the *'cofree' coalgebra* on V. The quotes around 'cofree' are intended to remind the reader that $\bar{T}'$ is not right adjoint to the forgetful functor from coalgebras to vector spaces. The functor $\bar{T}'$ is however cofree with respect to a smaller class of coalgebras. The *counital 'cofree' coalgebra* is given by $T'V = \bigoplus_{n\geq 0} V^{\otimes n}$, with the coproduct $\Delta(v_1 \otimes \ldots \otimes v_n) = \sum_{i=0}^{n}(v_1 \otimes \ldots \otimes v_k) \otimes (v_{k+1} \otimes \ldots \otimes v_n)$. Similarly, we define the *'cofree' cocommutative coalgebra* on V as $\bar{S}'V = \bigoplus_{n\geq 1}(V^{\otimes n})^{S_n}$, where the superscript indicates that we that invariants with respect to the action of kS_n. The *counital 'cofree' cocommutative coalgebra* is given by $S'V = \bigoplus_{n\geq 0}(V^{\otimes n})^{S_n} \subset T'V$. Note that $\bar{S}'V \subset \bar{T}'V$, whereas $\bar{S}V$ is a quotient of $\bar{T}V$. Nevertheless, $\bar{S}V$ and $\bar{S}'V$ are isomorphic as vector spaces (the characteristic of k is 0 throughout this thesis).

Let C be a coalgebra. A *left C-comodule* N is a vector space together with a linear map $\lambda : N \longrightarrow C \otimes N$ that satisfies $(\mathrm{id} \otimes \lambda) \circ \lambda = (\Delta \otimes \mathrm{id}) \circ \lambda$. A *right C-comodule* N is a vector space together with a linear map $\rho : N \longrightarrow N \otimes C$ that satisfies $(\rho \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \Delta) \circ \lambda$. A *C-bicomodule* is a vector space N together with a left and a right comodule structure that satisfy $(\lambda \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \lambda$.

0.2.3 DEFINITION If A is an unital coalgebra, then $A \otimes A$ is a counital coalgebra with respect to $\Delta(x, y) = \sum_{(x,y)}(x', y') \otimes (x'', y'')$ and $\varepsilon(x, y) = \varepsilon(x)\varepsilon(y)$. Similarly, if A is a unital algebra, then $A \otimes A$ is an algebra with respect to the multiplication $\mu((a, b), (c, d)) = (\mu(a, c), \mu(b, d))$ for $a, b, c, d \in A$ and the unit $(1, 1)$. A *(dg) bialgebra* is a unital (dg) algebra in the category of counital (dg) coalgebras. Explicitly a bialgebra is a vector space A together with an associative multiplication μ and a comultiplication Δ such that μ is a coalgebra map with respect to the coalgebra structure on $A \otimes A$, and similarly η is a coalgebra map from k (with its trivial coalgebra structure) to A. If A is a unital bialgebra, then the *primitive elements* $\mathrm{Prim}(A)$ are the elements $a \in A$ such that $\Delta(a) = 1 \otimes a + a \otimes 1$.

If A is a bialgebra, then the space of endomorphisms $k\text{-}\mathsf{Vect}(A, A)$ is an associative algebra with respect to the *convolution product* $\varphi * \psi = \mu \circ (\varphi \otimes \psi) \circ \Delta$. A unital bialgebra A is a *Hopf algebra* if $\mathrm{id} \in k\text{-}\mathsf{Vect}(A, A)$ is invertible with respect to the convolution product. The inverse S to $\mathrm{id} \in k\text{-}\mathsf{Vect}(A, A)$ is called the *antipode*.

Let A be a Hopf algebra. The group $\mathrm{Hom}_{\mathrm{Alg}}(A, k)$ with its convolution product and the inverse by pre-composition with the antipode is the *group of characters* of A.

§2 Lie algebras and pre-Lie algebras

Lie algebras are well known. More details can be found e.g. in Weibel [84]. Pre-Lie algebras are a more strict concept introduced by Gerstenhaber [24]. Pre-Lie algebras have the virtue that the axioms are sometimes more easy to check.

0.2.4 DEFINITION A *(dg) Lie algebra* is a (dg) vector space $\mathfrak{g}$ together with a (graded) skew symmetric bilinear bracket $[-, -]$, such that for (homogeneous) $X \in \mathfrak{g}$ the map $[X, -]$ is a derivation of the bracket (of degree $|X|$). The equality that expresses that $[X, -]$ is a derivation is called the *Jacobi identity*. On homogeneous elements $X, Y, Z \in \mathfrak{g}$ the Jacobi identity takes the form

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|}[Y, [X, Z]].$$

Let us give some examples. Left invariant vector fields on a Lie group have a natural Lie algebra structure. The primitive elements of a unital bialgebra form a Lie algebra with respect to the commutator bracket.
Two dg Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are quasi isomorphic iff there exists a chain of Lie algebra maps $\mathfrak{g} \longleftarrow \ldots \longrightarrow \mathfrak{h}$, such that each map is an isomorphism in cohomology. We will in later sections study solutions $X \in \mathfrak{g}^1$ of the *Maurer-Cartan equation* $dX + \frac{1}{2}[X, X] = 0$ in a dg Lie algebra $\mathfrak{g}$. A $\mathfrak{g}$-*module* is a vector space M together with a map $\lambda : \mathfrak{g} \longrightarrow \mathrm{Hom}_k(M, M)$ of Lie algebras, where the bracket on $\mathrm{Hom}_k(M, M)$ is the commutator of the composition product.

0.2.5 DEFINITION (GERSTENHABER [24]) A *(dg) pre-Lie algebra* is a (dg) vector space P together with a non-associative product $\bullet$ that satisfies the identity

$$\bullet \circ ((\bullet \otimes \mathrm{id}) - (\mathrm{id} \otimes \bullet)) = \bullet \circ ((\bullet \otimes \mathrm{id}) - (\mathrm{id} \otimes \bullet)) \circ (\mathrm{id} \otimes \tau) : P^{\otimes 3} \longrightarrow P,$$

where τ is the symmetry of the tensor product applied to the last 2 tensor factors. In terms of homogeneous elements $X, Y, Z \in P$ this reads

$$(X \bullet Y) \bullet Z - X \bullet (Y \bullet Z) = (-1)^{|Y||Z|}((X \bullet Z) \bullet Y - X \bullet (Z \bullet Y)).$$

The commutator $[-, -] = \bullet - (\bullet \circ \tau)$ of the product $\bullet$ defines the *associated Lie algebra* structure of P on the vector space P.

§3 Hochschild complex

The Hochschild complex was introduced by Hochschild [33]. Apart from its application to deformation theory the Hochschild complex is studied in relation to cyclic cohomology (cf. Loday [56]).

0.2.6 DEFINITION Let A be an associative algebra and let M be an A-bimodule. The *Hochschild complex* $C^*(A, M)$ of A (with coefficients in a bimodule M is defined as the complex associated to the cosimplicial object (cf. May [68], or Weibel [84] for the definition) which has $C^n(A, M) = k\text{-}\mathsf{Vect}(A^{\otimes n}, M)$ for $n \in \mathbb{N}$, and has for $\varphi \in C^n(A, M)$ coboundaries and cofaces

$$d^i\varphi(a_0, \ldots, a_n) = \begin{cases} a_0\varphi(a_1, \ldots, a_{n+1}) & \text{for } i = 0 \\ \varphi(a_0, \ldots, a_i a_{i+1}, \ldots, a_n) & \text{for } i = 1, \ldots, n-1 \\ \varphi(a_0, \ldots, a_{n-1})a_n & \text{for } i = n \end{cases}$$

$$s^i\varphi(a_1, \ldots, a_{n-1}) = \varphi(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1}) \qquad \text{for } i = 0, \ldots, n$$

The usual differential $d = \sum_i (-1)^i d_i$ is called the *Hochschild differential*. The cohomology of the Hochschild complex is called the *Hochschild cohomology* (with coefficients in M).
Consider A as an A-bimodule with respect to the map $\lambda = \mu = \rho$. There is a well known algebraic structure on the Hochschild complex $C^*(A, A)$ (cf. Gerstenhaber [24]). Define

$$\psi \circ_i \varphi = (-1)^{i(n-1)} \psi \circ (\mathrm{id}^{\otimes i} \otimes \varphi \otimes \mathrm{id}^{\otimes n-i-1})$$

for $\varphi \in C^m(A, A)$ and $\psi \in C^n(A, A)$ and $i \leq n$. These operations define a dg pre-Lie algebra structure on $C^*(A, A)[-1]$ by $\psi \bullet \varphi = \sum_{i=0}^{n-1} \psi \circ_i \varphi$ (using the Hochschild differential). The associated Lie bracket on the complex $C^*(A, A)[1]$ is called the *Gerstenhaber bracket*. The Gerstenhaber bracket and the Hochschild differential make $C^*(A, A)[1]$ a dg-Lie algebra. We denote the dg Lie subalgebra of all nonnegative degrees by $C^*_{\mathrm{Ass}}(A)$. Note that $C^n_{\mathrm{Ass}}(A) = k\text{-}\mathsf{dgVect}(A^{\otimes n+1}, A)$. To get the signs from the Koszul convention, one should write it rather as $C^n_{\mathrm{Ass}}(A) = k\text{-}\mathsf{dgVect}((A[-1])^{\otimes n}, A[-1])$. The cohomology of $C^*_{\mathrm{Ass}}(A)$ is denoted $H^*_{\mathrm{Ass}}(A)$.

0.3 Operads and cooperads

The Introduction describes operads, but does not give a rigorous definition. This section does give a complete definition, which differs a bit from the most common ones. Instead of numbering the inputs we take the coordinate free description. We take the inputs and the single output as labelled by a pointed set, such that the base-point corresponds to the output.

§4 Definitions

For a long time there were virtually no genuinely introductory texts on operads: the literature on the subject consisted exclusively of research papers. Recently, the appearance of some books (Markl-Shnider-Stasheff [67] and Smirnov [77]) changed this situation. Other standard references are Kriz-May [49], Loday [57], Getzler-Jones [26], and Ginzburg-Kapranov [29].
Let Fin_* be the category of pointed finite sets and bijections preserving the base-point. For $(X, x_0), (Y, y_0) \in \mathsf{Fin}_*$ and $x \in X - \{x_0\}$ define $Y \cup_x X$ to be the push-out

(in Set, the category of sets and functions)

$$\begin{array}{ccc} * & \xrightarrow{x} & X \\ {\scriptstyle y_0}\downarrow & & \downarrow \\ Y & \longrightarrow & Y \cup_x X, \end{array}$$

with the base-point x_0. Let $\tau \in \mathrm{Aut}(Y, y_0)$ and $\sigma \in \mathrm{Aut}(X, x_0)$. The map $\sigma \circ_x \tau : X \cup_x Y \longrightarrow X \cup_{\tau(x)} Y$ in Fin_* is the unique dotted arrow that makes the diagram below commute

$$\begin{array}{ccccc} Y & \longrightarrow & Y \cup_x X & \longleftarrow & X \\ \downarrow{\scriptstyle \sigma} & & \vdots{\scriptstyle \sigma \circ_x \tau} & & \downarrow{\scriptstyle \tau} \\ Y & \longrightarrow & Y \cup_{\tau(x)} X & \longleftarrow & X. \end{array} \tag{0.3.1}$$

In the remainder of this section I often drop the basepoint from the notation.

0.3.1 DEFINITION A *collection* is a contravariant functor $C : \mathsf{Fin}_* \longrightarrow k\text{-}\mathsf{dgVect}$. A *pseudo operad* is a collection together with a map

$$\circ_x : C(X, x_0) \otimes C(Y, y_0) \longrightarrow C(X \cup_x Y, x_0)$$

for each pair of pointed sets (X, x_0) and (Y, y_0) and each $x \in X$ such that $x \neq x_0$ that satisfies the following two axioms.

(i). With respect to the automorphisms of X and Y the map $\circ_x$ behaves as

$$p\tau \circ_x q\sigma = (p \circ_{\tau^{-1}x} q)(\tau \circ_{\tau^{-1}x} \sigma)$$

for $p \in C(X), q \in C(Y)$ and $x \in X - \{x_0\}, \tau \in \mathrm{Aut}(X, x_0), \sigma \in \mathrm{Aut}(Y, y_0)$.

(ii). The squares

$$\begin{array}{ccc} C(X) \otimes C(Y) \otimes C(Z) & \xrightarrow{\circ_x \otimes \mathrm{id}} & C(X \cup_x Y) \otimes C(Z) \\ \downarrow{\scriptstyle (\mathrm{id} \otimes \circ_{x'}) \circ (\mathrm{id} \otimes s)} & & \downarrow{\scriptstyle \circ_{x'}} \\ C(X \cup_{x'} Z) \otimes C(Y) & \xrightarrow{\circ_x} & C(X \cup_x Y \cup_{x'} Z) \end{array}$$

$$\begin{array}{ccc} C(X) \otimes C(Y) \otimes C(Z) & \xrightarrow{\mathrm{id} \otimes \circ_y} & C(X) \otimes C(Y \cup_y Z) \\ {\scriptstyle \mathrm{id} \otimes \circ_x}\downarrow & & \downarrow{\scriptstyle \circ_x} \\ C(X \cup_x Y) \otimes C(Z) & \xrightarrow{\circ_y} & C(X \cup_x Y \cup_y Z) \end{array}$$

commute for $y \in Y$ and $x \neq x' \in X$. We omit the base-points from the notation and the map s is the symmetry of the tensor product.

An *operad* is a pseudo operad together with a map $\mathrm{id} : k \longrightarrow C(\{x, x_0\})$ for a two element set $\{x, x_0\}$, such that id is a right/left identity with respect to any well defined right/left composition. (Compared to the intuitive description of operads in the Introduction, the elements of $C(X, x_0)$ correspond to operations with an input for each $x \in X - \{x_0\}$; x_0 to the output.)
Together with the natural notion of morphism (i.e. maps commuting with all algebraic structure) this defines the categories of collections, pseudo operads and operads. These categories are denoted Coll, PsOpd, and Opd respectively.

0.3.2 REMARK In some cases a different description is convenient. Consider the inclusion of the symmetric groupoid as a skeleton in Fin_* with objects $\mathbf{n} = \{0, \ldots, n\}$ and 0 as base-point. We write $P(n) = P(\mathbf{n})$. The automorphisms of $\mathbf{n}$ are naturally identified with S_n, the permutations on n letters. Operad structures can then uniquely be characterised by maps

$$\gamma : P(n) \otimes (P(k_1) \otimes \ldots \otimes P(k_n)) \longrightarrow P(k_1 + \ldots + k_n) \tag{0.3.2}$$

for any $n,\ k_1, \ldots, k_n$, and an identity map in $P(1)$. Here

$$\gamma(p, q_1, \ldots, q_n) := (\cdots((p \circ_n q_n) \circ_{n-1} q_{n-1}) \ldots \circ_1 q_1)$$

for $p \in P(n)$, and $q_i \in P(k_i)$. The operad axioms are equivalent to some compatibility relations for these maps (cf. May [69]).

0.3.3 EXAMPLE Let $V \in k\text{-}\mathsf{dgVect}$. Define the *endomorphism operad* End_V of V as $\mathrm{End}_V(n) = \mathrm{Hom}_k(V^{\otimes n}, V)$ with the right kS_n-action permuting tensor factors, and for $f \in \mathrm{End}_V(n)$, $g \in \mathrm{End}_V(m)$ and for $i \leq n$ the composition $f \circ_i g = f \circ (\mathrm{id}^{\otimes i-1} \otimes g \otimes \mathrm{id}^{\otimes n-i})$. If we spell this out (for once) in terms of the Koszul convention applied to homogeneous elements $x_j \in V$ for $j = 1, \ldots, m+n-1$ we get

$$\begin{aligned} f \circ_i g(x_1, \ldots, x_{m+n-1}) =& (-1)^{|g|(|x_1|+\ldots+|x_{i-1}|)} \\ & f(x_1, \ldots, x_{i-1}, g(x_i, \ldots, x_{i+m-1}), x_{i+m}, \ldots, x_{m+n-1}), \end{aligned}$$

which already clearly shows the advantage of the Koszul convention.

0.3.4 EXAMPLE Define the *associative operad* Ass by $\mathrm{Ass}(n) = kS_n$ for $n > 0$, where for $\tau \in S_n$ and $\sigma \in S_m$the composition $\tau \circ_k \sigma$ is the permutation such that

$$\tau \circ_k \sigma(i) = \begin{cases} \tau(i) & \text{if } \tau(i) < \tau(k) \\ \sigma(i-k+1) + \tau(k) & \text{if } k \leq i < k+m-1 \\ \tau(i) + m - 1 & \text{if } \tau(i) > \tau(k). \end{cases}$$

Intuitively, we first apply σ to the inputs $k, \ldots, k+m-1$ and then τ, where τ acts on blocks of inputs, that are of length 1 except for the k-th block which is of length m.
Define the *commutative operad* by $\mathrm{Com}(n) = k$ (the trivial kS_n-module) for $n > 0$, with the composition $\gamma : \mathrm{Com}(m) \otimes \mathrm{Com}(n_1) \otimes \ldots \otimes \mathrm{Com}(n_m) \longrightarrow C(n_1 + \ldots + n_m)$ defined by the natural identification of the $m+1$-th tensor power of k with k.

0.3.5 DEFINITION A *(pseudo) cooperad* is a (pseudo) operad in the opposite category $(k\text{-}\mathsf{dgVect})^{\mathrm{op}}$. That is, a covariant(!) functor $\mathsf{Fin}_* \longrightarrow k\text{-}\mathsf{dgVect}$ satisfying the dual axioms. Here the directions of arrows is inverted, the coinvariants change to invariants, sums change into products, and the right action of S_n changes to a left action. For convenience of the reader we twist this action by an inverse and obtain a right action. Therefore we do not have to distinguish between left and right collections. Let C be a cooperad. In addition to the coidentity $\mathrm{id}^* : C(1) \longrightarrow k$, there exist

$$\gamma^* : C(n) \longrightarrow \bigoplus_{\substack{k \\ n_1 + \ldots + n_k = n}} C(k) \otimes (C(n_1) \otimes \ldots \otimes C(n_k)),$$

that are equivariant with respect to the action of S_k and $S_{n_1} \times \ldots \times S_{n_k}$.The category of cooperads is denoted Coopd.

§5 Graphs and trees

This section sketches the approach of Getzler and Kapranov [28] to graphs.

0.3.6 DEFINITION A *graph* is a finite set of *flags* (or *half edges*) together with both an equivalence relation $\sim$, and an automorphism σ of order ≤ 2. The equivalence classes are called *vertices*, the orbits of length 2 *(internal) edges*, and the orbits of length 1 *external edges* or *legs*. We denote the vertices of a graph g by $\mathbf{v}(g)$, its (internal) edges by $\mathbf{e}(g)$ and its legs by $\mathbf{l}(g)$, especially we denote the half edges in a vertex v by $\mathbf{l}(v)$. The flags in a vertex are also called its *legs*. The vertex containing a flag f is called the vertex of f. To picture a graph, draw a node for each vertex with outgoing edges labelled by its flags. For each orbit of σ of length 2 (edge) draw a line connecting the two flags. For each orbit of length one this leaves an external edge.
A *tree* is a connected graph t such that $|\mathbf{v}(t)| - |\mathbf{e}(t)| = 1$. Equivalently, a connected graph is a tree iff it has no loops. A *rooted tree* is a tree together with a base point in the set of external edges. Note that on a rooted tree, the base point induces a canonical base point on the legs of any vertex. We denote by $T(X)$ the set of rooted trees with external edges labelled by the pointed set X. (An element t of $T(X)$ is a tree t together with a base-point preserving bijection $X \longrightarrow \mathbf{l}(t)$.)

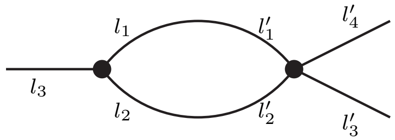


FIGURE 4: A graph with legs $l_1, l_2, l_3, l_1', l_2', l_3', l_4'$, and vertices $\{l_1, l_2, l_3\}$ and $\{l_1', l_2', l_3', l_4'\}$. The non-trivial orbits $\{l_1, l_1'\}$ and $\{l_2, l_2'\}$ of σ determine the edges.

For $n \in \mathbb{N}$, denote by $\mathbf{n}$ the pointed set $\{0, \ldots, n\}$ with base-point 0. A *planar rooted tree* is a rooted tree together with an isomorphism of pointed sets $\mathbf{n}_v \longrightarrow \mathbf{l}(v)$ for each $v \in \mathbf{v}(t)$, that sends 0 to the base point, where $n_v = |\mathbf{l}(v)| - 1$. The planar structure induces a labelling of $\mathbf{l}(t)$ by the pointed set $\mathbf{n}_t = \{0, \ldots, |\mathbf{l}(t)| - 1\}$. We denote planar trees with legs (external edges) labelled by a pointed set X by $PT(X)$.
A *morphism* of graphs is a morphism φ of the set of flags, such that for all flags f, f' both $\varphi(\sigma f) = \sigma\varphi(f)$, and $\varphi(f) \sim \varphi(f')$ iff $f \sim f'$ (where σ and $\sim$ are the defining automorphisms and equivalence relations). If the set of external edges has a base-point, we assume that morphisms preserve the base point. The *group of automorphisms* of a graph g is denoted $\mathrm{Aut}(g)$. Note that an automorphism defines an isomorphism of flags, an automorphism of edges, an automorphism of external edges, and an isomorphism of vertices. Moreover it defines an isomorphism of the legs of a vertex to the legs of its image under the map on vertices. For two rooted trees s and t and $x \in \mathbf{l}(t)$ define $s \circ_x t$ as the rooted tree obtained from s and t by grafting the root of s on leg x of t.

§6 Free operads and 'cofree' cooperads

0.3.7 DEFINITION The forgetful functor from pseudo operads to collections has a left adjoint T, which is the *free pseudo operad* functor. We can give an explicit formula for T. Let

$$C(t) = \bigotimes_{v \in \mathbf{v}(t)} C(\mathbf{l}(v)), \tag{0.3.3}$$

for any rooted tree t. There is a natural action of $\mathrm{Aut}(t)$ on $C(t)$ by permutation of the tensor factors according to the permutation of vertices in the tree t, by the associated isomorphism of $\mathbf{l}(v) \longrightarrow \mathbf{l}(\sigma(v))$ inside the tensor factor associated to v, and by the composition of the labelling of $\mathbf{l}(t)$ by the induced isomorphism of $\mathbf{l}(t)$. This application defines for a collection C and each pointed set X a functor $t \longmapsto C(t)$ from the groupoid $T(X)$ with objects rooted trees with legs labelled by X, and as maps isomorphisms to rooted trees to k-dgVect. Use this functor to define the underlying collection $\bar{T}C$ of the free pseudo operad by

$$\bar{T}C(X) = \operatorname*{colim}_{T(X)}(C(t)) = \bigoplus_{t \in \bar{T}(X)} C(t)_{\mathrm{Aut}(t)},$$

where the colimit is over the groupoid with rooted trees as objects and isomorphisms of trees as maps. The sum is over the isomorphism classes of rooted trees $\bar{T}(X)$ in $T(X)$. The pseudo operad structure on $\bar{T}C$ is given by grafting trees.
The forgetful functor from operads to pseudo operads has a left adjoint. This map is given by adjoining a unit with respect to composition in $P(X)$ for X such that $|X| = 2$. By definition the *free operad functor* T is the composition of both left adjoints.

0.3.8 DEFINITION We now define a cooperad that is cofree with respect to a restricted class of cooperads (compare Definition 0.2.3). Therefore we use quotes around 'cofree' when referring to this cooperad. The 'cofree' pseudo cooperad functor $\bar{T}'$ is defined by

$$\bar{T}'C(X, x_0) = \bigoplus_{t \in \bar{T}(X)} C(t)^{\mathrm{Aut}(t)}.$$

The pseudo cooperad structure is determined by cutting edges. Composition of the functor $\bar{T}'$ with the right adjoint to the forgetful functor from cooperads to pseudo cooperads defines the *'cofree' cooperad functor* T'.

0.3.9 REMARK There are some slightly different descriptions of free (pseudo) operads, which we briefly discuss. The construction of $\bar{T}C$ can equivalently be given as

$$\bar{T}C(X) = \operatorname*{colim}_{PT(X)} C(t),$$

where we use the groupoid with as objects planar trees with external edges labelled by X and maps that are isomorphisms of the underlying rooted trees that need not preserve the planar structure.
There is one more way to construct the free operad. Smirnov [76] defines a monoidal structure $(\Box, I, a, i_r, i_l)$ on the category of collections in k-dgVect

$$C \Box D(X) = \bigoplus_{Y} \bigoplus_{f: X \twoheadrightarrow Y} C(Y) \otimes_{\mathrm{Aut}(Y)} \left(\bigotimes_{y \in Y - y_0} D(f^{-1}(y) \cup \{y\}) \right),$$

where the sum is over one representing pointed sets Y for each isomorphism class, and where y is the base-point of $f^{-1}(y) \cup \{y\}$. (Shnider-Van Osdol [75] is a more accessible reference.) Operads are algebras in this monoidal category. The construction of the free algebra leads to the free operad functor. Getzler-Jones [26] defines a cooperad as a coalgebra with respect to this monoidal structure. Since $\mathrm{char}(k) = 0$, the natural isomorphism from coinvariants to invariants identifies both notions of cooperad.

§7 Cofibrant operads

The existence of a Quillen model category of operads is of great conceptual importance. The model category is determined by its weak equivalences and its fibrations. The first explicit statement of this result for dg operads is due to Hinich, who proves the existence of a model category for operads such that $P(0) = 0$. See also Berger-Moerdijk [2].

0.3.10 THEOREM (HINICH [31], BERGER-MOERDIJK [2]) *Operads without constants (i.e. $P(0) = 0$) in the category of dg vector spaces form a Quillen model category, where the weak equivalences are the quasi isomorphisms, and the fibrations are the surjections. The cofibrations are defined by the left lifting property with respect to acyclic fibrations.*

0.3.11 DEFINITION A complete treatment of Quillen model categories is not necessary here, since we only use part of the structure. The relevant structure is summarised here. An operad P is *cofibrant* if any (solid) diagram

$$\begin{array}{ccc} & & Q \\ & \nearrow & \big\downarrow \\ P & \longrightarrow & R \end{array}$$

of operads where $Q \longrightarrow R$ is a surjective quasi isomorphism, there exists a lift (dotted arrow) such that the resulting diagram commutes. Let P be an operad. A cofibrant operad $\tilde{P}$ together with a surjective quasi isomorphism $\pi : \tilde{P} \longrightarrow P$ is called a *cofibrant replacement* for P. The cofibrant operads used in this thesis are assumed to be of the special kind discussed in Example 0.3.12 below.

0.3.12 EXAMPLE We give the most important class of examples of cofibrant operads. Operads of this type will be used extensively in part II. Let P be an dg operad, such that P is free as a graded operad on a dg collection (C, d) together with a differential ∂ that is a derivation of the free operad TC. If in addition C is filtered as $0 = F^{\geq 0}C \subset F^{\geq 1}C \subset \ldots$, the associated graded is isomorphic to $T(C, d)$, and the differential ∂ on P decreases the filtration in the sense that $\partial(T(F^{\leq n}C)) \subset T(F^{\leq n-1}C)$, then the operad P is cofibrant. Cofibrant operads of this type are close to the elemental cofibrant operads introduced by Markl [65].

§8 Algebras and modules

0.3.13 DEFINITION Let P be an operad. A *P-algebra* is a vector space A together with a map of operads $\lambda : P \longrightarrow \mathrm{End}_A$. Equivalently we can define an algebra A by linear maps

$$\gamma_A : P(n) \otimes_{S_n} A^{\otimes n} \longrightarrow A,$$

such that $p \circ_i q(a_1, \ldots, a_{n+m-1}) = p(a_1, \ldots, q(a_i, \ldots, a_{i+m-1}), a_{n+i}, \ldots, a_{m+n-1})$ if we write

$$p(a_1, \ldots, a_n) = \gamma_A(p, a_1, \ldots, a_n),$$

where $a_1, \ldots, a_n \in A$. The category of algebras for Ass is the category of (non-unital) associative algebras. The category of algebras for Com is the category of non-unital commutative algebras. The category of algebras for Lie is the category of Lie algebras.

Let P be an operad, and let A be an algebra over P. An *A-module* is a vector space M together with linear maps

$$\gamma_M : P(n) \otimes_{S_{n-1}} A^{\otimes n-1} \otimes M \longrightarrow M,$$

such that $p \circ_i q(a_1, \ldots, a_{n+m-1}) = p(a_1, \ldots, q(a_i, \ldots, a_{i+m-1}), a_{n+i}, \ldots, a_{m+n-1})$ for $a_1, \ldots, a_{m+n-2} \in A$ and $a_{m+n-1} \in M$ when we write

$$p(a_1, \ldots, a_k) = \gamma_M(p, a_1, \ldots, a_k),$$

where $a_1, \ldots, a_{k-1} \in A$ and $a_k \in M$. The category modules for a Com-algebra is the usual category of modules of the commutative algebra, the category of modules for a Lie-algebras is the usual category of modules of the Lie algebra, and the category of modules for a Ass-algebra is the usual category of bimodules of the associative algebra.

0.3.14 DEFINITION Let P be an operad, and let A be a P-algebra. Then we define the *universal enveloping operad* P_A as the follows. $P_A(k)$ is the coequaliser

$$\begin{array}{c} \bigoplus_{k, m_1, \ldots m_k} P(k+n) \otimes_{S_{k+n}} \left(\left(\bigotimes_{l=1}^{k} P(m_l) \otimes_{S_{m_l}} A^{\otimes m_l} \right) \otimes_{S_m} kS_{m+n} \right) \\ \downarrow\downarrow \\ \bigoplus_k P(k+n) \otimes_{S_{k+n}} \left(A^{\otimes k} \otimes_{S_k} kS_{k+n} \right) \\ \vdots \\ P_A(n) \end{array}$$

in kS_n-Mod, where $m = m_1 + \ldots + m_k$, and where one of the parallel arrows is the operad composition $\gamma : P(k+n) \otimes P(m_1) \otimes \ldots \otimes P(m_k) \longrightarrow P(m+n)$ and the other consists of simultaneous actions $\gamma_A ; P(m_l) \otimes A^{\otimes m_l} \longrightarrow A$ for $l \leq k$. Intuitively, we add A freely to the space of constants $P(0)$ of P, and take the quotient by the relations stemming from the fact that A is a P-algebra. Algebras for P_A are P-algebras under A. The associative algebra $U_P(A) := P_A(1)$ is called the *universal enveloping algebra* of A as a P-algebra. There is an isomorphism of categories between modules for the P-algebra A, and and left $U_P A$-modules. Though the universal enveloping P-algebra was first defined in Ginzburg-Kapranov [29], the formulation via the universal enveloping operad is first given by Fresse [23].

0.3.15 EXAMPLE If A is a Com-algebra, then $U_{\mathrm{Com}} A = A^+$. If A is a Ass-algebra, then $U_{\mathrm{Ass}} A = A^+ \otimes A^+$. If $\mathfrak{g}$ is a Lie-algebra, then $U_{\mathrm{Lie}} \mathfrak{g} = U\mathfrak{g}$, the usual universal enveloping algebra of $\mathfrak{g}$.

Part I

Operads and Hopf Algebras

CHAPTER **1**

Operads and Renormalisation

This Chapter defines functors from (co)operads to bialgebras. It provides a new view on Hopf algebras related to renormalisation in perturbative quantum field theory. It simplifies the proof of the existence of the Hopf algebra structure, and characterises the group of characters in terms of the operad.
The examples include an operad of graphs that gives rise to the Connes-Kreimer Hopf algebra of graphs [16]. Formulas from quantum field theory lead to algebras for this operad.

1.1 Introduction

Kreimer [48] observes that a governing principle of renormalisation is given by the antipode of a Hopf algebra. More Hopf algebras related to renormalisation have been defined since then. In each of these cases one checks the Hopf algebra axioms, and computes the group of characters and the Lie algebra of primitive elements of the dual. The aim of this chapter is to provide a new approach to Hopf algebras of the type that occur in renormalisation. For these Hopf algebras the coproduct decomposes into more elementary operations which makes checking of the Hopf algebra axioms less cumbersome. Moreover, one obtains a direct description of the group of characters and the Lie algebra of primitive elements of the dual. I show that several of the Hopf algebras that occurred in renormalisation can indeed be understood in this way. Most examples occur in relation to well known operads, but I also describe a new operad of graphs that gives rise to the Connes-Kreimer Hopf algebra of graphs. Formulas from quantum field theory lead to algebras for this operad.

Section 1.2 defines functors $C \longmapsto H_C$ and $C \longmapsto \bar{H}_C$ from 1-reduced cooperads to connected complete Hopf algebras. This section shows that for C of finite type, the character groups $\mathrm{Hom}_{\mathrm{Alg}}(H_C, k)$ and $\mathrm{Hom}_{\mathrm{Alg}}(\bar{H}_C, k)$ of H_C and $\bar{H}_C$ are given as explicit functors from operads to groups applied to the dual operad of C. The Lie algebra of primitive elements of the dual Hopf algebra is given by well known functors from operads to Lie algebras applied to the the dual operad of C.
Section 1.3 explores some examples. The bitensor algebra of a bialgebra and its Pinter Hopf algebra are obtained using the constructions of section 1.2 (cf. Van der Laan-Moerdijk [82]). The Hopf algebra of higher order differentials of the line, and its non-commutative version (e.g. Brouder-Frabetti [6]) are obtained from the construction applied to the operad of commutative algebras.
Section 1.4 is devoted to one specific example. This section constructs an operad of graphs and shows that the Connes-Kreimer Hopf algebra of graphs [16] is the Hopf algebra constructed from the suboperad of one particle irreducible graphs.

1.2 Constructions on cooperads

§1 Bialgebras

To provide the constructions of this Chapter in a greater variety of situations, we introduce non-symmetric operads. These are obtained by omitting the symmetric group actions and the equivariance properties from the definition of an operad.

1.2.1 DEFINITION A *non-symmetric pseudo operad* P is a sequence $P(n)$ for $n \in \mathbb{N}$ of dg vector spaces together with linear operations $\circ_i : P(n) \otimes P(m) \longrightarrow P(m+n-1)$ for $i = 1, \ldots, n$ satisfying the same associativity constraint as the corresponding operations in an pseudo operad. A *non-symmetric operad* P is a non-symmetric pseudo operad together with an element $\mathrm{id} \in P(1)$ that is left/right identity with respect to all compositions $\circ_i$. Dually, one defines non-symmetric (pseudo) cooperads.
Throughout this chapter we always assume that $P(0) = 0$. There is a natural grading on the total space $\bigoplus_n P(n)$, defined by $(\bigoplus_n P(n))^m = P(m+1)$.

1.2.2 DEFINITION Let C be a non-symmetric cooperad. Define $B_C := T(\bigoplus_n C(n))$, the tensor algebra on the total space of C. Use the cocomposition

$$\gamma^* : C(n) \longrightarrow \bigoplus_{k, n_1 + \ldots + n_k = n} C(k) \otimes (C(n_1) \otimes \ldots \otimes C(n_k))$$

and the natural inclusions

$$i_1 : C(k) \longrightarrow T(\bigoplus_m C(m)), \quad \text{and} \quad i_2 : C(n_1) \otimes \ldots \otimes C(n_k) \longrightarrow T(\bigoplus_m C(m))$$

to define a map $\Delta : B_C \longrightarrow B_C \otimes B_C$ on generators as $\Delta = (i_1 \otimes i_2) \circ \gamma^*$. Extend Δ as an algebra morphism. Define the algebra morphism $\varepsilon : B_C \longrightarrow k$ as the map ε which vanishes on generators of degree $\neq 0$, and satisfies $\varepsilon|_{C(1)} = \varepsilon_C$.

1.2.3 LEMMA *Let C be a non-symmetric cooperad.*

(i). Comultiplication Δ and counit ε as defined above make $C \longmapsto B_C$ a functor from non-symmetric cooperads to graded bialgebras.

(ii). If C is a cooperad the bialgebra structure of B_C descends to $\bar{B}_C = S(\bigoplus_n C(n)^{S_n})$ (the symmetric algebra on the total space of invariants of C), and consequently $C \longmapsto \bar{B}_C$ defines a functor from cooperads to commutative graded bialgebras.

PROOF Let C be a non-symmetric cooperad and let B_C as above. Coassociativity of Δ is immediate from the coassociativity of γ^* since we only need to check it on generators. Similarly, the counit property of ε from the corresponding property of the coidentity of C. This shows that B_C is a bialgebra. Functoriality of the construction is obvious. Both product and coproduct are easily seen to respect the grading on the tensor algebra induced by the grading of $\bigoplus_n C(n)$.
Let C be a cooperad. For a cooperad, the cocomposition γ^* is equivariant with respect to the S_n-action. Thus

$$\gamma^* : C(n) \longrightarrow \bigoplus_k \left(\bigoplus_{n_1+\ldots+n_k=n} C(k) \otimes (C(n_1) \otimes \ldots \otimes C(n_k)) \right)$$

is equivariant with respect to S_k and $S_{n_1} \times \ldots \times S_{n_k}$ on the corresponding summands, where S_k permutes factors k blocks of consecutive length $n_1, \ldots, n_k$, and factors in $C(n_1) \otimes \ldots \otimes C(n_k)$. It follows that we can restrict γ^* to an S_k-equivariant map

$$\gamma^* : C(n)^{S_n} \longrightarrow \bigoplus_k \left(\bigoplus_{n_1+\ldots+n_k=n} C(k) \otimes (C(n_1)^{S_{n_1}} \otimes \ldots \otimes C(n_k)^{S_{n_k}}) \right).$$

Since γ^* takes values in the S_k-invariants, we therefore end up in the invariants $C(k)^{S_k}$ in the first tensor factor once we take the quotient (with respect to the action of S_k) of the other tensor factors. This gives a well defined as a map

$$\Delta : C(n)^{S_n} \longrightarrow \bigoplus_k \bigoplus_{n_1+\ldots+n_k=n} C(k)^{S_k} \otimes (C(n_1)^{S_{n_1}} \otimes \ldots \otimes C(n_k)^{S_{n_k}})_{S_k}.$$

We use this map to define the coproduct on generators and extend as an algebra morphism. The bialgebra axioms follow directly from the non-symmetric version. Functoriality of the constructions in trivial. QED

§2 Hopf algebras

A (co)operad P is called *1-reduced*, or *1-connected* if $P(0) = 0$, and $P(1) = k$. Let C be a 1-reduced non-symmetric cooperad. The space $\bigoplus_n C(n)$ has a base-point given by the inverse of the counit $\varepsilon : C(1) \longrightarrow k$. In the sequel we will use pointed tensor algebra and the pointed symmetric algebra

$$T_*(\bigoplus_n C(n)) = T(\bigoplus_n C(n))/(\varepsilon^{-1}(1) - \mathbf{1})$$
$$S_*(\bigoplus_n C(n)) = S(\bigoplus_n C(n))/(\varepsilon^{-1}(1) - \mathbf{1}),$$

where the unit in T and S is denoted by $\mathbf{1}$. In other words, T_* is the left adjoint to the forgetful functor $(A, \mu, u) \longmapsto (A, u)$ from unital associative algebras to vector spaces with a non-zero base-point, and S_* the left adjoint to the forgetful functor from unital commutative algebras to vector spaces with non-zero base-point.

