Representations up to homotopy and cohomology of classifying spaces

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Representations up to Homotopy and Cohomology of Classifying Spaces

Representaties modulo Homotopie en
Cohomologie van Classificerende Ruimtes

(met een samenvatting in het Nederlands)

Proefschrift

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Introduction

1.1 What this thesis is about

The classifying space BG of a Lie group G can be thought of as the homotopy quotient of the least free action imaginable, the action of G on a point. The cohomology of BG is related in a very precise sense to the representations on the polynomials in the Lie algebra. This relation makes it possible to construct infinitesimal models for the cohomology of BG, which in turn generalize to the models for equivariant cohomology. This thesis is the study of how these relations can be extended to more general homotopy quotients, classifying spaces of Lie groupoids. The conclusion is that most of the generalization goes through, provided one works in the context of representations up to homotopy.

1.1.1 Classifying spaces of Lie groups

The classifying space of a Lie group G is a space BG characterized up to homotopy by the property that it is the base of a principal G-bundle $\pi : EG \to BG$ in which EG is contractible. Given a space M and a function $f : M \to BG$, there is a pull back principal G-bundle $f^*(EG)$ over M. It turns out that all principal G-bundles over M are of this form. Thus, there is a bijective correspondence between homotopy classes of maps $f : M \to BG$ and isomorphism classes of principal G-bundles over M. Even though the function f is only well defined up to homotopy, the map induced cohomology $f^* : H(BG) \to H(M)$ depends only on the principal bundle P. The characteristic classes of the P are the cohomology classes that are in the image of this homomorphism. Therefore, the cohomology of BG is the universal algebra of characteristic classes of principal G-bundles.

The condition that G acts freely on a contractible space is very strong and can be realized only on infinite dimensional spaces. As a simple example, consider the case $G = U(1)$. We can take EG to be S^∞, the sphere in an infinite dimensional complex Hilbert space H. Since S^∞ is a contractible space on which $U(1)$ acts freely, we conclude that the quotient space, $\mathbb{CP}^\infty$, is the classifying space of $U(1)$. In this case the cohomology is $H(BU(1)) = \mathbb{R}[x]$, where the generator x of degree 2 is the Chern class of line bundles. The following list, which is taken from [13], illustrates other examples.

G	BG	Poincaré Series
$\mathbb{Z}$	S^1	$1+t$
$\mathbb{Z}^n$	$S^1 \times \cdots \times S^1$	$(1+t)^n$
$\mathbb{Z}_2$	$\mathbb{RP}^\infty$	1
$U(1)$	$\mathbb{CP}^\infty$	$(1-t^2)^{-1}$
$U(n)$	$Gr_n(\mathrm{H})$	$(1-t^2)^{-1} \dots (1-t^{2n})^{-1}$

Here $Gr_n(\mathrm{H})$ denotes the grassmannian of n-dimensional planes in H.

In general, for a compact Lie group G, a theorem of Borel [9] asserts that:

$$H(BG) \cong S(\mathfrak{g}^*)^G.$$

Here, the right hand side denotes the space of polynomials in the Lie algebra which are invariant under the action of G. This theorem is remarkable because it expresses the cohomology of BG purely in terms of infinitesimal data. The theorem of Borel becomes more natural when one considers the algebraic analog of the classifying bundle of G. The De Rham complex $\Omega(P)$ of a principal G-bundle P has the structure of a $\mathfrak{g}$-DG algebra, which means that it is a differential graded algebra that has an action of the Lie algebra $\mathfrak{g}$. Just like there is a universal principal G-bundle that all others map to, there is a universal $\mathfrak{g}$-DG algebra that maps to all others. It is called that Weil algebra of the Lie algebra $\mathfrak{g}$ and denoted by $W(\mathfrak{g})$. As a graded algebra it is the tensor product of the exterior and symmetric algebras:

$$W(\mathfrak{g}) = \Lambda(\mathfrak{g}^*) \otimes S(\mathfrak{g}^*),$$

where the degree of an element equals its exterior degree plus twice its polynomial degree. The differential, d_W is given by the sum of two differentials:

$$d_W = d_W^{\mathfrak{h}} + d_W^{\mathfrak{v}}.$$

Here, $d_W^{\mathfrak{v}}$ is the derivation that sends the generators of the exterior algebra to those of the symmetric algebra. On the other hand, $d_W^{\mathfrak{h}}$ is defined as a Koszul differential of $\mathfrak{g}$ with coefficients in the representations $S^q(\mathfrak{g}^*)$, the symmetric powers of the coadjoint representation. The interesting point about $W(\mathfrak{g})$ is that it behaves as the De Rham algebra of EG. The cohomology of $W(\mathfrak{g})$ is concentrated in degree zero, as one should expect from the fact that EG is contractible. More importantly, for a compact Lie group one can compute the cohomology of BG in terms of $W(\mathfrak{g})$. The differential forms on the base of a principal bundle are the basic forms on the total space, and in fact one sees that:

$$H(W(\mathfrak{g})_{\mathrm{bas}}) \cong S(\mathfrak{g}^*)^G.$$

Where $W(\mathfrak{g})_{\mathrm{bas}}$ are the elements of the Weil algebra which are horizontal and invariant with respect to the action of G. A more detailed description of this construction can be found in section 3.3.

In order to compute the cohomology of BG in the noncompact case, it is not enough to consider the representations of the Lie algebra. One needs to work with those of the group, as explained by Bott in [11]. Let us briefly discuss the theorem of Bott. Even though the classifying space is only defined up to homotopy, there is a combinatorial description of an explicit model using the geometric realization of simplicial spaces [24, 42]. This simplicial construction provides a way to compute the cohomology of BG as the total cohomology of the Bott-Shulman double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\Omega^2(G_0) & \xrightarrow{\delta} & \Omega^2(G_1) & \xrightarrow{\delta} & \Omega^2(G_2) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\Omega^1(G_0) & \xrightarrow{\delta} & \Omega^1(G_1) & \xrightarrow{\delta} & \Omega^1(G_2) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\Omega^0(G_0) & \xrightarrow{\delta} & \Omega^0(G_1) & \xrightarrow{\delta} & \Omega^0(G_2) & \xrightarrow{\delta} & \cdots
\end{array}$$

Here $\Omega^q(G_p)$ denotes the space of q-forms on $G \times \cdots \times G($ p times). The vertical differential d the is De Rham operator and the horizontal differential is the alternating sum of pullback maps:

$$\delta = \sum_{i=0}^{p} (-1)^i {d_i}^*,$$

where the map $d_i : G_p \to G_{p-1}$ is defined by:

$$d_i(g_1, \ldots, g_p) = \begin{cases} (g_2, \ldots, g_p) & \text{if } i = 0, \\ (g_1, \ldots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \ldots g_p) & \text{if } 0 < i \leq p, \\ (g_1, \ldots, g_{p-1}) & \text{if } i = p. \end{cases}$$

Bott computed the horizontal cohomologies of this double complex by proving the formula:

$$H^p_\delta(\Omega^q(G_\bullet)) \cong H^{p-q}_{\mathrm{diff}}(G, S^q(\mathfrak{g}^*)).$$

In particular, there is a spectral sequence relating the cohomology of G with coefficients in the polynomials in the Lie algebra and the cohomology of BG:

$$E_1^{pq} = H^{p-q}_{\mathrm{diff}}(G, S^q(\mathfrak{g}^*)) \Rightarrow H^{p+q}(BG). \tag{1.1}$$

In the compact case, the cohomology with coefficients in representations vanishes in positive degree and the first page of the spectral sequence becomes:

$\vdots$	$\vdots$	$\vdots$	$\vdots$	
0	0	0	$S^3(\mathfrak{g}^*)^G$	$\cdots$
0	0	$S^2(\mathfrak{g}^*)^G$	0	$\cdots$
0	$S^1(\mathfrak{g}^*)^G$	0	0	$\cdots$
$S^0(\mathfrak{g}^*)^G$	0	0	0	$\cdots$

This implies immediately that $H(BG) \cong S(\mathfrak{g}^*)^G$. Thus, the formula of Bott generalizes the theorem of Borel by taking into account the contributions from the cohomology in higher degrees for the non compact case.

1.1.2 Equivariant cohomology

The quotient of a manifold by a group action is often a pathological space whose cohomology is not a very interesting invariant of the action. The right cohomological invariant for this situation is the equivariant cohomology introduced by Borel in [10]. The singularity of the quotient space arises when the action is not free. The idea of equivariant cohomology is to replace the action of G on M by one that is free and equivalent to the original one from the point of view of homotopy. Since EG is a contractible space on which G acts freely, the space $M \times EG$ is homotopy equivalent to M and the diagonal action is free. The homotopy quotient of M by G is the space:

$$M_G = (M \times EG)/G.$$

The equivariant cohomology of M, denoted $H_G(M)$, is the ordinary cohomology of the homotopy quotient:

$$H_G(M) = H(M_G).$$

For free actions, the equivariant cohomology coincides with the cohomology of the quotient space. In general, it introduces information about the stabilizer groups which reflects the singularities of the quotient space. Let us illustrate this in a simple example. Consider the action of the circle $U(1)$ on the sphere S^2, by rotation around the z-axis. The quotient space of this action is the closed interval $[-1, 1]$, which has Poincare polynomial equal to 1. On the other hand, one can use Mayer-Vietoris to show that the equivariant Poincare polynomial of this action is:

$$P_t^G(M) = \frac{1}{1-t^2} + \frac{t^2}{1-t^2} \ .$$

The the fixed point information at the north and south poles is reflected in contributions that correspond to the cohomology of the classifying space of the stabilizer group at those points, which is $U(1)$.

As we have seen above, the cohomology of the classifying space of a compact Lie group can be computed in terms of infinitesimal information. The Weil algebra $W(\mathfrak{g})$ provides a model for the De Rham complex of EG and it is only natural the to think of the tensor product $\Omega(M) \otimes W(\mathfrak{g})$ as a model for the cohomology of $M \times EG$. These heuristics are justified by the fact that, when the group G is compact, that basic part of this algebra computes the equivariant cohomology:

$$H((\Omega(M) \otimes W(\mathfrak{g}))_{\mathrm{bas}}) \cong H_G(M).$$

The algebra $(\Omega(M) \otimes W(\mathfrak{g}))_{\mathrm{bas}}$ is known as the Weil model for equivariant cohomology. The Cartan model is the space $(\Omega(M) \otimes S(\mathfrak{g}^*))^G$ of invariant differential forms with values in polynomials on the Lie algebra, and is isomorphic to the Weil model. In his thesis [30], Kalkman introduced yet another infinitesimal model for equivariant cohomology, which is known as the BRST model.

The infinitesimal computations of equivariant cohomology only work for compact Lie groups. For noncompact group actions, Getzler [25] introduced a double complex which is a generalization of the spectral sequence of Bott for the cohomology of BG. As the in spectral sequence of Bott, the price to pay for the lack of compactness is that this double complex is not infinitesimal, it depends explicitly on Lie group.

1.1.3 Other homotopy quotients

Our discussion on equivariant cohomology shows that a good invariant of the action of a group on a manifold should take into account information about the stabilizer groups. The points where the action is not free are thought of as having automorphisms, which are part of the structure of the quotient space. Similarly, the leaf space of a foliation is a singular topological space whose properties can be well understood only by remembering the holonomy of the foliation, which again, means that the points in the quotient have automorphisms. These situations in which one is interested in quotients of smooth spaces by equivalence relations of Lie type are formalized by the concept of a groupoid. A groupoid is a category in which all arrows are isomorphisms. We often think of it as a generalized equivalence relation in which points can be equivalent in different ways, which are parametrized by a group. A Lie groupoid is a groupoid in which both the space of arrows and that of objects are smooth manifolds and all structure maps are smooth. The space of equivalence classes of a Lie groupoid is not only a topological space but a *stack*, which roughly means that we regard the automorphisms of the points as part of the structure. In the example of a group G acting on M, there is an action groupoid, denoted $G \ltimes M$, which represents the quotient. Haefliger has used Lie groupoids [26] to model leaf spaces of foliations and to give a precise meaning to the *transversal geometry*. In the same spirit, Lie groupoids are used to model orbifolds [37] and, more generally, orbispaces [24]. From this point of view, the classifying space BG of a Lie group is a space that encodes the information of the quotient of a point by G in terms of purely topological data. It turns out that, in general, Lie groupoids have classifying spaces which are generalized homotopy quotients. Indeed, for the transformation groupoid $G \ltimes M$ of a group action one verifies:

$$B(G \ltimes M) = M_G.$$

We will address the problem of how the method for the computation of the cohomology of classifying spaces of Lie groups can be extended to the case of Lie groupoids. Also, we study how this generalization relates to the models for equivariant cohomology. In the general case a new ingredient, representations up to homotopy, becomes necessary. The adjoint representations of G and $\mathfrak{g}$ play a central role in the computations mentioned above. The corresponding notions in the case of groupoids are only representations up to homotopy. This means that they are representations on cochain complexes for which the associativity conditions hold up to a coherent system of homotopies. Our aim will be to study representations up to homotopy for groupoids and use them to construct the analogs of the Weil algebra and the spectral sequence of Bott.

1.2 The content of this thesis

In this thesis we study representations up to homotopy in order to shed some light on the computation of the cohomology of classifying spaces of Lie groupoids. The first problem one encounters when trying to extend the computations we discussed for classifying spaces of Lie groups is that there is no obvious notion of adjoint representation for Lie groupoids. In the first two chapters we focus on the infinitesimal version of this issue and use representations up to homotopy to construct and study the Weil algebra of a Lie algebroid. The Lie algebroid of a Lie groupoid G is a vector bundle whose sections are the invariant vector fields on G. They play the role that Lie algebras do for Lie groups and most of the classical correspondence holds in this context. The second half of the thesis deals with representations up to homotopy at the global level and its main result is the generalization of the spectral sequence of Bott to the case of general Lie groupoids. We will now briefly describe the content of each chapter, a more detailed account of our results can be found in the individual introductions. This thesis is based on joint works in progress with Marius Crainic and Benoit Dherin.

Infinitesimal representations up to homotopy

We begin chapter 2 by introducing the notion of a representation up to homotopy of a Lie algebroid. Roughly, this is a connection on a complex of vector bundles whose curvature may fail to be zero, but is controlled by a system of higher homotopies. We show (proposition 2.2.10) that the adjoint representation of a Lie algebroid over M is a representation up to homotopy on the adjoint complex $A \to TM$, which has A in degree zero and TM in degree one. We prove (theorem 2.3.9) that the cohomology associated to the adjoint representation coincides with the deformation cohomology [19]. This is in agreement with the case of Lie algebras, where the deformations are controlled by the cohomology with coefficients in the adjoint representation [38, 39]. Once we have a good notion of an adjoint representation, we use it to define (proposition 2.4.1) the Weil algebra $W(A)$ of a Lie algebroid, by mimicking the construction for Lie algebras. We show that, when applied to the action of a Lie algebra on a manifold, the Weil algebra coincides with Kalkman's BRST algebra for equivariant cohomology (proposition 2.4.5). The Weil algebra $W(A)$ is isomorphic to the algebra $(C^\infty([-1]T([-1]A)), \mathcal{L}_{d_A} + d)$ studied in Mehta's thesis [35], using the language of supermanifolds. Unfortunately, unlike the case of group actions, it does not seem to be possible in general to compute the cohomology of BG using a basic subcomplex of the Weil algebra and therefore there is no good analog of the Cartan model in this case. A more satisfactory result regarding the cohomology of BG requires representations up to homotopy of the groupoid, as will be explained in chapter 5.

The Weil algebra and Van Est isomorphisms

In chapter 3 we construct (proposition 3.4.1) a homomorphism of double complexes from the Bott-Shulman complex of a Lie groupoid G to the Weil algebra of its Lie algebroid A,

$$V : \hat{\Omega}(G_\bullet) \to W(A),$$

which we call the Van Est map. The space $\hat{\Omega}(G_\bullet)$ models the cohomology of BG and $W(A)$ that of EG. The Van Est map corresponds to the projection $\pi : EG \to BG$.

We prove (theorem 3.5.1) that, if the fibers of the source of G are k-connected, the map induced in cohomology

$$V : H^p(\Omega^q(G_\bullet)) \to H^p(W^{\bullet,q}(A))$$

is an isomorphism for $p \leq k$ and is injective for $p = k + 1$. This isomorphism theorem is what one should expect from the topological interpretation of the situation. We then use (theorems 3.6.1 and 3.6.4) the isomorphism theorem to establish a correspondence between infinitesimal objects and multiplicative forms on the groupoid, generalizing the integration of Poisson manifolds [45] and Dirac structures [14].

Trees and Tensors

A representation up to homotopy of a group is an action on a cochain complex for which the associativity holds up to higher homotopies. Constructing the tensor product and symmetric powers of representations up to homotopy in the global case is a combinatorial problem similar to that of defining the tensor product of A_∞-algebras [33]. In Chapter 4 we give formulas for this construction. We introduce a differential graded algebra of planar trees $\mathbf{\Omega}$, which is universal for this problem. We then use the fact that the algebra is acyclic to construct (theorem 4.3.1) diagonal homomorphisms

$$\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}.$$

Finally, we prove (theorems 4.4.2 and 4.4.3) that the diagonal maps provide formulas for the tensor product and symmetric powers of representations up to homotopy.

The spectral sequence

In chapter 5 we show (theorem 5.2.2) that the adjoint representation of a Lie groupoid is a representation up to homotopy on the adjoint complex, exactly as in the infinitesimal case. We use the construction of the symmetric powers given in chapter 4 to prove (theorem 5.3.1) the formula of Bott

$$H^p_\delta(\Omega^q(G_\bullet)) \cong H^{p-q}_{\text{diff}}(G, S^q(\text{Ad}^*)),$$

for a general Lie groupoid G. As a consequence (theorem 5.4) we obtain a spectral sequence:

$$E_1^{pq} = H^{p-q}_{\text{diff}}(G, S^q(\text{Ad}^*)) \Rightarrow H^{p+q}(BG).$$

In the case of a Lie group acting on a manifold, this spectral sequence corresponds to the model of Getzler for equivariant cohomology [25]. In [7], Behrend generalized Getzler's work to the case of flat groupoids. In our terminology, this means groupoids that admit an adjoint representation which is strict, not only up to homotopy.

2

Infinitesimal representations up to homotopy

In this chapter we introduce the notion of representation up to homotopy of Lie algebroids and use it to define the adjoint representation. We prove that the corresponding cohomology controls the deformations of the structure, extending the well known result for Lie algebras. We explain the relation between representations up to homotopy and Kalkman's BRST algebra for equivariant cohomology.

2.1 Introduction

Lie algebroids are infinite dimensional Lie algebras which can be thought of as a *generalized tangent bundles* associated to various geometric situations. Apart from Lie algebras and (tangent bundles of) manifolds, examples of Lie algebroids come from foliation theory, equivariant geometry, Poisson geometry, riemannian foliations, quantization, etc. Lie algebroids are the infinitesimal counterparts of Lie groupoids exactly in the same sense in which Lie algebras are related to Lie groups, and much of the classical correspondence holds in this general setting. A representation of a Lie algebroid on a vector bundle is an action by derivations on the space of sections. In the case of tangent bundles, a representation corresponds to a flat vector bundle- note that vector bundles with flat connections serve as coefficient systems for cohomology theories, just like representations of Lie algebras do.
The aim of this chapter is to introduce and study a more general notion of representation, called *representation up to homotopy*. Our construction is inspired by Quillen's notion of superconnection [40] and fits into the general theory of structures up to homotopy. The idea of representations up to homotopy is to represent Lie algebroids in cochain complexes of vector bundles, rather than in vector bundles. We also allow the action to be flat only up to homotopy, that is, the curvature of the connection may not be zero, but it is controlled by some coherent homotopies which are part of the structure of the representation. The advantage of considering these representations is that they are flexible and general enough to contain interesting examples which are the correct generalization of the corresponding notions for Lie algebras. They also allow one to identify seemingly ad-hoc constructions and cohomology theories as instances of the cohomology with coefficients in representations. Our study of representations up to homotopy is motivated by applications to the following problems.

1. Cohomology of BG: The classifying space of a Lie groupoid G arises as the base of a universal principal G-bundle $EG \to BG$, which is typically infinite dimensional. When G is a Lie group, the standard model for $\Omega(EG)$ is the Weil algebra $W(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The algebra $W(\mathfrak{g})$ is built out of representations of $\mathfrak{g}$ on the symmetric powers of the coadjoint representation. We will use representations up to homotopy to construct the Weil algebra $W(A)$ of a general Lie algebroid A and show that it is a model for the cohomology of EG. For a Lie group G acting on a manifold M, there is an associated groupoid $\mathcal{G} = G \ltimes M$ and the cohomology of $B\mathcal{G}$ is precisely the equivariant cohomology of M. We will show that in this case the Weil algebra $W(\mathcal{A})$ of the Lie algebroid of $\mathcal{G}$ is Kalkman's BRST algebra for equivariant cohomology. There is a second way in which representations up to homotopy are related to the cohomology of BG. The corresponding notion of a representation up to homotopy of a Lie groupoid allows one to generalize Bott's spectral sequence [11] which is closely related to the models for equivariant cohomology of Getzler [25] and Cartan [15], this is the subject of chapter 5.

2. The adjoint representation: Ordinary representations are too restrictive to allow for a good notion of *adjoint representation* of a Lie algebroid. We prove that, using representations up to homotopy, a Lie algebroid has an adjoint representation with the same formal properties as that of a Lie algebra. For instance, we will see that the associated cohomology coincides with the deformation cohomology studied in [19], and thus controls the deformations of the structure. This is a generalization of the well known result for Lie algebras [38], [39].

3. Cyclic cohomology: Lie groupoids, via their convolution algebras, are the main source of examples of noncommutative spaces and provide a link between classical and noncommutative geometry. The cyclic cohomology of such convolution algebras appears as "the correct De Rham cohomology" in the non-commutative context. The computation of this cohomology for general Lie groupoids is a wide open problem, and one of the reasons is the lack of a representation theory which is general enough to handle the objects arising in the calculations. We believe that the notion of representation up to homotopy of Lie algebroids and of Lie groupoids may help deal with this issue.

4. Cocycle equations: One of the standard ways of understanding and handling complicated equations is to interpret them as cocycle equations and treat them as part of a cohomology theory. Sometimes it is not clear what the relevant cohomology theory is -for example when looking at the infinitesimal equations underlying multiplicative two forms (see the equations in the main theorem of [14]), or the equations satisfied by the k-differentials of [29]. Representations up to homotopy and the associated cohomologies provide a framework in which one can give a cohomological interpretation to various otherwise strange looking equations. There is a significant conceptual and practical advantage in this point of view. For instance, the infinitesimal equations associated to multiplicative two forms can be seen as cocycle equations in the Weil algebra and the k-differentials are cocycle equations for other representations (proposition 2.3.24). With this in mind, one can recover the main result of [14] on the integration of Dirac structures as a consequence of a Van Est isomorphism theorem- this new proof is essentially different and will be presented in chapter 3.

Let us now mention how this paper intersects related work that we have found in the literature. The question of constructing the adjoint representation of a Lie algebroid has already been addressed by Evens, Lu and Weinstein in [23]. Even though the two points of view are obviously related, we stress that our notion of representation up to homotopy differs fundamentally from the one used in their paper. Our construction of the Weil algebra is isomorphic to the one given by Mehta in his thesis [35], where using the language of supermanifolds, he described the algebra $(C^{\infty}([-1]T([-1]A)), \mathcal{L}_{d_A} + d)$. Some of the equations that appear in the definition of the adjoint representation where considered by Blaom in [8], which appeared while this work was being done. This chapter is organized as follows:

Section 2.2 begins by collecting the definitions of Lie algebroids, representations and associated cohomology theories. Then, the connections and curvatures underlying the adjoint representation are described and given a geometric interpretation. Finally, to fix our conventions, we recall some basic properties of graded algebra and complexes of vector bundles.

In Section 2.3 we give the definition of representations up to homotopy and introduce several examples, including the adjoint representation. We show that this notion has some of the usual properties of structures up to homotopy (proposition 2.3.36 and theorem 2.3.37). We study how to compute the associated cohomology and derive some invariance properties (proposition 2.3.33). In particular, we prove that the cohomology associated to

the adjoint representation is isomorphic to the deformation cohomology of [19] (theorem 2.3.9). We explain the relation between extensions of Lie algebroids and representations up to homotopy (proposition 2.3.11). The parallel transport arising from representations up to homotopy is explained in the case of tangent bundles (proposition 2.3.16).
In Section 2.4 we discuss several isomorphic models for the Weil algebra of a Lie algebroid-generalizing the standard Weil algebra of a Lie algebra. We show that, when applying this construction to the Lie algebroid associated to a Lie group action on a manifold, one obtains Kalkman's BRST algebra for equivariant cohomology (proposition 2.4.5).

2.2 Preliminaries

2.2.1 Representations and cohomologies

Here we review some known facts about Lie algebroids, representations and their cohomology. We recall some cocycle equations one finds in the literature (Remark 2.2.7) and make general comments regarding the notion of *adjoint representation* for Lie algebroids. Throughout this thesis, A denotes a Lie algebroid over a fixed base manifold M. As a general reference for Lie algebroids, we use [34].

Definition 2.2.1. *A Lie algebroid over M is a vector bundle $\pi : A \to M$ together with a bundle map $\rho : A \to TM$, called the anchor map, and a Lie bracket in the space $\Gamma(A)$ of sections of A satisfying the Leibnitz identity:*

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta,$$

for every $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$.

Given an algebroid A, there is an associated De Rham complex $\Omega(A) = \Gamma(\Lambda A^*)$, with De Rham operator given by the Koszul formula

$$\begin{aligned} d_A\omega(\alpha_1, \ldots, \alpha_{n+1}) &= \sum_{i<j}(-1)^{i+j}\omega([\alpha_i, \alpha_j], \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_{k+1}) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}\omega(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_{k+1}), \end{aligned}$$

where $L_X(f) = X(f)$ is the Lie derivative along vector fields. The operator d_A is a differential ($d_A^2 = 0$) and satisfies the derivation rule

$$d_A(\omega\eta) = d_A(\omega)\eta + (-1)^p \omega d_A(\eta),$$

for all $\omega \in \Omega^p(A), \eta \in \Omega^q(A)$.

Definition 2.2.2. *Let A be a Lie algebroid over M. An A-connection on a vector bundle E over M is a bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(\alpha, S) \mapsto \nabla_\alpha(S)$ such that:*

$$\nabla_{f\alpha}(s) = f\nabla_\alpha(s), \ \ \nabla_\alpha(fs) = f\nabla_\alpha(s) + L_{\rho(\alpha)}(f)(s)$$

for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. The A-curvature of ∇ is the tensor given by

$$R_\nabla(\alpha, \beta)(s) := \nabla_\alpha\nabla_\beta(s) - \nabla_\beta\nabla_\alpha(s) - \nabla_{[\alpha,\beta]}(s)$$

for $\alpha, \beta \in \Gamma(A)$, $s \in \Gamma(E)$. The A-connection ∇ is called flat if $R_\nabla = 0$. A representation of A is a vector bundle E together with a flat A-connection ∇ on E.

Given any A-connection ∇ on E, the space of E-valued A-differential forms, $\Omega(A, E) = \Gamma(\Lambda A^* \otimes E)$ has an induced operator d_∇ given by the Koszul formula

$$\begin{aligned} d_\nabla \omega(\alpha_1, \ldots, \alpha_{k+1}) &= \sum_{i<j} (-1)^{i+j} \omega([\alpha_i, \alpha_j], \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_{k+1}) \\ &+ \sum_i (-1)^{i+1} \nabla_{\alpha_i} \omega(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_{k+1}). \end{aligned}$$

In general, d_∇ satisfies the derivation rule

$$d_\nabla(\omega\eta) = d_A(\omega)\eta + (-1)^p \omega d_\nabla(\eta),$$

but squares to zero if and only if ∇ is flat. The following result is well known.

Proposition 2.2.3. *Given a Lie algebroid A and a vector bundle E over M, there is a 1-1 correspondence between A-connections ∇ on E and degree $+1$ operators d_∇ on $\Omega(A, E)$ which satisfy the derivation identity. Moreover, (E, ∇) is a representation if and only if $d_\nabla^2 = 0$.*

In a more algebraic language, every Lie algebroid has A has an associated differential graded algebra $(\Omega(A), d_A)$, and every representation E of A is a DG-module over it.

Definition 2.2.4. *Given a representation $E = (E, \nabla)$ of A, the cohomology of A with coefficients in E, denoted $H^\bullet(A, E)$, is the cohomology of the complex $(\Omega(A, E), d_\nabla)$. When E is the trivial representation (the trivial line bundle with $\nabla_\alpha = L_{\rho(\alpha)}$), we write $H^\bullet(A)$.*

The deformation cohomology of A arises in the study of deformations of the structure [19]. This cohomology cannot, in general, be realized as the cohomology associated to any representation. The deformation complex $(C^\bullet_{\text{def}}(A), \delta)$ is defined as follows. In degree k, it consists of antisymmetric, $\mathbb{R}$-multilinear maps

$$c : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{k-times} \to \Gamma(A),$$

which are multiderivations, in the sense that they come with a multilinear map, the symbol of c:

$$\sigma_c : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{k-1-times} \to \Gamma(TM),$$

satisfying:

$$c(\alpha_1, \ldots, f\alpha_k) = fc(\alpha_1, \ldots, \alpha_k) + L_{\sigma_c(\alpha_1, \ldots, \alpha_{k-1})}(f)\alpha_k,$$

for any function $f \in C^\infty(M)$ and sections $\alpha_i \in \Gamma(A)$. The differential

$$\delta : C^k_{\text{def}}(A) \to C^{k+1}_{\text{def}}(A)$$

given by

$$\begin{aligned} \delta(c)(\alpha_1, \ldots, \alpha_{k+1}) &= \sum_{i<j} (-1)^{i+j} c([\alpha_i, \alpha_j], \alpha_1, \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_{k+1}) \\ &+ \sum_{i=1}^{k+1} (-1)^{i+1} [\alpha_i, c(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_{k+1})]. \end{aligned}$$

Definition 2.2.5. *The deformation cohomology of the Lie algebroid A, denoted $H^\bullet_{def}(A)$ is defined as the cohomology of the cochain complex $(C^\bullet_{def}(A), \delta)$.*

Example 2.2.6.

1. In the extreme case when A is TM, its representations are the flat vector bundles over M, while the associated cohomology is the usual cohomology of M with local coefficients given by the flat sections of the vector bundle. The deformation cohomology is trivial in this case (cf. [19], Corollary 2).

2. When $A = \mathfrak{g}$ is a Lie algebra (hence M is a point), one recovers the standard notion of representation of Lie algebras, and Lie algebra cohomology. The deformation cohomology in this case is isomorphic to $H^\bullet(\mathfrak{g}, \mathfrak{g})$, the cohomology with coefficient the adjoint representation. This is related to the fact that deformations of the Lie algebra $\mathfrak{g}$ are controlled by $H^2(\mathfrak{g}, \mathfrak{g})$.

3. When $A = \mathcal{F}$ is a foliation on M, viewed as an involutive sub-bundle of TM, the Lie algebroid cohomology becomes the well-known foliated cohomology (see e.g. [1, 31]). Similar to the adjoint representation in case of Lie algebras, there is a canonical non-trivial representation of $\mathcal{F}$, the normal bundle $\nu = TM/\mathcal{F}$, endowed with the Bott connection [12]

$$\nabla_V(X \bmod \mathcal{F}) = [V, X] \bmod \mathcal{F}.$$

   The resulting cohomology $H^\bullet(\mathcal{F}, \nu)$ was introduced by Heitsch in his study of deformations of foliations [27], where he explains that such deformations are controlled by $H^1(\mathcal{F}, \nu)$. This is the first indication of the fact that, for foliations, $H^\bullet_{\mathrm{def}}(\mathcal{F})$ is isomorphic to $H^{\bullet-1}(\mathcal{F}, \nu)$ (Proposition 4 in [19]). We see that ν plays the role of the adjoint representation of Lie algebras, modulo a shift by one in the degree. In other words, we may declare the adjoint representation of $\mathcal{F}$ to be $\nu[-1]$, the graded vector bundle which is ν concentrated in degree one.

4. If A is a regular Lie algebroid, i.e. if $\rho : A \to TM$ has constant rank, then A has two canonical representations. They are the kernel of ρ, denoted $\mathfrak{g}(A)$, with the A-connection

$$\nabla^{\mathrm{adj}}_\alpha(\beta) = [\alpha, \beta],$$

   and the normal bundle $\nu(A) = TM/\rho(A)$ of the foliation induced by A, with the Bott-like connection

$$\nabla^{\mathrm{adj}}_\alpha(\overline{X}) = \overline{[\rho(\alpha), X]},$$

   where $\overline{X} = X \bmod \rho(A)$. It is tempting to define the adjoint representation of A to be the sum of these two representations, viewed as a graded vector bundle with $\mathfrak{g}(A)$ in degree zero and $\nu(A)$ in degree one. Is it reasonable to define

$$\mathrm{Ad} = \mathfrak{g}(A) \oplus \nu(A)[-1]? \tag{2.1}$$

   The situation is a bit subtle. The sum of the cohomologies associated to these two representations is not necessarily isomorphic to the deformation cohomology

$H^{\bullet}_{\text{def}}(A)$. Instead, there is a long exact sequence (Theorem 3 in [19]) relating these groups:

$$\cdots \to H^n(A, \mathfrak{g}(A)) \to H^n_{\text{def}}(A) \to H^{n-1}(A, \nu(A)) \xrightarrow{\delta} H^{n+1}(A, \mathfrak{g}(A)) \to \cdots$$

and the connecting map δ may be non-zero. As we shall see, definition (2.1) is correct provided we endow the objects involved with the right structure (and not just that of representations).

5. For general Lie algebroids, there may be disappointingly few representations. In particular, the representations $\mathfrak{g}(A)$ and $\nu(A)$ mentioned above no longer exist (they are not, in general, vector bundles). Hence the situation seems even worse when it comes to looking for the *adjoint representation*. However, the graded direct sum $\mathfrak{g}(A) \oplus \nu(A)[1]$ (which should be part of the adjoint representation, but which is non-smooth in general) can be viewed as the cohomology of a cochain complex of vector bundles, concentrated in two degrees:

$$A \xrightarrow{\rho} TM. \tag{2.2}$$

We will call this *the adjoint complex of* A. The idea of using this complex in order to make sense of the adjoint representation appeared already in [23], and is also present in [18]. We will work with cochain complexes of vector bundles as a replacement for their (possibly non-smooth) cohomology bundles.

Remark 2.2.7 (cocycle equations)**.** In the literature one often encounters equations which look like cocycle conditions, but which do not seem to have a cohomology theory behind them. Such equations arise naturally from various contexts (e.g. as infinitesimal manifestations of properties of interesting global objects) and it is often worth interpreting them as part of cohomology theories. This not only gives a conceptual point of view on the equations, but also allows one to handle them using powerful tools. We point out two such equations which will be shown later to fit into our framework.

The first one is related to the notion of k-differential studied in [29]. An almost k-differential on a Lie algebroid A is a pair of linear maps $\delta : C^\infty(M) \to \Gamma(\Lambda^{k-1}A)$, $\delta : \Gamma(A) \to \Gamma(\Lambda^k A)$ satisfying

(i) $\delta(fg) = \delta(f)g + f\delta(g)$

(ii) $\delta(f\alpha) = \delta(f) \wedge \alpha + f\delta(\alpha)$

for all $f, g \in C^\infty(M)$, $\alpha \in \Gamma(A)$. It is called a k-differential if

$$\delta[\alpha, \beta] = [\delta(\alpha), \beta] + [\alpha, \delta(\beta)]$$

for all $\alpha, \beta \in \Gamma(A)$.

The second example arises from the notion of IM (infinitesimally multiplicative) form, which is the infinitesimal counterpart of multiplicative 2-forms on groupoids [14]. An IM form on a Lie algebroid A is a bundle map $\sigma : A \to T^*M$ satisfying:

$$\begin{aligned} \langle \sigma(\alpha), \rho(\beta) \rangle &= -\langle \sigma(\beta), \rho(\alpha) \rangle, \\ \sigma([\alpha, \beta]) &= L_{\rho(\alpha)}(\sigma(\beta)) - L_{\rho(\beta)}(\sigma(\alpha)) + d\langle \sigma(\alpha), \rho(\beta) \rangle \end{aligned}$$

for all $\alpha, \beta \in \Gamma(A)$. Here ρ is the anchor of A and $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual.

2.2.2 Basic connections and the basic curvature

Keeping in mind our discussion on what the adjoint representation should be (Example 2.2.6), we want to *extend* the canonical flat A-connections ∇^{adj} (from $\mathfrak{g}(A)$ and $\nu(A)$) to A and TM or, even better, to the adjoint complex (2.2). This construction already appeared in the theory of exotic characteristic classes [18] and it was also used in [22].

Definition 2.2.8. *Given a Lie algebroid A over M and a connection ∇ on the vector bundle A, we define*

1. *The basic A-connection induced by ∇ on A:*

$$\nabla^{bas}_{\alpha}(\beta) = \nabla_{\rho(\beta)}(\alpha) + [\alpha, \beta].$$

2. *The basic A-connection induced by ∇ on TM:*

$$\nabla^{bas}_{\alpha}(X) = \rho(\nabla_X(\alpha)) + [\rho(\alpha), X].$$

Here α, β are sections of A and X is a vector field on M.

Note that $\nabla^{\text{bas}}_{\alpha} \circ \rho = \rho \circ \nabla^{\text{bas}}_{\alpha}$, i.e. ∇^{bas} is an A-connection on the adjoint complex (2.2). On the other hand, the existence of a connection ∇ such that ∇^{bas} is flat is a very restrictive condition on A. It turns out that the curvature of ∇^{bas} hides behind a more interesting tensor which we will call the basic curvature of ∇.

Definition 2.2.9. *Given a Lie algebroid A over M and a connection ∇ on the vector bundle A, we define the basic curvature of ∇ as the tensor*

$$R^{bas}_{\nabla} \in \Omega^2(A, Hom(TM, A))$$

given by

$$R^{bas}_{\nabla}(\alpha, \beta)(X) := \nabla_X([\alpha, \beta]) - [\nabla_X(\alpha), \beta] - [\alpha, \nabla_X(\beta)] - \nabla_{\nabla^{bas}_{\beta}X}(\alpha) + \nabla_{\nabla^{bas}_{\alpha}X}(\beta),$$

where α, β are sections of A and X, Y are vector fields on M.

This tensor appears when one looks at the curvatures of the A-connections ∇^{bas}. One may think of R^{bas}_{∇} as the expression $\nabla_X([\alpha, \beta]) - [\nabla_X(\alpha), \beta] - [\alpha, \nabla_X(\beta)]$ which measures the derivation property of ∇ with respect to $[\cdot, \cdot]$, *corrected* so that it becomes $C^{\infty}(M)$-linear on all arguments.

Proposition 2.2.10. *For any connection ∇ on A, one has:*

1. *The curvature of the A-connection ∇^{bas} on A equals to $-\rho \circ R^{bas}_{\nabla}$, while the curvature of the A-connection ∇^{bas} on TM equals to $-R^{bas}_{\nabla} \circ \rho$.*

2. *R^{bas}_{∇} is closed with respect to ∇^{bas} i.e. $d_{\nabla^{bas}}(R^{bas}) = 0$.*

Proof. For $\alpha, \beta, \gamma \in \Gamma(A)$,

$$\begin{aligned}
R^{\text{bas}}_{\nabla}(\alpha,\beta)\rho(\gamma) &= \nabla_{\rho(\gamma)}([\alpha,\beta]) - [\nabla_{\rho(\gamma)}(\alpha),\beta] - [\alpha, \nabla_{\rho(\gamma)}(\beta)] \\
&\quad -\nabla_{\nabla_{\beta}\rho(\gamma)}(\alpha) + \nabla_{\nabla_{\alpha}\rho(\gamma)}(\beta) \\
&= \nabla_{[\alpha,\beta]}(\gamma) - \nabla_{\alpha}(\nabla_{\beta}(\gamma)) + \nabla_{\beta}(\nabla_{\alpha}(\gamma)) \\
&= -R_{\nabla^{\text{bas}}}(\alpha,\beta)(\gamma)
\end{aligned}$$

For the second equation the computation becomes:

$$\begin{aligned}
\rho(R^{\text{bas}}_{\nabla}(\alpha,\beta)X) &= \rho(\nabla_X([\alpha,\beta]) - [\nabla_X(\alpha),\beta] - [\alpha,\nabla_X(\beta)] \\
&\quad -\nabla_{\nabla_{\beta}X}(\alpha) + \nabla_{\nabla_{\alpha}X}(\beta)) \\
&= \rho(\nabla_X([\alpha,\beta])) + [\rho([\alpha,\beta]),X] + [[\rho(\beta),X],\rho(\alpha)] \\
&\quad -\rho([\alpha,\nabla_X(\beta)]) - \rho(\nabla_{\nabla_{\beta}X}(\alpha)) + [\rho(\beta),[\rho(\alpha),X]] \\
&\quad +\rho(\nabla_{\nabla_{\alpha}X}(\beta)) - \rho([\nabla_X(\alpha),\beta]) \\
&= \nabla_{[\alpha,\beta]}(X) - \nabla_{\alpha}(\nabla_{\beta}(X)) + \nabla_{\beta}(\nabla_{\alpha}(X)) \\
&= -R_{\nabla^{\text{bas}}}(\alpha,\beta)(X).
\end{aligned}$$

The proof of the second part is a similar computation that we will omit. □

Let us now explain the relationship between the basic curvature R^{bas}_{∇} and the first jet algebroid of A. For the first part of the discussion we only use the vector bundle structure of A. We denote by $J^1(A)$ the first jet bundle of A and by $\pi : J^1(A) \to A$ the canonical projection. There is an extension of vector bundles over M,

$$0 \to \text{Hom}(TM, A) \xrightarrow{i} J^1(A) \xrightarrow{\pi} A \to 0, \tag{2.3}$$

where the inclusion i is uniquely determined by the condition

$$\text{Hom}(TM, A) \ni df \otimes \alpha \mapsto f j^1(\alpha) - j^1(f\alpha),$$

for all $f \in C^{\infty}(M)$, $\alpha \in \Gamma(A)$. Giving a connection ∇ on A is equivalent to choosing a splitting $j^{\nabla} : A \to J^1(A)$ of π. The relation between the two is given by

$$j^{\nabla}(\alpha) = j^1(\alpha) + \nabla.(\alpha),$$

for all $\alpha \in \Gamma(A)$. Here $\nabla.(\alpha) \in \text{Hom}(TM, A)$ is given by $X \mapsto \nabla_X(\alpha)$. This follows from the fact that, although $\alpha \mapsto j^1(\alpha)$ is not a bundle map, it defines a splitting of the sequence (2.3) at the level of sections. Hence any splitting j of our sequence differs from j^1 (at the level of sections) by elements coming from $\text{Hom}(TM, A)$. Moreover, the $C^{\infty}(M)$-linearity of j^{∇} translates precisely into the derivation property for ∇.

Let us now use the fact that A is a Lie algebroid. In this case, $J^1(A)$ admits a unique Lie algebroid structure such that for any section α of A,

$$\rho(j^1\alpha) = \rho(\alpha) \tag{2.4}$$

and for any sections $\alpha, \beta \in \Gamma(A)$:

$$[j^1\alpha, j^1\beta] = j^1([\alpha,\beta]). \tag{2.5}$$

Since π preserves the Lie bracket, it follows that $\text{Hom}(TM, A)$ inherits the structure of a bundle of Lie algebras, and (2.3) becomes an extension of Lie algebroids.

Proposition 2.2.11. *For any connection ∇ on A,*

$$R_{\nabla}^{bas}(\alpha,\beta) = j^{\nabla}([\alpha,\beta]) - [j^{\nabla}(\alpha), j^{\nabla}(\beta)].$$

Let us point out two results which indicate the geometric meaning of the basic curvature R_{∇}^{bas}. The first one refers to the characterization of Lie algebroids which arise from Lie algebra actions.

Proposition 2.2.12. *A Lie algebroid A over a simply connected manifold M is the algebroid associated to a Lie algebra action on M if an only if it admits a flat connection ∇ whose induced basic curvature R_{∇}^{bas} vanishes.*

Proof. If A is associated to a Lie algebra action, one chooses ∇ to be the obvious flat connection. Assume now that there is a connection ∇ as above. Since M is simply connected, the bundle is trivial. Choose a frame of flat sections $\alpha_1, \ldots, \alpha_r$ of A, and write

$$[\alpha_i, \alpha_j] = \sum_{k=1}^{r} c_{ij}^{k} \alpha_k,$$

with $c_{ij}^k \in C^{\infty}(M)$. Since

$$R_{\nabla}^{\text{bas}}(\alpha_i, \alpha_j)(X) = \sum_{k=1}^{r} \nabla_X(c_{ij}^k \alpha_k) = \sum_{k=1}^{r} X(c_{ij}^k)\alpha_k,$$

we deduce that the c_{ij}^k's are constant. The Jacobi identity for the Lie bracket on $\Gamma(A)$ implies that c_{ij}^k's are the structure constants of a Lie algebra, call it $\mathfrak{g}$. The anchor map defines an action of $\mathfrak{g}$ on M, and the trivialization of A induces the desired isomorphism. □

The condition that M is simply connected in the previous proposition is there to ensure that the vector bundle is trivial. Any nontrivial vector bundle with a flat connection and zero bracket and anchor provides a counterexample in the non simply connected case. Consider $A = TM$ for some compact simply connected manifold M and a flat connection ∇ on A. The condition that the basic curvature vanishes means precisely that the conjugated connection $\overline{\nabla}$ defined by:

$$\overline{\nabla}_X(Y) = [X, Y] + \nabla_Y(X)$$

is also flat. This implies that M is a Lie group.

The next result refers to the relation between bundles of Lie algebras (viewed as Lie algebroids with zero anchor map) and Lie algebra bundles, for which the fiber Lie algebra is fixed in the local trivializations.

