Chapter 10
Problem Posing as Providing Students with Content-Specific Motives

Kees Klaassen and Michiel Doorman

Abstract We interpret problem posing not as an end in itself, but as a means to add quality to students' process of learning content. Our basic tenet is that all along students know the purpose(s) of what they are doing. This condition is not easily and not often satisfied in education, as we illustrate with some attempts of other researchers to incorporate mathematical problem-posing activities in instruction. The emphasis of our approach lies on providing students with content-specific motives and on soliciting seeds in their existing ideas, in such a way that they are willing and able to extend their knowledge and skills in the direction intended by the course designer. This requires a detailed outlining of teaching–learning activities that support and build on each other. We illustrate and support our theoretical argument with results from two design-based studies concerning the topics of radioactivity and calculus.

Contents

K. Klaassen (✉) • M. Doorman
Freudenthal Institute for Science and Mathematics Education, Faculty of Science, Utrecht University, Utrecht, The Netherlands
e-mail: c.w.j.m.klaassen@uu.nl; m.doorman@uu.nl

F.M. Singer et al. (eds.), *Mathematical Problem Posing*,
Research in Mathematics Education, DOI 10.1007/978-1-4614-6258-3_10

Introduction

Among policy makers there is a growing appreciation of the educational relevance of mathematical problem posing. Given the central importance of problem posing in both pure and applied mathematics, it is argued that developing the ability to *pose* mathematical problems ought to be at least as important as developing the ability to *solve* them (e.g., National Council of Teachers of Mathematics [NCTM], 2000). This plea is reinforced by the recent UNESCO list of competences as challenges for basic mathematics education (UNESCO, 2012). Posing and solving mathematical problems is described as one of the eight major transverse competencies related to content acquisition in mathematics education. We are largely in sympathy with the plea that mathematical problem posing deserves a more prominent place in education. But we are cautious when it comes to the social and scientific benefits that mathematical problem posing is suggested to have according to many policy documents. For example, with reference to a statement of Einstein's, one often reads that by raising a new problem or by regarding an old problem from a new angle, many of the greatest scientists revolutionized their field of inquiry or even initiated an entirely new field of inquiry (Einstein & Infeld, 1938, p. 92). This is true, but it should be clear that problem posing at this level is and will remain the domain of exceptional genius, far beyond the reach of the vast majority. It may also be true that modern-day society requires flexible, creative, and mathematically able professionals. But we do not find it obvious that this demand will be met automatically by incorporating problem posing into mathematics education. Our aim for the incorporation of problem posing is much more humble, namely to increase the quality of students' process of learning mathematics.

Also among researchers there is a growing interest in mathematical problem posing. One area of research concerns the identification, characterization, operationalization, and framing of various aspects of mathematical problem posing (Christou, Mousoulides, Pittalis, Pitta-Pantazi, & Sriraman, 2005; English, 1997a; Silver & Cai, 1996; Stoyanova & Ellerton, 1996). Another area of research concerns the incorporation of mathematical problem-posing activities in mathematics education. This is done with a variety of partly overlapping aims. One aim is to gain insight into students' understanding of mathematical ideas and their perception of the nature of mathematics (Brown & Walter, 1983; Ellerton & Clarkson, 1996). This insight may function as a kind of formative assessment or perhaps even help the teacher to anticipate students' future understanding (Barlow & Cates, 2006; Van den Heuvel-Panhuizen, Middleton, & Streefland, 1995). Another aim is to develop students' actual problem-posing abilities by explicitly teaching them about what are considered to be key elements of mathematical problem posing (English, 1997a, 1997b, 1998). As a final goal, we mention the integration of mathematical problem posing within mathematical inquiry or modeling (Bonotto, 2010; Crespo & Sinclair, 2008; English, Fox, & Watters, 2005).

We do not pretend to have given a comprehensive overview, but it suffices to locate our own research. Our expertise is in secondary physics education (first author) and secondary mathematics education (second author). We are particularly interested in in-depth studies of teaching sequences about some physics or mathematics topics, for example mechanics or calculus. Our main concern differs from the ones mentioned above, which all relate to purposes of teachers or curriculum designers: *they* want to assess students' understanding, *they* want to establish what students like and dislike. In contrast, our aim concerns the purposes of students. The quality we want to add to *their* process of learning mathematics or physics is that all along they know the purpose of what they are doing.