1.2.4 DEFINITION Let C be a 1-reduced non-symmetric cooperad. Denote by H_C the pointed tensor algebra $T_*(\bigoplus_n C(n))$ on the total space of C, with respect to the base-point given by the inclusion of $C(1) = k$. The coalgebra structure on B_C induces maps $\Delta : H_C \longrightarrow H_C \otimes H_C$, and $\varepsilon : H_C \longrightarrow k$. If C is a 1-reduced cooperad, denote $\bar{H}_C = S_*(\bigoplus_n C(n)^{S_n})$, the pointed symmetric algebra on the total space of invariants of C. Lemma 1.2.3 now can be adapted to a pointed version.

1.2.5 THEOREM *Let C be a 1-reduced non-symmetric cooperad.*

(i). The application $C \longmapsto H_C$ defines a functor from non-symmetric cooperads to graded connected Hopf algebras.

(ii). If C is a 1-reduced cooperad the application $C \longmapsto \bar{H}_C$ defines a functor from cooperads to commutative graded connected Hopf algebras.

PROOF There is a natural surjection of algebras $T(\bigoplus_n C(n)) \longrightarrow T_*(\bigoplus_n C(n))$. We define a coproduct on $T_*(\bigoplus_n C(n))$ by the formula of the coproduct in Lemma 1.2.3. Since C is coaugmented and ε is the counit, the cocomposition γ^* respects the base point given by $\varepsilon^{-1} : k \longrightarrow C(1)$, and satisfies $(\varepsilon \otimes \mathrm{id}) \circ \gamma^* = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \gamma^*$. This implies the bialgebra structure is well defined on the pointed tensor algebra.

Functoriality is again trivial. To prove *(i)* it remains to check that H_C is in fact a graded connected Hopf algebra.
A bialgebra A is called *connected* if it is $\mathbb{Z}$-graded, concentrated in non-negative degree, and satisfies $A^0 = k \cdot 1$. The *augmentation ideal* of a connected bialgebra A is the ideal $\bigoplus_{n\geq 1} A^n$. Since the degree 0 part of H_C is $C(1) = k$, the bialgebra H_C is connected. It is well known (cf. Milnor and Moore [70]) that any graded connected bialgebra admits an antipode and is thus a Hopf algebra.
To pass to the symmetric version *(ii)* argue as in the proof of Lemma 1.2.3, and observe that $\bar{H}_C$ is also connected. QED

1.2.6 REMARK Note that the construction of H_C and $\bar{H}_C$ defines a graded bialgebra for any coaugmented cooperad C, but that we need C to be 1-reduced in order to get a graded connected Hopf algebra.
Let P be a (1-reduced) (non-symmetric) operad of finite type. Then the linear dual collection P^* is a (non-symmetric) cooperad. Regarding the bialgebras and Hopf algebras associated to the cooperad P^* we will use notation

$$B_P := B_{P^*}, \quad \bar{B}_P := \bar{B}_{P^*}, \quad H_P := H_{P^*}, \quad \text{and} \quad \bar{H}_P := \bar{H}_{P^*}.$$

Confusion of the notations is hardly possible, since in both cases we use the (co)operad structure to define the coproduct, and the product is free.

§3 Groups

1.2.7 DEFINITION Let P be a non-symmetric operad. Let

$$G_P = \left\{ \sum_{n=1}^{\infty} p_n \qquad \text{s.t.} \quad p_n \in P(n) \text{ and } p_1 = \mathrm{id} \right\} \subset \hat{\bigoplus_n} P(n)$$

and define a multiplication $\circ$ on this set by

$$\left(\sum_n p_n \right) \circ \left(\sum_m q_m \right) = \sum_{n, m_1, \ldots, m_n} \gamma(p_n, q_{m_1}, \ldots, q_{m_n}),$$

the completed sum. If P is an operad this multiplication defines a multiplication on

$$\bar{G}_P = \left\{ \sum_{n=1}^{\infty} p_n \qquad \text{s.t.} \quad p_n \in P(n)_{S_n} \text{ and } p_1 = \mathrm{id} \right\} \subset \hat{\bigoplus_n} (P(n)_{S_n}),$$

since the composition γ is equivariant with respect to th S_n-actions.

1.2.8 LEMMA *Let P be a non-symmetric operad.*

(i). The set G_P is a group with respect to the multiplication $\circ$ the unit element id $\in G_P$. *The application $P \longrightarrow G_P$ defines a functor from non-symmetric operads to groups.*

(ii). If P is an operad, the quotient $\bar{G}_P$ of G_P is a group. The application $P \longmapsto \bar{G}_P$ defines a functor from operads to groups

PROOF Associativity is clear from the associativity of γ. That $\mathrm{id} \in P(1)$ is the identity with respect to composition is also obvious. To prove the existence of a formal inverse one can use an inverse function theorem argument. QED

1.2.9 THEOREM *Let P be a 1-reduced non-symmetric operad of finite type.*

(i). The group G_P is isomorphic to the group of characters of the Hopf algebra H_P.

(ii). If P is a 1-reduced operad of finite type, then the group $\bar{G}_P$ is isomorphic to the group of characters of the Hopf algebra $\bar{H}_P$.

PROOF The comultiplication on $H_P = T_*(\bigoplus_n P^*(n))$ induces a multiplication on

$$\mathrm{Hom}_{\mathrm{Alg}}(H_P, k) \cong \mathrm{Hom}_k(\bigoplus_n P^*(n), k) \cong \hat{\bigoplus}_{n\geq 2} P(n).$$

The $n \geq 2$ comes from the unitality of algebra homomorphisms: the group consists of elements of $\hat{\bigoplus}_n P(n)$ such that the coefficient for id^* equals 1.
Choose a homogeneous basis $\{p\}$ of $\bigoplus_n P(n)$ and let $\{p^*\}$ be the dual basis. For such basis elements with p of degree $n-1$ and q and r in arbitrary degree we have

$$< r^*|p \cdot q >=< \gamma^*(r^*)|(p \otimes q^{\otimes n}) >=< r^*|\gamma(p \otimes q^{\otimes n}) > \qquad (1.2.1)$$

with respect to the standard pairing $< .|. >$ between P^* and P. This shows the first part of the result. To obtain the statement on coinvariants, note that

$$\mathrm{Hom}_{\mathrm{Alg}}(\bar{H}_P, k) = \hat{\bigoplus}_{n\geq 2} P(n)_{S_n},$$

and replace $< .|. >$ in Equation (1.2.1) by the pairing of $P(n)_{S_n}$ and $(P(n)^*)^{S_n}$. QED

1.2.10 REMARK Let C be a cooperad. We want to be a bit more specific on the relation between H_C and $\bar{H}_C$ and their groups of characters. Consider the symmetrisation $S_*(\bigoplus_n C(n))$ of H_C. The formulae for the structure on H_C make $S_*(\bigoplus_n C(n))$ a Hopf algebra. Moreover, we have maps

$$H_C \longrightarrow S_*(\bigoplus_n C(n)) \longleftarrow \bar{H}_C, \tag{1.2.2}$$

where the left map is a surjection and the right map is an injection. A similar diagram exists for B_C and $\bar{B}_C$. Since the algebra k is commutative, every character of H_P factorises through its symmetrisation $S_*(\bigoplus_n P^*(n))$. The quotient map $G_P \longrightarrow \bar{G}_P$ can thus be interpreted as the map of character groups

$$G_P = \mathrm{Hom}_{\mathrm{Alg}}(S_*(\bigoplus_n P^*(n)), k) \longrightarrow \mathrm{Hom}_{\mathrm{Alg}}(\bar{H}_P, k) = \bar{G}_P,$$

induced by the map of Hopf algebras $\bar{H}_P \longrightarrow S(\bigoplus_n P^*(n))$.

§4 Lie algebras

1.2.11 DEFINITION (KAPRANOV-MANIN [38]) Let P be a 1-reduced non-symmetric pseudo operad. The vector space $L_P = \bigoplus_{n\geq 2} P(n)$ is a Lie algebra with respect to the Lie bracket on $p \in P(n)$ and $q \in P(m)$ given by

$$\sum_{i=1}^{n} p \circ_i q - \sum_{j=1}^{m} q \circ_j p.$$

If P is a 1-reduced pseudo operad, this Lie algebra structure descends to the quotient $\bar{L}_P = \bigoplus_n P(n)_{S_n}$. Both Lie algebras are graded with respect to the grading $\deg(P(n)) = n - 1$ (no signs!), and the application $P \longmapsto L_P$ (resp. $P \longmapsto \bar{L}_P$) defines a functor from non-symmetric operads (resp. operads) to Lie algebras.

1.2.12 THEOREM *Let P be a 1-reduced non-symmetric operad of finite type.*

(i). The Lie algebra of primitive elements of $(H_P)^*$ is the Lie algebra L_P.

(ii). If P is a 1-reduced operad of finite type, the Lie algebra of primitive elements of $(\bar{H}_P)^*$ is the Lie algebra $\bar{L}_P$. Consequently, $(\bar{H}_P)^*$ is the universal enveloping algebra $U(\bar{L}_P)$ of the Lie algebra $\bar{L}_P$.

PROOF Let P be a 1-reduced non-symmetric operad of finite type. The Hopf algebra H_P^* is the pointed 'cofree' coalgebra on the total space of P with the multiplication defined on $(P(k)) \otimes (P(m_1) \otimes \ldots \otimes P(m_k))$ as the composition γ of the non-symmetric operad, and then extended as a coalgebra homomorphism. Projected to the cogenerators $\bigoplus_n P(n)$, the multiplication reduces to the sum of the circle-i operations:

$$\sum_{i=1}^{m} \circ_i : P(m) \otimes P(n) \longrightarrow P(m+n-1),$$

for $m, n > 1$. The Lie bracket on the primitive elements is the commutator of this (non-associative) product. This shows the first part of the result.
Assume that P is a 1-reduced operad of finite type. The Lie algebra of primitive elements of $(\bar{H}_P)^*$ is then the symmetric quotient $\bar{L}_P$ since we have the factorisation through $S_*(\bigoplus_n P(n))$ (cf. Remark 1.2.10). From the Milnor-Moore theorem [70] it follows $H_P^* = U(\bar{L}_P)$. QED

1.2.13 REMARK A group closely related to G_P and $\bar{G}_P$ was defined independently by Chapoton [13]. He uses the group for the study of the exponential map associated to the pre-Lie operad.

1.3 First examples

§5 Formal diffeomorphisms on the line

1.3.1 EXAMPLE Consider the operad Com, which has as algebras commutative associative algebras. This operad satisfies $\mathrm{Com}(n) = k$ for $n \geq 1$. Composition is the usual identification of tensor powers of k with k itself.
We now describe the Hopf algebra H_{Com}. Denote the generator of $\mathrm{Com}(n)$ by e_n. Thus H_{Com} is the pointed free associative algebra on generators $\{e_n\}_{n\geq 1}$ where e_1 is identified with the unit, and where e_n is of degree $n-1$, with coproduct

$$\Delta(e_n) = \sum_{k=1}^{n} \sum_{n_1+\ldots+n_k=n} e_k \otimes e_{n_1} \cdot \ldots \cdot e_{n_k},$$

where we sum over n_i such that the formula makes sense (i.e. $n_i \geq 1$).

1.3.2 DEFINITION Let $\mathcal{H}^{\mathrm{diff}}$ be the pointed free commutative algebra on variables a_i for $i \geq 0$. The base-point is $1 \longmapsto a_0$. Define a bilinear pairing between the space of generators and the group of formal power series $\varphi(x)$ with coefficients in k such that $\varphi(x) \equiv x(\mod x^2)$ by the formula

$$< a_n, f >= \frac{1}{(n+1)!} \frac{d^{n+1}}{dx^{n+1}} f(0).$$

Define a coproduct $\Delta : \mathcal{H}^{\text{diff}} \longrightarrow \mathcal{H}^{\text{diff}} \otimes \mathcal{H}^{\text{diff}}$ on generators by the formula $< \Delta(a_n), f \otimes g >=< a_n, f \circ g >$. These formulae define a Hopf algebra. This Hopf algebra is the *Hopf algebra of higher order differentials of the line*. The character group of $\mathcal{H}^{\text{diff}}$ is the group of power series $\varphi(x)$ with coefficients in k such that $\varphi(x) \equiv x$ (mod x^2), which is also called the *group of formal diffeomorphisms of the line*. The group multiplication is composition of power series.

Define the Hopf algebra $\mathcal{H}^{\text{ncdiff}}$ of *non-commutative higher order differentials of the line* (cf. eg. Brouder-Frabetti [6]). As an algebra $\mathcal{H}^{\text{ncdiff}}$ is the free unital algebra on variables a_i for $i \geq 0$, with respect to the base-point $1 \longmapsto a_0$. Define the coproduct on $\mathcal{H}^{\text{ncdiff}}$ by

$$\Delta(a_n) = \sum_{k, n-k = n_1 + \ldots + n_{k+1}} a_k \otimes a_{n_1} \cdot \ldots \cdot a_{n_{k+1}},$$

where $k, n_i \geq 0$ for $i \leq k+1$. (We got rid of the usual binomial coefficients by introducing $1 = a_0$ in the sum.) Note that the symmetric quotient of $\mathcal{H}^{\text{ncdiff}}$ is $\mathcal{H}^{\text{diff}}$.

1.3.3 THEOREM *The completion (wrt the grading by arity-1) of the Hopf algebra $\bar{H}_{Com}$ is the Hopf algebra of higher order differentials of the line. Its character group is $\bar{G}_{Com}$, the group of formal diffeomorphisms of the line. The Lie algebra of primitive elements of the dual $\bar{L}_{Com}$ is the Lie algebra of polynomial vector fields on the line without a constant term.*

PROOF To identify the group $\bigoplus_{n \geq 2} \mathsf{Com}(n)$ with the group of power series in one variable x with coefficients in k such that $\varphi(x) \equiv x(\text{ mod} x^2)$ and use composition of power series as multiplication. Use the isomorphism given by $e_n \longmapsto x^n$. The result on the Hopf algebras now follows since a graded complete commutative Hopf algebra of finite type is completely determined by its group of characters (Quillen [73]).

At the Hopf algebra level the isomorphism $\bar{H}_{\text{Com}} \longrightarrow \mathcal{H}^{\text{diff}}$ is given on the basis of generators by $e_n^*(\varphi) \longmapsto a_{n-1}$.

The Lie algebra $\bar{L}_{\text{Com}}$ with basis $\{e_n\}_{n \geq 1}$ satisfies the commutation relation $[e_n, e_m] = (n-m)e_{m+n-1}$. The explicit isomorphism is thus given by $e_n \longmapsto x^n \partial_x$. QED

1.3.4 COROLLARY *The map defined on generators by $a_i \longmapsto e_{i+1}^*$ is an isomorphism of graded Hopf algebras from $\mathcal{H}^{ncdiff}$ to H_{Com}.*

1.3.5 REMARK One can consider Com as the endomorphism operad of the one dimensional vector space k. Analogous if one considers the 1-reduced version of the endomorphism operad End_V of a finite dimensional vector space V, one obtains the higher dimensional analogue of the higher order differentials, formal diffeomorphisms, and polynomial vector fields.

§6 Associative and Lie operad

1.3.6 EXAMPLE To describe the structures we get from the operad Ass of associative algebra, first observe that the surjection Ass $\longrightarrow$ Com becomes an isomorphism on coinvariants: $\mathrm{Ass}(n)_{S_n} = \mathrm{Com}(n)_{S_n}$. thus $\bar{H}_{\mathrm{Ass}} = \bar{H}_{\mathrm{Com}}$, $\bar{L}_{\mathrm{Ass}} = \bar{L}_{\mathrm{Com}}$, and $\bar{G}_{\mathrm{Ass}} = \bar{G}_{\mathrm{Com}}$. However, on the non-symmetrised version there are some differences.
For the group G_{Ass} we write x^σ for the element corresponding to $\sigma \in S_n$. The group G_{Ass} is then the group of formal permutation-expanded series $\sum_\sigma c_\sigma x^\sigma$, where σ runs over permutations in S_n for all n and the coefficients c_σ are in the ground field. Moreover, the trivial permutation $(1) \in S_1$ has coefficient $c_{(1)} = 1$. Composition is the linear extension in x^σ of

$$x^\sigma \circ \left(\sum_\tau c_\tau x^\tau \right) = \sum_{\tau_1,\ldots,\tau_k} c_{\tau_1} \ldots c_{\tau_k} x^{\hat{\sigma}(\tau_1 \times \ldots \times \tau_k)},$$

where $\hat{\sigma}$ is the permutation that permutes k blocks on which the τ_i act. (Observe that $\hat{\sigma}$ thus depends on the degree of the τ_i.) The Lie algebra structure is given on $\sigma \in S_n$ and $\tau \in S_m$ by

$$[\sigma, \tau] = \sum_{i=1}^{n} \sigma \circ_i \tau - \sum_{j=1}^{m} \tau \circ_j \sigma.$$

Dually, the Hopf algebra $H_{\mathrm{Ass}} = T_*(\bigoplus_{n\geq 1} S_n)$ with S_n in degree $n-1$ has the coproduct

$$\Delta(\sigma) = \sum_k \sum_{\sigma = \hat{\tau}_0 \circ (\tau_1 \times \ldots \times \tau_k)} \tau_0 \otimes (\tau_1, \ldots, \tau_k),$$

where the sum is over all decompositions of σ as $\hat{\tau}_0 \circ (\tau_1, \ldots, \tau_k)$, where $(1) \in S_1$ is identified with the unit in H_{Ass}.

1.3.7 LEMMA *Write $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for the inputs. Then a basis of Lie(n) is given by*

$$\{[\mathbf{a}_{\sigma(1)}, [\mathbf{a}_{\sigma(2)}, \ldots, [\mathbf{a}_{\sigma(n-1)}, \mathbf{a}_n]\ldots]] \quad | \quad \sigma \in S_{n-1}\}, \tag{1.3.3}$$

where the brackets are in right-most position.

PROOF By a dimension argument ($\dim(\mathrm{Lie}(n)) = (n-1)!$) it suffices to show that these elements span Lie(n). Certainly Lie(n) is spanned by all bracketed expressions of the inputs $\mathbf{a}_1, \ldots, \mathbf{a}_n$. By anti-symmetry of the bracket, Lie(n) is spanned by all bracketed expressions such that the right-most input is $\mathbf{a}_n$. Write every element of this spanning set as $[\vec{\mathbf{a}}, \vec{\mathbf{b}}]$, where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are bracketed expressions of complementary non-empty subsets of the $\mathbf{a}_i$ such that the right-most input of $\vec{\mathbf{b}}$ is $\mathbf{a}_n$.

I prove the result by double induction on n and the number of inputs in $\vec{\mathbf{a}}$, which I denote $|\vec{\mathbf{a}}|$. Suppose that we know the result for $\mathrm{Lie}(k)$, where $k < n$. Then for $|\vec{\mathbf{a}}| = 1$, there is nothing to prove since $[\vec{\mathbf{a}}, \vec{\mathbf{b}}]$ is already an element of the set in Formula (1.3.3). For $|\vec{\mathbf{a}}| > 1$ we can assume without loss of generality $\vec{\mathbf{a}} = [\mathbf{a}_{i_0}, \vec{\mathbf{a}}']$. By the Jacobi identity,

$$[\vec{\mathbf{a}}, \vec{\mathbf{b}}] = [\mathbf{a}_{i_0}, [\vec{\mathbf{a}}', \vec{\mathbf{b}}]] - [\vec{\mathbf{a}}', [\mathbf{a}_{i_0}, \vec{\mathbf{b}}]].$$

Now apply the induction hypotheses to $|\vec{\mathbf{a}}'|$ and $k = n-1$ to see that $[\vec{\mathbf{a}}, \vec{\mathbf{b}}]$ is in the span of the set in Formula (1.3.3). QED

1.3.8 EXAMPLE There is a natural inclusion of operads $\mathrm{Lie} \subset \mathrm{Ass}$ which is defined by sending the bracket $\lambda \in \mathrm{Lie}(2)$ to the commutator $\mu - \mu^{\mathrm{op}}$ of the associative product $\mu \in \mathrm{Ass}(2)$. The group G_{Lie} is therefore a subgroup of G_{Ass}. To make the group more explicit it is useful to characterise the image of Lie in Ass. Write $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for the inputs, then one can describe the image of $[\mathbf{a}_1, [\mathbf{a}_2, \ldots, [\mathbf{a}_{n-1}, \mathbf{a}_n]...]]$ as the sum

$$\sum_{\sigma \in Z_n} (-1)^{n-\sigma^{-1}(n)} \mathbf{a}_{\sigma(1)} \cdot \mathbf{a}_{\sigma(2)} \cdot \ldots \cdot \mathbf{a}_{\sigma(n)},$$

where Z_n consists of those permutations $\sigma \in S_n$ such that for $i := \sigma^{-1}(n)$,

$$\sigma(1) < \sigma(2) \ldots < \sigma(i) \qquad \text{and} \qquad \sigma(i) > \sigma(i+1) > \ldots > \sigma(n).$$

By the Lemma above, we have the following. The images of elements in Lie are the series in G_{Ass} that are of the form

$$\sum_n \sum_{\tau \in S_{n-1}} c_{n,\tau} \sum_{\sigma \in Z_n} (-1)^{n-\sigma^{-1}(n)} x^{\sigma \circ (\tau \times (1))}.$$

§7 The Connes-Kreimer Hopf algebra of trees

1.3.9 DEFINITION Consider the set $T(*)$ of rooted trees without external edges different from the root. A *cut* of $t \in T(*)$ is a subset of edges. A cut is *admissible* if for every vertex $v \in \mathbf{v}(t)$ the path from v to the root contains at most one edge in the cut. If c is an admissible cut of t, denote by $R^c(t)$ the connected component of the graph obtained from t by removing the edges in c, where all new external edges are

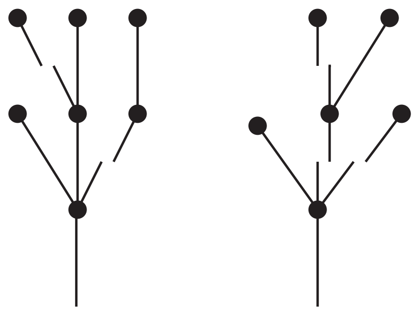


FIGURE 5: An admissible cut (on the left) and a non-admissible cut (on the right). Construct $R^c(t)$ as the connected component containing the root with all upward pointing external edges removed.

removed. Denote by P^c the disjoint union of the other components as elements of $T(*)$ with their new root.

The set $T(*)$ generates a commutative Hopf algebra $\mathcal{H}_R$. As an algebra it is the symmetric algebra on $T(*)$. Comultiplication Δ is defined on generators t, and extended as a algebra homomorphism:

$$\Delta(t) = \sum_{c \in C(t)} R^c(t) \otimes P^c(t), \tag{1.3.4}$$

where the sum is over admissible cuts. The counit $\varepsilon : \mathcal{H}_R \longrightarrow k$ takes value 1 on the empty tree and 0 on other trees. The Hopf algebra $\mathcal{H}_R$ is the *Hopf algebra of rooted trees*, introduced by Kreimer [48] and further studied in Connes and Kreimer [15]. The Lie algebra of primitive elements of $\mathcal{H}_R^*$ is isomorphic to the associated Lie algebra of the pre-Lie algebra $\mathcal{L}_R$ spanned by rooted trees without external edges other than the root, and the product

$$t \bullet s = \sum_{v \in \mathbf{v}(t)} t \circ_v s,$$

where $t \circ_v s$ is the rooted tree consisting of the connected subtrees t and s and one edge that connects the root of s to the vertex $v \in \mathbf{v}(t)$.

1.3.10 EXAMPLE Let M be the collection $M(n) = k$ for $n \geq 2$ and $M(n) = 0$ otherwise. Recall the free operad on M satisfies $TM(n) = \bigoplus_{t \in T_2(\mathbf{n})} k$, where $T_2(\mathbf{n})$ is the set of rooted trees with a a pointed bijection $\varphi : \mathbf{l}(t) \longrightarrow \{0, \ldots, n\}$ (i.e. the root leg is mapped to 0) such that every vertex has at least 3 legs. Thus TM is a graded operad of finite type with respect to the grading by $|\mathbf{v}(t)|$, the number of vertices. The symmetric group action permutes the labels on external edges.

1.3.11 PROPOSITION *There exists an inclusion $\mathcal{H}_R \longrightarrow \widehat{\bar{H}_{TM}}$, of the Connes-Kreimer Hopf algebra into the completion (w.r.t. the grading by eternal edges) of H_{TM}. This inclusion is graded with respect to the grading by the number of vertices.*

PROOF It suffices to show that there is a surjection φ of Lie algebras $\bar{L}_{TM} \longrightarrow \mathcal{L}_R$ onto the Lie algebra of primitive elements of $\mathcal{H}_R^*$.
A vertex $v \in \mathbf{v}(t)$ of a tree t is called saturated if it has no external edges other than the root attached to it. Denote by J the ideal in TM spanned by trees with a saturated vertex. Let φ be such that it factorises through TM/J. Thus is remains to define $\varphi : L_{TM} \longrightarrow \mathcal{L}_R$ on trees without a saturated vertex. On such a tree t define

$$\varphi(t) = \tilde{t} \cdot \prod_{v \in \mathbf{v}(t)} i_t(v)!,$$

where $i_t(v)$ is the number of external legs other than the root of t that are attached to vertex v and $\tilde{t}$ is the tree t with all external edges other than the root omitted. Recall that both of the the Lie algebras $\bar{L}_{TM}$ and $\mathcal{L}_R$ are pre-Lie algebras. To check that this is a Lie algebra homomorphism, write

$$\begin{aligned}
\varphi(t) \bullet \varphi(s) &= \sum_{u \in \mathbf{v}(t)} \widetilde{t \circ_u s} \cdot \prod_{v \in \mathbf{v}(t)} \prod_{w \in \mathbf{v}(s)} i_t(v)! i_s(w)! \\
&= \sum_{u \in \mathbf{v}(t)} \widetilde{t \circ_u s} \cdot i_{t \circ_u s}(u) \cdot \prod_{v \in \mathbf{v}(t \circ_u s)} i_{t \circ_u s}(v)! \\
&= \sum_{i=1}^{|\mathbf{l}(t)|-1} \widetilde{t \circ_i s} \cdot \prod_{v \in \mathbf{v}(t \circ_u s)} i_{t \circ_u s}(v)! \\
&= \varphi(t \bullet s).
\end{aligned}$$

To understand the second equality, note that $i_s(w) = i_{t \circ_u s}(w)$ for all $w \in \mathbf{v}(s)$, that $i_t(v) = i_{t \circ_u s}(v)$ if $v \neq u$, and that $i_t(u) = i_{t \circ_u s}(u) + 1$. The third equality rewrites the sum over vertices as a sum over external edges other than the root. QED

§8 The double symmetric algebra construction

The results in this section first appeared in Van der Laan and Moerdijk [82]. I start with a lemma. This lemma is not new, but I decided to include a sketch of the proof, since this point has led to some confusion in earlier drafts.

1.3.12 LEMMA (BERGER-MOERDIJK [2]) *Let A be a bialgebra.*

(i). The vector spaces $C_A(n) = A^{\otimes n}$ (for $n \geq 1$) form a coaugmented non-symmetric cooperad with as coidentity the counit $\varepsilon : A \longrightarrow k$ and the cocomposition γ^* defined on summands by the diagram

$$\begin{array}{ccc} A^{\otimes n} & \overset{\gamma^*}{\dashrightarrow} & A^{\otimes k} \otimes (A^{\otimes n_1} \otimes \ldots \otimes A^{\otimes n_k}) \\ \Delta \downarrow & & \uparrow (\mu_1 \otimes \ldots \otimes \mu_k) \otimes \mathrm{id} \\ A^{\otimes n} \otimes A^{\otimes n} & = & (A^{\otimes n_1} \otimes \ldots \otimes A^{\otimes n_k}) \otimes (A^{\otimes n_1} \otimes \ldots \otimes A^{\otimes n_k}), \end{array}$$

where Δ is the coproduct of $A^{\otimes n}$, and $\mu_i : A^{\otimes n_i} \longrightarrow A$ is the multiplication of the algebra A. A coaugmentation for this non-symmetric cooperad is given by the unit of A.

(ii). The collection C_A with the S_n-action by permuting tensor factors is a coaugmented cooperad with respect to the same structure maps if A is commutative.

PROOF Coassociativity follows directly from the fact that μ is a coalgebra morphism. Coidentity and coaugmentation are also direct from the bialgebra structure on A and $A^{\otimes n}$. Then it remains to consider the compatibility with the S_k and $(S_{n_1} \times \ldots \times S_{n_k})$ for

$$\gamma^* : C_A(n) \longrightarrow \bigoplus_{n_1+\ldots+n_k=n} C_A(k) \otimes C_A(n_1) \otimes \ldots \otimes C_A(n_k).$$

In Sweedler's notation one can write the cocomposition γ^* of C_A on a generator $(x_1, \ldots, x_n) \in C_A(n)$ as

$$\gamma^*(x_1, \ldots, x_n) = \sum \sum (x'_1 \star \cdots \star x'_{n_1}, \ldots, x'_{n-n_k+1} \star \cdots \star x'_n) \otimes \\ ((x''_1, \ldots, x''_{n_1}) \otimes \ldots \otimes (x''_{n-n_k+1} \ldots, x''_n)),$$

where the first sum is over all k and all partitions $n = n_1 + \ldots + n_k$, and the second sum is the sum of the Sweedler notation, and where $\star$ denotes the product of A. We study the action on the right hand side of this formula. The compatibility with the S_k-action is satisfied since the action permutes both the tensor factors in $A^{\otimes k}$, and the factors $A^{\otimes n_1}$ up to $A^{\otimes n_k}$. For compatibility with the S_{n_i}-action, consider $x'_{n_1+\ldots+n_{i-1}+1} \star \cdots \star x'_{n_1+\ldots+n_i} \otimes (x''_{n_1+\ldots+n_{i-1}+1}, \ldots, x''_{n_1+\ldots+n_i})$. The S_{n_i}-action only permutes elements in the right hand side. Thus, to obtain equivariance we should have that $x'_{n_1+\ldots+n_{i-1}+1} \star \cdots \star x'_{n_1+\ldots+n_i}$ is invariant under permutation of the indices. In other words, the product needs to be commutative. QED

Recall the bialgebras $T(\bar{T}'(A))$ and $S(\bar{S}'(A))$ (Brouder-Schmitt [8]), where T (resp. S) is the unital free associative (resp. commutative) algebra functor, and $\bar{T}'$ (resp.

$\bar{S}'$) is the non-unital free associative (resp. commutative) coalgebra functor. Comparing the equation for γ^* in Sweedler's notation with Brouder's formulae makes the following result a tautology.

1.3.13 THEOREM *Let A be a bialgebra. The bialgebra $T(\bar{T}'(A))$ is isomorphic to the opposite bialgebra of B_{C_A}. The bialgebra $S(\bar{S}'(A))$ is isomorphic to the opposite bialgebra of $\bar{B}_{C_A}$.*

Let C be a coaugmented cooperad. Then the collection $C^{>1}$ defined by $C^{>1}(1) = k$ and $C^{>1}(n) = C(n)$ for $n > 1$ is a 1-reduced cooperad with cocomposition induced by cocomposition in C through the surjection $C \longrightarrow C^{>1}$.

1.3.14 COROLLARY *The Pinter Hopf algebra associated to $T(\bar{T}'(A))$ (cf. Brouder and Schmitt [8] for terminology) is isomorphic to the opposite Hopf algebra of $H_{C_A^{>1}}$, and the Pinter Hopf algebra associated to $S(\bar{S}'(A))$ is isomorphic to the opposite Hopf algebra of $\bar{H}_{C_A^{>1}}$.*

1.4 Operads of graphs

The examples treated in the previous section treat Hopf algebras based on well known operads. However, a given Hopf algebra can also lead to a new operad structure. The operads of graphs to which this section is devoted are examples of this phenomenon.

§9 The operad Γ

1.4.1 DEFINITION In this section a labelled graph will be assumed to be a graph η in that sense of Definition 0.3.6 without edges from a vertex to itself, together with a numbering of the vertices (i.e. a bijection $\mathbf{v}(\eta) \longrightarrow \{1, \ldots, |\mathbf{v}(\eta)|\}$). If $k \leq |\mathbf{v}(\eta)|$ denote by $\mathbf{l}_\eta(k)$ the set of legs of the vertex numbered k in the labelled graph η. The restriction to labelled graphs without self-loops is not necessary, at this point, but it catalyses some of the arguments.
Define Graph(n) as the groupoid of labelled graphs η such that $|\mathbf{v}(\eta)| = n$ with isomorphisms of labelled graphs (compatible with the numbering of vertices) as maps, and let $\Gamma(1) = k$ and $\Gamma(n) = \text{colim}(\text{Graph}(n))$ for $n \geq 2$. Then $\Gamma(n)$ has a natural right S_n-action by permutation of the labels of the vertices. Below the structure of an operad on this collection Γ is defined.

Let η and ζ be two labelled graphs and let $k \leq |\mathbf{v}(\eta)|$. Assume that $|\mathbf{l}_\eta(k)| = |\mathbf{l}(\zeta)|$. For each bijection $b : \mathbf{l}_\eta(k) \longrightarrow \mathbf{l}(\zeta)$ define $\eta \circ_b \zeta$ as the labelled graph defined by replacing vertex k in η by the labelled graph ζ, and connecting the legs of ζ to the remaining part of η according to bijection b. The linear ordering of the vertices is obtained from the linear ordering on the vertices of η upon replacing vertex k by the linear order on vertices of ζ. For η, ζ and k as above define the circle-k operation of the operad as

$$\eta \circ_k \zeta = \begin{cases} \sum_{[b]} \eta \circ_b \zeta & \text{if } |\mathbf{l}_\eta(k)| = |\mathbf{l}(\zeta)| \\ 0 & \text{otherwise,} \end{cases} \tag{1.4.5}$$

where the sum is over equivalence classes of bijections $b : \mathbf{l}_\eta(k) \longrightarrow \mathbf{l}(\zeta)$ with respect to the equivalence relation $b \sim b'$ iff $\eta \circ_b \zeta \cong \eta \circ_{b'} \zeta$.

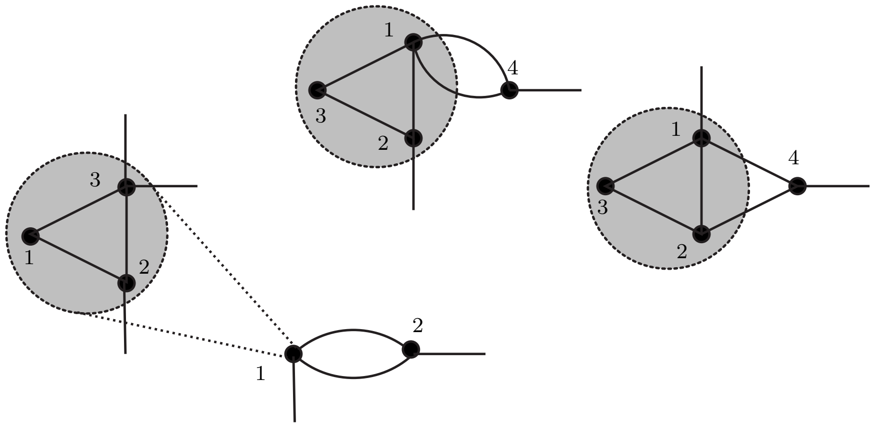


FIGURE 6: The composition $\circ_1$ in the operad Γ. The graph on the left is included in the two-vertex graphs. The result is the sum of the two other graphs.

1.4.2 LEMMA *The $\circ_k$-operations defined above, make the collection Γ an operad*

PROOF Since the identities with respect to the $\circ_k$-operations are trivial it remains to prove associativity. Consider $(\eta \circ_k \zeta) \circ_l \theta$ for η, ζ, θ labelled graphs in Γ. If l is a vertex of η in $\eta \circ_k \zeta$, the labelled graphs ζ and θ are plugged into different vertices of η. Thus we can write

$$\begin{aligned}(\eta \circ_k \zeta) \circ_l \theta &= \sum_{[b],[b']} (\eta \circ_b \zeta) \circ_{b'} \theta \\ &= \sum_{[b],[b']} (\eta \circ_{b'} \theta) \circ_b \zeta,\end{aligned}$$

which is sufficient to prove associativity in this case. Suppose on the contrary that l is a vertex of ζ in $\eta \circ_k \zeta$. Then

$$(\eta \circ_k \zeta) \circ_l \theta = \sum_{[b]} (\eta \circ_b (\sum_{[b']} \zeta \circ_{b'} \theta)) \\ = \sum_{[b],[b']} \eta \circ_b (\zeta \circ_{b'} \theta).$$

Regarding the second equality note that it is clear we can move the sum over $[b']$ out since the automorphisms of labelled graphs have to preserve the numbering of the vertices. This completes the proof. QED

1.4.3 REMARK In fact, Γ is a $\mathbb{N}$-coloured operad in the sense of Chapter 3, where each of the inputs is coloured by the number of legs of the corresponding vertex, and the output is coloured by the number of legs of the total graphs.

§10 The Connes-Kreimer Hopf algebra of graphs

1.4.4 DEFINITION Connes-Kreimer [16] introduces a Hopf algebra of graphs (without numbered vertices) associated to a particular QFT, the φ^3 theory in six dimensions. We define the Hopf algebra $\mathcal{H}_{\varphi^3}$, which is a slightly simplified version.
A *1 particle irreducible graph* η is a connected graph η such that η has at least two external edges, such that each vertex $k \in \mathbf{v}(\eta)$ satisfies $|\mathbf{l}_\eta(k)| \in \{2, 3\}$, and such that the graphs are still connected after removing one edge. This terminology comes from the physics. Let $\mathcal{H}_{\varphi^3}$ as an algebra be the unital free commutative algebra on the vector space spanned by 1 particle irreducible graphs modulo the numbering of vertices $\bar{\eta}$. The coproduct on a generator η is given by

$$\Delta(\bar{\eta}) = \sum_{\bar{\zeta} \subset \bar{\eta}} \bar{\eta}/\bar{\zeta} \otimes \bar{\zeta},$$

where the sum is over subgraphs $\bar{\zeta}$ such that $\bar{\zeta}$ is a disjoint union of generators and if $v \in \mathbf{v}(\bar{\zeta})$, then $\mathbf{l}_{\bar{\zeta}}(v) = \mathbf{l}_\eta(v)$. The disjoint union of generators ζ in the right tensor factor is interpreted as the product of generators. The counit is given by the usual augmentation of the unital free commutative algebra.