Proposition 2.2.13. *Let A be a bundle of Lie algebras over M. Then A is a Lie algebra bundle if and only if it admits a connection ∇ whose basic curvature vanishes.*

Proof. If the bundle of Lie algebras is locally trivial then locally one can choose connections with zero basic curvature. Then one can use partitions of unity to construct a global connection, which will have the same property. For the converse, assume there exists a ∇ with $R_{\nabla}^{\text{bas}} = 0$, and we need to prove that the Lie algebra structure on the fiber is locally trivial. We may assume that $A = R^n \times R^r$ as a vector bundle. The vanishing of the basic curvature means that ∇ acts as derivations of the Lie algebra fibers

$$\nabla_X([\alpha, \beta]) = [\nabla_X(\alpha), \beta] + [\alpha, \nabla_X(\beta)].$$

Since derivations are infinitesimal automorphisms, we deduce that the parallel transports induced by ∇ are Lie algebra isomorphisms, providing the necessary Lie algebra bundle trivialization. □

2.2.3 The graded setting

We have already seen (Example 2.2.6) that in order to make sense of the adjoint representation of a Lie algebroid, we will have to consider graded vector bundles. In this section we collect a few conventions and constructions regarding the graded setting. As a general rule, we will be constantly using the *standard sign convention*: whenever two graded objects x and y, say of degrees p and q, are interchanged, one introduces a sign $(-1)^{pq}$. For instance, the standard commutator $xy - yx$ is replaced by the graded commutator

$$[x, y] = xy - (-1)^{pq} yx.$$

Throughout this section, M is a fixed manifold, and all our vector bundles are over M.

1. Graded vector bundles. By a graded vector bundle over M we mean a vector bundle E together with a direct sum decomposition indexed by integers:

$$E = \bigoplus_n E^n.$$

An element $v \in E^n$ is called a homogeneous element of degree n and we write $|v| = n$. Most of the constructions on graded vector bundles follow by applying pointwise the standard constructions with graded vector spaces. Here are some of them.

1. Given two graded vector bundles E and F, their direct sum and their tensor product have natural gradings. On $E \otimes F$ we always use the total grading:

$$\deg(e \otimes f) = \deg(e) + \deg(f).$$

2. Given two graded vector bundles E and F one can also form the new graded space $\underline{\mathrm{Hom}}(E, F)$. Its degree k part, denoted $\underline{\mathrm{Hom}}^k(E, F)$, consists of vector bundle maps $T : E \to F$ which increase the degree by k. When $E = F$, we use the notation $\underline{\mathrm{End}}(E)$.

3. For any graded vector bundle, the associated tensor algebra bundle $T(E)$ is graded by the total degree

$$\deg(v_1 \otimes \ldots \otimes v_n) = \deg(v_1) + \ldots + \deg(v_n).$$

   The associated symmetric algebra bundle $S(E)$ is defined (fiberwise) as the quotient of $T(E)$ by forcing $[v, w] = 0$ for all $v, w \in E$, while for the exterior algebra bundle $\Lambda(E)$ one forces the relations $vw = -(-1)^{pq} wv$ where p and q are the degrees of v and w, respectively.

4. The dual E^* of a graded vector bundle is graded by

$$(E^*)^n = (E^{-n})^*.$$

2. Wedge products. We now discuss wedge products in the graded context. First of all, given a Lie algebroid A and a graded vector bundle E, the space of E-valued A-differential forms, $\Omega(A, E)$, is graded by the total degree:

$$\Omega(A, E)^p = \bigoplus_{i+j=p} \Omega^i(A, E^j).$$

Wedge products arise in the following general situation. Assume that E, F and G are graded vector bundles and

$$h : E \otimes F \to G$$

is a degree preserving vector bundle map. Then there is an induced wedge-product operation

$$\Omega(A, E) \times \Omega(A, F) \to \Omega(A, G),\ (\omega, \eta) \mapsto \omega \wedge_h \eta.$$

Explicitly, for $\omega \in \Omega^p(A, E^i)$, $\eta \in \Omega^q(A, F^j)$, $\omega \wedge_h \eta \in \Omega^{p+q}(A, G^{i+j})$ is given by

$$(\alpha_1, \ldots, \alpha_{p+q}) \mapsto \sum (-1)^{qi} \mathrm{sgn}(\sigma) h(\omega(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}), \eta(\alpha_{\sigma(p+1)} \ldots, \alpha_{\sigma(p+q)})),$$

where the sum is over all $(p-q)$-shuffles. One explanation for the sign is that it forces

$$\omega \wedge_h \eta = (-1)^{\deg(\omega)\deg(\eta)} \eta \wedge_{\overline{h}} \omega,$$

where $\overline{h}$ is the composition of h with the graded twist map:

$$F \otimes E \to E \otimes F,\ \ u \otimes v \mapsto (-1)^{\deg(u)\deg(v)} v \otimes u.$$

Here is a list of examples of this situation.

1. If h is the identity we get:

$$\cdot \wedge \cdot : \Omega(A, E) \otimes \Omega(A, F) \to \Omega(A, E \otimes F).$$

   In particular, we get two operations

$$\Omega(A) \otimes \Omega(A, E) \to \Omega(A, E),\ \Omega(A, E) \otimes \Omega(A) \to \Omega(A, E). \tag{2.6}$$

   which make $\Omega(A, E)$ into a (graded) $\Omega(A)$-bimodule. Note that, while the first one coincides with the wedge product applied to E viewed as a (ungraded) vector bundle, the second one involves a sign.

2. If h is the composition of endomorphisms of E we get an operation

$$\cdot \circ \cdot : \Omega(A, \underline{\mathrm{End}}(E)) \otimes \Omega(A, \underline{\mathrm{End}}(E)) \to \Omega(A, \underline{\mathrm{End}}(E)) \tag{2.7}$$

   which gives $\Omega(A, \underline{\mathrm{End}}(E))$ the structure of a graded algebra. Of course, this operation makes sense for general <u>Hom</u>'s instead of <u>End</u>.

3. If h is the evaluation map $\text{ev} : \underline{\text{End}}(E) \otimes E \to E,\ (T, v) \mapsto T(v)$, we get:

$$\cdot \wedge \cdot : \Omega(A, \underline{\text{End}}(E)) \otimes \Omega(A, E) \to \Omega(A, E), \tag{2.8}$$

while when h is the twisted evaluation map $\overline{\text{ev}} : E \otimes \underline{\text{End}}(E) \to E,\ (v, T) \mapsto (-1)^{|v||T|} T(v)$, we get:

$$\cdot \wedge \cdot : \Omega(A, E) \otimes \Omega(A, \underline{\text{End}}(E)) \to \Omega(A, E). \tag{2.9}$$

These operations make $\Omega(A, E)$ a graded $\Omega(A, \underline{\text{End}}(E))$-bimodule.

4. If $h : \Lambda^\bullet E \otimes \Lambda^\bullet E \to \Lambda^\bullet E$ is the multiplication, we get

$$\cdot \wedge \cdot : \Omega(A, \Lambda^\bullet E) \otimes \Omega(A, \Lambda^\bullet E) \to \Omega(A, \Lambda^\bullet E)$$

which makes $\Omega(A, \Lambda^\bullet E)$ a graded algebra.

Note that the ring $\Omega(A, \underline{\text{End}}(E))$ can be identified with the space of endomorphisms of the left $\Omega(A)$-module $\Omega(A, E)$ (in the graded sense). More precisely, we have the following simple lemma.

Lemma 2.2.14. *There is a 1-1 correspondence between degree n elements of $\Omega(A, \underline{End}(E))$ and operators F on $\Omega(A, E)$ which increase the degree by n and which are $\Omega(A)$-linear in the graded sense:*

$$F(\omega \wedge \eta) = (-1)^{n|\omega|} \omega \wedge F(\eta) \quad \forall\, \omega \in \Omega(A), \eta \in \Omega(A, E).$$

Explicitly, $T \in \Omega(A, \underline{End}(E))$ induces the operator $\hat{T}$ given by

$$\hat{T}(\eta) = T \wedge \eta.$$

Finally, there is one more interesting operation of type $\wedge_h$, namely the one where h is the graded commutator

$$h : \underline{\text{End}}(E) \otimes \underline{\text{End}}(E) \to \underline{\text{End}}(E),\ \ h(T, S) = T \circ S - (-1)^{|S||T|} S \circ T.$$

The resulting operation

$$\Omega(A, \underline{\text{End}}(E)) \otimes \Omega(A, \underline{\text{End}}(E)) \to \Omega(A, \underline{\text{End}}(E))$$

will be denoted by $[-, -]$. Note that

$$[T, S] = T \wedge S - (-1)^{|T||S|} S \wedge T.$$

2.2.4 Complexes of vector bundles

In this section we continue to construct the framework that will allow us to make sense of the adjoint representation of Lie algebroids. As we explained at the end of Example 2.2.6, in order to avoid working with non-smooth vector bundles, one realizes them as the cohomology of cochain complexes of (smooth) vector bundles. Here we bring together some rather standard constructions and facts about such complexes. We fix a manifold M, and all vector bundles will be over M.

Complexes. By a complex over M we mean a cochain complex of vector bundles over M, i.e. a graded vector bundle E over M endowed with a degree one endomorphism ∂ satisfying $\partial^2 = 0$:

$$(E, \partial) : \quad \cdots \xrightarrow{\partial} E^0 \xrightarrow{\partial} E^1 \xrightarrow{\partial} E^2 \xrightarrow{\partial} \cdots$$

We drop ∂ from the notation whenever there is no danger of confusion. A morphism between two complexes E and F over M is a vector bundle map $f : E \to F$ which preserves the degree and is compatible with the differentials. We denote by $\mathrm{Hom}(E, F)$ the space of all such maps. We denote by $\underline{\mathrm{Ch}}(M)$ the resulting category of complexes over M.

Definition 2.2.15. *We say that a complex (E, ∂) over M is regular if ∂ has constant rank. In this case one defines the cohomology of E as the graded vector bundle over M:*

$$\mathcal{H}^\bullet(E) := \mathit{Ker}(\partial)/\mathit{Im}(\partial).$$

Remark 2.2.16. *Note that $\mathcal{H}^\bullet(E)$ only makes sense (as a vector bundle) when E is regular. On the other hand, one can always take the pointwise cohomology. For each $x \in M$, there is a cochain complex of vector spaces (E_x, ∂_x) and one can take its cohomology $H^\bullet(E_x, \partial_x)$. The dimension of these spaces may vary as x varies, and it is constant if and only if E is regular, in which case they fit into a graded vector bundle over M- and that is $\mathcal{H}^\bullet(E)$.*

As for cochain complexes of vector spaces, we have the following terminology:

1. Given two complexes of vector bundles E and F and morphisms $f_1, f_2 : E \to F$, a homotopy between f and g is a degree -1 map $h : E \to F$ satisfying

   $$h\partial + \partial h = f_1 - f_2.$$

   If such an h exists, we say that f_1 and f_2 are homotopic.

2. A morphism $f : E \to F$ between two complexes of vector bundles E and F is called a homotopy equivalence if there exists a morphism $g : F \to E$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps. If such an f exists, we say that E and F are homotopy equivalent.

   We say that E is contractible if it is homotopic equivalent to the zero-complex or, equivalently, if there exists a homotopy between Id_E and the zero map.

3. A morphism $f : E \to F$ between two complexes of vector bundles is called a quasi-isomorphism if it induces isomorphism in the pointwise cohomologies.

   We say that E is acyclic if it is pointwise acyclic.

Lemma 2.2.17. *For complexes of vector bundles over M:*

(1) If $f : E \to F$ is a quasi-isomorphism at $x \in M$, then it is a quasi-isomorphism in a neighborhood of x. In particular, if a complex E is exact at $x \in M$, then it is exact in a neighborhood of x.

(2) *A morphism $f : E \to F$ is a quasi-isomorphism if and only if it is a homotopy equivalence. In particular, a complex E is acyclic if and only if it is contractible.*

(3) *If a complex E is regular then it is homotopy equivalent to its cohomology $\mathcal{H}^\bullet(E)$ endowed with the zero differential.*

Proof. For (1) and (2), it suffices to prove the apparently weaker statements in the lemma coming after "in particular". This follows from the standard mapping complex argument: given a morphism f, one builds a double complex with E as 0-th row, F as 1-st row, and f as vertical differential. The resulting double complex $M(f)$, has the property that it is acyclic, or contractible, if and only if f is a quasi-isomorphism, or a homotopy equivalence, respectively.

To prove the weaker statements of (1) and (2), we fix a complex (E, ∂) and choose a metric in each vector bundle E^i. Denote ∂^* the adjoint of ∂ with respect to the chosen metric and

$$\Delta = \partial\partial^* + \partial^*\partial$$

the corresponding "Laplacian". It is not difficult to see that the complex $(E^\bullet_x, \partial)$ is exact if and only if Δ_x is an isomorphism. Since the isomorphisms form an open set in the space of linear transformations, we get (1). When (E, ∂) is exact, a simple computation shows that $h := \Delta^{-1}\partial^*$ is a contracting homotopy for E, proving (2).

For (3) a linear version of Hodge decomposition gives us

$$E = Ker(\Delta) \oplus Im(\partial) \oplus Im(\partial^*)$$

and an identification $H^\bullet(E) = Ker(\Delta)$. The resulting projection $E \to Ker(\Delta)$ is a quasi-isomorphism. □

3. Operations. The operations with graded vector bundles discussed in the previous section extend to the setting of complexes. In other words, if E and F are complexes over M, then all the associated graded vector bundles $S(E)$, $\Lambda(E)$, E^*, $\underline{\mathrm{Hom}}(E, F)$, $E \otimes F$, inherit an operator ∂ making them complexes over M. The induced differentials are defined by requiring that they satisfy the (graded) derivation rule, written formally as

$$\partial(xy) = \partial(x)y + (-1)^{|x|}x\partial(y).$$

In each example, x and y live in the appropriate space, and the meaning of the *product* should be clear from the context. For instance, for $E \otimes F$,

$$\partial(v \otimes w) = \partial(v) \otimes w + (-1)^{|v|}v \otimes \partial(w).$$

Also, for $T \in \underline{\mathrm{Hom}}(E, F)$,

$$\partial(T(v)) = \partial(T)(v) + (-1)^{|T|}T(\partial(v)),$$

in terms of graded commutators:

$$\partial(T) = \partial \circ T - (-1)^{|T|}T \circ \partial = [\partial, T].$$

Next, if E is a complex over M, then its differential ∂ induces a differential ∂ on $\Omega(A, E)$ defined by:

$$\partial(\eta) = \partial \wedge \eta.$$

Explicitly, for $\eta \in \Omega^p(A, E^k)$, $\partial(\eta) \in \Omega^p(A, E^{k+1})$ is given by

$$(\alpha_1, \ldots, \alpha_p) \mapsto (-1)^p \partial(\eta(\alpha_1, \ldots, \alpha_p)).$$

The following simple lemma shows that the various differentials induced on $\Omega(A, \underline{\text{End}}(E))$ coincide.

Lemma 2.2.18. *For any* $T \in \Omega(A, \underline{End}(E))$*,*

$$\partial(T) = [\partial, T] = \partial \wedge T - (-1)^{|T|} T \wedge \partial.$$

Connections. Let A be a Lie algebroid. An A-connection on a graded vector bundle E is just an A-connection on the underlying vector bundle E which preserves the grading. Equivalently, it is a family of A-connections, one on each E^n. If (E, ∂) is a complex over M, an A-connection on (E, ∂) is a graded connection ∇ which is compatible with ∂ (i.e. $\nabla_\alpha \partial = \partial \nabla_\alpha$). Note that, in terms of the operators d_∇ and ∂ induced on $\Omega(A, E)$, the compatibility of ∇ and ∂ is equivalent to $[d_\nabla, \partial] = 0$.

Connections on E and F naturally induce connections on the associated bundles $S(E)$, $\underline{\text{End}}(E)$, $E \otimes F$, etc. The basic principle is, as before, the graded derivation rule. For instance, one has

$$d_\nabla(\eta_1 \wedge \eta_2) = d_\nabla(\eta_1) \wedge \eta_2 + (-1)^{|\eta_1|} \eta_1 \wedge d_\nabla(\eta_2),$$

for all $\eta_1 \in \Omega(A, E)$, $\eta_2 \in \Omega(A, F)$. Also, for $T \in \Omega(A, \underline{\text{End}}(E))$, $d_\nabla(T)$ is uniquely determined by

$$d_\nabla(T \wedge \eta) = d_\nabla(T) \wedge \eta + (-1)^{|T|} T \wedge d_\nabla(\eta),$$

for all $\eta \in \Omega(A, E)$. More explicitly,

$$d_\nabla(T) = [d_\nabla, T].$$

Lemma 2.2.19. *If a complex* (E, ∂) *admits an* A*-connection then, for any leaf* $L \subset M$ *of* A*,* $E|_L$ *is regular.*

Proof. When $A = TM$, there is only one leaf $L = M$, and we have to prove that E is regular. Since ∇ is compatible with ∂, it follows that the parallel transport with respect to ∇ commutes with ∂ and therefore induces isomorphisms between the point-wise cohomologies. The same argument applied to parallel transport along A-paths, as explained in [22], implies the general case. □

2.3 Representations up to homotopy

2.3.1 Definition and first examples

In this section we introduce the notion of representation up to homotopy and the adjoint representation of Lie algebroids. As before, A is a Lie algebroid over M. We start with the shortest, but less intuitive description of representations up to homotopy.

Definition 2.3.1. *A representation up to homotopy of A consists of a graded vector bundle E over M and an operator, called the structure operator,*

$$D : \Omega(A, E) \to \Omega(A, E)$$

which increases the total degree by one and satisfies $D^2 = 0$ and the graded derivation rule:

$$D(\omega\eta) = d_A(\omega)\eta + (-1)^k \omega D(\eta),$$

for all $\omega \in \Omega^k(A)$, $\eta \in \Omega(A, E)$. The cohomology of the resulting complex is denoted by $H^\bullet(A, E)$.

Intuitively, a representation up to homotopy of A is a complex endowed with an A-connection which is flat up to homotopy. We will make this precise in what follows.

Proposition 2.3.2. *There is a 1-1 correspondence between representations up to homotopy (E, D) of A and graded vector bundles E over M endowed with*

1. *A degree 1 operator ∂ on E making (E, ∂) a complex.*
2. *An A-connection ∇ on (E, ∂).*
3. *An End-valued 2-form ω_2 of total degree 1, i.e.*

   $$\omega_2 \in \Omega^2(A, \underline{End}^{-1}(E))$$

   satisfying

   $$\partial(\omega_2) + R_\nabla = 0,$$

   where R_∇ is the curvature of ∇.
4. *For each $i > 2$ an End(E)-valued i-form ω_i of total degree 1, i.e.*

   $$\omega_i \in \Omega^i(A, \underline{End}^{1-i}(E))$$

   satisfying

   $$\partial(\omega_i) + d_\nabla(\omega_{i-1}) + \omega_2 \circ \omega_{i-2} + \omega_3 \circ \omega_{i-3} + \ldots + \omega_{i-2} \circ \omega_2 = 0.$$

The correspondence is characterized by

$$D(\eta) = \partial(\eta) + d_\nabla(\eta) + \omega_2 \circ \eta + \omega_3 \circ \eta + \ldots.$$

We also write

$$D = \partial + \nabla + \omega_2 + \omega_3 + \ldots. \tag{2.10}$$

Proof. Due to the derivation rule and the fact that $\Omega(A, E)$ is generated as an $\Omega(A)$-module by $\Gamma(E)$, the operator D will be uniquely determined by what it does on $\Gamma(E)$. It will send each $\Gamma(E^k)$ into the sum

$$\Gamma(E^{k+1}) \oplus \Omega^1(A, E^k) \oplus \Omega^2(A, E^{k-1}) \oplus \ldots,$$

hence it will also send each $\Omega^p(A, E^k)$ into the sum

$$\Omega^p(A, E^{k+1}) \oplus \Omega^{p+1}(A, E^k) \oplus \Omega^{p+2}(A, E^{k-1}) \oplus \ldots,$$

and we denote by D_0, D_1, ... the components of D. From the derivation rule for D, we deduce that each D_i for $i \neq 1$ is a (graded) $\Omega(A)$-linear map and, by Lemma 2.2.14, it is the wedge product with an element in $\Omega(A, \underline{\text{End}}(E))$. On the other hand, D_1 satisfies the derivation rule on each of the vector bundles E^k and, by Proposition 2.2.3, it comes from A-connections on these bundles. The equations in the statement correspond to $D^2 = 0$. □

Next, one can define the notion of morphism between representations up to homotopy.

Definition 2.3.3. *A morphism $\Phi : E \to F$ between two representations up to homotopy of A is a degree zero linear map*

$$\Phi : \Omega(A, E) \to \Omega(A, F)$$

which is $\Omega(A)$-linear and commutes with the structure differentials D_E and D_F.

We denote by $\mathbb{R}\text{ep}^\infty(A)$ the resulting category, and by $\text{Rep}^\infty(A)$ the set of isomorphism classes of representations up to homotopy of A.

By the same arguments as above, one gets the following description of morphisms in $\mathbb{R}\text{ep}^\infty(A)$. Such a morphisms is necessarily of type

$$\Phi = \Phi_0 + \Phi_1 + \Phi_2 + \cdots$$

where Φ_i is an $\text{Hom}(E, F)$-valued i-form on A of total degree zero:

$$\Phi_i \in \Omega^i(A, \text{Hom}^{-i}(E, F))$$

satisfying

$$\partial(\Phi_n) + d_\nabla(\Phi^{n-1}) + \sum_{i+j=n, i\geq 2} [\omega_i, \Phi_j] = 0.$$

Note that, in particular, Φ_0 must be a map of complexes.

Example 2.3.4 (Usual representations)**.** Of course, any representation E of A can be seen as a representation up to homotopy concentrated in degree zero. More generally, for any integer k, one can form the representation up to homotopy $E[k]$, which is E concentrated in degree k.

Example 2.3.5 (Differential forms)**.** Any closed form $\omega \in \Omega^n(A)$ induces a representation up to homotopy on the complex which is the trivial line bundle in degrees 0 and n and zero otherwise. The structure operator is $\nabla^{\text{flat}} + \omega$ where ∇^{flat} is the flat connection on the trivial line bundle. If ω and ω' are cohomologous, then the resulting representations up to homotopy are isomorphic with isomorphism defined by $\Phi_0 = \text{Id}$, $\Phi_{n-1} = \theta \in \Omega^{n-1}(A)$ chosen so that $d(\theta) = \omega - \omega'$. In conclusion, we have a well defined map $H^\bullet(A) \to \text{Rep}^\infty(A)$.

Example 2.3.6 (Conjugation)**.** For any representation up to homotopy E with structure operator D given by (2.10), one can form a new representation up to homotopy $\overline{E}$, which has the same underlying graded vector bundle as E, but with the structure operator

$$\overline{D} = -\partial + \nabla - \omega_2 + \omega_3 - \omega_4 + \dots .$$

In general, E and $\overline{E}$ are isomorphic, with isomorphism $\Phi = \Phi_0$ equal to $(-1)^n\mathrm{Id}$ on E^n.

Remark 2.3.7. It is worth being more explicit on the building blocks of representations up to homotopy which are concentrated in two consecutive degrees, say 0 and 1. From Proposition 2.3.2 we see that such a representation consists of

1. Two vector bundles E and F, and a vector bundle map $f : E \to F$.
2. A-connections on E and F, both denoted ∇, compatible with ∂ ($\nabla_\alpha \partial = \partial \nabla_\alpha$).
3. A 2-form $K \in \Omega^2(A, \mathrm{Hom}(F, E))$ such that

$$R_{\nabla^E} = \partial \circ K,\ R_{\nabla^F} = K \circ \partial \text{ and } d_\nabla(K) = 0.$$

Example 2.3.8 (The adjoint representation)**.** It is now clear that the properties of the basic connections and the basic curvature given in Proposition 2.2.10 give the adjoint complex the structure of a representation up to homotopy. Choosing a connection on the vector bundle A, the chain complex

$$A \xrightarrow{\rho} TM,$$

together with the structure operator

$$D_\nabla := \rho + \nabla^{\mathrm{bas}} + R^{\mathrm{bas}},$$

becomes a representation up to homotopy of A, denoted Ad_∇. The isomorphism class of this representation is called the adjoint representation of A and is denoted

$$\mathrm{Ad} \in \mathrm{Rep}^\infty(A).$$

Theorem 2.3.9. $\mathrm{Ad} \in$ *Rep*$^\infty(A)$ *does not depend on the choice of the connection. Also, there is a natural isomorphism:*

$$H^\bullet(A, Ad) \cong H^\bullet_{def}(A).$$

Proof. Let ∇' be another connection. Then $\Phi = \Phi_0 + \Phi_1$ with

$$\Phi_0 = \mathrm{Id} \text{ and } \Phi_1(\alpha)X = \nabla_X(\alpha) - \nabla'_X(\alpha),$$

defines an isomorphism between the resulting representations. For the last part, note that there is an exact sequence

$$0 \to \Omega^n(A, A) \to C^n_{\mathrm{def}}(A) \xrightarrow{-\sigma} \Omega^{n-1}(A, TM) \to 0,$$

where σ is the symbol map (see Subsection 2.2). Moreover, a connection ∇ on A induces a splitting of this sequence, and then an isomorphism

$$\begin{array}{rcl} \Psi : C^n_{\mathrm{def}}(A) & \to & \Omega^n(A, A) \oplus \Omega^{n-1}(A, TM) = \Omega(A, \mathrm{Ad})^n \\ D & \mapsto & (c_D, -\sigma_D) \end{array}$$

where σ_D is the symbol of D and c_D is given by

$$c_D(\alpha_1, \ldots, \alpha_k) = D(\alpha_1, \ldots, \alpha_k) + (-1)^{k-1} \sum_{i=1}^{k} (-1)^i \nabla_{\sigma(D)(\alpha_1, \cdots, \widehat{\alpha_i}, \ldots, \alpha_k)}(\alpha_i).$$

This map is an isomorphism of $\Omega(A)$-modules. We need to prove that the operators δ and D_∇ coincide. Since these two operators are derivations with respect to the module structures, it is enough to prove that they coincide in low degrees and this is a simple check. □

Example 2.3.10 (The double of a vector bundle)**.** Let E be a vector bundle over M. For any A-connection ∇ on E with curvature $R_\nabla \in \Omega^2(A; \underline{\mathrm{End}}(E))$, the complex $E \stackrel{\mathrm{Id}}{\rightarrow} E$ concentrated in degrees 0 and 1, together with the structure operator

$$D_\nabla := \mathrm{Id} + \partial + R_\nabla$$

defines a representation up to homotopy of A denoted $\mathcal{D}_{E,\nabla}$. The resulting element

$$\mathcal{D}_E \in \mathrm{Rep}^\infty(A)$$

does not depend on the choice of the connection. To see this, remark that if ∇' is another A-connection, then there is an isomorphism

$$\Phi : (\mathcal{D}_E, D_\nabla) \rightarrow (\mathcal{D}_E, D_{\nabla'})$$

with two components:

$$\Phi_0 = \mathrm{Id} \text{ and } \Phi_1(\alpha) = \nabla_\alpha - \nabla'_\alpha.$$

Next, we will show how representations up to homotopy of length one are related to extensions.

Proposition 2.3.11. *For any representation up to homotopy of length one with vector bundles E in degree 0 and F in degree 1 and structure operator $D = \partial + \nabla + K$, there is an extension of Lie algebroids:*

$$\mathfrak{g}_\partial \rightarrow \tilde{A} \rightarrow A,$$

where

1. $\mathfrak{g}_\partial = Hom(F, E)$*, is a bundle of Lie algebras with bracket* $[S, T]_\partial = S\partial T - T\partial S$.
2. $\tilde{A} = \mathfrak{g}_\partial \oplus A$ *with anchor* $(S, \alpha) \mapsto \rho(\alpha)$ *and bracket*

$$[(S, \alpha), (T, \beta)] = ([S, T] + \nabla_\alpha(T) - \nabla_\beta(S) + K(\alpha, \beta), [\alpha, \beta]).$$

Proof. After a careful computation, we find that the Jacobi identity for the bracket of $\tilde{A}$ brakes into the following equations (cf. Theorem 7.3.7 in [34]):

$$\nabla_\alpha([S, T]) = [\nabla_\alpha(S), T] + [S, \nabla_\alpha(T)] \tag{2.11}$$

$$\nabla_{[\beta,\gamma]}(T) - \nabla_\beta \nabla_\gamma(T) + \nabla_\gamma \nabla_\beta(T) = [T, K(\beta, \gamma)] \tag{2.12}$$

$$K([\alpha,\beta],\gamma)+K([\beta,\gamma],\alpha)+K([\gamma,\alpha],\beta) = \nabla_\beta(K(\gamma,\alpha))+\nabla_\alpha(K(\beta,\gamma))+\nabla_\gamma(K(\alpha,\beta)) \tag{2.13}$$

for $\alpha, \beta, \gamma \in \Gamma(A)$ and $S, T \in \Gamma(\mathfrak{g}_\partial)$. These equations are not equivalent to, but they follow from the equations satisfied by ∂, ∇ and K. The first equation follows from the compatibility of ∂ and ∇, the second equation follows from the two equations satisfied by the curvature, while the last equation is precisely $d_\nabla(K) = 0$. □

Remark 2.3.12. One can show that isomorphic length one representations up to homotopy induce isomorphic extensions.

Example 2.3.13. When $A = TM$ and E is a vector bundle, the extension associated to the double of E (Proposition 2.3.10) is isomorphic to the "Atiyah extension" induced by E:

$$\text{End}(E) \to \mathfrak{gl}(E) \to TM.$$

This extension is discussed e.g. in Section 1 of [32] (where $\mathfrak{gl}(E)$ is denoted by $\mathcal{D}(E)$). Recall that $\mathfrak{gl}(E)$ is the vector bundle over M whose sections are the derivations of E, i.e. pairs (D, X) consisting of a linear map $D : \Gamma(E) \to \Gamma(E)$ and a vector field X on M, such that $D(fs) = fD(s) + L_X(f)s$ for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. The Lie bracket of $\mathfrak{gl}(E)$ is the commutator

$$[(D, X), (D', X')] = (D \circ D' - D' \circ D, [X, X']),$$

while the anchor sends (D, X) to X. A connection on E is the same thing as a splitting of the Atiyah extension, and it induces an identification

$$\mathfrak{gl}(E) \cong \text{End}(E) \oplus TM.$$

Computing the bracket (or consulting [32]) we find that, after this identification, the Atiyah extension becomes the extension associated to the double $\mathcal{D}_E$.

Example 2.3.14. The extension associated to the adjoint representation is the one given by the first jet algebroid $J^1(A)$. This follows from a computation similar to the previous one, or by using the identities in [8]. In our view, this is the precise relation between the adjoint representation and $J^1(A)$. We emphasize here that there doesn't seem to be any construction which associates to a Lie algebroid extension of A a representation up to homotopy so that, applying it to $J^1(A)$ one recovers the adjoint representation. In other words, $J^1(A)$ with its structure of extension of A does not contain all the information about the structure of the adjoint representation.

Example 2.3.15 (Representations up to homotopy of TM)**.** The representations up to homotopy of TM are connections on complexes of vector bundles which are *flat up to homotopy*. Indeed, at least the first equation in Proposition 2.3.2 says that the curvature of ∇ is trivial cohomologically (up to homotopy).

On the other hand, a flat connection ∇ on a vector bundle E can be integrated to a representation of the fundamental groupoid of M. This correspondence is induced by parallel transport. To be more precise, given a vector bundle E endowed with a connection ∇, for any path γ in M from x to y, the parallel transport along γ with respect to ∇ induces a linear isomorphism

$$T_\gamma : E_x \to E_y.$$

This construction is compatible with path concatenation. When ∇ is flat, T_γ only depends on the homotopy class of γ, and this defines an action of the homotopy groupoid of M on E. What do connections which are flat up homotopy integrate to?

Proposition 2.3.16. *Let (E, D) be a representation up to homotopy of TM. Then*

1. *For any path γ in M from x to y, there is an induced chain map*

$$T_\gamma : (E_x, \partial) \to (E_y, \partial)$$

and this construction is compatible with path concatenation. More precisely, T_γ is the parallel transport with respect to the connection underlying D.

2. *If γ_0 and γ_1 are two homotopic paths in M from x to y then T_{γ_0} and T_{γ_1} are chain homotopic. More precisely, for any homotopy h between γ_0 and γ_1 there is an associated map of degree -1, $T_h : E_y \to E_x$, such that*

$$T_{\gamma_1} - T_{\gamma_0} = [\partial, T_h].$$

Proof. The compatibility of ∇ with the grading and ∂ implies that the parallel transport T_γ is a map of chain complexes. We now prove (2). Given a path $u : I \to E$ ($I = [0, 1]$), sitting over some base path $\gamma : I \to M$, we denote by

$$\frac{Du}{Dt} = \nabla_{\frac{d\gamma}{dt}}(u)$$

the derivative of u with respect to the connection ∇. Then, for any path γ, and any s, t, the parallel transport

$$T_\gamma^{s,t} : E_{\gamma(s)} \to E_{\gamma(t)}$$

is defined by the equation

$$\frac{D}{Dt} T_\gamma^{s,t}(u) = 0, \;\; T_\gamma^{s,s}(u) = u.$$

The global parallel transport along γ, $T_\gamma : E_x \to E_y$ is obtained for $s = 0$, $t = 1$. Note that, for a path in the fiber above $\gamma(s)$, $\phi : I \to E_{\gamma(s)}$, one has

$$\frac{D}{Dt} T_\gamma^{s,t}(\phi(t)) = T_\gamma^{s,t}(\frac{d\phi}{dt}(t)).$$

This implies that, for a path $v : I \to E$ above γ and $u_0 \in E_x$, the (unique) solution of the equation

$$\frac{Du}{dt} = v, \;\; u(0) = u_0$$

can be written in terms of the parallel transport as

$$u(t) = T_\gamma^{0,t}(u_0 + \int_0^t T_\gamma^{t',0}(v(t'))dt').$$

For any map $u : I \times I \to E$ sitting above some $h : I \times I \to M$, we have

$$\frac{D^2 u}{DtD\epsilon} - \frac{D^2 u}{D\epsilon Dt} = R(\frac{dh}{dt}, \frac{dh}{d\epsilon})(u(\epsilon, t)),$$

where $R = R_\nabla$ is the curvature of ∇. Let now h be as in the statement, $u_0 \in E_x$, and consider in the previous equation applied to

$$u(\epsilon, t) = T^{0,t}_{\gamma_\epsilon}(u_0).$$

We find

$$\frac{D}{Dt}(\frac{Du}{D\epsilon}) = R(\frac{dh}{dt}, \frac{dh}{d\epsilon})u.$$

where $\gamma_\epsilon = h(\epsilon, \cdot)$. Since $\frac{Du}{D\epsilon}(\epsilon, 0) = 0$, we find

$$\frac{Du}{D\epsilon} = T^{0,t}_{\gamma_\epsilon} \int_0^t T^{t',0}_{\gamma_\epsilon} R(\frac{dh}{dt}, \frac{dh}{d\epsilon})u(\epsilon, t')dt'.$$

Fixing the argument t, since

$$u(0, t) = T^{0,t}_{\gamma_0}(u_0),$$

by the same argument as above, we deduce that

$$u(\epsilon, t) = T^{0,\epsilon}_{h_t}[T^{0,t}_{\gamma_0}(u_0) + \int_0^\epsilon T^{\epsilon',0}_{h_t} T^{0,t}_{\gamma_{\epsilon'}} \int_0^t T^{t',0}_{\gamma_{\epsilon'}} R(\frac{dh}{dt}, \frac{dh}{d\epsilon})u(\epsilon', t')dt'd\epsilon'],$$

where $h_t(\cdot) = h(\cdot, t)$. Taking $\epsilon = 1$, $t = 1$, we find

$$T_{\gamma_1}(u_0) = T_{\gamma_0}(u_0) + \int_0^1 \int_0^1 T^{t,1}_{\gamma_\epsilon} R(\frac{dh}{dt}, \frac{dh}{d\epsilon}) T^{0,t}_{\gamma_\epsilon}(u_0) dt d\epsilon.$$

Using now that $R + \partial(\omega_2) = 0$, we deduce that

$$T_{\gamma_1}(u_0) - T_{\gamma_0}(u_0) = [\partial, T_h]u_0,$$

where $T_h \in \text{Hom}(E_y, E_x)$ is

$$T_h = -\int_0^1 \int_0^1 T^{t,1}_{\gamma_\epsilon} \omega_2(\frac{dh}{dt}, \frac{dh}{d\epsilon}) T^{0,t}_{\gamma_\epsilon} dt d\epsilon \ .$$

Note that the expression under the integral is in the (ϵ, t)-independent vector space $\text{Hom}(E_y, E_x)$. □

Remark 2.3.17. In order to pass to the homotopy groupoid, one can choose a representative γ_a for any element a in the fundamental groupoid of M and a homotopy $h_{a,b}$ between γ_{ab} and $\gamma_a * \gamma_b$ for any two a and b. Define the *action* of a on E by

$$a \cdot v = T_{\gamma_a}(v).$$

This is not a true action because the associativity is not respected. However, the previous proposition implies that

$$a \cdot (b \cdot v) - (ab) \cdot v = [\partial, T(a, b)]u,$$

where $T(a, b) = T_{h_{a,b}}$. This is just the first equation in an infinite list of equations. Although the precise notion of representations up to homotopy of groupoids has not been introduced yet, this shows, at least intuitively, that a connection which is flat up to homotopy integrates to "a representation up to homotopy of the fundamental groupoid of M".

2.3.2 Operations and more examples

We have already seen in Subsections 2.2.3 and 2.2.4 that the standard operations with vector spaces such as

$$E \mapsto E^*, E \mapsto \Lambda(E), E \mapsto S(E),$$

$$(E, F) \mapsto E \oplus F, (E, F) \mapsto E \otimes F, (E, F) \mapsto \underline{\mathrm{Hom}}(E, F)$$

extend to the setting of graded vector bundles, complexes of vector bundles and complexes of vector bundles endowed with a connection. The extension to representations up to homotopy is now straightforward.

Example 2.3.18 (Taking duals)**.** For $E \in \mathbb{R}\mathrm{ep}^{\infty}(A)$ with associated structure operator D, the operator D^* corresponding to the dual E^* is uniquely determined by the condition

$$d_A(\eta \wedge \eta') = D^*(\eta) \wedge \eta' + (-1)^{|\eta|}\eta \wedge D(\eta'),$$

for all $\eta \in \Omega(A, E^*)$, $\eta' \in \Omega(A, E)$, where $\wedge$ is the operation

$$\Omega(A, E^*) \otimes \Omega(A, E) \to \Omega(A)$$

induced by the pairing between E^* and E (see Subsection 2.2.3). In terms of the components of D, if $D = \partial + \nabla + \sum_{i \geq 2} \omega_i$, we find $D^* = \partial^* + \nabla^* + \sum_{i \geq 2} \omega_i^*$, where ∇^* is the connection dual to ∇ and, for $\eta_k \in (E^k)^*$,

$$\partial^*(\eta) = -(-1)^k \eta \circ \partial, \;\; \omega_p^*(\alpha_1, \ldots, \alpha_p)(\eta_k) = -(-1)^{k(p+1)}\eta_k \circ \omega_p(\alpha_1, \ldots, \alpha_p).$$

In particular, when starting with a representation up to homotopy of length one, $D = \partial + \nabla + K$ on $E \xrightarrow{\partial} F$ (E in degree 0 and F in degree 1), the dual complex will be $F^* \xrightarrow{\partial^*} E^*$ (F^* in degree -1 and E^* in degree 0), with $D^* = \partial^* + \nabla^* - K^*$. The fact that some signs appear when taking duals is to be expected since, for any connection ∇, the curvature of ∇^* equals the negative of the dual of the curvature of ∇.

Example 2.3.19 (Tensor products)**.** For $E, F \in \mathbb{R}\mathrm{ep}^{\infty}(A)$, with associated structure operators D^E and D^F, the operator D corresponding to $E \otimes F$ is uniquely determined by the condition

$$D(\eta_1 \wedge \eta_2) = D^E(\eta_1) \wedge \eta_2 + (-1)^{|\eta_1|}\eta_1 \wedge D^F(\eta_2),$$

for all $\eta_1 \in \Omega(A, E)$, $\eta_2 \in \Omega(A, F)$. More explicitly, if $D^E = \partial^E + \nabla^E + \omega_2^E + \ldots$ and similarly for D^F, then $D = \partial + \nabla + \omega_2 + \ldots$ where

1. ∂ is just the tensor product of ∂^E and ∂^F: $\partial = \partial^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial^F$,

$$\partial(u \otimes v) = \partial^E(u) \otimes v + (-1)^{|u|}u \otimes \partial^F(v).$$

2. ∇ is just the tensor product connection of ∇^E and ∇^F: $d_\nabla = d_{\nabla^E} \otimes \mathrm{Id} + \mathrm{Id} \otimes d_{\nabla^F}$,

$$\nabla_\alpha(u \otimes v) = \nabla_\alpha^E(u) \otimes v + u \otimes \nabla_\alpha^F(v).$$

3. $\omega_p = \omega_p^E \otimes \mathrm{Id} + \mathrm{Id} \otimes \omega_p^F$.

Example 2.3.20 (Pull-back's)**.** A Lie algebroid A over M can be pulled-back along submersion $\tau : N \to M$ or, more generally, along smooth maps τ which satisfy certain condition (see below). Recall [32] that the pull-back algebroid $\tau^! A$ has the fiber at $x \in N$:

$$\tau^!(A)_x = \{(X, \alpha) : X \in T_x N, \alpha \in A_{\tau(x)}, (d\tau)(X) = \rho(\alpha)\}.$$

The condition mentioned above is that this is a smooth vector bundle over N, which certainly happens if τ is a submersion or the inclusion of an orbit of A. The anchor of $\tau^! A$ sends (X, α) to X, while the bracket is uniquely determined by the derivation rule and

$$[(X, \tau^*\alpha), (Y, \tau^*\beta)] = ([X, Y], \tau^*[\alpha, \beta]).$$

In general, there is a pull-back map (functor)

$$\tau^* : \mathbb{R}\mathrm{ep}^\infty(A) \to \mathbb{R}\mathrm{ep}^\infty(\tau^!(A))$$

which sends E with structure operator $D = \partial + \nabla + \sum \omega_i$ to $\tau^*(E)$ endowed with $D = \partial + \tau^*(\nabla) + \sum \tau^*(\omega_i)$ where $\tau^*\nabla$ is the pull-back connection

$$(\tau^*\nabla)_{(X,\alpha)}(\tau^*(s)) = \tau^*(\nabla_\alpha(s)),$$

while

$$\tau^*(\omega_i)((X_1, \alpha_1), \ldots, (X_i, \alpha_i)) = \omega_i(\alpha_1, \ldots, \alpha_i).$$

Remark 2.3.21 (DG-algebras)**.** The notion of representation up to homotopy can be recasted in the language of differential graded modules over differential graded algebras [28], and some of the constructions in this paper may be seen as particular cases of the general theory. We emphasize however that we insist on working with DG-modules which are sections of vector bundles.

Example 2.3.22 (Semidirect products with representations up to homotopy)**.** If $\mathfrak{g}$ is a Lie algebra and $V \in \mathbb{R}\mathrm{ep}^\infty(\mathfrak{g})$, the operator making $\Lambda(V^*)$ a representation up to homotopy of $\mathfrak{g}$ is a derivation on the algebra

$$\Lambda(\mathfrak{g}^*) \otimes \Lambda(V^*) = \Lambda((\mathfrak{g} \oplus V)^*),$$

i.e. defines the structure of L_∞-algebra (see [43]) on the direct sum $\mathfrak{g} \oplus V$. This L_∞-algebra deserves the name "semi-direct product of $\mathfrak{g}$ and V", and is denoted $\mathfrak{g} \ltimes V$. It is the usual semi-direct product if V is just a usual representation.

Example 2.3.23 (Exterior powers of Ad and k-differentials)**.** Applying the exterior power construction to the adjoint representation, we find new elements

$$\Lambda^k \mathrm{Ad} \in \mathrm{Rep}^\infty(A),$$

one for each positive integer k. These are given by the representations up to homotopy $\Lambda^k(\mathrm{Ad}_\nabla)$, where ∇ is an arbitrary connection on A. Generalizing the case of the cohomology of A with coefficients in the adjoint representation Ad, we now show that the cohomology with coefficients in $\Lambda^k \mathrm{Ad}$ can be computed by an intrinsic complex (which does not require the use of a connection). More precisely, we define $(C^\bullet(A, \Lambda^k \text{Ad}), d)$ as follows. An element $c \in C^p(A, \Lambda^k\text{Ad})$ is a string $c = (c_0, c_1, \dots)$ where

$$c_i : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{(p-i) \text{ times}} \to \Gamma(\Lambda^{k-i} A \otimes S^i TM),$$

are multilinear, antisymmetric maps related by:

$$c_i(\alpha_1, \dots, f\alpha_{p-i}) = fc_i(\alpha_1, \dots, f\alpha_{p-i}) + i(df)(c_{i+1}(\alpha_1, \dots, \alpha_{p-i-1}) \wedge \alpha_{p-i})),$$

where $i(df) : S^{i+1}(TM) \to S^i(TM)$ is the contraction by df. We think of $c_1, c_2, \dots$ as *the tail* of c_0, which measures the failure of c_0 to be $C^\infty(M)$-linear. For instance, to define the differential dc, we first define its leading term by the Koszul formula

$$\begin{aligned} (dc)_0(\alpha_1, \dots, \alpha_{p+1}) &= \sum_{i<j} (-1)^{i+j} c_0([\alpha_i, \alpha_j], \dots, \hat{\alpha_i}, \dots, \hat{\alpha_j}, \dots, \alpha_{p+1}) + \\ &+ \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c_0(\alpha_1, \dots, \hat{\alpha_i}, \dots, \alpha_{p+1})), \end{aligned}$$

and then the higher order terms can be computed by applying the principle we have mentioned above. The following proposition provides a cohomological interpretation of the notion of a k-differential (see Remark 2.2.7).