The aim that students know what they are doing and why is not often satisfied in education, nor is it easily satisfiable. Gunstone (1992) writes in this respect: "This problem of students not knowing the purpose(s) of what they are doing, even when they have been told, is perfectly familiar to any of us who have spent time teaching. The real issue is why the problem is so common and why it is so very hard to avoid." As we will illustrate in the next section, this problem also applies to many attempts to incorporate mathematical problem-posing activities in instruction. Even when it is clear to us what the designer of such an activity wanted to achieve, we often feel that students will be at a loss as to why they are to engage in the activity. At best they will only in retrospect be able to appreciate what it has been good for.

Nearly two decades ago, we introduced an educational approach, the basic tenet of which was to bring students to such a position that, not only in retrospect, but already beforehand, they know the purpose(s) of what they are going to do. We have dubbed this approach *problem posing* because it would be a clear case of students knowing what they are doing and why, when they can be brought to such a position that (a) they themselves come to pose the main problems they are going to work on, and (b) in the process also come to appreciate the main means by which to tackle those problems. In this chapter, our approach will be further described, illustrated with two teaching sequences, and discussed. In the final section, we return to mathematical problem posing and reflect on it from the point of view of our problem-posing approach.

What Is the Point of Mathematical Problem Posing for Students?

In order to illustrate the problem of students not knowing the purpose(s) of what they are doing, Gunstone (1992) wrote: "In the following typical example, the student (P) has been asked by the interviewer (O) about the purpose of the activity they have just completed.

P: He [the teacher] talked about it…that's about all…

O: What have you decided it [the activity] is all about?

P: I dunno, I never really thought about it …. just doing—doing what it says … it's 8.5 …. just got to do different numbers and the next one we have to do is this [points in text to 8.6]."

Note that it is not the case that the student has no answer at all to the question "Why are you doing this?" The student did have an answer. More fully articulated it may be something like this: "I am now working on 8.5, because I just finished 8.4; and after I finish 8.5, I am going to do 8.6; these are the numbers the teacher told us we got to do and we are supposed to do as the teacher says." Although this fragment in itself proves nothing, we hope it will strike the reader as familiar, as exemplifying the implicit didactical contract (Tiberghien, 2000) that the teacher knows what is best for students and that students simply are to follow suit. What we especially want to draw attention to is the absence of content-specific features in the student's answer. There is not even an indication of the topic or subject he or she is working on. We do not blame the student for this. Nevertheless, it is hard to suppress a feeling of disappointment. One would have hoped for more.

The problem of students not knowing the purpose(s) of what they are doing also applies to attempts to incorporate mathematical problem-posing activities in instruction. In Figure 10.1, we have collected from the literature a variety of kinds of

Write a problem to the following story so that the answer to the problem is "385 pencils." "Alex has 180 pencils while Chris has 25 pencils more than Alex."

Write an appropriate problem for the following:
$(2300 + 1100) - 790 = n$

Last night there was a party and the host's doorbell rang 10 times. The first time the doorbell rang only one guest arrived. Each time the doorbell rang after that, three more guests arrived than had arrived on the previous ring.
Ask as many questions as you can. Try to put them in a suitable order.

Write a problem that involves use of the concept of a right-angled triangle.

Write a problem based on the following picture:

Write a problem that you would find difficult to solve.

Figure 10.1. A variety of examples of mathematical problem-posing activities.

mathematical problem-posing activities involving a variety of cognitive processes (Christou et al., 2005; Stoyanova & Ellerton, 1996).

Now, think about these examples from the point of view suggested by Gunstone (1992). What could be the point for children to engage in these activities? When asked “Why are you doing this?”, would students be able to give an answer other than “Because the teacher told us so”? Would they be able to give any content-directed reasons for being involved in the activities? We are not suggesting that students will dislike such activities or that they will not learn anything from them. We are suggesting, however, that it would add quality to an activity if students had reasons for being involved that are specifically directed at topical content.

It may be said that in order to judge whether or not an activity is purposeful for students, more must be done than just to consider a single activity in isolation (such as the ones in Figure 10.1). A particular activity may rather get its point from the way it is embedded in a series of activities. We agree, but wish to make two observations. First, we know of very little research in which mathematical problem posing is embedded in a series of connected activities. Second, in the few cases we are aware of, the problem of students not knowing the purposes of what they are doing receives very little explicit or systematic attention. Let us discuss some examples.