1.4.5 REMARK The construction of Γ is extremely flexible. It can be adapted to labelled graphs with coloured edges, labelled graphs with coloured vertices, labelled graphs satisfying certain coherence conditions, connected labelled graphs, planar graphs, etc. There is one example with is of our special interest. Let Γ_{PI} be the suboperad of Γ spanned by the 1 particle irreducible labelled graphs. It is easily checked that this is indeed a suboperad.

1.4.6 THEOREM *The Hopf algebras $\bar{H}(\Gamma_{PI})$ and $\mathcal{H}_{\varphi^3}$ are isomorphic.*

PROOF By the Milnor-Moore Theorem, it suffices to compare the Lie algebras of primitive elements of the dual Hopf algebras. In both cases the Lie algebra of primitive elements is spanned by 1 particle irreducible graphs without numbering of the vertices.

If $\eta \in \Gamma_{PI}(n)$, denote by $\bar{\eta} \in \Gamma(n)_{S_n}$ the graph η obtained by forgetting the numbering of vertices. In $P(\bar{H}(\Gamma_{PI})^*)$, the Lie bracket (cf. Theorem 1.2.12) is given by

$$[\bar{\eta}, \bar{\zeta}] = \sum_{v \in \mathbf{v}(\bar{\eta})} \sum_{[b]} \bar{\eta} \circ_b \bar{\zeta} - \sum_{w \in \mathbf{v}(\bar{\zeta})} \sum_{[b']} \bar{\zeta} \circ_{b'} \bar{\eta}$$

where the sums in the first line are over equivalence classes of bijections $b : \mathbf{l}_{\bar{\eta}}(v) \longrightarrow \mathbf{l}(\bar{\zeta})$ for $v \in \mathbf{v}(\bar{\eta})$ and $b : \mathbf{l}_{\bar{\zeta}}(v) \longrightarrow \mathbf{l}(\bar{\eta})$ for $w \in \mathbf{v}(\bar{\zeta})$. Theorem 2 in Connes-Kreimer [16] states that this Lie algebra is isomorphic to the Lie algebra of primitive elements in $\mathcal{H}_{\varphi^3}$. QED

1.4.7 REMARK A remark on the isomorphism is in place. Connes-Kreimer [16] uses the notation $\bar{\eta} \circ_v \bar{\zeta} := \sum_{v \in \mathbf{v}(\bar{\eta})} \sum_{[b]} \bar{\eta} \circ_b \bar{\zeta}$, but do not state explicitly that they sum over equivalence classes of bijections. The linear space $P(\mathcal{H}^*_{\varphi^3})$ of primitive elements has a natural basis spanned by graphs (dual to the basis of the generators of $\mathcal{H}_{\varphi^3}$ given above).

The Connes-Kreimer isomorphism $P(\mathcal{H}^*_{\varphi^3}) \longrightarrow \bar{L}_{\Gamma_{PI}}$ mentioned in the proof of Theorem 1.4.6 is given by

$$\bar{\eta} \longmapsto S(\bar{\eta}) \cdot \bar{\eta},$$

where $S(\bar{\eta})$ is the symmetry factor of $\bar{\eta}$ which takes the form

$$S(\bar{\eta}) = \frac{|\mathrm{Aut}(\bar{\eta})|}{|\mathrm{Aut}(\eta)|},$$

where $\mathrm{Aut}(\eta)$ is the isomorphism group of η (i.e. automorphisms need to preserve the numbering of vertices), and $\mathrm{Aut}(\bar{\eta})$ is the automorphism group of $\bar{\eta}$ (i.e. the automorphism need not preserve the numbering of the vertices). In other words, $S(\bar{\eta})$ is the cardinality of the automorphism group of $\bar{\eta}$ divided by the subgroup of automorphisms that induce the identity map on the vertices of $\bar{\eta}$. Note that $S(\bar{\eta})$ is indeed independent of the representative η of $\bar{\eta}$.

1.4.8 COROLLARY *The group of characters of $\mathcal{H}_{\varphi^3}$ is isomorphic to $\bar{G}_{\Gamma_{PI}}$. It is the group of formal series of connected 1 particle irreducible graphs $\sum_{\bar{\zeta}} c_{\bar{\zeta}} x^{\bar{\zeta}}$ with $c_\eta \in k$ and coefficient $c_{\bar{\xi}} = 1$ for each 1-vertex graphs $\bar{\xi}$. The composition is the linear extension of*

$$x^{\bar{\eta}} \cdot \sum_{\bar{\zeta}} d_{\bar{\zeta}} x^{\bar{\zeta}} = \sum_{\bar{\zeta}_1,\ldots,\bar{\zeta}_{|v(\bar{\eta})|}} d_{\zeta_1} \cdot \ldots \cdot d_{\zeta_k} x^{\gamma(\bar{\eta};\bar{\zeta}_1,\ldots,\bar{\zeta}_{|v(\bar{\eta})|})},$$

in terms of the composition of graphs in the operad Γ_{PI}, where the one vertex trees are identified with the unit in the algebra.

1.4.9 REMARK In order to reproduce the Hopf algebra originally defined by Connes and Kreimer exactly, one needs to introduce a second colour for each vertex with two legs, and one needs to label external edges by elements of a certain space of distributions.

§**11** Cyclic operads

1.4.10 DEFINITION A *cyclic collection* is a contravariant functor from the category of non-empty finite sets (without base-point) to the category of vector spaces (compare Definition 0.3.1). A *cyclic pseudo operad* is a cyclic collection P together with for finite sets X and Y, and elements $x \in X$ and $y \in Y$ a composition map

$$\circ_{xy} : P(X) \otimes P(Y) \longrightarrow P(X \cup_{x=y} Y),$$

that is equivariant with respect to automorphisms of X and Y, and is associative in the sense of Definition 0.3.1. A *cyclic operad* is a cyclic pseudo operad P together with for a 2 element set $\{x, x'\}$ a two-sided identity that is invariant under the non-trivial automorphism.
Equivalently we can define a cyclic operad as follows. Write $P(n) = P(\{0,\ldots,n\})$ and interpret S_{n+1} as the automorphism group of the set $\{0,\ldots,n\}$, and let $S_n \subset S_{n+1}$ as the automorphisms that leave 0 fixed. Given an S_n-module, a extension of the structure to an S_{n+1}-module is completely determined by the action of $\tau_n = (01\ldots n) \in S_{n+1}$. A cyclic operad P is an operad, such that the S_n-module structure on $P(n)$ extends to an S_{n+1}-module structure. Moreover, these data should satisfy the following axioms

(i). The identity is invariant under τ_1.

(ii). For $p \in P(m)$ and $q \in P(m)$, $\tau_{m+n-1}(q \circ_1 p) = \tau_n(p) \circ_n \tau_m(q)$.

(iii). For each $2 \le i \le m$ and p and q as above, $\tau_{m+n-1}(q \circ_i p) = \tau_m(q) \circ_{i-1} p$.

Note that (iii) is my correction of the original definition by Getzler and Kapranov [27] (cf. Markl, Shnider and Stasheff [67]).

1.4.11 PROPOSITION *The operad structure on Γ can be extended to an cyclic operad structure.*

PROOF For any graph η define a graph without external edges by joining all external edges of η in a new vertex numbered 0. This defines an inclusion of $C\Gamma(n)$ in $\Gamma(n+1)$ and makes $C\Gamma$ a cyclic $A_{\mathbb{N}}$-collection with respect to the cyclic action $\tau = (0, 1 \ldots n)$.
The operations $\eta \circ_i \zeta$ are defined by first forgetting the vertex 0 of ζ and then applying the $\circ_i$ operation from the operad Γ. The compatibility of the cyclic permutation τ with composition follows directly from the definitions above since the cyclic collection $C\Gamma$. QED

§**12** The operad $\tilde{\Gamma}$

1.4.12 DEFINITION The definition of the operad Γ above is not the only natural choice. Let $\tilde{\Gamma}(n) = \Gamma(n)$ as an S_n-module for $n \in \mathbb{N}$. Define $\circ_k$-operations on $\tilde{\Gamma}$ as

$$\eta \circ_k \zeta = \begin{cases} \sum_b \eta \circ_b \zeta & \text{if } |\mathbf{1}_\eta(k)| = |\mathbf{1}(\zeta)| \\ 0 & \text{otherwise,} \end{cases} \quad , \tag{1.4.6}$$

where b runs over all bijections $b : \mathbf{1}(\zeta) \longrightarrow \mathbf{1}_\eta(k)$, instead of equivalence classes (as in Formula (1.4.5)).

1.4.13 PROPOSITION *The structure defined above makes $\tilde{\Gamma}$ an operad isomorphic to Γ.*

PROOF Checking the operad axioms for $\tilde{\Gamma}$ is not difficult, the argument is analogous to the argument for Γ. One can skip this exercise if one gives an isomorphism $\varphi : \Gamma \longrightarrow \tilde{\Gamma}$ compatible with the $\circ_i$-compositions. This will complete the proof of the Proposition.
For a graph η, denote by $\mathrm{Aut}(\eta)$ the automorphisms of η that preserve the numbering of the vertices. Define $\varphi(\eta) = |Aut(\eta)| \cdot \eta$. This is an isomorphism of collections from Γ to $\tilde{\Gamma}$ (in characteristic 0). To see that it commutes with the $\circ_i$-operations, one needs for a bijection $b : \mathbf{1}_\eta(i) \longrightarrow \mathbf{1}(\zeta)$ the equality

$$|\mathrm{Aut}(\eta \circ_b \zeta)| \cdot |\{b' : \mathbf{1}_\eta(i) \longrightarrow \mathbf{1}(\zeta) \text{ s.t. } [b'] = [b]\}| = |\mathrm{Aut}(\eta)| \cdot |\mathrm{Aut}(\zeta)|.$$

It suffices to construct a bijection

$$\psi : \mathrm{Aut}(\eta) \times \mathrm{Aut}(\zeta) \longrightarrow \coprod_{b' \sim b} \mathrm{Iso}(\eta \circ_b \zeta, \eta \circ_{b'} \zeta).$$

Given $(\tau,\sigma) \in: \mathrm{Aut}(\eta) \times \mathrm{Aut}(\zeta)$, construct $\psi(\tau,\sigma)$ as follows. There are natural inclusions of $\zeta - \mathbf{l}(\zeta) \hookrightarrow \eta \circ_{b'} \zeta$ and $\eta - \mathbf{l}_\eta(i) \hookrightarrow \eta \circ_{b'} \zeta$ for every b'. Let $b' = \sigma_{\mathbf{l}(\zeta)} \circ b$, and define $\psi(\tau,\sigma) : \eta \circ_b \zeta \longrightarrow \eta \circ_{b'} \zeta$ by $\psi(\tau,\sigma)|_{\zeta - \mathbf{l}(\zeta)} := \sigma|_{\zeta-\mathbf{l}(\zeta)}$ and $\psi(\tau,\sigma)|_{\eta-\mathbf{l}(i)} = \tau|_{\eta-\mathbf{l}(i)}$ on legs that are not glued. Denote the leg obtained from glueing $l \in \mathbf{l}_\eta(i)$ to $b(l) \in \mathbf{l}(\zeta)$ by $\{l, b(l)\}$. On such legs define $\psi(\tau,\sigma)(\{l,b(l)\}) := \{\tau(l), b'(\tau(l)\}$. It remains to see that ψ is a bijection.

If $\psi(\tau,\sigma) = \psi(\xi,\chi)$, then obviously $\tau|_{\eta-\mathbf{l}(i)} = \xi|_{\eta-\mathbf{l}(i)}$, and $\sigma|_{\zeta-\mathbf{l}(\zeta)} = \chi|_{\zeta-\mathbf{l}(\zeta)}$. Moreover, $\sigma|_{\mathbf{l}(\zeta)} = b' \circ b^{-1} = \chi|_{\mathbf{l}(\zeta)}$, and since $\{\tau(l), b'(\tau(l))\} = \{\xi(l), b'(\xi(l))\}$ it follows that ψ is an injection.

Given $\theta : \eta \circ_b \zeta \longrightarrow \eta \circ_{b'} \zeta$, restriction to $\eta - \mathbf{l}_\eta(i)$ and $\zeta - \mathbf{l}(\zeta)$ determine $(\tau,\sigma) \in \mathrm{Aut}(\eta)\times\mathrm{Aut}(\zeta)$ except for glued legs. On these, let $\sigma_{\mathbf{l}(\zeta)} = b' \circ b^{-1}$, and if $\theta(\{l,b(l)\}) = \{l', b'(l')\}$ define $\tau(l) = l'$. Then $\psi(\tau,\sigma) = \theta$ shows ψ is surjective. QED

§13 The Wick algebra

1.4.14 DEFINITION Let V be a finite dimensional vector space together with a symmetric quadratic form $b \in (V \otimes V)^*$. Let η be a graph with n vertices. For $i = 1, \ldots, n$ denote $k_i = |\mathbf{l}_\eta(i)|$, the number of legs of vertex i, and write $S'^\eta V = \bigotimes_{i=1}^n S'^{k_i} V$. For an edge $e \in \mathbf{e}(\eta)$ with vertices $j_1, j_2 \in \mathbf{v}(\eta)$ let $\eta - e$ denote the graph with edge e omitted, and define

$$b(e) : S'^\eta \longrightarrow S'^{(\eta - e)}$$

by applying b to one tensor factor in $S'^{k_{j_1}} V$ and one tensor factor in $S'^{k_{j_2}} V$. If $e' \neq e$ is an other edge of η, then $e' \in \mathbf{e}(\eta - e)$ implies that $b(e') \circ b(e) : S'^\eta \longrightarrow S'^{(\eta - e - e')}$ is well defined and independent of the order of e and e'.

Let η be a graph with n vertices and edges $\mathbf{e}(\eta) = \{e_1, \ldots, e_m\}$. For $i = 1, \ldots, n$ denote $k_i = |\mathbf{l}_\eta(i)|$, the number of legs of vertex i, and $l_i = |\mathbf{e}_\eta(i)|$ the number of legs of vertex i that are part of an edge. Define $\tau^\eta : S'^\eta \longrightarrow S'^{|\mathbf{l}(\eta|)}$ as

$$\tau^\eta = \sum_{\sigma \in S_{|\mathbf{l}(\eta)|}} \sigma \circ b(e_m) \circ \ldots \circ b(e_1).$$

That is, first we contract tensor factors corresponding to all edges, and on the result in $\bigotimes_{i=1}^n S'^{l_i} V$ we apply the sum of all permutations to symmetrise the remaining tensor factors, which means that we end up in $S'^{(l_1 + \cdots + l_n)} V$. The result is again independent of the order of the edges. Extend τ^η as zero to a map

$$\tau^\eta : (S'V)^{\otimes n} \longrightarrow S'V.$$

1.4.15 THEOREM *Let V be a vector space with a quadratic form $b \in (V \otimes V)^*$. The symmetric coalgebra $S'V$ enjoys a Γ-algebra structure with respect to the maps*

$$\gamma(\eta, p_1, \ldots, p_{|v(\eta)|}) = \frac{1}{|\mathrm{Aut}(\eta)|}\tau^\eta(p_1, \ldots, p_{|v(\eta)|}).$$

PROOF It suffices to show that $\tilde{\gamma}(\eta, p_1, \ldots, p_n) = \tau^\eta(p_1, \ldots, p_n)$ defines a $\tilde{\Gamma}$-algebra structure. Since compatibility with the symmetric group action is obvious, it remains to check that $\tau^\eta \circ_1 \tau^\zeta = \tau^{\eta \circ_1 \zeta}$. Let $\mathbf{e}(\eta) = \{e_1, \ldots, e_n\}$ be the set of edges of η and let $\mathbf{e}(\zeta) = \{e_{n+1}, \ldots, e_{n+m}\}$ be the set of edges of ζ. For any bijection $b : \mathbf{l}(\zeta) \longrightarrow \mathbf{l}_\eta(1)$ from the external legs of ζ to the legs of vertex i in η we can write canonically $\mathbf{e}(\eta \circ_b \zeta) = \{e_1, \ldots, e_{n+m}\}$. Then

$$\tau^{\eta \circ_1 \zeta} = \sum_{\sigma \in S_{|\mathbf{l}(\eta)|}} \sum_{b : \mathbf{l}(\zeta) \longrightarrow \mathbf{l}_\eta(1)} \sigma \circ b(e_{n+m}) \circ \ldots \circ b(e_1),$$

and

$$\tau^\eta \circ_1 \tau^\zeta = \sum_{\sigma \in S_{|\mathbf{l}(\eta)|}} \sum_{\tau \in S_{|\mathbf{l}(\zeta)|}} \sigma \circ b(e_n) \circ \ldots b(e_1) \circ \tau \circ b(e_{n+m}) \circ \ldots \circ b(e_{n+1}).$$

It is not hard to see that summing over the $|\mathbf{l}(\zeta)|!$ possibilities for τ and summing over the $|\mathbf{l}(\zeta)|!$ possibilities for b have the same effect. QED

1.4.16 REMARK Theorem 1.4.15 is suggested by the Wick rotation formula from quantum field theory, more precisely, the asymptotic series for the functional integral of a QFT in a neighbourhood of a free QFT (cf. Kazhdan [39]).

CHAPTER 2

Hopf operads and trees

This Chapter shows how Hopf operads can be used to construct many Hopf algebras of trees. Some of these Hopf algebras have interesting additional properties such as a pre-Lie algebra structure on the primitive elements of the dual. In special cases we recover previously studied Hopf algebras of trees. This chapter is based on [81].

2.1 Introduction

In Mathematics and Physics several Hopf algebras described in terms of trees have been studied. For example, Loday-Ronco [60] describes a Hopf algebra of planar binary trees that is of interest in Combinatorics. More in particular, to non-commutative versions of quasi symmetric functions. Kreimer [48] defined a Hopf algebra of rooted trees that proved useful in renormalisation (cf. Chapter 1 for the definition). The antipode of the Hopf algebra provides counter terms when computing Feynman integrals in perturbative quantum field theory. Connes and Kreimer [15] showed that this Hopf algebra is also related to non-commutative geometry. These results fit in a wider interest in Hopf algebras based on combinatorial objects (cf. for example Chapoton [12]).
Chapter 2 studies Hopf algebras of this type. The Hopf algebras are constructed by a generalisation of the operadic methods in Moerdijk [71]. Hopf algebras generated by (planar) rooted trees with coloured edges get special attention. The chapter shows that the coproducts of these Hopf algebras can be described by a closed formula based on the geometry of the trees. Moreover, it is possible to characterise the Lie algebra of primitive elements in terms of trees. Some of these Lie algebras

are in fact associated to pre-Lie algebras.
The construction is based on the notion of a Hopf operad and a Hopf P-algebra (Getzler-Jones [26]). We construct a free extension of such a Hopf operad P, and give some conditions that assure a Hopf P-algebras structure on the initial algebra of this extended operad. This approach is an extension of Moerdijk [71].
Section 2.2 gives the theoretical results on Hopf operad, Hopf P-algebras and extensions. If we restrict to the Hopf algebra Com_* of unitary commutative algebras, Hopf P-algebras are commutative bialgebras. This approach shows that there exists a special family of Hopf algebra structures on the symmetric algebra on rooted trees with coloured edges. Section 2.3 shows that the coproducts in this family have a description in terms of the geometry of the trees. Initially, these coproducts where defined inductively. It is shown which of the coproducts in this family induce a pre-Lie algebra structure. In addition, it is shown that there is a non-commutative analog of the construction, which uses planar trees. It is well known that the primitive elements of the graded dual of a Hopf algebra H form a Lie algebra. If the Hopf algebra is commutative, the dual Hopf algebra H^* is the universal enveloping algebra of its Lie algebra of primitive elements. We compute the Lie algebra of primitive elements for the family of Hopf algebras we constructed.
Section 2.4 shows that Section 2.2 and the explicit formulae in Section 2.3 enable the construction of some Hopf algebras on trees that have been studied in the literature. In addition to the construction of the Connes-Kreimer Hopf algebra of rooted trees and its non-commutative analogue (cf. Moerdijk [71]), the more general approach yields other examples of Hopf algebras known in the mathematical literature (cf. Loday-Ronco [60], and Brouder-Frabetti [7]).

2.2 Hopf operads and their Hopf algebras

This section aims to introduce Hopf operads and Hopf P-algebras, and to discuss free extension of Hopf operads. Extensions can be used to construct many Hopf P-algebras.

§1 Hopf operads

A *pointed operad* is an operad P (in k-Vect) together with a base point $u : k \longrightarrow P(0)$. Morphisms of pointed operads preserve base points. Throughout this chapter we will restrict to pointed operads. The reason for this is that we prefer the initial

algebras are non-empty (which is true if $u \neq 0$).
The operads Ass and Com of associative and commutative algebras can be made into pointed operads Ass_* and Com_*, redefining $\mathrm{Com}_*(0) = \mathrm{Ass}_*(0) = k$ and $u = \mathrm{id} : k \longrightarrow k$. Algebras over Ass_* (resp. Com_*) are unital associative (resp. commutative) algebras.
A pointed vector space is a vector space V together with a linear map $u : k \longrightarrow V$. The endomorphism operad End_V of a pointed vector space V has a natural structure of a pointed operad. We only use $u : k \longrightarrow V$ to define the base-point if one defines $\mathrm{End}_V(0) := V$. The endomorphisms need not preserve base points.

2.2.1 DEFINITION A *Hopf operad* is a pointed operad in the category of counital coalgebras such that $\varepsilon \circ u = \mathrm{id}$. Here ε the counit of the coalgebra $P(0)$. A Hopf operad P is a (k-linear) operad together with for each $n \in \mathbb{N}$ morphisms

$$k \xleftarrow{\varepsilon} P(n) \xrightarrow{\Delta} P(n) \otimes P(n),$$

such that Δ is S_n-equivariant, and Δ and ε satisfy the usual axioms that make $P(n)$ a coalgebra in k-Vect (cf. Definition 0.2.2). The action of S_n on $P(n) \otimes P(n)$ is defined through the coproduct on the group ring kS_n (induced by the diagonal on S_n). Using Sweedler's notation, we can write for $p_i \in P(n_i)$ for $i = 0 \ldots m$ as

$$\Delta(\gamma(p_0; p_1, \ldots, p_n)) = \sum_{(p_0,\ldots,p_n)} \gamma(p_0'; p_1', \ldots p_n') \otimes \gamma(p_0''; p_1'', \ldots p_n'').$$

For a (dg) vector spaces V and W let $\tau : (V \otimes W)^{\otimes n} \longrightarrow V^{\otimes n} \otimes W^{\otimes n}$ be the map that separates the odd and the even tensor factors. Let A and B be algebras over a Hopf operad P. The tensor product $A \otimes B$ has a natural structure of a P-algebra by

$$\gamma_{A\otimes B} = (\gamma_A \otimes \gamma_B) \circ \tau \circ (\Delta(p) \otimes \mathrm{id}^{\otimes 2n}) : P(n) \otimes A^{\otimes n} \otimes B^{\otimes n} \longrightarrow A \otimes B.$$

2.2.2 EXAMPLE The operads Com_* and Ass_* in k-Vect are Hopf operads With respect to the usual bialgebra structure on the group algebra ($\Delta(\sigma) = \sigma \otimes \sigma$ for $\sigma \in S_n$). Both coproduct and counit on $\mathrm{Com}_*(n) = k$ are the identity.
The initial pointed operad k satisfies $\mathrm{k}(0) = k = \mathrm{k}(1)$ and $\mathrm{k}(n) = 0$ for $n > 1$. The map $u : k \longrightarrow k$ is the identity. This is a Hopf operad with respect to the obvious structure maps.

§2 Free extensions

2.2.3 DEFINITION A *free extension* of a pointed operad is the coproduct (in the category of operads) of a pointed operad with a free pointed operad on a pointed collection. If P is a pointed operad and C is a collection, we use notation $P[C]$ for the push-out of P and the free pointed operad T_*C over the initial pointed operad k (cf. Example 2.2.2)

$$\begin{array}{ccc} \mathrm{k} & \longrightarrow & P \\ \downarrow & & \downarrow \\ T_*C & \longrightarrow & P[C]. \end{array}$$

Denote λ_n the collection given by $\lambda_n(n) := kS_n$ and $\lambda_n(m) = 0$ if $m \neq n$, then consequently we denote by $P[\lambda_n]$ the operad P with one free n-ary operation adjoint. Let C be a collection. Obviously, an algebra over $P[C]$ is a P-algebra with more structure. For example, a $P[\lambda_n]$-algebra is a pair (A, α) of a P-algebra A together with a linear map $\alpha : A^{\otimes n} \longrightarrow A$.

An *n-coloured tree* is a rooted tree with the root as only external edge, together with a colouring of the internal edges by $\{1, \ldots, n\}$. denote the set of n-coloured trees by T_n.
An *n-coloured planar tree* is a planar rooted tree with the root as only output together with a labeling of the internal edges by $\{1, \ldots, n\}$ such that the linear order of the incoming edges at each vertex extends the partial ordering given by the colouring. An n-coloured forest is a linearly ordered set of n-coloured trees. Denote the set of planar n-coloured trees by P_n.
Define a *planar n-ary tree* as a non-empty oriented graph in which each vertex which is not a leaf has exactly n direct predecessors and one direct successor. Denote the vector space spanned by the set of n-ary planar trees by $k[Y_\infty^{(n)}]$.

2.2.4 PROPOSITION *Consider the cases $P = \mathrm{Com}$ and $P = \mathrm{Ass}$.*

(i). The initial algebra $C_n = \mathrm{Com}_*[\lambda_n](0)$ is isomorphic to the free commutative algebra on n-coloured trees.

(ii). The initial algebra $A_n = Ass_*[\lambda_n](0)$ is isomorphic to the free associative algebra on planar n-coloured trees.

(iii). The vector space of n-ary planar trees is isomorphic to $k[\lambda_n](0)$.

PROOF The symmetric algebra $S(T_n)$ on n-coloured trees can be identified with the vector space spanned by n-coloured forests, with the uit represented by the empty forest. There is a natural n-ary operation $\lambda : S(T_n)^{\otimes n} \longrightarrow S(T_n)$ which takes an n-tuple $f_1, \ldots, f_n$ of n-coloured forests and froms an n-coloured tree by adding a new root to the disjoint union of the forests, attaching the roots in the forest f_i to the new root by an edge of colour i. To see that $(S(T_n), \lambda)$ is the initial commutative algebra together with an n-ary operation, let (A, α) be such an algebra. Define a morphism

$$\varphi : (S(T_n), \lambda) \longrightarrow (A, \alpha)$$

by induction. If $f = t_1 \cdot \ldots \cdot t_k$ a forest consisting of k trees, then $\varphi(f) = \varphi(t_1) \cdot \ldots \cdot \varphi(t_k)$. Thus it remains to define φ on trees. Define $\varphi(\bullet) := 1$, if $\bullet$ is the 1-vertex tree. Inductively, let

$$\varphi(\lambda(f_1, \ldots, f_n)) := \alpha(\varphi(f_1)), \ldots, \varphi(f_n))$$

for the tree $\lambda(f_1, \ldots, f_n)$. Obviously, φ is a morphism compatible with n-ary maps, and it is easy to see that φ is unique with this property. This proves part *(i)*.
This proof of part *(ii)* is almost identical, replacing $S(T_n)$ by $T(P_n)$, and $T(P_n)$ is identified spanned by the set of n-coloured forests. There is a natural operation $\lambda : T(P_n)^{\otimes n} \longrightarrow T(P_n)$, which takes forests $f_1, \ldots, f_n$ and constructs a tree from these by adding a new root, and attaching the roots in f_i to the new root by edges of colour i. The linear order on edges of colour i coming into the root is induced by the linear order of trees in the forests and the linear order of the colours. Given an associative algebra together with an n-ary map (A, α), we can construct again a morphism $\varphi : (T(P_n), \lambda) \longrightarrow (A, \alpha)$. Again it suffices to define this on trees, where it is given by the same formula $\varphi(\lambda(f_1, \ldots, f_n)) := \alpha(\varphi(f_1)), \ldots, \varphi(f_n))$. Again uniqueness is clear.
Finally, to prove *(iii)*, note that $k[Y_n]$ is isomorphic to the subspace of $T(P_n)$ spanned by non-empty trees that can be constructed by induction steps that do not involve multiplication, i.e. it is spanned by trees where each input of a vertex has a different colour. QED

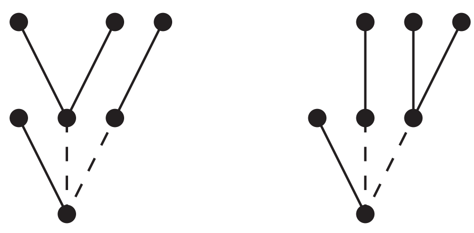


FIGURE 7: Trees not equal in A_2, but equal in C_2. Permutation of the two edges coloured 2 that are attached to the root replaces μ by μ^{op}, which is the same in Com_*.

2.2.5 EXAMPLE As a special case of *(iii)*, the isomorphism between $\mathrm{k}[\lambda_2](0)$ and the vector space spanned by binary trees $k[Y_\infty^{(2)}]$ is given by identifying a binary tree in

$k[Y_\infty^{(2)}]$ with the tree in $\mathrm{k}[\lambda_2](0)$ consisting of the vertices that are not leaves and the edges connecting those. In figure 8 below edges of colour 1 are indicated by solid lines and edges of colour 2 by dotted lines. For clarity the edges in the binary tree (to the left) are dotted as well.

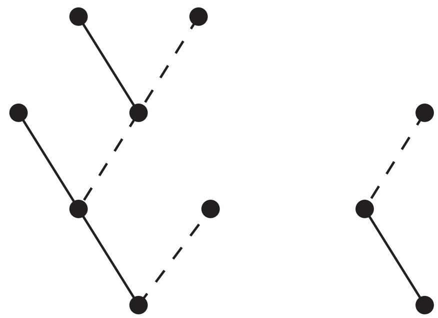


FIGURE 8: Corresponding trees in $k[Y_\infty^{(2)}]$ (space of planar binary trees) and $\mathrm{k}[\lambda_2](0)$ (initial algebra of the free extension of k).

§3 Hopf P-algebras

2.2.6 DEFINITION Let P be a Hopf operad. A *Hopf P-algebra A* is a P-algebra A in the category of counital coalgebras. (We can not use the description of algebras in terms of the endomorphism operad in the category of coalgebras, since coalgebra homomorphisms do not form a linear space.) A Hopf k-algebra is a coaugmented coalgebra. A Hopf Ass_*-algebra is just a bialgebra, A Hopf Com-algebra is a commutative bialgebra. The initial algebra $P(0)$ of a Hopf operad P is a Hopf P-algebra, it is the initial object in the category of Hopf P-algebras. The term 'Hopf' might thus be somewhat misleading. In general, it is not even clear what an antipode should be in this context.
Let $\varphi : Q \longrightarrow P$ be a map of Hopf operads. The map φ induces functors

$$\varphi^* : P\text{-Alg} \longrightarrow Q\text{-Alg} \qquad \text{and} \qquad \bar{\varphi}^* : P\text{-HopfAlg} \longrightarrow Q\text{-HopfAlg}.$$

The map φ^* has a left adjoint $\varphi_!$ (we work with k-vector spaces). Note that $\varphi^*(I) = I$ and for P-algebras A and B we have $\varphi^*(A \otimes B) = \varphi^*(A) \otimes \varphi^*(B)$ (the map φ is compatible with Δ). Using this observation, conclude that the adjunction induces algebra maps $\varphi_!(I) = \varphi_!\varphi_*(I) \longrightarrow I$ and $\varphi_!(A \otimes B) \longrightarrow \varphi_!(\varphi^*\varphi_! A \otimes \varphi^*\varphi_! B) \longrightarrow \varphi_!(A) \otimes \varphi_!(B)$. Consequently, $\varphi_!$ lifts to a left adjoint of $\bar{\varphi}^*$

$$\bar{\varphi}_! : Q\text{-HopfAlg} \longrightarrow P\text{-HopfAlg}.$$

Let P be a pointed operad. The *pointed free P-algebra functor* assigns to a pointed vector space (V, u) the quotient of the free P-algebra $F_P V$ on V modulo the ideal generated by the condition that the unique P-algebra map $P(0) \longrightarrow V$ preserves base-points.

For any Hopf operad P, the inclusion $i : \mathrm{k} \longrightarrow P$ is a map of Hopf operads. The functor $i_* : P\text{-}\mathsf{Alg} \longrightarrow k\text{-}\mathsf{Vect}$ is the forgetful functor, and $i_! : k\text{-}\mathsf{Vect} \longrightarrow P\text{-}\mathsf{Alg}$ is the pointed free P-algebra functor.

§4 Free extensions and Hopf P-algebras

The results in this section are obtained along the same lines as the results in Moerdijk [71]. Moerdijk considered only the case $n = 1$ of Theorem 2.2.7.

Let P be an operad, and let (A, α) be a $P[\lambda_n]$-algebra. For any two n-ary maps $\sigma_1, \sigma_2 : A^{\otimes n} \longrightarrow A$, define the linear map $(\sigma_1, \sigma_2) : (A \otimes A)^{\otimes n} \longrightarrow A \otimes A$ as

$$(\sigma_1, \sigma_2) = (\sigma_1 \otimes \alpha + \alpha \otimes \sigma_2) \circ \tau.$$

This section aims to study in what cases we can define Hopf algebra structures on $P_n = P[\lambda_n](0)$ using this kind of maps. Let (P_n, λ_n) be the initial $P[\lambda_n]$-algebra. For any pair of n-ary linear maps σ_1, σ_2 there is a unique $P[\lambda_n]$-algebra morphism $\Delta : (P_n, \lambda_n) \longrightarrow (P_n \otimes P_n, (\sigma_1, \sigma_2))$. That is, a unique P-algebra morphism Δ such that the diagram

$$\begin{array}{ccc} P_n^{\otimes n} & \xrightarrow{\lambda} & P_n \\ \downarrow{\scriptstyle \Delta^{\otimes n}} & & \downarrow{\scriptstyle \Delta} \\ (P_n \otimes P_n)^{\otimes n} & \xrightarrow{(\sigma_1, \sigma_2)} & P_n \otimes P_n \end{array}$$

commutes. Define $\varepsilon : P_n \longrightarrow k$ as the unique $P[\lambda]$-algebra morphism $(P_n, \lambda_n) \longrightarrow (k, 0)$, which extends $\varepsilon : P(0) \longrightarrow k$. The following Theorem is a generalisation of Moerdijk's construction [71].

2.2.7 THEOREM *Let $n \in \mathbb{N}$, λ_n and P_n be defined as above. Let $\sigma_i : P_n^{\otimes n} \longrightarrow P_n$ for $i = 1, 2$ be linear maps. If σ_i satisfies*

$$\epsilon \circ \sigma_i = \epsilon^{\otimes n}, \quad \textit{and}$$
$$(\sigma_i \otimes \sigma_i) \circ \tau \circ \Delta^{\otimes n} = \Delta \circ \sigma_i;$$

then there exists a unique Hopf-P algebra structure on P_n extending the Hopf P-algebra structure on $P(0)$ such that $\Delta \circ \lambda = (\sigma_1, \sigma_2) \circ \tau \circ \Delta^{\otimes n}$ and $\varepsilon \circ \lambda = 0$.

PROOF To verify the counit property, we need to check

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id}_{P_n} = (\varepsilon \otimes \mathrm{id}) \circ \Delta$$

as a map of P-algebras. We check that the equality is in fact an equality of maps of $P[\lambda]$-algebras. Since the equality is obvious on $P(0)$, the universal property of the pair $(P[\lambda_n](0), \lambda_n)$ implies that it suffices to check the equalities after pre-composition with λ

$$\begin{aligned}(\varepsilon \otimes \mathrm{id}) \circ \Delta \circ \lambda &= ((\varepsilon \circ \sigma_1) \otimes \mathrm{id} + (\varepsilon \circ \lambda) \otimes \sigma_2) \circ \tau \circ \Delta^{\otimes n} \\ &= (\varepsilon^{\otimes n} \otimes \lambda + 0) \circ \tau \circ \Delta^{\otimes n} \\ &= \lambda \circ (\varepsilon \otimes \mathrm{id}) \circ \Delta)^{\otimes n} \\ &= \lambda.\end{aligned}$$

A similar argument implies $(\mathrm{id} \otimes \varepsilon) \circ \Delta \circ \lambda = \lambda$. This proves the counit axiom for ε. Regarding coassociativity we can argue along the same lines: The equality $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ is an equality of maps of $P[\lambda]$-algebras. Coassociativity in guaranteed on $P(0)$. By the universal property it suffices to check the commutation relation with λ:

$$\begin{aligned}(\Delta \otimes \mathrm{id}) \circ \Delta \circ \lambda &= (\Delta \circ \sigma_1) \otimes \lambda + (\Delta \circ \lambda) \otimes \sigma_2 \circ \tau \circ \Delta^{\otimes n} \\ &= (\sigma_1^{\otimes 2} \otimes \lambda + \sigma_1 \otimes \lambda \otimes \sigma_2 + \lambda \otimes \sigma_2^{\otimes 2}) \circ \tau \circ ((\Delta \otimes \mathrm{id}) \circ \Delta)^{\otimes n}.\end{aligned}$$

Similarly,

$$(\mathrm{id} \otimes \Delta) \circ \Delta \circ \lambda = (\sigma_1^{\otimes 2} \otimes \lambda + \sigma_1 \otimes \lambda \otimes \sigma_2 + \lambda \otimes \sigma_2^{\otimes 2}) \circ \tau \circ ((\Delta \otimes \mathrm{id}) \circ \Delta)^{\otimes n}.$$

Therefore both compositions agree. This proves the theorem. QED

2.3 Hopf algebras of trees

This section is devoted to a special case of the last section. A specific family of coproducts on the symmetric algebra on n-coloured trees C_n is studied. Explicit formulae for a priori inductively defined coproducts are given. We give the antipode for these coproducts and the Lie bracket on the primitive elements of the dual. Specific elements of this family of coproducts yield a pre-Lie algebra.

§5 A family of Hopf algebras

2.3.1 EXAMPLE We start with an example of the construction in Theorem 2.2.7. Let P be a Hopf operad with multiplication. That is, and Hopf operad together with a morphism of Hopf operads $\mathsf{Ass}_* \longrightarrow P$. we denote the associative multiplication induced by the map by $\cdot$. For any choice of $q_{i\,j} \in k$ for all $j \leq n$, the maps

$$\sigma_i(t_1, \dots, t_n) = q_{i\,1}^{|t_1|} \cdot \dots \cdot q_{i\,n}^{|t_n|} t_1 \cdot \dots \cdot t_n \qquad \text{for } i = 1, 2$$

define a Hopf P-algebra structure on P_n. Here and in the sequel $|t|$ denotes the number of applications of λ_n in an element $t \in P_n = P[\lambda_n](0)$.