Proposition 2.3.24. *The cohomology $H^\bullet(A, \Lambda^k Ad)$ is naturally isomorphic to the cohomology of the complex $(C^\bullet(A, \Lambda^k Ad), d)$. Moreover, the 1-cocycles of this complex are precisely the k-differentials on A.*

Proof. For the first part, we pick a connection ∇ to realize the adjoint representation and we claim that the complexes $(C^\bullet(A, \Lambda^k\text{Ad}), d)$ and $(\Omega^\bullet(A, \Lambda^k\text{Ad}), D_\nabla)$ are isomorphic. For $k = 0$ the statement is trivial while the case $k = 1$ follows from theorem 2.3.9. The general statement follows from these two cases if one observes that, both $\oplus_k C^\bullet(A, \Lambda^k\text{Ad})$ and $\oplus_k\Omega^\bullet(A, \Lambda^k\text{Ad})$ are algebras generated in low degree for which the corresponding differentials are derivations. For the second part, remark that an element in $C^1(A, \Lambda^k\text{Ad})$ is a pair (c_0, c_1) where $c_0 : \Gamma(A) \to \Gamma(\Lambda^k A)$ and $c_1 \in \Gamma(\Lambda^{k-1}A \otimes TM)$ satisfy the corresponding equation. Viewing c_1 as the map $C^\infty(M) \to \Gamma(\Lambda^{k-1}A)$, $f \mapsto i(df)(c_0)$, we see that the elements of $C^1(A, \Lambda^k\text{Ad})$ are precisely the almost k-differentials on A. The fact that the cocycle equation is precisely the k-differential equation follows by a simple computation. □

Example 2.3.25 (The coadjoint representation)**.** The dual of the adjoint representation of a Lie algebroid A is called the coadjoint representation of A, denoted Ad*. Using a connection ∇ on A, it is given by the representation up to homotopy

$$\text{Ad}^* : \underbrace{T^*M}_{\text{degree } -1} \xrightarrow{\rho^*} \underbrace{A^*}_{\text{degree } 0}, \quad D = \rho^* + (\nabla^{\text{bas}})^* - (R^{\text{bas}}_\nabla)^*.$$

As in the case of the adjoint representation, the resulting cohomology can be computed by an intrinsic complex which does not require the choice of a connection. This complex, denoted $C^\bullet(A, \mathrm{Ad}^*)$, is defined as follows. An element in $C^p(A, \mathrm{Ad}^*)$ is a pair $c = (c_0, c_1)$ with

$$c_0 : \underbrace{\Gamma(A) \times \ldots \times \Gamma(A)}_{p \text{ times}} \to \Omega^1(M),$$

which is multilinear and antisymmetric, and

$$c_1 \in \Omega^{p-1}(A, A^*)$$

such that

$$c_0(\alpha_1, \ldots, \alpha_{p-1}, f\alpha_p) = fc_0(\alpha_1, \ldots, \alpha_{p-1}, \alpha_p) - df \wedge c_1(\alpha_1, \ldots, \alpha_{p-1})(\alpha_p),$$

for all $f \in C^\infty(M)$, $\alpha_i \in \Gamma(A)$. The differential of c, $d(c) \in C^{p+1}(A, \mathrm{Ad}^*)$ is given by the Koszul-type formulas

$$\begin{aligned}(dc)_0(\alpha_1, \ldots, \alpha_{p+1}) &= \sum_{i<j}(-1)^{i+j}c_0([\alpha_i, \alpha_j], \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{p+1}) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c_0(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{p+1})),\end{aligned}$$

$$\begin{aligned}(dc)_1(\alpha_1, \ldots, \alpha_p) &= \sum_{i<j}(-1)^{i+j}c_0([\alpha_i, \alpha_j], \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_p) \\ &\quad + \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c_0(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_p)) \\ &\quad + (-1)^{p+1}c(\alpha_1, \ldots, \alpha_p) \circ \rho.\end{aligned}$$

Proposition 2.3.26. *$(C^\bullet(A, Ad^*), d)$ is a cochain complex whose cohomology is canonically isomorphic to $H^\bullet(A, Ad^*)$.*

Proof. As in the case of the adjoint representation, even the complexes are isomorphic. □

Remark 2.3.27. More generally, for any q, the representation up to homotopy $S^q(\mathrm{Ad}^*)$ and its cohomology can be treated similarly. This will be made more explicit in our discussion on the Weil algebra.

Example 2.3.28 (restricted deformation cohomology)**.** Given a Lie algebroid A and a leaf $L \subset M$ of A, the restriction of the adjoint representation of A defines a representation up to homotopy

$$\mathrm{Ad}|_L \in \mathrm{Rep}^\infty(A_L).$$

The resulting cohomology $H^\bullet(A_L, \mathrm{Ad}|_L)$ plays an important role in the generalization of Thurston's stability of leaves of foliations, and is called the restricted deformation cohomology in [17], where is denoted $H^*_{\mathrm{def},L}(A)$. Note that, unlike the previous examples, there does not seem to be any intrinsic complex computing this cohomology.

Proposition 2.3.29. *There is a long exact sequence:*

$$\ldots \to H^n_{def}(A_L) \to H^n_{def,L}(A) \to H^{n-1}(A_L, \nu_L) \to H^{n+1}_{def}(A) \to \ldots$$

Proof. Once we choose a connection on A, using it to represent Ad_A, and using its restriction to L to represent the adjoint representation of A_L, we get a short exact sequence of representations up to homotopy

$$\mathrm{Ad}_{A_L} \to \mathrm{Ad}_A|_L \to \nu_L[1],$$

where ν_L is the representation of A_L given by the Bott-type connection. □

2.3.3 Cohomology, the derived category, and some more examples

As we already mentioned several times (e.g. in Example 2.2.6), one of the reasons we work with complexes is that we want to avoid non-smooth vector bundles. The basic idea was that a complex represents its cohomology bundle (typically a graded non-smooth vector bundle). To complete this idea, we need to allow ourselves more freedom when comparing two complexes so that, morally, if they have the same cohomology bundles, then they become *isomorphic*. This will happen in the derived category. For a more general discussion on the derived category of a DG-algebra we refer the reader to [41].

Definition 2.3.30. *A morphism Φ between two representations up to homotopy E and F is called a quasi-isomorphism if the first component of Φ, the map of complexes*

$$\Phi_0 : (E, \partial) \to (F, \partial),$$

is a quasi-isomorphism. We denote by $\mathbb{D}er(A)$ the category obtained from $\mathbb{R}ep^{\infty}(A)$ by formally inverting the quasi-isomorphisms, and by $Der(A)$ the set of isomorphisms classes of objects of $\mathbb{D}er(A)$.

Remark 2.3.31 (Hom in the derived category)**.** *Since we work with vector bundles, there is the following simple realization of the derived category. Given two representations up to homotopy E and F of A, there is a notion of homotopy between maps from E to F. To describe it, we remark that morphisms in $\mathbb{R}ep^{\infty}(A)$ from E to F correspond to 0-cycles in the complex with coefficients in the induced representation up to homotopy $\underline{Hom}(E, F)$:*

$$Hom_{\mathbb{R}ep^{\infty}(A)}(E, F) = Z^0(\Omega(A, \underline{Hom}(E, F))).$$

Two maps $\Phi, \Psi : E \to F$ in $\mathbb{R}ep^{\infty}(A)$ are called homotopic if there exists a degree -1 map $H : \Omega(A, E) \to \Omega(A, F)$ which is $\Omega(A)$-linear and satisfies $D^E H + H D^F = \Phi - \Psi$, where D^E and D^F are the structure operators of E and F, respectively. We denote by $[E, F]$ the set of homotopy classes of such maps. Hence,

$$[E, F] := H^0(\Omega(A, \underline{Hom}(E, F))).$$

As in the case of complexes of vector bundles (see part (2) of Lemma 2.2.17), and by the same type of arguments, we see that a map $\Phi : E \to F$ is a quasi-isomorphism if and only if it is a homotopy equivalence. From this, we deduce the following realization of $\mathbb{D}er(A)$. Its objects are the representations up to homotopy of A, while

$$Hom_{\mathbb{D}er(A)}(E, F) = [E, F].$$

Note that, in this language, for any $F \in \mathbb{R}ep^{\infty}(A)$,

$$H^n(F) = [\mathbb{R}[n], F].$$

Also, the mapping cone construction gives a function

$$Map : [E, F] \to Rep^{\infty}(A)$$

which, when applied to $E = \mathbb{R}[n]$, gives the construction from Example 2.3.5.

Example 2.3.32. If $A = \mathcal{F} \subset TM$ is a foliation on M, then the projection from the complex $\mathcal{F} \hookrightarrow TM$ underlying the adjoint representation into $\nu[1]$ (the normal bundle $\nu = TM/\mathcal{F}$ concentrated in degree 1) is clearly a quasi-isomorphism of complexes. It is easy to see that this projection is actually a morphism of representations up to homotopy, when ν is endowed with the Bott connection (see Example 2.2.6). Hence, as expected,

$$\mathrm{Ad}_{\mathcal{F}} \cong \nu[1] \quad (\text{in } \mathrm{Der}(\mathcal{F})).$$

Similarly, for a transitive Lie algebroid A, i.e. one for which the anchor is surjective,

$$\mathrm{Ad}_A \cong \mathfrak{g}(A) \quad (\text{in } \mathrm{Der}(A)).$$

Next, we will show that the cohomology $H^{\bullet}(A; -)$, viewed as a functor from the category of representations up to homotopy, descends to the derived category.

Proposition 2.3.33. *Any quasi-isomorphism $\Phi : E \to F$ between two representations up to homotopy of A induces an isomorphism in cohomology $\Phi : H(A, E) \to H(A, F)$.*

Proof. If E is a representation up to homotopy of A, one can form a decreasing filtration on $\Omega(A, E)$ induced by the form-degree

$$\cdots \subset F^2(\Omega(A, E)) \subset F^1(\Omega(A, E)) \subset F^0(\Omega(A, E)) = \Omega(A, E),$$

where

$$F^p(\Omega(A, E)) = \Omega^p(A, E) \oplus \Omega^{p+1}(A, E) \oplus \cdots$$

This filtration induces a spectral sequence $\mathcal{E}$ with

$$\mathcal{E}_0^{p,q} = \Omega^p(A, E^q) \Rightarrow H^{p+q}(A, E),$$

where the differential $d_0^{p,q} : \mathcal{E}_0^{p,q} \to \mathcal{E}_0^{p,q+1}$ is induced by the differential ∂ of E. Given a morphism $\Phi : E \to F$, there is a map of spectral sequences which, at the first level is induced by Φ_0. The assumption that Φ is a quasi-isomorphism implies that the map induced at the level of spectral sequences is an isomorphism at the second level. We conclude that Φ induces an isomorphism in cohomology. □

Example 2.3.34. Applying this result to the first quasi-isomorphism discussed in Example 2.3.32, we find that the deformation cohomology of a foliation $\mathcal{F}$ is isomorphic to the shifted cohomology of $\mathcal{F}$ with coefficients in ν, this is Proposition 4 in [19].

Example 2.3.35 (Acyclic complexes)**.** As a corollary of the previous proposition, we find that $H^{\bullet}(A, E) = 0$ for any $E \in \mathrm{Rep}^{\infty}(A)$ whose underlying complex (E, ∂) is acyclic. However, one can say much more in this case, exact complexes can naturally be viewed as representations up to homotopy, and they are precisely the representations up to homotopy which are quasi-isomorphic to the trivial (zero) representation. Of course, the double $\mathcal{D}_E$ of a vector bundle (Example 2.3.10) belongs to this class.

Proposition 2.3.36. *Any exact complex (E, ∂) induces a well-defined element in Rep$^\infty(A)$. More precisely, given (E, ∂),*

1. *There exist A-connections ∇ on (E, ∂).*
2. *For any A-connection ∇ on the complex (E, ∂), one can find ω_i for $i \geq 2$ such that*
$$D = \partial + d_\nabla + \sum_i \omega_i$$
gives E the structure of a representation up to homotopy of A.
3. *Any two representations up to homotopy obtained in this way are isomorphic.*

Proof. Choosing a metric on the complex as in Lemma 2.2.17 we obtain a decomposition $E^k = Im(\partial)^k \oplus Im(\partial^*)^k$. Now we choose arbitrary A-connections on all the subbundles $Im(\partial^*)^k$ and use the isomorphisms $\partial : Im(\partial^*)^{k-1} \to Im(\partial)^k$ to extend them to E^k. It is clear from the construction that such connections are compatible with ∂.

For the existence of D, we have to solve the equations

$$\partial(\omega_k) + d_\nabla(\omega_{k-1}) + \omega_2 \circ \omega_{k-2} + \omega_3 \circ \omega_{k-3} + \ldots + \omega_{k-2} \circ \omega_2 = 0. \qquad (2.14)$$

This can be done by an inductive cohomological argument, but we can be more explicit here if we choose a contracting homotopy h of (E, ∂) (cf. Lemma 2.2.17)

$$h\partial + \partial h = -Id.$$

It is not difficult to check that

$$\omega_n = h \circ d_\nabla(h)^{n-2} \circ R_\nabla$$

satisfy the desired equations. Assume now that D and D' define two structures of representations up to homotopy. To distinguish them, we denote $E = (E, D)$ and $E' = (E, D')$. We consider the resulting representation up to homotopy

$$\mathcal{E} := \mathrm{Hom}(E, E'),$$

with associated operator $D^{\mathcal{E}}$. Due to the exactness assumption, $\mathcal{E}$ will be acyclic. Looking at the associated filtration (see the proof of Proposition 2.3.33), we deduce that the subcomplex $F^1\Omega(A, \mathcal{E})$ must be exact. On the other hand, since D and D' have the same underlying complex, we have

$$D - D' \in F^1\Omega(A, \mathcal{E}).$$

A simple computation shows that this is actually a cocycle:

$$D^{\mathcal{E}}(D - D') = D'(D - D') + (D - D')D = D'D + D'^2 - D^2 - D'D = 0,$$

hence we can find $T \in F^1(\Omega(A, \mathcal{E}))$ such that $D^{\mathcal{E}}(T) = D - D'$ or, equivalently, $(Id + T)D = D'(Id + T)$. Since T is in F^1, it is nilpotent and therefore $Id + T$ is the desired isomorphism. □

Next, we look at the case where E is a regular representation up to homotopy of A in the sense that the underlying complex (E, ∂) is regular. In this case, the cohomology $\mathcal{H}^\bullet(E)$ is a graded vector bundle over M (see subsection 2.2.4).

Theorem 2.3.37. *Let E be a regular representation up to homotopy of A. Then the equation*

$$\overline{\nabla}_\alpha([S]) := [\nabla_\alpha(S)],$$

for $\alpha \in \Gamma(A)$, $[S] \in \mathcal{H}^\bullet(E)$ makes $\mathcal{H}^\bullet(E)$ a representation of A. Moreover, the complex $(\mathcal{H}^\bullet(E), 0)$ with connection $\overline{\nabla}$ can be given the structure of a representation up to homotopy of A which is quasi-isomorphic to E.

Also, there is a spectral sequence

$$\mathcal{E}_2^{pq} = H^p(A, \mathcal{H}^q(E)) \Rightarrow H^{p+q}(A, E).$$

Proof. That $\overline{\nabla}$ is flat follows from the fact that the curvature of ∇ is exact. Also, the spectral sequence is just the one appearing in the previous proof. Next, we have to construct the structure of representation up to homotopy on $\mathcal{H}^\bullet(E)$. We will use the notations from the proof of Lemma 2.2.17 (part 3). The linear Hodge decomposition $E = Ker\Delta + Im(\partial) + im(\partial^*)$ provides quasi-isomorphisms $p : E \to \mathcal{H}(E)$ and $i : \mathcal{H}(E) \to E$. The restriction of the Laplacian to $Im(\partial) \oplus Im(\partial^*)$ denoted $\diamondsuit$ is an isomorphism. Then

$$h = -\diamondsuit^{-1}\partial^*$$

satisfies the following equations:

1. $p\partial = 0, \partial i = 0, ip = \mathrm{Id} + h\partial + \partial h.$
2. $h^2 = 0.$
3. $ph = 0.$

We denote by the same letters the maps induced at the level of forms ∂ goes from $\Omega(A, E)$ to $\Omega(A, E)$, etc, where we use the standard sign conventions. Note that these maps still satisfy the previous equations. We consider

$$\delta := D - \partial : \Omega(A, E) \to \Omega(A, E),$$

where D is the structure operator of A. With these,

$$D_{\mathcal{H}} := p(1 + (\delta h) + (\delta h)^2 + (\delta h)^3 + \dots)\delta i,$$

is a map

$$D_{\mathcal{H}} : \Omega(A, \mathcal{H}) \to \Omega(A, \mathcal{H})$$

of degree one which squares to zero, and

$$\Phi := p(1 + \delta h + (\delta h)^2 + (\delta h)^3 + \dots) : (\Omega(A, E), D) \to (\Omega(A, \mathcal{H}), D_{\mathcal{H}})$$

is a cochain map. The assertions about $D_{\mathcal{H}}$ and Φ follow by direct computation, or by applying the homological perturbation lemma (see e.g. [16]), which also explains our choices. Finally, equations $h^2 = 0$ and $ph = 0$ imply that Φ is $\Omega(A)$-linear and that $D_{\mathcal{H}}$ is a derivation. □

Example 2.3.38. Any extension of Lie algebras

$$\mathfrak{l} \to \tilde{\mathfrak{g}} \to \mathfrak{g}$$

induces a representation up to homotopy of $\mathfrak{g}$ with underlying complex the Chevalley-Eilenberg complex $(C^\bullet(\mathfrak{l}), d_\mathfrak{l})$ of $\mathfrak{l}$. To describe this representation, we use a splitting $\sigma : \mathfrak{g} \to \tilde{\mathfrak{g}}$ of the sequence. This induces

1. For $u \in \mathfrak{g}$, a degree zero operator

$$\nabla^\sigma_u := ad^*_{\sigma(u)} : C^\bullet(\mathfrak{l}) \to C^\bullet(\mathfrak{l}),$$

hence a $\mathfrak{g}$-connection ∇^σ on $C^\bullet(\mathfrak{l})$.

2. The *curvature* of σ, $R_\sigma \in C^2(\mathfrak{g}, \mathfrak{l})$ given by

$$R^\sigma(u, v) = [\sigma(u), \sigma(v)] - \sigma([u, v]).$$

To produce an $\underline{\text{End}}^{-1}(C^\bullet(\mathfrak{l}))$-valued cochain, we use the contraction operator $i : \mathfrak{l} \to \underline{\text{End}}^{-1}(C^\bullet(\mathfrak{l}))$ (but mind the sign conventions!).

It is now straightforward to check that

$$D := d_\mathfrak{l} + \nabla^\sigma + i(R^\sigma)$$

makes $C^\bullet(\mathfrak{l})$ a representation up to homotopy of $\mathfrak{g}$, with associated cohomology complex isomorphic to $C^\bullet(\tilde{\mathfrak{g}})$. Note that Theorem 2.3.37 implies that the cohomology of $\mathfrak{l}$ is naturally a representation of $\mathfrak{g}$, and there is spectral sequence with

$$\mathcal{E}_2^{pq} = H^p(\mathfrak{g}, H^q(\mathfrak{l})) \Rightarrow H^{p+q}(\tilde{\mathfrak{g}}).$$

This is precisely the Serre spectral sequence (see e.g. [44]). The same argument applies to any extension of Lie algebroids, giving us the spectral sequence of Theorem 7.4.6 in [34].

Corollary 2.3.39. *If (E, D) is a representation up to homotopy of A with underlying regular complex $E = E^0 \oplus E^1$ then there is a long exact sequence*

$$\cdots \to H^n(A, \mathcal{H}^0(E)) \to H^n(A, E) \to H^{n-1}(A, \mathcal{H}^1(E)) \to H^{n+1}(A, \mathcal{H}^0(E)) \to \cdots$$

Proof. The map

$$H^{n-1}(A, \mathcal{H}^1(E)) \to H^{n+1}(A, \mathcal{H}^0(E))$$

is $d_2^{pq} : E_2^{p,1} \to E_2^{p+2,0}$ in the spectral sequence and is given by the wedge product with ω_2. The map

$$H^n(A, \mathcal{H}^0(E)) \to H^n(A, E)$$

is determined by the natural inclusion at the level of cochains.
Finally,

$$H^n(A, E) \to H^{n-1}(A, \mathcal{H}^1(E))$$

is given at the level of complexes by:

$$(\omega_0 + \omega_1) \mapsto \overline{\omega_1}$$

where $\omega_0 \in \Omega^n(A, E^0)$ and $\omega_1 \in \Omega^{n-1}(A, E^1)$. Since the spectral sequence collapses at the third stage, we conclude that the sequence is exact. □

Example 2.3.40. Applying the previous corollary to the adjoint representation of a regular Lie algebroid, we obtain the long exact sequence which is theorem 3 in [19].

2.4 The Weil algebra and the BRST model for equivariant cohomology

In this section we will make use of representations up to homotopy to define the Weil algebra associated to a Lie algebroid A, generalizing the Weil algebra of a Lie algebra and Kalkman's BRST algebra for equivariant cohomology. We will discuss a version of the Weil algebra which uses a connection and is made of representations up to homotopy. In chapter 3 we will give an intrinsic version of this algebra and use it to define the Van Est map. This algebra has also been considered, in the language of supermanifolds, by Mehta [36].

2.4.1 The connection dependent version

Let ∇ be a connection on the vector bundle A. We define the algebra

$$W(A,\nabla) = \bigoplus_{u,v,w} \Gamma(\Lambda^u T^*M \otimes S^v(A^*) \otimes \Lambda^w(A^*)).$$

It is graded by the total degree $u+2v+w$, and has an underlying bi-grading $p = v+w, q = u+v$, so that $W^{p,q}(A,\nabla)$ is the sum over all u, v and w satisfying these equations. The connection ∇ will be used to define a differential on $W(A,\nabla)$. This will be a total differential

$$d_\nabla = d^{\mathrm{h}}_\nabla + d^{\mathrm{v}}_\nabla,$$

where, to define d^{h}_∇ and d^{v}_∇ we look at $W(A,\nabla)$ from two different points of view. First of all remark that, according to our conventions, the symmetric powers of the dual of the graded vector bundle $\mathcal{D} = A \oplus A$ (concentrated in degrees 0 and 1) is

$$S^p\mathcal{D}^* = (\underbrace{\Lambda^p A^*}_{\text{degree } -p}) \oplus (\underbrace{A^* \otimes \Lambda^{p-1}A^*}_{\text{degree } -p+1}) \oplus \ldots \oplus (\underbrace{S^{p-1}A^* \otimes A^*}_{\text{degree } -1}) \oplus (\underbrace{S^p A^*}_{\text{degree } 0}),$$

hence

$$W^{p,q}(A,\nabla) = \Omega(M, S^p\mathcal{D}^*)^{q-p}.$$

We now use the representation up to homotopy structure induced by ∇ on the double $\mathcal{D}$ of A (see Example 2.3.10), which we dualize and extend to $S\mathcal{D}^*$. The resulting structure operator will be our vertical differential

$$d^{\mathrm{v}}_\nabla : W^{p,q}(A,\nabla) \to W^{p,q+1}(A,\nabla).$$

Similarly, for the coadjoint complex Ad^* one has

$$S^q\mathrm{Ad}^* = (\underbrace{\Lambda^q T^*M}_{\text{degree } -q}) \oplus (\underbrace{A^* \otimes \Lambda^{q-1}T^*M}_{\text{degree } -q+1}) \oplus \ldots \oplus (\underbrace{T^*M \otimes S^{q-1}A^*}_{\text{degree } -1}) \oplus (\underbrace{S^q A^*}_{\text{degree } 0}),$$

hence

$$W^{p,q}(A,\nabla) = \Omega(A, S^q\mathrm{Ad}^*)^{p-q}.$$

We now use the connection ∇ to form the coadjoint representation Ad^*_∇. To obtain a horizontal operator which commutes with the vertical one, we consider the conjugation of the coadjoint representation, i.e. Ad^* with the structure operator $-\rho^* + (\nabla^{\mathrm{bas}})^* + (R_\nabla^{\mathrm{bas}})^*$. The symmetric powers will inherit a structure operator, and this will be our horizontal differential

$$d_\nabla^{\mathrm{h}} : W^{p,q}(A, \nabla) \to W^{p+1,q}(A, \nabla).$$

Proposition 2.4.1. *Endowed with d_∇^h and d_∇^v, $W(A, \nabla)$ becomes a differential bi-graded algebra whose cohomology is isomorphic to the cohomology of M. Moreover, up to isomorphisms of differential bi-graded algebras, $W(A, \nabla)$ does not depend on the choice of the connection ∇.*

Proof. To prove that the two differentials commute in the graded sense, one first remarks that it suffices to check the commutation relation on functions and on sections of T^*M, $\Lambda^1 A^*$ and $S^1 A^*$, which generate the entire algebra. This follows by direct computation (one can also use the local formulas below, but the computations are much more involved). The independence of ∇ will follow from the intrinsic description of the Weil algebra, discussed in the next chapter. Finally we need to prove that the cohomology equals that of M. Note that for $p > 0$ the column $(W^{p,\bullet}, d_\nabla^{\mathrm{v}})$ is acyclic, because it corresponds to an acyclic representation up to homotopy of TM. Next, we note that the first column $(W^{0,\bullet}, d_\nabla^{\mathrm{v}})$ is the De Rham complex of M and the usual spectral sequence argument provides the desired result. □

Remark 2.4.2 (Local coordinates)**.** Since the operators are local, it is worth looking at their expressions in coordinates. Let us assume that we are over a chart (x_a) of M on which we have a trivialization (e_i) of A. Over this chart, the Weil algebra will be the bi-graded commutative algebras over the space of smooth functions, generated by elements ∂^a of bidegree $(0, 1)$ (1-forms), elements θ^i of bi-degree $(1, 0)$ (the dual basis of (e_i), viewed in $\Lambda^1 A^*$), and elements μ^i of bi-degree $(1, 1)$ (the dual basis of (e_i), viewed in $S^1 A^*$). A careful but straightforward computation shows that, on these elements,

$$\begin{aligned}
d_\nabla^{\mathrm{v}}(\partial^a) &= 0,\\
d_\nabla^{\mathrm{v}}(\theta^i) &= \mu^i - \Gamma_{aj}^i \partial^a \theta^j,\\
d_\nabla^{\mathrm{v}}(\mu^i) &= -\Gamma_{aj}^i \partial^a \mu^j + \frac{1}{2} r_{abj}^i \partial^a \partial^b \theta^j,\\
d_\nabla^{\mathrm{h}}(\partial^a) &= -\rho_i^a \mu^i + \left(\frac{\partial \rho_i^a}{\partial x_b} - \Gamma_{bi}^j \rho_j^a\right) \theta^i \partial^b,\\
d_\nabla^{\mathrm{h}}(\theta^i) &= -\frac{1}{2} c_{jk}^i \theta^j \theta^k,\\
d_\nabla^{\mathrm{h}}(\mu^i) &= -(c_{jk}^i + \sum_a \rho_k^a \Gamma_{aj}^i) \theta^j \mu^k + \frac{1}{2} R_{jka}^i \theta^j \theta^k \partial^a.
\end{aligned}$$

Here we use the Einstein summation convention, ρ_i^a are the coefficients of the anchor ρ, c_{jk}^i are the structure functions of A, r_{abj}^i are the coefficients of the curvature of ∇ and R_{jka}^i are the coefficients of the basic curvature:

$$\rho(e_i) = \sum \rho_i^a \partial_a, \ \ [e_j, e_k] = \sum c_{jk}^i e_i,$$

$$R_\nabla(\partial_a, \partial_b) e_j = r_{abj}^i e_i, \ \ R_\nabla^{\mathrm{bas}}(e_j, e_k) \partial_a = R_{jka}^l e_l.$$

Moreover, for a smooth function f,

$$d^{\mathrm{v}}_{\nabla}(f) = \partial_a(f)\partial^a, \;\; d^{\mathrm{h}}_{\nabla}(f) = \partial_a(f)\rho^a_i\omega^i.$$

Note also that, in the case that the connection ∇ is flat, all the Γ and r-terms above vanish, while the R-terms are given by the partial derivatives of the structure functions c^i_{jk}.

Example 2.4.3 (The standard Weil algebra)**.** *In the case of Lie algebras $\mathfrak{g}$, one can immediately see that we recover the standard Weil algebra $W(\mathfrak{g})$. In particular, the local coordinates description becomes*

$$\begin{aligned} d^{v}(\theta^i) &= \mu^i, \\ d^{v}(\mu^i) &= 0, \\ d^{h}(\theta^i) &= -\frac{1}{2}\sum_{j,k} c^i_{jk}\theta^j\theta^k, \\ d^{h}(\mu^i) &= -\sum_{j,k} c^i_{jk}\theta^j\mu^k. \end{aligned}$$

which is one of the standard descriptions of the Weil algebra. In this case, d^v is usually called the Koszul differential, denoted d_K, d^h is called the Cartan differential, denoted d_C, and the total differential is denoted by d_W.

Example 2.4.4 (The BRST algebra)**.** *Recall Kalkman's BRST algebra associated to a $\mathfrak{g}$-manifold M. It is $W(\mathfrak{g}, M) := W(\mathfrak{g}) \otimes \Omega(M)$ with differential:*

$$\delta = d_W \otimes 1 + 1 \otimes d_{DR} + \sum_{a=1}^{n} \theta^a \otimes \mathcal{L}_a - \sum_{b=1}^{n} \mu^b \otimes \iota_b.$$

Proposition 2.4.5. *Let $A = \mathfrak{g} \ltimes M$ be the action algebroid associated to a $\mathfrak{g}$-manifold M. Then*

$$W(\mathfrak{g}, M) = W(A, \nabla^{\mathit{flat}}),$$

where ∇^{flat} is the canonical flat connection on A.

Proof. One only has to prove that the differentials coincide, which follows immediately from the formulas in local coordinates. □

3

The Weil algebra and Van Est isomorphisms

We describe a version of the Weil algebra $W(A)$ which does not depend on the choice of a connection. As in the case of Lie algebras, $W(A)$ is a model for the cohomology of the universal bundle EG of G, the groupoid integrating A. We construct a Van Est map $V : \hat{\Omega}^\bullet(G_\bullet) \to W(A)$, from the Bott-Shulman complex of G to the Weil algebra of A, which corresponds to the map induced in cohomology by the projection from EG to BG. We prove an isomorphism theorem for the Van Est map depending on the connectedness of the source fibers of G. Finally, we show that the integration of Poisson and Dirac manifolds can be derived as a consequence of the isomorphism theorem for the Van Est map.

3.1 Introduction

The purpose of this chapter is to describe the relation between the cohomology of the classifying bundle of a Lie groupoid and the integration of Poisson and Dirac manifolds. Lie groupoids are a tool to model singular spaces which arise as quotients by smooth equivalence relations. They are used to describe leaf spaces of foliations [26], orbifolds [37] and more generally, orbispaces [24]. The classifying space BG of a Lie groupoid G is a topological space which reflects the singularities of the quotient. The invariants of the singular space represented by G are defined via the classifying space and one is particularly interested in the cohomology. For instance, the action of a Lie group H on a manifold M has an associated groupoid $G = H \ltimes M$ whose classifying space BG is the homotopy quotient of the action, M_H. In this case, the cohomology of BG is the equivariant cohomology of the action, $H^\bullet(BG) \cong H^\bullet(M_H)$. The homotopy type of an orbifold represented by a proper étale groupoid G is the homotopy type of BG [37]. The classifying space of G arises as the base space of a principal G-bundle $EG \to BG$ with contractible fibers (over G). In this chapter we describe an algebraic model for the cohomology map of this universal bundle. We associate to a Lie algebroid A a differential graded algebra $W(A)$, which generalizes the Weil algebra of a Lie algebra. As in the case of Lie algebras, $W(A)$ is a model for the cohomology of EG. Once a connection has been chosen, the Weil algebra can be identified with the algebra of symmetric powers of the coadjoint representation A, described in chapter 2. Mehta [36] studied isomorphic algebras using the language of supermanifolds and these algebras have also been considered in unpublished work of D. Roytenberg. In the case of Lie algebra actions, the algebra $W(A)$ is Kalkman's BRST algebra for equivariant cohomology [30].

We define a Van Est map:

$$V : \hat{\Omega}^\bullet(G_\bullet) \to W(A),$$

from the Bott-Shulman complex of G to the Weil algebra of A. The Bott-Shulman complex is a model for the cohomology of BG and V corresponds to the map induced in cohomology by the projection from EG to BG. The map V is an extension of the Van Est map from differentiable to algebroid cohomology constructed in [20].
We prove the following isomorphism theorem, which answers a question posed in Mehta's thesis [36].

Theorem. 3.5.1 *Let G be a Lie groupoid with Lie algebroid A and k-connected source fibers. The homomorphism induced in cohomology by the Van Est map:*

$$V : H^p(\hat{\Omega}^q(G_\bullet)) \to H^p(W^{\bullet,q}(A)),$$

is an isomorphism for $p \leq k$ and is injective for $p = k+1$.

As a consequence, we prove the following generalization of the reconstruction result for multiplicative 2-forms which appears in [14].

Theorem. 3.6.1 *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\phi \in \Omega^{k+1}(M)$ be a closed form. Then there is a one to one correspondence between:*

1. *Multiplicative forms $\omega \in \Omega^k(G)$ which are relatively closed with respect to ϕ.*

2. *$C^\infty(M)$-linear maps $\tau : \Gamma(A) \to \Omega^{k-1}(M)$ satisfying the equations:*

$$i_{\rho(\beta)}(\tau(\alpha)) = -i_{\rho(\alpha)}(\tau(\beta)),$$

$$\tau([\alpha,\beta]) = L_\alpha(\tau(\beta)) - L_\beta(\tau(\alpha)) + d_{DR}(i_{\rho(\beta)}\tau(\alpha)) + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi).$$

The correspondence is given by:

$$\tau(\alpha) = i_{\alpha^1}(\omega)|_M,$$

where α^1 is the right invariant vector field on G determined by α and the restriction to M makes use of the inclusion $M \hookrightarrow G$ as units.

Theorem 3.6.1 reveals the relation between the Van Est map V and the integrability of Poisson and Dirac structures. It establishes a bijective correspondence between cocycles in the Bott-Shulman complex and cocycles in the Weil algebra that generalizes the correspondence between twisted Dirac structures and twisted presymplectic groupoids explained by Bursztyn et. al. in [14]. From the point of view of the Van Est map, the integrability results can be explained as follows. Poisson and Dirac structures determine cocycles in the Weil algebra of the associated Lie algebroid. In view of the isomorphism theorem, these cocycles can be integrated to cocycles in the Bott-Shulman complex which give the groupoid the corresponding symplectic or presymplectic structure.

This chapter is organized as follows. In section 3.2 we begin by recalling some facts about classifying spaces of Lie groups, the Weil algebra of a Lie algebra and equivariant cohomology. Then we introduce Lie groupoids and their classifying spaces, and describe how the cohomology of BG can be computed using the Bott-Shulman complex. In section 3.3 we describe a version of the Weil algebra of a Lie algebroid that does not depend on the choice of a connection (definition 3.3.1). We prove (proposition 3.3.8) that this algebra is isomorphic to the one described, using representations up to homotopy, in chapter 2. Section 3.4 contains the definition of the Van Est map $V : \hat{\Omega}^\bullet(G_\bullet) \to W(A)$ (theorem 3.4.1). In section 3.5 we prove an isomorphism theorem for the homomorphism induced in cohomology by the Van Est map (theorem 3.5.1). We explain the relation between the Van Est map and the integration of Poisson and Dirac structures in section 3.6 (theorems 3.6.1 and 3.6.4). Finally, in the appendix 3.7 we describe an infinite dimensional version of Kalkman's BRST algebra which is used throughout this chapter.

3.2 Preliminaries

3.2.1 Classifying spaces and the Weil algebra: the case of Lie groups

We will recall some standard facts about classifying spaces of Lie groups and equivariant cohomology.

The universal principal bundle: First of all, associated to any Lie group G there is a classifying space BG and an universal principal G-bundle $EG \longrightarrow BG$. These have the following universal property. For any space M there is a bijective correspondence

$$[M; BG] \overset{1-1}{\longleftrightarrow} \mathrm{Bun}_G(M)$$

between homotopy classes of maps $f : M \longrightarrow BG$ and isomorphism classes of principal G-bundles over a M. This correspondence sends a function f to the pull-back bundle $f^*(EG)$. The universal property determines $EG \longrightarrow BG$ uniquely up to homotopy. Another property that determines EG, hence also BG, uniquely, is that EG is a free, contractible G-space. There are explicit combinatorial constructions of the classifying bundle of a group, we will say more about this in a moment.

The cohomology of BG is the universal algebra of characteristic classes for principal G-bundles. Indeed, from the universal property, any such bundle $P \longrightarrow M$ is classified by a map $f_P : M \longrightarrow BG$. Although f_P is unique only up to homotopy, the map induced in cohomology

$$f_P^* : H^\bullet(BG) \longrightarrow H^\bullet(M)$$

only depends on P and is called the characteristic map of P. Any element $c \in H^\bullet(BG)$ will induce the c-characteristic class of P, $c(P) := (f_P)^*(c) \in H^\bullet(M)$. This is one of the reasons one is often interested in explicit models for the cohomology of BG. When G is compact, a theorem of Borel asserts that

$$H^\bullet(BG) \cong S(\mathfrak{g}^*)^G.$$

Moreover, the map f_P^* can be described geometrically. This is the Chern-Weil construction of characteristic classes in $H^\bullet(M)$ out of invariant polynomials on $\mathfrak{g}$, viewed as a map

$$S(\mathfrak{g}^*)^G \longrightarrow H(M).$$

The Weil algebra: Regarding the cohomology of BG and the construction of characteristic classes, the full picture is achieved only after finding a related model for the *De Rham cohomology* of EG. This is the Weil algebra $W(\mathfrak{g})$ of the Lie algebra of G. As a graded algebra, it is defined as

$$W^n(\mathfrak{g}) = \bigoplus_{2p+q=n} S^p(\mathfrak{g}^*) \otimes \Lambda^q(\mathfrak{g}^*).$$

We interpret its elements as polynomials P on $\mathfrak{g}$ with values in $\Lambda(\mathfrak{g}^*)$, but keep in mind that the polynomial degree counts twice. The Weil algebra can also be made into bigraded algebra, with

$$W^{p,q}(\mathfrak{g}) = S^q(\mathfrak{g}^*) \otimes \Lambda^{p-q}(\mathfrak{g}^*),$$

and its differential d_W can be decomposed into two components:

$$d_W = d_W^{\mathrm{h}} + d_W^{\mathrm{v}}.$$

Here, d_W^{v} increases q and is given by

$$d_W^{\mathrm{v}}(P)(v) = i_v(P(v)),$$

while d_W^{h} increases p and is defined as the Koszul differential of $\mathfrak{g}$ with coefficients in $S^q(\mathfrak{g}^*)$, the symmetric powers of the coadjoint representation. To be more explicit, it is customary to use coordinates. A basis $e^1, \ldots, e^n$ for $\mathfrak{g}$ gives structure functions c^i_{jk}. With this choice, $W(\mathfrak{g})$ can be described as the free graded commutative algebra generated by elements $\theta^1, \ldots, \theta^n$ of degree 1, $\mu^1, \ldots, \mu^n$ of degree 2, with differential:

$$\begin{aligned} d_W^{\mathrm{v}}(\theta^i) &= \mu^i, \\ d_W^{\mathrm{v}}(\mu^i) &= 0, \\ d_W^{\mathrm{h}}(\theta^i) &= -\frac{1}{2}\sum_{j,k} c_{jk}^i \theta^j \theta^k, \\ d_W^{\mathrm{h}}(\mu^i) &= -\sum_{j,k} c_{jk}^i \theta^j \mu^k. \end{aligned}$$

Of course, $\theta^1, \ldots, \theta^n$ is just the induced basis of $\Lambda^1(\mathfrak{g}^*)$, while $\mu^1, \ldots, \mu^n$ is the one of $S^1(\mathfrak{g}^*)$. The cohomology of $W(\mathfrak{g})$ is $\mathbb{R}$ concentrated in degree zero, as one should expect from the fact that EG is a contractible space.

$\mathfrak{g}$-DG algebras: To understand why $W(\mathfrak{g})$ is a model for the De Rham complex of EG, one has to look at the structure present in the De Rham complexes of principal G-bundles. This brings us to the notion of $\mathfrak{g}$-DG algebras. A $\mathfrak{g}$-DG algebra is a differential graded algebra $(\mathcal{A}, d)$ (for us $\mathcal{A}$ lives in positive degrees and d increases the degree by one), together with

- degree zero derivations L_v on the DG algebra $\mathcal{A}$, depending linearly on $v \in \mathfrak{g}$, which induce an action of $\mathfrak{g}$ on $\mathcal{A}$.
- degree -1 derivations i_v on the DG algebra $\mathcal{A}$ such that for all $v, w \in \mathfrak{g}$

  $$[i_v, i_w] = 0, \;\; [L_v, i_w] = i_{[v,w]}$$

  and such that they determine the Lie derivatives by Cartan's formula

  $$di_v + i_v d = L_v.$$

The basic subcomplex of a $\mathfrak{g}$-DG algebra $\mathcal{A}$ is defined as

$$\mathcal{A}_{\text{bas}} := \{\omega \in \mathcal{A} : i_v\omega = 0, L_v\omega = 0, \;\; \forall v \in \mathfrak{g}\}.$$

Of course, the De Rham complexes $\Omega(P)$ of G-manifolds are the typical examples of $\mathfrak{g}$-DG algebras. In this case L_v and i_v are just the usual Lie derivative and interior product with respect to the vector field $\rho(v)$ on P induced from v via the action of G. If P is a principal G-bundle over M, then $\Omega(P)_{\text{bas}}$ is canonically isomorphic to $\Omega(M)$.

The Weil algebra $W(\mathfrak{g})$ is a model for the De Rham complex of EG. Indeed, $W(\mathfrak{g})$ is canonically a $\mathfrak{g}$-DG algebra. The operators L_v are the unique derivations which, on $S^1(\mathfrak{g}^*)$ and on $\Lambda^1(\mathfrak{g}^*)$, are just the coadjoint action. The operators i_v are just the standard interior products on the exterior powers hence they act trivially on $S(\mathfrak{g}^*)$.

Equivariant cohomology: The Weil algebra is useful because it provides explicit models that compute equivariant cohomology. Given a G-space M, the pathological quotient M/G is often replaced by the homotopy quotient

$$M_G = (EG \times M)/G.$$

Here, EG should be thought of as a replacement of the one-point space pt with a free G-space which has the same homotopy as pt. The equivariant cohomology of M is defined as

$$H^{\bullet}_{G}(M) := H^{\bullet}(M_G).$$

When M is a manifold, one would like to have a geometric De Rham model computing this cohomology. This brings us to Cartan's model for equivariant cohomology. One defines the equivariant De Rham complex of a G-manifold M as

$$\Omega_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G,$$

the space of G-invariant polynomials on $\mathfrak{g}$ with values in $\Omega(M)$. The differential d_G on $\Omega_G(M)$ is very similar to the one of $W(\mathfrak{g})$:

$$d_G(P)(v) = d_{DR}(P(v)) + i_v(P(v)).$$

Let us see how the Weil algebra leads naturally to the Cartan model. The idea is quite simple. $W(\mathfrak{g})$ is a model for the De Rham complex of EG, the similar model for $EG \times M$ is $W(\mathfrak{g}) \otimes \Omega(M)$- viewed as a $\mathfrak{g}$-DG algebra with operators

$$i_v = i_v \otimes 1 + 1 \otimes i_v, \;\; L_v = L_v \otimes 1 + 1 \otimes L_v,$$

and with differential

$$d = d_W \otimes 1 + 1 \otimes d_{DR}.$$

The resulting basic subcomplex should provide a model for the cohomology of the homotopy quotient. Indeed, there is an isomorphism:

$$(W(\mathfrak{g}) \otimes \Omega(M))_{\text{bas}} \cong \Omega_G(M).$$

This is best seen using Kalkman's BRST model, which is a perturbation of $W(\mathfrak{g}) \otimes \Omega(M)$. As a $\mathfrak{g}$-DG algebra, it has

$$i^K_v = i_v \otimes 1, \; L^K_v = L_v \otimes 1.$$

To describe its differential, we use a basis for $\mathfrak{g}$ as above and set:

$$d^K = d + \theta^a \otimes L_{e^a} - \omega^a \otimes i_{e^a}.$$

The resulting basic subcomplex is $\Omega_G(M)$. In fact, there is an explicit automorphism Φ of $W(\mathfrak{g}) \otimes \Omega(M)$ (the Mathai-Quillen isomorphism) which transforms i_v, L_v and d into Kalkman's i^K_v, L^K_v, d^K.

3.2.2 Groupoids, classifying spaces and the Bott-Shulman complex

Lie groupoids: A groupoid is a category in which all arrows are isomorphisms. A Lie groupoid is a groupoid in which the space of objects G_0 and the space of arrows G_1 are smooth manifolds and all the structure maps are smooth. More explicitly, a Lie groupoid is given by a manifold of objects G_0 and a manifold of arrows G_1 together with smooth maps $s, t : G_1 \to G_0$ the source and target map, a composition map $m : G_1 \times_{G_0} G_1 \to G_1$, an inversion map $\iota : G \to G$ and an identity map $\epsilon : G_0 \to G_1$ that sends an object to the corresponding identity. These structure maps should satisfy the usual identities for a category. The source and target maps are required to be surjective submersions and therefore the domain of the composition map is a manifold. We will usually denote the space of objects of a Lie groupoid by M and say that G is a groupoid over M. We say that a groupoid is source k-connected if the fibers of the source map are k-connected.

Example 3.2.1. A Lie group G can be seen as a Lie groupoid in which the space of objects is a point. Associated to any manifold M there is the pair groupoid $M \times M$ over M for which there is exactly one arrow between each pair of points. If a Lie group G acts on a manifold M there is an associated action groupoid over M denoted $G \ltimes M$ whose space of arrows is $G \times M$. Other important examples of groupoids are the holonomy and monodromy groupoids of foliations, the symplectic groupoids of Poisson geometry- some of which arise via their infinitesimal counterparts, Lie algebroids.

Lie algebroids: As we already mentioned in chapter 2, a Lie algebroid over a manifold M is a vector bundle $\pi : A \to M$ together with a bundle map $\rho : A \to TM$, called the anchor map and a Lie bracket in the space $\Gamma(A)$ of sections of A satisfying Leibnitz identity:

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta,$$

for every $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$. Every Lie groupoid G has an associated Lie algebroid $A = A(G)$ defined as follows. As a vector bundle, it is the restriction of the kernel of the differential of the source map to M. Hence its fiber at $x \in M$ is the tangent space at the identity arrow 1_x of the source fiber $s^{-1}(x)$. The anchor map is the differential of the target map. To describe the bracket, we need to discuss invariant vector fields. A right invariant vector field on a Lie groupoid G is a vector field α which is tangent to the fibers of s and such that, if g, h are two composable arrows and we denote by R_{h} the right multiplication by h, then

$$\alpha(gh) = D_g(R_{\mathrm{h}})(\alpha(g)).$$

The space of right invariant vector fields is closed under the Lie bracket of vector fields and is isomorphic to $\Gamma(A)$. Thus, we get the desired Lie bracket on $\Gamma(A)$.
Unlike the case of Lie algebras, Lie's third theorem does not hold in general. Not every Lie algebroid can be integrated to a Lie groupoid. The precise conditions for the integrability are described in [22]. However, Lie's first and second theorem do hold. Due to the first one- which says that if a Lie algebroid is integrable then it admits a canonical source simply connected integration- one may often assume that the Lie groupoids under discussion satisfy this simply-connectedness assumption.