For third-, fifth-, and seventh-grade, English (1997a, 1997b, 1998) designed and evaluated problem-posing programs comprising about 10 weeks for about 1 hour per week. The programs consisted of a sequence of main activities: exploring attitudes towards problems; classifying problems; separating problem structures from contextual features; modeling new problems on existing structures; creating new problems from given components; transforming given problems into new problems. The rationale behind this sequence seems clear enough. It was based on what in the literature were identified as key elements of mathematical problem posing. But let us now reflect on the sequence from the point of view suggested by Gunstone. What could be the point for children to engage in these activities in this order? Children may in some general sense be (made) aware that you learn more from creating and solving your own problems than from solving ones the teacher makes up. But even given this general motive, we still doubt whether it is “logical” for them subsequently to go on to classify problems or to separate contextual features from structural elements in given problems. We do not wish to underrate the efforts of English, if only because her findings show that, with some guidance from the teacher, key components of mathematical problem posing are well within reach of students. Students may also be able to tell at a later stage of the sequence, for example when modeling new problems on existing structures, why in an earlier stage they had to classify problems. That is, in retrospect they may see the reason for what they had to do earlier. Let us also stress that one need not be moved by our considerations that center on students’ advance content-directed motives. But if one is, we conjecture that the sequence designed by English will not appear so “logical” any more.

Whereas mathematical problem posing is often promoted because it is part and parcel of mathematical inquiry, in educational settings the bond between mathematical problem posing and mathematical inquiry very often is broken. Crespo and Sinclair (2008) detected as symptoms of this broken bond an emphasis on de-contextualized problem posing (such as the examples in Figure 10.1), and on prescriptive problem-posing strategies that permit an almost effortless generation of new problems. Although we sympathize with this criticism, we think that in their study Crespo and Sinclair (2008) do not make real progress towards making mathematical problem posing functional for students within some worthwhile mathematical inquiry. As in the approaches they criticize, they too very much focus on mathematical problem posing per se, though in their case with an emphasis on the quality instead of the quantity of the problems posed. Again, we do not wish to underrate their efforts, in particular their finding that an easily implemented measure such as allowing students some exploration time will increase the quality of the problems they pose. Nevertheless, our point remains that students still are not provided with a purpose to pose mathematical problems in the first place.

Let us take stock. We have drawn attention to the problem of students not knowing the purpose(s) of what they are doing, and we have also illustrated that with respect to mathematical problem posing this is quite common. We agree that it is useful for a curriculum designer to have a clear idea of key elements of mathematical problem posing. We also recognize the temptation to design an educational program of which the rationale is that students first need to be trained in each of these elements as prerequisites to later mathematical problem posing. But we also urge course designers to resist this temptation if one explicitly aims to provide students with advance content-directed reasons for what they are going to do. Finally, we agree that the natural context for mathematical problem posing is mathematical inquiry. But we also note that we have found no convincing examples of weaving mathematical problem posing, in a for-students purposeful way, into an ongoing process of mathematical inquiry.

Providing Students with Content-Specific Motives as an Educational Ideal

The issue of students not knowing the purpose of what they are doing is a major concern within our problem-posing approach. Our basic tenet is that all along students know what they are doing and why, as much as possible on content-specific grounds. This ideal serves as a quality standard that as designers we aim to meet when concretely designing teaching-learning activities. Since the basic way to answer the question “Why am I doing this?” is by citing a motive (or reason or purpose), it is an essential ingredient of our approach to think of ways to induce motives in students for engaging in particular activities. In order to get a coherent sequence of activities, moreover, students’ reasons for being involved in a particular activity are to be induced by preceding activities, while that particular activity in turn, together with the preceding ones, are to induce the reasons for being involved in subsequent activities. One way to achieve this coherence is by designing activities with the explicit educational function of making students pose certain content-specific problems, in particular problems that more or less coincide with the tasks they are going to work on next or that at least provide the next tasks with a clear purpose. The overall aim is to increase the quality of students' learning process by enabling students to perceive their learning process as an internally coherent one, which in important respects is driven by their own questions (either existing or induced), over which they have some control, and which point in a certain direction. Of course, there is also the traditional demand that the direction of the learning process is worthwhile from the course designer's point of view, in that it leads to specified attainment targets.

Before discussing our approach any further in general terms, we think it is illustrative to first further clarify it with concrete cases. The cases concern teaching sequences about the physics topic of radioactivity and the mathematics topic of calculus, which were developed and tested in the Ph.D. studies of Klaassen (1995) and Doorman (2005), respectively. The cases represent our efforts to meet, for the two topics at hand, the problem-posing ideal of providing students with content-specific motives. The two cases differ with respect to the vigor with which the ideal is striven for and the extent to which it is attained. But this does not matter for our main aim with presenting the cases, which is to clarify our problem-posing approach as much as possible. For this purpose, partial failures may be as illuminating as partial successes.