2.3.2 THEOREM *For the operads Com$_*$ and Ass$_*$, Example 2.3.1 above takes the following form:*

(i). The symmetric algebra C_n on n-coloured trees has a natural family of graded connected Hopf algebra structures, indexed by two sequences $(q_{11}, \dots, q_{1n}) \in k^n$, and $(q_{21}, \dots, q_{2n}) \in k^n$. The grading is with respect to the number of vertices of the trees.

(ii). The tensor algebra A_n on planar n-coloured trees has a natural family of graded connected Hopf algebra structures, indexed by two sequences $(q_{11}, \dots, q_{1n}) \in k^n$, and $(q_{21}, \dots, q_{2n}) \in k^n$.

In both cases the coproduct satisfies

$$\begin{aligned}\Delta(\lambda(t_1, \dots, t_n)) = & \sum q_{11}^{|t'_1|} \cdot \dots \cdot q_{1n}^{|t'_n|} \cdot t'_1 \cdot \dots \cdot t'_n \otimes \lambda(t''_1, \dots, t''_n) \\ & + \sum \lambda(t'_1, \dots, t'_n) \otimes q_{21}^{|t''_1|} \cdot \dots \cdot q_{2n}^{|t''_n|} \cdot t''_1 \cdot \dots \cdot t''_n,\end{aligned}$$

where $\lambda(t_1, \dots, t_n)$ is the (planar) rooted tree obtained for n products of (planar) rooted trees $t_1, \dots, t_n$ by adding a new root and connecting each of the roots of trees in t_i to the new root by an edge of colour i, and where $|t_i|$ is the number of vertices of the trees constituting t_i.

PROOF The bialgebra structures are a direct translation of the special cases of Example 2.3.1 to the more usual terminology. We use Example 2.2.4 to make the identification of the initial algebras with (planar) n-coloured trees.
The bialgebras C_n and A_n are graded connected with respect to the grading $|.|$. It is well known (cf. Milnor and Moore [70]) that any graded connected bialgebra admits an antipode. QED

§6 Hopf algebras of rooted trees with coloured edges

Below we make a more detailed study of the Hopf algebra structures on the n-coloured trees C_n described by Theorem 2.3.2. We fix $n \in \mathbb{N}$ throughout this section.

For $i = 1, 2$ and $1 \leq j \leq n$, let $q_{ij} \in k$, and define for n-coloured trees $t_1, \ldots, t_n$

$$\sigma_i(t_1, \ldots t_n) = t_1 \cdot \ldots \cdot t_n \cdot \prod_j q_{ij}^{|t_j|}. \tag{2.3.1}$$

Any rooted tree has a natural partial ordering on its vertices in which the root is the minimal element. A *subforest* s of a rooted tree t is a subset of the partially ordered set $\mathbf{v}(t)$ with the induced partial ordering. For coloured trees, the colour of the edge connecting $v > w$ in s is the colour of edge connecting w to its direct predecessor in the unique path from v to w in t. For $v \in \mathbf{v}(s)$ we denote by $p_k(v, s, t)$ the number of edges of colour k in the path in t from v to the root of t that have their lower vertex in s^c. For forests t we define $p_k(v, s, t)$ as $p_k(v, s \cap t', t')$, where t' is the connected component of t containing v. There is an easy but useful lemma on the calculus of the p_k.

2.3.3 LEMMA *Let t and s be subforests of a forest u. Let $v \in \boldsymbol{v}(s)$ and set $t' = t \cup v$, $s' = s \cap t'$, $t'' = t^c \cup v$ and $s'' = s \cap t''$. Then*

$$p_k(v, s, u) = p_k(v, s', t') + p_k(v, s'', t''),$$

where t', t'', s' and s'' are interpreted as subforests of u.

PROOF The lemma follows at once when we observe that a vertex in the path from v to the root in u that is not in s is either in t' or in t''. QED

Define for $s \subset t$ a subforest

$$q(s, t) := \prod_j q_{1\,j}^{\sum_{v \in s} p_j(v, s, t)} \cdot \prod_j q_{2\,j}^{\sum_{v \in s^c} p_j(v, s^c, t)}. \tag{2.3.2}$$

More intuitively, $q(s, t)$ counts for $v \in \mathbf{v}(s)$ the number of edges of colour j are in the path from v to the root that have their lower vertex in s^c and adds a factor q_{1j} for each of these, and $q(s, t)$ counts for $v \in \mathbf{v}(s^c)$ the number of edges of colour j are in the path from v to the root that have their lower vertex in s and adds a factor $q_{2,j}$ for each of these.

2.3.4 THEOREM *Let C_n be the symmetric algebra on n-coloured trees as in Theorem 2.3.2.*

(i). For a forest $t \in C_n$ the coproduct defined by $(q_{11}, \ldots, q_{2n}) \in k^{2n}$ is given by the formula

$$\Delta(t) = \sum_{s \subset t} q(s, t)\, s \otimes s^c,$$

where the sum is over all subforests s of t.

(ii). The antipode on Hopf algebra C_n with the coproduct of part (i) is given by

$$S(t) = \sum_{k=1}^{|t|} \sum_{\cup_i s_i = t} (-1)^k s_1 \cdot \ldots \cdot s_k \prod_{1 \le j < k} q(s_j, s_{j+1} \cup \ldots \cup s_k, s_j \cup \ldots \cup s_k),$$

where we only sum over partitions $t = s_1 \cup \ldots \cup s_k$ of the forest t with all forests s_i non-empty.

PROOF We use induction with respect to the number of applications of λ to show the first result. The formula is trivial for the empty tree. Let $t = \lambda(x_1, \ldots, x_n)$ be a tree and suppose (as the induction hypothesis) that the formula holds for all forests with less then $|t|$ vertices. Subforests of t are either of the form $s = \cup_i s_i$, a (disjoint) union of subforests of the x_i, or of the form $s = r \cup (\cup_i s_i)$, a (disjoint) union of subforests of the x_i together with the root. By definition,

$$\begin{aligned} \Delta(t) = & \sum_{s_i \subset x_i} s_1 \cdot \ldots \cdot s_n \otimes \lambda(s_1^c, \ldots, s_n^c) \cdot \prod_i q_{1i}^{|s_i|} q(s_i, x_i) \\ & + \sum_{s_i \subset x_i} \lambda(s_1, \ldots, s_n) \otimes s_1^c \cdot \ldots \cdot s_n^c \cdot \prod_i q_{2i}^{|s_i^c|} q(s_i, x_i). \end{aligned}$$

But by the lemma above,

$$\prod q_{1i}^{|s_i|} q(s_i, x_i) = \prod_j q_{1\,j}^{\sum_{v \in s} p_j(v,s,t)} \cdot \prod_j q_{2\,j}^{\sum_{v \in s^c} p_j(v,s^c,t)}$$

for $s = \cup_i s_i = s_1 \cdot \ldots \cdot s_n$ and $s^c = r \cup (\cup_i s_i^c) = \lambda(s_1^c, \ldots, s_n^c)$; and

$$\prod q_{2i}^{|s_i^c|} q(s_i, x_i) = \prod_j q_{1\,j}^{\sum_{v \in s} p_j(v,s,t)} \cdot \prod_j q_{2\,j}^{\sum_{v \in s^c} p_j(v,s^c,t)}$$

for $s = r \cup (\cup_i s_i) = \lambda(s_1, \ldots, s_n)$ and $s^c = \cup_i s_i = s_1^c \cdot \ldots \cdot s_n^c$. Putting these together proves the formula for the coproduct.

Let A be a graded connected bialgebra. The augmentation ideal of A is $\bar{A} = \bigoplus_{n \ge 1} A^n$ The antipode on A applied to $x \in \bar{A}$ is given by

$$S(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \mu^{(k)} \circ \bar{\Delta}^{(k)}(x),$$

where $\bar{\Delta} = \Delta - (\mathrm{id} \otimes 1 + 1 \otimes \mathrm{id})$, and $\mu^{(0)} = \mathrm{id} = \bar{\Delta}^{(0)}$, and $\mu^{(k)} : A^{\otimes k+1} \longrightarrow A$ and $\bar{\Delta}^{(k)} : A \longrightarrow A^{\otimes k+1}$ are defined using (co)associativity for $k > 0$ and $\mu^{(0)} = \mathrm{id} = \bar{\Delta}^{(0)}$. As a special case, consider C_n with the coproduct just described. This proves the result. QED

§7 The dual Hopf algebras

Since C_n is commutative, we know by the Milnor-Moore Theorem [70] that the linear dual C_n^* is the universal enveloping algebra of the Lie algebra of its primitive elements. The result below provides an explicit formula for the Lie bracket on the these primitive elements.

2.3.5 COROLLARY *Let C_n be a the symmetric algebra on n-coloured trees, and let Δ be the coproduct defined by $(q_{11}, \ldots, q_{2n}) \in k^{2n}$ (cf. Theorem 2.3.4). The graded dual C_n^* is the universal enveloping algebra of the Lie algebra which as a vector space is spanned by elements D_t, where t is a rooted tree in C_n. The bracket is given by $[D_s, D_t] = D_s \bullet D_t - D_t \bullet D_s$, where*

$$D_s \bullet D_t = \sum_w \sum_{s \subset w,\ s^c = t} q(s, w)\, D_w.$$

The first sum ranges over all n-coloured trees w and the second sum over all subtrees $s \subset w$ such that the complementary subtree satisfies $s^c = t$.

PROOF For any cocommutative Hopf algebra we can define an operation $\bullet$ on the primitive elements, such that its commutator is the Lie bracket on primitive elements: Simply define $\bullet$ as the truncation of the product at degree > 1, with respect to the primitive filtration F. In this case, $F_m C_n^*$ is spanned by the elements D_u dual to forests u consisting of at most m trees. The product in C_n^* is determined by the coproduct in C_n. Explicitly, for forest u, we can write the multiplication in C_n^* as

$$\begin{aligned} D_s D_t(u) &= (D_s \otimes D_t)\Delta(u) \\ &= \sum_{u = w \cup w^c} q(w, u) D_s(w) D_t(w^c). \end{aligned}$$

When we then restrict to the primitive part, we conclude that $D_s \bullet D_t$ is given by the desired formula. QED

§8 Related operads

With some minor changes, there is an analogue of the previous section for the cases $P =$ Ass$_*$ and $P =$ k.

2.3.6 COROLLARY *Let A_n be the free associative algebra on the planar n-coloured trees.*

(i). The coproducts on A_n of Corollary 2.3.2 are given by the closed formula

$$\Delta(t) = \sum_{s \subset t} \prod_j q_{1j}^{\sum_{v \in s} p_j(v,s,t)} \cdot \prod_j q_{2j}^{\sum_{v \in t} p_j(v,s^c,t)} s \otimes s^c,$$

where the product of trees is associative, non-commutative. The order of multiplication is given by the linear order on the roots of the trees defined by the linear order on the incoming edges at each vertex and the partial order on vertices.

(ii). Dually, the primitive elements of the dual are spanned by elements dual to single trees (with linear ordering on incoming edges in each vertex). The Lie bracket is the commutator of the (non-associative) product $*$ given by

$$D_s \bullet D_t = \sum_w \sum_{s \subset w,\ s^c = t} \prod_j q_{1\,j}^{\sum_{v\in s} p_j(v,s,w)} \cdot \prod_j q_{2\,j}^{\sum_{v\in t} p_j(v,t,w)} D_w,$$

where w, s and t are trees with a linear ordering on the incoming edges of the same colour at each vertex and the inclusions of s and t in w have to respect these orderings.

PROOF The only change is that we have to remember the ordering of up going edges at each vertex. Then one can copy the proof of the commutative case verbatim. QED

2.3.7 REMARK Independently, Foissy [21] has found this formula for the Lie bracket in the case where $n = 1,\ q_1 = 1$ and $q_2 = 0$ (and Ass_* is the underlying operad). He uses this formula to give an explicit isomorphism between the Hopf algebras A_1 and A_1^* with this coproduct.

2.3.8 COROLLARY *Consider once more the symmetric algebra on n-coloured trees C_n with its family of Hopf algebra structures. If for fixed $i = 1, 2$ there is at most one j such that $q_{ij} \neq 0$, the formula of Theorem 2.3.4 for the coproduct on C_n defines a coproduct on $k[\lambda_n](0)$.*

PROOF The vector space $\mathrm{k}[\lambda_n](0)$ is naturally identified with the subspace of C_n consisting of trees whose corresponding expression in $\mathrm{Com}[\lambda_n](0)$ does not involve multiplication. For the values of the parameters q_{ij} mentioned above, the formula of Theorem 2.3.4 for the coproduct on C_n defines a coproduct on $\mathrm{k}[\lambda_n](0)$. We just observe that the coproduct in this case preserves trees that do not involve multiplication. QED

Consequently, the coproduct defines an associative multiplication $*$ on the dual. A formula for this is given by the formula for Com_* modulo trees involving multiplication (their span is denoted μ): the quotient $\mathrm{Com}_*[\lambda_n](0)^*/(\mu)$ is canonically isomorphic to $\mathrm{k}[\lambda_n](0)^*$.

§9 Pre-Lie algebras

This section shows that in some cases the Lie bracket on the primitive elements of the Hopf algebras C_n^* (the linear duals to the Hopf algebras on the symmetric algebra of rooted trees of Theorem 2.3.4) is the associated Lie algebra of a pre-Lie algebra (cf. Definition 0.2.5). Recall (cf. Chapoton and Livernet [14]) that the *free pre-Lie algebra* L_p on p generators is given by the vector space spanned by rooted trees with vertices labelled with elements of $\mathbf{p} = \{1, 2, \ldots, p\}$. The pre-Lie algebra product is given by grafting trees. That is,

$$t \bullet s = \sum_{v \in t} s \circ_v t$$

for trees s and t, where the colour of the vertices is preserved. For any tree $t \in L_p$ define $r_i(t)$ to be the tree t with the colour of the root changed to i. Thus r_i defines a linear endomorphism of the vector space L_p.
Denote by χ_S the characteristic function of a subset $S \subset X$ which has value 1 on S and value 0 on $X - S$, and denote the primitive elements of a coalgebra C by $P(C)$.

2.3.9 THEOREM *Let* $\mathbf{p} \subset \{1, \ldots, n\}$ *and define* $q_{1j} = 0$ *and* $q_{2j} = \chi_{\mathbf{p}}(j)$ *for* $j = 1, \ldots, n$. *Consider the Hopf algebra structure on the symmetric algebra* C_n *on* n*-coloured trees that corresponds to this choice of* q_{ij}.

(i). The product $D_s \bullet D_t = \sum_{w=s\cup t} q(s,w)\, D_w$ *of Corollary 2.3.5 defines a pre-Lie algebra structure on the primitive elements* $P(C_n^*)$ *of* C_n^*.

(ii). If $\mathbf{p} = \{1, \ldots, n\}$, *then there is a natural inclusion of this pre-Lie algebra into the free pre-Lie algebra on* n *generators. The image in the free pre-Lie algebra is spanned by all sums* $\sum_{i \in \mathbf{p}} t_i$, *of trees that only differ in that the colour of the root of* t_i *is* i.

PROOF Consider the general formula for $D_t \bullet D_s$ in Corollary 2.3.5. Note that for $q_{ij} \in \{0, 1\}$ the coefficients $q(s,w)$ are either 0 or 1. We can be more precise. Let w and t be trees. An subtree $s \subset w$ is t-admissible if s^c contains the root of w while s is grafted onto $t = s^c$ by an edge of colour $i \in \mathbf{p}$ to a vertex connected to the root by edges each of which has a colour in $\mathbf{p}$. Thus, in the case we consider $q(s,w) = 0$ unless the the corresponding subtree s is t-admissible.
Call a subforest $t \cdot u \subset w$ in a tree w admissible if it restricts to an admissible subtrees of each of the trees. The pre-Lie identity follows from

$$(D_s \bullet D_t) \bullet D_u - D_s \bullet (D_t \bullet D_u) = \sum_w \sum_{t \cdot u \subset w} D_w,$$

where the second sum is over s-admissible subtrees. This proves part *(i)*.

In the remainder of the proof, let $\mathbf{p} = \{1, 2, \ldots, n\}$. Let $m(t, s, w)$ be the number of t-admissible subtrees $s \subset w$. The operation $\bullet$ is then

$$D_t \bullet D_s = \sum_w m(t, s, w) D_w,$$

where the sum is over all rooted trees. To define an inclusion of this pre-Lie algebra in the free pre-Lie algebra L_n, we first adapt the strategy of Hoffman [34] to get a different description of the pre-Lie structure to n-coloured trees. Note that if $m(t, s, w) \neq 0$, it is exactly the order of the orbit of the root of the subtree s under the action of the group $\mathrm{Aut}(w)$ of automorphisms of w. If s and t are n-coloured trees and v is a vertex in t, denote by $t \circ_{(v,i)} s$ the tree obtained from t and s by connecting the root of s to the vertex v by an edge of colour i. Let $n(t, s, w)$ be the number if vertices $v \in t$ such that $t \circ_{(v,i)} s = w$ for some $i \in \mathbf{p}$. The for any such vertex v, the order of the orbit of v in t under the action of $\mathrm{Aut}(t)$ is exactly $n(t, s, w)$. Define an ohter pre-Lie algebra structure $\bullet'$ on the same vector space $P(C_n^*)$, by

$$D_t \bullet' D_s = \sum_w n(t, s, w)\, D_w,$$

and denote this pre-Lie algebra by $P(C_n^*)'$. For a subtree $s \subset w$, denote by $\mathrm{Aut}^s(w)$ the automorphisms of w that pointwise fix s. Then, if $m(s, t, w) \neq 0$ we can write, following Hoffman [34],

$$m(t, s, w) = \frac{|\mathrm{Aut}(w)|}{|\mathrm{Aut}^s(w)| \cdot |\mathrm{Aut}(t)|}$$
$$n(t, s, w) = \frac{|\mathrm{Aut}(s)|}{|\mathrm{Aut}^{\{v\}}(t)|},$$

for a vertex v such that $t \circ_{(v,i)} s = w$ for some i. Since $|\mathrm{Aut}^s(w)| = |\mathrm{Aut}^{\{v\}}(t)|$ it follows that $D_t \longmapsto |\mathrm{Aut}(t)| D_t$ defines an isomorphism of pre-Lie algebras $P(C_n^*) \longrightarrow P(C_n^*)'$ (in characteristic 0).

We prove *(ii)* by constructing an inclusion $P(C_n^*)' \longrightarrow L_n$. Note that

$$D_t \bullet' D_s = \sum_{v \in t} \sum_{i \in \mathbf{p}} D_{t \circ_{(v,i)} s}$$

For an n-coloured tree t, denote by $\uparrow_i(t)$ the tree with coloured vertices obtined by moving the colour of ech edge up to the vertex directly above it and colouring the root by i. Note that $\uparrow_j(s \circ_{(v,i)} t) = \uparrow_i(s) \circ_v \uparrow_j(t)$. Let $\mathbf{p} = \{1, \ldots, n\}$ and consider $S(T_n)$ with the corresponding Hopg algebra structure as defined above. Define $\varphi : P(S(T_n)^*)' \longrightarrow L_n$ from the pre-Lie algebra of primitives to the free pre-Lie algebra on n generators by

$$\varphi(D_t) = \sum_{j=1}^{n} \uparrow_j(t).$$

Then φ is a linear enbedding. Moreover, φ preserves the pre-Lie algebra structure since

$$\varphi(D_t \bullet D_s) = \sum_{(v,i)} \varphi(D_{s\circ_{(v,i)}t}) = \sum_{(v,i)} \sum_j \uparrow(s_{(v,i)}t) = \sum_v \sum_{i,j} \uparrow_i(s) \circ_v \uparrow_j(t),$$

which equals $\varphi(D_s) \bullet \varphi(D_t)$. QED

2.4 Examples

This section aims to show that some of the Hopf algebras we studied in the Section §5 are in fact well known mathematical objects that are studied in recent literature.

§10 Motivation

Recall the Hopf algebra $\mathcal{H}_R$ (cf. Definition 1.3.9). Moerdijk [71] proves that $\mathcal{H}_R$ is the Hopf algebra C_1 with the coproduct of characterised by Formula (2.3.1) for $q_{11} = 0$ and $q_{21} = 1$. The analogous Hopf algebra on planar trees $\mathcal{H}_{\not\Sigma R}$, whose existence is shown in Moerdijk [71] has been studied in detail in Foissy [21].
The Hopf algebra of rooted trees is a graded Hopf algebra of finite type with respect to the number of vertices. The homogeneous elements of degree m are spanned by products $t_1 \cdot \ldots \cdot t_n$ of trees, such that $\sum_i |t_i| = m$. A basis for the graded dual $(\mathcal{H}_R)_n^*$ is $\{D_{t_1\cdot\ldots\cdot t_k} | \sum_i |t_i| = n\}$, the dual basis to the basis for $(H_R)_n$ given by products of trees. As proved by Connes and Kreimer [15], the Lie algebra which has H_R^* as its universal enveloping algebra is the linear span of rooted trees with the Lie bracket

$$[D_t, D_s] = \sum_u (n(t,s,u) - n(s,t,u))D_u,$$

where $n(t,s,u)$ is the number of admissible cuts c of u such that $P^c(u) = t$ and $R^c(u) = s$. Chapoton and Livernet [14] show that this is the Lie algebra associated to the free pre-Lie algebra on one generator. (This result now also follows from our Theorem 2.3.9.) It is natural to ask if one can interpret other well known Hopf algebras in a similar way as examples of the Hopf algebra construction by free extensions.

§11 Pruning Hopf algebras

Recall the vector space spanned by the set of n-ary planar trees by $k[Y_\infty^{(n)}]$. The base-point in $k[Y_\infty^{(n)}]$ is the tree with one leaf. In the special case of planar binary trees we write $k[Y_\infty] := k[Y_\infty^{(2)}]$.

2.4.1 PROPOSITION *Let $n \geq 0$. The unitary free associative algebra on the pointed vector space spanned by planar n-ary trees has a family of Hopf algebra structures parametrised by $i \in \{1, \dots, n\}$.*

PROOF Let $i \in \{1, \dots, n\}$. The coproducts induces by $q_{1j} = \chi_{\{i\}}(j)$ and $q_{2j} = 0$ are well defined on $\mathrm{k}[\lambda_n](0)$ by Corollary 2.3.8. This shows that we have a family of coproducts on $\mathrm{k}[\lambda_n](0)$. The result *(ii)* is now a direct consequence of the deformation in remark 2.3.6 and functorial properties. The functor $\bar{i}_! : \mathrm{k}[\lambda_n]$-HopfAlg $\longrightarrow$ $\mathrm{Ass}_*[\lambda_n]$-HopfAlg induced by the inclusion of Hopf operads $i : \mathrm{k} \longrightarrow \mathrm{Ass}_*$ is the unitary free algebra functor and the result follows since the σ_i are a special case of example 2.3.1. QED

2.4.2 COROLLARY (BROUDER AND FRABETTI [7]) *Let $P : k[Y_\infty] \longrightarrow k[Y_\infty] \otimes k[Y_\infty]$ be the on the vector space spanned by planar binary trees $k[Y_\infty]$, defined inductively by*

$$P(\lambda(T,1)) = 0, \qquad P(\lambda(T,S)) = \sum_i \lambda(T, S_i') \otimes S_i'' + \lambda(T,1) \otimes S,$$

if $P(S) = \sum_i S_i' \otimes S_i''$. Then the formulae

$$\Delta^P(1) = 1 \otimes 1, \qquad \Delta^P(T) = 1 \otimes T + P(T) + T \otimes 1,$$

define a coassociative coproduct on $k[Y_\infty]$. Thus the unitary free associative algebra $T(k[Y_\infty])$ is a Hopf algebra with the coproduct described on generators by Δ^P.

PROOF Let $n = 2$, and $i = 2$. Apply proposition 2.4.1. QED

2.4.3 EXAMPLE The coproduct on generators (i.e. trees) can again be described in terms of cuts. The only cuts we allow in this context are cuts containing at most one edge, such that colour of the edge is 1 and that the edge is connected to the root by edges of colour 1. That is to say, we can only cut the left most path from the root upward. Figure 9 shows a binary tree together with the two edges that give a non-trivial contribution to the coproduct (apart from the primitive part).

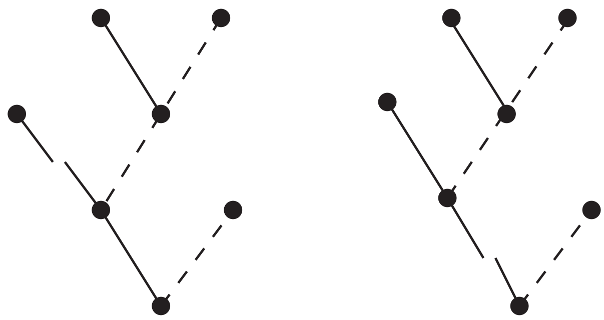


FIGURE 9: The two non-trivial cuts that are allowed in this binary tree. Cutting both edges at the same time is not allowed.

§12 Interlude: dendriform algebras

2.4.4 DEFINITION A *dendriform algebra* (cf. Loday [58], and Loday-Ronco [60]) is a vector space together with two bilinear (non-associative) products $\prec$ and $\succ$, satisfying the identities

$$(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$$
$$a \succ (b \prec c) = (a \succ b) \prec c$$
$$a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c.$$

A dendriform algebra D defines an associative algebra on the same vector space with associative product $*$ defined by $a*b = a \prec b + a \succ b$. A *dendriform Hopf algebra* (Ronco [74]) is a dendriform algebra together with a coproduct $\bar{\Delta}$ that satisfies

$$\bar{\Delta}(x \prec y) = \sum x' * y' \otimes x'' \prec y'' + x' * y \otimes x'' + y' \otimes x \prec y'' + y \otimes x$$
$$\bar{\Delta}(x \succ y) = \sum x' * y' \otimes x'' \succ y'' + x * y' \otimes y'' + y' \otimes x \succ y'' + x \otimes y.$$

2.4.5 REMARK One motivation for studying dendriform algebras is the classification of Hopf algebras analogous to the Milnor-Moore theorem (cf. Milnor and Moore [70]). The Milnor-Moore theorem gives an equivalence of categories between the category of cocommutative Hopf algebras and the category of Lie algebras (in characteristic 0). A Milnor-Moore theorem for dendriform Hopf algebras can be found in Ronco [74]. This theorem states the equivalence of dendriform Hopf algebras and brace algebras.

§13 The Loday-Ronco construction

Throughout this section let $k, l \in \mathbb{N}$. Use $P =$k, and take

$$q_{1j} = \delta_{kj}, \qquad \text{and} \qquad q_{2j} = \delta_{lj} \tag{2.4.3}$$

to define σ_1 and σ_2 as in Example 2.3.1. According to Corollary 2.3.8 this choice of parameters indeed defines a coproduct on $\mathrm{k}[\lambda_p](0)$. We use the augmentation of $u^* : \mathrm{k}[\lambda](0)^* \longrightarrow k$. The coalgebra structure on $\mathrm{k}[\lambda_n](0)$ defines an associative multiplication $*$ on $(\mathrm{k}[\lambda_n](0))^*$.

2.4.6 PROPOSITION *Consider $(k[\lambda_p](0))^*$ which is isomorphic to the vector space spanned by planar p-ary trees, as a vector space graded by the number of applications of λ_p. The augmentation ideal of $(k[\lambda_p](0))^*$ is a graded dendriform algebra with $\prec + \succ = *$.*

PROOF Let us give an inductive characterisation of the operations $\prec$ and $\succ$. To avoid awesome notation, we identify trees t with their dual D_t. The product is characterised by its unit $1 = \emptyset$ $(= D_\emptyset)$, and the identity

$$s * t = \lambda(s_1, \ldots, s_k * t, \ldots, s_n) + \lambda(t_1, \ldots, s * t_l, \ldots, t_n),$$

as is readily checked. Define dendriform operations $s \prec t = \lambda(s_1, \ldots, s_k * t, \ldots, s_n)$ and $s \succ t = \lambda(t_1, \ldots, s * t_l, \ldots, t_n)$. Let $t = \lambda(t_1, \ldots, t_n)$, and $u = \lambda(u_1, \ldots, u_n)$ and $s = \lambda(s_1, \ldots, s_n)$ and suppose that $\prec$ and $\succ$ satisfy the dendriform axioms for all trees with in total $< |t| + |s| + |u|$ vertices. Then we have

$$\begin{aligned}(s \prec t) \prec u &= \lambda(s_1, \ldots, (s_k * t) * u, \ldots, s_n)\\ &= \lambda(s_1, \ldots, s_k * (t * u), \ldots, s_n)\\ &= s \prec (t * u)\\ (s \succ t) \prec u &= \lambda(t_1, \ldots, s * t_l, \ldots, t_k * u, \ldots, t_n)\\ &= s \succ (t \prec u).\end{aligned}$$

The first identity follows by the definition of $*$ and associativity on smaller trees. The second one uses associativity on smaller trees in case $k = l$. The proof of the third dendriform identity can be copied from the first almost verbatim. It is then left to show that the dendriform operations preserve the grading. This follows again by induction. QED

There is a isomorphism of graded vector spaces between $\mathrm{k}[\lambda_n](0)$ and $(\mathrm{k}[\lambda_n](0))^*$ that maps t to D_t. This defines an associative multiplication on $\mathrm{k}[\lambda_n](0)$, which we will also denoted by $*$. Denote by $*_{(n)} : \mathrm{k}[\lambda_n](0)^{\otimes n} \longrightarrow \mathrm{k}[\lambda_n](0)$ the multiplication of n element, and by $\varepsilon_{(n)} : \mathrm{k}[\lambda_n](0)^{\otimes n} \longrightarrow \mathrm{k}[\lambda_n](0)$ the map $u \circ \varepsilon^{\otimes n}$.

2.4.7 THEOREM *The maps $\sigma_1 = *_{(n)}$ and $\sigma_2 = \varepsilon_{(n)}$ define a Hopf algebra structure on the space of planar n-ary trees $k[\lambda_n](0)$ iff $k = n$ and $l = 1$ in Formula (2.4.3). Moreover, the augmentation ideal of $k[\lambda_n](0)$ is a dendriform Hopf algebra.*

PROOF Check the requirements of Theorem 2.2.7. The requirement for the counit is fulfilled for any element of $(\mathrm{k}[\lambda_p](0))^*$. We prove the claim for coassociativity inductively. Let σ_i as above and assume that the requirement for Δ holds up to degree n. Then coassociativity of Δ holds up to degree n. Compute for $k = n$ and $l = 1$ (using Sweedler notation)

$$\begin{aligned}\Delta(a * b) &= \Delta(\lambda(a_1, \ldots, a_n) * \lambda(b_1, \ldots, b_n))\\ &= \Delta(\lambda(a_1, \ldots, a_n * b) + \Delta(\lambda(a * b_1, \ldots, b_n))\\ &= \sum a' * \ldots * (a_n * b)' \otimes \lambda(a_1'', \ldots, (a_n * b)'')\\ &+ \sum (a * b_1)' * \ldots * b_n' \otimes \lambda((a * b_1)'', \ldots, b_n'')\\ &+ (a * b) \otimes 1\end{aligned}$$

for $a = \lambda(a_1, \ldots, a_n)$ and $b = \lambda(b_1, \ldots, b_n)$ such that the sum of their degrees is m. Then use the induction hypothesis for degrees smaller then m (in particular $\Delta(a_n * b) = \Delta(a_n) * \Delta(b)$ and $\Delta(a * b_1) = \Delta(a) * \Delta(b_1)$). Deduce

$$\begin{aligned}\Delta(a \prec b) &= \sum a' * b' \otimes a'' \prec b'', \qquad \text{and}\\ \Delta(a \succ b) &= \sum a' * b' \otimes a'' \succ b''.\end{aligned}$$

Conclude that $(\mathrm{k}[\lambda_p](0))^*$ is a Hopf algebra if $k = n$ and $l = 1$. For other choices of k and l the induction fails and coassociativity does not hold since $*$ is not commutative. The dendriform Hopf identities for $\bar{\Delta}$ follow from the above. QED

2.4.8 COROLLARY (LODAY-RONCO [60]) *Let $T, T' \in k[Y_\infty]$ be planar binary trees. If $T = \lambda(T_1, T_2)$ and $S = \lambda(S_1, S_2)$, then the product*

$$T * S = \lambda(T_1, T_2 * S) + \lambda(T * S_1 S_2)$$

defines an associative product on the space of planar binary trees $k[Y_\infty]$ with as unit the tree with one leaf. Moreover the coproduct defined inductively by

$$\Delta(T) = \sum (T_1' * T_2') \otimes \lambda(T_1'', T_2'') + T \otimes 1,$$

makes the space of planar binary trees a Hopf algebra. The counit ε is the obvious augmentation of the product by projection on the span of the unit. The augmentation ideal of this Hopf algebra is a dendriform Hopf algebra.

PROOF Recognise the star product on $\mathrm{k}[\lambda_2](0)$ given by $q_{12} = 1 = q_{21}$ and $q_{11} = 0 = q_{22}$, and the coproduct associated to $*$. Apply Theorem 2.4.7. QED

Part II

Coloured Koszul Duality

CHAPTER 3

Algebroids and Koszul duality

This chapter shows that Lie algebroids are algebras over a self-dual Koszul coloured operad, and gives a characterisation of algebroids in terms of dg Lie algebra structures. To this end the theory of Koszul duality has to be extended to coloured operads. As an intermediate result, the chapter shows new relations between operads and classical homological algebra that extend results of Ginzburg-Kapranov [29].

3.1 Introduction

Lie Algebroids (cf. Cannas Da Silva-Weinstein [9], MacKenzie [62]) are geometric structures that have a wide range of applications. A Lie algebroid is a vector bundle together with a Lie algebra structure on the space of sections and a Lie algebra map to the vector fields on the base manifold. Of course there are some compatibility conditions that have to be satisfied. Natural examples of Lie algebroids include the tangent bundle of a manifold, the trivial bundle with fibre $\mathfrak{g}$ for a manifold with an action of a Lie algebra $\mathfrak{g}$, the source-invariant tangent vectors on a smooth groupoid, and the cotangent bundle of a Poisson manifold.

Algebroids can be considered as generalisations of Lie algebras. This led to interesting results on the integration of Lie algebroids to Lie groupoids (cf. Moerdijk-Mrčun [72], and Crainic-Fernandez [19]), and the Van Est-map between basic cohomology of the Lie algebroid and the differentiable cohomology of the Lie groupoid (cf. Crainic [18]).

However, to interpret algebroids just as Lie algebras with some base manifold does not do justice to these objects. This is probably most prominent in applications to

Poisson geometry (eg. [20] Cattaneo-Felder [10] and Landsman [53]).
Lie algebroids have been studied as purely algebraic concepts under various names. The most common term for these structures is Lie-Rinehart algebra (cf. Huebschmann [36]). A Lie-Rinehart algebra is a pair of a commutative algebra and a Lie algebra together with mutual actions that satisfy certain compatibility relations. A Lie algebroid defines a Lie-Rinehart algebra when one takes the functions on the base manifold for the commutative algebra, and the sections of the vector bundle as the Lie algebra. Notably, the universal enveloping algebra of a Lie-Rinehart algebra allows to interpret the cohomology of an algebroid in term of derived functors (Huebschmann [36]).
This purely algebraic approach suggests to use operads to solve questions on Lie-Rinehart algebras regarding both combinatorics and homological algebra. This is a new approach in the study of Lie-Rinehart algebras. This chapter shows that Lie-Rinehart algebras are algebras for a self dual quadratic coloured operad and exhibits new algebraic structure on the bar complex of the commutative algebra with coefficients in the Lie algebra.
In this Chapter I am a bit sloppy with the terminology, and refer to both Lie algebroids and Lie-Rinehart algebras as *algebroids*. If I want to stress the smooth nature, I use the term *smooth algebroids*.

3.2 The coloured operad for Algebroids

§1 Algebroids

3.2.1 DEFINITION Let M be a smooth manifold, and let $\mathfrak{g}$ be a vector bundle over M together with a Lie algebra structure on the space of sections $\Gamma^\infty(\mathfrak{g})$ and a bundle map $\rho : \mathfrak{g} \longrightarrow TM$ from $\mathfrak{g}$ to the tangent bundle TM on M. If ρ defines a Lie algebra morphism $\rho : \Gamma^\infty(\mathfrak{g}) \longrightarrow \Gamma^\infty(TM)$ on smooth sections and the bracket satisfies the Leibniz identity

$$[fX, Y] = f[X, Y] + \rho(Y)(f)X, \tag{3.2.1}$$

then $\mathfrak{g}$ with its additional structure is a *Lie algebroid* (or *smooth algebroid*, or *algebroid*). More details on these structures can be found in Cannas Da Silva-Weinstein [9], and MacKenzie [62].

3.2.2 EXAMPLE Many classical structures in differential geometry are examples of

Lie algebroids. If a Lie algebra $\mathfrak{h}$ acts on a manifold M, then the trivial bundle $\mathfrak{g} = M \times \mathfrak{h}$ is a Lie algebroid with the bracket

$$[X,Y](m) := [X(m),Y(m)] + \rho(X)Y(m) - \rho(Y)X(m),$$

for $X, Y \in \Gamma(\mathfrak{g}) = C^\infty(M,\mathfrak{h})$. and the action map ρ.
Another important class of examples is given by Poisson manifolds. If P is a Poisson manifold, then the Poisson tensor $\pi : T^*P \longrightarrow TP$ and the Koszul bracket $[df, dg] = d\{f, g\}$ on covector fields make T^*P a Lie algebroid with respect to $\rho = -\pi$.
Algebroids also occur as infinitesimal structure of smooth groupoids. If G is a smooth groupoid, then the source-invariant tangent vectors $u^*(T^sG)$ pulled back to the units $u : M \longrightarrow G$ have a natural structure of a Lie algebroid. The tangent bundle TM of a manifold M is a special case, corresponding to the pair-groupoid $G = M \times M$ over M.

3.2.3 DEFINITION We now turn to the algebraic counterparts of these geometric structures. An *algebroid* (or *Lie algebroid*) is a pair (R, A) of a commutative algebra R and a Lie algebra A such that A act on R by derivations and R acts on A and moreover for $a, b \in R$ and $X, Y \in A$ the following identities hold:

$$\begin{aligned} a(X(b)) &= (aX)(b) \\ [X, aY] &= a[X,Y] + X(a)Y, \end{aligned} \tag{3.2.2}$$

where $aX \in A$ and $X(a) \in R$ denote the actions of a on X and X on a respectively. Huebschmann [36] prefers to call these structures Lie-Rinehart algebras. Our motivation to call them algebroids is the similarity to smooth algebroids.

3.2.4 EXAMPLE Some examples are in place here. Let $\mathfrak{g}$ be a Lie algebroid over M. Define $R = C^\infty(M)$ and $A = \Gamma\mathfrak{g}$. Then (R, A) forms a Lie algebroid in the algebraic sense. (On the other hand, if $R = C^\infty(M)$ for a manifold M, the Serre-Swan theorem implies that any algebroid (R, A) such A is a finitely generated projective R-module is in fact induced by a Lie algebroid structure in the vector bundle sense.) There are also purely algebraic examples. Let $\mathfrak{g}$ be a Lie algebra that acts by derivations on a commutative algebra R, then the pair $(R, R^+ \otimes \mathfrak{g})$ is an algebroid with the bracket

$$[f \otimes X, g \otimes Y] = fg \otimes [X,Y] + fX(g) \otimes Y - Y(f)g \otimes X, \tag{3.2.3}$$

and the obvious actions. This construction should remind one of the algebroid structure on the cotangent bundle of a Poisson manifold (eg. Landsman [52]).