Actions: A left action of a groupoid G on a space $P \xrightarrow{\nu} M$ over M is a map $G_1 \times_M P \to P$ defined on the space $G_1 \times_M P$ of pairs (g, p) with $s(g) = \nu(p)$, which satisfies $\nu(gp) = t(g)$ and the usual conditions for actions. Associated to the action of G on $P \to M$ there is the action groupoid, denoted $G \ltimes P$. The base space is P, the space of arrows is $G_1 \times_M P$, the source map is the second projection and the target map is the action. The multiplication in this groupoid is $(g, p)(h, q) = (gh, q)$.

Example 3.2.2. For a Lie groupoid G, we denote by G_k the space of strings of k composable arrows of G. When we write a string of k composable arrows $(g_1, \ldots, g_k)$ we mean that $t(g_i) = s(g_{i-1})$. Since the source and target maps are submersions, all the G_k are manifolds. Each of the G_k's carries a natural left action. First of all, we view G_k over M via the map

$$t : G_k \to M, \;\; (g_1, \ldots, g_k) \mapsto t(g_1).$$

The left action of G on $G_k \xrightarrow{t} M$ is just

$$g(g_1, g_2, \ldots, g_k) = (gg_1, g_2, \ldots, g_k).$$

We denote by $P_{k-1}(G)$ the corresponding action groupoid.

Analogous to actions of Lie groupoids, there is the notion of infinitesimal actions. An action of a Lie algebroid A on a space $P \xrightarrow{\nu} M$ over M is a Lie algebra map $\rho_P : \Gamma(A) \to \mathfrak{X}(P)$, into the Lie algebra of vector fields on P, which is $C^\infty(M)$-linear in the sense that

$$\rho_P(f\alpha) = (f \circ \nu)\rho_P(\alpha),$$

for all $\alpha \in \Gamma(A)$, $f \in C^\infty(M)$. Note that this last condition is equivalent to the fact that ρ_P is induced by a bundle map $\nu^* A \to TP$.

As in the case of Lie groupoids, associated to an action of A on P there is an action algebroid $A \ltimes P$ over P. As a vector bundle, it is just the pull-back of A via μ. The anchor is just the infinitesimal action ρ_P. Finally, the bracket is uniquely determined by the Leibniz identity and

$$[\mu^*(\alpha), \mu^*(\beta)] = \mu^*([\alpha, \beta]),$$

for all $\alpha, \beta \in \Gamma(A)$.

As expected, an action of a groupoid G on a space $P \xrightarrow{\nu} M$ over M induces an action of the Lie algebroid A of G on P. As a bundle map $\rho_P : \nu^* A \to TP$ it is defined fiberwise as the differential at the identity of the map

$$s^{-1}(\nu(p)) \to P, \;\; g \mapsto gp.$$

Moreover, the Lie algebroid of $G \ltimes P$ is equal to $A \ltimes P$.

Example 3.2.3. For the action of G on G_k (Example 3.2.2), the resulting algebroid $A \ltimes G_k$ is just the foliation $\mathcal{F}_k$ of G_k by the fibers of the map $d_0 : G_k \longrightarrow G_{k-1}$, which deletes g_1 from $(g_1, \dots, g_k)$.

Classifying spaces: We now recall the construction of the classifying space of a Lie groupoid, as the geometric realization of its nerve. First of all, the nerve of G, denoted $N(G)$, is the simplicial manifold whose k-th component is $N_k(G) = G_k$, with the simplicial structure given by the face maps:

$$d_i(g_1, \dots, g_k) = \begin{cases} (g_2, \dots, g_k) & \text{if } i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_k) & \text{if } 0 < i < k, \\ (g_1, \dots, g_{k-1}) & \text{if } i = k, \end{cases}$$

and the degeneracy maps:

$$s_i(g_1, \dots, g_k) = (g_1 \dots, g_i, 1, g_{i+1}, \dots, g_k)$$

for $0 \leq i \leq k$.

The thick geometric realization of a simplicial manifold $X_\bullet$ [24, 42], is defined as the quotient space

$$||X_\bullet|| = (\coprod_{k \geq 0} X_k \times \Delta^k) / \sim,$$

obtained by identifying $(d_i(p), v) \in X_k \times \Delta^k$ with $(p, \delta_i(v)) \in X_{k+1} \times \Delta^{k+1}$ for any $p \in X_{k+1}$ and any $v \in \Delta^k$. Here Δ^k denotes the standard topological k-simplex and $\delta_i : \Delta^k \longrightarrow \Delta^{k+1}$ is the inclusion as the i-th face. The classifying space of a Lie groupoid G is defined as

$$\mathrm{B}G = ||N(G)||.$$

Definition 3.2.4. *The universal G-bundle of a Lie groupoid G is defined as*

$$EG = B(\mathrm{P}_1(G)).$$

The nerve of $P_1(G)$ has $(\mathrm{P}_1(G))_k = G_{k+1}$ which, for each k, is a (principal) G-space over G_k. Moreover, each face map is G-equivariant with respect to the right action, see Example 3.2.2. It follows that EG is a principal G-bundle over BG

$$\begin{array}{ccc} G_1 & & \mathrm{E}G \\ \downarrow\downarrow & \overset{\mu}{\swarrow} & \downarrow \pi \\ G_0 & & \mathrm{B}G \end{array}$$

Example 3.2.5. When G is a Lie group, one recovers (up to homotopy) the usual classifying space of G and the universal principal G-bundle $EG \to BG$. More generally, for the groupoid $G \ltimes M$ associated to an action of G on M, $B(G \ltimes M)$ is a model for the homotopy quotient

$$B(G \ltimes M) \cong M_G = (EG \times M)/G.$$

The Bott-Shulman complex: In general, the geometric realization of a simplicial manifold $X_\bullet$ is infinite dimensional and in particular, it is not a manifold. However, there is a De Rham theory that allows one to compute the cohomology of the geometric realization $||X_\bullet||$ with real coefficients using differential forms. Given a simplicial manifold $X_\bullet$ the Bott-Shulman complex, denoted $\Omega(X_\bullet)$, is the double complex

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \\
\Omega^2(X_0) & \xrightarrow{\delta} & \Omega^2(X_1) & \xrightarrow{\delta} & \Omega^2(X_2) & \xrightarrow{\delta} & \cdots \\
\mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \\
\Omega^1(X_0) & \xrightarrow{\delta} & \Omega^1(X_1) & \xrightarrow{\delta} & \Omega^1(X_2) & \xrightarrow{\delta} & \cdots \\
\mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \mathrm{d}\uparrow & & \\
\Omega^0(X_0) & \xrightarrow{\delta} & \Omega^0(X_1) & \xrightarrow{\delta} & \Omega^0(X_2) & \xrightarrow{\delta} & \cdots
\end{array}$$

where the vertical differential is just the De Rham differential and the horizontal differential δ is given by the simplicial structure,

$$\delta = \sum_{i=0}^{p+1} (-1)^i d_i^*.$$

The total complex of $\Omega(X_\bullet)$ is the De Rham model for the cohomology of $||X_\bullet||$. We will also consider the normalized Bott-Shulman complex of $X_\bullet$, denoted $\hat{\Omega}(X)$, which is the subcomplex of $\Omega(X_\bullet)$ that consists of forms $\eta \in \Omega^q(X_p)$ such that $s_i^*(\eta) = 0$ for all $i = 0, \ldots, p-1$. The inclusion $\hat{\Omega}(X_\bullet) \to \Omega(X_\bullet)$ induces an isomorphism in cohomology.

Theorem 3.2.6. (Dupont, Bott, Shulman, Stasheff,. . .) *There is a natural isomorphism*

$$H(\mathrm{Tot}(\Omega(X_\bullet))) \cong H(||X_\bullet||),$$

where $\mathrm{Tot}(\Omega(X_\bullet))$ denotes the total complex of the double complex $\Omega(X_\bullet)$.

For a Lie groupoid G we will write $\Omega(G_\bullet)$ instead of $\Omega(\mathrm{N}(G))$. Note that the Bott-Shulman complex $\Omega(G_\bullet)$ provides us with an explicit model computing $H^\bullet(BG)$. However, it is rather big and unsatisfactory compared with the infinitesimal models available for Lie groups.

We would like to emphasize another aspect of the Bott-Shulman complex. It is the natural place on which several geometric structures live. The best example is probably that of multiplicative forms. We first recall the definition (see for instance [14]).

Definition 3.2.7. *A multiplicative k-form on a Lie groupoid G is a k-form $\omega \in \Omega^k(G)$ satisfying*

$$d_1^*(\omega) = d_0^*(\omega) + d_2^*(\omega).$$

Given $\phi \in \Omega^{k+1}(M)$ closed, we say that ω is relatively ϕ-closed if $d\omega = s^*\phi - t^*\phi$.

In terms of the Bott-Shulman complex, the conditions appearing in the previous definition can be put together into just one: $\omega + \phi$ is a cocycle in the Bott-Shulman complex of G.

Example 3.2.8. With this terminology, a symplectic groupoid is a Lie groupoid G endowed with a symplectic form ω which is multiplicative. This corresponds to the case $k = 2$, $\phi = 0$ in the previous definition. Symplectic groupoids arise in Poisson geometry, the global geometry of a Poisson manifold is encoded in a topological groupoid which is a symplectic groupoid provided it is smooth. In turn, smoothness holds under relatively mild topological conditions. The case $k = 2$ and ϕ-arbitrary arises from various generalizations of Poisson geometry which, in turn, show up in the study of Lie-group valued momentum maps. With these motivations, relatively closed multiplicative two forms have been intensively studied in [14] culminating with their infinitesimal description which we now recall. Given a Lie algebroid A over M and a closed 3-form ϕ on M, an IM (infinitesimally multiplicative) form on A relative to ϕ is, by definition, a bundle map

$$\sigma : A \longrightarrow T^*M,$$

satisfying

$$\begin{aligned}
\langle \sigma(\alpha), \rho(\beta) \rangle &= -\langle \sigma(\beta), \rho(\alpha) \rangle, \\
\sigma([\alpha, \beta]) &= L_{\rho(\alpha)}(\sigma(\beta)) - L_{\rho(\beta)}(\sigma(\alpha)) + d\langle \sigma(\alpha), \rho(\beta) \rangle + i_{\rho(\alpha \wedge \beta)}(\phi),
\end{aligned}$$

for all $\alpha, \beta \in \Gamma(A)$. Here $\langle \cdot, \cdot \rangle$ denotes the pairing between a vector space and its dual. If A is the Lie algebroid of a Lie groupoid G, then any multiplicative 2-form ω on G which is closed relative to ϕ induces such a σ:

$$\sigma(\alpha) = i_\alpha(\omega)|_M.$$

The main result of [14] says that, if the s-fibers of G are 1-connected, then the correspondence $\omega \mapsto \sigma$ is a bijection.

The basic example comes from Poisson geometry. The cotangent bundle T^*M of a Poisson manifold M carries an induced algebroid structure and the identity map is an IM form. If T^*M is integrable and $\Sigma(M)$ is the (unique) Lie groupoid with 1-connected s-fibers integrating it, the corresponding multiplicative two form ω is precisely the one that makes $\Sigma(M)$ a symplectic groupoid.

3.3 The Weil algebra

In this section, A is a fixed Lie algebroid over M. We will define the Weil algebra of A, denoted $W(A)$. An element $c \in W^{p,q}(A)$ is a string $c = (c_0, c_1, \ldots)$ of operators that satisfy some compatibility relation. Before explaining what each c_i is, we want to emphasize that c_0 should be viewed as the leading term of c, while the remaining terms $c_1, c_2, \ldots$ should be viewed as correction terms for c_0. The leading term c_0 is just an antisymmetric $\mathbb{R}$-multilinear map

$$c_0 : \underbrace{\Gamma(A) \times \cdots \times \Gamma(A)}_{p \text{ times}} \to \Omega^q(M).$$

As a general principle, the role of the higher order terms is to measure the failure of c_0 to be $C^\infty(M)$ linear. With this in mind, one can often compute the higher terms from c_0.

Definition 3.3.1. *A element in $W^{p,q}(A)$ is a sequence of operators $c = (c_0, c_1, \ldots)$, where each c_i is an antisymmetric map*

$$c_i : \underbrace{\Gamma(A) \times \ldots \times \Gamma(A)}_{p-i \text{ times}} \to \Omega^{q-i}(M, S^i(A^*)),$$

satisfying

$$c_i(\alpha_1, \ldots, f\alpha_{p-i}) = fc_i(\alpha_1, \ldots, \alpha_{p-i}) - df \wedge \partial_{\alpha_{p-i}}(c_{i+1}(\alpha_1, \ldots, \alpha_{p-i-1})),$$

for all $f \in C^\infty(M)$, $\alpha_i \in \Gamma(A)$.

Here we use the notation from the appendix. In particular, for $\alpha \in \Gamma(A)$, $\partial_\alpha : S^k(A^*) \longrightarrow S^{k-1}(A^*)$ is the partial derivative along α. Also, viewing elements of $\Omega(M, S(A^*))$ as polynomial functions on A with values in ΛT^*M, we use the notation:

$$c_i(\alpha_1, \ldots, \alpha_{p-i}|\alpha) := c_i(\alpha_1, \ldots, \alpha_{p-i})(\alpha) \in \Omega(M) \quad (\text{for } \alpha \in \Gamma(A)).$$

Remark 3.3.2. Suppose that c, c' are elements of $\mathrm{W}^{p,q}(A)$. If $c_0 = c'_0$ then $c = c'$ provided $q \leq \dim(M)$.

We will now discuss the differential d on $W(A)$. As in the case of the Weil algebra of a Lie algebra, we write it as the sum of two differentials

$$d = d^{\mathrm{v}} + d^{\mathrm{h}}.$$

The vertical differential d^{v}: The vertical differential d^{v} increases q. It is induced by the De Rham differential on M in the following sense. Given $c \in W^{p,q}(A)$, the leading term of $d^{\mathrm{v}}(c)$ is, up to a sign, just the De Rham differential of the leading term of c:

$$(d^{\mathrm{v}}c)_0(\alpha_1, \ldots, \alpha_p|\alpha) = (-1)^p d_{DR}(c_0(\alpha_1, \ldots, \alpha_p|\alpha)).$$

The other components $(d^{\mathrm{v}}c)_k$ $(k \geq 1)$ can be found by applying the general principle mentioned above, by looking at the failure of $C^\infty(M)$-linearity. For instance, replacing α_p with $f\alpha_p$ in the previous formula, one finds the following formula for the next component of $d^{\mathrm{v}}c$:

$$(d^{\mathrm{v}}c)_1(\alpha_1, \cdots, \alpha_{p-1}|\alpha) = (-1)^{p-1}(d_{DR}(c_1(\alpha_1, \ldots, \alpha_{p-1}|\alpha)) + c_0(\alpha_1, \ldots, \alpha_{p-1}, \alpha|\alpha)).$$

Proceeding inductively, one can find the explicit formulas for all the other components. The final result, which will be taken as the complete definition of $(d^{\mathrm{v}}c)$, is:

$$(d^{\mathrm{v}}c)_k(\alpha_1,\cdots,\alpha_{p-k}|\alpha)=(-1)^{p-k}(d_{DR}(c_k(\alpha_1,\cdots,\alpha_{p-k}|\alpha))+c_{k-1}(\alpha_1,\ldots,\alpha_{p-k},\alpha|\alpha)).$$

The horizontal differential d^{h}: The horizontal differential d^{h} increases p. As above, it is induced by the Koszul differential in the following sense. Given $c\in W^{p,q}(A)$, the leading term of $d^{\mathrm{h}}(c)$ is given just by the Koszul differential of the leading term of c, where we use $\Omega(M,SA^*)$ as a representation of the Lie algebra $\Gamma(A)$ (see the Appendix):

$$\begin{aligned}(d^{\mathrm{h}}c)_0(\alpha_1,\cdots,\alpha_{p+1}) &= \sum_{i<j}(-1)^{i+j}c_0([\alpha_i,\alpha_j],\cdots,\hat{\alpha_i},\cdots,\hat{\alpha_j},\cdots,\alpha_{p+1})+\\ &+ \sum_i(-1)^{i+1}L_{\rho(\alpha_i)}(c_0(\alpha_1,\cdots,\hat{\alpha_i},\cdots,\alpha_{p+1})).\end{aligned}$$

As above, replacing α_{p+1} with $f\alpha_{p+1}$ and applying again the general principle, one finds the formula for the next component of $d^{\mathrm{h}}c$:

$$(d^{\mathrm{h}}c)_1(\alpha_1,\cdots,\alpha_p|\alpha)=\delta(c_1)(\alpha_1,\ldots,\alpha_p|\alpha)+(-1)^{p-1}i_{\rho(\alpha)}c_0(\alpha_1,\ldots,\alpha_p).$$

Proceeding inductively, one finds the explicit formulas for all the other components. The final result, which will be taken as the complete definition of $(d^{\mathrm{h}}c)$, is:

$$(d^{\mathrm{h}}c)_k(\alpha_1,\cdots,\alpha_{p-k+1}|\alpha)=\delta(c_k)(\alpha_1,\cdots,\alpha_{p-k+1}|\alpha)+(-1)^{p-k}i_{\rho(\alpha)}c_{k-1}(\alpha_1,\ldots,\alpha_{p-k+1}|\alpha).$$

Remark 3.3.3. Our signs were chosen so that they coincide with the standard ones for Lie algebra actions. Admittedly, they do not look very natural.

The algebra structure: Given $c\in W^{p,q}(A)$, $c'\in W^{p',q'}(A)$, define $cc'\in W^{p+p',q+q'}(A)$ as follows. The leading term is

$$(cc')_0(\alpha_1,\ldots,\alpha_{p+p'}|\alpha)=(-1)^{qp'}\sum\mathrm{sgn}(\sigma)c_0(\alpha_{\sigma(1)},\ldots,\alpha_{\sigma(p)}|\alpha)c_0^{'}(\alpha_{\sigma(p+1)},\ldots,\alpha_{\sigma(p+p')}|\alpha),$$

where the sum is over all (p,p')-shuffles. The other components can be deduced, again, by applying the general principle we have already used. The general formula also follows from the relation with Kalkman's BRST algebra (proposition 3.3.5 below).

Theorem 3.3.4. *d^v, d^h and the product are well defined, d^v, d^h are derivations and*

$$d^vd^v=0,d^hd^h=0,d^vd^h+d^hd^v=0.$$

In conclusion, $W(A)$ becomes a bigraded bidifferential algebra.

Proof. The fact that d^{v}, d^{h} and the product are well defined follows by a straightforward computation. □

For the remaining part of the theorem, as well as in order to shed some light into the formulas, we point out the relation with the infinite dimensional version of Kalkman's algebra (see the Appendix). We will consider the algebra $W(\mathfrak{g},\Omega(M))$ applied to the Lie algebra

$$\mathfrak{g}_A:=\Gamma(A),$$

acting on M via the anchor map. We will use the canonical inclusion

$$W(A) \hookrightarrow W(\mathfrak{g}_A, \Omega(M))$$

to realize $W(A)$ as a subspace of Kalkman's complex. From the explicit formulas, we deduce the following:

Proposition 3.3.5. *$W(A)$ is a sub-algebra of $W(\mathfrak{g}_A, \Omega(M))$ and the horizontal and the vertical differentials of $W(\mathfrak{g}_A, \Omega(M))$ restrict to those of $W(A)$.*

Remark 3.3.6. The previous proposition can be taken as a definition of the differentials and the product on $W(A)$. The converse is more interesting, Kalkman's formulas can be recovered from the De Rham and Koszul differentials by computing the higher order terms.

Example 3.3.7. When $A = \mathfrak{g}$ is a Lie algebra one recovers the usual Weil algebra. Also, when $A = \mathfrak{g} \ltimes M$, one recovers Kalkman's differentials.

Local coordinates: Since all the operators involved are local, it is possible to describe $W(A)$ in coordinates. Let (x_a) be the local coordinates in a chart for M on which we also have a trivialization (e_i) of the vector bundle A. Over this chart, we obtain the following model $W_{\text{flat}}(A)$. As a bi-graded algebra, it is the commutative algebra over the space of smooth functions generated by elements ∂^a of bidegree $(0, 1)$, elements θ^i of bi-degree $(1, 0)$, and elements μ^i of bi-degree $(1, 1)$. The isomorphism between $W_{\text{flat}}(A)$ and $W(A)$ (over the trivializing chart) sends

1. ∂^a into $dx_a \in \Omega^1(M) = W^{0,1}(A)$.
2. θ^i into the duals of e_i, viewed as elements in $\Gamma(\Lambda^1 A^*) = W^{1,0}(A)$.
3. μ^i into the elements $\widehat{\mu}^i \in W^{1,1}(A)$, where $\widehat{\mu}^i$ is determined by the fact that $\widehat{\mu}^i_0$ vanishes on the e_i's, while $\widehat{\mu}^i_1$ is the dual of e_i, viewed as an element of $\Gamma(S^1 A^*)$.

The map $W_{\text{flat}}(A) \longrightarrow W(A)$ is an isomorphism. The differentials can now be computed explicitly on generators and one finds:

$$\begin{aligned}
d^{\text{v}}_{\text{flat}}(\partial^a) &= 0,\\
d^{\text{v}}_{\text{flat}}(\theta^i) &= \mu^i,\\
d^{\text{v}}_{\text{flat}}(\mu^i) &= 0,\\
d^{\text{h}}_{\text{flat}}(\partial^a) &= -\rho^a_i \mu^i + \frac{\partial \rho^a_i}{\partial x_b} \theta^i \partial^b,\\
d^{\text{h}}_{\text{flat}}(\theta^i) &= -\frac{1}{2} c^i_{jk} \theta^j \theta^k,\\
d^{\text{h}}_{\text{flat}}(\mu^i) &= -c^i_{jk} \theta^j \mu^k + \frac{1}{2} \frac{\partial c^l_{jk}}{\partial x_a} \theta^j \theta^k \partial^a,
\end{aligned}$$

where we use the Einstein summation convention, ρ^a_i are the coefficients of ρ and c^i_{jk} are the structure functions of A. Namely,

$$\rho(e_i) = \sum \rho^a_i \partial_a, \;\; [e_j, e_k] = \sum c^i_{jk} e_i.$$

Note that, on smooth functions:

$$d^{\mathrm{v}}_{\mathrm{flat}}(f) = \partial_a(f)\partial^a,\ d^{\mathrm{h}}_{\mathrm{flat}}(f) = \partial_a(f)\rho_i^a\theta^i.$$

Connection version A global version of the previous remark is possible with the help of a connection ∇, which allows one to identify $W(A)$ with the Weil algebra described in chapter 2. Recall that as a bigraded algebra it is just:

$$W^{p,q}_{\nabla}(A) = \bigoplus_k \Gamma(\Lambda^{q-k}T^*M \otimes S^k(A^*) \otimes \Lambda^{p-k}(A^*).$$

However, the associated operators d^{h}_{∇}, d^{v}_{∇} acting on $W_{\nabla}(A)$ are more involved and are computed in chapter 2. Working with the global ∇, one can write down the explicit local formulas for d^{h}_{∇}, d^{v}_{∇} on generators. The resulting equations will be similar to the ones for $W_{\mathrm{flat}}(A)$, but they have extra terms which involve the coefficients of the connection and two types of curvature tensors. The explicit map

$$I_{\nabla} : W_{\nabla}(A) \longrightarrow W(A),$$

is defined as follows. It is the unique algebra map which is $C^{\infty}(M)$-linear and has the properties:

- on $\Omega(M)$ and $\Gamma(\Lambda A^*)$, which are subspaces of both $W_{\nabla}(A)$ and $W(A)$, I_{∇} is the identity.
- for $\xi \in \Gamma(S^1 A^*)$, $I_{\nabla}(\xi) = \widehat{\xi}$, where

$$\widehat{\xi}_0(\alpha) = -\xi(\nabla(\alpha)), \widehat{\xi}_1 = \xi.$$

Proposition 3.3.8. *I_{∇} is an isomorphism of differential bigraded algebras.*

Proof. We view I_{∇} as a map of sheaves. It suffices to show that I_{∇} is an isomorphism locally. We then use the generators ∂^a, θ^i and μ^i as above. These elements also belong to the Kalkman algebra $W(\mathfrak{g}_A, \Omega(M))$, and the map I_{∇} can be seen as a map from $W_{\nabla}(A)$ into $W(\mathfrak{g}_A, \Omega(M))$ which leaves ∂^a and θ^i invariant, but which sends μ^i into $\widehat{\mu}^i$. Since the Kalkman algebra is free commutative and the map is injective on the generators, we conclude that I_{∇} is injective. Surjectivity is a consequence of the fact that $W(A)$ is generated by the elements in $W^{0,0}(A)$, $W^{1,0}(A)$, $W^{0,1}(A)$ and the map I_{∇} is clearly a bijection in those degrees. Finally, since the differentials are derivations, it is enough to prove that they coincide in low degree, and this is a simple check. □

The adjoint representation: Let us give a short summary of chapter 2 and explain the connection with the Weil algebra. In order to be able to talk about the adjoint representation of a Lie algebroid, one has to enlarge the category Rep(A) of (standard) representations and work in the category Rep$^{\infty}(A)$ of representations up to homotopy. Such representations, by their nature, serve as coefficients for the cohomology of A. Underlying any object of Rep$^{\infty}(A)$ there is a cochain complex (E, ∂) of vector bundles over A; the extra-structure present on (E, ∂) is a linear operation of A on E, which is not quite an action- but the failure is precisely measured and there are higher and higher coherence conditions. For instance, for the adjoint representation, the underlying complex is:

$$A \stackrel{\rho}{\longrightarrow} TM, \tag{3.1}$$

with A in degree zero, TM in degree one. However, to complete this to a representation up to homotopy, one needs to use a connection ∇ on the vector bundle A. The resulting object $\mathrm{Ad}_\nabla \in \mathrm{Rep}^\infty(A)$ does not depend on ∇ up to isomorphisms. Its isomorphism class is denoted by Ad. This indicates in particular that the resulting cohomologies with coefficients in Ad_∇ or other associated representations (such as symmetric powers, duals etc) do not depend on ∇ and can be computed by a ∇-independent (intrinsic) complex. The rows $(W^{\bullet,q}(A), d^{\mathrm{h}})$ of the Weil algebra are the intrinsic complexes computing the cohomology of A with coefficients in $S^q(\mathrm{Ad}^*)$:

$$H^\bullet(W^{\bullet,q}(A)) \cong H^\bullet(A, S^q(\mathrm{Ad}^*)). \tag{3.2}$$

From this description it immediately follows that the cohomology of $W(A)$ is isomorphic to the cohomology of M- which should be viewed as the algebraic counterpart of the fact that the fibers of the map $EG \longrightarrow M$ are contractible.

Example 3.3.9 (Multiplicative forms)**.** Closed multiplicative forms on groupoids are related to homogenous cocycles of the Weil algebra. To illustrate this, let A be the Lie algebroid of a Lie groupoid G over M. Then any 2-form $\omega \in \Omega^2(G)$ induces an element $c \in W^{1,2}(A)$ with leading term

$$c_0 : \Gamma(A) \longrightarrow \Omega^2(M), c_0(|\alpha) = L_\alpha(\omega)|_M,$$

where $\alpha \in \Gamma(A)$ is identified with the induced right invariant vector field on G and we use the inclusion $M \hookrightarrow G$ as units. The other component, $c_1 \in \Omega^1(M, S^1A^*)$, is given by

$$c_1(|\alpha) = -i_\alpha(\omega)|_M.$$

When ω is closed c is d^{v} closed and when ω is multiplicative c is d^{h}-closed. This is an instance of the Van Est map that will be explained in the next section.

Example 3.3.10 (IM forms)**.** In turn, $(1,2)$ cocycles on the Weil algebra of a Lie algebroid A can be identified with the IM forms on A (see Example 3.2.8 in the case when $\phi = 0$). To see this, we first remark that an element $c \in W^{1,2}(A)$ which is d^{v}-closed is uniquely determined by its component c_1, which we interpret as a bundle map $A \longrightarrow T^*M$ as before and denote it by σ. Indeed, the condition $(d^{\mathrm{v}}c)_1 = 0$ gives us

$$c_0(|\alpha) = -d_{DR}(\sigma(\alpha)).$$

If c is also d^{h}-closed one has in particular that $(d^{\mathrm{h}}c)_2 = 0$ and $(d^{\mathrm{h}}c)_1 = 0$. These two conditions coincide with the conditions for σ to be an IM form (see Example 3.2.8). One can check directly that, conversely, these conditions also imply $(d^{\mathrm{h}}c)_0 = 0$.

As a conclusion of the last two examples, the correspondence between multiplicative two-forms on groupoids and IM-forms on algebroids described in Example (3.2.8) factors through the Weil algebra. This will be generalized to arbitrary forms on the nerve of G in the next section. We will show that the main result of [14] can be derived as a consequence of a general Van Est isomorphism theorem.

We will now describe a version of the Weil algebra with coefficients which will be used in the proof of our main theorem. The coefficients that appear are the generalizations of the notion of $\mathfrak{g}$-DG algebras.

Definition 3.3.11. *Given a Lie algebroid A over M, an A-DG algebra is a DG-algebra $(\mathcal{A}, d)$ together with*

- *a structure of $\Gamma(A)$-DG algebra, with Lie derivatives and interior products denoted by L_α and i_α, respectively.*
- *a graded multiplication $\Omega(M) \otimes \mathcal{A} \longrightarrow \mathcal{A}$ which makes $(\mathcal{A}, d)$ into a DG algebra over the De Rham algebra $\Omega(M)$ and which is compatible with L_α and i_α*

such that

$$i_{f\alpha}(a) = f i_\alpha(a), \;\; L_{f\alpha}(a) = f L_\alpha(a) + (df) i_\alpha(a),$$

for all $\alpha \in \Gamma(A)$, $f \in C^\infty(M)$, $a \in \mathcal{A}$.

Given such an A-DG algebra, we define $W(A, \mathcal{A})$ as follows. An element $c \in W^{p,q}(A; \mathcal{A})$ is a sequence $(c_0, c_1, \ldots)$ where

$$c_i : \underbrace{\Gamma(A) \times \ldots \times \Gamma(A)}_{p-i \text{ times}} \times \underbrace{\Gamma(A) \times \ldots \times \Gamma(A)}_{i \text{ times}} \to \mathcal{A}^{q-i},$$

$$(\alpha_1, \ldots, \alpha_{p-i}, \alpha_{p-i+1}, \ldots, \alpha_p) \mapsto c_i(\alpha_1, \ldots, \alpha_{p-i} | \alpha_{p-i+1}, \ldots \alpha_p)$$

is $\mathbb{R}$-multilinear and antisymmetric on $\alpha_1, \ldots, \alpha_{p-i}$ and is $C^\infty(M)$-multilinear and symmetric on $\alpha_{p-i+1}, \ldots, \alpha_p$. Moreover, c_i and c_{i+1} are required to be related in the same way as in Definition 3.3.1. As before, $W(A, \mathcal{A})$ sits inside Kalkman's $W(\mathfrak{g}_A, \mathcal{A})$ and we use this inclusion to induce the algebra structure and the two differentials on $W(A, \mathcal{A})$.

Example 3.3.12. *The basic example of an A-DG algebra is the De Rham complex of M, in which case we recover $W(A)$. More generally, if A acts on a space $P \xrightarrow{\mu} M$ over M, $\Omega(P)$ has the structure of an A-DG algebra: L_α and i_α are the usual Lie derivatives and interior products with respect to the vector fields on P induced by α, while the $\Omega(M)$-module structure is*

$$\Phi \cdot \omega = \mu^*(\Phi) \wedge \omega,$$

for $\Phi \in \Omega(M)$, $\omega \in \Omega(P)$. In this case we can describe the Weil algebra as follows.

Lemma 3.3.13. *Consider an action of A on $P \xrightarrow{\mu} M$ and the induced A-DG algebra structure on $\Omega(P)$. Then, the algebra $W(A, \Omega(P))$ is isomorphic (as a bigraded differential algebra) to $W(A \ltimes P)$, where $A \ltimes P$ is the induced action Lie algebroid.*

3.4 The Van Est map

In this section we discuss the Van Est map which relates the Bott-Shulman complex of a groupoid to the Weil algebra of its algebroid. Any section $\alpha \in \Gamma(A)$ induces a vector field α^p on each of the spaces G_p of strings of p-composable arrows. Explicitly, for $g = (g_1, \ldots, g_p) \in G_p$ with $t(g) = x$, α^p_g is the image of $\alpha_x \in A_x$ (i.e. in the tangent space at 1_x of $s^{-1}(x)$) by the differential of the map

$$R_g : s^{-1}(x) \longrightarrow G_p, \;\; a \mapsto ag := (ag_1, g_2, \ldots, g_p).$$

The map $\alpha \longrightarrow \alpha^p$ is nothing but the infinitesimal action induced by the canonical right action of G on G_p (see subsection 3.2.2 and in particular example 3.2.2). When no confusion arises, we will denote the vector field α^p simply by α. The induced Lie derivative acting on $\Omega(G_p)$, combined with the simplicial degeneracy map $s_0 : G_{p-1} \longrightarrow G_p$ (which inserts a unit on the first place) induces a map

$$R_\alpha : \Omega^q(G_p) \longrightarrow \Omega^q(G_{p-1}).$$

Intuitively, $R_\alpha(\omega)$ is the derivative on the first argument along α, at the units.

Proposition 3.4.1. *Let G be a Lie groupoid over M with Lie algebroid A. For any normalized form in the Bott-Shulman complex of G, $\omega \in \hat{\Omega}^q(G_p)$, the map*

$$\underbrace{\Gamma(A) \times \ldots \times \Gamma(A)}_{p \text{ times}} \longrightarrow \Omega^q(M),$$

$$(\alpha_1, \ldots, \alpha_p) \mapsto (-1)^{\frac{p(p+1)}{2}} \sum_{\sigma \in S_p} sgn(\sigma) R_{\alpha_{\sigma(1)}} \ldots R_{\alpha_{\sigma(p)}}(\omega)$$

is the leading term of a canonical element $V(\omega) \in W^{p,q}(A)$ induced by ω. Moreover, the resulting map

$$V : \hat{\Omega}^q(G_p) \to W^{p,q}(A),$$

is compatible with the horizontal and the vertical differentials in the sense that

$$V\mathrm{d} = (-1)^p d^v V, \tag{3.3}$$

$$V\delta = d^h V. \tag{3.4}$$

Remark 3.4.2 (More standard Van Est maps)**.** The standard Van Est map for a Lie groupoid G relates the differentiable cohomology $H^\bullet_d(G)$ with the cohomology $H^\bullet(A)$ of the associated Lie algebroid. These cohomologies can be identified in our case as follows. $H^\bullet_d(G)$ is the cohomology of the first row $\Omega^0(G_\bullet)$ of the Bott-Shulman complex of G. On the other hand $H^\bullet(A)$ is the cohomology of the first row $W^{\bullet,0}(A)$ of the Weil algebra. Our Van Est map extends the ordinary one to a map of double complexes.

As in the discussions in the previous section, one can heuristically derive all the components of $V(\omega)$ out of the formula for the leading term. However, strictly speaking we do have to specify the higher order terms for $V(\omega)$ to be well defined. To achieve this, we need an operation similar to R_α, but which uses interior products instead of Lie derivatives:

$$J_\alpha : \Omega^q(G_p) \longrightarrow \Omega^{q-1}(G_{p-1}), \;\; J_\alpha(\omega) := s_0^*(i_\alpha(\omega)).$$

The component $V(\omega)_i$ evaluated on sections of $\Gamma(A)$,

$$V(\omega)_i(\alpha_1, \ldots, \alpha_{p-i}|\alpha) \in \Omega^{q-i}(M)$$

will be a sum in which each term arises by applying the operators R_{α_k} $p - i$ times and J_α i times in all possible ways, with the appropriate sign. The summation is over all permutations $\sigma \in S_p$ such that

$$\sigma^{-1}(p - i + 1) < \ldots < \sigma^{-1}(p - 1) < \sigma^{-1}(p).$$

We denote by $S_p(i)$ the set of all such permutations. For each $\sigma \in S_p(i)$, we consider the expression

$$V(\omega)^{\sigma}_{i}(\alpha_1, \ldots, \alpha_{p-i}|\alpha) := (-1)^i D_1 \ldots D_p(\omega)$$

where the ordered sequence $D_1, \ldots, D_p$ is obtained as follows. One starts with the sequence

$$R_{\alpha_{\sigma(1)}}, \ldots, R_{\alpha_{\sigma(2)}}, R_{\alpha_{\sigma(p)}}$$

and one replaces R_{α_k} by J_α whenever $k \in \{p-i+1, \ldots, p\}$. Define

$$V(\omega)_i = (-1)^{\frac{p(p+1)}{2}} \sum_{\sigma \in S_p(i)} \mathrm{sgn}(\sigma) V(\omega)^{\sigma}_{i}.$$

Proof. (of Proposition 3.4.1) We first point out the following properties of the operators R_α and J_α, which follow immediately from similar properties of the operators L_α and i_α:

$$R_\alpha = J_\alpha \mathrm{d} + \mathrm{d} J_\alpha, \tag{3.5}$$
$$R_\alpha(\eta\omega) = R_\alpha(\eta) s_0^*(\omega) + s_0^*(\eta) R_\alpha(\omega), \tag{3.6}$$
$$J_\alpha(\eta\omega) = J_\alpha(\eta) s_0^*(\omega) + (-1)^q s_0^*(\eta) J_\alpha(\omega), \tag{3.7}$$
$$R_{f\alpha}(\eta) = \mathrm{d}(f) J_\alpha(\eta) + f R_\alpha(\eta), \tag{3.8}$$
$$J_{f\alpha} = f J_\alpha. \tag{3.9}$$
$$\tag{3.10}$$

Next, R_α and J_α interact with the degeneracy maps s_i as follows:

$$s_j^* J_\alpha = J_\alpha s_{j+1}^*, \tag{3.11}$$
$$s_j^* R_\alpha = R_\alpha s_{j+1}^*. \tag{3.12}$$

The second equation follows formally from the first one and formula (5.19). The first one follows from the simplicial relations and the equation

$$s_{j+1}^* i_{\alpha^q} = i_{\alpha^{q-1}} s_{j+1}^* \tag{3.13}$$

In order to prove this last equation it is enough to evaluate it on a one form $\omega \in \Omega^1(G_q)$. We will use the formula

$$(ds_{j+1})_g(\alpha^{q-1}) = \alpha^q_{s_{j+1}(g)},$$

which follows from the definition of α^q and the fact that $s_{j+1} R_g = R_{s_{j+1}(g)}$. We compute:

$$\begin{aligned} i_{\alpha^{q-1}} s_{j+1}^*(\omega)_g &= s_{j+1}^*(\omega)(\alpha_g^{q-1}) \\ &= \omega(ds_{j+1})_g(\alpha_g^{q-1})) \\ &= \omega(\alpha^q_{s_{j+1}(g)}) \\ &= s_{j+1}^* i_{\alpha^q}(\omega)_g. \end{aligned}$$

In particular, the equations above imply that R_α and J_α preserve the normalized subcomplex. We will also use the $\Omega(M)$-module structure on $\Omega(G_p)$ given by

$$\Phi\omega = t^*(\Phi) \wedge \omega.$$

As a consequence of the previous formulas we have:

$$R_\alpha(\Phi\omega) = \Phi R_\alpha(\omega),\ J_\alpha(\Phi\omega) = (-1)^{\deg(\Phi)}\Phi J_\alpha(\omega),$$

for all $\Phi \in \Omega(M), \omega \in \hat{\Omega}(G_\bullet)$. From these and (3.8) and (3.9), it immediately follows that the components $V(\omega)_i$ satisfy the desired $C^\infty(M)$-linearity in the symmetric variables while on the other variables we obtain the equation which expresses the relation between $V(\omega)_i$ and $V(\omega)_{i+1}$. In other words, $V(\omega)$ does belong to $W(A)$.

We will need the following remark on the functoriality of the Van Est map. Given an action of G on a space $P \xrightarrow{\mu} M$, we have the action groupoid $G \ltimes P$ over P with associated Lie algebroid $A \ltimes P$. We use the Van Est map applied to the induced action groupoid $G \ltimes P$ and the induced Lie algebroid $A \ltimes P$. The pull-back from M to P induces inclusions of the Weil algebra of A and of the Bott-Shulman complex of G, into the ones corresponding to $A \ltimes P$ and $G \ltimes P$, respectively (see also Proposition 3.3.13), which is compatible with the Van Est map:

$$\begin{array}{ccc} \hat{\Omega}(G) & \xrightarrow{V} & W(A) \\ \downarrow{\scriptstyle \text{incl}} & & \downarrow{\scriptstyle \text{incl}} \\ \hat{\Omega}(P \ltimes G) & \xrightarrow{V} & W(A \ltimes P) \end{array}$$

Also, the inclusion maps are compatible with the vertical and the horizontal differentials. We will use this diagram in order to simplify the proof of the compatibility of V with the differentials. For instance, for $\omega \in \hat{\Omega}^q(G_p)$ and $q \leq \dim(M)$, in order to prove that

$$V(\mathrm{d}(\omega)) = (-1)^p d^{\mathrm{v}}(V(\omega)),$$

it suffices to show that their leading terms coincide (see Remark 3.3.2). However, using a G-space P with the dimension of P big enough (for a fixed ω!), the previous diagram shows that all we have to show is that:

$$V(\mathrm{d}(\omega))_0 = (-1)^p d^{\mathrm{v}}(V(\omega))_0,$$

for all algebroids and all ω's. In turn, this formula follows immediately from the definition of $d^{\mathrm{v}}(\omega)_0$ and the fact that the operations R_α commute with De Rham differentials.

For the compatibility of V with the horizontal differentials we will use the following formulas.

$$R_\alpha d_i^* = \begin{cases} d_{i-1}^* R_\alpha & \text{if } i > 1, \\ L_\alpha & \text{if } i = 1, \\ 0 & \text{if } i = 0. \end{cases} \tag{3.14}$$

$$R_\alpha L_\beta - R_\beta L_\alpha = R_{[\alpha,\beta]}. \tag{3.15}$$

Equations (3.14) follow from the simplicial equations and the following formula, which can be proven in the same way in which (3.13) was proved:

$$i_{\alpha^{q+1}} d_i^* = \begin{cases} d_i{}^* i_{\alpha^q} & \text{if } i > 0, \\ 0 & \text{if } i = 0. \end{cases}$$

Equation (3.15) follows immediately from $[L_\alpha, L_\beta] = L_{[\alpha,\beta]}$. We now prove that V commutes with the horizontal differentials. As before, it suffices to show that

$$V(\delta(\omega))_0 = d^{\mathrm{h}}(V(\omega))_0. \tag{3.16}$$

Assume that $\omega \in \hat{\Omega}^q(G_{p-1})$, and we evaluate the right hand side on $(\alpha_1, \ldots, \alpha_p)$. We have two types of terms. The first type is

$$\sum_{i=1}^{p}(-1)^{i+1}L_{\alpha_i}(V(\omega)(\alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \alpha_p)).$$

Writing out V (for each i fixed) we get a sum over permutations σ_0 of $1, \ldots, \hat{i}, \ldots, p$. To (i, σ_0) it corresponds the permutation $\sigma = (i, \sigma_0(1), \ldots) \in S_p$. Note that the number $\tau(\sigma)$ of transpositions of σ equals to $i - 1 + \tau(\sigma_0)$, so the sum above equals to

$$\sum_{i=1}^{p}(-1)^{\frac{p(p-1)}{2}}\mathrm{sgn}(\sigma)L_{\alpha_{\sigma(1)}}(R_{\alpha_{\sigma(2)}}\ldots R_{\alpha_{\sigma(p)}}\omega). \tag{3.17}$$

The other term is

$$\sum_{i<j}(-1)^{i+j}(V(\omega))([\alpha_i, \alpha_j], \alpha_1, \ldots, \widehat{\alpha_i}, \ldots, \widehat{\alpha_j}, \ldots, \alpha_p).$$

Again, for each i and j, writing out V we get a sum over permutations σ_1 of the list

$$0, 1, \ldots, \hat{i}, \ldots, \hat{j}, \ldots, p,$$

(where 0 is used to index the position of $[\alpha_i, \alpha_j]$). To (i, j, σ_1) we associate

- a number $k \in \{1, \ldots, p-1\}$ defined by the condition that 0 is on the $(p-k)^{\mathrm{th}}$-position of σ_1.
- a permutation $\sigma \in S_p$ which is obtained from σ_1 by inserting i on the $(p-k)^{\mathrm{th}}$ place, and j on the $(p-k+1)^{\mathrm{th}}$ (so that $\sigma(p-k) = i$, $\sigma(p-k+1) = j$), and the ordered sequence $\sigma(1), \sigma(2), \ldots$ from which i and j are deleted coincides with ordered sequence $\sigma_1(0), \sigma_1(1), \ldots$ from which 0 is deleted.