The details of the two cases—radioactivity and calculus—take the form of argumentative accounts rather than reports of empirical evaluations. The main steps of each teaching sequence are outlined at several intertwined levels of description:

- Descriptions of what happened in classrooms when the design was put to the test;
- Indications of why the designer expected that this would happen;
- Explanations of the cases in which the expectations did not come out; and
- Clarifying remarks and notes.

The interested reader is referred to Klaassen (1995, Chapters 6–10) and to Doorman (2005, Chapters 5 and 6) for more conventional presentations of the several cycles of small-scale in-depth developmental research involved in each case, as well as for extensive discussion of methodological issues, and for detailed information about textbooks, other materials, in-service programs, and so on.

The Case of Radioactivity

In this section, we illustrate our problem-posing approach with a teaching sequence about the topic of radioactivity. In order to better highlight the defining aspects of our approach, by way of contrast we first sketch the "traditional" way of teaching the topic. Both the traditional approach and our alternative approach are aimed at middle-ability students of about 15 years of age, and take about ten 50-minute lessons. We close with a reflection on the problem-posing features.

Traditional Treatment of the Topic of Radioactivity

Figure 10.2 represents the main structure of how the topic of radioactivity is typically taught. Given the aura of danger and mystery surrounding the topic, it is easily introduced in such a way that students are really motivated to begin with it. For this purpose it suffices to simply announce that safety measures and applications in health care will be covered.

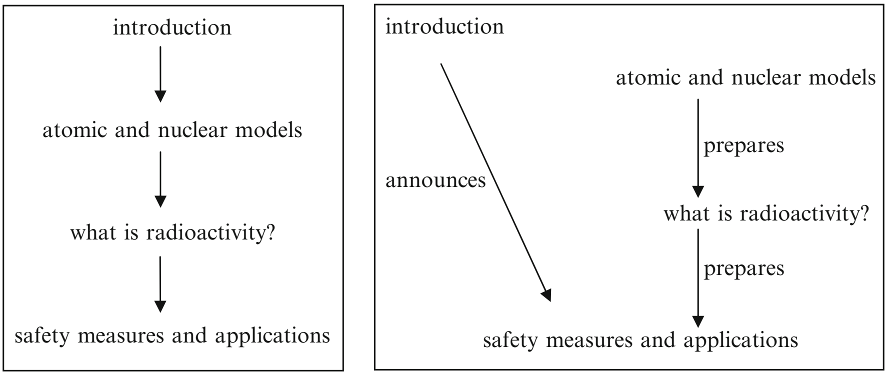

Figure 10.2. Structure of the common treatment of radioactivity. *Left*: temporal order. *Right*: rationale.

The motivating introduction is followed by a presentation of atomic and nuclear models along the following lines. Substances consist of molecules, molecules consist of atoms, atoms consist of .... At the level of middle-ability students the "models" typically take the form of pictorial representations as in Figure 10.3.

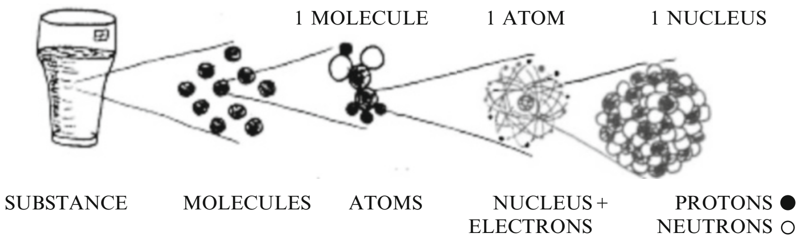

Figure 10.3. From substances to protons, neutrons, and electrons.

A subsequent step in the traditional treatment concerns the introduction of isotopes and an answer to the question "What is radioactivity?" in terms of unstable isotopes that decay while emitting radiation. Finally, safety measures and applications of radiation are treated.

The arrows on the left in Figure 10.2 represent temporal order. On the right, they represent the rationale behind the structure. The rationale seems clear enough. In order to be able to understand safety measures and applications, students should first know what radioactivity is: what a radioactive substance is, what radiation is, how it emerges, and so on. And in order to be able to understand what radioactivity is, students should first know about isotopes, helium nuclei, electrons, and so on. And in order to be able to understand that, they must first know, albeit at a simplified level, about nuclear and atomic models.