Another algebraic example is given by Poisson algebras. A *Poisson algebra* P is a commutative associative algebra together with a Lie algebra structure that make $[a,-]$ a derivation of the product for any $a \in P$. If P is a Poisson algebra, $R = P$ and $A = P$ define an algebroid with respect to the obvious algebraic structures

§2 Coloured operads

3.2.5 DEFINITION Let I be a set, and denote by I-Fin the category of pointed, non-empty finite sets (X, x_0) together with a map $i : X \longrightarrow I$, together with pointed bijections commuting with i. A *I-coloured collection* (or I-collection) is contravariant functor $C : I\text{-}\mathsf{Fin} \longrightarrow k\text{-}\mathsf{dgVect}$. A *pseudo I-coloured operad* (or *I-pseudo operad*) is a collection together with a map

$$\circ_x : C(X, x_0, i_X) \otimes C(Y, y_0, i_Y) \longrightarrow C(X \cup_x Y, x_0, i_X \cup_x i_Y),$$

for each pair of pointed sets (X, x_0) and (Y, y_0) and each $x \in X$ such that $x \neq x_0$ and $i_X(x) = i_Y(y_0)$, where $i_X \cup_x i_Y : X \cup_x Y \longrightarrow I$ is the map induced by i_X and i_Y. When defined, compositions of these structure maps satisfy the same equivariance and associativity axioms as the corresponding structures for pseudo operads (cf. Definition 0.3.1). (Recall that the elements of $X - \{x_0\}$ correspond to the inputs and x_0 to the output. Thus each of the inputs and the output have a colour, and compositions are defined when the colours match.)
An *I-coloured operad* (or I-operad) is an I-pseudo operad together with for any two-element set $(\{x, x_0\}, i) \in I\text{-}\mathsf{Fin}$ with $i(x) = i(x_0) = \alpha \in I$, a map $\mathrm{id} : k \longrightarrow C(\{x, x_0\}, i)$ such that id is a right/left identity with respect to any well defined right/left composition. The natural notion of morphism defines the categories I-Coll of collections, I-PsOpd of pseudo operads, and I-Opd of operads. An *(pseudo) I-cooperad* is an I-operad in the opposite category $(k\text{-}\mathsf{dgVect})^{\mathrm{op}}$.
The full subcategory $I\text{-}\mathsf{Perm} \subset I\text{-}\mathsf{Fin}$ has as objects $(\mathbf{n}, i)$ where $\mathbf{n} = \{0, \ldots, n\}$ with base-point 0 for $n \in \mathbb{N}$. If I has a linear ordering define the full subcategory $I\text{-}\mathsf{Ord} \subset I\text{-}\mathsf{Perm}$, with objects $(\mathbf{n}, i)$ in I-Perm such that the restriction $i : \{1, \ldots, n\} \longrightarrow I$ preserves the ordering (i.e. $x < y$ in $\{1, \ldots, n\}$ if $i(x) = \alpha$, $i(y) = \beta$, and $\alpha < \beta \in I$). Both inclusions are equivalences of categories. Consequently, one might restrict collections to I-Perm or even I-Ord, and obtain equivalent notions.

3.2.6 DEFINITION Let I be a set, and let $V = \{V_\alpha\}_{\alpha \in I}$ be an set of vector spaces indexed by elements of I. Define the *I-endomorphism operad* of V as the I-coloured operad

$$\mathrm{End}_V(n, i) = k\text{-}\mathsf{dgVect}(V_{i(1)} \otimes \ldots \otimes V_{i(n)}, V_{i(0)}),$$

for $n > 0$, with the action of $\mathrm{Aut}(n, i)$ by permutation of inputs, and with the usual composition of multi-linear maps. This defines the coloured endomorphism op-

erad of V. The existence of an endomorphism operad allows us to define algebras over a coloured operad in terms of operad maps.
Let P be an I-coloured operad. A *P-algebra* is a collection of vector spaces $V = \{V_\alpha\}_{\alpha\in I}$ together with an I-operad map $\varphi : P \longrightarrow \mathrm{End}_V$.

3.2.7 EXAMPLE Many coloured operads can be constructed by changing the set of colours. Let $f : I \longrightarrow J$ be a map of sets. This map defines a functor $f^* : J\text{-}\mathsf{Coll} \longrightarrow I\text{-}\mathsf{Coll}$ which on a collection P is given by $f^*P(n,i) = P(n, f \circ i)$. Moreover, this map defines a functor from J-(pseudo) operads to I-(pseudo) operads

$$f^* : J\text{-}\mathsf{Opd} \longrightarrow I\text{-}\mathsf{Opd},$$

since composition of the J-operad P can be used to define composition in f^*P. If we work with augmented operads, this construction is understood to mean applying f^* to the augmentation ideal and adding the units we need for the I-operad. This assures that the images are augmented I-operads.
We sketch how this construction can be applied. Let P be an operad, and let $\pi : \{1,2\} \longrightarrow \{1\}$ be the unique map. Define $P_{\bullet\to\bullet}$ as the $\{1,2\}$-suboperad of the $\{1,2\}$-coloured operads π^*P that consists of the spaces $\pi^*P(n,i)$ for i such that if $i(0) = 1$, then $i(k) = 1$ for all k. If i is a map from $\mathbf{n}$ to $\{1,2\}$ and $p \in P(n)$, then we denote by p_i the element p in $\pi^*P(n,i)$. For i such that $i(k) = 2$. Algebras for $P_{\bullet\to\bullet}$ are triples (A,B,f) of two P-algebras A and B together with a P-algebra morphism $f : A \longrightarrow B$.

§3 The operad Abd

3.2.8 DEFINITION When $I = \{1,2\}$, call I-operads 2-operads. We can define algebroids as the algebras for a coloured operad Abd. This operad is a quadratic 2-operad generated by a collection E concentrated in arity 2, such that

$$E(2,i) = \begin{cases} k & \text{if } i \equiv 1 \\ \mathrm{sgn} & \text{if } i \equiv 2 \\ k & \text{if } i(0) = i(1) = 1,\ i(2) = 2 \\ k & \text{if } i(0) = i(1) = 2,\ i(2) = 1, \end{cases}$$

and $E(2,i) = 0$ otherwise. Here the first k is the trivial representation of kS_2, and sgn is the sign representation of kS_2. Thus, $\mathrm{Abd}_{i\equiv 1}$ is generated by an S_2-invariant operation μ, and $\mathrm{Abd}_{i\equiv 2}$ is generated by an antisymmetric operation λ. The I-operad Abd is generated by these and operations $l \in \mathrm{Abd}(2,i)$ and $m \in \mathrm{Abd}(2,j)$ where $i(0) = i(1) = 1,\ i(2) = 2$ and $j(0) = j(2) = 2,\ j(1) = 1$. For an algebroid (S,A), the operation μ corresponds to multiplication in S, and m to the action of S on A, the operation λ to the bracket in A, and l to the action of A on S. From this we can deduce the ideal R such that $TE/R = \mathrm{Abd}$. The ideal R is generated by the relations

$$
\begin{aligned}
&\mu \circ_1 \mu = \mu \circ_2 \mu && \text{(associativity)}\\
&m \circ_1 \mu = m \circ_2 m && \text{(algebra action)}\\
&\lambda \circ_2 \lambda = \lambda \circ_1 \lambda - (\lambda \circ_1 \lambda)(23) && \text{(Jacobi identity)}\\
&l \circ_2 \lambda = l \circ_1 l(23) - (l \circ_1 l) && \text{(Lie algebra action)},
\end{aligned}
$$

that guarantee that and Abd-algebra is a pair of a Commutative and a Lie algebra with an action of the commutative algebra on the Lie algebra and vice versa, and the relations

$$
\begin{aligned}
l \circ_1 \mu &= \mu \circ_2 l + (\mu \circ_1 l)(23) && \text{(derivation)}\\
(l \circ_1 m)(23) &= \mu \circ_2 l && \text{(linearity)}\\
\lambda \circ_1 m &= m \circ_2 \lambda + (m \circ_1 l)(23) && \text{(Leibniz rule)},
\end{aligned}
$$

that give the desired compatibility conditions of Formula (3.2.2).

Using this description we can characterise the free algebroid on a pair of vector spaces (V, W). To give the result we need to recall some notation. Let P be an operad. Then we denote by $F_P(V)$ the free P-algebra on V. For an I-operad P (where $I = \{1, \ldots, n\}$) we write $F_P(V_1, \ldots, V_n)_{(i)}$ for the vector space with colour i in the free P-algebra.

3.2.9 PROPOSITION *The free algebroid on (V, W) is the algebroid defined by Formula (3.2.3) for $R = F_{\mathrm{Abd}}(V, W)_{(1)}$ and $\mathfrak{g} = F_{\mathrm{Lie}}(W)$ where*

$$
\begin{aligned}
F_{\mathrm{Abd}}(V, W)_{(1)} &= F_{\mathrm{Com}}(F^{+}_{\mathrm{Ass}}(W) \otimes V) \qquad \textit{and thus}\\
F_{\mathrm{Abd}}(V, W)_{(2)} &= F_{\mathrm{Lie}}(W) \otimes F^{+}_{\mathrm{Com}}(F^{+}_{\mathrm{Ass}}(W) \otimes V).
\end{aligned}
$$

(The superscript $+$ denotes the addition of a unit.)

PROOF We need to give the commutative algebras structure on $F_{\mathrm{Abd}}(V, W)_{(1)}$, the Lie algebra structure on $F_{\mathrm{Abd}}(V, W)_{(2)}$, and the mutual actions. These need to satisfy the universal property of the free algebra over the operad Abd.
The commutative algebra structure on $F_{\mathrm{Abd}}(V, W)_{(1)}$ is the obvious one. Recall that if V is a vector space and $\mathfrak{g}$ is a Lie algebra, the commutative algebra $F_{\mathrm{Com}}(U(\mathfrak{g}) \otimes V)$ is free on the vector space V as a commutative algebra with an action of $\mathfrak{g}$ by derivations. Recall also, that the free associative algebra $F^{+}_{\mathrm{Ass}}(W)$ is the universal enveloping algebra of the free Lie algebra on the vector space W. Consequently, $R = F_{\mathrm{Abd}}(V, W)_{(1)}$ is the universal commutative algebra with a action of $F_{\mathrm{Lie}}(W)$ by derivations. That is, for any commutative algebra S with an action of $F_{\mathrm{Lie}}(W)$ by derivations, there is a natural 1 - 1 correspondence between linear

maps k-Vect(V, S), and algebra maps $F_{\mathrm{Abd}}(V, W)_{(1)} \longrightarrow S$ that preserve the module structure.
Note that if R is a commutative algebra, and $\mathfrak{g}$ is a Lie algebra that acts by derivations on R, that the algebroid $(\mathfrak{g} \otimes R^+, R)$ defined by Formula (3.2.3) is the universal algebra with respect to this structure. That is, given an Lie algebroid (S, A), there is a natural 1 - 1 correspondence between maps pairs $(\varphi_{(1)}, \varphi_{(2)})$ where $\varphi_{(1)} : R \longrightarrow S$ is an algebra homomorphism and $\varphi_{(2)} : \mathfrak{g} \longrightarrow A$ is a Lie algebra homomorphism, and $\varphi_{(1)}$ is a map of $\mathfrak{g}$-modules on the one hand; and morphisms of algebroids $(\mathfrak{g} \otimes R^+) \longrightarrow (S, A)$ on the other hand.
The observations above imply the desired universal property of the free algebroid: Let (R, A) be an algebroid, and $V \longrightarrow R$ and $W \longrightarrow A$ be linear maps. These maps define a unique map $F_{\mathrm{Lie}}(W) \longrightarrow A$ of Lie algebras and a unique map $F^+_{\mathrm{Ass}}(W) \otimes V \longrightarrow R$ of $F_{\mathrm{Lie}}(W)$-modules. This map induces a unique map $F_{\mathrm{Abd}}(V, W)_{(1)} \longrightarrow R$ of commutative algebras. Since the action of $F_{\mathrm{Lie}}(W)$ is by derivations, this map is compatible with the $F_{\mathrm{Lie}}(W)$-module structure. The maps $F_{\mathrm{Lie}}(W) \longrightarrow A$ and $F_{\mathrm{Abd}}(V, W)_{(1)} \longrightarrow R$ induce a unique map $F_{\mathrm{Abd}}(V, W)_{(2)} \longrightarrow A$ of $F_{\mathrm{Abd}}(V, W)_{(1)}$-modules. The algebroid axioms assure that this is a Lie algebra map. QED

3.3 Coloured Koszul duality

§4 Coloured operads as operads

To develop some theory of coloured operads, we would like to apply powerful tools from operad theory (in particular bar-cobar duality and Koszul duality) to coloured operads. One might spell out all proofs in the suitable coloured context where all arguments are somewhat more involved. It is preferable to use a trick. One needs to allow operads over a semi-simple algebra instead of over the ground field k (cf. Ginzburg and Kapranov [29]).

3.3.1 DEFINITION Let A be a semi-simple (associative) algebra. An *operad under A* is an pseudo operad P together with an inclusion of associative algebras $A \longrightarrow P(1)$. Consequently, $P(n)$ is a left A, right $A^{\otimes n}$ module. The kS_n-action and the $A^{\otimes n}$-action on $P(n)$ satisfy a natural compatibility relation, and composition is equivariant with respect to these actions. An A-operad P is *augmented* if the inclusion

$A \longrightarrow P$ is split.
An pseudo operad under A is a pseudo operad P together with a left A and right $A^{\otimes n}$ module structure on $P(n)$ that satisfies the same compatibility relations as in the case of operads under A. (The difference is that these relations are not induced by an inclusion of A in the unitary operations.)
We now introduce a semi-simple algebra which will be of interest. Let I be a set. Define A_I as the algebra $\bigoplus_{\alpha \in I} k[\alpha]$ with the product defined by the linear extension of the Kronecker delta on I (i.e. $[\alpha][\beta] = \delta_{\alpha\beta}[\alpha]$ for $\alpha, \beta \in I$). An A_I-module M is *orthogonal* if it is the direct sum of eigenspaces M_α for the $[\alpha]$ with $\alpha \in I$. Thus $M = \bigoplus_{\alpha \in I} M_\alpha$. A module M for $A_I^{\otimes n}$ is *orthogonal* if it has a decomposition in eigenspaces $M_{\alpha_1 \dots \alpha_n}$ for $([\alpha_1], \dots, [\alpha_n])$.

3.3.2 PROPOSITION (MARKL [66]).) *There is an isomorphism of categories between the category I-PsOpd of I-coloured pseudo operads, and the category of pseudo operads under A_I such that each $P(n)$ is a orthogonal left A- right $S_n \propto A^{\otimes n}$-bimodule. A similar statement holds for I-cooperads.*

We sketch the proof. Let P be an object in I-PsOpd. Write $\mathbf{n} := \{0, \dots, n\}$. Orthogonality of $P(n)$ means that $P(n)$ has a decomposition $P(n) = \bigoplus_{i:\mathbf{n} \longrightarrow I} P(n, i)$ as a bimodule, such that the left A_I-action is nontrivial only for the idempotent $[i(0)]$, and the right $A_I^{\otimes n}$-action is nontrivial only for the idempotent $\bigotimes_{k=1}^{n} [i(k)]$. The operad structure is given by extending the $\circ_j$-operations as 0 when they are not well defined (i.e. $i_{\mathbf{m}}(j) \neq i_{\mathbf{n}}(0)$). The inverse construction is given by restriction of the $\circ_i$-operations to the suitable summands. There is a dual statement for cooperads.

§5 Bar and cobar construction

3.3.3 DEFINITION Let C be a graded collection. A *differential* on the 'cofree' pseudo cooperad $\bar{T}'C$ is a coderivation ∂ of degree $+1$ that satisfies $\partial^2 = 0$. A *differential* on the free pseudo operad $\bar{T}C$ is a derivation ∂ of degree $+1$ that satisfies $\partial^2 = 0$. Recall that both $\bar{T}C$ and $\bar{T}'C$ are graded by the number of vertices in the trees (cf. Formula 0.3.3). Getzler and Jones [26] showed that for a graded collection P, there is a 1-1 correspondence between pseudo operad structures on P and differentials on $\bar{T}'(P[-1])$ that are of degree ≥ -1 in vertices. Similarly, there is a 1-1 correspondence between pseudo cooperad structures on P and differentials on $\bar{T}(P[1])$ that are of degree $\leq +1$ in vertices.
Let P be an operad. Define the bar construction BP to be the corresponding pseudo cooperad $(\bar{T}'(P[1]), \partial)$. Let C be a cooperad. Define the cobar construction $\text{Ɑ}C$ to be the corresponding operad $(\bar{T}(C[1]), \partial)$. These constructions define functors

$$B : \mathsf{PsOpd} \longrightarrow \mathsf{PsCoopd} \qquad \text{and} \qquad \text{Ɑ} : \mathsf{PsCoopd} \longrightarrow \mathsf{PsOpd}.$$

For a semi simple algebra A, there exists a version for pseudo (co)operads over A,

$$B_A : A\text{-PsOpd} \longrightarrow A\text{-PsCoopd} \qquad \text{and} \qquad \Omega_A : A\text{-PsCoopd} \longrightarrow A\text{-PsOpd}.$$

If $A = A_I$, it will be convenient to use the notation $B_I := B_{A_I}$, and $\Omega_I := \Omega_{A_I}$.

3.3.4 THEOREM (GINZBURG AND KAPRANOV [29]) *Let P be a pseudo operad over A. The natural projection $\Omega_A B_A P \longrightarrow P$ is a quasi isomorphism of pseudo operads. Similarly, let C be a pseudo cooperad. The natural inclusion $C \longrightarrow B_A \Omega_A C$ is a quasi isomorphism of pseudo cooperads over A.*

3.3.5 THEOREM *Let I be a set. If P is an I-operad, then the bar construction $B_I P$ is an I-cooperad. If C is an I-cooperad, then the cobar construction $\Omega_I C$ is an I-operad.*

PROOF To prove *(i)*, it remains to show that $B_I P$ and $\Omega_I C$ are orthogonal as A_I-modules. Let C be an I-collection. Recall the description of the free operad in terms of rooted trees $\bar{T}_I C = \bigoplus_t C(t)$, where $C(t) = \bigotimes_{v \in \mathbf{v}(t)} C(\mathbf{l}_t(v))$. If one considers operads over A_I, the tensor products are tensor products over A_I. For a coloured collection C, recall the decomposition $C(n) = \bigoplus_{i:\mathbf{n} \longrightarrow I} C(n,i)$ and consider for some rooted tree t the induced decomposition

$$P(t) = \bigoplus_{\{i_v\}_{v \in \mathbf{v}(t)}} \bigotimes_{v \in \mathbf{v}(t)} P(\mathbf{l}_t(v), i_v),$$

where we sum over all $i_v : \mathbf{l}_t(v) \longrightarrow I$. The divisors of 0 in A_I assure that a summand vanishes if an output colour and an input colour corresponding to one edge in t do not match. Consequently, one can write TC as a sum over rooted trees with edges (both internal and external) coloured by elements of I. The desired decomposition of $\Omega_I C(t)$ is given by the decomposition given by the external edges. The proof of the result on $B_I P$ is similar. QED

§6 Koszul duality

3.3.6 DEFINITION (GINZBURG AND KAPRANOV [29]) If $P = T_A E/R$, a quadratic operad under a a semi-simple algebra A, the *quadratic dual cooperad* $P^\perp$ is the kernel of the surjection $T'_A(E[1]) \longrightarrow T'_A(R')$, where $R' = T(E[-1])(3)/(R(3)[-2])$. An operad under A is *Koszul* if the natural map $P^\perp \longrightarrow B_A P$ is a quasi isomorphism. Recall that Ginzburg and Kapranov constructed a homology theory for algebras over a quadratic operad P under A, and showed that the operad P is Koszul iff the homology vanishes for every free algebra.

3.3.7 EXAMPLE We examine this situation in more detail for the case of I-coloured operads, where the algebra A in the Definition above is A_I and the A and $A^{\otimes n}$ module structures on $P(n)$ are orthogonal. A I-operad P is called a *quadratic I-operad* if $P = T_I E/R$, the quotient of a free I-operad on a finite dimensional collection E concentrated in arity 2 by an I-operad ideal R generated by elements of arity 3. Since R is orthogonal, the Koszul dual of an quadratic I-coloured operad P is then a I-coloured cooperad $P^\perp$. In examples it will often be convenient to work not with the *quadratic dual I-operad* $P^!$ instead of $P^\perp$. The I-operad $P^!$ is defined as $(P^\perp)^* \otimes \Lambda$, where $(P^\perp)^*$ is the linear dual of $P^\perp$ with its natural operad structure, and Λ is the determinant operad $\Lambda(n,i) = \mathrm{sgn}_n[1-n]$ (cf. Ginzburg and Kapranov [29]).
For any quadratic I-coloured operad P and any P-algebra K this defines the P-algebra homology $H_*^P(K)$ of K. Recall that for a quadratic operad P under K, and a P-algebra K, the *P-algebra homology complex* is given by $C_*^P(K) = F'_{P^\perp}(K)$, with the differential induced by the P-algebra structure on K. We study this complex in somewhat greater detail for coloured operads.
Let P be an I-coloured operad, and let K be an P-algebra. The decomposition $P^\perp(n) = \bigoplus_i P^\perp(n,i)$ gives a decomposition of the homology complex $C_*^P(K)$ as

$$C_{n-1}^P(K) = \bigoplus_{(n,i)} P^\perp(n,i) \otimes_{kS_n} K^{(n,i)}, \tag{3.3.4}$$

where $K^{(n,i)} = \bigotimes_{k=1}^n K_{i(k)}$. The differential is induced by the P-algebra structure on K. This differential can be given explicitly as the composition

$$\begin{array}{c} P^\perp(n) \otimes K^{\otimes n} \\ \Big\downarrow \gamma^* \\ \left(\bigoplus_k P^\perp(n-1) \otimes_{S_{n-1}} (\mathrm{id}^{\otimes k} \otimes P(2)^\perp \otimes \mathrm{id}^{\otimes n-k-2})\right) \otimes_{S_n} K^{\otimes n} \\ \Big\downarrow \gamma_K \\ P^\perp(n-1) \otimes_{S_{n-1}} K^{\otimes n-1}. \end{array}$$

Note that we only consider the value of γ^* on the summands

$$P^\perp(n-1) \otimes_{S_{n-1}} (\bigoplus_{i=0}^{n-2} k^{\otimes i} \otimes P^\perp(2) \otimes k^{\otimes n-i-2}),$$

(here k is identified with $k \cdot \mathrm{id} \subset P(1)$) and that we identify $P^\perp(2)$ with $P(2)$ in order to apply γ_K. This identification accounts for the degree 1 of the differential. Recall $P(n) = \bigoplus_i P(n,i)$ often can be used to provide a further decomposition of the P-algebra homology complex. For example, for any quadratic I-operad P and any P-algebra A, the complex $C_*^P(K)$ decomposes as

$$C_*^P(K) = \bigoplus_{\alpha \in I} C_{*(\alpha)}^P(K),$$

where $\alpha = i(0)$ stemming from $P(n) = \bigoplus_i P(n,i)$.
If P is a quadratic I-operad concentrated in degree 0, a P-algebra K concentrated in degree 0, the complex $C_*^P(K) = F'_{P^\perp}(K)$ is concentrated in cohomological degree ≤ 0, which yields a homological complex $C_*^P(K)$, concentrated in degree ≥ 0. The results of Ginzburg and Kapranov now specialise to the following Corollary.

3.3.8 COROLLARY *Let P be a quadratic I-coloured operad.*

(i). The I-operad P is Koszul iff $P^!$ is Koszul

(ii). The homology $H_*^P(K)$ vanishes for every free P-algebra K iff P is Koszul.

3.3.9 EXAMPLE Recall Koszul duality for the most standard non-coloured operads:

$$\mathrm{Com}^! = \mathrm{Lie}, \qquad \mathrm{Ass}^! = \mathrm{Ass}.$$

Let A be an associative algebra. Then the complex C_*^{A} is the shifted truncated bar complex $B_*(A)$ that satisfies $B_p(A) = A^{p+1}$ for $p \geq 0$, and $B_p(A) = 0$ otherwise, the differential is $d(x_1, \ldots, x_n) = \sum_{i=1}^{n-1}(-1)^{i+1}(x_1, \ldots, x_i x_{i+1}, \ldots, x_n)$.
Let A be a commutative algebra. The complex $C_*^{\mathrm{Com}}(A)$ is the Harrison complex of A, described as the indecomposables of the bar complex $B_*(A)$ of A with respect to the shuffle product.
Let $\mathfrak{g}$ be a Lie algebra. The the complex $C_*^{\mathrm{Lie}}(\mathfrak{g})$ is the shifted-truncated Chevalley-Eilenberg complex $C_p(\mathfrak{g}) = \Lambda^{p+1}\mathfrak{g}$ for $p \geq 0$ and zero otherwise.

3.4 Example: algebras with a module

This section describes a particular example of Koszul duality for coloured operads. The results in this section are necessary to deduce Koszul duality of the operad Abd.

§7 The operad for algebras with a module

3.4.1 DEFINITION Let P be a non-coloured operad, and let $\pi : \{1,2\} \longrightarrow \{1\}$ be the unique map. Define $MP \subset \pi^* P$ as the suboperad consisting of the spaces $P(n,i)$ such that either (i). $i \equiv 1$, or (ii). $i(0) = 2$ and $\exists! k \neq 0 : \ i(k) = 2$. Algebras for MP are P-algebras together with a module. For a cooperad C we can similarly define a $\{1,2\}$-cooperad such that coalgebras for $M'C$ are C-coalgebras together with a comodule.
If $R \subset P$ is an ideal, then $MR \subset MP$ is an ideal as well. If $P = TE/R$ is a quadratic operad, then $MP = M(TE)/MR$ is a quadratic 2-operad.

3.4.2 THEOREM *Let $P = TE/R$ be a quadratic operad, and MP the coloured operad for P-algebras with a module.*

(i). The Koszul dual cooperad of MP is given by $(MP)^\perp = M'(P^\perp)$.

(ii). The operad MP is Koszul iff the operad P is Koszul.

PROOF The identification as $\mathrm{Aut}(\mathbf{n}, i)$-modules is obvious from the defining pullback

$$\begin{array}{ccc} (MP)^\perp & \longrightarrow & M'(T'(E[1]) \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & (M(T'R(3)[2])^\perp), \end{array}$$

where of course $M'(T'E) = T'(E(\mathbf{2}, i) \oplus E(\mathbf{2}, j))$ for $i \equiv 1$, and $j(0) = j(2) = 2$, $j(1) = 1$. Cocomposition follows from the defining pullback of $(MP)^\perp$ as well. This shows $(MP)^\perp = M'(P^\perp)$.
If MP is Koszul, the inclusion $P \subset MP$ as the summands $P(n,i)$ such that $i \equiv 1$ shows that P is Koszul. Suppose on the other hand that P is Koszul. Then the result follows from part *(i)* and the isomorphism

$$B(MP) \cong M'(BP).$$

To understand the isomorphism, observe that $B(MP)$ consists of monochromatic trees of colour 1 and trees with exactly one path from a leaf to the root of colour 2. Similarly, $M'(BP)$ consists of monochromatic trees and trees with one input and the output of colour 2. Observe that there is a 1-1 correspondence between paths and end points. QED

3.4.3 REMARK Theorem 3.4.2 has been found independently by Longoni and Tradler, in their soon expected preprint [61].

§8 Homology with coefficients

Let P be a quadratic operad. From the definition of the homology of an algebra over a coloured quadratic operad (cf. Formula (3.3.4)), it is clear that the homology of an MP-algebra (A, M) splits in two direct summands. The first summand is the homology of A as a P-algebra. The second summand, corresponding to $i(0) = 2$, is a natural candidate for the homology of of A with coefficients in M.

3.4.4 DEFINITION Let P be an operad, and let (A, M) be an MP-algebra. The subcomplex

$$C_*^P(A, M) := C_{*(2)}^{MP}((A, M)) \subset C_*^{MP}((A, M))$$

consisting of the spaces $(MP)^\perp(n, i) \otimes_{kS_n} A^{(n,i)}$ with output colour $i(0) = 2$ is the *P-algebra homology complex of A with coefficients in M*. Its homology $H_*^P(A, M)$ is called the *homology of A with coefficients in M*.
Theorem 3.4.8 shows that the complex indeed specifies to well known complexes from homological algebra if we restrict to well known operads. But first we study the complex at a more conceptual level.

Recall the definition of the universal enveloping algebra of a P-algebra. We have a dual notion $U'_C(A)$ of a *universal enveloping coalgebra* for a cooperad C and a C-comodule A. One easily identifies $C_{*(2)}^{MP}((A, M)) \subset C_*^{MP}(A, M)$ as a graded vector space with the space $(U'_{P^\perp}(F'_{P^\perp}(A))) \otimes M$. However, this does not yet describe the differential on $C_{*(2)}^{MP}((A, M)) \subset C_*^{MP}(A, M)$. To give a nice description of this differential, we recall that $A \oplus M$ is a P-algebra with respect to $p((a, m), (b, n)) := p(a, b) + p(a, n) + p(m, b)$.

3.4.5 LEMMA *There is a natural inclusion of complexes*

$$C_{*(2)}^{MP}((A, M)) \subset C_*^{MP}(A, M) \subset C_*^P(A \oplus M)$$

induced by the inclusions $P(n) \otimes_{S_n} (A^{\otimes n-1} \otimes M) \subset P(n) \otimes_{S_n} (A \oplus M)^{\otimes n}$.

PROOF The identification as graded vector spaces as stated in the Lemma is obvious. Just compare the definition of the differentials. Both are defined by first applying the cocomposition in $P^\perp$, and then the composition in A, or the action of A on M (depending on whether both tensor factors come from A or whether one of the tensor factors comes from M). QED

Getzler and Jones [26] show that for a quadratic Koszul operad P and an P-algebra A, the homology $H_*^P(A)$ is the left derived functor of the functor Q that takes indecomposables with respect to the $U_P A$-module structure on A. The result generalises to algebras with a module.

3.4.6 THEOREM *Let P be a Koszul operad, and let (A, M) be an MP-algebra. Denote $U_P A$ the universal enveloping algebra of A, and denote both the functor taking indecomposables in M as an A-module and the functor that takes indecomposables of (A, M) as a MP-algebra by $Q(A, M)$. Then the homology $H_*^P(A, M)$ can be written as a left derived functor in both the category $U_P A$-Mod of left $U_P A$-modules, and the category MP-Alg of MP-algebras:*

$$H_*^P(A, M) = \mathbb{L}Q_*(A, M)_{(2)}. = \mathbb{L}Q_*^{U_P A}(M),$$

where $Q^{U_P A}$ is the functor taking indecomposables as an $U_P A$-module.

PROOF Let (A, M) be a MP-algebra. Then $F_P(F'_{P^\perp}((A, M))) \longrightarrow (A, M)$ is a quasi isomorphism. Omitting the differential in the notation, we have the resolution

$$F_P(F'_{P^\perp}((A, M)))_{(2)} = U_P(F_P(F'_{P^\perp}(A)) \otimes F'_{P^\perp}(A, M)_{(2)} \longrightarrow M$$

of M in the category of $(F_P(F'_{P^\perp}(A)), \partial)$-modules. The left derived functor of Q from P-algebras to k-Vect is invariant under replacing A with the cofibrant replacement $F_P(F'_{P^\perp}(A))$. When we take the indecomposables of $U_P(F_P(F'_{P^\perp}(A)) \otimes F'_{P^\perp}(A, M)_{(2)}$ we get the Koszul complex $C^P_{*(2)}(A, M)$. This proves the first equality.

Let $C = (F_P F'_{P^\perp}(A), \partial)$. Clearly, $H_*(A, M)$ computes the left derived functor of the functor taking indecomposables of M as a $U_P(C)$-module. Lemma 3.4.7 below shows $U_P(C) \longrightarrow U_P(A)$ is a cofibrant replacement in the model category of dg algebras, which implies that $H_*(A, M)$ computes the left derived functor of indecomposables of M as an $U_P(A)$ module. QED

3.4.7 LEMMA *If $f : C \longrightarrow A$ is an surjective quasi isomorphism of P-algebras, then f induces a surjective quasi isomorphism $U_P(C) \longrightarrow U_P(A)$.*

PROOF Write $U_P(C)$ as a coequaliser in k-dgVect (cf. Definition 0.3.14)

$$\bigoplus_{n,m_1,\ldots m_{n-1}} P(n) \otimes_{S_n} \left(\left(\bigotimes_{k=1}^{n-1} P(m_k) \otimes_{S_{m_k}} C^{\otimes m_k}\right) \otimes_{S_{n-1}} kS_n\right)$$
$$\downarrow\downarrow$$
$$\bigoplus_m P(m) \otimes_{S_m} \left(C^{\otimes m-1} \otimes_{S_{m-1}} kS_m\right)$$
$$\downarrow$$
$$U_P(C).$$

For a surjection $f : C \longrightarrow A$, this yields a pushout diagram in k-dgVect

$$\begin{array}{ccc}
\bigoplus_m P(m) \otimes_{S_m} \left(C^{\otimes m-1} \otimes_{S_{m-1}} kS_m\right) & \longrightarrow & U_P(C) \\
\downarrow & & \downarrow \\
\bigoplus_m P(m) \otimes_{S_m} \left(A^{\otimes m-1} \otimes_{S_{m-1}} kS_m\right) & \longrightarrow & U_P(A).
\end{array}$$

Assume in addition that $f : C \longrightarrow A$ is a quasi isomorphism. Apply the Künneth formula and the fact that every kS_n module is a retract of a free module (characteristic 0!), to see that the leftmost vertical map is a surjective quasi isomorphism. The pushout property than guarantees that the map in universal enveloping algebras is a surjective quasi isomorphism. QED

§9 Complexes from classical homological algebra

Theorem 3.4.6 expresses that one can find a small complex to compute the total left derived functor of $Q^{U_P A}$ from modules of a P-algebra A to dg vector spaces. For some well known operads P this procedure recovers classical complexes from homological algebra.

Let A be a (non-unital) associative algebra. Let L be a left A module and let R be a right A-module. Denote by $B(R, A, L)$ the 2-sided bar complex with coefficients in R and L.

3.4.8 THEOREM *Let* Com, , Lie, *and* Ass *denote the operad for commutative, Lie and associative algebras respectively.*

(i). Let A be a commutative algebra, and let M be a A-module. Then $C_*^{\mathrm{Com}}(A, M)$ is the bar complex $B_*(k, A, M)$ of A with coefficients in the right module k and the left module M.

(ii). Let $\mathfrak{g}$ be a Lie algebra and let N be a $\mathfrak{g}$-module. Then $C_*^{\mathrm{Lie}}(\mathfrak{g}, N)$ is the Chevalley-Eilenberg complex $C_*^{CE}(\mathfrak{g}, N)$ of $\mathfrak{g}$ with coefficients in N.

(iii). Let A be an associative algebra, and let M be a A-bimodule. Then $C_*^{\mathrm{Ass}}(A, M)$ is the bar complex of A with coefficients in the dg right module $B_*(k, A, M)$ and the left module k, where the right action of A on $B_*(k, A, M)$ is the right action of A on M. Thus $C_*^{\mathrm{Ass}}(A, M) = \mathrm{Tot}(B_*(B_*(k, A, M), A, k))$.

PROOF Let Com be the operad for commutative algebras and let A be a commutative algebra. The Harrison complex $C_*^{\mathrm{Com}}(A)$ is a dg Lie^*-coalgebra. The

bar complex $B_*(A)^+$ (w.r.t. shuffle product and 'cofree' coproduct) is the universal enveloping coalgebra of the Harrison complex of A with coefficients in (Balavoine [1] has all the details of this classical result). Write (as graded vector spaces) $C^{MP}_{*(2)}((A,M)) = B_*(A) \otimes M$. This identifies $C^P_*(A,M)$ and $B(A,M)$ as complexes. Let Lie be the operad for Lie algebras and $(\mathfrak{g}, N)$ a $M\mathrm{Lie}$-algebra. One easily identifies $F'_{\mathrm{Com}}(\mathfrak{g}) \otimes N$ with the Chevalley-Eilenberg complex.
Let Ass be the operad of associative algebras. Recall that $U_{\mathrm{Ass}}A = A^+ \otimes A^{\mathrm{op}+}$, where A^+ is A with a unit freely added). The complex $C^{\mathrm{Ass}}_*(A,M)$ then takes the form $T'(A[-1]) \otimes M[-1] \otimes T'(A[-1])$, as a graded vector space. The differential consists of contractions of two adjacent tensor factors via the dual to multiplication in A or the action of A on M. That is, $C^{\mathrm{Ass}}_*(A,M) = \mathrm{Tot}(B_*(B_*(k,A,M),A,k))$.
QED

3.4.9 REMARK The homology of the complexes described in Theorem 3.4.8 is well known.

$$H^{\mathrm{Com}}_*(A,M) = \mathrm{Tor}^{A^+}_*(k,M)$$
$$H^{\mathrm{Ass}}_*(A,M) = \mathrm{Tor}^{A^+\otimes A^{\mathrm{op}+}}_*(k,M)$$
$$H^{\mathrm{Lie}}_*(\mathfrak{g},M) = \mathrm{Tor}^{U\mathfrak{g}}_*(k,M).$$

The only point that needs explanation is the associative case. There we use an Eilenberg-Zilber argument on the double bar complex for the algebra A^+ and identify the bar complex of A with the reduced bar complex of A^+. This agrees with the results of Theorem 3.4.6.

3.5 Koszul duality for algebroids

§10 The homology complex

This section describes the homology complex of an algebroid in general terms, using algebraic structures on $B_*(R,A)$ and $C^{\mathrm{Com}}_*(R)$.

3.5.1 LEMMA *The coloured operad* Abd *is self-dual up to an automorphism:*

$$\mathrm{Abd}^! \cong \mathrm{Abd}.$$

The automorphism is given by interchange of colours.