Note that, modulo 2, the number of transpositions of these permutations satisfy:

$$\begin{aligned}\tau(\sigma_1) &= \tau(\sigma(1), \ldots, \sigma(p-k-1), 0, \sigma(p-k+2), \ldots, \sigma(p)) = \\ &= p-k+1+\tau(\sigma(1), \ldots, \sigma(p-k-1), \sigma(p-k+2), \ldots, \sigma(p)),\end{aligned}$$

and

$$\begin{aligned}\tau(\sigma) &= \tau(\sigma(1), \ldots, \sigma(p-k-1), i, j, \sigma(p-k+2), \ldots, \sigma(p)) \\ &= \tau(i, j, \sigma(1), \ldots, \sigma(p-k-1), \sigma(p-k+2), \ldots, \sigma(p)) \\ &\quad +\tau(p-k, p-k+1, 1, 2, \ldots) \\ &= \tau(i, j, \sigma(1), \ldots, \sigma(p-k-1), \sigma(p-k+2), \ldots, \sigma(p)) \\ &= (i-1)+(j-2)+\tau(\sigma(1), \ldots, \sigma(p-k-1), \sigma(p-k+2), \ldots, \sigma(p)).\end{aligned}$$

Thus,

$$(-1)^{i+j}\mathrm{sgn}(\sigma_1) = (-1)^{p-k}\mathrm{sgn}(\sigma).$$

We conclude that the second term that comes from the right hand side of (3.16) is:

$$(-1)^{\frac{p(p+1)}{2}}\sum_k \sum_{\sigma:\sigma(p-k)<\sigma(p-k+1)} (-1)^k\mathrm{sgn}(\sigma)R_{\alpha_{\sigma(1)}}\dots R_{\alpha_{\sigma(p-k-1)}}R_{[\alpha_{\sigma(p-k)},\alpha_{\sigma(p-k+1)}]}\dots R_{\alpha_{\sigma(p)}}. \tag{3.18}$$

The left hand side of (3.16) applied to $(\alpha_1,\dots,\alpha_p)$ is

$$(-1)^{\frac{p(p+1)}{2}}\sum_\sigma\sum_{k=0}^{p}\mathrm{sgn}(\sigma)(-1)^kR_{\alpha_{\sigma(1)}}\dots R_{\alpha_{\sigma(p)}}d_k^*\omega.$$

Using (3.14), this is equal to

$$(-1)^{\frac{p(p+1)}{2}}\sum_{k=1}^{p}\mathrm{sgn}(\sigma)(-1)^kR_{\alpha_{\sigma(1)}}\dots R_{\alpha_{\sigma(p-k)}}L_{\alpha_{\sigma(p-k+1)}}\dots R_{\alpha_{\sigma(p)}}.$$

When $k=p$ we obtain precisely (3.17). It remains to show that the remaining terms give us (3.18). In that sum (over σ and $k\le p-1$) we distinguish two cases:

- (k,σ) satisfies: $\sigma(p-k)<\sigma(p-k+1)$.
- (k,σ)-satisfies: $\sigma(p-k)>\sigma(p-k+1)$.

Note that, using the transposition $\tau_k := (p-k,p-k+1))$, we have a bijection $(k,\sigma)\mapsto(k,\sigma\circ\tau_k)$ between the first and second cases. Hence, both cases can be indexed by (k,σ) which satisfy $\sigma(p-k)<\sigma(p-k+1)$, but the second case will produce terms of type:

$$(-1)^{\frac{p(p+1)}{2}}(-\mathrm{sgn}(\sigma))(-1)^kR_{\alpha_{\sigma(1)}}\dots R_{\alpha_{\sigma(p-k+1)}}L_{\alpha_{\sigma(p-k)}}\dots R_{\alpha_{\sigma(p)}},$$

where we used that $\mathrm{sgn}(\sigma\circ\tau_k)=-\mathrm{sgn}(\sigma)$. Putting together the two cases, we obtain:

$$(-1)^{\frac{p(p+1)}{2}}(-1)^k\mathrm{sgn}(\sigma)R_{\alpha_{\sigma(1)}}\dots(R_{\alpha_{\sigma(p-k)}}L_{\alpha_{\sigma(p-k+1)}}-R_{\alpha_{\sigma(p-k+1)}}L_{\alpha_{\sigma(p-k)}})\dots R_{\alpha_{\sigma(p)}},$$

which in view of (3.15) is precisely (3.18). □

3.5 The Van Est isomorphism

In this section we will prove the following Van Est isomorphism theorem.

Theorem 3.5.1. *Let G be a Lie groupoid with Lie algebroid A and k-connected source fibers. The homomorphism induced in cohomology by the Van Est map:*

$$V: H^p(\hat{\Omega}^q(G_\bullet))\to H^p(W^{\bullet,q}(A)),$$

is an isomorphism for $p\le k$ and is injective for $p=k+1$.

Remark 3.5.2. When $q = 0$ one recovers the Van Est isomorphism of [20]. In view of the isomorphism (3.2), the theorem gives an isomorphism between $H^p(\Omega^q(G_\bullet))$ and $H^p(A, S^q(\mathrm{Ad}^*))$. When G is a Lie group and $A = \mathfrak{g}$ is a Lie algebra, this should be compared with the result of Bott [11] which gives an isomorphism between $H^p(\Omega^q(G_\bullet))$ and the differentiable cohomology $H^{p-q}_d(G; S^q\mathfrak{g}^*)$. In chapter 4 we will show that the result of Bott holds for arbitrary Lie groupoids.

We will divide the proof in two steps. First we will prove that there is a homomorphism in cohomology which is an isomorphism in the required degrees and then we will prove that this map is equal to the one induced by the Van Est map.

The first step is organized in the following co-augmented double complex:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \\
\Omega^q(G_2) & \xrightarrow{d_0^*} & \mathrm{W}^{0,q}(\mathcal{F}_2) & \xrightarrow{d^{\mathrm{h}}} & \mathrm{W}^{1,q}(\mathcal{F}_2) & \longrightarrow & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \\
\Omega^q(G_1) & \xrightarrow{d_0^*} & \mathrm{W}^{0,q}(\mathcal{F}_1) & \xrightarrow{d^{\mathrm{h}}} & W^{1,q}(\mathcal{F}_1) & \longrightarrow & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \\
\Omega^q(G_0) & \xrightarrow{d_0^*} & \mathrm{W}^{0,q}(\mathcal{F}_0) & \xrightarrow{d^{\mathrm{h}}} & W^{1,q}(\mathcal{F}_0) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \uparrow{\scriptstyle \delta^{\mathrm{v}}} & & \\
 & & \mathrm{W}^{0,q}(A) & \xrightarrow{d^{\mathrm{h}}} & \mathrm{W}^{1,q}(A) & \longrightarrow & \cdots
\end{array}
$$

which we now explain. First of all, G_k is the space of strings of k-composable arrows. Next, $\mathcal{F}_k$ is the foliation on G_{k+1} given by the fibers of the map $d_0 : G_{k+1} \longrightarrow G_k$. We interpret $\mathcal{F}_k$ as an integrable sub-bundle of TG_{k+1} (namely the kernel of the differential of d_0), hence also as a Lie algebroid over G_{k+1}, with the inclusion as anchor. With this, $W(\mathcal{F}_k)$ is just the associated Weil algebra, and the d^{h}'s are the corresponding horizontal differentials. We also define $\mathcal{F}_{-1} := A$.

The maps $d_0^* : \Omega^q(G_k) \longrightarrow W^{0,q}(\mathcal{F}_k) = \Omega^q(G_{k+1})$ are the pull-back by d_0. To explain δ^{v}, we view $W(\mathcal{F}_k)$ as follows. First of all, using the action of G on G_{k+1} (Example 3.2.2) and the induced infinitesimal action of A on G_{k+1}, we have already remarked that the associated Lie algebroid $A \ltimes G_{k+1}$ can be identified with $\mathcal{F}_k$ (see Example 3.2.3). Hence, by Lemma 3.3.13, we have a canonical isomorphism

$$W(\mathcal{F}_k) \cong W(A, \Omega(G_{k+1})),$$

where the left hand side is the Weil algebra with coefficients in the A-DG algebra $\Omega(G_{k+1})$ associated to the action of A on G_{k+1} (see Example 3.3.12). Since the simplicial maps $d_i : G_{k+1} \longrightarrow G_k$ are maps of G-spaces for $i \geq 1$, they induce maps

$$d_i^* : W(A, \Omega(G_k)) \longrightarrow W(A, \Omega(G_{k+1}))$$

which commute with d^{h}. We define

$$\delta^{\mathrm{v}} = \sum_{i \geq 1} (-1)^i d_i^*.$$

This completes the description of the complex. Next, we claim that the coaugmented columns of the double complex

$$0 \longrightarrow W^{p,q}(A) \xrightarrow{\delta^{\rm v}} W^{p,q}(\mathcal{F}_0) \xrightarrow{\delta^{\rm v}} W^{p,q}(\mathcal{F}_1) \xrightarrow{\delta^{\rm v}} W^{p,q}(\mathcal{F}_2) \xrightarrow{\delta^{\rm v}} \cdots$$

are exact. This is a rather standard argument. Since $\delta^{\rm v}$ comes from a simplicial structure arising from the nerve of G by deleting the first face map (d_0 is not used in the definition of $\delta^{\rm v}$), the first degeneracy map s_0 can be used to produce a contraction. More precisely, since $s_0^* : \Omega(G_k) \longrightarrow \Omega(G_{k-1})$ respects the $\Omega(M)$-module structure, induces a map

$$s_0^* : W(A, \Omega(G_k)) \longrightarrow W(A, \Omega(G_{k-1})).$$

The fact that s_0^* is a contracting homotopy follows formally from the simplicial identities:

$$\begin{aligned} s_0^*\delta^{\rm v} + \delta^{\rm v} s_0^* &= \sum_{i=1}^{j+2}(-1)^{i+1}s_0^*\delta_i^* + \sum_{i=1}^{j+1}(-1)^{i+1}\delta_i^* s_0^* \\ &= Id + \sum_{i=2}^{j+2}(-1)^{i+1}s_0^*\delta_i^* + \sum_{i=1}^{j+1}(-1)^{i+1}\delta_i^* s_0^* \\ &= Id - \sum_{i=1}^{j+1}(-1)^{i+1}\delta_i^* s_0^* + \sum_{i=1}^{j+1}(-1)^{i+1}\delta_i^* s_0^* = Id. \end{aligned}$$

This proves that the coaugmented columns of the double complex are exact. The standard homological algebra of double complexes implies that the maps induced by the coaugmentation maps are isomorphisms:

$$a : H(W^{\bullet,q}(A)) \cong H(\mathrm{Tot}(W^{\bullet,q}(\mathcal{F}_\bullet))).$$

Next, we look at the coaugmented rows

$$0 \longrightarrow \Omega^q(G_j) \xrightarrow{d_0^*} \mathrm{W}^{0,q}(\mathcal{F}_j) \longrightarrow \mathrm{W}^{1,q}(\mathcal{F}_j) \longrightarrow \cdots,$$

and we show that, under the assumption in the theorem, these are exact up to degree k. Again, homological algebra implies that the map induced by the coaugmentation of the columns

$$b : H^p(\Omega^q(G_\bullet)) \to H^p(\mathrm{Tot}(W^{\bullet,q}(\mathcal{F}_\bullet)),$$

is an isomorphism for $p < k+1$ and is injective for $p = k+1$. To prove the (partial) acyclicity of the rows, we will use the following lemma.

Lemma 3.5.3. *Let $\mathcal{F}$ be a foliation given by the fibers of a submersion with homologically k-connected fibers. Then, for $0 < p \leq k$,*

$$H^p(W^{\bullet,q}(\mathcal{F})) = 0.$$

Proof. Here we will use the interpretation in terms of representations up to homotopy given in remark 3.1. The adjoint complex of a foliation is quasi-isomorphic to the complex $\nu[1]$ which consists of the normal bundle $\nu = TM/\mathcal{F}$ concentrated in degree 1. We now use the properties of representations up to homotopy from chapter 2. We observe that the representation up to homotopy $S^q(\mathrm{Ad}^*)$ is quasi-isomorphic to the ordinary representation $\Lambda^q(\nu^*)$. Passing to cohomology and using the isomorphism (3.2), we deduce that

$$H^p(W^{\bullet,q}(\mathcal{F})) \cong H^p(\mathcal{F}, S^q(\nu^*)) = 0$$

where the last equation is a direct application of theorem 2 from [20]. □

We still have to show that the map

$$a^{-1} \circ b : H^p(\Omega^q(G_\bullet)) \to H^p(W^{\bullet,q}(A)),$$

which is an isomorphism in the desired degrees, is the same as the Van Est map. More precisely we will show that, in cohomology,

$$V = \pm a^{-1} \circ b \circ e,$$

where e is the isomorphism induced by the inclusion $\hat{\Omega}(G_\bullet) \hookrightarrow \Omega(G_\bullet)$. First we observe that the homotopy operator s_0^* for the columns of the double complex gives a formula for the map a^{-1}. Consider an element $c \in W^{j,q}(\mathcal{F}_p)$ such that $\delta^{\mathrm{v}}(c) = d^{\mathrm{h}}(c) = 0$. Then, chasing the diagram we obtain that $a^{-1}(c) = (-1)^p s_0^*(d^{\mathrm{h}} s_0^*)^p(c)$. We will use this formula to compute $a^{-1} \circ b \circ e$. Take an element $\eta \in \hat{\Omega}^q(G_p)$ such that $\delta(\eta) = 0$ and compute:

$$a^{-1} \circ b \circ e(\eta) = (-1)^p s_0^*(d^{\mathrm{h}} s_0^*)^p d_0^*(\eta) = (-1)^p (s_0^* d^{\mathrm{h}})^p(\eta).$$

We claim that for each $0 \le l \le p$:

$$(s_0^* d^{\mathrm{h}})^l(\eta)_0(\alpha_1, \dots, \alpha_l) = (-1)^{lq} \sum_{\lambda \in S_l} (-1)^{|\lambda|} R_{\alpha_{\lambda_1}} \dots R_{\alpha_{\lambda_l}}(\eta),$$

and also that $s_0^*((s_0^* d^{\mathrm{h}})^l(\eta)_0(\alpha_1, \dots, \alpha_l)) = 0$. We will prove our claim by induction on l. For $l = 0$ the claim is true because η is normalized. We assume now that the condition holds for $l - 1$ and compute:

$$\begin{aligned}
(s_0^* d^{\mathrm{h}})^l(\eta)_0(\alpha_1, \dots, \alpha_l) &= s_0^*(d^{\mathrm{h}}((s_0^* d^{\mathrm{h}})^{l-1}(\eta)(\alpha_1, \dots, \alpha_l) \\
&= s_0^*(\sum_{i=1}^{l} (-1)^{i+q+1} L_{\alpha_i}((s_0^* d^{\mathrm{h}})^{l-1}(\eta))_0(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_l)) \\
&= (-1)^q (\sum_{i=1}^{l} (-1)^{i+1} R_{\alpha_i}((s_0^* d^{\mathrm{h}})^{l-1}(\eta))_0(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_l)) \\
&= (-1)^{lq} \sum_{\lambda \in S_l} (-1)^{|\lambda|} R_{\alpha_{\lambda_1}} \dots R_{\alpha_{\lambda_l}}(\eta).
\end{aligned}$$

In particular, for $l = p$ this means that $V = \pm a^{-1} \circ b \circ e$. Since all e, b, a^{-1} are isomorphisms in the required degrees, this completes the proof.

Corollary 3.5.4. *Let G be a Lie groupoid with Lie algebroid A and k-connected source fibers. Then, the Van Est map $V : H^p(\Omega(G_\bullet)) \to H^p(\mathrm{Tot}(W(A)))$ is an isomorphism for $p \le k$ and is injective for $k = p + 1$.*

Remark 3.5.5. *The isomorphism theorem 3.5.1 is the result one would expect from a topological point of view. The map V corresponds to the projection $EG \to BG$, whose fibers are isomorphic to the fibers of the source of G. In case this map is a fibration, the Leray-Serre spectral sequence gives isomorphisms in cohomology in degrees less than the connectedness of the fiber.*

3.6 An application

In this section we discuss multiplicative forms from the point of view of the Van Est isomorphism theorem. We generalize the main result of [14] on integration of Dirac structures and give a more conceptual proof as an application of theorem 3.5.1. The first result that we discuss is the following.

Theorem 3.6.1. *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\phi \in \Omega^{k+1}(M)$ be a closed form. Then there is a one to one correspondence between:*

1. *Multiplicative forms $\omega \in \Omega^k(G)$ which are relatively closed with respect to ϕ.*
2. *$C^\infty(M)$-linear maps $\tau : \Gamma(A) \to \Omega^{k-1}(M)$ satisfying the equations:*

$$i_{\rho(\beta)}(\tau(\alpha)) = -i_{\rho(\alpha)}(\tau(\beta)), \tag{3.19}$$

$$\tau([\alpha,\beta]) = L_\alpha(\tau(\beta)) - L_\beta(\tau(\alpha)) + d_{DR}(i_{\rho(\beta)}\tau(\alpha)) + i_{\rho(\alpha)\wedge\rho(\beta)}(\phi). \tag{3.20}$$

The correspondence is given by

$$\tau(\alpha) = i_{\alpha^1}(\omega)|_M,$$

where α^1 is the right invariant vector field on G determined by α and the restriction to M makes use of the inclusion $M \hookrightarrow G$ as units.

First we show that the correspondence $\omega \mapsto \tau$ is well-defined, i.e. τ satisfies the equations (3.19) and (3.20). For that, we first remark that τ is precisely $V(\omega)_1 \in \Omega^{k-1}(M, S^1(A^*))$, viewed as a map $\Gamma(A) \longrightarrow \Omega^{k-1}$. Since $\omega + \phi$ is a cocycle in the Bott-Shulman complex (see subsection 3.2.2), it follows that $V(\omega) + \phi$ is a cocycle in the Weil algebra. The desired equations for τ will then be implied by the following:

Proposition 3.6.2. *Given $\phi \in \Omega^{k+1}(M)$, $\sigma = (\sigma_0, \sigma_1) \in W^{1,k}(A)$, $\sigma + \phi$ is a cocycle in the Weil algebra if and only if:*

1. *ϕ is a closed form and σ_1 satisfies equations (3.19), (3.20).*
2. *$\sigma_0(\alpha) = i_{\rho(\alpha)}(\phi) - d_{DR}(\sigma_1(\alpha))$.*

Proof. Let $\sigma \in W^{1,k}(A)$. That $\sigma+\phi$ is a cocycle means, first of all, that $d^{\mathrm{v}}(\sigma)+d^{\mathrm{h}}(\phi) = 0$. But

$$d^{\mathrm{v}}(\sigma)_0(\alpha) = -d_{DR}(\sigma_0(\alpha)), (d^{\mathrm{v}}\sigma)_1(|\alpha) = d_{DR}(\sigma_1(|\alpha)) + \sigma_0(\alpha),$$
$$(d^{\mathrm{h}}\phi)_0(\alpha) = L_{\rho(\alpha)}(\phi), (d^{\mathrm{h}}\phi)_1(|\alpha) = -i_{\rho(\alpha)}(\phi),$$

hence we obtain the equations:

$$\sigma_0(\alpha) = i_{\rho(\alpha)}(\phi) - d_{DR}(\sigma_1(\alpha)), d_{DR}(\sigma_0(\alpha)) = L_{\rho(\alpha)}(\phi).$$

Since the second equation is obtained by applying d_{DR} to the first one and using that ϕ is closed, we only have to keep in mind the first equation.

The other condition for $\sigma + \phi$ to be a cocycle is $d^{\mathrm{h}}(\sigma) = 0$. We write the components:

$$(d^{\mathrm{h}}\sigma)_0(\alpha,\beta) = -\sigma_0([\alpha,\beta]) + L_{\rho(\alpha)}(\sigma_0(\beta)) - L_{\rho(\beta)}(\sigma_0(\alpha)),$$

$$(d^{\mathrm{h}}\sigma)_1(\alpha|\beta) = L_{\rho(\alpha)}(\sigma_1(\beta)) - \sigma_1([\alpha,\beta]) + i_{\rho(\beta)}(\sigma_0(\alpha)),$$
$$(d^{\mathrm{h}}\sigma)_2(|\alpha) = -i_{\rho(\alpha)}(\sigma_1(\alpha)).$$

Clearly, $(d^{\mathrm{h}}\sigma)_2 = 0$ is equivalent to (3.19). Also, $(d^{\mathrm{h}}\sigma)_1 = 0$ is equivalent to

$$\sigma_1([\alpha,\beta]) = L_{\rho(\alpha)}(\sigma_1(\beta)) - L_{\rho(\beta)}(\sigma_1(\alpha)) + d_{DR}i_{\rho(\beta)}(\sigma_1(\alpha)) + i_{\rho(\beta)}i_{\rho(\alpha)}(\phi),$$

which is equivalent to (3.20). Finally, a simple computation shows that if $(d^{\mathrm{h}}\sigma)_0 = 0$ and $(d^{\mathrm{h}}\sigma)_1 = 0$ then $(d^{\mathrm{h}}\sigma)_2 = 0$. □

Next, we prove that the correspondence $\omega \mapsto \tau$ in the theorem is injective. Since $\tau = V(\omega)_1$, this part follows from the following:

Lemma 3.6.3. *If G is a Lie groupoid with connected source-fibers and $\omega \in \Omega^k(G)$ is closed and multiplicative, then the following are equivalent:*

(i) $V(\omega) = 0$.

(ii) $V(\omega)_1 = 0$.

(iii) $\omega_x = 0$ *for all* $x \in M$.

(iv) $\omega = 0$.

Proof. Since $V(\omega)$ is a cocycle in the Weil algebra, proposition 3.6.2 tells us that $V(\omega)$ is determined by $V(\omega)_1$, hence (i) and (ii) are equivalent. In turn, from the definition of $V(\omega)_1$, (ii) means that

$$\omega(\alpha_x, V_x^2, \ldots, V_x^k) = 0$$

for all $x \in M$, $V_x^k \in T_xM$, $\alpha \in \Gamma(A)$, where we identify α with the induced right invariant vector field on G. In other words, ω_x is zero when applied to one vector tangent to the s-fiber and $(k-1)$ vectors tangent to the base. We have to show that this implies ω_x is zero when applied to all vectors. But T_xG splits as the sum of the tangent space to the s-fiber and the tangent space of M (both at x), hence it remains to show that $\omega|M = 0$. But this follows immediately from $\omega|_M = s_0^* d^{\mathrm{h}}\omega$. Finally, we have to show that (iii) implies $\omega = 0$. For this we evaluate expressions of type

$$\omega_g(\alpha_g, V_g^2, \ldots, V_g^k) \tag{3.21}$$

for $g \in G$, $\alpha \in \Gamma(A)$, $V_g^i \in T_gG$ arbitrary. To make use of the multiplicativity of ω, we write

$$\alpha_g = (dm)_{y,g}(\alpha_y, 0), v_g^i = (dm)_{y,g}((dt)_g(V_g^i), V_g^i)$$

and we find that (3.21) is equal to

$$\omega(\alpha_y, (dt)_g(V_g^2), \ldots, (dt)_g(V_g^k)),$$

which, by assumption, is zero. We conclude that $i_\alpha(\omega) = 0$ for all α hence, since ω is also closed, it is basic with respect to the submersion $s : G \longrightarrow M$. Since the s-fibers are connected, we find θ on M such that $\omega = s^*\theta$. But $\omega|_M = 0$ implies $\theta = 0$ and then $\omega = 0$. □

Finally, we prove that the correspondence $\omega \mapsto \tau$ in the theorem is surjective. Note that the case $k = 2$ was proved in [14] and surjectivity was the most difficult part of the proof, while, using the Van Est map, it is simple diagram chasing. Given τ, take $\sigma \in W^{1,k}(A)$ as in proposition 3.6.2 with $\sigma_1 = \tau$. Since σ is d^{h}-closed, theorem 3.5.1 implies that there exist some multiplicative form $\omega' \in \Omega^p(G)$ such that $V(\omega') = d^{\mathrm{h}}(\theta)+\sigma$, for some $\theta \in \Omega^k(M)$. It is then clear that $\omega = \omega' - \delta(\theta)$ is a multiplicative form satisfying $V(\omega) = \sigma$. In particular, $V(\mathrm{d}\omega + \delta(\phi)) = d^{\mathrm{v}}\sigma + d^{\mathrm{h}}\phi = 0$ and $\mathrm{d}(\omega) + \delta(\phi)$ is both multiplicative and closed. Using the previous lemma, we find that ω is ϕ-relatively closed. This concludes the proof of theorem 3.6.1.

Next, we generalize a result of [21] which answers the following question. Given $\omega \in \Omega^k(G)$ multiplicative and closed, when can one write $\omega = d_{DR}\theta$ with $\theta \in \Omega^{k-1}(G)$ multiplicative?

Theorem 3.6.4. *Let G be a source simply connected Lie groupoid over M with Lie algebroid A and let $\omega \in \Omega^k(G)$ be a closed multiplicative k-form. Then there is a 1-1 correspondence between:*

1. *$\theta \in \Omega^{k-1}(G)$ multiplicative satisfying $d_{DR}(\theta) = \omega$.*
2. *$C^\infty(M)$-linear maps $l : \Gamma(A) \longrightarrow \Omega^{k-2}(M)$ satisfying*

$$i_{\rho(\beta)}(l(\alpha)) = -i_{\rho(\alpha)}(l(\beta)), \tag{3.22}$$

$$c_\omega(\alpha, \beta) = -l([\alpha, \beta]) + L_{\rho(\alpha)}(l(\beta)) - L_{\rho(\beta)}(l(\alpha)) + d_{DR}(i_{\rho(\beta)}l(\alpha)). \tag{3.23}$$

where $c_\omega = \rho^*(\omega|_M)$ ($c_\omega(\alpha, \beta) = \omega(\rho(\alpha), \rho(\beta))$). The correspondence is given by

$$l(\alpha) = -i_\alpha(\theta)|_M.$$

Remark 3.6.5. The case $k = 2$ is answered in [21] and involves the De Rham complex of A i.e. $\Omega(A) = \Gamma(\Lambda A^*)$ with the Chevalley-Eilenberg differential δ. That is possible because, in that case, $\Omega^{k-2}(M) = C^\infty(M)$, so that (3.22) is void while (3.22) simply becomes $c_\omega = \delta(l)$. The outcome is that, if $\omega \in \Omega^2(G)$ is multiplicative and closed then $c_\omega \in \Omega^2(A)$ is a cocycle, and there is a 1-1 correspondence between $\theta \in \Omega^1(G)$ multiplicative such that $d_{DR}\theta = \omega$ and $l \in \Omega^1(A)$ satisfying $\delta(l) = \omega$.

We now discuss the proof in general. Let $\sigma = V(\omega)$ and let us first look at solutions $\xi \in W^{1,k-1}(A)$ of the equations:

$$d^{\mathrm{v}}(\xi) = \sigma, d^{\mathrm{h}}(\xi) = 0. \tag{3.24}$$

We use the same formulas as in the proof of proposition 3.6.2 (but applied to ξ instead of σ) to write out explicitly the equations. For $d^{\mathrm{v}}(\xi) = \sigma$ we find

$$-d_{DR}(\xi_0(\alpha)) = \sigma_0(\alpha), d_{DR}(\xi_1(\alpha)) + \xi_0(\alpha) = \sigma_1(\alpha)$$

where, as in the proof of proposition 3.6.2, we only have to remember the second one. In other words, $d^{\mathrm{v}}(\xi) = \sigma$ tells us that ξ_0 is determined by ξ_1:

$$\xi_0(\alpha) = \sigma_1(\alpha) - d_{DR}(\xi_1(\alpha)). \tag{3.25}$$

the condition $d^{\mathrm{h}}(\xi) = 0$ gives three equations, corresponding to the three components. For $(d^{\mathrm{h}}(\xi))_2 = 0$ we find that ξ_1 must satisfy the anti-symmetry condition (3.22). For $(d^{\mathrm{h}}(\xi))_1 = 0$ we find:

$$\xi_1([\alpha, \beta]) = L_{\rho(\alpha)}(\xi_1(\beta)) + i_{\rho(\beta)}(\xi_0(\alpha)).$$

Using the formula for ξ_0 in terms of ξ_1, we find that ξ_1 must satisfy (3.23).

Next, if θ is as in the theorem, we have $\delta(\theta) = 0$ and $\mathrm{d}(\theta) = -\omega$, i.e. $\xi := -V(\theta)$ must satisfy (3.24). The previous discussion shows that $l = \xi_1$ must satisfy the equations above. From the definition of the Van Est map it follows that $l(\alpha) = -J_\alpha(\theta) = -i_\alpha(\theta)|_M$.

Assume now that l satisfies the equations from the statement. Let $\xi \in W^{1,k-1}(A)$ with $\xi_1 = l$ and ξ_0 defined by (3.25), so that ξ satisfies (3.24). Using the Van Est isomorphism, we find $\theta' \in \Omega^{k-1}(G)$ multiplicative and $\eta \in \Omega^{k-1}(M)$ such that $V(\theta') = -\xi + d^{\mathrm{h}}(\eta)$. Define $\theta = \theta' - \delta(\eta)$. Then

$$V(d_{DR}(\theta) - \omega) = -d^{\mathrm{v}}(V(\theta)) - V(\omega) = -d^{\mathrm{v}}(-\xi + d^{\mathrm{h}}\eta - V(d^{\mathrm{h}}\eta)) - \sigma = d^{\mathrm{v}}\xi - \sigma = 0.$$

On the other hand, $d_{DR}(\theta) - \omega$ is both multiplicative and closed and therefore lemma 3.6.3 implies that $\omega = d_{DR}(\theta)$. By construction, the l corresponding to θ is the l we started with, concluding the proof of the surjectivity. For the injectivity, one proceeds exactly as in the proof of theorem 3.6.1. If θ and θ' have the same associated l and transgress ω then $\theta - \theta'$ will be multiplicative and closed with $V(\theta - \theta')_1 = l - l = 0$. In this case lemma 3.6.3 implies that $\theta = \theta'$.

3.7 Appendix: Kalkman's BRST algebra in the infinite dimensional case

In this chapter we need some constructions which, although standard in the finite dimensional case, need some clarification in the infinite dimensional setting. We will use this appendix to give the details of these constructions and to we fix our notations. In particular, we will give a description of Kalkman's BRST complex which applies also to infinite dimensional Lie algebras and more general coefficients.

Chevalley-Eilenberg complex: For a representation V of a Lie algebra $\mathfrak{g}$, the action $\mathfrak{g} \otimes V \longrightarrow V$ is denoted by $(\alpha, v) \mapsto L_\alpha(v)$. The Chevalley-Eilenberg complex with coefficients in V is

$$\Lambda(\mathfrak{g}^*, V) := \mathrm{Hom}_{\mathbb{R}}(\Lambda\mathfrak{g}, V),$$

where "$\mathrm{Hom}_{\mathbb{R}}$" stands for the space of $\mathbb{R}$-linear maps. In degree k, $\Lambda^k(\mathfrak{g}^*, V)$ consists of antisymmetric multilinear maps depending on k-variables from $\mathfrak{g}$, with values in V. The Chevalley-Eilenberg differential,

$$\delta : \Lambda^p(\mathfrak{g}^*, V) \longrightarrow \Lambda^{p+1}(\mathfrak{g}^*, V)$$

is given by the Koszul formula:

$$\begin{aligned}(\delta(c))(\alpha_1, \ldots, \alpha_{p+1}) &= \sum_{i<j} (-1)^{i+j} c([\alpha_i, \alpha_j], \ldots, \hat{\alpha_i}, \ldots, \hat{\alpha_j}, \ldots, \alpha_{p+1}) + \\ &+ \sum_i (-1)^{i+1} L_{\rho(\alpha_i)}(c(\alpha_1, \ldots, \hat{\alpha_i}, \ldots, \alpha_{p+1})).\end{aligned}$$

Symmetric powers: We now specify our conventions and notations regarding symmetric powers. For any two vector spaces E and V, the space of V-valued polynomials on E is

$$S(E^*, V) := \mathrm{Hom}_{\mathbb{R}}(SE, V).$$

where "$\mathrm{Hom}_{\mathbb{R}}$" stands for the space of $\mathbb{R}$-linear maps. A polynomial of degree k will be viewed either as a symmetric k-multilinear map

$$P : \underbrace{E \times \ldots \times E}_{k \text{ times}} \longrightarrow V$$

or as an actual function on E with values in V, $P(\alpha) = P(\alpha, \ldots, \alpha)$. We will also use the following operation. For $\alpha \in E$, there is the induced partial derivative

$$\partial_\alpha : S^k(E^*, V) \longrightarrow S^{k-1}(E^*, V),$$

$$\partial_\alpha(P)(\alpha_0) := \frac{d}{dt}|_{t=0} P(\alpha_0 + t\alpha),$$

where the derivative should be interpreted formally. In the multilinear notation, this becomes

$$\partial_\alpha(P)(\alpha_0) = kP(\alpha, \alpha_0, \ldots, \alpha_0).$$

Some representations: For any representation V of $\mathfrak{g}$, the vector space $S(\mathfrak{g}^*, V)$ is itself a representation in a canonical way. The action

$$\mathfrak{g} \otimes S(\mathfrak{g}^*, V) \longrightarrow S(\mathfrak{g}^*, V), \;\; (\alpha, P) \mapsto L_\alpha(P)$$

is induced from the coadjoint action on $\mathfrak{g}^*$ and the given action on V. These, together with the Leibniz identity for L_α, determine the action uniquely in the finite dimensional case. We take the resulting explicit formula as the definition in the general case:

$$L_\alpha(P)(\alpha_1, \ldots, \alpha_p) = L_\alpha(P(\alpha_1, \ldots, \alpha_p)) - \sum_i P(\alpha_1, \ldots, [\alpha, \alpha_i], \ldots, \alpha_p).$$

A similar representation arises in the case when

$$\mathfrak{g} = \Gamma(A)$$

is the Lie algebra of sections of a Lie algebroid A over M. It is the algebra $\Omega(M, SA^*)$ of forms on M with values in the symmetric algebra of A. The action $(\alpha, \omega) \mapsto L_\alpha(\omega)$ is uniquely determined by the following conditions:

- The Leibniz derivation identity: for all $\omega, \omega' \in \Omega(M, SA^*)$,

$$L_\alpha(\omega\omega') = L_\alpha(\omega)\omega' + \omega L_\alpha(\omega').$$

- On $\Omega(M)$, L_α coincides with the usual Lie derivative $L_{\rho(\alpha)}$ along the vector field $\rho(\alpha)$.
- For ξ in $\Gamma(A^*)$, L_α is given by $L_\alpha(\xi)(\beta) = L_\alpha(\xi(\beta)) - \xi(L_\alpha(\beta))$.

Actually, $\Omega(M, SA^*)$ is just a sub-representation of $S(\mathfrak{g}^*; V)$ with $V = \Omega(M)$.

The Weil algebra with coefficients: Assume now that $\mathfrak{g}$ is a Lie algebra and $\mathcal{A}$ is an $\mathfrak{g}$-DG algebra. We define

$$W(\mathfrak{g}, \mathcal{A}) := \Lambda(\mathfrak{g}^*, S(\mathfrak{g}^*, \mathcal{A})),$$

with the following bidegree:

$$W^{p,q}(\mathfrak{g}, \mathcal{A}) := \bigoplus_k \Lambda^{p-k}(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^{q-k})).$$

For an element $c \in \Lambda^{p-k}(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^{q-k}))$ we use the notation

$$c(\alpha_1, \ldots, \alpha_{p-k}|\alpha) := c(\alpha_1, \ldots, \alpha_{p-k})(\alpha) \in \mathcal{A}^{q-k},$$

which is an expression multilinear antisymmetric on the first entries and polynomial in the last one. $W(\mathfrak{g}; \mathcal{A})$ has a product structure compatible with the bi-grading, for

$$c \in \Lambda^p(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^q)), c' \in \Lambda^{p'}(\mathfrak{g}^*, S^{k'}(\mathfrak{g}^*, \mathcal{A}^{q'})),$$

cc' is given by

$$(cc')(\alpha_1, \ldots, \alpha_{p+p'}|\alpha) = (-1)^{qp'} \sum \mathrm{sgn}(\sigma) c(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}|\alpha) c'(\alpha_{\sigma(p+1)}, \ldots, \alpha_{\sigma(p+p')}|\alpha),$$

where the sum is over all (p, p')-shuffles. The sign in front of the sum comes from the finite dimensional case and the standard sign conventions, in $W(\mathfrak{g}) \otimes \mathcal{A}$,

$$(w \otimes a)(w' \otimes a') = (-1)^{\deg(a)\deg(w')}(ww' \otimes aa').$$

Kalkman's BRST differentials: As in the case of the standard Weil algebra, there are two differentials. The first one, d^{h}, increases p and is given by

$$d^{\mathrm{h}}(c) = \delta(c) + i_{\mathcal{A}}(c).$$

Here δ is the Koszul differential, while

$$i_{\mathcal{A}} : \Lambda^p(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^q)) \longrightarrow \Lambda^p(\mathfrak{g}^*, S^{k+1}(\mathfrak{g}^*, \mathcal{A}^{q-1}))$$

is given by

$$i_{\mathcal{A}}(c)(\alpha_1, \ldots, \alpha_p|\alpha) = (-1)^{p+1} i_{\alpha}(c(\alpha_1, \ldots, \alpha_p|\alpha)).$$

Both δ and $i_{\mathcal{A}}$ are derivations (and that motivates the sign in $i_{\mathcal{A}}$).

The second differential, d^{v}, increases q and is given by

$$d^{\mathrm{v}}(c) = d_{\mathcal{A}}(c) + i_{\mathfrak{g}}(c).$$

Here $d_{\mathcal{A}}$ is given by

$$d_{\mathcal{A}}(c)(\alpha_1, \ldots, \alpha_p|\alpha) = (-1)^p d_{\mathcal{A}}(c(\alpha_1, \ldots, \alpha_p|\alpha)),$$

while

$$i_{\mathfrak{g}} : \Lambda^p(\mathfrak{g}^*, S^k(\mathfrak{g}^*, \mathcal{A}^q)) \longrightarrow \Lambda^{p-1}(\mathfrak{g}^*, S^{k+1}(\mathfrak{g}^*, \mathcal{A}^q))$$

is given by
$$i_{\mathfrak{g}}(c)(\alpha_1,\dots,\alpha_{p-1}|\alpha) = (-1)^{p+1}c(\alpha_1,\dots,\alpha_{p-1},\alpha|\alpha).$$
Again, both $d_{\mathcal{A}}$ and $i_{\mathfrak{g}}$ are derivations (and this motivates the sign in $d_{\mathcal{A}}$).

In the case that $\mathfrak{g}$ is finite dimensional, $W(\mathfrak{g},\mathcal{A}) = W(\mathfrak{g}) \otimes \mathcal{A}$ and we can use a basis e^a of $\mathfrak{g}$ to write the formulas more explicitly. We denote by θ^a the induced basis of $\Lambda^1\mathfrak{g}^*$, by μ^a the induced basis of $S^1\mathfrak{g}^*$ and by d_W the differential of $W(\mathfrak{g})$. From the derivation property of all the operators $\delta, i_{\mathcal{A}}$, $d_{\mathcal{A}}$, $i_{\mathfrak{g}}$ and after a straightforward checking on generators, we deduce that
$$\delta = d_W^{\mathrm{h}} \otimes 1 + \theta^a \otimes L_{e^a},\ \ i_{\mathcal{A}} = -\mu^a \otimes i_{e^a},$$
$$d_{\mathcal{A}} = 1 \otimes d_{\mathcal{A}},\ \ i_{\mathfrak{g}} = d_W^{\mathrm{v}} \otimes 1.$$
Hence, in this case, $d^{\mathrm{h}} + d^{\mathrm{v}}$ coincides with Kalkman's differential on $W(\mathfrak{g}) \otimes \mathcal{A}$.

4

Trees and tensor products

In order to define the tensor product of representations up to homotopy of groups, we introduce a differential graded algebra $\mathbf{\Omega}$*, generated by a family of planar trees. We construct diagonal maps* $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$*, which provide formulas for the tensor product and symmetric powers of representations up to homotopy.*

4.1 Introduction

In the previous chapters we explained the extent to which the relation between the representations of a Lie algebra $\mathfrak{g}$ and the cohomology of BG generalizes to the case of Lie algebroids in the context of -infinitesimal- representations up to homotopy. There is also a very strong relation between the representations of a Lie group G and the cohomology of BG. In [11], Bott constructed a spectral sequence

$$E_1^{pq} = \mathrm{H}^{p-q}(G, S^q(\mathfrak{g}^*)) \Rightarrow \mathrm{H}^{p+q}(BG),$$

whose E^1 term is the cohomology of G with coefficients in the symmetric powers of the coadjoint representation. The generalization of this spectral sequence to the case of arbitrary Lie groupoids and its relation with the models of equivariant cohomology is the subject of chapter 5. In the case of Lie groupoids, there is a combinatorial aspect to this construction which is not present for groups. In general, the spectral sequence only exists when one allows representations up to homotopy and it is necessary to consider the symmetric powers of this kind of representations. This is a combinatorial problem of the same nature as that of taking tensor products of A_∞-algebras [33].

The main purpose of this chapter is to produce formulas for the construction of symmetric powers of representations up to homotopy. Even though we are interested in representations of groupoids, the discussion in this chapter is limited to groups. Working with groupoids would only overload the notation and is completely irrelevant for the combinatorics involved. The general idea of our construction is as follows. We introduce a differential graded algebra $\mathbf{\Omega}$, generated by planar trees, which is universal for representations up to homotopy in some precise sense (theorem 4.4.1). Then, solving a cohomological recurrence, we construct diagonal maps $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$, which provide formulas for the tensor products and symmetric powers of representations up to homotopy. The diagonal maps are not unique and we consider here one of many possible choices. It would be interesting to know whether the tensor products depend up to isomorphism on the choice of the diagonal maps. However, we do not address this question here. Let us now be more specific about the contents of this chapter.

In section 4.2 we introduce the notion of a representation up to homotopy of a group. This is done in complete analogy with the infinitesimal case explained in the previous chapters. We compute (proposition 4.2.1) some explicit formulas which explain the structure of representations up to homotopy. Section 4.3 is a rather detailed study of the algebra $\mathbf{\Omega}$. We prove that its cohomology is concentrated in degree zero (proposition 4.3.2). Indeed, we show that the algebra $\mathbf{\Omega}$ is the direct sum of complexes that compute the homology of the cubes with respect to the natural cell decomposition (remark 4.3.2). We use the computation of the cohomology to construct diagonal maps $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$ (theorem 4.3.1). In section 4.4 we explain the relation between the algebra $\mathbf{\Omega}$ and representations up to homotopy (theorem 4.4.1). We then show that the diagonal maps $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$ provide formulas for the tensor products and the symmetric powers (theorems 4.4.2 and 4.4.3).

4.2 Representations up to homotopy of groups

4.2.1 Representations up to homotopy

Here we introduce the notion of a representation up to homotopy of a group. Throughout this chapter G denotes a group. We will start by recalling some well known facts about ordinary representations.

Definition 4.2.1. *Given a vector space E, the space of E-valued cochains of G, denoted $C^\bullet(G,E)$, is the graded vector space whose degree k part consists of functions:*

$$C^k(G,E) = \{\eta : \underbrace{G\times\cdots\times G}_{k-times} \to E\}.$$

A representation $\lambda : G \to GL(E)$, induces a differential D_λ on the graded vector space $C^\bullet(G,E)$, given by the formulas:

$$D_\lambda(\eta)(g_1,\dots,g_{k+1}) = \lambda_{g_1} d_0^*(\eta)(g_1,\dots,g_{k+1}) + \sum_{i=1}^{k+1}(-1)^i {d_i}^*(\eta)(g_1,\dots,g_{k+1}), \quad (4.1)$$

where the map $d_i : G_{k+1} \to G_k$ is defined by:

$$d_i(g_1,\dots,g_{k+1}) = \begin{cases} (g_2,\dots,g_{k+1}) & \text{if } i=0,\\ (g_1,\dots,g_ig_{i+1},\dots g_{k+1}) & \text{if } 0<i<k,\\ (g_1,\dots,g_k) & \text{if } i=k+1. \end{cases}$$

The cohomology of the resulting complex is called the cohomology of G with coefficient E and we denote it by $H^\bullet(G,E)$. When there is no risk of confusion we will write D instead of D_λ.

For the trivial representation of G in $\mathbb{R}$ we denote the complex simply by $C^\bullet(G)$, the cohomology by $H^\bullet(G)$ and the differential by δ. The space $C^\bullet(G)$ has an algebra structure given by:

$$(f*h)(g_1,\dots,g_{k+p}) = f(g_1,\dots,g_k)h(g_{k+1},\dots,g_{k+p}),$$

for $f\in C^k(G)$ and $h \in C^p(G)$. We will use the symbol $*$ to distinguish this product from the pointwise product defined when $k=p$. The differential δ is a (graded) derivation with respect to the algebra structure, namely, it satisfies:

$$\delta(f*h) = \delta(f)*h + (-1)^k f*\delta(h).$$

The space $C^\bullet(G,E)$ has the structure of a right $C^\bullet(G)$-module. Given $\eta\in C^p(G,E)$ and $f\in C^k(G)$ their product $\eta * f \in C^{p+k}(G,E)$ is defined by:

$$(\eta * f)(g_1,\dots,g_{k+p}) = \eta(g_1,\dots,g_p)f(g_{p+1},\dots,g_{k+p}).$$

For any representation $\lambda : G \to GL(E)$, the operator D_λ is a graded derivation with respect to this module structure:

$$D(\eta * f) = D(\eta)*f + (-1)^p \eta * \delta(f).$$

In fact, there is a precise correspondence between derivations of $C^\bullet(G,E)$ and representations of G on E.

Definition 4.2.2. *A nonassociative action λ of G on E is a function $\lambda : G \to GL(E)$.*

Lemma 4.2.1. *There is a bijective correspondence between nonassociative actions of G on the vector space E and degree one linear operators:*

$$D : C^\bullet(G, E) \to C^{\bullet+1}(G, E),$$

which are graded derivations with respect to the $C^\bullet(G)$-module structure. The correspondence is given by the relation:

$$\lambda_g(v) = v + D(v)(g).$$

A nonassociative action λ is a representation if and only if $\lambda_1(v) = v$, and the corresponding operator D_λ squares to zero.

Proof. A straightforward computation shows that for any nonassociative action λ, the operator D_λ defined by formula (4.1) is a derivation with respect to the $C^\bullet(G)$-module structure. This shows that the correspondence is well defined. Injectivity follows from the fact that, as a $C^\bullet(G)$-module, $C^\bullet(G, E)$ is generated by $C^0(G, E)$. Surjectivity is also easy, given a derivation D one can recover a nonassociative action by setting:

$$\lambda_g(v) = v + D(v)(g).$$

Finally, we need to prove that λ is associative if and only if $D_\lambda^2 = 0$. Suppose that λ is a representation. Since D_λ is a derivation, D_λ^2 satisfies:

$$D_\lambda^2(\eta * f) = D_\lambda^2(\eta) * f.$$

Thus, it is enough to show that $D_\lambda^2(v) = 0$ for $v \in C^0(G, E) = E$. Now we compute:

$$\begin{aligned} D_\lambda^2(v)(g_1, g_2) &= \lambda_{g_1}(D_\lambda(v)(g_2)) - D_\lambda(v)(g_1 g_2) + D_\lambda(v)(g_1) \\ &= \lambda_{g_1}(\lambda_{g_2}(v) - v) - \lambda_{g_1 g_2}(v) + v + \lambda_{g_1}(v) - v \\ &= \lambda_{g_1}\lambda_{g_2}(v) - \lambda_{g_1 g_2}(v) = 0. \end{aligned}$$

The previous computation also shows that if $D_\lambda^2 = 0$ then λ is a representation. □

Next, we will extend the previous discussion to the graded setting.