Some Comments on the Traditional Treatment

Like many traditional curricula in general, the standard treatment of the topic of radioactivity is cast in the form of a simplified rational reconstruction. Apart from a simplification appropriate to the target group, the content is sequenced in the way in which someone who has already mastered it may in hindsight conveniently reconstruct or summarize it, or build it up from first principles. For those who have not yet mastered it, however, following a simplified rational construction may not be a particularly useful route towards mastering it. What we especially want to draw attention to, in contrast to the problem-posing approach to be described later, is that following a simplified rational construction is not very suited for making students understand the purpose(s) of what they are doing. Before they are going to do what they were motivated for in the introduction (safety measures, etc.), there are five or six lessons about rather tough material (atomic models, etc.). But since middle-ability students are not familiar with (sub)microscopic models, it is not at all obvious for them to begin with such models. While observing some middle-ability classes in which the topic of radioactivity was taught in the traditional way, Klaassen (see also 1995, section 3.3) found that after 2–3 lessons on atomic models students became impatient and somewhat rebellious. The more assertive students began to complain why they were spending so much time on atoms, and when they would at last begin with radioactivity.

A further comment on the traditional treatment is that in order to arrive at a useful understanding of safety measures and applications, it is not at all necessary to first understand at a fundamental level what radioactivity is. The question if an irradiated object poses a radiation hazard to its environment, for example, is most relevantly answered by probing with a Geiger counter—which would be more relevant than by a theoretical treatment of the processes involved in the absorption of helium nuclei, electrons, and so on.

We hope to have made clear that the arrows in Figure 10.2 cannot be taken to represent motives for students to make a transition from one block to the next. Even in retrospect students may have a hard time trying to say why they have done what they did. A possible exception concerns the blocks "atomic and nuclear models" and "what is radioactivity?". While working on the block "what is radioactivity?", presumably it will be clear to students that use is made of concepts and models that were introduced in the block "atomic and nuclear models." This is why in Figure 10.4 we have drawn a backward pointing "retrospective arrow."

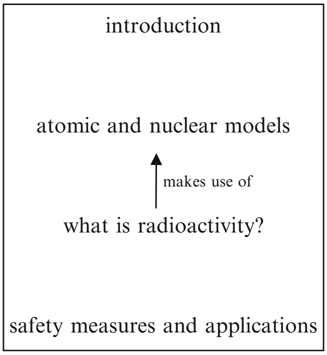

Figure 10.4. For students there is at best only a weak retrospective coherence.

Some Preliminaries About an Alternative Approach

We tried to design an alternative approach to the topic of radioactivity, such that for students there is a solid coherence and such that they do have advance motives for making a transition to the next block. We arrived at a structure that is almost a complete reversal of the traditional structure (compare Figure 10.4 with Figure 10.5, and in particular note the forward pointing arrows in Figure 10.5). Whereas the

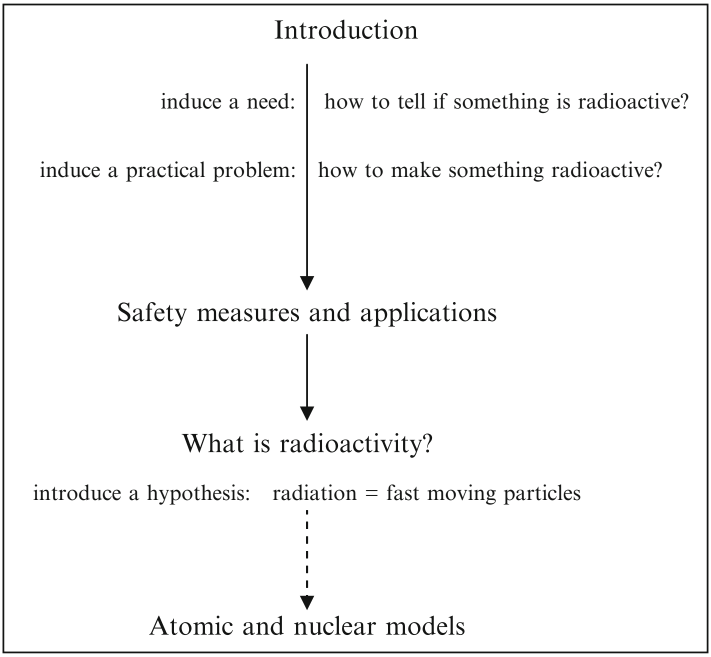

Figure 10.5. A didactical structure of radioactivity with a solid coherence.

introduction in the alternative approach is similar to the one in the traditional structure, after that the alternative approach proceeds in the direction announced in the introduction. Students are made to experience that they do already know quite a lot about radioactivity, but not enough to gain a genuine understanding of safety measures and applications.