PROOF The generating collection has $\dim(E) = 4$, $\dim(FE(3)) = 22$ and $\dim(R(3)) = 11$. Check the identities per $i : \mathbf{3} \longrightarrow I = \{1, 2\}$. If i is constant, this is obvious. Let $i : (0, 1, 2, 3) \mapsto (1, 1, 1, 2)$. We write the relations in terms of the generators applied to inputs, where the bold lower case and upper case letters denote inputs of colour 1 and 2 respectively with the ordering $\mathbf{a} < \mathbf{b} < \mathbf{X}$ on inputs. Then we have in Abd the relations

$$\lambda(m(\mathbf{a}, \mathbf{b}), \mathbf{X}) = m(\mathbf{b}, \lambda(\mathbf{a}, \mathbf{X})) + m(\mathbf{a}, \lambda(\mathbf{b}, \mathbf{X})) \qquad \text{and ,}$$
$$\lambda(\mathbf{b}, (\mu(\mathbf{a}, \mathbf{X})) = m(\mathbf{a}, \lambda(\mathbf{b}, \mathbf{X})).$$

Write the relations in the Koszul dual for this i in terms of the dual generators. The ideal $R^{\perp}(3, i)$ is the intersection

$$< \lambda^{\vee}(\mathbf{a}, \mu^{\vee}(\mathbf{b}, \mathbf{X}));\ \lambda^{\vee}(m^{\vee}(\mathbf{a}, \mathbf{b}), \mathbf{X}) - m^{\vee}(\mathbf{a}, \lambda^{\vee}(\mathbf{b}, \mathbf{X})) >$$
$$\cap < \lambda^{\vee}(\mathbf{a}, \mu^{\vee}(\mathbf{b}, \mathbf{X})) - m^{\vee}(\mathbf{a}, \lambda^{\vee}(\mathbf{b}, \mathbf{X}));\ \lambda^{\vee}(m^{\vee}(\mathbf{a}, \mathbf{b}), \mathbf{X}) >$$
$$= < \lambda^{\vee}(m^{\vee}(\mathbf{a}, \mathbf{b}), \mathbf{X}) - m^{\vee}(\mathbf{a}, \lambda^{\vee}(\mathbf{b}, \mathbf{X})) + \lambda^{\vee}(\mathbf{b}, \mu^{\vee}(\mathbf{a}, \mathbf{X})) >,$$

where we use $< \ldots >$ to denote an ideal by its generators. This is the ideal generated by the Leibniz rule in the dual $\mathrm{Abd}^!$. Note that this interchanges the colours. If $i : (0, 1, 2, 3,) \mapsto (2, 1, 1, 2)$, then we have the action relations. Using $(\mathrm{Abd}^!)^! = \mathrm{Abd}$, we have a complete set of relations. QED

3.5.2 THEOREM *Let (R, A) be an algebroid.*

(i). Then the complex $C^{\mathrm{Abd}}_*(R, A)_{(2)}$ defines the structure of a dg Lie algebra on the bar complex $B_*(R, A)$.

(ii). The complex $C^{\mathrm{Abd}}_{*(1)}(R, A)$ defines on the Harrison complex $C^{\mathrm{Com}}_*(R)$ of R the structure of a dg module for the dg Lie algebra $B_*(R, A)$.

(iii). Furthermore, $C^{\mathrm{Abd}}_{*(2)}(R, A)$ is the Chevalley-Eilenberg complex of the dg Lie algebra $B_*(R, A)$, and $C^{\mathrm{Abd}}_{*(1)}(R, A)$ is the Chevalley-Eilenberg complex of $B_*(R, A)$ with coefficients in the Harrison complex $C^{\mathrm{Com}}_*(R)$. That is

$$C^{\mathrm{Abd}}_{*(2)}(R, A) = C^{Lie}_*(B_*(R, A)), \qquad \text{and}$$
$$C_{*(1)}(R, A) = C^{\mathrm{Lie}}_*(B_*(R, A), C^{\mathrm{Com}}_*(R)).$$

PROOF As a graded vector space, $C^{\mathrm{Abd}}_*(R, A)_{(2)} = \bar{S}'(B_*(R, A)[-1])$. The differential is a coderivation with respect to the free $\mathrm{Abd}^{\perp}$-structure. In particular it is a

coderivation of the symmetric coalgebra $\bar{S}'(B_*(R,A)[-1])$ and thus an L_∞-algebra structure on $B_*(R,A)$ as a graded vector space. Since the differential is quadratic, this L_∞-algebra is a dg Lie algebra. The inclusion $M\mathrm{Com}^\perp \subset \mathrm{Abd}^\perp$ and Example 3.3.9 show that the internal differential is indeed the bar differential on the dg vector space $B_*(R,A)$. This proves *(i)*.
As a graded vector space,

$$C^{\mathrm{Abd}}_{*(1)}(R,A) = C^{\mathrm{Com}}_*(R) \otimes \bar{S}'(B_*(R,A)[-1]).$$

Since the differential is again a quadratic coderivation of the $\mathrm{Abd}^\perp$-algebra, it is a coderivation of the $M\mathrm{Com}^\perp$-algebra. Once we observe that the differential restricts to the usual differential on $C^{\mathrm{Com}}_*(R)$ (by $\mathrm{Com}^\perp \subset \mathrm{Abd}^\perp$) this shows *(ii)*.
Finally, *(iii)* is a direct corollary to *(i)* and *(ii)*. QED

§**11** Koszul duality

3.5.3 THEOREM *The operad* Abd *is Koszul.*

PROOF Since Abd is an **2**-operad, the homology complex of an algebroid splits into a sum

$$C^{\mathrm{Abd}}(R,A) = C^{\mathrm{Abd}}_{(1)}(R,A) \oplus C^{\mathrm{Abd}}_{(2)}(R,A),$$

and each complex is a bicomplex with the bigrading induced by the colours of the inputs. The two lemmas below compute the homology of these two summands for a free algebroid (R,A). Corollary 3.3.8 shows that these propositions suffice to complete the proof. QED

3.5.4 LEMMA *Let* $(R,A) = F_{\mathrm{Abd}}(V,W)$. *Then* $H^{\mathrm{Abd}}_{*(2)}(R,A) = W$, *concentrated in degree 0.*

PROOF The complex $C^{\mathrm{Abd}}_{(2)}(R,A)$ allows a filtration with respect to the number of inputs of type A:

$$\mathcal{F}^{\leq m} C^{\mathrm{Abd}}_{(2)}(R,A) = \bigoplus_{|i^{-1}(2)|\leq m+1,\ i(0)=2} \mathrm{Abd}^\perp(n,i) \otimes_{\mathrm{Aut}(n,i)} (R,A)^{(n,i)}.$$

In other words, if we interpret $C^{\mathrm{Abd}}_{(2)}(R,A)$ as the Chevalley-Eilenberg complex of $B_*(R,A)$, it is a bicomplex, where one of the gradings is given by the internal degree of $B_*(R,A)$ and the other by the Chevaley-Eilenberg degree and we take the induced filtration by internal degree. We compute the corresponding spectral sequence. In E_0 we only have the part of the differential that comes from multiplying elements of R or applying the action of R on A. Example 3.4.8 shows that

$$E^0_{pq} = (S'^q(B_*(k, R, A)))_{p+q},$$

the q-fold symmetric tensor power of the bar complex of the left R-module A with the bar differential extended as a coderivation to the cofree coalgebra. Since A is free as an R-module the homology of the bar complex consists solely of the indecomposables $F_{\mathrm{Lie}}(W)$ of A in degree 0 (cf. Weibel [84]). Therefore (Künneth formula)

$$E^1_{pq} = \begin{cases} S'^q(F_{\mathrm{Lie}}(W))_{p+q} & \text{for } p = 0 \\ 0 & \text{otherwise,} \end{cases}$$

The differential on E^1 is the bracket on $F_{\mathrm{Lie}}(W)$ extended as a coderivation. Thus, E^1_{0*} is the Chevalley-Eilenberg homology complex of the Lie algebra $F_{\mathrm{Lie}}(W)$ with coefficients in k. Its cohomology is W concentrated in degree 0 (cf. Weibel [84]), and the spectral sequence collapses in E_2. QED

3.5.5 LEMMA *Let $(A, R) = F_{\mathrm{Abd}}(V, W)$. Then $H^{\mathrm{Abd}}_{(1)}(R, A) = V$, concentrated in degree 0.*

PROOF As a complex, $C^{\mathrm{Abd}}_{(1)}(R, A)$ again enjoys a filtration by inputs of type A:

$$\tilde{\mathcal{F}}^{\leq m} C^{\mathrm{Abd}}_{(1)}(R, A) = \bigoplus_{|i^{-1}(2)| \leq m,\ i(0)=1} \mathrm{Abd}^{\perp}(n, i) \otimes_{\mathrm{Aut}(n,i)} (R, A)^{(n,i)},$$

which comes from the internal degree of the dg Lie algebra $B_*(R, A)$ and the dg module $C^{\mathrm{Com}}_*(R)$. Construct the spectral sequence $\tilde{E}$ associated to this filtration. Since A is a free R module and R is itself free as a commutative algebra, we conclude that E^1 is the Lie algebra homology of the indecomposables of A with coefficients in the indecomposables of R. More explicitly,

$$\tilde{E}^1_{pq} = \begin{cases} C^{\mathrm{Lie}}_q(F_{\mathrm{Lie}}(R), F^+_{Ass}(R) \otimes W) & \text{if } p = 0 \\ 0 & \text{otherwise,} \end{cases}$$

where $F^+_{\mathrm{Ass}}(R) \otimes W$ is the free module. Consequently, $\tilde{E}_{**}$ is W concentrated in degree 0. QED

§12 The explicit formulae

This section explores in more detail the algebraic structures of Theorem 3.5.2. In particular, we give explicit formulae for the dg Lie and dg module structures and check the axioms by classical means. Moreover, we include explicit versions of the proof of the dg Lie algebra structure. Strictly speaking, this is superfluous, but it is reassuring to the author to see the explicit cancellations.

3.5.6 LEMMA *Let (R, A) be a pair of a commutative algebra and a Lie algebra together with mutual actions of R on A and of A on R. Then the bracket*

$$\begin{aligned}
[(f_1,\dots,f_n,X),&(g_1,\dots,g_m,Y)] = \\
&(\mathrm{Sh}(f_1,\dots,f_n;g_1,\dots,g_m),[X,Y]) \\
&\pm\sum_i(\mathrm{Sh}(f_1,\dots,f_n;g_1,\dots,g_{i-1}),X(g_i),\dots,g_m),Y), \\
&\mp\sum_i(\mathrm{Sh}(f_1,\dots,f_{i-1};g_1,\dots,g_m),Y(f_i),\dots,f_n,X)
\end{aligned}$$

makes the bar construction $B_*(R, A)$ of R with coefficients in A a graded Lie algebra. Here $\pm$ denotes the sign of the permutation applied on $(f_1,\dots,f_n,g_1,\dots,g_m)$ to order these elements the way they occur in the corresponding term, and $\mp$ denotes the opposite sign (these signs incorporate the signs of the shuffles).

PROOF To compute the bracket $[[(f_1,\dots,f_n,X),(g_1,\dots,g_m,Y)],(h_1,\dots,h_k,Z)]$ use notation $f_{n',n} = (f_{n'},\dots,f_n)$. The terms fall apart in several groups. The sign $\pm$ (resp. $\mp$) is the sign of the permutation applied to $(f_1,\dots,f_n,g_1,\dots,g_m,h_1,\dots,h_k)$ (resp. its opposite). There is one term with two brackets in the A-factor:

$$\pm(\mathrm{Sh}(f_{1,n};g_{1,m};h_{1,k}),[[X,Y],Z]).$$

The terms with one bracket in the A-factor:

$$\begin{aligned}
&\pm\sum_{k',m'}(\mathrm{Sh}(f_{1,n};g_{1,m'-1};h_{1,k'}),X(g_{m'}),\mathrm{Sh}(g_{m'+1,m};h_{k'+1,k}),[Y,Z]) \\
&\pm\sum_{k',n'}(\mathrm{Sh}(f_{1,n'-1};g_{1,m};h_{1,k'}),Y(f_{n'}),\mathrm{Sh}(f_{n'+1,n};h_{k'+1,k}),[X,Z]) \\
&\mp\sum_{n',m,}(\mathrm{Sh}(f_{1,n'-1};g_{1,m'};h_{1,k}),Z(f_{n'}),\mathrm{Sh}(f_{n'+1,n};g_{m'+1,m}),[X,Y]) \\
&\mp\sum_{n',m,}(\mathrm{Sh}(f_{1,n'};g_{1,m'-1};h_{1,h_k}),Z(g_{m'}),\mathrm{Sh}(f_{n'+1,n};g_{m'+1,m}),[X,Y]).
\end{aligned}$$

The terms where two elements of A or a bracket of these acts on one element of R:

$$\pm \sum_{k'} (\mathrm{Sh}(f_{1,n}; g_{1,m}; h_{1,k'-1}), [X,Y](h_{k'}), h_{k'+1,k}, Z)$$
$$\mp \sum_{m'} (\mathrm{Sh}(f_{1,n}; g_{1,m'-1}; h_{1,k}), ZX(g_{m'}), g_{m'+1,m}, Y)$$
$$\pm \sum_{n'} (\mathrm{Sh}(f_{1,n'-1}; g_{1,m}; h_{1,k}), ZY(f_{n'}), f_{n'+1,n}, X).$$

Terms where two elements of A act on elements of R of the same type:

$$\mp \sum_{n',m'<m''} (\mathrm{Sh}(f_{1,n'}; g_{1,m'-1}; h_{1,k}), Z(g_{m'}), \mathrm{Sh}(f_{n'+1,n}; g_{m'+1,m''-1}), X(g_{m''}), g_{m''+1,m}, Y)$$
$$\mp \sum_{m'<m'',k'} (\mathrm{Sh}(f_{1,n}; g_{1,m'}; h_{1,k'}), X(g_{m'}), \mathrm{Sh}(g_{m'+1,m''-1}; h_{k',k}), Z(g_{m''}), g_{m''+1,m}, Y)$$
$$\pm \sum_{n'<n'',k'} (\mathrm{Sh}(f_{1,n'-1}; g_{1,m}; h_{1,k'}), Y(f_{n'}), \mathrm{Sh}(f_{n'+1,n''-1}; h_{k'+1,k}), Z(f_{n''}), f_{n''+1,n}, X)$$
$$\pm \sum_{n'<n'',m'} (\mathrm{Sh}(f_{1,n'-1}; g_{1,m'}; h_{1,k}), Z(f_{n'}), \mathrm{Sh}(f_{n'+1,n''-1}; g_{m'+1,m}), Y(f_{n''}), f_{n''+1,n}, X).$$

The remaining terms:

$$\pm \sum_{k'<k'',m'} (\mathrm{Sh}(f_{1,n}; g_{1,m'-1}; h_{1,k'}), X(g_{m'}), \mathrm{Sh}(g_{m'+1,m}; h_{k',k''-1}), Y(h_{k''}), h_{k''+1,k}, Z)$$
$$\mp \sum_{k'<k'',n'} (\mathrm{Sh}(f_{1,n'}; g_{1,m}; h_{1,k'}), Y(f_{n'}), \mathrm{Sh}(f_{n'+1,n}; h_{k',k''-1}), X(h_{k''}), h_{k''+1,k}, Z)$$
$$\mp \sum_{n',m'<m''} (\mathrm{Sh}(f_{1,n'-1}; g_{1,m'}; h_{1,k}), Z(f_{n'}), \mathrm{Sh}(f_{n'+1n}; g_{m'+1,m''-1}), X(g_{m''}), g_{m''+1,m}, Y)$$
$$\pm \sum_{n'<n'',m'} (\mathrm{Sh}(f_{1,n'}; g_{1,m'-1}; h_{1,k}), Z(g_{m'}), \mathrm{Sh}(f_{n'+1,n''-1}; g_{m',m}), Y(f_{n''}), f_{n''+1,n}, X).$$

To prove the result we should show that this expression plus the same expression for the cyclic permutations of $(f_1, \ldots, f_n, X)$, $(g_1, \ldots, g_m, Y)$, and $(h_1, \ldots, h_k, Z)$ equals zero. (Do not forget the signs coming from the grading here!) The terms above are ordered in groups of terms that cancel together with their cyclic translates. The result follows from some nasty book keeping. QED

3.5.7 THEOREM *If (R, A) is a pair of a commutative algebra together with a module A. Then a Lie algebra structure on A and an action of A on R make (R, A) an algebroid iff the bar complex $B_*(R, A)$ of R with coefficients in A is a dg Lie algebra with respect to the bracket on Lemma 3.5.6.*

PROOF By Lemma 3.5.6 remains to check the compatibility with the differential is equivalent to the algebroid axioms. The proof uses the alternative formula

$$\begin{aligned}
[(f_{1,n}, X); (g_{1,m}, Y)] = {} & (\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m}), [X, Y]) \\
& + \sum_{m'} (\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m'-1}), X(g_{m'}), g_{m'+1,m}, Y) \\
& - (-1)^{mn} \sum_{n'} (\mathrm{Sh}^{\pm}(g_{1,m}, f_{1,n'-1}, Y(f_{n'}), f_{n'-1,n}, X),
\end{aligned}$$

for the bracket of Lemma 3.5.6. Here $\mathrm{Sh}^{\pm}$ is the sum of all shuffles twisted by the sign on the shuffle. Write $df_{n',n} = \sum_{i=n'}^{n-1}(-1)^{i-n'}(f_{n'}, \ldots, f_i f_{i+1}, \ldots, f_n)$ (the differential of the bar complex applied to $f_{n',n}$) whenever convenient.

Next we show that compatibility of differential and bracket on $B_*(R, A)$ follows from the algebroid axioms. Compute that $d[(f_{1,n}, X), g_{1,m}, Y)]$ is the sum of the following terms:

$$\begin{aligned}
& + (d\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m}), [X, Y]) \\
& + \sum_{m'} (d\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m'-1}), X(g_{m'}), g_{m'+1,m}, Y) \\
& - \sum_{n'} (d\mathrm{Sh}^{\pm}(g_{1,m}; f_{1,n'-1}), Y(f_{n'}), f_{n'+1,n}, X)(-1)^{mn} \\
& + \sum_{m'} (\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m'-1}), X(g_{m'}), dg_{m'+1,m}, Y)(-1)^{n+m'+1} \\
& - \sum_{n'} (\mathrm{Sh}^{\pm}(g_{1,m}; f_{1,n'-1}), Y(f_{n'}), df_{n'+1,n}, X)(-1)^{n'+m+1+mn},
\end{aligned}$$

and

$$\begin{aligned}
& + \sum_{m'} (\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m'-1}), X(g_{m'}), g_{m'+1,m-1}, g_m Y)(-1)^{n+m+1} \\
& - \sum_{n'} (\mathrm{Sh}^{\pm}(g_{1,m}; f_{1,n'-1}), Y(f_{n'}), f_{n'+1,n-1}, f_n X)(-1)^{mn+m+n+1}, \\
& + \sum_{m'} (\mathrm{Sh}^{\pm}(f_{1,n-1}; g_{1,m'-1}), f_n X(g_{m'}), g_{m'+1,m}, Y)(-1)^{n+1} \\
& - \sum_{n'} (\mathrm{Sh}^{\pm}(g_{1,m-1}; f_{1,n'-1}), g_m Y(f_{n'}), f_{n'+1,n}, X)(-1)^{mn+m+1},
\end{aligned}$$

and

$$\begin{aligned}
& + (\mathrm{Sh}^{\pm}(f_{1,n-1}; g_{1,m}), f_n[X, Y])(-1)^{n+1} \\
& - (\mathrm{Sh}^{\pm}(g_{1,m}; f_{1,n-1}), Y(f_n)X)(-1)^{n+m+mn} \\
& + (\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m-1}), g_m[X, Y])(-1)^{n+m+1} \\
& + (\mathrm{Sh}^{\pm}(f_{1,n}; g_{1,m-1}), X(g_m)Y)(-1)^{n+m},
\end{aligned}$$

and

$$
\begin{aligned}
&+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m'-2}),g_{m'-1}X(g_{m'}),g_{m'+1,m},Y)(-1)^{n+m'}\\
&+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m'-1}),X(g_{m'})g_{m'+1},g_{m'+2,m},Y)(-1)^{n+m'+1}\\
&-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m};f_{1,n'-2}),f_{n'-1}Y(f_{n'}),f_{n'+1,n},X)(-1)^{mn+n'+m}\\
&-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m};f_{1,n'-1}),Y(f_{n'})f_{n'+1},f_{n'+2,n},X)(-1)^{mn+n'+m+1}.
\end{aligned}
$$

Similarly, we can compute $[d(f_{1,n},X);(g_{1,m},Y)]+(-1)^n[(f_{1,n},X);(g_{1,m},Y)]$ as the sum of the following terms:

$$
\begin{aligned}
&+(\mathrm{Sh}^{\pm}(df_{1,n};g_{1,m}),[X,Y])\\
&+(\mathrm{Sh}^{\pm}(f_{1,n};dg_{1,m}),[X,Y])(-1)^n\\
&+\sum_{m'}(\mathrm{Sh}^{\pm}(df_{1,n};g_{1,m'-1}),X(g_{m'}),g_{m'+1,m},Y)\\
&-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m};df_{1,n'-1}),Y(f_{n'}),f_{n'+1,n},X)(-1)^{m+(n-1)m}\\
&-\sum_{n'}(\mathrm{Sh}^{\pm}(dg_{1,m};f_{1,n'-1}),Y(f_{n'}),f_{n'+1,n},X)(-1)^{n+n(m-1)}\\
&+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n};dg_{1,m'-1}),X(g_{m'}),g_{m'+1,m},Y)(-1)^{n}\\
&+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m'-1}),X(g_{m'}),dg_{m'+1,m},Y)(-1)^{n+m'+1}\\
&-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m};f_{1,n'-1}),Y(f_{n'}),df_{n'+1,n},X)(-1)^{n'+1+m(n-1)},
\end{aligned}
$$

and

$$
\begin{aligned}
&-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m-1};f_{1,n'-1}),g_mY(f_{n'}),f_{n'+1,n},X)(-1)^{n(m-1)+m+n+1}\\
&-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m};f_{1,n'-1}),Y(f_{n'}),f_{n'+1,n-1},f_nX)(-1)^{(n-1)m+n+1},
\end{aligned}
$$

and

$$
\begin{aligned}
&+(\mathrm{Sh}^{\pm}(f_{1,n-1};g_{1,m}),[f_nX,Y])(-1)^{n+1}\\
&+(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m-1}),[X,g_mY])(-1)^{n(m-1)+m+1},
\end{aligned}
$$

and

$$
\begin{aligned}
&+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n-1};g_{1,m'-1}),f_nX(g_{m'}),g_{m'+1,m},Y)(-1)^{n+1}\\
&+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m'-1}),X(g_{m'}),g_{m'+1,m-1},g_mY)(-1)^{n+m+1},
\end{aligned}
$$

and

$$+\sum_{m'}(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m'-2}),X(g_{m'}g_{m'+1}),g_{m'+1,m},Y)(-1)^{n+m'+1}$$
$$-\sum_{n'}(\mathrm{Sh}^{\pm}(g_{1,m};f_{1,n'-2}),Y(f_{n'}f_{n'+1}),f_{n'+1,n},X)(-1)^{n'+1+(n-1)m}.$$

The way we ordered the terms it is easy to conclude that bracket and differential are indeed compatible if (R, A) is an algebroid.
On the other hand, assuming that $B_*(R, A)$ is a dg Lie algebra, the algebroid axioms follow. The algebroid axioms follows from the expressions above for $n = 1$ and $m = 0$. Then, the linearity of the action for $n = 1$ and $m = 1$, and the derivation property follows for $n = 2$, $m = 0$. QED

3.5.8 THEOREM *Let (R, A) be an algebroid, and consider $B_*(R, A)$ as a dg Lie algebra.*

(i). The bar complex $B_*(R)$ is a dg module for the dg Lie algebra $B_*(R, A)$ with respect to the action

$$(f_1,\dots,f_n,X)(g_1,\dots,g_m) = \sum_{m'=1}^{m}(\mathrm{Sh}^{\pm}(f_1,\dots,f_n;g_1,\dots,g_{m'-1}),X(g_{m'}),\dots,g_m).$$

(ii). The dg Lie algebra $B_*(R, A)$ acts by derivations with respect to the shuffle product on $B_*(R)$. Consequently, the action descends to a dg $B_*(R, A)$-module structure on the Harrison complex $C^{\mathrm{Com}}_*(R)$.

PROOF The proof that the formula above makes $B_*(R)$ a graded module is a somewhat less involved version of the argument in the proof of Lemma 3.5.6. Similarly, the compatibility with the differential follows from an easier version of the proof of Theorem 3.5.7. This shows *(i)*.
To prove *(ii)*, observe that

$$(f_{1,n},X)(\mathrm{Sh}^{\pm}(g_{1,m};h_{1,k})) = \sum_{m',k'}\pm(\mathrm{Sh}(f_{1,n};g_{1,m'-1};h_{1,k'}),X(g_{m'}),\mathrm{Sh}^{\pm}(g_{m'+1,m};h_{k'+1,k}))$$
$$+\sum_{m',k'}\pm(\mathrm{Sh}^{\pm}(f_{1,n};g_{1,m'};h_{1,k'-1}),X(h_{k'}),\mathrm{Sh}^{\pm}(g_{m'+1,m};h_{k'+1,k})),$$

again with the sign obtained from the permutation of elements of R. The result then follows from associativity of the shuffle product. QED

3.6 Algebroids and Poisson algebras

§13 An adjunction

3.6.1 DEFINITION A Poisson algebra P is a commutative algebra P together with a Lie bracket $[,] : P \otimes P \longrightarrow P$ that is a derivation of the commutative product on P. That is, for all $a \in P$, the map $[a, -] : P \longrightarrow P$ is a derivation of the algebra structure. The most classical example of a Poisson algebra is the algebra of functions on a Poisson manifold.

3.6.2 PROPOSITION *Let $\varphi : \{1, 2\} \longrightarrow *$ be the unique map of sets, and denote by* Pois *the operad whose algebras are Poisson algebras.*

(i). There exists a map of $\{1, 2\}$-operads $\mathrm{Abd} \longrightarrow \varphi^*\mathrm{Pois}$ from the operad of algebroids to the $\{1, 2\}$-operad associated to the operad for Poisson algebras.

*(ii). The map induces the adjunction Abd-**Alg** $\leftrightarrows$ Pois-**Alg**, given by the functor from Poisson algebras to Algebroids given by $P \longmapsto (P, P)$ (cf. Example 3.2.4) and its right adjoint $(R, A) \longmapsto R \oplus A$, where the multiplication and Lie bracket on $R \oplus A$ are given by*

$$(a, X)(b, Y) = (ab, aY + bX)$$
$$[(a, X), (b, Y)] = (X(b) - Y(a), [X, Y]).$$

PROOF The Poisson operad Pois is the quadratic operad generated by an antisymmetric bracket l and a symmetric multiplication m with the relations expressing associativity of m, the Jacobi identity for l, and the fact that the adjoint representation for the Lie algebra structure l is an action by derivations with respect to m. The morphism from Abd is given by sending λ and l in $\mathrm{Abd}(2)$ to l and m and μ in $\mathrm{Abd}(2)$ to m in $\varphi^*\mathrm{Pois}(2)$, in such a way that the colours of input and outputs are preserved by the map. It is immediate that this defines an morphism of operads, which proves *(i)*. The second part of the proposition then follows immediately by a general argument. QED

§14 Poisson homology

Poisson algebras are algebras over a self dual quadratic Koszul operad. The Poisson homology complex has been studied by Fresse [22]. In fact the adjunction in the previous section gives a relation between the homology of Poisson algebras and algebroids. We will only give the definition of the complex $C_*^{\mathrm{Pois}}(P)$.

3.6.3 DEFINITION (FRESSE [22]) Let P be a Poisson algebra. Then the Poisson homology complex satisfies

$$C_*^{\mathrm{Pois}}(P) = F'_{\mathrm{Com}}(F'_{\mathrm{Lie}}(P)).$$

The differential is the differential that makes $C_*^{\mathrm{Pois}}(P)$ the Chevalley-Eilenberg complex of the dg Lie algebra $(F'_{\mathrm{Lie}}(P)) = C^*_{\mathrm{Com}}(P)$ with the bracket given by the bracket on P and extended as a coderivation.

3.6.4 PROPOSITION *Let (R, A) be an algebroid. Then there is a natural inclusion of complexes*

$$C_{*(2)}^{\mathrm{Abd}}(R, A) \longrightarrow C_*^{\mathrm{Pois}}(R \oplus A),$$

induced by a natural inclusion of dg Lie algebras $B_*(R, A) \subset C_*^{\mathrm{Com}}(R \oplus A)$.

PROOF This result follows directly from the operad map in Proposition 3.6.2. However, we can also give an direct argument. According to Lemma 3.4.5, there is a natural inclusion of complexes $B_*(R, A) \subset B_{*+1}(R \oplus A)$, which descends to an inclusion $B_*(R, A) \subset C_{*+1}^{\mathrm{Com}}(R \oplus A)$. The dg Lie algebra structure on this last dg Lie algebra ($R \oplus A$ is a Poisson algebra) is given by $[(a, X), (b, Y)] = (ab, [X, Y])$ extended as a derivation of the cofree coproduct on $B_{*+1}(R \oplus A)$. One checks directly that this inclusion is in fact an inclusion of dg Lie algebras. QED

CHAPTER **4**

Strongly homotopy operads

This chapter starts by showing that the $\mathbb{N}$-coloured operad of non-sigma pseudo operads is a self dual quadratic Koszul operad. This suggests a homotopy invariant version of operads. Singular $\mathbb{Q}$-chains on configuration spaces of ordered points in the unit disk in $\mathbb{R}^2$ form a strongly homotopy operad homotopic to the operad of $\mathbb{Q}$-chains on little disks. In addition, the chapter exhibits a functor from strongly homotopy operads to L_∞-algebras. This can be applied to show that Markl's cotangent complex [64] enjoys an L_∞-structure, and to prove an invariance result for the L_∞-algebra governing deformations of operad algebras (cf. Kontsevich-Soibelman [46]).

4.1 Introduction

This chapter defines operads up to homotopy on the basis of Koszul duality and shows how this concept can be applied as a new tool to study topics related to homotopy theory of operads. The advantage of working with operads up to homotopy is comparable to the advantage of L_∞-algebras over dg Lie algebras (cf. §8 and §12 in this chapter).

One of the annoying points in studying the interplay between homotopy theory of algebras and homotopy theory of operads is that the application $V \longmapsto \mathrm{End}_V$ from dg vector spaces to operads is not functorial. This Chapter derives the existence homomorphisms of operads up to homotopy $\mathrm{End}_W \rightsquigarrow \mathrm{End}_V$ under some conditions. As a consequence we show that the singular $\mathbb{Q}$-chains on configuration spaces of distinct numbered points in the unit disk in $\mathbb{R}^2$ from a s.h. operad homotopy equivalent to the singular $\mathbb{Q}$-chains on the little disk operad. It should be stressed that we do not need to compactify the configurations spaces.

As a second application, the Chapter gives a functorial construction of an L_∞-algebra form a s.h. operad. This construction (which in the strict case reduces to the well known construction of a dg Lie algebra from an operad) allows us to show that Markl's cotangent complex does possess a natural L_∞-algebra structure. Moreover, if we choose as coefficients the endomorphism operad of an algebra this complex is shown to be a subcomplex of the Kontsevich-Soibelman complex for deformation of operad algebras.
This is not the first homotopy invariant version of operads. For example, weak operads were introduced by Brinkmeier [5]. These are a homotopy invariant version of operads in the topological category. A weak operad is a collection together with circle-i operations. The operations are almost equivariant and associative, in the sense that there exists a coherent system of homotopies for all possible compositions of $\circ_i$-operations and permutations. The result is an ingenious system of cellular spaces, that contains associahedra, permutahedra, and non-trivial mixtures of these. The complexity of the structure makes it hard to grasp the concept in its full generality. The present approach is less involved. The price we have to pay is that we obtain a less general concept, where the compatibility with the symmetric group actions is strict.

Section 4.2 proves that the $\mathbb{N}$-coloured operad of non-symmetric pseudo operads is a self dual Koszul operad. This fact is used to define strongly homotopy operads and morphisms between them as an equivariant version of the usual definition of strongly homotopy algebras for a Koszul operad. Section 4.3 shows how this approach can be applied to prove homotopy invariance properties of homotopy Q-algebras for an augmented operad Q, and applies this to singular $\mathbb{Q}$-chains on configuration spaces which form an operad up to homotopy. Section 4.4 provides the construction of an L_∞-algebra associated to a s.h. operad. This construction reduces to the usual Lie algebra construction when we start from an operad, and to the commutator L_∞-algebra if we restrict to A_∞-algebras. Section 4.5 applies the L_∞-algebra construction to convolution s.h. operads. As an application, it is shown that Markl's cotangent complex has a natural L_∞-structure. Study of the solution of the Maurer-Cartan equation shows that the cotangent complex should be interpreted as a L_∞-algebra controlling deformation of operad maps. With a slight modification, we recover the Kontsevich-Soibelman approach to deformation of operad algebras.

4.2 Operads up to homotopy

§1 An operad of non-symmetric pseudo operads

Recall the conventions on graphs, trees, and planar trees of Definition 0.3.6. In particular, for a graph t the set of vertices is denoted $\mathbf{v}(t)$, $\mathbf{e}(t)$ is the set of internal edges, $\mathbf{l}(t)$ is the set of external edges or legs, and $\mathbf{l}_t(v)$ the set of edges or legs attached to a vertex v.

4.2.1 DEFINITION The $\mathbb{N}$-operad PsOpd is the $\mathbb{N}$-operad such that $\mathrm{PsOpd}(n, i)$ is spanned by planar rooted trees t (cf. Definition 0.3.6) with n vertices numbered 1 up to n, that satisfy $|\mathbf{l}_t(k)| - 1 = i(k)$ for $k = 1, \ldots, n$, and $i(0) = |\mathbf{l}(t)| - 1$. Composition $s \circ_k t$ is defined by replacing vertex k in s by the planar rooted tree t (cf. Figure 10). More precisely, $s \circ_k t$ has vertices $\mathbf{v}(s) - \{k\} \cup \mathbf{v}(t)$, and edges $\mathbf{e}(s) \cup \mathbf{e}(t)$, where the elements of $\mathbf{l}_s(k)$ necessary to define the edges of s are interpreted as elements of $\mathbf{l}(t)$. This is well defined since the planar structure gives a natural isomorphism between $\mathbf{l}(t)$ and $\mathbf{l}_s(k)$.

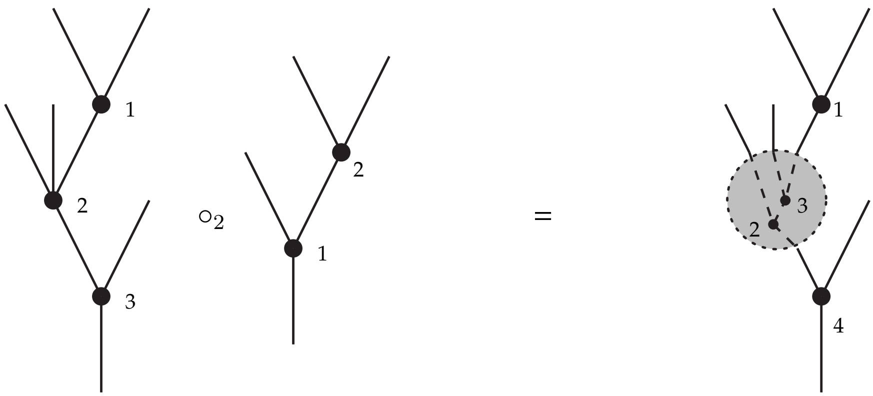


FIGURE 10: Composition $\circ_2$ in PsOpd: vertex 2 of the left tree is replaced by a tree with matching number of legs.

Recall non-symmetric pseudo operads as discussed in Definition 1.2.1. Non-Σ operads are easy to handle when compared to operads, since one need not worry about compatibility with symmetric group actions.

4.2.2 PROPOSITION *The* $\mathbb{N}$*-operad* PsOpd *is quadratic. Algebras for* PsOpd *are non-symmetric pseudo operads.*

PROOF Every planar rooted tree can be constructed from 2-vertex trees by compositions in PsOpd, adding one edge at a time. Denote by $(\mathbf{m} \circ_i \mathbf{n})$ the 2-vertex planar rooted tree with the root vertex having legs $\{0, \ldots, m\}$, and the other vertex having legs $\{0, \ldots, n\}$. The unique internal edge connects leg i of the root vertex to leg 0 of the other vertex. These generators satisfy the quadratic relations

$$(\mathbf{k} \circ_j (\mathbf{m} \circ_i \mathbf{n})) = \begin{cases} ((\mathbf{k} \circ_i \mathbf{n}) \circ_{j+n-1} \mathbf{m}) & \text{if} i < j \\ (\mathbf{k} \circ_j (\mathbf{m} \circ_{i-j+1} \mathbf{n})) & \text{if } j \leq i < j+m \\ ((\mathbf{k} \circ_{i-m-1} \mathbf{n}) \circ_j \mathbf{m})) & \text{if } j \leq i+m \end{cases} \tag{4.2.1}$$

The generators and relations above define a quadratic Σ-free $\mathbb{N}$-operad which has non-symmetric pseudo operads as algebras, as follows from the definition (cf. Definition 1.2.1). Denote this quadratic operad TE/R.
To identify the two 1-reduced Σ-free operads PsOpd and TE/R it suffices to identify the free algebras on 1 generator in each colour. The free non-symmetric pseudo operad on $A = \{A_n\}_{n \in \mathbb{N}}$ is given as $\bigoplus_t A(t)$, where $A(t) = \bigotimes_{v \in \mathbf{v}(t)} A_{|\mathbf{l}_t(v)|-1}$ (cf. Loday [58], Appendix B), which is exactly the free PsOpd-algebra on these generators. QED

§2 Koszul duality for PsOpd

4.2.3 THEOREM *The* $\mathbb{N}$*-operad* PsOpd *is a self dual Koszul operad.*

PROOF The dimension of $R(3)$ is exactly half the dimension of $\mathrm{PsOpd}(3)$, since the associativity relations divide the basis elements of $\mathrm{PsOpd}(3)$ in pairs which satisfy a non-trivial relation. Observe that the dual relations $R^{\perp}(3)$ certainly are contained in the ideal generated by

$$(\mathbf{k} \circ_j (\mathbf{m} \circ_i \mathbf{n})) = \begin{cases} ((\mathbf{k} \circ_i \mathbf{n}) \circ_{j+n-1} \mathbf{m}) & \text{if } i < j \\ -(\mathbf{k} \circ_j (\mathbf{m} \circ_{i-j+1} \mathbf{n})) & \text{if } j \leq i < j+m \\ ((\mathbf{k} \circ_{i-m-1} \mathbf{n}) \circ_j \mathbf{m})) & \text{if } j \leq i+m \end{cases}$$

By a dimension argument these relations must exactly be all the relations. Then a base change shows that $(\mathrm{PsOpd})^!$ is isomorphic to PsOpd. The base change is given by multiplying a basis element corresponding to a planar rooted tree t with the sign $(-1)^{c(t)}$, where $c(t)$ is the number of internal axils of t. That is, the number of distinct subsets $\{v, w, u\} \subset \mathbf{v}(t)$ such that two of the three vertices are direct predecessors of the third. This shows that PsOpd is self dual.