Definition 4.2.3. *Given a graded vector space $E = \bigoplus_{i \in \mathbb{Z}} E^i$, we will denote by $C(G, E)$ the space of E-valued cochains:*

$$C(G, E) = \bigoplus_{i,j} C^i(G, E^j).$$

The total degree of an element $\eta \in C^i(G, E^j)$ is $|\eta| = i + j$. There is a decomposition with respect to the total degree $C(G, E) = \bigoplus_p C(G, E)^p$ where:

$$C(G, E)^p = \bigoplus_{i+j=p} C^i(G, E^j).$$

The cocycle degree of η is $\langle \eta \rangle = i$ and there is a decomposition $C^p(G, E) = \bigoplus_p C^p(G, E)$ where:

$$C^p(G, E) = \bigoplus_{i \in \mathbb{Z}} C^p(G, E^i).$$

As we have seen before, representations of a group on a vector space correspond to derivations on the space of cochains which square to zero. With this in mind, we define a representation up to homotopy as the corresponding structure in the graded setting

Definition 4.2.4. *A representation up to homotopy of G on a graded vector space E is a linear degree one operator $D : C(G,E)^\bullet \to C(G,E)^\bullet$, such that:*

(a) $D \circ D = 0$,

(b) $D(\eta * f) = D(\eta)f + (-1)^k \eta * \delta(f)$ *for* $\eta \in C(G,E)^k$ *and* $f \in C^\bullet(G)$.

The cohomology computed with respect to this operator will be denoted by $H^\bullet(G,E)$.

Since the space $C(G,E)$ is generated as a $C^\bullet(G)$-module by $C^0(G,E) = E$, the fact that the operator D is a derivation with respect to the $C^\bullet(G)$-module structure implies that it is determined by what it does on E. One can decompose the operator D with respect to the cocycle grading:

$$D = D_0 + D_1 + D_2 + \dots, \tag{4.2}$$

where each D_i raises the cocycle degree by i,

$$D_i : C^k(G, E^p) \to C^{k+i}(G, E^{p-i+1}).$$

Note that, since D is a derivation, the operators in the decomposition above do not lower the cocycle degree. We obtain the following decomposition for representations up to homotopy.

Lemma 4.2.2. *Any representation up to homotopy D of G on the graded vector space E decomposes as:*

$$D = D_0 + D_1 + D_2 + \cdots,$$

where the operator D_i increases the cocycle degree by i. For $i \neq 1$ the operator D_i satisfies:

$$D_i(\eta * f) = D_i(\eta) * f,$$

while:

$$D_1(\eta * f) = D_1(\eta) * f + (-1)^{|\eta|} \eta * \delta(f).$$

For each $k > 0$ the operators satisfy the equation:

$$D_0 D_k + \cdots + D_k D_0 = 0. \tag{4.3}$$

Proof. By looking at the cocycle degree we obtain the decomposition $D = D_0 + D_1 + \cdots$. Now we compute:

$$\begin{aligned}
\sum_{i \geq 0} D_i(\eta * f) &= D(\eta * f) \\
&= D(\eta) * f + (-1)^{|\eta|} \eta * \delta f \\
&= \sum_{i \geq 0} D_i(\eta) * f + (-1)^{|\eta|} \eta * \delta f.
\end{aligned}$$

Collecting the homogeneous elements we conclude that D_1 is a derivation while the other operators are $C^\bullet(G)$-module maps. Equations (4.3) follow by looking at the homogeneous components of:

$$D \circ D = (D_0 + D_1 + D_2 + \cdots) \circ (D_0 + D_1 + D_2 + \cdots) = 0.$$

□

Remark 4.2.1. *There is a one to one correspondence between:*

1. *Cochains* $\Phi \in C^k(G, \mathrm{End}^p(E))$.
2. *Maps* $\hat{\Phi} : C^i(G, E^j) \to C^{i+k}(G, E^{j+p})$ *such that* $\hat{\Phi}(\eta * f) = \hat{\Phi}(\eta) * f$.

The correspondence is given by $\Phi \mapsto \hat{\Phi}$ *where:*

$$\hat{\Phi}(\eta)(g_1, \dots, g_{k+i}) = (-1)^{(|\eta| + \langle \eta \rangle)} \Phi(g_1, \dots, g_k)(\eta(g_{k+1}, \dots, g_{k+i})).$$

The correspondence is bijective because one can obviously recover Φ *from* $\hat{\Phi}$.

Now we can give a different characterization of a representation up to homotopy.

Proposition 4.2.1. *There is a one to one correspondence between representations up to homotopy of G on the graded vector space* E *and sets of data as follows:*

1. *A differential* d_E *giving* E *the structure of a cochain complex.*
2. *A nonassociative action* F_1 *on each* E^k.
3. *For each* $i > 1$, *a cochain* $F_i \in C^i(G, \mathrm{End}^{1-i}(E))$.

Subject to the equations:

$$\begin{aligned} [d_E, F_k](g_1, \dots, g_k) &= \sum_{j=1}^{k-1} (-1)^{j+1} F_j(g_1, \dots, g_j) \circ F_{k-j}(g_{j+1}, \dots, g_k) \\ &+ \sum_{i=1}^{k-1} (-1)^j F_{k-1}(g_1, \dots, g_i g_{i+1}, \dots, g_k). \end{aligned}$$

Proof. We use the decomposition $D = D_0 + D_1 + D_2 + \dots$. For $i \neq 1$ define F_i to be the element in $C^i(G, \mathrm{End}^{1-i}(E))$ corresponding to D_i and set $d_E = F_0$. If one forgets the degree of E^p, the operator $(-1)^p D_1 : C^\bullet(G, E^p) \to C^{\bullet+1}(G, E^p)$ satisfies the equation:

$$(-1)^p D_1(\eta * f) = (-1)^p D_1(\eta) * f + (-1)^{\langle \eta \rangle} \eta * \delta(f),$$

and therefore corresponds to a nonassociative action F_1 on E^p. Finally, a simple computation shows that the equation:

$$D_k D_0 + \dots + D_0 D_k = 0,$$

holds if and only if:

$$\begin{aligned} [d_E, F_k](g_1, \dots, g_k) &= \sum_{j=1}^{k-1} (-1)^{j+1} F_j(g_1, \dots, g_j) \circ F_{k-j}(g_{j+1}, \dots, g_k) \\ &+ \sum_{j=1}^{k-1} (-1)^j F_{k-1}(g_1, \dots, g_i g_{i+1}, \dots, g_k). \end{aligned}$$

□

Since the operator d_E in the decomposition above does not depend on G, we will take the point of view that the group G is represented in the complex of vector bundles (E, d_E).

Definition 4.2.5. *We will say that a representation up to homotopy is unital if* $F_1(1) = Id$.

4.3 The algebra Ω and the diagonal maps

4.3.1 Trees

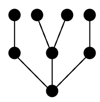
In the sequel, $\mathbf{T}$ and $\mathbf{F}$ will stand for the set of isomorphism classes of planar rooted trees and forests, respectively. Note that $\mathbf{T} \subset \mathbf{F}$. In our pictures, we will join the roots of a forest by a line to emphasize that the trees in a forrest do not commute.

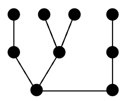
Definition 4.3.1. *Let F be a forest in $\mathbf{F}$. The height of a leaf l in F is the minimum number of edges needed to join l with a root. The set of short forests, denoted by $\mathbf{S}$, is the set of forest all of whose leaves have height 2. A branch of a forest $F \in \mathbf{S}$ is an edge that goes from a root to a vertex which is not a root. We will use the following conventions:*

$$\begin{aligned} l_F &= \#\text{ of leaves of F.} \\ b_F &= \#\text{ of branches of F.} \\ r_F &= \#\text{ of roots of } F. \end{aligned}$$

It is obvious that $l_F \geq b_F \geq r_F$.

Example 4.3.1. *This illustrates the conventions above.*

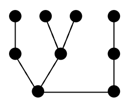
For $F =$  , $l_F = 4,\ b_F = 3,\ r_F = 1.$

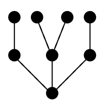
For $F =$  , $l_F = 4,\ b_F = 3,\ r_F = 2.$

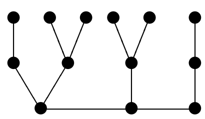
Remark 4.3.1. *The set $\mathbf{S}$ of short forests can be canonically identified with the set of expressions of the form:*

$$[t_1^1, \dots, t_{n_1}^1] \dots [t_1^k, \dots, t_{n_k}^k],$$

where each t_ν^μ is a nonzero natural number. We interpret each bracket as a tree. The number of entries inside the bracket is the number of branches and the entries specify the number of leaves in each branch, for instance:

 $= [1, 2][1],$

 $= [1, 2, 1],$

 $= [1, 2][2][1].$

4.3.2 The algebra Ω

We will construct a differential graded algebra $\mathbf{\Omega}$ whose algebraic structure provides formulas for the tensor product and symmetric powers of representations up to homotopy.

Definition 4.3.2. *As a vector space, $\mathbf{\Omega}$ is the vector space over $\mathbb{Q}$ generated by $\mathbf{S}$. $\mathbf{\Omega}$ has an algebra structure, it is the free algebra generated by the trees in $\mathbf{S}$. We will denote by $\circ$ the multiplication in $\mathbf{\Omega}$. The degree $\overline{F}$ of an element $F \in \mathbf{S}$ equals the number of roots minus the number of branches:*

$$\overline{F} = r_F - b_F.$$

The order $\langle F \rangle$ of F equals the number of leaves.

$$\langle F \rangle = l_F.$$

Clearly:

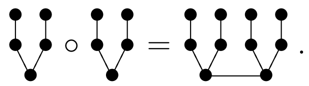
$$\overline{F \circ F'} = \overline{F} + \overline{F'}.$$

and

$$\langle F \circ F' \rangle = \langle F \rangle + \langle F' \rangle.$$

We denote by $\mathbf{\Omega}^p$ the subspace generated by forests of degree p and by $\mathbf{\Omega}(m)$ the subspace generated by forests of order m. Also, $\mathbf{\Omega}^p(m) = \mathbf{\Omega}^p \cap \mathbf{\Omega}(m)$.

Example 4.3.2. *This illustrates the algebra structure in $\mathbf{\Omega}$.*



Definition 4.3.3. *The operators $\delta_\star, \delta_\Delta : \mathbf{\Omega}^p \to \mathbf{\Omega}^{p+1}$ are defined on generators of the algebra $\mathbf{\Omega}$ by the formulas:*

$$\delta_\Delta[t_1, \ldots, t_k] := \sum_{i=1}^{k-1} (-1)^{i+1} [t_1, \ldots, t_i] \circ [t_{i+1}, \ldots, t_k],$$

and

$$\delta_\star[t_1, \ldots, t_k] := \sum_{i=1}^{k-1} (-1)^{i} [t_1, \ldots, t_i + t_{i+1}, \ldots, t_k].$$

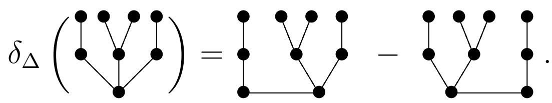
They extend uniquely as graded derivations on the algebra $\mathbf{\Omega}$, namely satisfying:

$$\delta_\Delta(F \circ F') = \delta_\Delta(F) \circ F' + (-1)^{\overline{F}} F \circ \delta_\Delta(F'),$$

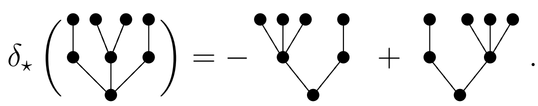
and

$$\delta_*(F \circ F') = \delta_*(F) \circ F' + (-1)^{\overline{F}} F \circ \delta_*(F').$$

The operators δ_Δ and $\delta_\star$ have a simple graphic description. The operator δ_Δ separates the branches with alternating signs. For instance:



The operator $\delta_\star$ collapses the branches with alternating signs, as in the example:



Lemma 4.3.1. *$\delta_\triangle$ and $\delta_\star$ have degree one and satisfy the equations:*

$$\delta_\triangle\delta_\triangle = \delta_\star\delta_\star = 0, \quad \delta_\triangle\delta_\star + \delta_\triangle\delta_\star = 0.$$

Proof. Since the operators $\delta_\triangle$ and $\delta_\star$ are derivations in the algebra $\mathbf{\Omega}$, it is enough to prove the equations evaluated on a generator $[t_1, \ldots, t_k] \in \mathbf{\Omega}$. For this we compute:

$$\begin{aligned}
\delta_\triangle\delta_\triangle([t_1,\ldots,t_k]) &= \sum_{i=1}^{k-1}(-1)^{i+1}\delta_\triangle([t_1,\ldots,t_i]\circ[t_{i+1},\ldots,t_k]) \\
&= \sum_{i=1}^{k-1}(-1)^{i+1}\delta_\triangle([t_1,\ldots,t_i])\circ[t_{i+1},\ldots,t_k] \\
&\quad + \sum_{i=1}^{k-1}[t_1,\ldots,t_i]\circ\delta_\triangle([t_{i+1},\ldots,t_k]) \\
&= \sum_{i>j}(-1)^{i+j}[t_1,\ldots,t_j]\circ[t_{j+1},\cdots,t_i]\circ[t_{i+1},\ldots,t_k] \\
&\quad + \sum_{i<j}(-1)^{i+j+1}[t_1,\ldots,t_i]\circ[t_{i+1},\cdots,t_j]\circ[t_{j+1},\ldots,t_k] = 0.
\end{aligned}$$

Next we have:

$$\begin{aligned}
\delta_\star\delta_\star([t_1,\ldots,t_k]) &= \sum_{i=1}^{k-1}(-1)^{i}\delta_\star[t_1,\ldots,t_i+t_{i+1},\ldots,t_k] \\
&= \sum_{j<i}(-1)^{i+j}[t_1,\ldots,t_j+t_{j+1},\ldots,t_i+t_{i+1},\ldots,t_k] \\
&\quad + \sum_{j>i}(-1)^{i+j+1}[t_1,\ldots,t_i+t_{i+1},\ldots,t_j+t_{j+1},\ldots,t_k] = 0.
\end{aligned}$$

Then, we compute:

$$\begin{aligned}
\delta_\star\delta_\triangle([t_1,\ldots,t_k]) &= \sum_{i=1}^{k-1}(-1)^{i+1}\delta_\star([t_1,\ldots,t_i]\circ[t_{i+1},\ldots,t_k]) \\
&= \sum_{i=1}^{k-1}(-1)^{i+1}\delta_\star([t_1,\ldots,t_i])\circ[t_{i+1},\ldots,t_k] \\
&\quad + \sum_{i=1}^{k-1}[t_1,\ldots,t_i]\circ\delta_\star([t_{i+1},\ldots,t_k]) \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{i-1}(-1)^{i+j+1}[t_1,\ldots,t_j+t_{j+1},\ldots,t_i]\circ[t_{i+1},\ldots,t_k] \\
&\quad + \sum_{i=1}^{k-1}\sum_{j=i+1}^{k-1}(-1)^{i+j}[t_1,\ldots,t_i]\circ[t_{i+1},\ldots,t_j+t_{j+1},\ldots,t_k].
\end{aligned}$$

Finally,

$$\begin{aligned}
\delta_\Delta\delta_\star([t_1,\ldots,t_k]) &= \sum_{i=1}^{k-1}(-1)^i\delta_\Delta[t_1,\ldots,t_i+t_{i+1},\ldots,t_k] \\
&= \sum_{i=1}^{k-1}\sum_{j=1}^{i-1}(-1)^{j+i+1}[t_1,\ldots,t_j]\circ[t_{j+1},\ldots,t_i+t_{i+1},\ldots,t_k] \\
&\quad+\sum_{i=1}^{k-1}\sum_{j=i+1}^{k-1}(-1)^{j+i}[t_1,\ldots,t_i+t_{i+1},\ldots,t_j]\circ[t_{j+1},\ldots,t_k] \\
&= -\delta_\star\delta_\Delta([t_1,\ldots,t_k]).
\end{aligned}$$

□

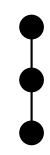
In conclusion, we have proved the following.

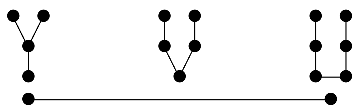
Proposition 4.3.1. *The algebra* $\mathbf{\Omega}$ *with differential* $\delta = \delta_\Delta + \delta_\star$ *is a differential graded algebra.*

4.3.3 The cohomology of $\mathbf{\Omega}$

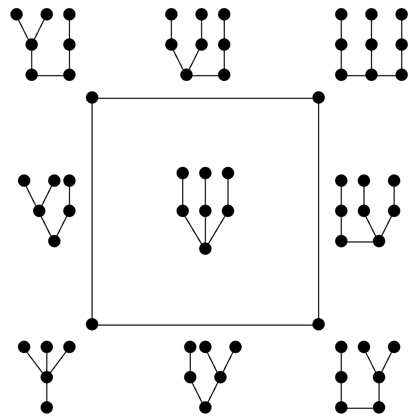
The purpose of this section is to show that the cohomology of the algebra $\mathbf{\Omega}$ is concentrated in degree zero. We will use this computation in the construction of diagonal maps $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$.

Remark 4.3.2. *We will prove by direct computation that the cohomology of the algebra* $\mathbf{\Omega}$ *is concentrated in degree zero. There is also a topological explanation for this fact. The forests with* m *leaves in the algebra* $\mathbf{\Omega}$ *correspond to the cells in the natural cell decomposition of the* $m-1$ *dimensional cube. The differential* δ *in* $\mathbf{\Omega}$ *computes the homology of the cube, which is of course concentrated in degree zero. In this correspondence, the trees of degree* k *represent cells of dimension* $-k$*. We will illustrate this in some low degree examples.*

- *There is only one tree in* $\mathbf{\Omega}$ *with one leaf, namely* *, which represents the point.*
- *For two leaves, we have the following decomposition of the interval:*



- *The decomposition of the square looks like this:*



One can easily check that the differential δ computes the homology of the cube with respect to this cell decomposition.

Remark 4.3.3. *For each number of leaves m fixed, there is a finite dimensional fourth quadrant cohomology double complex $\mathbf{\Omega}^{\bullet,\bullet}(m)$:*

$$
\begin{array}{ccccccc}
\mathbf{\Omega}^{1,-1}(m) & & & & & & \\
\uparrow{\scriptstyle \delta_\star} & & & & & & \\
\mathbf{\Omega}^{1,-2}(m) & \xrightarrow{\delta_\Delta} & \mathbf{\Omega}^{2,-2}(m) & & & & \\
\uparrow{\scriptstyle \delta_\star} & & \uparrow{\scriptstyle \delta_\star} & & & & \\
\mathbf{\Omega}^{1,-3}(m) & \xrightarrow{\delta_\Delta} & \mathbf{\Omega}^{2,-3}(m) & \xrightarrow{\delta_\Delta} & \mathbf{\Omega}^{3,-3}(m) & & \\
\uparrow{\scriptstyle \delta_\star} & & \uparrow{\scriptstyle \delta_\star} & & \uparrow{\scriptstyle \delta_\star} & & \\
\mathbf{\Omega}^{1,-4}(m) & \xrightarrow{\delta_\Delta} & \mathbf{\Omega}^{2,-4}(m) & \xrightarrow{\delta_\Delta} & \mathbf{\Omega}^{3,-4}(m) & \xrightarrow{\delta_\Delta} & \mathbf{\Omega}^{4,-4}(m) \\
\uparrow{\scriptstyle \delta_\star} & & \uparrow{\scriptstyle \delta_\star} & & \uparrow{\scriptstyle \delta_\star} & & \uparrow{\scriptstyle \delta_\star} \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

Here $\mathbf{\Omega}^{p,q}(m)$ denotes the space generated by forests with p roots, $-q$ branches and m leaves.

Definition 4.3.4. *For trees $T = [t_1, \dots, t_k]$ and $S = [s_1, \dots, s_l]$ in $\mathbf{\Omega}$, we denote by $T \star S$ the concatenation:*

$$T \star S := [t_1, \dots, t_k, s_1, \dots, s_l].$$

In order to compute the cohomology of $\mathbf{\Omega}$ we will first prove that the cohomology with respect to δ_Δ vanishes and then use a homological perturbation argument.

Lemma 4.3.2. *Define $h_\Delta : \mathbf{\Omega}^{\bullet,l} \to \mathbf{\Omega}^{\bullet-1,l}$ by the formula:*

$$h_\Delta([t_1^1, \dots, t_{n_1}^1] \dots [t_1^k, \dots, t_{n_k}^k]) =$$
$$\frac{1}{l+1} \sum_{i=1}^{k-1} (-1)^{n_1+\cdots+n_i+i+1} [t_1^1, \dots, t_{n_1}^1] \dots [t_1^i, \dots, t_{n_i}^i, t_1^{i+1}, \dots, t_{n_{i+1}}^{i+1}] \dots [t_1^k, \dots, t_{n_k}^k]),$$

for $l < -1$ and we set $h_\Delta(T) = 0$, if $T \in \mathbf{\Omega}^{\bullet,-1}$. Then, for $T_1 \in \mathbf{\Omega}^{\bullet,l_1}$, $T_2 \in \mathbf{\Omega}^{\bullet,l_2}$ and $l = l_1 + l_2$, the following equations are satisfied:

$$\delta_\Delta(T_1 * T_2) = \delta_\Delta(T_1) * T_2 + (-1)^{\overline{T}_1} T_1 \circ T_2 + (-1)^{\overline{T}_1+1} T_1 * \delta_\Delta(T_2), \tag{4.4}$$

$$h_\Delta(T_1 \circ T_2) = \frac{l_1+1}{l+1} h_\Delta(T_1) \circ T_2 + \frac{(-1)^{\overline{T}_1}(l_2+1)}{l+1} T_1 \circ h_\Delta(T_2) + \frac{(-1)^{\overline{T}_1+1}}{l+1} T_1 * T_2. \tag{4.5}$$

Proof. Straightforward induction. □

Lemma 4.3.3. *For any $T \in \mathbf{\Omega}^{k,l}$ with $l < -1$ we have:*

$$(h_\Delta \delta_\Delta + \delta_\Delta h_\Delta)(T) = T.$$

In particular, each of the complexes $(\mathbf{\Omega}^{\bullet,l}(m), \delta_\Delta)$ is acyclic.

Proof. We will use induction on k. For $k = 1$ the statement is a simple check. Any $T \in \mathbf{\Omega}^{k+1,l}$ can be written $T = T_1 \circ T_2$ for $T_1 \in \mathbf{\Omega}^{k,l_1}$ and $T_2 \in \mathbf{\Omega}^{1,l_2}$. We use lemma 4.3.2 to compute:

$$\begin{aligned} h_\Delta \delta_\Delta(T_1 \circ T_2) &= h_\Delta\left(\delta_\Delta(T_1) \circ (T_2) + (-1)^{\overline{T}_1} T_1 \circ \delta_\Delta(T_2)\right) \\ &= \frac{l_1+1}{l+1} h_\Delta(\delta_\Delta(T_1)) \circ (T_2) + \frac{(-1)^{\overline{T}_1}}{l+1} \delta_\Delta(T_1) * T_2 \\ &\quad + \frac{(-1)^{\overline{T}_1}(l_1+1)}{l+1} h_\Delta(T_1) \circ \delta_\Delta(T_2) + \frac{l_2+1}{l+1} T_1 \circ h_\Delta \delta_\Delta(T_2) \\ &\quad - \frac{1}{l+1} T_1 * \delta_\Delta(T_2). \end{aligned}$$

On the other hand,

$$\begin{aligned} \delta_\Delta h_\Delta(T_1 \circ T_2) &= \delta_\Delta\left(\frac{(l_1+1)}{l+1} h_\Delta(T_1) \circ T_2 + \frac{(-1)^{\overline{T}_1+1}}{l+1} T_1 * T_2\right) \\ &= \frac{(l_1+1)}{l+1} \delta_\Delta h_\Delta(T_1) \circ (T_2) + \frac{(-1)^{\overline{T}_1+1}(l_1+1)}{l+1} h_\Delta(T_1) \circ \delta_\Delta(T_2) \\ &\quad + \frac{(-1)^{\overline{T}_1+1}}{l+1} \delta_\Delta(T_1) * T_2 + \frac{1}{l+1} T_1 * \delta_\Delta(T_2) - \frac{1}{l+1} T_1 \circ T_2. \end{aligned}$$

Adding these two expressions and using the inductive hypothesis we obtain:

$$h_\Delta \delta_\Delta(T_1 \circ T_2) + \delta_\Delta h_\Delta(T_1 \circ T_2) = \left(\frac{l_1+1}{l+1} + \frac{l_2+1}{l+1} - \frac{1}{l+1}\right) T_1 \circ T_2 = T_1 \circ T_2.$$

□

Now we will use a general homological perturbation argument to find a contacting homotopy for the complex $(\mathbf{\Omega}, \delta)$.

Proposition 4.3.2. *Consider the operator $h : \mathbf{\Omega}^\bullet \to \mathbf{\Omega}^{\bullet-1}$ defined by:*

$$h = h_\Delta[1 - (\delta_2 h_\Delta) + (\delta_2 h_\Delta)^2 - (\delta_2 h_\Delta)^3 + \dots]. \tag{4.6}$$

For any $T \in \mathbf{\Omega}^p$ with $p < 0$ we have:

$$(\delta h + h\delta)(T) = T.$$

In particular, the cohomology of $(\mathbf{\Omega}, \delta)$ is concentrated in degree zero and it is generated by the classes of the trees $[k]$ with $k \in \mathbb{N}^*$.

Proof. First, one proves by induction that:

$$(\delta_\star h_\Delta)^k \delta_\Delta = \delta_\Delta (\delta_\star h_\Delta)^k + (\delta_\star h_\Delta)^{k-1}\delta_\star. \tag{4.7}$$

Then we compute:

$$\begin{aligned}
\delta h + h\delta &= (\delta_\Delta h_\Delta + \delta_\star h_\Delta)\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\\
&\quad + h_\Delta\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\delta_\Delta\\
&\quad + h_\Delta\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\delta_\star\\
&= (\delta_\Delta h_\Delta + \delta_\star h_\Delta)\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\\
&\quad + h_\Delta\delta_\Delta\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\\
&\quad - h_\Delta\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\delta_\star\\
&\quad + h_\Delta\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big)\delta_\star\\
&= (1 + \delta_\star h_\Delta)\big(1 - (\delta_\star h_\Delta) + (\delta_\star h_\Delta)^2 - (\delta_\star h_\Delta)^3 + \dots\big) = 1.
\end{aligned}$$

This equation holds only when evaluated on elements of negative degree because in the computation we used $(\delta_\Delta h_\Delta + h_\Delta \delta_\Delta) = 1$, which fails in $\mathbf{\Omega}^{1,-1}$. □

4.3.4 Ω differential graded algebras

Here we will axiomatize the structure that the algebra $\mathbf{\Omega}$ has by introducing the notion of an Ω differential graded algebra. We will show that the category of Ω differential graded algebras has a natural tensor product. This will simplify the exposition of the relation between representations up to homotopy and the algebra $\mathbf{\Omega}$. Also, it will provide a natural setting for our discussion on the tensor product of representations up to homotopy.

Definition 4.3.5. *An Ω differential graded algebra A is a bigraded vector space:*

$$A = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{m=1}^{\infty} A^k(m),$$

together with the following structure:

- *An associative multiplication:*

  $$\mu : A^k(m) \otimes A^{k'}(m') :\to A^{k+k'}(m+m'),$$

  that gives A the structure of a bigraded algebra.

- *A differential:*
$$\delta : A^k(m) \to A^{k+1}(m),$$
which squares to zero and satisfies:
$$\delta(ab) = \delta(a)b + (-1)^k a\delta(b),$$
for $a \in A^k(m)$.

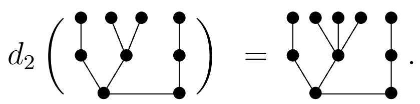
- *For each m there are maps $d_1, \ldots, d_m$:*
$$d_i : A^k(m) \to A^k(m+1),$$
with the following properties:
$$d_j d_i = d_i d_{j-1}, \text{ if } i < j,$$
$$\delta d_i = d_i \delta,$$
and, for $a \in A^k(m)$:
$$d_i(ab) = \begin{cases} d_i(a)b & \text{if } m \geq i, \\ a d_{i-m} b & \text{if } m < i. \end{cases}$$

A morphism, $\psi : A \to B$, between Ω differential graded algebras is a linear map that preserves both degrees and commutes with all the structure maps.

Of course, the algebra $\mathbf{\Omega}$ is our model for this definition. Let us spell out what the operators d_i are in this case.

Remark 4.3.4. *The algebra $\mathbf{\Omega}$ is an Ω differential graded algebra. For each $i = 1, \ldots, m$, we define the operator*
$$d_i : \mathbf{\Omega}(m) \to \mathbf{\Omega}(m+1),$$
by
$$d_i([t_1^1, \ldots, t_{n_1}^1] \ldots [t_1^k, \ldots, t_{n_k}^k]) = [t_1^1, \ldots, t_{n_1}^1] \ldots [t_1^j, \ldots, t_s^j + 1, \ldots t_{n_j}^j] \ldots [t_1^k, \ldots, t^k n_k],$$
where:
$$\sum_{\mu=1}^{j-1} \sum_{\nu=1}^{n_\mu} t_\nu^\mu + \sum_{\nu=1}^{s-1} t_\nu^j < i \leq \sum_{\mu=1}^{j-1} \sum_{\nu=1}^{n_\mu} t_\nu^\mu + \sum_{\nu=1}^{s} t_\nu^j.$$

Contrary to what the formula suggests, the operator d_i is very simple. It acts by replacing the i-th leaf of a forest, counting from the left, by two leaves. For instance:



Proposition 4.3.3. *The category of Ω differential graded algebras has a natural tensor product.*

Proof. The tensor product of two Ω differential graded algebras A and B is the Ω differential graded algebra $A \otimes B$ defined as follows. As a vector space:

$$(A \otimes B)^k(m) = \bigoplus_{i+j=k} A^j(m) \otimes B^j(m).$$

The multiplication in $A \otimes B$ is the usual multiplication of graded algebras:

$$(a \otimes b)(a' \otimes b') = (-1)^{\overline{ba'}} aa' \otimes bb'.$$

The differential is given by the usual formula:

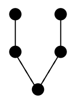
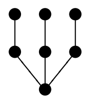
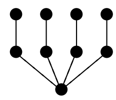
$$\delta^{A \otimes B} = \delta^A \otimes Id + Id \otimes \delta^B.$$

Finally,

$$d_i^{A \otimes B} = d_i^A \otimes d_i^B.$$

If $\psi : A \to B$ and $\psi' : A' \to B'$ are maps of Ω differential graded algebras then, $\psi \otimes \psi' : A \otimes A' \to B \otimes B'$ is also a morphism in the category. $\square$

Definition 4.3.6. *We will denote by L_n the tree in $\mathbf{\Omega}$ which has one root, n branches and n leaves. For instance:*

$L_1 =$  , $L_2 =$  , $L_3 =$  , $L_4 =$  .

Remark 4.3.5. *A simple induction shows that every forest $F \in \mathbf{\Omega}$ can be written uniquely as:*

$$F = d_{i_1} \dots d_{i_l}(L_{n_1} \circ \dots \circ L_{n_s}),$$

with $i_1 > \dots > i_l$. We will denote by $\mathbf{\Omega}[n]$ the subalgebra generated by the forests such that only $L_1, \dots, L_n$ appear in the factorization above. It is clear that all the structure maps of $\mathbf{\Omega}$ restrict to $\mathbf{\Omega}[n]$, which inherits the structure of a Ω differential graded algebra. Also,

$$\mathbf{\Omega} = \bigcup_{n>0} \mathbf{\Omega}[n].$$

4.3.5 The diagonal maps

Here we will construct morphisms of Ω differential graded algebras:

$$\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k},$$

which provide formulas for the tensor product of representations up to homotopy. By $\mathbf{\Omega}^{\otimes k}$ we will always mean the k-fold tensor product of $\mathbf{\Omega}$ in the category of Ω differential graded algebras. We will denote by δ_k the differential on the algebra $\mathbf{\Omega}^{\otimes k}$, explicitly:

$$\delta_k = \sum_{i=1}^{k} Id \otimes \dots \otimes \delta \otimes \dots \otimes Id.$$

Remark 4.3.6. *The Kunneth formula implies that the cohomology of the differential graded algebra $(\mathbf{\Omega}^{\otimes k}, \delta_k)$ is concentrated in degree zero. More precisely,*

$$H(\mathbf{\Omega}^{\otimes k}(m), \delta_k) = H^0(\mathbf{\Omega}(m), \delta)^{\otimes k} \cong \mathbb{Q}.$$

There is an action of the symmetric group S_k on the algebra $\mathbf{\Omega}^{\otimes k}$, which is determined by the following rule. Let $\sigma_{i,i+1}$ be the transposition that interchanges i and $i+1$. Then, for homogeneous elements $T_1, \ldots, T_k \in \mathbf{\Omega}$, we define the action:

$$\hat{\sigma}_{i,i+1}(T_1 \otimes \cdots \otimes T_k) = (-1)^{\overline{T}_i \overline{T}_{i+1}} T_1 \otimes \cdots \otimes T_{i+1} \otimes T_i \otimes \cdots \otimes T_k, \tag{4.8}$$

which extends uniquely to an action of S_k. A simple computation shows that the differential δ_k is equivariant with respect to this action. This means that for any $\sigma \in S_k$ we have:

$$\hat{\sigma}\delta_k = \delta_k \hat{\sigma}. \tag{4.9}$$

We will describe a contracting homotopy for each of the complexes $\mathbf{\Omega}^{\otimes k}(m)$. This will be important in our construction of the diagonal maps. We start with the following simple lemma.

Lemma 4.3.4. *Let (E, d) be a cochain complex of finite dimensional vector spaces. Suppose that, for each k, the vector space E^k has an inner product and denote by d^* the adjoint of d. Then, the Laplacian $\mathbf{\Delta} = (dd^* + d^*d) : E^k \to E^k$, is an isomorphism if and only if $H^k(E) = 0$.*

Proof. It is enough to notice that there is a natural isomorphism $\mathrm{Ker}(\mathbf{\Delta}) \cong H^k(E)$. □

Definition 4.3.7. *Each of the finite dimensional vector spaces $\mathbf{\Omega}^{\otimes k}(m)$ has an inner product $\langle\ ,\ \dot{\rangle}$, induced by the natural basis. Denote by δ_k^*, the adjoint of δ_k with respect to $\langle\ ,\ \dot{\rangle}$. We define the operator*

$$h_k : (\mathbf{\Omega}^{\otimes k})^p(m) \to (\mathbf{\Omega}^{\otimes k})^{p-1}(m),$$

by:

$$h_k = \mathbf{\Delta}^{-1}\delta_k^*,$$

where $\mathbf{\Delta} = \delta_k\delta_k^* + \delta_k^*\delta_k$, is the Laplacian. This is well defined in view of lemma 4.3.4 and the fact that the cohomology of $(\mathbf{\Omega}^{\otimes k}(m), \delta_k)$ is concentrated in degree zero.

Lemma 4.3.5. *For any $T \in \mathbf{\Omega}^{\otimes k}(m)$ of negative degree, the following formula holds:*

$$(h_k\delta_k + \delta_k h_k)(T) = T.$$

Moreover, for every $\sigma \in S_k$ we have:

$$h_k\hat{\sigma} = \hat{\sigma}h_k.$$

Proof. Clearly, $\mathbf{\Delta}$ commutes with both δ_k and δ_k^* and therefore so does $\mathbf{\Delta}^{-1}$. With this in mind we compute:

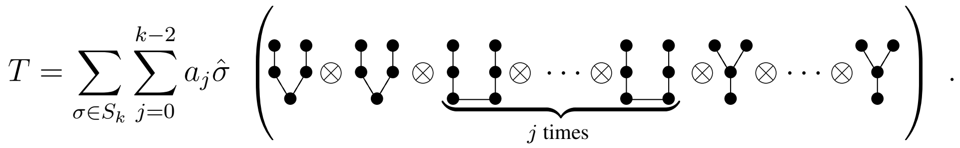
$$(h_k\delta_k + \delta_k h_k)(T) \;=\; (\mathbf{\Delta}^{-1}\delta_k^*\delta_k + \delta_k\mathbf{\Delta}^{-1}\delta_k^*)(T) = \mathbf{\Delta}^{-1}(\mathbf{\Delta})(T) = T.$$

For the second part we observe that the action of S_k preserves the inner product $\langle\ ,\ \dot{\rangle}$. Since δ_k is equivariant we conclude that so are δ_k^* and $\mathbf{\Delta}$. This clearly implies that $h_k = \mathbf{\Delta}^{-1}\delta_k^*$, is equivariant with respect to the action of S_k. □

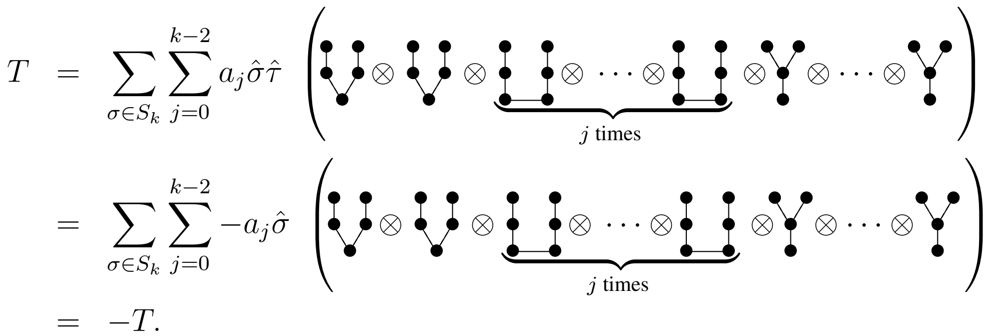
Lemma 4.3.6. *The space of elements in* $\left(\boldsymbol{\Omega}^{\otimes k}\right)^{-2}(2)$ *which are invariant under the action of* S_k *is zero. In symbols,*

$$\left(\left(\boldsymbol{\Omega}^{\otimes k}\right)^{-2}(2)\right)^{inv} = 0.$$

Proof. Suppose that $T \in \left(\boldsymbol{\Omega}^{\otimes k}\right)^{-2}(2)$ is invariant with respect to the action of S_k. Then, T can be written in the form:



Now, let τ denote the transposition that interchanges one and two and compute:



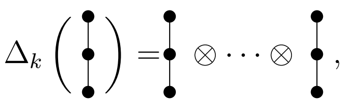
□

Now we can prove the main result of this section, which is the existence of diagonal maps.

Theorem 4.3.1. *For each* $k \geq 0$ *there exists a map of* Ω *differential graded algebras:*

$$\Delta_k : \boldsymbol{\Omega} \to \boldsymbol{\Omega}^{\otimes k},$$

such that:

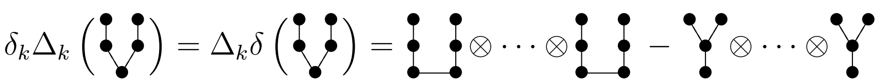
 (4.10)

and

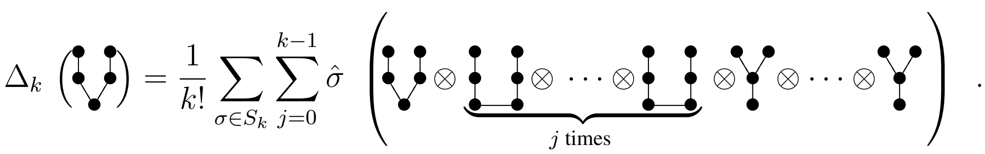
$$\hat{\sigma}\Delta_k = \Delta_k, \tag{4.11}$$

for all $\sigma \in S_k$.

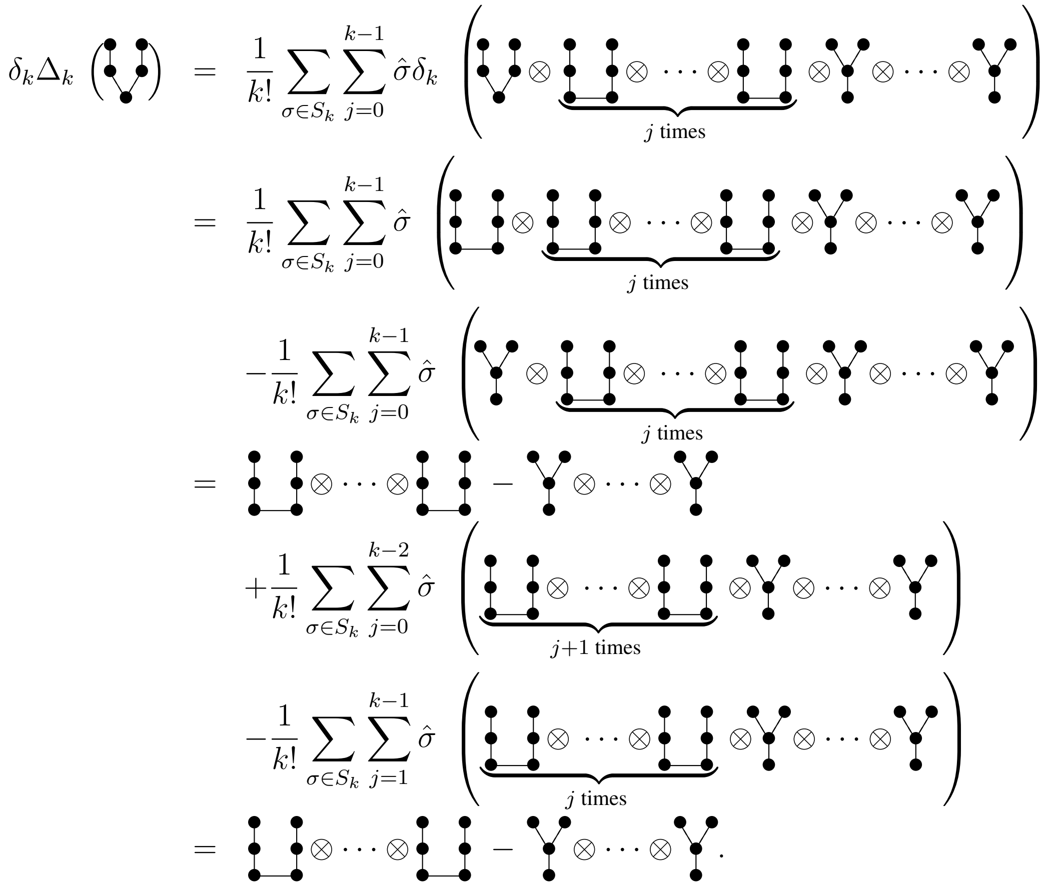
Proof. We will construct the map Δ_k inductively on each subalgebra $\boldsymbol{\Omega}[n]$. Clearly, there is a unique map $\Delta_k : \boldsymbol{\Omega}[1] \to \boldsymbol{\Omega}^{\otimes k}$, as above. We claim that this map can be extended uniquely to $\boldsymbol{\Omega}[2]$. Indeed, if we impose the symmetry condition $\hat{\sigma}\Delta_k = \Delta_k$, the equation:

 (4.12)

has the unique solution:



To check that this gives a solution, we compute:



Now, suppose that there is a different symmetric solution to equation (4.12) and let S be the difference between the solutions. Then, $S \in \left(\left(\mathbf{\Omega}^{\otimes k}\right)^{-1}(2)\right)^{inv}$ and $\delta_k(S) = 0$. This implies that $S = \delta_k h_k(S)$. However, by lemma 4.3.6, $\left(\left(\mathbf{\Omega}^{\otimes k}\right)^{-2}(2)\right)^{inv} = 0$ and therefore, since h_k is equivariant with respect to the symmetric group, $S = 0$. We conclude that there is a well defined map of Ω differential graded algebras $\Delta_k : \mathbf{\Omega}[2] \to \mathbf{\Omega}^{\otimes k}$. Now we proceed inductively. Suppose we have defined Δ_k on $\mathbf{\Omega}[n]$ for n at least two. Since $\delta(L_{n+1}) \in \mathbf{\Omega}[n]$, we can define:

$$\Delta_k(L_{n+1}) = h_k\Delta_k(\delta(L_{n+1})).$$

By inductive hypothesis:

$$\delta_k(\Delta_k\delta(L_{n+1})) = \Delta_k(\delta\delta(L_{n+1})) = 0,$$

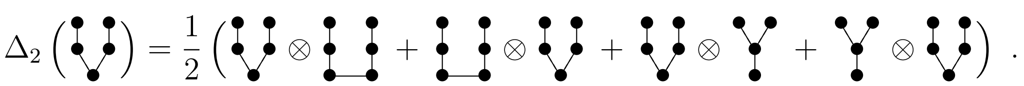
and therefore:

$$\delta_k\Delta_k(L_{n+1}) = \Delta_k\delta(L_{n+1}).$$

In view of remark 4.3.5, this formula determines Δ_k on $\mathbf{\Omega}[n+1]$. Finally, since h_k is equivariant with respect to S_k, the symmetry is preserved.

$\square$

Example 4.3.3. *As an instance of the map defined above we have:*



Remark 4.3.7. *We work over $\mathbb{Q}$ because there are no symmetric diagonal maps over $\mathbb{Z}$. Ideally, one would like to construct the diagonal maps Δ_k by iterating Δ_2. However, a simple computation shows that it is not possible to find a symmetric coassociative diagonal in $\mathbf{\Omega}$. The properties required in theorem 4.3.1 guarantee that the diagonal maps give formulas for the tensor products, but they do not determine the maps Δ_k that we constructed. We expect that the tensor products do not depend up to isomorphism on the choice of diagonal maps, but we will not discuss this issue here.*

4.4 Tensor products

4.4.1 Representations up to homotopy and the algebra $\mathbf{\Omega}$

Here we will explain the relation between the algebra $\mathbf{\Omega}$ and representations up to homotopy. We will show that a representation up to homotopy can be described as a morphism of Ω differential graded algebras and then we will use this description to construct tensor products.