Klaassen (1995, sections 2.3–2.5) first did some research on students' existing knowledge about radioactivity.[1] The findings were that students' existing knowledge could to a large extent be understood in terms of very basic notions concerning causation. In essence, an affector harms an object by means of an instrument.[2] In the case at hand, X-ray machines, radioactive waste, irradiated food, Chernobyl, and so on have the potential to harm something or someone because in one way or another they can make it happen that something harmful enters the thing or person. Students often call this something harmful "radiation" or "radioactivity." It functions as the instrument. In the case at hand, it is invisible, transportable, and penetrating. The Chernobyl accident was an affector because huge amounts of the instrument were released. According to many students, irradiated food is a potential affector because it contains the instrument and by eating the food we get the instrument inside. An object or person is affected as long as it contains the instrument. The effects may be reduced by applying a resistance, i.e., something that counteracts the instrument. A resistance, such as a lead wall or a special suit, prevents the instrument from entering an object or person. Furthermore, students applied semiquantitative relationships such as: the stronger the affector is, the more the object is affected; the longer the affector harms the object, the more the object is affected; the more affectors harm an object, the more the object is affected; the nearer the affector is to the object, the more the object is affected; the greater the resistance, the less the object is affected.

Sketch of an Alternative Approach

Partly based on the preceding analysis of students' existing knowledge, an alternative treatment of the topic of radioactivity was designed and tested. The structure is outlined in Figure 10.5. In the following description, we will especially focus on the way content-specific motives are induced for making a transition to the next block.

Inducing a need: How to tell if something is radioactive? After a motivating introduction, students discuss what has and what has not got to do with radioactivity. They all know that nuclear power plants and X-ray machines have got to do

[1] Klaassen's research was carried out in the late 1980s and early 1990s. At that time, Dutch students all knew about the accident that had happened just a few years earlier (in 1986) with a nuclear power plant in Chernobyl. Also in the Netherlands the accident had consequences. For example, fresh products such as milk and spinach had become radioactive and had to be withdrawn from the market. Our alternative approach also draws heavily on students' familiarity with the Chernobyl accident.

[2] We do not mean to suggest that terms such as "affector" or "instrument" are used by students. It is we who use these terms to talk about their ideas.

with it. In the terminology introduced above, students are sure that nuclear power plants and X-ray machines are affectors. But they are not so sure, or mutually disagree, about whether or not a battery has got to do with radioactivity, or a laser, or a magnet. We had foreseen these doubts and disagreements because many people find batteries, lasers, and magnets somewhat mysterious or dangerous, just as radioactivity. By referring to students' doubts and disagreements, it is relatively easy to induce a need for an objective criterion of telling when something is radioactive (as was our explicit intention).[3] This need is eventually met by a Geiger counter, which in the sequel also makes it possible for students to check their predictions and expectations experimentally.

Inducing a practical problem: How to make something radioactive? We had also foreseen that students' existing knowledge would enable them to simulate the Chernobyl accident. The teacher introduces a weak radioactive source, e.g., a mantle of a gas lamp, as the radioactive material that was stored in the power plant before the accident, and asks students to store it in such a way that it poses no radiation hazard to its immediate environment ("the people living nearby"). Students had no trouble doing this. They immediately built "walls" of lead around the mantle until a Geiger counter on the outside no longer ticked above the background rate. We expected such proposals. In the terminology introduced above, the proposals amounted to applying a resistance. Students also believed that they knew what would have to happen in order that radiation could be measured at the other side of the classroom ("the Netherlands"). They proposed that the "walls" must be broken, that there must be a wind blowing towards "the Netherlands," and that it must rain above "the Netherlands." These proposals were also expected. In the terms introduced above, the proposals all amounted to a means of transporting the instrument from the affector to the affected. As was our explicit intention, students were really surprised when it turned out that their proposals did not work. They broke down the "walls," used a fan to produce a flow of air towards "the Netherlands," sprinkled some water above a Geiger counter in "the Netherlands"—whatever they tried, the counter did not begin to tick any faster.

In another part of the simulation activity, students were asked to make an apple radioactive with the materials present in the classroom.[4] This, too, they thought they knew how to achieve. For example, they proposed to put the apple next to the mantle or to X-ray the apple for a while. Such proposals were also foreseen. In the terms introduced above, the proposals all amounted to a means to get the instrument from the affector into the apple. Students were baffled even more when these proposals

[3] Here, we have a first major example where a reason is induced in students for what they are going to do next. It is not of a general nature, such as: we are going to do this, because we want to please the teacher, get a good grade, or stay out of trouble. Instead the reason directly and specifically concerns the topical content: in order to reach mutual agreement and secure knowledge about safety measures and applications of radioactivity, we first of all need an objective criterion of telling when something is radioactive, and that is what we are going to find out now. Because this reason is specifically directed at topical content, we call it content specific or content directed.