The PsOpd-algebra homology complex of P is as a graded vector space the free non-symmetric pseudo cooperad on the collection $P[-1]$. That is,

$$C_*^{\mathrm{PsOpd}}(P) = F'_{\mathrm{PsOpd}^\perp}(P) = \bigoplus_{t \text{ planar}} \bigotimes_{v \in t} P(\mathbf{l}_t(v)).$$

The differential is given by contracting edges using the $\circ_i$-compositions in P. In other words, this complex is the non-symmetric bar construction $B_{\not\Sigma}P$ (cf. Loday [58], appendix B). The Theorem follows since the homology of this complex vanishes in the case where $P = T_{\not\Sigma}C$, the free non-symmetric operad on a collection C.

QED

4.2.4 REMARK Theorem 4.2.3 invites the reader to a conceptual excursion. As explained in the proof, the homology complex

$$C_*^{\mathrm{PsOpd}}(P) = (F'_{\mathrm{PsOpd}^\perp}(P), \partial)$$

of a non-symmetric pseudo operad P is the non-symmetric bar complex of P. This shows how bar/cobar duality for non-symmetric operads is an example of Koszul duality for the coloured operad PsOpd. The bar construction $B_{\not\Sigma}P$ of a non-symmetric operad is nothing but the PsOpd-algebra complex of P, computing the PsOpd-algebra homology of the algebra P. The homology of the bar complex is the left derived functor of taking indecomposables.

4.2.5 DEFINITION Let $P = \{P(n)\}_{n \geq 0}$ be a sequence of vector spaces. The formalism of Koszul duality defines a *homotopy algebra for* PsOpd (or a *homotopy* PsOpd-*algebra*) as a square zero coderivation ∂ of the 'cofree' $\mathrm{PsOpd}^\perp$-coalgebra on P of cohomological degree +1 (cf. Ginzburg-Kapranov [29]).
For a planar rooted tree t define $P(t) = \bigotimes_{v \in \mathbf{v}(t)} P_{|\mathbf{l}(t)|-1}$. A homotopy PsOpd-algebra structure on P is determined by operations

$$\circ_t : P[-1](t) \longrightarrow P[-1]$$

of degree $+1$, one for each planar rooted tree t. The condition on $\partial^2 = 0$ on the differential is equivalent to a sequence of relations on these operations. For each planar rooted tree t, we obtain a relation of the form

$$\sum_{s \subset t} \pm(\circ_{t/s}) \circ (\circ_s) = 0, \qquad (4.2.2)$$

where the sum is over (connected) planar subtrees s of t and t/s is the tree obtained from t by contracting the subtree s to a point, and the signs involved are induced by a choice of ordering on the vertices of the planar rooted trees t and s in combination with the Koszul convention. Here a connected planar subtree is a subset of vertices together with all their legs and edges such that the graph they constitute is connected. One term of the sum is illustrated in Figure 11.

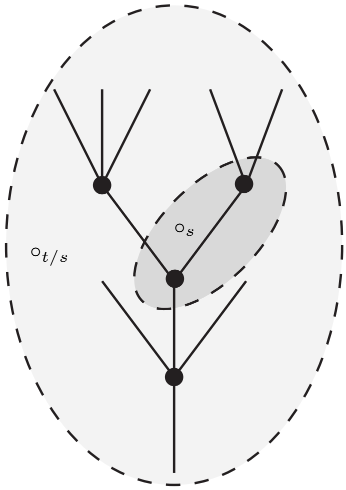


FIGURE 11: One summand of Equation (4.2.2): $\circ_s$ contracts the darker part, covering subtree s of t and $\circ_{t/s}$ contracts the remaining tree.

§3 Operads up to homotopy

4.2.6 DEFINITION Let $P = \{P(n)\}_{n\in\mathbb{N}}$ be a collection such that the vector spaces $P(n)$ form a homotopy PsOpd-algebra (cf. Definition 4.2.5). Let t and t' be planar rooted trees. If $\sigma : t \longmapsto t'$ is an isomorphism of the underlying rooted trees, then σ induces $\sigma : P(t) \longmapsto P(t')$ through the maps of $\mathrm{Aut}(\mathbf{l}(v))$-modules in the tensor factors of $A(t)$, and it induced $\mathbf{l}(\sigma) : \mathbf{l}(t) \longrightarrow \mathbf{l}(t')$ and consequently a map of $\mathrm{Aut}(\mathbf{l}(t))$-modules $\mathbf{l}(\sigma) : P(\mathbf{l}(t)) \longrightarrow P(\mathbf{l}(t'))$. (One should bear in mind that the planar structure of t induces a natural identification of $P(\mathbf{l}(t))$ with $P(n)$, where $n = |\mathbf{l}(t)| - 1$.)
A homotopy PsOpd-algebra P is *equivariant* if for every planar rooted tree t and every automorphism σ as above

$$\mathbf{l}(\sigma) \circ (\circ_t) = (\circ_{t'}) \circ \sigma.$$

An *strongly homotopy operad* (or *s.h. operad*) is an equivariant homotopy PsOpd-algebra.
A differential ∂ on $F'_{\mathrm{PsOpd}^\perp}(P)$ induces a differential on $T'(P[-1])$ iff ∂ defines a

s.h. operad. To see this, note that one can write

$$T'(P[-1])(n) = \lim_{t \in PT(n)} P(t),$$

where the limit is over the groupoid $PT(n)$ of planar rooted trees with n leafs different from the root, and $P(t) = \bigotimes_{v \in \mathbf{v}(t)} P(|\mathbf{l}(v)| - 1)$.
Dually, a *strongly homotopy cooperad* (or *s.h. cooperad*) C is a collection C together with an equivariant differential ∂ on the free nonsymmetric operad on the collection $C[1]$. Many statements on s.h. operads can be dualised to s.h. cooperads by 'inverting arrows'. Proofs can be copied almost verbatim. The sequel of this section restricts to s.h. operads. Denote

$$BP = (T'(P[-1]), \partial), \qquad \text{and} \qquad \reflectbox{B}C = (T(C[1]), \partial),$$

where in both cases ∂ is the induced differential.

4.2.7 EXAMPLE The correspondence between (co)operad structures and differentials given by bar and cobar constructions makes (co)operads a special case of s.h. (co)operads, as is suggested by the notations BP and $\reflectbox{B}C$. In the operad case, the trees with $|\mathbf{v}(t)| = 1$ define the internal differential, and the trees with $|\mathbf{v}(t)| = 2$ the compositions $\circ_i$. The $\circ_t$ operations vanish if $|\mathbf{v}(t)| \geq 3$. The conditions on the $\circ_t$-operations translate into the operad axioms. Operads are exactly s.h. operads such that $\circ_s$ vanishes if $\mathbf{v}(s) \geq 3$. One can interpret s.h. operads as a generalisation of operads where one needs 'higher homotopies' that measure the failure of associativity of the $\circ_i$ operations.

4.2.8 EXAMPLE Let P be a s.h. operad. When $|\mathbf{v}(t)| = 1$, $\circ_t$ defines an internal differential on $P(\mathbf{l}(t))$. When $|\mathbf{v}(t)| = 2$, the operation $\circ_t$ defines a circle-i operation. In general these operations need no longer be associative. If $\circ_s$ does not vanish for $|\mathbf{v}(s)| = 3$, then Equation (4.2.2) expresses that $\circ_s$ serves as a homotopy for associativity as follows. Denote the internal differential by d and the two contractions of the internal edges e or e' of s by $\circ_e$ and $\circ_{e'}$ that correspond to operad compositions. The formula

$$(\circ_e) \circ (\circ_{e'}) - (\circ_{e'}) \circ (\circ_e) = d \circ (\circ_s) + (\circ_s) \circ d$$

shows that associativity of the $\circ_e$ compositions holds up to the homotopies $\circ_s$ with $|\mathbf{v}(s)| = 3$. More explicitly (with the signs), for a linear tree labelled with elements p, q, r in P we have

$$\begin{aligned}&(p \circ_e q) \circ_{e'} r - p \circ_e (q \circ_{e'} r) \\ &\quad = d(\circ_s(p, q, r)) + \circ_s(dp, q, r) + (-1)^{|p|} \circ_s (p, dq, r) + (-1)^{|p|+|q|} \circ_s (p, q, dr).\end{aligned}$$

Consequently, if P is a s.h. operad, then the cohomology H^*P with respect to the internal differential d is a graded operad. The $\circ_i$-compositions are induced by the operations $\circ_t$ for trees t with 2 vertices.

§4 Strongly homotopy homomorphisms

4.2.9 DEFINITION In the spirit of Koszul duality, we define a *homotopy morphism of homotopy* $\mathrm{Opd}^\perp$*-algebras* to be a morphism of cofree $\mathrm{Opd}^\perp$-coalgebras compatible with the differentials. Such a morphism is a quasi isomorphism if the underlying map of vector spaces is a quasi isomorphism. Recall that by the moves of Markl [65] (we need to extend the theory to coloured operads but this is no problem) such a quasi isomorphism has a quasi inverse. A homotopy homomorphism of homotopy PsOpd-algebras $\varphi : A \rightsquigarrow B$ is completely determined by its restrictions

$$\varphi_t : (A[-1])(t) \longrightarrow B(\mathbf{1}(t))[-1].$$

The condition that φ is compatible with the differential can be described in terms of conditions about compatibility with the $\circ_t$ operations:

$$\sum_{s\subset t} \pm\varphi_{t/(s)} \circ (\circ_s) = \sum_{n,s_1,\ldots,s_n\subset t} \pm(\circ_{t/(s_1,\ldots,s_n)}) \circ (\varphi_{s_1} \otimes \ldots \otimes \varphi_{s_n}). \tag{4.2.3}$$

where the sum in the left hand side is over subtrees, and the sum in the right hand side for each n is over n-tuples of (connected) subtrees of t with disjoint sets of internal edges that together cover all vertices of t. The $\pm$ is the sign induced by the Koszul convention.

4.2.10 DEFINITION A s.h. morphism $\varphi : A \rightsquigarrow B$ of equivariant homotopy PsOpd-algebras is *equivariant* if for any planar rooted trees t and t' and any isomorphism $\sigma : t \longrightarrow t'$ of the underlying rooted trees, the equation

$$\mathbf{1}(\sigma) \circ (\varphi_t) = \varphi_{t'} \circ \sigma$$

is satisfied (compare Definition 4.2.6).A *morphism of s.h. operads* (or *s.h. morphism*) $\varphi : P \rightsquigarrow Q$ is an equivariant homotopy homomorphism $\varphi : P \rightsquigarrow Q$ of equivariant homotopy PsOpd-algebras. An equivariant homotopy homomorphism $\varphi : P \rightsquigarrow Q$ induces a morphism of dg cooperads $\varphi : BP \longrightarrow BQ$.
Note tat φ is determined by maps $\varphi_t : (P[-1])(t) \longrightarrow P[-1]$. Denote by $\varphi_\bullet$ the restriction of φ to the 1-vertex trees. A s.h. morphism is a *quasi isomorphism* if the morphism $\varphi_\bullet$ of dg collections is a isomorphism in cohomology.
Dually, a *morphism of s.h. cooperads* $\psi : A \rightsquigarrow C$ is a equivariant morphism of s.h. $\mathrm{PsOpd}^\perp$-coalgebras. It induces $\psi : ᗺA \longrightarrow ᗺC$. The sequel of this section focuses on the s.h. operad case and leaves the formulation of the dual statements to the reader.

4.2.11 THEOREM *Let P and Q be operads, and let $\varphi : P \rightsquigarrow Q$ be a homotopy quasi isomorphism. Then there exists a quasi inverse $\psi : Q \rightsquigarrow P$ to φ.*

PROOF It follows from generalities on Koszul duality that a homotopy quasi isomorphism of operads φ has a quasi inverse when interpreted as a morphism of homotopy PsOpd-algebras (cf. Markl [65]). Let ψ denote this quasi inverse. This quasi inverse can be symmetrised as follows. Let t be a planar rooted tree with n vertices. Define

$$\psi'(t) = \frac{1}{|\mathrm{Aut}(t)|} \sum_{\sigma \in \mathrm{Aut}(t)} \psi_{\sigma(t)} \circ \sigma.$$

Since φ is equivariant, ψ' still is a quasi inverse to φ. Moreover, for $\tau \in \mathrm{Aut}(t)$

$$|\mathrm{Aut}(t)| \cdot \psi'_t \circ \tau = \sum_{\sigma \in \mathrm{Aut}(t)} \psi_{\sigma(t)} \circ \sigma \circ \tau = \sum_{\sigma' \in \mathrm{Aut}(t)} \psi_{\sigma' \circ \tau(t)} \circ \sigma',$$

where we use $\sigma' = \sigma \circ \tau^{-1}$ to compare the sums. Then ψ' is an equivariant quasi inverse to φ. QED

4.2.12 COROLLARY *Two augmented operads P and Q are quasi isomorphic iff there exists a quasi isomorphism $P \rightsquigarrow Q$ of operads up to homotopy.*

PROOF By definition P and Q are quasi isomorphic iff there exists a sequence of quasi isomorphisms of operads $P \longleftarrow \cdots \longrightarrow Q$. The previous theorem can be applied to make all arrows point in the same direction if we allow s.h. maps.
On the other hand, if there exists an s.h. quasi isomorphism $P \rightsquigarrow Q$, then the bar-cobar adjunction gives a strict quasi isomorphim $\mho(BP) \longrightarrow Q$, and a quasi isomorphism $\mho B(P) \longrightarrow P$. QED

4.3 Strongly homotopy algebras

§5 Endomorphism operads

A somewhat dissatisfying fact is that the map from vector spaces to operads $V \longmapsto \mathrm{End}_V$ is not functorial. This section constructs s.h. morphisms between endomorphism operads, some even compatible with the identity. Well known boundary conditions turn up naturally in this context (compare Huebschmann-Kadeishvili [37]).

4.3.1 DEFINITION a s.h. operad is *strictly unital* if there is a $\mathrm{id} \in P(1)$ that is a left and right identity with respect to the $\circ_t$ operations where $|\mathbf{v}(t)| = 2$ and such that the other compositions $\circ_t$ vanish when applied to id in one coordinate. A s.h. morphism φ of two strictly unital s.h. operads is *strictly unital* if the underlying morphism of collections preserves the identity, and if for $|\mathbf{v}(t)| > 1$, the map $\varphi(t)$ vanishes when applied to id in one coordinate.

4.3.2 DEFINITION Let V and W be dg vector spaces. V is a *strict deformation retract* of W if there exist an inclusion $i : V \longrightarrow W$ and a retraction $r : W \longrightarrow V$ such that both i and r are dg maps, $r \circ i = \mathrm{id}_V$, and there exists a chain homotopy H between $i \circ r$ and id_W, satisfying the boundary conditions $H \circ i = 0$, $r \circ H = 0$, and $H \circ H = 0$.

4.3.3 THEOREM *Let V and W be dg vector spaces. Let $i : V \longrightarrow W$ and $r : W \longrightarrow V$ be dg linear maps, and $H : W \longrightarrow W[1]$ a chain homotopy between $i \circ r$ and id_W.*

(i). There exists a (non-unital) s.h. morphism $\varphi : \mathrm{End}_W \rightsquigarrow \mathrm{End}_V$ (defined by the Formula (4.3.4) below).

(ii). If i and r are quasi isomorphisms, then φ is a quasi isomorphism.

(iii). If the data above make V a strict deformation retract of W, then φ is strictly unital.

PROOF We first prove the second part of the Theorem. The map $\varphi_\bullet$ corresponding to 1 vertex trees is $f \longmapsto r \circ f \circ i^{\otimes n}$ for $f \in \mathrm{End}_V(n)$. Define an alternative composition $\hat{\gamma}$ on End_V by $f \hat{\circ}_i g = f \circ_1 H' \circ_i g$, where $H'(x) = (-1)^{|x|} H(x)$. This composition makes End_V a pseudo operad. For a planar rooted tree t, the map

$$\varphi(t) = \varphi_\bullet \circ \hat{\gamma}_t, \tag{4.3.4}$$

where $\hat{\gamma}_t : \mathrm{End}_W(t) \longrightarrow \mathrm{End}_W(\mathbf{l}(t))$ is the composition based on $\hat{\gamma}$. This is visualised in Figure 12.

It remains to check Formula (4.2.3). For a fixed tree t this reduces to

$$\sum_{e \in \mathbf{e}(t)} (\circ_e) \circ (\varphi(t^e) \otimes \varphi(t_e)) + d \circ \varphi(t) = \sum_{e \in \mathbf{e}(t)} \varphi(t/e) \circ (\circ_e) + \varphi(t) \circ d. \tag{4.3.5}$$

The argument that this hold is the following. Since r and i commute with the differential d, and the internal differentials act as a derivation with respect to composition of multi-linear maps, the formula follows from the equalities $d \circ H + H \circ d = \mathrm{id} - i \circ r$ applied to the summand for each edge e. This shows part *(ii)*.

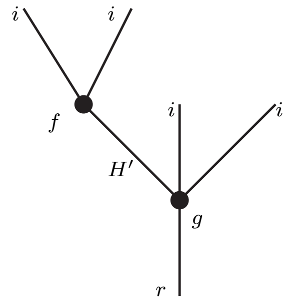


FIGURE 12: The map $\varphi(t)(f, g, h) = r \circ g \circ ((H' \circ f \circ i^{\otimes 2}), i^{\otimes 2})$, represented by a tree with labelled internal and external edges.

To assure that $\varphi_\bullet$ preserves the identity, use $r \circ i = \mathrm{id}_V$. The conditions on compositions with H assure that higher operations applied to the identity vanish. QED

4.3.4 REMARK Let us take a closer look at the proof above. Since the cancellation of terms is local with respect to the geometry of the tree t (i.e. cancellation per edge), it suffices to check the signs for a tree with one edge as in Figure (12). Let us do the calculation with the signs for this tree. We leave out the pre-composition with i and post composition with r in the final terms. The usual degree of f is denoted by $|f|$. The left hand side of Equation (4.3.5) reads

$$g \circ_k i \circ r \circ f + d \circ g \circ_k H \circ f + (-1)^{|f|+|g|+1} g \circ_k H \circ f \circ d.$$

The right hand side equals

$$g \circ_k f + d \circ g \circ_k H \circ f + (-1)^{|g|} g \circ d \circ_k H \circ f + (-1)^{|g|+1} g \circ_k H \circ d \circ f + (-1)^{|f|+|g|+1} g \circ_k H \circ f \circ d.$$

To obtain these signs, note that we have a sign from moving d in, and note that these signs are with respect to the shifted grading on End_V and End_W, while the sign in $d(f) = d \circ f + (-1)^{|f|} f \circ d$ is with respect to the usual grading. The signs are correct if we replace H by $H'(x) = (-1)^{|x|} H(x)$.

§6 Strongly homotopy Q-algebras

We already discussed homotopy algebras for Koszul operads. This section discusses the more general approach to strongly homotopy algebras. It shows how s.h. operads can be used to give a different interpretation of the usual definition.

4.3.5 DEFINITION Let Q be an augmented operad. A *strongly homotopy Q-algebra* (or *s.h. Q-algebra*) structure on a dg vector space V is a s.h. morphism $Q \rightsquigarrow \mathrm{End}_V$. Recall that this iinduces a map of dg cooperads $BQ \longrightarrow B\mathrm{End}_V$. To such a morphism corresponds by bar/cobar duality a operad morphism $\mathcal{B}(BQ) \longrightarrow \mathrm{End}_V$, and vice versa.

4.3.6 PROPOSITION *Let Q be an augmented operad.*

(i). let W be a homotopy Q-algebra, and W a dg vector space. If $i : V \longrightarrow W$, $r : W \longrightarrow V$ are quasi isomorphisms, and $H : r \circ i \sim \mathrm{id}_W$, then V has the structure of a homotopy Q-algebra such that the induced maps in cohomology $H(r)$ and $H(i)$ are isomorphisms of Q-algebras.

(ii). Let V be a homotopy Q-algebra, and let W be a dg vector space. If $i : V \longrightarrow W$, $r : W \longrightarrow V$ are quasi isomorphisms, and $H : r \circ i \sim \mathrm{id}_W$, then W has the structure of a homotopy Q-algebra such that $H(r)$ and $H(i)$ are isomorphisms of Q-algebras.

PROOF Suppose that W is a homotopy Q-algebra. Recall that we constructed from the data in the Theorem a quasi isomorphism $\mathrm{End}_W \rightsquigarrow \mathrm{End}_V$ in Theorem 4.3.3. The composition

$$BQ \longrightarrow B\mathrm{End}_W \longrightarrow B\mathrm{End}_V$$

defines the desired s.h. morphism $Q \rightsquigarrow \mathrm{End}_V$, where the map $BQ \longrightarrow B\mathrm{End}_W$ is the map defined by the homotopy Q-algebra structure on W, which proves *(i)*.
Suppose that V is a homotopy Q-algebra. The quasi isomorphism $\mathrm{End}_W \rightsquigarrow \mathrm{End}_V$ has a quasi inverse (by Theorem 4.2.11), and thus we can construct the composition $BQ \longrightarrow B\mathrm{End}_V \longrightarrow B\mathrm{End}_W$, which defines a homotopy Q-algebra structure on W. QED

4.3.7 REMARK Observe that all the results up to now can be generalised to coloured operads. Notably, for sequences of vector spaces $V = \{V_\alpha\}_{\alpha \in I}$ and $W = \{W_\alpha\}_{\alpha \in I}$ such that each W_α is a strict deformation retract of V_α we can find a quasi isomorphism $\mathrm{End}_W \rightsquigarrow \mathrm{End}_V$, which yields the analogue of Proposition 4.3.6 for algebras over I-operads.

§7 Example: configuration spaces

4.3.8 DEFINITION Let D_2 be the *operad of little disks*. That is, D_2 is the topological operad such that $D_2(n)$ is the space of ordered n-tuples of disjoint embedding of the unit disk D_2 in D_2 that preserve horizontal and vertical directions. The operations $\circ_k$ are defined by compositions of embeddings.

Let $F(n)$ denote the *configuration space* of n distinct ordered points in the open unit disk in $\mathbb{R}^2$. Thus $F(n)$ is the n-fold product of the unit disk with the (sub)diagonals cut out. Consider $F = \{F(n)\}_{n\geq 1}$ as a collection with respect to permutation of the order of the points.

For a topological space X, denote by $S_*(X)$ the singular k-chain complex on X with coefficients in k.

4.3.9 THEOREM *The singular k-chains $S_*(F)$ on configuration spaces form an operad up to homotopy quasi isomorphic (in the sense of Proposition 4.3.6) to the operad $S_*(D_2)$ of singular k-chains on the little disks operad.*

PROOF We first sketch the line of argument. We construct an S_n-equivariant homotopy between the little disks and the configuration spaces. It then follows that $S_*(F)$ is a homotopy algebra for PsOpd homotopy equivalent to $S_*(D_2)$. Since the homotopy algebra $S_*(F)$ is equivariant it, $S_*(F)$ is a s.h. operad. This argument uses the observation in Remark 4.3.7 that the all results go through for coloured operads.

Then there exists an inclusion $i : F(n) \longrightarrow D_2(n)$ and a retraction $r : D_2(n) \longrightarrow F(n)$ such that $\mathrm{id} \sim i \circ r$ by a homotopy H, and $r \circ i = \mathrm{id}$. Consider points in $D_2(n)$ as given by a n-tuple $(x_1, \ldots, x_n)$ of points in the interior of D_2 and an n-tuple $(r_1, \ldots, r_n)$ of radii, and a point in $F(n)$ by a n-tuple $(x_1, \ldots, x_n)$ of points in the interior of D_2. One might take the retraction r by defining all radii in $r(x_1, \ldots, x_n)$ equal to

$$\frac{1}{3}(\min(\{|x_i - x_j| \quad (i \neq j)\} \cup \{1 - |x_i|\})).$$

A homotopy H between $i \circ r$ and the identity is readily defined by drawing a tube of configurations with the two configurations at the boundary disks, connection the little disks by straight lines. (cf. Figure 13).

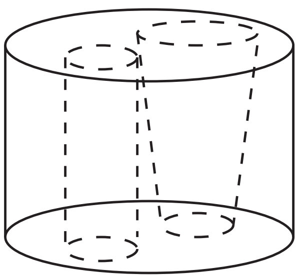


FIGURE 13: Construction of the homotopy H. The tubes do not intersect since the centres of the disks are fixed.

The homotopy H induces a chain homotopy between $S_*(i) \circ S_*(r)$ and the identity. Theorem 4.3.3 then shows that there exists a homotopy homomorphism of $\mathbb{N}$-operads $\mathrm{End}_{S_*(D_2)} \longrightarrow \mathrm{End}_{S_*(F)}$. By composition with the morphism $\mathrm{PsOpd} \longrightarrow \mathrm{End}_{S_*(D_2)}$, the $\mathbb{N}$-collection $S_*(F)$ is a homotopy algebra for the $\mathbb{N}$-operad PsOpd (cf. Proposition 4.3.6). Both i and r (and thus $S_*(i)$ and $S_*(r)$) are compatible with the symmetric group actions on $D_2(n)$ and $F(n)$. Consequently, this makes the singular chains $S_*(F)$ an equivariant homotopy PsOpd-algebra, and thus an operad up to homotopy. QED

4.4 The L_∞-algebra of a s.h. operad

§8 Résumé on L_∞-algebras

L_∞-algebras should be considered as a generalisation of Lie algebras. This new concept provides more freedom in manipulation than Lie algebras, but preserve the necessary information on solutions of the Maurer-Cartan equation.

4.4.1 DEFINITION (LADA AND STASHEFF [51]) An *L_∞-algebra* is a dg vector space $\mathfrak{g}$, together with a differential ∂ on the free cocommutative coalgebra $S'(\mathfrak{g}[-1])$ of cohomological degree $+1$, where by a differential we mean a coderivation s.t. $\partial^2 = 0$. The differential ∂ is completely determined by its restrictions

$$l_n : S'^n(\mathfrak{g}[-1]) \to \mathfrak{g}[-1],$$

where $S'^n(\mathfrak{g}[-1]) \subset S'(\mathfrak{g}[-1])$ of course denotes the subspace of elements of tensor degree n. A *morphism of L_∞-algebras* is a morphism of cofree coalgebras compatible with differentials. Such a morphism is denoted $f : \mathfrak{g} \rightsquigarrow \mathfrak{h}$, and is completely determined by its restrictions

$$f_n : S'^n(\mathfrak{g}[-1]) \to \mathfrak{h}[-1].$$

A morphism of L_∞-algebras is a *quasi isomorphism* if f_1 is a quasi isomorphism of dg vector spaces.

4.4.2 EXAMPLE The Chevalley-Eilenberg complex with trivial coefficients makes every Lie algebra an L_∞-algebra. The Chevalley-Eilenberg complex $C^{CE}_*(\mathfrak{g}, k)$ equals

$S'(\mathfrak{g}[-1])$ as a graded vector space (in homological convention we have positive degrees). The Lie bracket is a map $\Lambda^2\mathfrak{g} \longrightarrow \mathfrak{g}$ and can be interpreted as a map $l_2 : S'^2(\mathfrak{g}[-1]) \longrightarrow \mathfrak{g}[-1]$ of homological degree -1 (the shift changes the S_2-action by the sign representation). The map l_2 uniquely extends to a coderivation on $S'(\mathfrak{g}[-1])$. This coderivation is the Chevalley-Eilenberg differential ∂. The shift assures that the derivation is of homological degree -1, and the equation $\partial^2 = 0$ is equivalent to the Jacobi identity.

4.4.3 LEMMA (KONTSEVICH [41, 42], MARKL [65]) *Two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are quasi isomorphic as Lie algebras iff there exists a quasi isomorphism $f : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ of L_∞-algebras.*

§9 Theorem and examples

Introduce two functors $\oplus$ and $\oplus_S$ from collections to k-dgVect. On an object C these are given by

$$\oplus(C) = \bigoplus_{n\in\mathbb{N}} C(n) \qquad \text{and} \qquad \oplus_S(C) = \bigoplus_{n\in\mathbb{N}} (C(n))_{S_n}.$$

4.4.4 THEOREM *Let P be a s.h. operad.*

(i). There exists a natural L_∞-algebra structure on $\oplus P$. This L_∞-structure descends to the symmetric quotient $\oplus_S P$.

(ii). The operations l_n in the L_∞-algebra structure are given by

$$l_n = \sum_{\substack{t \text{ planar}, \\ |v(t)|=n}} (\circ_t) \circ i : S'^n(\oplus P[-1]) \longrightarrow \oplus P[-1]. \tag{4.4.6}$$

(iii). Let $\varphi : P \rightsquigarrow Q$ be a s.h. morphism. Then φ induces morphisms of L_∞-algebras

$$\oplus P \rightsquigarrow \oplus Q \qquad \textit{and} \qquad \oplus_S P \rightsquigarrow \oplus_S Q.$$

Moreover, if φ is a quasi isomorphism, then so are the induced L_∞-maps.

Before we turn to the proof of this result we study some special cases. We first restrict our attention to A_∞-algebras, where we obtain a well known result (Corollary 4.4.7) as a corollary to Theorem 4.4.4. Corollary 4.4.7 is based on the observation that A_∞-algebras are exactly operads up to homotopy concentrated in arity 1. As a second special case we recover the Lie algebra structure on the total space $\oplus P$ of an operad P in Corollary 4.4.8.

4.4.5 DEFINITION (STASHEFF [78]) An A_∞-*algebra* is a vector space A, together with a differential ∂ (i.e. a coderivation s.t. $\partial^2 = 0$) on the free coalgebra $T'(A[-1])$. The differential is completely determined by its restrictions

$$m_n : T'^n(A[-1]) \longrightarrow A[-1],$$

where $T'^n(A[-1])$ of course denotes the elements of tensor degree n in $T'(A[-1])$.

4.4.6 LEMMA *Let $(T'(P[1]), d)$ be a s.h. operad. Then the restriction of d to the part of arity 1 makes $P(1)$ an A_∞-algebra. On the other hand, any A_∞-algebra can be interpreted as an s.h. operad concentrated in arity 1.*

PROOF Let $A = P(1)$. the differential on the 'cofree' cooperad $T'(P[-1])$ restricts to a differential on the subspace of $T'(P[-1])(1)$ consisting of the summands corresponding to trees in which $|\mathbf{l}(v)| = 2$ for each vertex v (i.e. trees without branches). This sub-cooperad equals the cofree coalgebra $T'(A[-1])$ on $A[-1]$. Moreover, a map is a coderivation on $T'(A[-1])$ iff it is a coderivation on $T'(P[-1])$. This proves the result. QED

4.4.7 COROLLARY (LADA AND MARKL [50]) *An A_∞-algebra A defines an L_∞-algebra structure on A by higher order commutators $l_n = m_n|_{S'^n(A[-1])}$ (we use invariants instead of coinvariants!).*

PROOF Apply Theorem 4.4.4 to the situation of Lemma 4.4.6. QED

4.4.8 COROLLARY (KAPRANOV-MANIN [38]) *Let P be an operad.*

(i). The total space $\oplus P$ is a Lie algebra with respect to the commutator of the (pre-Lie) multiplication $\bullet = \sum_i \circ_i$.

(ii). The Lie algebra structure descends to quotient $\oplus_S P$.

PROOF Note that in this case the only non-trivial $\circ_t$-operations correspond to trees with 2 vertices. Consequently the only non-trivial l_n-operation is l_2. The formula for l_2 reduces to the usual Lie algebra structure (compare Definition 0.2.4). QED

§10 A symmetrisation lemma

The remainder of this section proves Theorem 4.4.4. The strategy of the proof can be outlined as follows. We can identify the symmetric coalgebra $S'(\oplus(P[-1])$ with the invariant part of the tensor coalgebra $T'(\oplus(P[-1]))$ (with respect to permutation of tensor factors). We construct a map of graded vector spaces $S'(\oplus(P[-1])) \longrightarrow \oplus(T'(P[-1]), \partial)$ into the total space of the 'cofree' pseudo cooperad on P. Then it remains to show that a differential on the 'cofree' cooperad does induce a differential on the symmetric coalgebra $S'(\oplus C)$.

Recall the 'cofree' coassociative coalgebra functor T' from k-dgVect to cooperads, and the 'cofree' non-counital cocommutative coalgebra functor S' from k-dgVect to operads. Let C be a collection. We define an inclusion of vector spaces from the tensor coalgebra $T'(\oplus C)$ into the total space of the 'cofree' non-symmetric cooperad on C,

$$T'(\oplus C) \xrightarrow{i} \oplus(T'_{\not\Sigma} C) = \bigoplus_{t \text{ planar}} \bigotimes_{v \in \mathbf{v}(t)} C(\mathbf{l}(t)). \tag{4.4.7}$$

Consider the map i restricted to $C(n_1) \otimes \ldots \otimes C(n_m) \subset T'(\oplus C)$. The image of this restriction in the summand of t is non-zero only on summands corresponding to planar trees t such that $|\mathbf{v}(t)| = m$ and $\mathbf{l}(v_l) = \{0, \ldots, n_l\}$ for all $l \leq |\mathbf{v}(t)|$. On these summands its value is given by the natural identification of $C(n_1) \otimes \ldots \otimes C(n_m)$ with $C(\mathbf{l}(v_1)) \otimes \ldots \otimes C(\mathbf{l}(v_m))$.

4.4.9 LEMMA *Let C be a collection, and let i be the map of Equation (4.4.7).*

(i). The map i induces a map (again denoted by i) from the symmetric coalgebra on the total space $S'(\oplus C)$ into the total space of the'cofree' cooperad $\oplus(T'C)$. That is, i defines a dg linear map $S'(\oplus C) \xrightarrow{i} \oplus(\bar{T}'C)$.

(ii). The map i of (i) descends to the coinvariants with respect to the kS_n-actions ($n \in \mathbb{N}$) on the collection C. That is, i defines a map $S'(\oplus_S C) \xrightarrow{i} \oplus_S(\bar{T}'C)$.

PROOF The 'cofree' cooperad inside the 'cofree' non-symmetric cooperad consists of the the invariants with respect to the action of automorphisms of rooted trees. It thus suffices to show that the image of the map i in Equation (4.4.7) is invariant under the $\mathrm{Aut}(t)$-action for each rooted tree t. We study i on elements $\sum_{\sigma \in S_m}(p_{\sigma 1}, \ldots, p_{\sigma m}) \in S^m(\oplus C)$ with $p_j \in C(n_j)$. The map i then takes the form

$$\sum_{\sigma \in S_m}(p_{\sigma 1}, \ldots, p_{\sigma m}) \longmapsto \sum_{j=1}^{m} \sum_{(s_i)_{i=1}^{n_j}} \sum_{(P_i)_{i=1}^{n_j}} ((\ldots(v_0(p_j) \circ_{n_j} s_{n_j}(P_{n_j})) \circ_{n_j-1} \ldots) \circ_1 s_1(P_1)) \tag{4.4.8}$$

where the second sum is over n_j-tuples of planar trees $(s_1, \ldots, s_{n_j})$ that satisfy the quality $\sum_k |\mathbf{v}(s_k)| = m - 1$ (for the sake of simplicity of the formula a planar tree might be empty here) and the third sum is over partitions $(P_1, \ldots, P_{n_j})$ of $\{1, \ldots, m\} - \{j\}$ in n_j subsets with $|P_k| = |\mathbf{v}(s_k)|$ for all k. The symbolic notation

$$((...(v_0(p_j) \circ_{n_j} s_{n_j}(P_{n_j})) \circ_{n_j - 1} \ldots) \circ_1 s_1(P_1))$$

denotes the sum of all possible ways to label vertices of s_k by p_l with $l \in P_k$ (for all k) and the root vertex v_0 by p_j. The summand of $(s_1, \ldots, s_{n_j})$ corresponds to the tree with root r such that $\mathbf{l}(r) = \{0, \ldots, n_j\}$ and with planar tree s_k grafted to the root along leg k (cf. Figure 14).

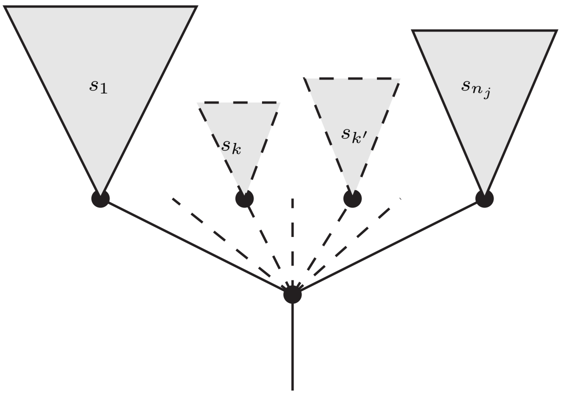


FIGURE 14: The planar tree corresponding to a summand, with root of arity n_j. For each such tree we have to sum over all ways to assign tensor factors to vertices.

We prove the invariance by induction on the number of vertices of a tree. Let $n \in \mathbb{N}$ and suppose that the image of (4.4.7) is contained in the invariants when restricted to the summands corresponding to a rooted tree t with $|\mathbf{v}(t)| < n$. Let s be a rooted tree such that $|\mathbf{v}(s)| = n$ such that the root vertex $v_0 \in \mathbf{v}(s)$ has j legs other than the root. Every automorphism of s can be decomposed in automorphisms of each of the the subtrees $s_1, \ldots, s_j$ attached to the root vertex v_0 and a permutation of the incoming edges of v_0: Let $\tau \in \mathrm{Aut}(\mathbf{l}(v_0))$. Denote by $\hat{\tau}$ the corresponding automorphism of s. It suffices (by the induction hypothesis) to show that the image is invariant with respect to elements $\hat{\tau} \in \mathrm{Aut}(s)$. This will be done for each summand j in Formula (4.4.8). Let $\tau \in \mathrm{Aut}(\mathbf{l}(p_j))$. Then, τ applied to a summand of the right hand side of Formula (4.4.8) gives

$$((...(v_0(p_j) \circ_{n_j} s_{\tau^{-1} n_j}(P_{\tau^{-1} n_j}))) \circ_{n_j - 1} \ldots \circ_1 s_{\tau^{-1} 1}(P_{\tau^{-1} 1})) \tag{4.4.9}$$

where we compensate for the omitted action of τ on p_j by not permuting the legs

of the tree according to the induced permutation. This gives a permutation of summands in Equation (4.4.8) and invariance follows, which proves *(i)*.
Then *(ii)* follows by the equivariance property of cocompositions in $\bar{T}'C$, which we can express by compensating for the action of τ on the label of a vertex of a tree t by a permutation of $\mathbf{l}(t)$, as in Formula (4.4.9). QED

§11 Proof of Theorem 4.4.4

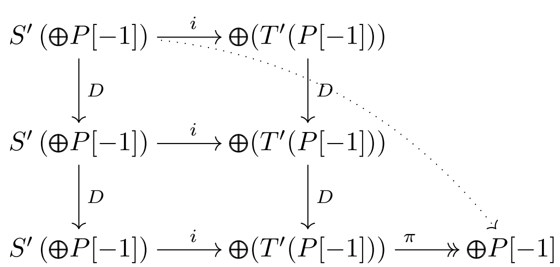
PROOF (OF THEOREM 4.4.4) Now that we have established the map $i : S'(\oplus C) \longrightarrow \oplus(\bar{T}'C)$ and the induced map on coinvariants (Lemma 4.4.9) it remains to show that when $C = P[-1]$ a differential on $\oplus(\bar{T}'(P[-1]))$ gives rise to a differential on $S'(\oplus(P[-1]))$ (and mutatis mutandi on coinvariants), since this endows $\oplus P$ and $\oplus_S P$ with an L_∞-structure.
The differential on $T'(P[-1])$ induces a derivation D on $S'(\oplus P[-1])$. To show that D defines an L_∞-algebra, we need to show that $D^2 = 0$. Let C be a collection and let D be a coderivation on $T'C$. Observe that then the diagram

$$\begin{CD} S'(\oplus C) @>{i}>> \oplus(T'C) \\ @V{D}VV @VV{\oplus D}V \\ S'(\oplus C) @>{i}>> \oplus(T'C) \end{CD}$$

commutes, where $D : S' \oplus C \longrightarrow S' \oplus C$ is the coderivation defined by the composition

$$S'(\oplus C) \xrightarrow{\ i\ } \oplus(T'C) \xrightarrow{\oplus D} \oplus(T'C) \twoheadrightarrow \oplus C.$$

(This is a direct corollary of the definition of coderivation for coalgebras and cooperads.) Therefore there exists a commutative diagram



with the composition from $S'(\oplus P[-1])$ at the top left to $\oplus P[-1]$ (dotted arrow) equal to 0. Therefore, $D^2 = 0$ on $S'(\oplus P[-1])$.