Remark 4.4.1. *For a group G and a differential graded algebra (A, d), the vector space:*

$$\overline{C}(G, A) = \bigoplus_{m>0} \bigoplus_{k\in\mathbb{Z}} C(G_m, A^k),$$

has the structure of an Ω differential graded algebra, as follows. For $F \in C(G_m, A^k)$ and $F' \in C(G_p, A^q)$, the product $F \circ F' \in C(G_{m+p}, A^{k+q})$ is defined by:

$$F \circ F'(g_1, \dots g_{m+p}) = F(g_1, \dots, g_m)F'(g_{m+1}, \dots, g_{m+p}).$$

The differential δ is defined by:

$$\delta(F)(g_1, \dots, g_m) = d(F(g_1, \dots, g_m)).$$

Finally, the operators $d_i : \overline{C}(G, A)(m) \to \overline{C}(G, A)(m + 1)$ are defined by:

$$d_i(F)(g_1, \dots, g_{m+1}) = F(g_1, \dots, g_i g_{i+1}, \dots, g_{m+1}).$$

A simple computation shows that these operators give $\overline{C}(G, A)$ the structure of a Ω differential graded algebra. A morphism of differential graded algebras $\varphi : A \to A'$ induces a morphism of Ω differential graded algebras $\hat{\varphi} : \overline{C}(G, A) \to \overline{C}(G, A')$ in an obvious way. We will mainly be interested in the case in which A is the algebra of endomorphisms of a complex of vector spaces.

Theorem 4.4.1. *There is a natural one to one correspondence between representations up to homotopy of the group G on the complex of vector spaces (E, d_E) and maps of Ω differential graded algebras:*

$$\Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E)).$$

Proof. Recall that, by proposition 4.2.1, a representation up to homotopy of G on (E, d_E) is the same thing as a sequence of operators $F_k \in C(G_k, \mathrm{End}^{1-k}(E))$ satisfying the equations:

$$\begin{aligned}[d_E, F_k](g_1, \ldots, g_k) &= \sum_{j=1}^{k-1}(-1)^{j+1} F_j(g_1, \ldots, g_j) \circ F_{k-j}(g_{j+1}, \ldots, g_k) \\ &\quad + \sum_{j=1}^{k-1}(-1)^j F_{k-1}(g_1, \ldots, g_i g_{i+1}, \ldots, g_k).\end{aligned}$$

In terms of the Ω differential graded algebra structure on $\overline{C}(G, \mathrm{End}(E))$, the equation above can be written:

$$\delta(F_k) = \sum_{j=1}^{k-1}(-1)^{j+1} F_j \circ F_{k-j} + \sum_{j=1}^{k-1}(-1)^j d_j(F_{k-1}).$$

On the other hand, in the algebra $\mathbf{\Omega}$:

$$\delta(L_k) = \sum_{j=1}^{k-1}(-1)^{j+1} L_j \circ L_{k-j} + \sum_{j=1}^{k-1}(-1)^j d_j(L_{k-1}),$$

where the trees L_k are those in definition 4.3.6. Now, suppose we have a morphism $\Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E))$ and define $F_k = \Phi(L_k)$. Clearly, $F_k \in C(G_k, \mathrm{End}^{1-k}(E))$ and:

$$\begin{aligned}\delta(F_k) = \delta(\Phi(L_k)) = \Phi(\delta(L_k)) &= \Phi(\sum_{j=1}^{k-1}(-1)^{j+1} L_j \circ L_{k-j} + \sum_{j=1}^{k-1}(-1)^j d_j(L_{k-1})) \\ &= \sum_{j=1}^{k-1}(-1)^{j+1} F_j \circ F_{k-j} + \sum_{j=1}^{k-1}(-1)^j d_j(F_{k-1}).\end{aligned}$$

For the converse, suppose we are given operators F_k as above. By remark 4.3.5, every tree $T \in \mathbf{\Omega}$ can be factored uniquely as:

$$T = d_{i_1} \ldots d_{i_l}(L_{n_1} \circ \cdots \circ L_{n_s}),$$

where $i_1 > \cdots > i_l$. Define the morphism $\Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E))$ by:

$$T \mapsto d_{i_1} \ldots d_{i_l}(F_{n_1} \circ \cdots \circ F_{n_s}).$$

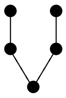
Clearly, the map Φ preserves the multiplication and the operators d_i. We need to show that it also preserves the differential δ. Since δ is a derivation and commutes with the d_i's, it is enough to show that $\delta(\Phi(L_k)) = \Phi(\delta(L_k))$. For this we compute:

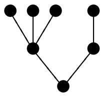
$$\delta(\Phi(L_k)) = \delta(F_k) = \sum_{j=1}^{k-1}(-1)^{j+1} F_j \circ F_{k-j} + \sum_{j=1}^{k-1}(-1)^j d_j(F_{k-1}) = \Phi(\delta(L_k)).$$

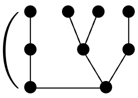
□

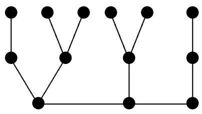
The previous theorem shows that, given a representation up to homotopy of G on the complex (E, d_E), one can interpret an element $T \in \mathbf{\Omega}$ as an operator $\Phi(T) \in \overline{C}(G, \mathrm{End}(E))$. We associate the operator F_k to the tree L_k, view the multiplication as composition and the operation of doubling the leaves as multiplication of the variables.

Example 4.4.1. *This illustrates how one associates operators to elements of* $\mathbf{\Omega}$.

Φ () $(g_1, g_2) = F_2(g_1, g_2),$

Φ () $(g_1, g_2, g_3, g_4) = F_2(g_1 g_2 g_3, g_4),$

Φ () $(g_1, g_2, g_3, g_4) = F_1(g_1) \circ F_2(g_2 g_3, g_4),$

Φ () $(g_1, g_2, g_3, g_4, g_5, g_6) = F_2(g_1, g_2 g_3) \circ F_1(g_4 g_5) \circ F_1(g_6).$

4.4.2 Tensor products and symmetric powers

Here we will show that the diagonal maps $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$, constructed above, provide formulas for the tensor product and the symmetric powers of representations up to homotopy. We will use the following natural homomorphism.

Lemma 4.4.1. *Suppose that G is a group and $(E_1, d_{E_1}), \dots, (E_n, d_{E_n})$ are complexes of vector spaces. Then, there is a natural map of Ω differential graded algebras:*

$$\Psi : \overline{C}(G, \mathrm{End}(E_1)) \otimes \cdots \otimes \overline{C}(G, \mathrm{End}(E_k)) \to \overline{C}(G, \mathrm{End}(E_1 \otimes \cdots \otimes E_k)).$$

Proof. For an element $P_1 \otimes \cdots \otimes P_n \in \big(\overline{C}(G, \mathrm{End}(E_1)) \otimes \cdots \otimes \overline{C}(G, \mathrm{End}(E_k))\big)(m)$ we define $\Psi(P_1 \otimes \cdots \otimes P_k) \in \big(\overline{C}(G, \mathrm{End}(E_1 \otimes \cdots \otimes E_k))\big)(m)$ by the formula:

$$\Psi(P_1 \otimes \cdots \otimes P_k)(g_1, \dots, g_m) = P_1(g_1, \dots, g_m) \otimes \cdots \otimes P_k(g_1, \dots, g_m) \in \mathrm{End}(E_1) \otimes \cdots \otimes \mathrm{End}(E_k),$$

and we use the identification:

$$\mathrm{End}(E_1) \otimes \cdots \otimes \mathrm{End}(E_n) \cong \mathrm{End}(E_1 \otimes \cdots \otimes E_n).$$

A straightforward computation shows that this is a map of Ω differential graded algebras. □

Theorem 4.4.2. *Let G be a group and $(E_1, d_{E_1}), \dots, (E_k, d_{E_k})$ be representations up to homotopy of G with structures given by maps: $\Phi_i : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E_i))$. Then, the complex of vector spaces $(E_1, d_{E_1}) \otimes \cdots \otimes (E_k, d_{E_k})$ is a representation up to homotopy with structure:*

$$\Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E_1 \otimes \cdots \otimes E_k)),$$

where:

$$\Phi = \Psi \circ (\phi_1 \otimes \cdots \otimes \phi_k) \circ \Delta_k.$$

Proof. Since each of the maps Δ_k, $(\phi_1 \otimes \cdots \otimes \phi_k)$ and Ψ are morphisms of good differential graded algebras, so is Φ. Thus, by theorem 4.4.1, we conclude that $(E_1, d_{E_1}) \otimes \cdots \otimes (E_k, d_{E_k})$ is a representation up to homotopy of G. □

As one would expect, when restricted to ordinary representations this is the usual tensor product of representations. We will now show that the symmetry of the diagonal maps $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$ implies that the category of representations up to homotopy is closed under the operation of taking symmetric powers.

Remark 4.4.2. *Let (E, d_E) be a cochain complex of finite dimensional vector spaces. The symmetric group S_k acts on the complex $(E^{\otimes k}, d_{E^{\otimes k}})$. The action of the transposition $\sigma_{i,i+1}$ is given by:*

$$\hat{\sigma}_{i,i+1}(v_1 \otimes \cdots \otimes v_k) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k.$$

For any permutation $\sigma \in S_k$, one verifies $\hat{\sigma} d_{E^{\otimes k}} = d_{E^{\otimes k}} \hat{\sigma}$. Therefore, both the space of invariants $S_k(E)$ and the space of coinvariants $S^k(E)$ are cochain complexes. There is a natural homomorphism of differential graded algebras:

$$\kappa : S_k(\mathrm{End}(E)) \to \mathrm{End}(S^k(E)),$$

given by the formula:

$$\sum_{j=1}^{n} (\phi_1^j \otimes \cdots \otimes \phi_k^j)(v_1 \otimes \cdots \otimes v_k) = \sum_{j=1}^{n} (-1)^{\sum_{i=1}^{k} |v_i|(|\phi_{i+1}^j| + \cdots + |\phi_k^j|)} \phi_1^j(v_1) \otimes \cdots \otimes \phi_k^j(v_k).$$

A simple computation shows that if $\sum_{j=1}^{n}(\phi_1^j \otimes \cdots \otimes \phi_k^j)$ is invariant under the action of S_k, then the formula above gives a well defined endomorphism of $S^k(E)$.

Theorem 4.4.3. *Let G be a group and (E, d_E) a representation up to homotopy of G with structure given by $\Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E))$. Then, the complex of vector spaces $(S^k(E), d_{S^k(E)})$ is a representation up to homotopy of G.*

Proof. The map $\kappa : S_k(\mathrm{End}(E)) \to \mathrm{End}(S^k(E))$ induces a morphism of good differential graded algebras:

$$\hat{\kappa} : \overline{C}(G, S_k(\mathrm{End}(E))) \to \overline{C}(G, \mathrm{End}(S^k(E))).$$

Also, the diagonal map $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$ satisfies the equation:

$$\hat{\sigma} \Delta_k = \Delta_k,$$

for every permutation σ. Therefore, the image of the composition:

$$\Psi \circ (\Phi \otimes \cdots \otimes \Phi) \circ \Delta_k : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E)^{\otimes k}),$$

is contained in $\overline{C}(G, S_k(\mathrm{End}(E)))$ so that it makes sense to define the map:

$$\hat{\kappa} \circ \Psi \circ (\Phi \otimes \cdots \otimes \Phi) \circ \Delta_k : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(S^k(E))),$$

which gives $(S_k(\mathrm{End}(E)), d_{S^k(E)})$ the structure of a representation up to homotopy. □

5

The spectral sequence

Given a Lie group G, Bott [11] constructed a spectral sequence converging to the cohomology of the classifying space BG:

$$E_1^{pq} = H^{p-q}(G, S^q(\mathfrak{g}^*)) \Rightarrow H^{p+q}(BG),$$

which should be thought of as a generalization of the Chern-Weil homomorphism. In this chapter we show that this spectral sequence exists for an arbitrary Lie groupoid. The representations that appear in the spectral sequence are representations up to homotopy, which we introduce in order to make sense of the adjoint representation of a Lie groupoid. When our spectral sequence is applied to the transformation groupoid of a group action, one recovers the spectral sequence associated to Getzler's model for equivariant cohomology. Our work is closely related to and inspired by that of Behrend [7], Bott [11] and Getzler [25].

5.1 Introduction

5.1.1 The formula of Bott

The cohomology of the classifying space BG of a Lie group G can be computed as the total cohomology of the double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \\
\Omega^2(G_0) & \xrightarrow{\delta} & \Omega^2(G_1) & \xrightarrow{\delta} & \Omega^2(G_2) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \\
\Omega^1(G_0) & \xrightarrow{\delta} & \Omega^1(G_1) & \xrightarrow{\delta} & \Omega^1(G_2) & \xrightarrow{\delta} & \cdots \\
\uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \uparrow{\scriptstyle \mathrm{d}} & & \\
\Omega^0(G_0) & \xrightarrow{\delta} & \Omega^0(G_1) & \xrightarrow{\delta} & \Omega^0(G_2) & \xrightarrow{\delta} & \cdots
\end{array}$$

in which $\Omega^q(G_p)$ denotes the space of q-forms in $\underbrace{G \times \cdots \times G}_{p-\mathrm{times}}$. The vertical differential d is De-Rham differential and the horizontal differential is the alternating sum of pullback maps:

$$\delta = \sum_{i=0}^{p} (-1)^i {d_i}^*,$$

where the map $d_i : G_p \to G_{p-1}$ is defined by:

$$d_i(g_1, \ldots, g_p) = \begin{cases} (g_2, \ldots, g_p) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots g_p) & \text{if } 0 < i \leq p, \\ (g_1, \ldots, g_{p-1}) & \text{if } i = p. \end{cases}$$

In [11], Bott computed the horizontal cohomologies of this double complex by proving the formula:

$$H^p_\delta(\Omega^q(G_\bullet)) \cong H^{p-q}_{\mathrm{diff}}(G, S^q(\mathfrak{g}^*)), \tag{5.1}$$

where the right hand side is the differentiable cohomology with coefficients in the symmetric powers of the coadjoint representation (see section 5.2 for the definition of differentiable cohomology). Filtering the double complex by the differential form degree, he obtained a spectral sequence:

$$E_1^{pq} = H^{p-q}_{\mathrm{diff}}(G, S^q(\mathfrak{g}^*)) \Rightarrow H^{p+q}(BG). \tag{5.2}$$

In particular, the first page of the spectral sequence vanishes above the diagonal. On the other hand, for any representation E of a compact Lie group G:

$$H^k_{\mathrm{diff}}(G, E) = 0, \qquad \text{for } k > 0,$$

because one can use integration with respect to a Haar measure to construct an explicit contraction for the differential. Thus, in the compact case, the E_1 terms of the spectral sequence vanish outside the diagonal and Bott concludes:

$$S^p(\mathfrak{g}^*)^G \cong H^{2p}(BG),$$

which is a classical theorem of Borel [9].

5.1.2 This chapter

We generalize Bott's formula (5.1) and spectral sequence (5.2) for an arbitrary Lie groupoid G. Our main results is:

Theorem. 5.3.1 *Let G be a Lie groupoid. Then:*

$$H^p_\delta(\Omega^q(G_\bullet)) \cong H^{p-q}_{\text{diff}}(G, S^q(\text{Ad}^*)). \tag{5.3}$$

In order to make sense of the right hand side we work with representations up to homotopy. The complexes in which we represent may have negative degree and therefore, in the general case, the spectral sequence is not as simple as in the case of groups. Working in this graded context is necessary in the case of groupoids because the adjoint representation only makes sense as a representation up to homotopy. The general idea of a representation up to homotopy is to represent the groupoids in complexes of vector bundles rather than vector bundles. Consider a groupoid G and a complex of vector bundles $E^\bullet$ over M. In a representation up to homotopy, an arrow $g : x \to y$, acts as a map of complexes

$$\lambda_g : E^\bullet_x \to E^\bullet_y,$$

but we allow this action not to respect the associativity. That is, in general $\lambda_{g_1 g_2}$ and $\lambda_{g_1}\lambda_{g_2}$ are not the same map of chain complexes. However, they are homotopic maps, and there is a controlled and coherent way of choosing the homotopies. Our definition of representations up to homotopy is analogous to the infinitesimal case of Lie algebroids discussed in chapter 2, which in turn is based on Quillen's superconnections [40].
There is a second issue that one needs to deal with in order to give meaning to the right hand side of equation 5.3. It is the construction of the symmetric powers of a given representation up to homotopy. This is solved by the combinatorial construction given in chapter 4. We would like to emphasize that in the proof we only use general properties of the symmetric powers construction, so that the formula holds for any other diagonal map with the same formal properties. Also, we expect that the isomorphism class of the representation will not depend on the choice of the diagonal map. However, we will work here with the choice we made in the previous chapter.
Once theorem (5.3.1) is established, we use the De Rham model for the classifying space BG to obtain the generalization of Bott's spectral sequence:

Theorem. 5.3.2 *Let G be a Lie groupoid. There is a spectral sequence converging to the cohomology of BG:*

$$E_1^{pq} = H^{p-q}_{\text{diff}}(G, S^q(\text{Ad}^*)) \Rightarrow H^{p+q}(BG). \tag{5.4}$$

The vanishing of the spectral sequence above the diagonal does not occur in the general case because the complexes in which we represent have negative degree. In the case of a Lie group G, the vanishing below the diagonal depended on the compactness of G. We still do not know whether this vanishing is in general true for proper groupoids. However, we do know the answer to this question in two interesting cases.

Corollary. 5.3.1 *Let G be either a regular proper groupoid or the transformation groupoid of a proper action. Then, the spectral sequence*

$$E_1^{pq} = H^{p-q}_{\text{diff}}(G, S^q(\text{Ad}^*)) \Rightarrow H^{p+q}(BG),$$

vanishes below the diagonal.

Besides [11], there are two papers which are closely related to our work and have been a source of inspiration. In [25], Getzler constructed a model for equivariant cohomology of a noncompact group G. He constructed a spectral sequence:

$$E_1 \cong H^\bullet(G, S(\mathfrak{g}^*) \otimes \Omega(M)) \Rightarrow H^\bullet_G(M) \cong H^\bullet(B(G \ltimes M))),$$

which coincides with our spectral sequence (5.4). Later, Behrend [6] extended Getzler's work to stacks with a cofoliation or, in simpler terms, flat groupoids. The spectral sequence of Behrend corresponds to the case in which the groupoid is such that the adjoint representation is actually a representation, not only up to homotopy.

5.2 Representations up to homotopy of groupoids

Before giving the precise definition of a representation up to homotopy, let us briefly recall some basic facts about ordinary representations of groupoids. Given a vector bundle E over M there is a groupoid $\mathrm{GL}(E)$ over M. The arrows between $x, y \in M$ in $\mathrm{GL}(E)$ are:

$$\mathrm{GL}(E)(x,y) = \{\phi : E_x \to E_y \mid \phi \text{ is a linear isomorphism}\}.$$

This space has an obvious smooth structure that turns it into a Lie groupoid. A representation of G on E is a smooth functor $\tau : G \to \mathrm{GL}(E)$ which is the identity over M. This means that each $g : x \to y$ can be realized as a linear map:

$$\lambda_g : E_x \to E_y,$$

and this association is smooth and compatible with compositions.

Given a representation E of G, the differentiable cohomology with coefficients in E is the cohomology computed by the complex $C^\bullet(G, E)$, whose degree k part is

$$C^k(G, E) = \Gamma(G_k, t^*(E)),$$

and has differential D given by the formulas:

$$D(\eta)(g_1, \dots, g_{k+1}) = \lambda_{g_1} d_0^*(\eta) + \sum_{i=1}^{k+1} (-1)^i {d_i}^*(\eta).$$

The differentiable cohomology is denoted by $\mathrm{H}^\bullet_{\mathrm{diff}}(G, E)$. There is an obvious representation of G on the trivial line bundle over M for which all the arrows map to the identity. In this case we denote the complex simply by $C^\bullet(G)$, the cohomology by $\mathrm{H}^\bullet_{\mathrm{diff}}(G)$ and the differential by δ. The space $C^\bullet(G)$ has a natural algebra structure given by:

$$(f * h)(g_1, \dots, g_{k+p}) = f(g_1, \dots, g_k) f(g_{k+1}, \dots, g_{k+p}),$$

for $f \in C^k(G)$ and $h \in C^p(G)$. We will use the $*$ to distinguish this product from the pointwise product defined when $k = p$. The differential δ is a (graded) derivation with respect to the algebra structure, that is it satisfies:

$$\delta(f * h) = \delta(f) * h + (-1)^k f * \delta(h).$$

If E is any representation of G, the space $C^\bullet(G,E)$ has the structure of a right $C^\bullet(G)$-module. Given $\eta \in C^p(G,E)$ and $f \in C^k(G)$ their product $\eta * f \in C^{p+k}(G,E)$ is defined by:

$$(\eta * f)(g_1,\dots,g_{k+p}) = \eta(g_1,\dots,g_p)f(g_{p+1},\dots,g_{k+p}).$$

Again, we use the symbol $*$ to distinguish this product form the pointwise product. The operator D is a graded derivation with respect to this module structure:

$$D(\eta * f) = D(\eta) * f + (-1)^p \eta * \delta(f)$$

In fact, such a differential completely determines the representation of G on E, see remark 5.2.1.

Definition 5.2.1. *A nonassociative action π of a groupoid G on a vector bundle E is a smooth map $\pi : G \to \mathrm{GL}(E)$, which commutes with the target and source maps. In other words, given any $g \in G$ there is a linear map $\lambda_g : E_{s(g)} \to E_{t(g)}$ which depends smoothly on g.*

Remark 5.2.1. *There is a bijective correspondence between nonassocitive actions of G on the vector bundle E and degree one linear operators*

$$D : C^\bullet(G,E) \to C^{\bullet+1}(G,E),$$

which are graded derivations with respect to the $C^\bullet(G)$-module structure. The nonassociative action is a representation if and only if the corresponding operator squares to zero and satisfies $D(v)(1) = 0$, for $v \in C^0(G,E) = E$. We already saw that given a representation π one can construct such an operator. On the other hand, given D one can recover the action by setting:

$$\lambda_g(v) = \alpha(t(g)) + D(\alpha)(g).$$

For $g \in G$, $v \in E_{s(g)}$ and α any section of E whose value at $s(g)$ is v. It is easy to check that in this correspondence, the fact that $D^2 = 0$ corresponds precisely to the associativity of the action. The other equation guarantees that the identities in G act as the identity.

Given a graded vector bundle $E = E^0 \oplus \cdots \oplus E^l$ over M, we will denote by $C(G,E)$ the space of E-valued cochains. This space has two natural gradings. The total grading, whose degree p part is denoted by $C(G,E)^\bullet$ and defined by:

$$C(G,E)^p = \bigoplus_{i+j=p} C^i(G,E^j).$$

And the cocycle grading, whose degree p part is denoted $C^p(G,E)$ and defined by:

$$C^p(G,E) = \bigoplus_{i\geq 0} C^p(G,E^i).$$

Unless otherwise specified, we will be interested in the total grading.

Definition 5.2.2. *A representation up to homotopy of G on a graded vector bundle E over M is a linear degree one operator $D : C(G,E)^\bullet \to C(G,E)^\bullet$, such that:*

(a) $D^2 = 0$,

(b) $D(\eta * f) = D(\eta)f + (-1)^k \eta * \delta(f)$ *for* $\eta \in C(G,E)^k$ *and* $f \in C^\bullet(G)$.

The cohomology computed with respect to this operator will be denoted by $\mathrm{H}^\bullet_{\text{diff}}(G,E)$.

Since the space $C(G,E)$ is generated as a $C^\bullet(G)$-module by $\Gamma(M,E)$, the fact that the operator D is a derivation with respect to the $C^\bullet(G)$-module structure implies that it is determined by the action on sections of E. One can decompose the operator D with respect to the cocycle grading:

$$D = D_0 + D_1 + \cdots + D_{l+1}, \tag{5.5}$$

where each D_i raises the total degree by 1 and the cocycle degree by i,

$$D_i : C^k(G, E^p) \to C^{k+i}(G, E^{p-i+1}).$$

We know that the operator D_1 in the decomposition above corresponds to a nonassociative action. As in chapter 4, the fact that the other operators are linear with respect to $C^\bullet(G)$ implies that they are given by $F_i \in \Gamma(G_i, \mathrm{Hom}(s^*(E^\bullet), t^*(E^{\bullet-i+1})))$, and there is the following decomposition.

Proposition 5.2.1. *There is a one to one correspondence between representations up to homotopy of G on the graded vector space E and sets of data as follows:*

- *A differential* d_E *giving* E *the structure of a cochain complex of vector bundles.*

- *A nonassociative action* F_1 *on each* E^k.

- *For each* $i > 1$, *a cochain* $F_i \in \Gamma(G_i, \mathrm{Hom}(s^*(E^\bullet), t^*(E^{\bullet-i+1})))$.

Subject to the equations:

$$\begin{aligned}[]
[d_E, F_k](g_1, \ldots, g_k) &= \sum_{j=1}^{k-1} (-1)^{j+1} F_j(g_1, \ldots, g_j) \circ F_{k-j}(g_{j+1}, \ldots, g_k) \\
&\quad + \sum_{i=1}^{k-1} (-1)^j F_{k-1}(g_1, \ldots, g_i g_{i+1}, \ldots, g_k).
\end{aligned}$$

Definition 5.2.3. *We will say that a representation up to homotopy is unital if* $F_1(1) = Id$.

5.2.1 The category of representations up to homotopy

Given a Lie groupoid G, there is a category $\mathrm{Rep}^\infty(G)$ of representations up to homotopy of G. Suppose that E and E' are two representations up to homotopy with structure given by operators D^E and $D^{E'}$, respectively. Then, a map $\Phi \in \mathrm{Hom}_{\mathrm{Rep}^\infty(G)}(E, E')$ is a degree zero $C^\bullet(G)$-linear map

$$\Phi : C(G,E)^\bullet \to C(G,E')^\bullet,$$

that commutes with the differentials. Clearly, the composition of two such maps is a map of representations, thus $\mathrm{Rep}^\infty(G)$ is a category. This assignment of the category of representations is functorial.

Remark 5.2.2. *Suppose $\phi : H \to G$ is a morphism of Lie groupoids and E is a representation up to homotopy of G. Then, the pullback complex of vector bundles $\phi^*(E)$ has the structure of a representation up to homotopy of H.*

Proof. The structure of representation up to homotopy of E is given by operators

$$F_1, F_2, F_3 \dots,$$

where $F_k \in C(G_k, \mathrm{Hom}^{1-k}(s^*(E), t^*(E)))$ and they satisfy the equations:

$$\begin{aligned}[d_E, F_k](g_1, \dots, g_k) &= \sum_{j=1}^{k-1}(-1)^{j+1} F_j(g_1, \dots, g_j) \circ F_{k-j}(g_{j+1}, \dots, g_k) \\ &\quad + \sum_{i=1}^{k-1}(-1)^j F_{k-1}(g_1, \dots, g_i g_{i+1}, \dots, g_k).\end{aligned}$$

Then, the operators $\phi^*(F_k) \in C(H_k, \mathrm{Hom}^{1-k}(s^*(\phi^*(E)), t^*(\phi^*(E))))$ defined by:

$$\phi^*(F_k)(h_1, \dots, h_k) = F_k(\phi(h_1), \dots, \phi(h_k)),$$

satisfy the corresponding equations:

$$\begin{aligned}[d_{\phi*(E)}, \phi^*(F_k)](h_1, \dots, h_k) &= \sum_{j=1}^{k-1}(-1)^{j+1}\phi^*(F_j)(h_1, \dots, h_j) \circ \phi^*(F_{k-j})(h_{j+1}, \dots, h_k) \\ &\quad + \sum_{i=1}^{k-1}(-1)^j \phi^*(F_{k-1})(h_1, \dots, h_i h_{i+1}, \dots, h_k),\end{aligned}$$

and therefore give the complex $\phi^*(E)$ the structure of a representation up to homotopy of H. □

Maps between representation have a very simple structure. They can be decomposed into small pieces much in the same way in which we did it for representations. Let $\Phi : (E, D^E) \to (E', D^{E'})$ be a map between representations of G. Then, there is a decomposition:

$$\Phi : \Phi_0 + \Phi_1 + \Phi_2 + \dots,$$

where Φ_k is a $C^\bullet(G)$-linear map that increases the cocycle degree by k. The commutation with the differentials translates into the equations:

$$\sum_{i+j=k} \Phi_j D_i^E = \sum_{i+j=k} D_i^{E'} \Phi_j, \tag{5.6}$$

for $k \geq 0$. As we have seen before, the fact that Φ_k is $C^\bullet(G)$-linear implies that it is given by an element $\Psi_k \in C(G_k, \mathrm{Hom}^{-k}(s^*(E), t^*(F)))$. Denote by ${F_k}^E$ and $F_k^{E'}$ the operators corresponding to ${D_k}^E$ and ${D_k}^{E'}$, respectively. Then, equation (5.6) becomes:

$$\begin{aligned}\sum_{i+j=k}(-1)^{j+1}\Psi_j(g_1, \dots, g_j) \quad &\circ \quad {F^E}_i(g_{j+1}, \dots, g_k) = \\ &= \sum_{i+j=k}(-1)^j {F_j}^{E'}(g_1, \dots, g_j) \circ \Psi_i(g_{j+1}, \dots, g_k) \\ &\quad + \sum_{j=1}^{k-1}(-1)^{j+1}\Psi_{k-1}(g_1, \dots, g_j g_{j+1}, \dots, g_k).\end{aligned} \tag{5.7}$$

Remark 5.2.3. *We will also need to consider maps between representations of different groupoids. Let G and H be Lie groupoids with representations up to homotopy E and E', respectively. A map Ψ from E' to E consists of a smooth functor $\varphi : G \to H$ together with a map of representations $\xi : \varphi^*(E') \to E$. The functor φ gives the spaces $C(G, \varphi^*(E'))^\bullet$ and $C^\bullet(G, E)$ the structure of a right $C^\bullet(H)$-modules. Moreover, the pull-back map:*

$$\varphi^* : C(H, E')^\bullet \to C(G, \varphi^*(E'))^\bullet,$$

is $C^\bullet(H)$-linear and commutes with the differentials. Thus, there is a map:

$$\xi \circ \varphi^* : C^\bullet(H, E') \to C^\bullet(G, E),$$

which is $C^\bullet(H)$-linear and commutes with the differentials. Sometimes, when the map ξ is clear from the context, will write ϕ^* instead of $\xi \circ \varphi^*$.

Definition 5.2.4. *A map Φ between representations up to homotopy of a groupoid G is called a quasi-isomorphism if the map of vector bundles Φ_0 induces isomorphisms in cohomology pointwise. Two representations are quasi-isomorphic if there is a quasi-isomorphism between them (in either direction).*

It is a result of [20] that the categories of ordinary representations of Morita equivalent Lie groupoids are equivalent, and thus the category of representations is a transversal invariant. It is then natural to ask how much can the categories of representations up to homotopy of Morita equivalent groupoids differ. One important feature of the category $\mathrm{Rep}^\infty(G)$ is that it has a distinguished class of arrows, the quasi-isomorphism. Since we want to think of quasi-isomorphic representations as being equivalent, we can formally introduce inverses for the quasi-isomorphisms in $\mathrm{Rep}^\infty(G)$ and denote the resulting *localized* category by $\mathbb{D}er^\infty(G)$. Does a Morita equivalence $\phi : G \to H$ induce an equivalence of categories $\phi^* : \mathbb{D}er^\infty(H) \to \mathbb{D}er^\infty(G)$? At the moment we do not have a proof of this, but we expect that it is true.

5.2.2 Duals and symmetric powers

Here we show that representations up to homotopy can be dualised in a natural way. Also, we recall the construction of the symmetric powers from chapter 4 and add some remarks that will be used in the proof of our main theorem.

Proposition 5.2.2. *Let G be a groupoid and E a representation up to homotopy of G with structure given by operators $d_E = {F^E}_0, {F^E}_1, {F^E}_2, {F^E}_3, \ldots$. Then, the dual vector bundle is also a representation up to homotopy of G.*

Proof. We need to construct operators $F_k^{E^*} \in C(G_k, \mathrm{Hom}^{1-k}(s^*(E^*), t^*(E^*))$. Namely, given $(g_1, \ldots, g_k) \in G_k$ we should have:

$$F_k^{E^*}(g_1, \ldots, g_k) \in \mathrm{Hom}((E^*)^p_{s(g_k)}, {(E^*)^{p-k+1}}_{t(g_1)}) \;\cong\; \mathrm{Hom}((E^{-p})^*_{s(g_k)}, {(E^{k-p-1})^*}_{t(g_1)}).$$

We define

$$F_k^{E^*}(g_1, \ldots, g_k)(\phi)(v) = (-1)^{k+1}\phi\left(F_k^E({g_k}^{-1}, \ldots, {g_1}^{-1})(v)\right),$$

for $\phi \in (E^{-p})^*_{s(g_k)}$ and $v \in E^{k-p-1}_{t(g_1)}$. We need to prove that these operators satisfy the equations

$$\sum_{j=0}^{k}(-1)^{j+1}F_j^{E^*}(g_1,\dots,g_j) \circ F_{k-j}^{E^*}(g_{j+1},\dots,g_k) + \sum_{j=1}^{k-1}(-1)^j F_{k-1}^{E^*}(g_1,\dots,g_j g_{j+1},\dots,g_k) = 0.$$

For this we compute:

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^{j+1}F_j^{E^*}(g_1,\dots,g_j) \circ F_{k-j}^{E^*}(g_{j+1},\dots,g_k)(\phi)(v)\\
&+\sum_{j=1}^{k-1}(-1)^j F_{k-1}^{E^*}(g_1,\dots,g_j g_{j+1},\dots,g_k)(\phi)(v)\\
=\;&\sum_{j=0}^{k} F_{k-j}^{E^*}(g_{j+1},\dots,g_k)(\phi)\left(F_j^E({g_j}^{-1},\dots,{g_1}^{-1})(v)\right)\\
&+\sum_{j=1}^{k-1}(-1)^{j+k}\phi\left(F_{k-1}^E({g_k}^{-1},\dots,{g_{j+1}}^{-1}{g_j}^{-1},\dots,{g_1}^{-1})(v)\right)\\
=\;&\sum_{j=0}^{k}(-1)^{k-j+1}\phi\left(F_{k-j}^E({g_k}^{-1},\dots,{g_{j-1}}^{-1}) \circ F_j^E({g_j}^{-1},\dots,{g_1}^{-1})(v)\right)\\
&+\sum_{j=1}^{k-1}(-1)^{j+k}\phi\left(F_{k-1}^E({g_k}^{-1},\dots,{g_{j+1}}^{-1}{g_j}^{-1},\dots,{g_1}^{-1})(v)\right)\\
=\;&\phi\left(\sum_{i=0}^{k}(-1)^{i+1}\phi\left(F_i^E({g_k}^{-1},\dots,{g_{k-i-1}}^{-1}) \circ F_{k-i}^E({g_{k-i}}^{-1},\dots,{g_1}^{-1})(v)\right)\right)\\
&+\phi\left(\sum_{i=1}^{k-1}(-1)^i\phi\left(F_{k-1}^E({g_k}^{-1},\dots,{g_{k-i+1}}^{-1}{g_{k-i}}^{-1},\dots,{g_1}^{-1})(v)\right)\right)\\
=\;&\phi(0)=0.
\end{aligned}$$

□

The main result of chapter 4 is the computation of formulas that give tensor products and symmetric powers of representations up to homotopy. We showed that, given a representation up to homotopy E of G, there is a representation up to homotopy on the complex $S^k(E)$. The components of this structure are given by a diagonal map $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}$, where $\mathbf{\Omega}$ is an algebra of planar trees which is universal for that problem. There are two properties of our construction of the symmetric powers that will be important in what follows. They are the fact that the construction is natural with respect to pull-back and functoriality.

Proposition 5.2.3. *Let E be a representation up to homotopy of a groupoid H and suppose that there is a map of groupoids $\varphi : G \to H$. Then, the representations up to homotopy $\varphi^*(S^k(E))$ and $S^k(\varphi^*(E))$ coincide.*

Proof. This property follows from the general form of the construction of the symmetric powers. Let $F_1, F_2, \ldots$ be the operators giving E the structure of representation up to homotopy of H. Also, denote by $F_1^A, F_2^A, \ldots$ and $F_1^B, F_2^B, \ldots$ the operators corresponding to $\varphi^*(S^k(E))$ and $S^k(\varphi^*(E))$, respectively. We need to prove that $F_i^A = F_i^B$ for all $i = 1, 2, \ldots$. Recall that since E is a representation up to homotopy of H, there is a map:

$$\Phi : \mathbf{\Omega} \to \overline{C}(H, \mathrm{End}(E)),$$

which allows one to view the elements of $\mathbf{\Omega}$ as operators in $\overline{C}(H, \mathrm{End}(E))$. On the other hand, the map $\varphi : G \to H$ induces a map:

$$\varphi^* : \overline{C}(H, \mathrm{End}(E)) \to \overline{C}(G, \mathrm{End}(\varphi^*(E))).$$

Namely,

$$\varphi^*(\eta)(g_1, \ldots, g_i) = \eta(\varphi(g_1), \ldots, \varphi(g_i)).$$

The structure of representation up to homotopy of $\varphi^*(E)$ is given by the map

$$\varphi^* \circ \Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(\varphi^*(E))).$$

The operator F_i^B has the form:

$$F_i^B = \sum_{T_1, \ldots, T_k} \mu_{T_1, \ldots, T_k} (\varphi^* \circ \Phi)(T_1) \otimes \cdots \otimes (\varphi^* \circ \Phi)(T_k),$$

where the sum runs over all sequences of forests in $\mathbf{\Omega}$ and the coefficients $\mu_{T_1, \ldots, T_k}$ are given by the diagonal map $\Delta_k : \mathbf{\Omega} \to \mathbf{\Omega}^{\otimes k}$. On the other hand, the operator F_i^A is given by:

$$\begin{aligned} F_i^A &= \varphi^* \left(\sum_{T_1, \ldots, T_k} \mu_{T_1, \ldots, T_k} \Phi(T_1) \otimes \cdots \otimes \Phi(T_k) \right) \\ &= \sum_{T_1, \ldots, T_k} \mu_{T_1, \ldots, T_k} (\varphi^* \circ \Phi)(T_1) \otimes \cdots \otimes (\varphi^* \circ \Phi)(T_k) \\ &= F_i^B. \end{aligned}$$

□

In our construction of the tensor products and symmetric powers we used the algebra $\mathbf{\Omega}$ to find some universal formulas which work for all groupoids and representations. For this reason, we expect the construction to be very functorial with respect to maps of representations. However, proving the functoriality in general would require to solve a new series of equations in the algebra $\mathbf{\Omega}$. It turns out that for our purposes it is enough to have functoriality with respect to a very special class of maps.

Definition 5.2.5. *Let G be a groupoid E and E' representations up to homotopy of G and $\zeta : E \to E'$ a map of representations. We say that the map ζ is simple if in the decomposition given be the cocycle degree:*

$$\zeta = \zeta_0 + \zeta_1 + \zeta_2 \ldots,$$

all the operators ζ_k are zero for $k > 0$. In other words, a simple map of representations up to homotopy is just a map of graded vector bundles which commutes with all the structure.

Proposition 5.2.4. *Suppose that $\zeta : E \to E'$ is a simple map of representations up to homotopy. Then, the induced map of vector bundles $\zeta_0^{\otimes k} : S^k(E) \to S^k(E)$ is a simple map of representations up to homotopy.*

Proof. Since $\zeta = \zeta_0$ is a map of representations, it satisfies the equations:

$$D_i^{E'}\zeta_0 = \zeta_0 D_i^E.$$

Equivalently, in terms of the operators $F_i^E, F_i^{E'}$ and Ψ_0 that appear in equation (5.7) this equations become:

$$F_i^{E'}\Psi_0 = (-1)^{i-1}\Psi_0 F_i^E.$$

We want to prove that:

$$F_i^{S^k(E')}\Psi_0^{\otimes k} = (-1)^{i-1}\Psi_0^{\otimes k}F_i^{S^k(E)}.$$

As in the proof of the previous proposition, the fact that E and E' are representations up to homotopy of G gives maps:

$$\Phi : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E)),$$

and

$$\Phi' : \mathbf{\Omega} \to \overline{C}(G, \mathrm{End}(E')).$$

One easily shows that, for any $T \in \mathbf{\Omega}$:

$$\Psi_0\Phi(T) = (-1)^{|T|}\Phi'(T)\Psi_0.$$

We can now compute:

$$\begin{aligned}
\Psi_0^{\otimes k}F_i^{S^k(E)} &= \Psi_0^{\otimes k}\left(\sum_{T_1,\ldots,T_k}\mu_{T_1,\ldots,T_k}\Phi(T_1)\otimes\cdots\otimes\Phi(T_k)\right)\\
&= \sum_{T_1,\ldots,T_k}\mu_{T_1,\ldots,T_k}\Psi_0\Phi(T_1)\otimes\cdots\otimes\Psi_0\Phi(T_k)\\
&= \sum_{T_1,\ldots,T_k}(-1)^{|T_1|+\cdots+|T_k|}\mu_{T_1,\ldots,T_k}\Phi'(T_1)\Psi_0\otimes\cdots\otimes\Phi'(T_k)\Psi_0\\
&= \sum_{T_1,\ldots,T_k}(-1)^{i+1}\mu_{T_1,\ldots,T_k}\Phi'(T_1)\Psi_0\otimes\cdots\otimes\Phi'(T_k)\Psi_0\\
&= (-1)^{i-1}F_i^{S^k(E')}\Psi_0^{\otimes k}.
\end{aligned}$$

□

5.2.3 Basic properties and cohomology

The subject of this subsection is the cohomology with coefficients in representations up to homotopy. We pay particular attention to the case in which the complex is regular. In this case, we describe a spectral sequence and deduce some consequences of it. The results we obtain here are the global counterpart of those of section 3 of chapter 2.
Recall that we say that a complex of vector bundles (E, d_E) is regular if d_E has constant

rank. We say that it is acyclic if $\mathcal{H}^\bullet(E_x, d_E) = 0$ for all $x \in M$. Given a representation up to homotopy of G on the complex (E, d_E), there is a decreasing filtration

$$\cdots \subseteq F^2(C(G,E)) \subseteq F^1(C(G,E)) \subseteq F^0(C(G,E)) = C(G,E),$$

where

$$F^k(C(G,E)) = C(G,E) \cap (C^k(G,E) \oplus C^{k+1}(G,E) \oplus C^{k+2}(G,E) \oplus \cdots).$$

This filtration gives a spectral sequence converging to $H^{p+q}_{\text{diff}}(G,E)$ whose second page can be computed when the complex is regular.

Proposition 5.2.5. *Let E be a unital representation up to homotopy of a Lie groupoid G on a regular complex. Then, the cohomology vector bundles $\mathcal{H}^q(E)$ have the structure of ordinary representations of G and there is a spectral sequence:*

$$\mathcal{E}_2^{pq} \cong H^p_{\text{diff}}(G, \mathcal{H}^q(E)) \Rightarrow H^{p+q}_{\text{diff}}(G,E).$$

Proof. The formula $\lambda_g(\overline{v}) = \overline{\lambda_g(v)}$ defines the representation structure on $\mathcal{H}^q(E)$. On the other hand, one easily shows that $\mathcal{E}_1^{p,q} \cong C^p(G, \mathcal{H}^q(E))$ and that the differential $d_1^{p,q}$ is the one given by the representation structure. □

Corollary 5.2.1. *For any representation up to homotopy on an acyclic complex (E, d_E),*

$$H^\bullet_{\text{diff}}(G,E) = 0.$$

Corollary 5.2.2. *For a proper Lie groupoid G and a regular unital representation up to homotopy E,*

$$H^q_{\text{diff}}(G,E) \cong \Gamma(\mathcal{H}^q(E))_{inv}.$$

Proof. We know that $\mathcal{E}_2^{pq} \cong H^p_{\text{diff}}(G, \mathcal{H}^q(E))$. Since the groupoid is proper, proposition 1 of [20] implies that $H_{\text{diff}}(G, \mathcal{H}^q(E)) \cong H^0_{\text{diff}}(G, \mathcal{H}^q(E)) = \Gamma(\mathcal{H}^q(E))_{inv}$. We conclude that the spectral sequence degenerates at the second page. □

We can also use the spectral sequence to prove that quasi-isomorphisms induce isomorphisms in cohomology.

Proposition 5.2.6. *Let E, F be representations up to homotopy of G.*
If $\Phi : C(G,E) \to C(G,F)$ is a quasi-isomorphism, then the map induced in cohomology

$$\Phi : H^\bullet_{\text{diff}}(G,E) \to H^\bullet_{\text{diff}}(G,F),$$

is an isomorphism.

Proof. Φ induces a map of spectral sequences $\Phi : (\mathcal{E}_r^{p,q}(E), d_r^{p,q}) \to (\mathcal{E}_r^{p,q}(F), d_r^{p,q})$ which is an isomorphism for $r = 1$. The result follows from the comparison theorem of spectral sequences. □

The following is the global analog of theorem 3.37 from chapter two. Since the proof can be copied word by word from the one of that theorem, we omit it here.

Theorem 5.2.1. *Let E be a regular unital representation up to homotopy of a groupoid G. Then, there is a representation up to homotopy structure in the cohomology complex $\mathcal{H}(E)$, with zero differential, and a quasi-isomorphism:*

$$\Phi : C(G,E) \to C(G, \mathcal{H}(E)).$$

5.2.4 The adjoint representation

We will see that for any Lie groupoid G, the adjoint complex $A \to TM$ has the structure of a representation up to homotopy which is well defined up to isomorphism. This construction makes precise the analogy between the adjoint representation of a Lie group and the tangent bundle of a manifold, which appear as extreme cases. For this reason, the adjoint representation will be thought of as being the *vector fields on the quotient space*. In order to define the adjoint representation some choices have to be made. However, we will see later that it is well defined up to isomorphism. Consider the following short exact sequence of vector bundles over G:

$$0 \longrightarrow t^*(A) \xrightarrow{\chi_t} TG \xrightarrow{ds} s^*(TM) \longrightarrow 0,$$

where the map χ_t is given by right multiplication. When restricted to M, this sequence is canonically split. One can use partitions of unity to construct a splitting $\sigma_s : s^*(TM) \to TG$ which restricts to the given one on M.