[4] Apart from some weak radioactive sources, also a small X-ray machine was present in the classroom.

also did not work. The problem of how to make something radioactive thus thrust itself upon the students, and with quite some force given its practical relevance, as was our explicit intention.[5]

Solving the practical problem in the context of safety measures. As is often the case in situations where one oneself has framed a problem that has a clear meaning to oneself, the students were already on the way to solving the practical problem once they have framed it. For one thing, they had an open eye and mind for possible contributions to its solution. In the process that had led to their formulation of the problem, they were implicitly also provided with the conceptual equipment that was appropriate to recognize possible solutions as such. This is not to say that it was obvious to students how they might find a solution to the problem. They needed guidance. Lack of space prevents us from going into details here. We merely mention that gradually students developed what might be called a macroscopic theory of radioactivity. It consisted of relationships between the core concepts of radiation, radioactive, irradiation, and contamination. For example, objects do not get radioactive from being irradiated. Students also learned to apply the theory in the context of safety measures, for example when they thought about whether or not the prevention of irradiation required the same sort of safety measures as the prevention of contamination.

Inducing theoretical problems: What is radioactivity? The macroscopic theory answers the practical problem, as well as related questions such as how the spinach in the Netherlands did become radioactive. But the macroscopic theory also raises new questions, such as the following. Why is it that an object does not emit radiation after it has been irradiated? What, then, happens to the radiation when it enters an object and, in particular, why is it that receiving radiation *does* have harmful effects? And what is radiation anyway? We did not expect all students to raise all of these questions or to find such questions very exciting. But we did expect that at least some such questions would be raised by at least some students, and that, once raised, the other students would at least recognize that the macroscopic theory does not provide answers. This typically happened.

Note that questions such as those just mentioned do not demand an improved understanding of situations that are of practical interest, but rather require a deeper understanding than is offered by the macroscopic theory. In short, they are questions of a more theoretical nature, of the kind: what is radioactivity? Such theoretical questions were also at the forefront in the traditional treatment. But whereas in the traditional approach the questions were prematurely raised by the textbook or the

[5] This is a second major example, where a reason is induced in students for what they are going to do next. This reason is content specific: we do not yet know how to make something radioactive, but clearly this is at least one thing we need to know in order to properly understand safety measures and applications of radioactivity. So what we are going to do next is find out why all of our proposals did not work and how something *can* be made radioactive.

teacher, this time they were either raised by the students themselves or at least fell on fertile soil, as was our explicit intention.[6]

"Solving" the theoretical problems: Atomic and nuclear models. Students received some hints with which they could tackle the theoretical problems, such as the suggestion to think of radiation as consisting of very small and very fast moving particles. The challenge then was to think of some micro-level account of what happens when the particles enter an object, that explains why food is affected while it is being irradiated (e.g., the bacteria in it are killed), but no longer poses a radiation hazard after it is irradiated, also not when it is eaten. Along these lines students get a flavor of how micro-level mechanisms might enable a deeper understanding.[7]

Reflection on Problem-Posing Features

As will have become clear from the previous sketch, it is not coincidental that:

- At one stage students felt a need for an objective criterion for telling whether or not something was radioactive;
- At a later stage students came to appreciate as urgent the practical problem of how to make something radioactive; and
- At a still later stage students came to pose theoretical problems that invited an account of what radiation does in terms of what radiation is.

All of this was carefully planned and outlined, by making productive use of students' existing knowledge and by tuning activities to one another in considerable detail. The main difference with the traditional approach was that it is *not* unquestioningly assumed that students simply stand ready to absorb new knowledge, such that all one has to do is present them with this new knowledge. The main difference with conceptual-change approaches is that it is *not* deemed necessary first to delete existing knowledge in order to create a place for the knowledge to be taught to occupy. Our emphasis rather lies on providing students with content-directed motives and on soliciting seeds in their existing ideas, in such a way that they are willing and able to extend their knowledge and skills in a certain direction. This direction, moreover, from the perspective of the designer must be such that by

[6] This is a third major example, where a reason is induced in students for what they are going to do next. This reason is content specific and of a theoretical rather than practical nature: we now know a lot about safety measures and applications, but some questions are left open, especially concerning the interaction of radiation with matter and living tissue; we are going to find out more about that now. The theoretical questions invite an account of what radiation does in terms of what radiation is.

[7] It was not expected that students' theoretical questions would provide a basis that was strong enough to support the introduction of full-fledged nuclear models. The bottom arrow in Figure 10.5 is drawn dotted because it represents only a weak content-directed reason suggested by students. A rather detailed nuclear model was only included to meet the requirements of the then examination program.

following it students can be expected to get closer to the intended attainment targets. The designer must explain, for example, how in a process that is given an initial purpose and direction by the practical problem of how to make something radioactive, students can come to establish, and to value as a solution to the practical problem, what above is called the macroscopic theory of radioactivity.