Since the differential on $T'(P[-1])$ preserves the symmetric group action, it induces a differential on $\oplus_S T'(P[-1])$. The map $i : S'(\oplus_S P[-1]) \longrightarrow \oplus_S T'(P[-1])$ is well defined. The induced differential on $S'(\oplus_S P[-1])$ commutes with the map i. The remaining statement of Part *(i)* follows by the commutation relation of the $\circ_t$ with the $\mathrm{Aut}(\mathbf{1}(t))$-actions. The construction is quite explicit due to the map i we constructed. Given a s.h. operad we can write down the higher operations l_n of the L_∞-algebra in terms of the operations $\circ_t$ in which we described the s.h. operad structure. This implies part *(ii)*, in which we applied the degree shift to avoid writing the signs.

It thus remains to prove *(iii)*. Let C and E be collections, and let $\varphi : T'C \longrightarrow T'E$ be a morphism of cooperads. Then the diagram

$$\begin{array}{ccc} S' \oplus C & \xrightarrow{\ i\ } & \oplus T'C \\ \Big\downarrow{\scriptstyle \tilde{\varphi}} & & \Big\downarrow{\scriptstyle \oplus\varphi} \\ S' \oplus E & \xrightarrow{\ i\ } & \oplus T'E \end{array}$$

is commutative, where $\tilde{\varphi} : S' \oplus C \longrightarrow S' \oplus E$ is the coalgebra morphism induced by the composition

$$S'(\oplus C) \xrightarrow{\ i\ } \oplus(T'C) \xrightarrow{\ \oplus\varphi\ } \oplus(T'E) \twoheadrightarrow (\oplus E).$$

Consequently, in the situation of the Theorem, the universal property of $S'(\oplus Q[-1])$ defines a coalgebra morphism φ. It suffices to check that the coalgebra map φ commutes with the differential. Since the differential is a coderivation it suffices to show that $\pi \circ d \circ \varphi = \pi \circ \varphi \circ d$, where $\pi : S'(\oplus Q[-1]) \longrightarrow \oplus Q$ is the natural projection. The result now follows since i and φ commute and since φ is equivariant with respect to the S_n-actions. QED

4.5 Cotangent cohomology and deformations

This section defines for any 1-reduced operad Q and each operad $\varphi : Q \longrightarrow P$ under Q the operad cohomology $H_Q(P)$ of Q with coefficients in P. The cohomology is Markl's cotangent cohomology. The results from the previous section show the existence of an L_∞-algebra structure on the cotangent complex. This section also shows how these results relate to the work on deformations of algebras over an operad in Balavoine [1], and Kontsevich-Soibelman [46].

§12 Résumé on the Maurer-Cartan equation

4.5.1 DEFINITION Let $(S'(\mathfrak{g}[-1]), \partial)$ be an L_∞-algebra. An element $\varphi \in \mathfrak{g}^1$ (i.e. of degree 1 in the underlying complex) is said to satisfy the (generalised) *Maurer-Cartan equation* if

$$\sum_{n\geq 1} l_n(\varphi^{\otimes n}) = 0. \tag{4.5.10}$$

Alternatively, one says that φ is a *Maurer-Cartan element*. In the case of a dg Lie algebra this equation indeed takes the form $d\varphi + \frac{1}{2}[\varphi,\varphi] = 0$, where the factor $\frac{1}{2}$ comes from using coinvariants instead of invariants. This agrees with Definition 0.2.4.

Some fact on Maurer-Cartan elements are necessary. These Lemmas are well known (cf. Kontsevich [41, 42]): a solution of the Maurer-Cartan equation can be interpreted as a perturbation of the L_∞-structure; a solution of the Maurer-Cartan equation can be transported along an L_∞-map; in this last situation there exists a morphism of the perturbed L_∞-algebras. Each of these facts can be proved in a straightforward way, by rearranging the summations and using associativity of the shuffle product.

4.5.2 LEMMA *Let $(S'(\mathfrak{g}[-1]), \partial)$ and $S'(\mathfrak{h}[-1]), \partial)$ be a L_∞-algebras, let $\varphi \in \mathfrak{g}^1$ be a solution of the Maurer-Cartan equation, and let $f : (S'(\mathfrak{g}[-1]), \partial) \longrightarrow (S'(\mathfrak{h}[-1]), \partial)$ be a map of L_∞-algebras.*

(i). The dg vector space $\mathfrak{g}$ enjoys a second L_∞-structure $(S'(\mathfrak{g}[-1]), \partial_\varphi)$, defined by its components

$$\tilde{l}_j(x_1, \ldots, x_j) = \sum_{p\geq 0} l_{j+p}(\mathrm{Sh}_{p,j}(\varphi^{\otimes p}; x_1, \ldots, x_j)). \tag{4.5.11}$$

(ii). There exists a solution ψ of the Maurer-Cartan in $(S'(\mathfrak{h}[-1]), \partial)$, defined by

$$\psi = \sum_n f_n(\varphi^{\otimes n}). \tag{4.5.12}$$

(iii). There exists a morphism of L_∞-algebras $f_\varphi : (S'(\mathfrak{g}[-1]), \partial_\varphi) \longrightarrow (S'(\mathfrak{h}[-1]), \partial_\psi)$ defined by

$$\tilde{f}_n(x_1, \ldots, x_n) = \sum_{p\geq 0} f_{n+p}(\mathrm{Sh}_{p,n}(\varphi^{\otimes p}; x_1, \ldots, x_n)). \tag{4.5.13}$$

4.5.3 REMARK The Lemma above works formally. The formulae contain a priori infinite series, and the result is understood to hold modulo convergence of the relevant series. Introducing a filtration and passing to completed L_∞-algebras is one possibility to make things work.
For an L_∞-algebra $(S'(\mathfrak{g}[-1]), \partial)$, denote by d the internal differential on $\mathfrak{g}$ induced by ∂. If φ is a solution of the Maurer-Cartan equation we denote by $\tilde{d}$ the internal differential of the φ-perturbed L_∞-algebra $(S'(\mathfrak{g}[-1]), \partial)$.

4.5.4 DEFINITION A *filtered L_∞-algebra* is an L_∞-algebra $\mathfrak{g}$ with a filtration

$$\ldots \subset F^{\geq n}\mathfrak{g} \subset \ldots \subset F^{\geq 1}\mathfrak{g} \subset F^{\geq 0}\mathfrak{g} = \mathfrak{g}$$

by L_∞-subalgebras such that (i). $\bigcap_n F^{\geq n}\mathfrak{g} = 0$, and (ii). the differential ∂ on the symmetric coalgebra preserves the induced filtration. The *completion* $\hat{\mathfrak{g}}$ (with respect to the topology induced by the filtration) is an L_∞-algebra with respect to the completed tensor product. A *morphism of filtered complete filtered L_∞-algebras* is assumed to preserve the filtration. Completeness assures that all the series in the last paragraph are well defined for filtered complete L_∞-algebras with a solution φ of the Maurer-Cartan equation in $F^{\geq 1}\hat{\mathfrak{g}}$.

4.5.5 PROPOSITION *Let $(S'(\mathfrak{g}[-1]), \partial)$ and $(S'(\mathfrak{h}[-1]), \partial)$ be filtered L_∞-algebras such that the associated graded $\mathrm{gr}(\mathfrak{g}, d)$ (resp. $\mathrm{gr}(\mathfrak{h}, d)$) with the internal differential is isomorphic to $(\mathfrak{g}, d)$ (resp. $(\mathfrak{h}, d)$). Let $\varphi \in F^{\geq 1}\hat{\mathfrak{g}}$ be a solution to the Maurer-Cartan equation in the completed L_∞-algebra. Let $f : (\mathfrak{g}, d) \rightsquigarrow (\mathfrak{h}, d)$ be a quasi isomorphism of filtered L_∞-algebras. Then the map f_φ of Lemma 4.5.2 is a quasi isomorphism of the completed L_∞-algebras.*

PROOF Let $(S'(\mathfrak{g}[-1], \partial)$ be a bigraded L_∞-algebra, and let $\varphi \in F^{\geq 1}\hat{\mathfrak{g}}$ be a solution of the Maurer-Cartan equation in the completed L_∞-algebra. Then the spectral sequence induced by the filtration $F^{\geq n}(\mathfrak{g}, \tilde{d})$ has E_1^{**}-term

$$E_1^{pq}(\hat{\mathfrak{g}}) = \left(F^{\geq p}H(\hat{\mathfrak{g}}, d)/F^{\geq p+1}H(\hat{\mathfrak{g}}, d)\right)^{p+q}, \tag{4.5.14}$$

due to $\tilde{d} = \sum_{m\geq 0} l_{m+1}(\mathrm{Sh}_{m,1}(\varphi^{\otimes m}, -))$ with φ increasing the filtration, and the condition on the associated graded.
Denote $\psi = \sum_n f_n(\varphi^{\otimes n})$ the solution of the Maurer-Cartan equation in $\mathfrak{h}$ corresponding to φ (cf. Lemma 4.5.2). Since the f is a morphism of filtered L_∞-algebras, the morphism $\tilde{f}_1 : \mathfrak{g} \longrightarrow \mathfrak{h}$ commutes with the perturbed internal differentials $\tilde{d}$ in $\mathfrak{g}$ and $\mathfrak{h}$, and preserves the filtration (cf. Lemma 4.5.2). Therefore it induces a morphism of spectral sequences. Since f_1 is a quasi isomorphism by definition, the explicit form of E_1 given in Formula (4.5.14) shows the induced morphism $E_1^{**}(f_1)$

of spectral sequences $E_1^{**}(\mathfrak{g}) \longrightarrow E_1^{**}(\mathfrak{h})$ is an isomorphism, and thus it induces a quasi isomorphism on E_k^{**} for all $k \geq 1$. This proves the result. QED

§13 Maurer-Cartan for convolution operads up to homotopy

Let C and P be collections. There is an obvious structure of a collection on the spaces $P^C(n) = \mathrm{Hom}(C(n), P(n))$, with their natural action for $\sigma \in S_n$ and $\varphi \in P^C(n)$ defined by $(\varphi\sigma)(c) = (\varphi(c\sigma^{-1}))\sigma$, the differential is defined by $\varphi \circ d - d \circ \varphi$, in terms of the differentials on C and P.

4.5.6 THEOREM *Let A be a s.h. cooperad, and let P be an operad.*

(i). The collection P^A has the natural structure of an operad up to homotopy.

(ii). Solutions φ of the Maurer-Cartan equation in $(\hat{\bigoplus}_S P^A)^1$ of the L_∞-algebra $\oplus_S P^A$ (defined in Theorem 4.4.4) are in 1 - 1 correspondence with dg operad maps $\hat{\varphi}$: $\mathcal{B} A \longrightarrow P$.

PROOF Let $\partial_A + d$ be the differential on the 'cofree' non-symmetric operad on $A[1]$, where d is the part that defines the internal differential on the collection A. Write

$$T'_{\Sigma}(P^A[-1]) = \bigoplus_{t\text{planar}} P^A[-1](t) = \bigoplus_{t \text{ planar}} Hom_k(A[1](t), P(t)).$$

Define a differential ∂ on $T'_{\Sigma}(P^C)$ by its components for planar rooted trees t

$$\circ_t : \mathrm{Hom}_k(A[1](t), P(t)) \longrightarrow \mathrm{Hom}_k(A(\mathbf{l}(t))[1], P(\mathbf{l}(t)))$$

for $\varphi \in \mathrm{Hom}_k(A[1](t), P(t))$ as

$$\circ_t : \varphi \longmapsto \gamma_t \circ \varphi \circ (\partial_{A\,t})$$

if $|\mathbf{v}(t)| > 1$, and $\varphi \longmapsto d \circ \varphi + \varphi \circ d$ if $|\mathbf{v}(t)| = 1$. Here γ_t is the operad composition $\gamma_t : P(t) \longrightarrow P$, and $\partial_{A\,t} : A[1] \longrightarrow A[1](t)$ of degree $+1$ is the summand of the differential of A (and d is used to denote both the internal differential of P and of A). We need to check that the differential ∂ squares to zero. In the case $|\mathbf{v}(t)| = 1$ this follows from the definition of the internal differential. In the other cases, compute for $\varphi \in P^A(t)$:

$$\begin{aligned}\partial^2(\varphi) = &\sum_{s\subset t,\ |\mathbf{v}(s)|\geq 2} \gamma_{t/s} \circ \gamma_s \circ \varphi \circ \partial_{A\,s} \circ \partial_{A\,t/s}\\ &+ \gamma_t \circ \varphi \circ d \circ \partial_{A\,t} + \gamma_t \circ \varphi \circ \partial_{A\,t} \circ d\\ &+ d \circ \gamma_t \circ \varphi \circ \partial_{A\,t} + \gamma_t \circ d \circ \varphi \circ \partial_{A\,t}.\end{aligned}$$

The terms on the last line cancel since P is a dg operad. The terms on the first two lines can be rewritten as

$$\gamma_t \circ \varphi \circ \left(\sum_{s \subset t,\ |\mathbf{v}(s)| \geq 2} \partial_{A\, t/s} \circ \partial_{A\, s} - d \circ \partial_{A\, t} - \partial_{A\, t} \circ d \right).$$

The bracketed expression cancels since it equals $(\partial_A + d)^2$, which vanishes since A is a s.h. cooperad. This proves part *(i)*.
Note that collection morphisms $\mathrm{Hom}(A, P)$ are contained in $\hat{\bigoplus} P^C$, whereas $(\hat{\bigoplus}_S P^A)^n$ Identify the collection morphisms from A to P of degree n with the objects of $(\hat{\bigoplus}_S P^A)^n$ (i.e. elements of internal degree n) using the section $\frac{1}{n!} \sum_\sigma \sigma$. Elements φ in $(\hat{\bigoplus}_S P^A)^n$ of internal degree 1 are in 1 - 1 correspondence with collection morphisms $A[1] \longrightarrow P$. Due to the free/forgetful adjunction for graded collections and graded operads, these are in turn in 1 - 1 correspondence with graded operad maps $\hat{\varphi} : T(A[1]) \longrightarrow P$.
Thus it remains to prove that $\hat{\varphi}$ is compatible with the differentials iff φ satisfies the Maurer-Cartan equation. Compatibility of $\tilde{\varphi}$ with the differential can be stated as

$$\tilde{\varphi} \circ \partial_A = d \circ \tilde{\varphi} - \tilde{\varphi} \circ d = -d(\tilde{\varphi}), \tag{4.5.15}$$

where the internal differential of P is denoted d as well. We further examine the left hand side of this formula, and write it as the sum of the maps

$$A \xrightarrow{\partial_{A\, t}} A(t) \xrightarrow{T(\varphi)} P(t) \xrightarrow{\gamma_t} P$$

for all rooted trees t with at least two vertices (i.e. $\mathbf{v}(t) \geq 2$). In Equation (4.5.15) we bring $d(\tilde{\varphi})$ to the left hand side, which gives the Maurer-Cartan by the definition of l_n in terms of the $\circ_t$ operations on the operad up to homotopy P^A (cf. Theorem 4.4.4). QED

4.5.7 EXAMPLE Let P be an operad and let C be a cooperad. Consider the cooperad structure on C as a coalgebra for the the triples T'. The collection P^C enjoys an operad structure given by

$$\gamma(\varphi; \psi_1, \ldots, \psi_n) = \gamma_P \circ (\varphi \otimes (\psi_1 \otimes \ldots \otimes \psi_n)) \circ \gamma_C^*.$$

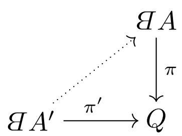
(The identity is given by $\mathrm{id} = u \circ \varepsilon$, the composition of the coidentity of C with the identity of P.) The operad P^C is called the *convolution operad* of C and P (cf. Berger and Moerdijk [2]). The L_∞-algebra of the Theorem 4.5.6 is then a dg Lie algebra.

§14 Total cotangent cohomology

4.5.8 DEFINITION Let A be a s.h. cooperad and let P be an operad. Let $\psi : ᗺA \longrightarrow P$ be a map of dg operads. Consider the space $(\hat{\oplus}_S P^A$ as the completion of the L_∞-algebra $(S'(\oplus_S P^A[-1]), \partial)$, of the operad up to homotopy P^A (cf. Theorems 4.4.4 and 4.5.6). To the map $\psi : ᗺA \longrightarrow P$ corresponds a solution of the Maurer-Cartan equation, which we (by a abuse of notation) denote ψ as well. Then the differential ∂_ψ on $S'(\oplus_S P^A[-1])$ endows $(\hat{\oplus}_S P^A)$ with the corresponding perturbed L_∞-structure (cf. Lemma 4.5.2). This L_∞-algebra is denoted $L_A(\psi)$.
Suppose in addition that $\pi : ᗺA \longrightarrow Q$ is a cofibrant resolution of an augmented operad Q, and that $\varphi : Q \longrightarrow P$ is a map of dg operads. The cohomology of $L_A(\varphi \circ \pi)$ is called the *total cotangent cohomology* of φ and denoted $H^*_Q(\varphi)$. To justify this notation, we need to show that the cohomology $H^*_Q(P)$ is independent of the cofibrant replacement $\pi : ᗺA \longrightarrow Q$.

4.5.9 PROPOSITION *Let Q be an augmented operad, and let $\varphi : Q \longrightarrow P$ be an operad morphism. Let A and A' be s.h. cooperads, such that $\pi : ᗺA \longrightarrow Q$ and $\pi' : ᗺA' \longrightarrow Q$ are cofibrant replacements for the augmented operad Q. Then $L_A(\varphi)$ and $L_{A'}(\varphi)$ are quasi isomorphic as L_∞-algebras. Consequently, the total cotangent cohomology $H_Q(P)$ is independent of the choice of A.*

PROOF Let $\pi : ᗺA \longrightarrow Q$ and $\pi' : ᗺA' \longrightarrow Q$ be two cofibrant replacements for Q. Since $ᗺA$ is cofibrant, and π is a surjective quasi isomorphism, there exists a quasi isomorphism of operads (dotted arrow) that makes the diagram



commute. Consequently, A and A' are quasi isomorphic as cooperads up to homotopy by the map defined by the dotted arrow $A' \rightsquigarrow A$. This map therefore defines a quasi isomorphism $P^A \rightsquigarrow P^{A'}$, which in turn induces a quasi isomorphism $\oplus_S P^A \rightsquigarrow \oplus_S P^{A'}$. Since the diagram above commutes, this quasi isomorphism indeed maps to a solution $\varphi \circ \pi$ of the Maurer-Cartan equation to $\varphi \circ \pi'$.
Since Q is 1-reduced, the solutions of the Maurer-Cartan equation strictly increase the arity-grading. There is a perturbed L_∞-map $L_A(\varphi) \rightsquigarrow L_{A'}(\varphi)$ and since this perturbed map is perturbed from a quasi isomorphism of L_∞-algebras, it is itself again a quasi isomorphism (Proposition 4.5.5). QED

4.5.10 REMARK Recall that the category Opd of dg operads is a model category with the quasi isomorphisms as weak equivalences and surjections as fibrations. Let Q be an augmented operad, and let $\varphi : Q \longrightarrow P$ be a morphism of operads. Then the cohomology $H_Q(P)$ computes the hyper right derived functor from the opposite model category of operad $\mathsf{Opd}^{\mathrm{op}}$ to the category k-dgVect,

$$H_Q(P) = \mathbb{R}(\mathrm{Der})(-, P)(Q). \tag{4.5.16}$$

We only outline the proof of this result. Let $(T(A[-1]), \partial)$ be a cofibrant replacement for Q. First we identify $L_A(\varphi)$ with the derivations $\mathrm{Der}((T(A[-1]), \partial), P)$. The identification as graded vector spaces is obvious. The differential on derivations is the restriction of the usual differential on collection morphisms: $d(\varphi) = d \circ \varphi - \varphi \circ d$. The isomorphism from derivations to $L_A(\varphi)$ restricts derivations to the value on generators of $\mho A$, which completes the identification of the complexes $L_A(\varphi)$ and $\mathrm{Der}(\mho A, P)$.
Then, since every cofibrant operad is a retract of an operad of the form $\mho A$, we can conclude from Proposition 4.5.9 that the right derived functor $\mathbb{R}(\mathrm{Der})(-, P)$ exists and is computed as in Equation (4.5.16).

§15 Cotangent complex

4.5.11 DEFINITION (MARKL [64]) Let Q be a 1-reduced operad. A *bigraded minimal model* for Q is a minimal s.h. cooperad M together with a second grading on the underlying collection M and an operad map $\pi_M : (T(M[1]), \partial) \longrightarrow Q$, such that the operad $(T(M[1]), \partial)$ is graded with respect to the second grading and π_M is a quasi isomorphism. For a 1-reduced operad Q there exists a minimal bigraded model that is unique up to isomorphism.
We give the definition of the cotangent complex by Markl in our terminology. Let Q be a 1-reduced operad, and let $\varphi : Q \longrightarrow P$ be an operad morphism, and let $(T(M[-1]), \partial)$ be a bigraded minimal model for Q. The *cotangent complex* of Q with coefficients in P defined by Markl is the complex $L_M(\varphi)$. Since M is bigraded, the cotangent complex is in fact a bicomplex. It's bigraded homology is called the *cotangent cohomology* $H^{**}(Q; P)$. This new construction of the cotangent complex, relates the cotangent cohomology to the total cotangent cohomology. In particular we constructed the complex from a solution of the Maurer-Cartan equation in an L_∞-algebra.

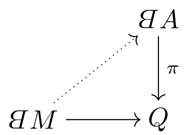
4.5.12 THEOREM *Let Q be a 1-reduced operad, and let $\varphi : Q \longrightarrow P$ be an operad under Q. Then the following hold.*

(i). The cotangent complex enjoys an L_∞-algebra structure.

(ii). The total cotangent cohomology equals the cotangent cohomology once we forget the second grading:

$$H^p_Q(P) = \bigoplus_q H^{pq}(Q; P).$$

PROOF To prove *(i)*, it suffices to observe that we constructed $L_M(\varphi)$ as an L_∞-algebra.
To prove *(ii)*, observe that the complex $L_M(\varphi)$ computes $H_Q(\varphi)$ according to the invariance of cofibrant resolution. The only point is that $\text{ꓭ}M \longrightarrow Q$ need not be surjective, but since $\text{ꓭ}M$ is cofibrant we can still find a lift



for any cofibrant replacement $\pi : \text{ꓭ}A \longrightarrow Q$ (where π is indeed surjective). The construction of a quasi isomorphism on the convolution operads up to homotopy in the proof of Proposition 4.5.9 still applies, and shows that we still can define a quasi isomorphism $L_A(\varphi) \rightsquigarrow L_M(\varphi)$. QED

4.5.13 REMARK Markl [64] states that the cotangent cohomology is 'the best possible cohomology governing deformations of operads'. We can now add that for an operad morphism $\varphi : Q \longrightarrow P$, and the bigraded minimal model $\text{ꓭ}M \longrightarrow Q$, the complex $L_M(\varphi)$ is an L_∞-algebra controlling deformations of the operad map φ.

§16 Deformation complex

4.5.14 DEFINITION The *non-reduced bar construction* $\underline{B}P$ of an operad P is the bar construction $B(P^+)$ of the operad P^+, which is P with an identity freely added. Similarly, for a cooperad up to homotopy A define $\underline{\text{ꓭ}}A = \text{ꓭ}(A^+)$, where A^+ is the s.h. cooperad obtained from A be freely adjoining a coidentity. Let Q be an augmented operad. Let $\varphi : Q \longrightarrow P$ be a morphism of operads. Let A be a s.h. cooperad such that $\text{ꓭ}(A) = (T(A[-1]), \partial)[1]$ is a cofibrant replacement of Q. Define the *deformation complex* $\underline{L}_A(\varphi) := L_{A^+}(\varphi)$ as the L_∞-algebra of Theorem 4.5.9. Note that as a graded vector space $\underline{L}_A(\varphi)$ is isomorphic to $L_A(\varphi) \oplus P(1)$. The cohomology of $\underline{L}_A(\varphi)$ is denoted $\underline{H}_Q(\varphi)$.

4.5.15 REMARK The deformation complex was first defined by Kontsevich and Soibelman [46]. Our definition provide a bit more detail on the L_∞-algebra structure.

Note that in the definition above we indeed have the equation of graded vector spaces

$$\underline{L}_A(\mathrm{End}_V) = \mathrm{Der}(\underline{\mathcal{F}}A, \mathrm{End}_V) = \mathrm{End}_V \oplus \mathrm{Der}(\mathcal{F}A, \mathrm{End}_V)$$

(cf. Remark 4.5.10). The right hand side of which is the starting point of Kontsevich and Soibelman.

4.5.16 THEOREM *Let Q be a 1-reduced operad, and let $\pi : \mathcal{F}A \longrightarrow Q$ be a cofibrant replacement. In addition, let V and W be Q-algebras, and let $i : V \longrightarrow W$, $r : W \longrightarrow V$ and $H : i \circ r \sim \mathrm{id}_W$ form a strict deformation retract, such that either i or r is an algebra map. Then $\underline{L}_A(\mathrm{End}_V)$ and $\underline{L}_A(\mathrm{End}_W)$ are isomorphic as L_∞-algebras.*

PROOF The s.h. morphism $\mathrm{End}_W \rightsquigarrow \mathrm{End}_V$ constructed in this situation in Proposition 4.3.3 induces a quasi isomorphism of L_∞-algebras from $\oplus(\mathrm{End}_W)^A$ to the total space $\oplus(\mathrm{End}_V)^A$. If W is a Q-algebra and i a map of Q-algebras, the composition $Q \rightsquigarrow \mathrm{End}_W$ is in fact an operad map, due to the boundary conditions and the fact that we can write $q \circ i^{\otimes n} = i \circ q$, for any q the image in $\mathrm{End}_V(n)$ of an element in $Q(n)$. Indeed, it is exactly the defining operad map of the Q-algebra structure on W, again by the same trick. It follows that the solution of the Maurer-Cartan equation in $\oplus(\mathrm{End}_W)^A$ corresponding to the Q-algebra W is mapped to the solution in $\oplus(\mathrm{End}_V)^A$ corresponding to the Q-algebra structure on V. The result now follows since the perturbed map of a quasi isomorphism is again a quasi isomorphism by Proposition 4.5.5. The argument if r is a Q-algebra map is similar. QED

4.5.17 EXAMPLE Let $Q = \mathrm{Ass}$, and $A = \mathrm{Ass}^\perp$. Let V be an associative algebra. Then $\underline{L}_A(\mathrm{End}_V)$ is the shifted truncated Hochschild complex $C^*_{\mathrm{Ass}}(V, V)$ controlling deformation deformations of V (cf. Definition 0.2.6). Suppose that V and W in k-dgVect such that W is a strict deformation retract of V. Then there is a natural 1-1 correspondence between homotopy types of homotopy Q-algebra structures on V and homotopy types of homotopy Q-algebra structures on W, given by the homotopy quasi isomorphism $\mathrm{End}_W \longrightarrow \mathrm{End}_V$ (cf. Proposition 4.3.3), which induces a quasi isomorphism of L_∞-algebras between $\underline{L}_{BQ}(\mathrm{End}_W)$ and $\underline{L}_{BQ}(\mathrm{End}_V)$.

4.5.18 COROLLARY *Let $i : V \longrightarrow W$ be an inclusion of associative algebras such that i can be completed to a strict deformation retract of dg vector spaces. Then the shifted truncated Hochschild complexes controlling deformations of V and W are quasi isomorphic as dg Lie algebras.*

4.5.19 EXAMPLE Let Q be a quadratic operad concentrated in degree 0, and let $\varphi : Q \longrightarrow \mathrm{End}_V$ be an operad map. Let $Q^\perp$ be the dual cooperad of Q. Thus we can form the operad $P^{Q^\perp}$ and the L_∞-algebra $\underline{L}_{Q^\perp}(\varphi^\perp)$, based on the map

$$\varphi^\perp : \reflectbox{B} Q^\perp \longrightarrow \reflectbox{B} BQ \longrightarrow Q \longrightarrow P. \tag{4.5.17}$$

If Q is Koszul, then $\reflectbox{$B$} Q^\perp$ is a cofibrant replacement for Q, and $\underline{L}_{Q^\perp}(\varphi^\perp)$ computes the total cotangent cohomology. Note that Corollary 4.5.18 has a natural generalisation to a Q-algebra V for any Q with the complex $\underline{L}_{Q^\perp}(\varphi)$ instead of the shifted truncated Hochschild complex.
However, Balavoine [1] shows that for any (not necessary Koszul) quadratic operad Q, the cohomology in degrees 1 and 2 of $\underline{L}_{Q^\perp}(\varphi^\perp)$ controls formal deformations in the sense of Gerstenhaber [25]. This result can now be understood as follows. The map of complexes $\underline{L}_{BQ}(\varphi) \longrightarrow \underline{L}_{Q^\perp}(\varphi^\perp)$ induced by $Q^\perp \longrightarrow BQ$ defines an isomorphism

$$H^k_Q(\varphi) \longrightarrow H^k(\underline{L}_{Q^\perp}(\varphi^\perp)) \qquad (k \leq 2).$$

4.5.20 REMARK Again, all constructions can be generalised to the category of I-coloured operads. We briefly mention some of the results.
Let P be an I-operad, and let C be an I-cooperad. We construct the convolution I-operad $P^C(n,i) = \mathrm{Hom}_k(C(n,i), P(n,i))$ as in Definition 4.5.7. There is a 1-1 correspondence between solutions of the Maurer-Cartan equation in the Lie algebra $\hat{\oplus}_S(\overline{P^C})$ of coinvariants, and maps of I-operads $\reflectbox{$B$} C \longrightarrow P$.
Let Q be an I-operad, let $\varphi : Q \rightarrow P$ be an I-operad map, and $\pi : \reflectbox{$B$} A \longrightarrow Q$ a cofibrant resolution. The composition $\varphi \circ \pi$ defines a solution of the Maurer-Cartan equation in P^A. If Q is 1-reduced, then the solution is strictly positive with respect to the arity filtration. We denote the φ-perturbed dg L_∞-algebra by $L_A(P)$. The total cotangent cohomology $H_Q(P)$ of Q with coefficients in P in the cohomology of $L_A(P)$. Similarly, we define the non-reduced versions $\underline{L}_A(\varphi)$ and $\underline{H}_Q(\varphi)$. We call $\underline{L}_A(\varphi)$ the *deformation complex* of φ. The cohomology is again independent of the choice of cofibrant model. For $P = \mathrm{End}_V$, the L_∞-algebra $\underline{L}_A(\varphi)$ is a L_∞-algebra controlling deformations of the Q-algebra V. A different cofibrant model yields a quasi isomorphic L_∞-algebra.

4.5.21 EXAMPLE Let Q be a quadratic Koszul operad, and let (A, M) be an MQ-algebra. The complex $\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})$ is the direct sum of two complexes, according to the colour of the output. The part $\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})_{(1)}$ corresponding to colour 1 is the dg Lie algebra controlling deformations of A,

$$\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})_{(1)} = \underline{L}_{Q^\perp}(\mathrm{End}_A).$$

Similarly, the summand $\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})_{(2)}$ tells us something about deformations of A-module structures on M. We give some examples.
Let $Q = \mathrm{Com}$, and let (A, M) be an MQ-algebra. Then

$$\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})_{(2)} = C^*(A, \mathrm{End}(M)),$$

the Hochschild complex with coefficients in the associative algebra $\mathrm{End}(M)$ (no shift here!). The bimodule structure on $\mathrm{End}(M)$ is induced by the algebra map $\lambda : A \longrightarrow \mathrm{End}(M)$ defining the action of A on M.
Let $Q = \mathrm{Ass}$. If M is a bimodule for an associative algebra A, the Hochschild complex $C^*(A, \mathrm{End}(M))$ is again a bimodule with respect to the structure defined by $\rho : A^{\mathrm{op}} \longrightarrow \mathrm{End}(M)$. The complex $\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})_{(2)}$ is the total complex of the Hochschild complex of A with coefficients in $C^*(A, \mathrm{End}(M))$. That is,

$$\underline{L}_{MQ^\perp}(\mathrm{End}_{(A,M)})_{(2)} = \mathrm{Tot}(C^*(A^{\mathrm{op}}, C^*(A, \mathrm{End}(M)))).$$

4.5.22 REMARK Let $Q = \mathrm{PsOpd}$ and let $\varphi : Q \longrightarrow \mathrm{End}_V$ define the structure of a non-symmetric pseudo operad on the collection of vector spaces V. The deformation complex is the non-symmetric version of the complex $L_{BQ}(\varphi)$ computing the total cotangent complex.

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Samenvatting

Operaden zijn concepten uit de algebraïsche topologie. In de algebraïsche topologie wil je eigenschappen van meetkundige (topologische) objecten in algebraïsche termen te beschrijven. Uiterst naief kun je denken aan een hoeveelheid klei, waarvan je de vorm probeert uit te leggen: door te vertellen uit hoeveel stukken het bestaat en of het hol is of niet, of door te zeggen hoeveel gaten er in zitten (een getal, algebraïsch), maar zonder je meteen over subtielere details als uitstulpingen of deuken te bekommeren.

Als je getallen vermenigvuldigt kijk je meestal naar vermenigvuldigingen van twee getallen. Vermenigvuldiging is dan een operatie die van twee getallen (bijvoorbeeld 2 en 3) een nieuw getal (in het voorbeeld 6) maakt. Als je eigenschappen van de vermenigvuldiging wilt onderzoeken, kan het handig zijn naar vermenigvuldigingen van meerdere getallen te kijken als operatie die van aantal getallen één nieuw getal maakt (de vier getallen 2, 3, 4, 5 geven $2 \times 3 \times 4 \times 5 = 120$). Dit gezichtspunt leidt uiteindelijk tot de definitie van een operade.

Deze verandering van perspectief heeft verregaande gevolgen als zij consequent wordt doorgevoerd. Operaden worden onder andere gebruikt om te kijken of een meetkundig object equivalent zijn aan een object met een bepaalde extra eigenschap. Tegenwoordig zijn er vele belangwekkende toepassingen van operaden in uiteenlopende takken van de wiskunde en de theoretische natuurkunde.

In dit proefschrift worden diverse toepassingen van operaden ontwikkeld, die betrekking hebben op Hopf-algebra's en homologische algebra. Hopf-algebra's zijn recentelijk toegepast om een aspect van de quantummechanica (renormalisatie) beter te begrijpen. Ik laat zien dat deze toepassing wiskundig behoorlijk te vereenvoudigen is door operaden te gebruiken. Daarnaast geven operaden aanleiding tot nieuwe Hopf-algebra's die op zichzelf interessant zijn. In de homologische algebra zijn operaden vaak gerelateerd aan de vraag hoe je een gegeven algebraïsche structuur kunt vervormen (deformatietheorie). Ik breid de bekende theorie op dit gebied uit, hetgeen leidt tot nieuwe structuren en nieuwe resultaten over bekende objecten.

Een uitgebreidere samenvatting van de belangrijkste resultaten staat in het inleidende hoofdstuk.

Dankwoord

Ieke, bedankt. Je was een inspirerende begeleider. Iedereen in Ieke's groep (Andreas, Benno, Claire, Federico, Jaap, Janez, Marius, Pieter, Tibor) bedankt voor prettige lunches, koffiepauzes, en zelfs soms wiskundige gesprekken. I thank my other collegues in Utrecht, Paris, and Barcelona (Liesbeth, Martijn, Martijn, Ramon, and all the others) for making life as a mathematician enjoyable. In particular, I wish to mention the *baloncesto* and the *futbol* in Barcelona, and the *voetbal* in Utrecht. De UU-UvA seminaria waren uitermate motiverend. Daarvoor bedank ik naast de Utrechtse deelnemers ook de Amsterdammers (Hessel, Klaas, Michael, en Rogier die toen nog bij 'ons' hoorde).

Martin, thanks for inviting me to Prague. Muriel, merci. Ma visite à Paris fût très agréable et tu m'as trouvé une habitation magnifique à St. Denis. Muriel et Christian, merci pour les discussions sur les arbres, les algèbres d'Hopf, et la renormalisation de la théorie des champs quantiques. Carles, thanks for inviting me to stay in Barcelona which broadened my mathematical view. Maria, thanks for the invitation to Strasbourg and your constant interest. Alessandra, thanks for the invitation to Lyon and the nice discussions there, and in Paris and Montreal.

Ret en Bert bedankt voor alles. Sebastiaan en al mijn andere vrienden (W94, Feps, en alle anderen), bedankt. Jullie hebben voorkomen dat het niet-wiskundige deel van mijn brein kon afsterven. Ik ben erg blij dat Daan (namens W94) en Sebastiaan mijn paranimfen willen zijn.

En Else, bedankt! Je bent geweldig.

Curriculum Vitae

Geboren op 27 juni 1976 te Amsterdam. Na in 1994 het gymnasiumdiploma (Reviuslyceum, Doorn) behaald te hebben, op 31 augustus 1998 bij Prof. dr. Tom Koornwinder afgestudeerd in de Wiskundige Analyse (Universiteit van Amsterdam). Aansluitend TwAiO Wiskundige Beheers- en Beleidsmodellen aan de Technische Universiteit Delft. Deze positie werd in mei 1999 ingeruild voor een AiO-schap bij Prof. dr. Ieke Moerdijk aan het Mathematisch Instituut van de Universiteit Utrecht, met dit proefschrift tot gevolg. Met Marie Curie Training Site-beurzen drie maanden aan de Université Paris Nord XIII en vier maanden aan het Centre de Recerca Matemàtica te Barcelona verbonden geweest.