Definition 5.2.6. *An Ehresmann connection on a Lie groupoid G is a map of vector bundles $\sigma_s : s^*(TM) \to TG$ such that $ds \circ \sigma_s = id$ and, when restricted to M, is the canonical map given by the inclusion of M into G.*

Let us fix an Ehresmann connection σ_s on G. Obviously, σ_s induces a projection map $\pi_t : TG \to t^*(A)$. We will call the distribution $\sigma_s(s^*(TM))$ the horizontal distribution and denote it by H. There is another short exact sequence:

$$0 \longrightarrow s^*(A) \xrightarrow{\chi_s} TG \xrightarrow{dt} t^*(TM) \longrightarrow 0,$$

where $\chi_s(g) = d\iota(g^{-1}) \circ \chi_t(g^{-1})$. This sequence also becomes split once σ_s has been chosen. The splitting σ_t of this sequence is given by:

$$\sigma_t(g) = d\iota(g^{-1}) \circ \sigma_s(g^{-1}).$$

Of course, this also gives the corresponding map $\pi_s : TG \to s^*(A)$. These operators will be used in the definition of the adjoint representation of G. Let us organize some of the relations that they satisfy in order to simplify computations.

Remark 5.2.4. *The operators σ_s, σ_t, π_s and π_t satisfy the following equations:*

$$\pi_t = \pi_s \circ d\iota, \tag{5.8}$$
$$\pi_s = \pi_t \circ d\iota, \tag{5.9}$$
$$d\iota(g^{-1}) \circ \sigma_t(g^{-1}) = \sigma_s(g), \tag{5.10}$$
$$\sigma_s \circ ds + \chi_t \circ \pi_t = id, \tag{5.11}$$
$$\sigma_t \circ dt + \chi_s \circ \pi_s = id, \tag{5.12}$$
$$d\iota(g^{-1}) \circ \chi_t(g^{-1}) = \chi_s(g), \tag{5.13}$$
$$d\iota(g^{-1}) \circ \chi_s(g^{-1}) = \chi_t(g), \tag{5.14}$$
$$dR_{g_2}(g_1) \circ \chi_t(g_1) = \chi_t(g_1 g_2), \tag{5.15}$$
$$dL_{g_1}(g_2) \circ \chi_s(g_2) = \chi_s(g_1 g_2), \tag{5.16}$$
$$dm(g_1, g_2)(\chi_s(g_1)(v), \chi_t(g_2)(v)) = 0. \tag{5.17}$$

Here R_g and L_g denote the right and left multiplication by g, m the multiplication map and ι the inversion map.
The adjoint complex of a Lie groupoid G is the complex of vector bundles $\rho : A \to TM$. Where ρ denotes the anchor map, A has degree zero and TM degree one. We will denote it by $\mathrm{Ad}(G)$ or simply by Ad when there is no risk of confusion.

Definition 5.2.7. *The adjoint representation of G (induced by σ_s), denoted $\mathrm{Ad}(G)_{\sigma_s}$, is the representation up to homotopy on the adjoint complex $A \to TM$ given by the following data:*

(a) The non-associative action is defined by

$$\lambda_g(v) = -(\pi_t \circ \chi_s)(g)(v),$$

for $v \in A_{s(g)}$, and by

$$\lambda_g(X) = (dt \circ \sigma_s)(g)(X).$$

for $X \in TM_{s(g)}$.

(b) The operator $F_2 \in \Gamma(G_2, Hom(s^*(TM), t^*(A)))$ is defined by:

$$F_2(g_1, g_2)(X) = \pi_t(g_1g_2)(dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(X)), \sigma_s(g_2)(X))),$$

where dm denotes the derivative of the multiplication map.

Equivalently, the operator F_2 can be determined by the equation:

$$\chi_t(g_1g_2) \circ F_2(g_1, g_2)(X) = dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(X)), \sigma_s(g_2)(X)) - \sigma_s(g_1g_2)(X). \tag{5.18}$$

Theorem 5.2.2. *Let G be a Lie groupoid and σ_s an Ehresmann connection on G.*

(1) The operators λ_g, F_2 defined above give the adjoint complex Ad the structure of a unital representation up to homotopy.

(2) If γ_s is any other Ehresmann connection on G, then the representations up to homotopy $\mathrm{Ad}(G)_{\sigma_s}$ and $\mathrm{Ad}(G)_{\gamma_s}$ are naturally isomorphic.

Proof. We only need to prove the associativity equations (4.3). The first equation says that ρ is a differential for the graded vector bundle, which becomes empty in this case because there are only two nontrivial degrees. The next equation is:

$$\lambda_g(\rho(v)) = \rho(\lambda_g(v)),$$

for any $v \in A_{s(g)}$. Let us compute using equation (5.12):

$$\begin{aligned}\rho(\lambda_g(v)) &= -dt(g)\chi_t(g)\pi_t(g)\chi_s(g)(v) \\ &= -dt(g)\chi_s(g)(v) + dt(g)\sigma_s(g)ds(g)\chi_s(g)(v) \\ &= \lambda_g(\rho(v)).\end{aligned}$$

Next, we need to prove the equation:

$$\lambda_{g_1}(\lambda_{g_2}(w)) - \lambda_{g_1g_2}(w) = F_2(g_1, g_2)(\rho(w)) + \rho(F_2(g_1, g_2)(w)), \tag{5.19}$$

for any $w \in \mathrm{Ad}_{s(g_2)}$. For $v \in \mathrm{Ad}^0_{s(g_2)} = A_{s(g_2)}$, the equation becomes:

$$\lambda_{g_1}(\lambda_{g_1}(v)) - \lambda_{g_1 g_2}(v) = F_2(g_1, g_2)(\rho(v)). \tag{5.20}$$

We will use the equation:

$$\sigma_s(g)(\rho(v)) = \chi_s(g)(v) + \chi_t(g)(\lambda_g(v)). \tag{5.21}$$

With this in mind, we can compute:

$$\begin{aligned} F_2(g_1, g_2)(\rho(v)) &= \pi_t(g_1 g_2)(dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(\rho(v))), \sigma_s(g_2)(\rho(v)))) \\ &= \pi_t(g_1 g_2)(dm(g_1, g_2)(\sigma_s(g_1)(\rho(\lambda_{g_2}(v))), \sigma_s(g_2)(\rho(v)))) \\ &= \pi_t(g_1 g_2)(dm(g_1, g_2)(\chi_t(g_1)(\lambda_{g_1}\lambda_{g_2}(v)), 0)) \\ &\quad + \pi_t(g_1 g_2)(dm(g_1, g_2)(0, \chi_s(g_2)(v))) \\ &\quad + \pi_t(g_1 g_2)(dm(g_1, g_2)(\chi_s(g_1)(\lambda_{g_2}(v)), \chi_t(g_2)(\lambda_{g_2}(v)))) \\ &= \lambda_{g_1}\lambda_{g_2}(v) + \pi_t(g_1 g_2)\chi_s(g_1 g_2)(v) = \lambda_{g_1}\lambda_{g_2}(v) - \lambda_{g_1 g_2}(v). \end{aligned}$$

In the computation we used that

$$dm(g_1, g_2)(\chi_s(g_1)(\lambda_{g_2}(v)), \chi_t(g_2)(\lambda_{g_2}(v))) = 0,$$

which is an instance of equation (5.17).

For $X \in \mathrm{Ad}^1_{s(g_2)} = TM_{s(g_2)}$, the equation (5.19) becomes:

$$\lambda_{g_1}(\lambda_{g_2}(X)) - \lambda_{g_1 g_2}(X) = \rho(F_2(g_1, g_2)(w)). \tag{5.22}$$

Again, we can explicitly compute:

$$\begin{aligned} \rho(F_2(g_1, g_2)(X)) &= dt(g_1 g_2)\chi_t(g_1 g_2)\pi_t(g_1 g_2)dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(X)), \sigma_s(g_2)(X)) \\ &= dt(g_1 g_2)dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(X)), \sigma_s(g_2)(X)) \\ &\quad - dt(g_1 g_2)\sigma_s(g_1 g_2)ds(g_1 g_2)dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(X)), \sigma_s(g_2)(X)) \\ &= dt(g_1 g_2)\sigma_s(g_1)(\lambda_{g_2}(X)) - dt(g_1 g_1)\sigma_s(g_1 g_2)(X) \\ &= \lambda_{g_1}(\lambda_{g_2}(X)) - \lambda g_1 g_2(X). \end{aligned}$$

The last of the associativity conditions that we need to prove is:

$$\lambda_{g_1}(F_2(g_2, g_3)(X)) - F_2(g_1 g_2, g_3)(X) = F_2(g_1, g_2)(\lambda_{g_3}(X)) - F_2(g_1, g_2 g_3)(X).$$

Using equations (5.22) and (5.21) one can easily show that:

$$F_2(g_1, g_2 g_3)(X) + \lambda_{g_1} \circ F_2(g_2, g_3)(X),$$

is equal to

$$\pi_t(g_1 g_2 g_3) \circ dm(g_1, g_2 g_3)(\sigma_s(g_1)(\lambda_{g_2} \circ \lambda_{g_3})(X) - \chi_s(g_1)(F_2(g_2 g_3)(X)), \sigma_s(g_2 g_3)(X)).(\clubsuit)$$

Next, we can use equations (5.18) and (5.17) to prove that:

$$(\clubsuit) = \pi_t(g_1 g_2 g_3) \circ dm(g_1, g_2 g_3)(\sigma_s(g_1)(\lambda_{g_2} \circ \lambda_{g_3}(X)), \mathrm{d}m(g_2, g_3)(\sigma_s(g_2)(\lambda_{g_3}(X)), \sigma_s(g_3)(X))).(\maltese)$$

Then, using the associativity of the multiplication and equation (5.18), one shows that:

$$(\maltese) = F_2(g_1g_2, g_3)(X) + F_2(g_1g_2)(\lambda_{g_3}(X)).$$

This completes the proof of (1).
Let us now prove the second claim. A map between the representations Ad_{σ_s} and Ad_{γ_s} is given by operators $\Psi_k \in C(G_k, Hom^k(s^*(TM), t^*(A)))$ satisfying the equations:

$$\begin{aligned}\sum_{i+j=k}(-1)^{j+1}\Psi_j(g_1,\ldots,g_j) \quad \circ \quad & F^{\mathrm{Ad}^{\sigma_s}}{}_i(g_{j+1},\ldots,g_k) = \\ = \quad & \sum_{i+j=k}(-1)^j F_j^{\mathrm{Ad}^{\gamma_s}}(g_1,\ldots,g_j)\circ\Psi_i(g_{j+1},\ldots,g_k) \\ + \quad & \sum_{j=1}^{k-1}(-1)^{j+1}\Psi_{k-1}(g_1,\ldots,g_jg_{j+1},\ldots,g_k).\end{aligned}$$

We define the bundle map:

$$\Psi_0(v) = (-1)^{|v|}v,$$

and

$$\Psi_1(g)(X) = \sigma_s(g)(X) - \gamma_s(g)(X).$$

Since in this case the representations have length one, we only need to prove the equations for $k = 0, 1, 2$, namely:

$$F_0^{\gamma_s}\circ\Psi_0 + \Psi_0\circ F_0^{\sigma_s} = 0, \tag{5.23}$$

$$\Psi_1(g)\circ F_0^{\sigma_s} - \Psi_0\circ F_1{}^{\sigma_s}(g) = F_0^{\gamma_s}\circ\Psi_1(g) - F_1{}^{\gamma_s}(g)\circ\Psi_0, \tag{5.24}$$

and

$$\Psi_1(g_1)\circ F_1^{\sigma_s}(g_2) - \Psi_0\circ F_2^{\sigma_s}(g_1,g_2) = F_2^{\gamma_s}(g_1,g_2)\circ\Psi_0 - F_1^{\gamma_s}(g_1)\circ\Psi_1(g_2) + \Psi_1(g_1,g_2). \tag{5.25}$$

The first equation is obviously satisfied. Equation (5.24) applied to $X \in T_{s(g)}M$ becomes:

$$\lambda_g^{\sigma_s}(X) - \gamma_g^{\sigma_s}(X) = \rho\circ\Psi_1(g)(X),$$

which is an easy computation:

$$\begin{aligned}\rho\circ\Psi_1(g)(X) &= \rho(\sigma_s(g)(X) - \gamma_s(g)(X)) \\ &= \lambda_g^{\sigma_s}(X) - \lambda_g^{\gamma_s}(X).\end{aligned}$$

On the other hand, when applied to $v \in A_{s(g)}$, equation (5.24) becomes:

$$\lambda_g^{\sigma_s}(v) - \lambda_g^{\gamma_s}(v) = \Psi_1(g)(\rho(v)). \tag{5.26}$$

Using equation (5.21) one computes:

$$\begin{aligned}\Psi_1(g)(\rho(v)) &= \sigma_s(g)(\rho(v)) - \gamma_s(\rho(v)) \\ &= \chi_s(g)(v) + \chi_t(g)(\lambda_g^{\sigma_s}(v)) - \chi_s(g)(w) - \chi_t(g)(\lambda_g^{\gamma_s}(w)) \\ &= \chi_t(g)(\lambda_g^{\sigma_s}(v)) - \chi_t(g)(\lambda_g^{\gamma_s}(v)) = \lambda_g^{\sigma_s}(v) - \lambda_g^{\gamma_s}(v).\end{aligned}$$

Equation (5.25) is trivial when applied to $v \in A_{s(g)}$. When applied to $X \in TM$, it becomes:

$$F_2^{\sigma_s}(g_1, g_2)(X) - F_2^{\gamma_s}(g_1, g_2)(X) = \lambda^{\gamma_s}{}_{g_1}(\Psi_1(g_2)(v)) + \Psi_1(g_1)(\lambda^{\sigma_s}{}_{g_2}(X)) - \Psi_1(g_1 g_2)(X),$$

which can also be verified by a simple computation. Finally, it is clear that the map is an isomorphism with inverse given by $\Psi_0 - \Psi_1$. □

Given a Lie groupoid G over M, one can apply the functor T that takes tangent bundles and obtain a groupoid TG over TM whose structure maps are the differentials of the structure maps of G. The horizontal distribution H contains all the identities of TG. However, it need not be a subgroupoid. This is precisely the reason why the adjoint representation is only a representation up to homotopy. In case the horizontal distribution H is a subgroupoid of TG, the adjoint representation is just a pair of ordinary representations on A and TM that commute with the anchor. Behrend [7] calls a horizontal distribution which is a subgroupoid of TG a connection on G. He uses connections to define the Cech complexes that appear in his Hodge to De Rham spectral sequence. In general, one can only guarantee the existence of horizontal distributions which are not subgroupoids of TG and this amounts to working with representations up to homotopy.

Remark 5.2.5. *The horizontal distribution H is a subgroupoid of TG if and only if $F_2 = 0$.*

Proof. It is clear from the definition that $F_2 = 0$ if and only if

$$dm(g_1, g_2)(\sigma_s(g_1)(\lambda_{g_2}(X)), \sigma_s(g_2)(X)) \in H_{g_1 g_2}, \tag{5.27}$$

for all $(g_1, g_2) \in G_2$. Relation (5.27) is necessary for H to be a subgroupoid, since the product of elements of H should be in H.
Let us now see that relation (5.27) is also sufficient. We need to prove that the inverse of a horizontal vector is horizontal and that the composition of horizontal vectors is horizontal. Take a vector $w \in H_g$. Then, $w = \sigma_s(g)(X)$ for some $X \in T_{s(g)}(M)$. Now consider the product:

$$dm(g^{-1}, g)(\sigma_s(g^{-1})(\lambda_g(X)), \sigma_s(g)(X)) \in H_{s(g)}. \tag{5.28}$$

It is clear that:

$$ds(s(g)) \circ dm(g^{-1}, g)(\sigma_s(g^{-1})(\lambda_g(X)), \sigma_s(g)(X)) = ds(g)(\sigma_s(g)(X)) = X.$$

This implies that:

$$dm(g^{-1}, g)(\sigma_s(g^{-1})(\lambda_g(X)), \sigma_s(g)(X)) = \sigma_s(s(g))(X),$$

which is an identity in TG. This shows that the inverse of w is horizontal.
Next, take $(w, v) = (\sigma_{g_1}(X), \sigma_{g_2}(Y)) \in H \times_{TM} H$. We claim that $w = \sigma_s(g_1)(\lambda_{g_2}(Y))$. It is obvious that $w, \sigma_{g_1}(\lambda_{g_2}(Y))$ are both horizontal and also

$$ds(g_1)(w) = dt(g_2)(v) = \lambda_{g_2}(Y) = ds(g_1)(\sigma_s(g_1)(\lambda_{g_2}(Y))).$$

Thus, the product of w and v is horizontal because of relation (5.27). □

Example 5.2.1. *This illustrates the adjoint representation in some simple cases.*

1. *When G is a Lie group one recovers the usual adjoint representation.*
2. *If M is a manifold seen as a unit groupoid, then the adjoint complex is concentrated in degree one. Moreover, we have:*
$$H(M, \mathrm{Ad}) \cong H(M, \mathrm{Ad})^1 \cong \mathfrak{X}(M).$$
3. *Let G be a Lie group and $\pi : P \to B$ a principal G-bundle. Denote by $G \ltimes P$ the transformation groupoid. Then, $\pi^*(TB)$ is a representation of $G \ltimes P$ in a tautological way. The representation $\pi^*(TB)$ is quasi-isomorphic to the adjoint representation.*
4. *If the anchor map $\rho : A \to TM$ is injective, then the adjoint representation is quasi-isomorphic to the representation on the normal bundle $\nu = TM/A$ concentrated in degree one.*

5.3 The formula of Bott

In [11] Bott proves the following formula for a Lie group G,

$$H^p(\Omega^q(G_\bullet)) \cong H^{p-q}_{\text{diff}}(G, S^q(g^*)). \tag{5.29}$$

The right hand side of the formula is the differentiable cohomology with coefficients in the symmetric powers of the coadjoint representation. The left hand side is the cohomology of the complex:

$$\Omega^q(G_0) \xrightarrow{\delta} \Omega^q(G_1) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^q(G_p) \xrightarrow{\delta} \cdots$$

in which the differential δ is the sum of pull-back maps:

$$\delta = \sum_{i=0}^{p} (-1)^i {d_i}^*,$$

where the map $d_i : G_p \to G_{p-1}$ is given by:

$$d_i(g_1, \cdots, g_p) = \begin{cases} (g_2, \ldots, g_p) & \text{if } i = 0, \\ (g_1, \ldots, g_i g_{i+1}, \ldots g_p) & \text{if } 0 < i < p, \\ (g_1, \ldots, g_{p-1}) & \text{if } i = p. \end{cases}$$

We will prove that formula (5.29) holds for an arbitrary Lie groupoid. The right hand side is only defined in terms of representations up to homotopy, as explained in previous sections. Let us start with some preliminary results that will be needed in the proof.

Recall that a Lie groupoid G is said to be proper if the map:

$$(s, t) : G_1 \to G_0 \times G_0,$$

is proper. This definition is of course based on the notion of a proper action on a manifold, the transformation groupoid of a group action is proper if and only if the action is proper. Also, a Lie group is proper if and only if it is compact.

Definition 5.3.1. *Given a surjective submersion $\pi : M \to B$, there is a Lie groupoid G over M with $G_1 = M \times_B M$. In this case, the obvious quotient map $\pi : G \to B$ is a Morita equivalence. Here we see B as a groupoid $B \rightrightarrows B$ in the trivial way. The Lie algebroid of such a groupoid is the foliation given by the fibers of π. Following the notation of [6], we will say that a groupoid associated in this way to a submersion is a banal groupoid. Banal groupoids are proper.*

Proposition 5.3.1 (Crainic [20]). *For any proper Lie groupoid G and any representation E,*

$$H^p_{\text{diff}}(G,E) = \begin{cases} \Gamma(M,E)_{inv} & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

Proof. This is proposition 1 from [20]. □

Proposition 5.3.2. *Let P be a banal groupoid associated to a surjective submersion $\pi : M \to B$. Then,*

$$H^p_{\text{diff}}(P, S^q(\text{Ad}^*)) = \begin{cases} \Omega^q(B) & \text{if } p = -q, \\ 0 & \text{if } p > -q. \end{cases}$$

More precisely, the sequence

$$0 \longrightarrow \Omega^q(B) \xrightarrow{\pi^*} C(P, S^q(\text{Ad}^*))^{-q} \xrightarrow{D} C(P, S^q(\text{Ad}^*))^{1-q} \xrightarrow{D} \cdots$$

is exact.

Proof. The degree $-q$ vector bundle $\pi^*(\Lambda^q(T^*B))$ is a representation of P in a tautological way, let us denote by D_B operator corresponding to this structure. On the other hand, from theorem 5.2.1, we know that the representation up to homotopy structure on the complex $S^q(\text{Ad}^*)$ is quasi-isomorphic to a representation D_H on its cohomology $\mathcal{H}(S^q(\text{Ad}^*))$. Let us denote by Φ this quasi-isomorphism. There is a canonical identification:

$$\mathcal{H}(S^q(\text{Ad}^*)) \cong \pi^*(\Lambda^q(T^*B)),$$

and under this identification the operators D_B and D_H coincide. Since Φ is a quasi-isomorphism, it induces isomorphisms in cohomology and therefore:

$$H^p_{\text{diff}}(P, S^q(\text{Ad}^*)) \cong H^p_{\text{diff}}(P, \mathcal{H}(S^q(\text{Ad}^*))) \cong H^p_{\text{diff}}(P, \pi^*(\Lambda^q(T^*B))).$$

Also, since P is proper and the vector bundle $\pi^*(\Lambda^q(T^*B))$ has degree $-q$, we know from proposition 5.3.1 that:

$$H^\bullet_{\text{diff}}(P, \pi^*(\Lambda^q(T^*B))) = H^{-q}_{\text{diff}}(P, \pi^*(\Lambda^q(T^*B))) = \Gamma(M, \pi^*(\Lambda^q(T^*B)))_{\text{inv}} = \Omega^q(B).$$

□

In the remaining of this section, G is a fixed Lie groupoid over M on which an Ehresmann connection σ_s has been chosen.

Remark 5.3.1. *For each $k \geq 0$, we will denote by P^k the banal groupoid associated to the surjective submersion $d_0 : G_{k+1} \to G_k$ and we set $P^{-1} = G$. Explicitly, P^k is the groupoid:*

$$\begin{array}{c} G_{k+2} \\ d_0 \Big\downarrow\Big\downarrow d_1 \\ G_{k+1} \end{array}$$

We will denote the Lie algebroid of P^k by A^k.

Remark 5.3.2. *The collection of groupoids P^k for $k \geq 0$, has the structure of a semi-simplicial groupoid i.e. a semi-simplicial object in the category of Lie groupoids. This structure is given by the maps*

$$\flat_i : P^k \to P^{k-1}, \text{ for } i = 0, \ldots, k,$$

were $\flat_i$ is equal to d_{i+2} at the level of arrows and to d_{i+1} at the level of objects. The map $\flat_0 : P^0 \to P^{-1}$ is also defined and will be used later.

There is a natural identification:

$$A^k_{(g_1,\ldots,g_{k+1})} \cong A_{t(g_1)}. \tag{5.30}$$

Also, for the tangent space of G_{k+1}, there is a decomposition:

$$T(G_{k+1})_{(g_1,\ldots,g_{k+1})} = \{(Y_1, Y_2) \in TG_{g_1} \oplus T(G_k)_{(g_2,\ldots,g_{k+1})} : ds(Y_1) = dt(Y_2)\}.$$

Using the identification $TG \cong t^*(A) \oplus s^*(TM)$ given by the Ehresmann connection, the decomposition above becomes:

$$T(G_{k+1})_{(g_1,\ldots,g_{k+1})} \cong A_{t(g_1)} \oplus T(G_k)(g_2, \ldots, g_{k+1}). \tag{5.31}$$

We will now describe the adjoint representation of the groupoid P^k in terms of the operators defining that of G.

Lemma 5.3.1. *The adjoint representation $\mathrm{Ad}(P^k)$ of the groupoid P^k with respect to the Ehresmann connection induced by σ_s is determined by the following formulas, which we write using the identifications given in equations (5.30) and (5.31).*

- *The anchor map :*
$$\rho^k : A^k_{(g_1,\ldots,g_{k+1})} \to T(G_{k+1})_{(g_1,\ldots,g_{+1})}$$
is given by $\alpha \mapsto (\alpha, 0)$, for $\alpha \in A^k_{(g_1,\ldots,g_{k+1})} = A_{t(g_1)}$.

- *The non-associative action on A^k is given by:*
$$\lambda_{(g_0,\ldots,g_{k+1})}(\alpha) = \lambda_{g_0}(\alpha),$$
for $\alpha \in A_{t(g_1)} \cong A^k_{(g_1,\ldots,g_{k+1})}$.
The general element of $T(G_{k+1})_{(g_1,\ldots,g_{k+1})}$ is of the form $(\alpha + X, Y)$ where $\alpha \in A_{t(g_1)}$, $X \in TM_{s(g_1)}$ and $Y \in T(G_k)_{(g_2,\ldots,g_{k+1})}$. On such a vector, the nonassociative action is given by:
$$\lambda_{(g_0,\ldots,g_{k+1})}(\alpha + X, Y) = (\lambda_{g_0}(\alpha) + F_2(g_0, g_1)(X) + X, Y).$$

- *Finally, there is the operator:*

$$\begin{aligned} F_2^k((g_0,\dots,g_{k+1}),(h_0,\dots,h_{k+1}))(\alpha+X,Y) &= F_2(g_0,h_0)(\lambda_{h_1}(X)) \\ &+ F_2(g_0,h_0)(\rho(\alpha)), \end{aligned}$$

for $X \in TM_{s(h_1)}$ *and* $Y \in T(G_k)_{(h_2,\dots,h_{k+1})}$ *and* $\alpha \in A_{t(h_1)}$.

Proof. This is a simple computation with the formulas for the adjoint representation given in definition 5.2.7. □

We now have a good description of the adjoint representation of the groupoids P^k. This will be useful in showing that the operators $\flat_i$ induce maps of representations between the corresponding coadjoint representations.

Proposition 5.3.3. *The map* $\flat_i : P^k \to P^{k-1}$ *induces naturally a simple map of representations:*

$$\flat_i{}^* : C(P^{k-1}, S^q(\mathrm{Ad}^*)) \to C(P^k, S^q(\mathrm{Ad}^*)).$$

Proof. In propositions 5.2.3 and 5.2.4 we proved that the operation of taking symmetric powers is natural with respect to pull-back and functorial with respect to simple maps. Therefore, it is enough to prove the statement for the case $q = 1$. We need to exhibit a map of representations of P^k between the pullback representation $\flat_i^*(\mathrm{Ad}^*(P^{k-1}))$ and $\mathrm{Ad}^*(P^k)$. This is given by the map of graded vector bundles:

$$\zeta : \flat_i^*(T^*(G_k)) \oplus \flat_i^*((A^{k-1})^*) \mapsto T^*(G_{k+1}) \oplus (A^k)^*,$$

which is the pullback by $\flat_i$ in the fist component and the identity in the first one (using the identification (5.30)). We only need to prove that this map commutes with the differentials. It easy to check that it commutes with the respective anchor maps. Let us now see that it commutes with the non-associative actions. Take $\phi \in \flat_i{}^*((A^{k-1})^*)_{(g_1,\dots,g_{k+1})} \cong A^*{}_{t(g_1)}$ and $\alpha \in A^k_{(g_0g_1,\dots,g_{k+1})} \cong A_{t(g_0)}$. In what follows we will use lemma 5.3.1 and the formulas for the dual representation given in proposition 5.2.2. We set $g = (g_0, g_1, \dots, g_{k+1}) \in G_{k+2}$ and compute:

$$\begin{aligned} \zeta(\lambda_g(\phi))(\alpha) &= \zeta(\phi(\lambda_{\flat_i(g)^{-1}}(\alpha))) \\ &= \zeta(\phi(\lambda_{(g_0{}^{-1},g_0g_1,g_2,\dots,g_{i+1}g_{i+2},\dots,g_{k+1})}(\alpha))) \\ &= \zeta(\phi(\lambda_{g_0{}^{-1}}(\alpha))) \\ &= (\lambda_g(\zeta(\phi))(\alpha). \end{aligned}$$

Next, take

$$\eta \in \flat^*{}_i(T^*(G_k))_{(g_1,\dots,g_{k+1})} = T^*(G_k)_{(g_1,\dots,g_{i+1}g_{i+2},\dots,g_{k+1})}$$

and

$$W \in T(G_{k+1})_{(g_0g_1,\dots,g_{k+1})}.$$

Using the identification (5.31) we can write $W = (\alpha + X, \beta + Y, Z)$ with $\alpha \in A_{t(g_0)}$,$\beta \in A_{t(g_2)}$, $X \in TM_{s(g_1)}$, $Y \in TM_{s(g_2)}$ and $Z \in T(G_{q-1})_{(g_3,\dots,g_{k+1})}$. We will only consider the case $i = 0$, the others are trivial. We will constantly use the following formula for the derivative of the multiplication:

$$dm_{(g_1,g_2)}(\alpha + X, \beta + Y) = \alpha + F_2(g_1, g_2)(Y) + \lambda_{g_1}(\beta) + Y. \tag{5.32}$$

Now we can prove that the map ζ commutes with the non-associative action on η. As before, we set $g = (g_0, g_1, \ldots, g_{k+1}) \in G_{k+2}$ and compute:

$$
\begin{aligned}
\zeta(\lambda_g(\eta))(W) &= \lambda_{\flat_0(g)}(\eta)(D(\flat_0)(W)) \\
&= \lambda_{\flat_0(g)}(\eta)(D(\flat_0)(\alpha + X, \beta + Y, Z)) \\
&= \lambda_{\flat_0(g)}(\eta)(\alpha + F_2(g_0g_1, g_2)(Y) + \lambda_{(g_0g_1)}(\beta) + Y, Z) \\
&= \eta((\lambda_{\flat_0(g)^{-1}}(\alpha + F_2(g_0g_1, g_2)(Y) + \lambda_{(g_0g_1)}(\beta) + Y, Z)) \\
&= \eta((\lambda_{g_0^{-1}}(\alpha) + \lambda_{g_0^{-1}}(F_2(g_0g_1, g_2)(Y)) + \lambda_{g_0^{-1}} \circ \lambda_{(g_0g_1)}(\beta), 0)) \\
&\quad + \eta(F_2({g_0}^{-1}, g_0g_1g_2)(Y) + Y, Z) \\
&= \eta((\lambda_{g_0^{-1}}(\alpha) + F_2(g_1, g_2)(Y) + \lambda_{g_0^{-1}} \circ \lambda_{(g_0g_1)}(\beta), 0)) \\
&\quad + \eta(F_2({g_0}^{-1}, g_0g_1)(\lambda_{g_2}Y) + Y, Z).
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\lambda_g(\zeta(\eta))(W) &= \zeta(\eta)(\lambda_{g^{-1}}(\alpha + X, \beta + Y, Z)) \\
&= \zeta(\eta)((\lambda_{g_0^{-1}}(\alpha) + F_2({g_0}^{-1}, g_0g_1)(X) + X, \beta + Y, Z)) \\
&= \eta(D(\flat_0)((\lambda_{g_0^{-1}}(\alpha) + F_2({g_0}^{-1}, g_0g_1)(X) + X, \beta + Y, Z)) \\
&= \eta((\lambda_{g_0^{-1}}(\alpha) + F_2({g_0}^{-1}, g_0g_1)(X) + F_2(g_1, g_2)(Y) + \lambda_{g_1}(\beta) + Y, Z)) \\
&= \eta((\lambda_{g_0^{-1}}(\alpha) + F_2({g_0}^{-1}, g_0g_1)(\lambda_{g_2}(\beta)) + F_2(g_0^{-1}, g_0g_1)(\rho(\beta)), 0)) \\
&\quad + \eta(F_2(g_1, g_2)(Y) + \lambda_{g_1}(\beta) + Y, Z) \\
&= \eta((\lambda_{g_0^{-1}}(\alpha) + F_2(g_1, g_2)(Y) + \lambda_{g_0^{-1}} \circ \lambda_{(g_0g_1)}(\beta), 0)) \\
&\quad + \eta(F_2({g_0}^{-1}, g_0g_1)(\lambda_{g_2}Y) + Y, Z).
\end{aligned}
$$

It only remains to prove that ζ commutes with the respective F_2 operators. As before, we will give the details of the case $i = 0$, which is the interesting one. Consider composable elements $g = (g_0, \ldots, g_{k+1}), h = (h_0, \ldots, h_{k+1}) \in G_{k+2}$, $\phi \in {\flat_0}^*((A^{k-1})^*)_{(h_1,\ldots,h_{k+1})} \cong A^*_{t(h_1)}$, and $W = (\alpha + X, \beta + Y, Z) \in T(G_{k+1})_{(g_0h_0,h_1,\ldots,h_{k+1})}$. Now let us compute:

$$
\begin{aligned}
F_2(g, h)(\zeta(\phi))(W) &= -\zeta(\phi)(F_2(h^{-1}, g^{-1})(W)) \\
&= -\zeta(\phi)(F_2({h_0}^{-1}, {g_0}^{-1})(\rho(\alpha)) + F_2({h_0}^{-1}, {g_0}^{-1})(\lambda_{(g_0g_1)}(X))) \\
&= -\phi\left(F_2({h_0}^{-1}, {g_0}^{-1})(\rho(\alpha)) + F_2({h_0}^{-1}, {g_0}^{-1})(\lambda_{(g_0g_1)}(X))\right).
\end{aligned}
$$

On the other hand:

$$
\begin{aligned}
\zeta(F_2(g, h)(\phi))(W) &= F_2(\flat_0(g), \flat_0(h))(\phi)(D(\flat_0)(W)) \\
&= F_2(\flat_0(g), \flat_0(h))(\phi)((\alpha + F_2(g_0g_1, g_2)(Y) + \lambda_{g_0g_1}(\beta) + Y, Z) \\
&= -\phi\left(F_2(\flat_0(h)^{-1}, \flat_0(g)^{-1})(\alpha + F_2(g_0g_1, g_2)(Y) + \lambda_{g_0g_1}(\beta) + Y, Z)\right) \\
&= -\phi\left(F_2({h_0}^{-1}, {g_0}^{-1})(\rho(\alpha)) + F_2({h_0}^{-1}, {g_0}^{-1})(\rho(F_2(g_0g_1, g_2)(Y)))\right) \\
&\quad -\phi\left(F_2({h_0}^{-1}, {g_0}^{-1})(\rho(\lambda_{g_0g_1}(\beta))) + F_2({h_0}^{-1}, {g_0}^{-1})(\lambda_{g_0g_1g_2}(Y))\right) \\
&= -\phi\left(F_2({h_0}^{-1}, {g_0}^{-1})(\rho(\alpha)) + F_2({h_0}^{-1}, {g_0}^{-1})(\lambda_{g_0g_1}(\rho(\beta)))\right) \\
&\quad -\phi\left(F_2({h_0}^{-1}, {g_0}^{-1})(\lambda_{g_0g_1} \circ \lambda_{g_2}(Y))\right) \\
&= -\phi\left(F_2({h_0}^{-1}, {g_0}^{-1})(\rho(\alpha)) + F_2({h_0}^{-1}, {g_0}^{-1})(\lambda_{(g_0g_1)}(X))\right).
\end{aligned}
$$

This completes the proof. $\square$

We will use the maps we just constructed in the following proposition, which is the key to the proof the theorem.

Proposition 5.3.4. *For each $q \geq 0$ there is a coaugmented double complex with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
0 & \longrightarrow & C(G, S^q(\mathrm{Ad}^*))^{2-q} & \xrightarrow{\flat_0{}^*} & C(P^0, S^q(\mathrm{Ad}^*))^{2-q} & \xrightarrow{\flat^*} & C(P^1, S^q(\mathrm{Ad}^*))^{2-q} & \xrightarrow{\flat^*} & \cdots \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
0 & \longrightarrow & C(G, S^q(\mathrm{Ad}^*))^{1-q} & \xrightarrow{\flat_0{}^*} & C(P^0, S^q(\mathrm{Ad}^*))^{1-q} & \xrightarrow{\flat^*} & C(P^1, S^q(\mathrm{Ad}^*))^{1-q} & \xrightarrow{\flat^*} & \cdots \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
0 & \longrightarrow & C(G, S^q(\mathrm{Ad}^*))^{-q} & \xrightarrow{\flat_0{}^*} & C(P^0, S^q(\mathrm{Ad}^*))^{-q} & \xrightarrow{\flat^*} & C(P^1, S^q(\mathrm{Ad}^*))^{-q} & \xrightarrow{\flat^*} & \cdots \\
 & & & & \uparrow d_0{}^* & & \uparrow d_0{}^* & & \\
 & & & & \Omega^q(M) & \xrightarrow{\delta} & \Omega^q(G) & \xrightarrow{\delta} & \cdots \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Proof. The maps

$$\flat^* : C(P^{k-1}, S^q(\mathrm{Ad}^*)) \to C(P^k, S^q(\mathrm{Ad}^*))$$

are defined by:

$$\flat^* = \sum_{i=0}^{k} (-1)^i \flat_i{}^*.$$

The fact that $(\flat^*)^2 = 0$ follows from the simplicial relations and remark 5.3.2. The vertical operator D is the differential corresponding to the q^{th} symmetric power of the coadjoint representation. Since, by proposition 5.3.3, each of the maps $\flat_i^*$ commutes with D, so does $\flat^*$. Proposition 5.3.2 implies that the columns of the double complex are exact. All we need to prove now is that the rows are also exact. For this we will construct an explicit contracting homotopy. Consider the maps:

$$\begin{array}{rcl}
\sigma_0 : P^{k-1}_m \cong G_{k+m} & \to & P^k_m \cong G_{k+m+1}, \\
(g_1, \dots, g_{k+m}) & \mapsto & (1, g_1, \dots, g_{k+m}).
\end{array}$$

They induce pullback maps of vector bundles:

$$\sigma_0^* : C(P^k_m, S^q(\mathrm{Ad}^*)) \to C(P^{k-1}_m, S^q(\mathrm{Ad}^*)),$$

which satisfy the equations:

$$\sigma_0^* \flat_i^* = \begin{cases} Id & \text{if } i = 0, \\ \flat_{i-1}^* \sigma_0^* & \text{if } i > 0. \end{cases}$$

We now use these relations to compute:

$$\begin{aligned}
\sigma_0^*\flat^* + \flat^*\sigma_0^* &= \sum_{i=0}^{k}(-1)^i\sigma_0^*\flat_i^* + \sum_{i=0}^{k-1}(-1)^i\flat_i^*\sigma_0^* \\
&= Id + \sum_{i=1}^{k}(-1)^i\flat_{i-1}^*\sigma_0^* + \sum_{i=0}^{k-1}(-1)^i\flat_i^*\sigma_0^* \\
&= Id - \sum_{i=0}^{k-1}(-1)^i\flat_i^*\sigma_0^* + \sum_{i=0}^{k-1}(-1)^i\flat_i^*\sigma_0^* \\
&= Id.
\end{aligned}$$

We conclude that the rows of the double complex are exact. □

Theorem 5.3.1. *Let G be a Lie groupoid. Then:*

$$H_\delta^p(\Omega^q(G_\bullet)) \cong H_{\mathrm{diff}}^{p-q}(G, S^q(\mathrm{Ad}^*)).$$

Proof. By proposition 5.3.4, the inclusion of the coaugmentations in the double complex above induce isomorphisms in cohomology. Thus, there are canonical isomorphisms:

$$H_{\mathrm{diff}}^{p-q}(G, S^q(\mathrm{Ad}^*)) \cong H^{p-q}(\mathrm{Tot}(C(P^\bullet, S^q(\mathrm{Ad}^*))^{\bullet - q})) \cong H_\delta^p(\Omega^q(G_\bullet)).$$

□

5.3.1 The spectral sequence

Recall that the cohomology of the classifying space of a Lie groupoid G can be computed as the total cohomology of the double complex:

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow \mathrm{d} & & \uparrow \mathrm{d} & & \uparrow \mathrm{d} & & \\
\Omega^2(M) & \xrightarrow{\delta} & \Omega^2(G_1) & \xrightarrow{\delta} & \Omega^2(G_2) & \xrightarrow{\delta} & \cdots \\
\uparrow \mathrm{d} & & \uparrow \mathrm{d} & & \uparrow \mathrm{d} & & \\
\Omega^1(M) & \xrightarrow{\delta} & \Omega^1(G_1) & \xrightarrow{\delta} & \Omega^1(G_2) & \xrightarrow{\delta} & \cdots \\
\uparrow \mathrm{d} & & \uparrow \mathrm{d} & & \uparrow \mathrm{d} & & \\
\Omega^0(M) & \xrightarrow{\delta} & \Omega^0(G_1) & \xrightarrow{\delta} & \Omega^0(G_2) & \xrightarrow{\delta} & \cdots
\end{array}$$

where the vertical differential d is De-Rham operator and the horizontal differential δ is the sum of pullback maps defined before. Filtering the double complex by the vertical degree, one obtains a spectral sequence:

$$E_1^{pq} \cong H_\delta^p(\Omega^q(G_p)) \Rightarrow H^{p+q}(\mathrm{Tot}(\Omega^\bullet(G_\bullet))) \cong H^{p+q}(BG).$$

The relationship with theorem 5.3.1 is now evident.

Theorem 5.3.2. *Let G be a Lie groupoid. There is a spectral sequence converging to the cohomology of BG:*

$$E_1^{pq} = H_{\text{diff}}^{p-q}(G, S^q(\text{Ad}^*)) \Rightarrow H^{p+q}(BG).$$

Proof. Direct application of theorem 5.3.1. □

Remember that for Lie groups the shift in degree immediately implies that the first page of the spectral sequence vanishes above the diagonal. In the general case, one has representations up to homotopy on vector bundles that may have negative degree and therefore there is no vanishing above the diagonal. This should come as no surprise, already in the case of compact group actions there are terms above the diagonal.
Recall that the cohomology of a compact Lie group with coefficient in a representation vanishes in degree greater than zero. This implies that Bott's spectral sequence vanishes bellow the diagonal for compact groups. A similar vanishing result occurs for the action groupoid associated to a proper action of a Lie group on a manifold. This is how Getzler's model for equivariant cohomology relates to the Cartan model [25]. We expect these facts to be particular cases of a vanishing result for cohomology of proper groupoids with coefficients in representations up to homotopy. At the moment we can not prove such general result, however, in some special cases, the cohomology vanishes as expected.

Corollary 5.3.1. *Let G be either a regular proper groupoid or the transformation groupoid of a proper action. Then, the spectral sequence*

$$E_1^{pq} = H_{\text{diff}}^{p-q}(G, S^q(\text{Ad}^*)) \Rightarrow H^{p+q}(BG),$$

vanishes below the diagonal.

Proof. In case G is regular proper, we apply corollary 5.2.2 to compute:

$$E_1^{pq} = H_{\text{diff}}^{p-q}(G, S^q(\text{Ad}^*)) \cong \Gamma(\mathcal{H}^{p-q}(S^q(\text{Ad}^*)))_{\text{inv}},$$

which vanishes if $p > q$ because the complex $S^q(Ad^*)$ is concentrated in non positive degree. If G is the groupoid of a proper action then, $D = D_0 + D_1$, and the claim follows from a standard double complex argument. □

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Samenvatting

De classificerende ruimte van een Liegroep G is een topologische ruimte BG die veel interessante eigenschappen van de groep weerspiegelt. Deze ruimte is gedefiniëerd als het quotiënt van EG met G, met EG een samentrekbare ruimte waar G vrij op werkt. De naam "classificerende ruimte" komt van het feit dat BG alle G-hoofdvezelbundels classificeert. Om precies te zijn, voor iedere ruimte X is er een bijectie tussen G- hoofdvezelbundels over X en homotopieklassen van afbeeldingen van X naar BG. Een gevolg van deze classificatie is dat gegeven een G-hoofdvezelbundel P, er een afbeelding is van de cohomologie van BG naar die van X. De klassen in het beeld van deze afbeelding heten karakteristieke klassen van P, en daarom is de cohomologie van BG de universele algebra van karakteristieke klassen van G-bundels. Het is dan een interessant probleem om de cohomologie van BG te berekenen in termen van intrinsieke invarianten van G. Het is een stelling van Borel dat deze cohomologie voor compacte Liegroepen bestaat uit de polynomen in de Lie-algebra die invariant zijn onder de actie van G. Later heeft Bott bewezen dat de cohomologie van BG in het niet-compacte geval nauw verbonden is, via een spectraalrij, met de differentieerbare cohomologie van G met waarden in een canonieke representatie van G.

De quotiëntruimte van de actie van een Liegroep G op een differentiëerbare variëteit M is vaak een erg pathologische ruimte waarvan de cohomologie geen goede invariant voor de actie is. De juiste invariant voor deze situatie is de equivariante cohomologie van M. Vervang de actie van G op M door de diagonale actie van G op $M \times EG$, die vrij is. Het homotopiequotiënt van M en G is dan het gewone quotiënt van de actie van G op $M \times EG$, en de equivariante cohomologie van M is de gewone cohomologie van dit homotopiequotiënt. De stellingen van Borel en Bott over het berekenen van de cohomologie van de classificerende ruimtes kunnen worden uitgebreid naar het berekenen van equivariante cohomologie. Dit zijn de modellen van Cartan en Getzler voor equivariante cohomologie. Het homotopiequotiënt van de actie van G op M kun je zien als de classificerende ruimte van de Liegroepoïde die van de actie afkomt. In het algemeen hebben Liegroepoïden classificerende ruimtes die de ruimte van banen van de groepoïde representeren. De belangrijkste vraag die we in dit proefschrift bestuderen is of de modellen voor equivariante cohomologie kunnen worden uitgebreid om de cohomologie van deze algemenere homotopiequotiënten uit te rekenen. Onze conclusie is dat generalisatie meestal goed gaat, mits de gewone representaties vervangen worden door representaties "modulo homotopie". In het bijzonder bewijzen we dat Botts spectraalrij bestaat voor algemene Liegroepoïden.

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Curriculum Vitae

I was born on April 22 1980 in Medellín, Colombia. In the period between 1998 and 2003 I studied mathematics at the Universidad Nacional de Colombia sede Medellín. I spent the year 2003-2004 at Utrecht University, taking part in the MRI Master Class on noncommutative geometry. I started my PhD studies at Utrecht University in 2004, under the supervision of Ieke Moerdijk and Marius Crainic. During the Fall semester 2007 I was a visiting student at the University of California at Berkeley under the supervision of Alan Weinstein. Starting January 2009 I will be a postdoc at the University of Zurich, working in the group of Alberto Cattaneo.