Perhaps it is good to add that there is no contradiction between, on the one hand, students' bottom-up control and, on the other hand, the designer's carefully outlined plan that the process will proceed in a particular way and will lead to the attainment of certain preset targets. The students may be well aware that this was all prearranged, but still feel that they are contributing substantially to the direction taken by the process.

The Case of Calculus

This section concerns a teaching sequence about the topic of calculus. Here too we first sketch the "traditional" way of teaching the topic. Both the traditional and our alternative approach were aimed at academically streamed students in upper secondary education (Grade 10). Our approach took about ten 50-minute lessons. We again close with a reflection on the problem-posing features.

Traditional Treatment of Calculus

The traditional setup of a calculus course is presented in Figure 10.6. It builds upon an early treatment of the limit concept. The gradient of a graph is introduced as the limit of a difference quotient. This notion is extrapolated to a function that describes all gradients of the graph.

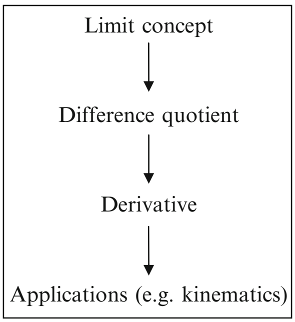

Figure 10.6. Structure of the common treatment of calculus.

The rationale is that in order to understand the derivative $f'(x)$, one has to have the concept of a limit at one's disposal because the derivative is the limit of the difference quotient $\left(f\left(x+h\right)-f\left(x\right)\right)/h$, where h tends to zero. This process is visualized with a decreasing chord on a graph, tending to the local slope of the graph. Traditionally, the variable x is initially replaced by a number or a placeholder a to define and calculate the slope of a graph at a point. The next step is to let a vary, or to replace a by an x, and to introduce the idea of the derivative of a function. Finally, this process is used to derive $f\left(x\right)=x^2$ and some other relatively simple functions, and to proceed quickly to techniques for differentiation such as product and chain rules. In this approach, the students will find at a late stage—after dealing with the concept of and techniques for integration—the connection between differentiation for grasping change and integration for finding "totals." This connection is mainly expressed as a kind of inverse relationship. The emphasis is on the techniques, and the tasks and applications are mainly meant to practice the techniques.

Some Comments on the Traditional Treatment

The late attention for applications in the traditional treatment of calculus creates difficulties for students to see connections with different notations and approaches in other disciplines. The mathematical language of functions (f, x, y, …) and chords in graphs are hardly used in secondary school science, while the tangent method (i.e., sketching a tangent and determining its slope) is important in science but hardly treated in mathematics. The introduction of the limit concept prior to the difference quotient suddenly appears for no reason to the students. Also, the conceptual step from a limit with a fixed x to a varying x is rather difficult, since taking a limit in one point is substantially different from perceiving $f'(x)$ as a function, the values of which describe the gradient of a graph of $f(x)$.

The traditional treatment of calculus is the result of a similar rational reconstruction as in the case of radioactivity. Tall (1991) suggested that it was no wonder that mathematicians especially tended to make this typical error when they designed instructional sequences. The general approach of a mathematician is to try to simplify a complex mathematical topic by breaking it up into smaller parts which can be ordered in a sequence that is logical from a mathematical point of view. From the expert's viewpoint the components may be seen as part of a whole. But the student may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available (Tall, 1991, p. 17). It may be even worse if the student does not realize that there is a total picture.

Freudenthal's interest in mathematics education started with his critique of such rational reconstructions. He was fiercely opposed to what he called an anti-didactical inversion (Freudenthal, 1973), where the end results of the work of mathematicians are taken as starting points for mathematics education. Mach (1976) had already pointed out this inversion in the presentation of mathematical theorems: "mathematicians more than others tend to eliminate all trace of development as soon as they present their findings. The perfectly clear recognition of mathematical propositions is by no means attained all at once, but is preceded and prepared by incidental observations, surmises, thought-experiments and physical experiments with counters and geometrical constructions" (pp. 182–183).

As an alternative for this inversion, Freudenthal advocated that mathematics education should take its starting point in mathematics as an activity, and not in mathematics as a ready-made system (Freudenthal, 1973, 1991). For him the core mathematical activity was mathematizing, i.e., organizing from a mathematical perspective. Mathematizing involves both mathematizing everyday-life subject matter, and mathematizing the mathematical activity itself. The main idea is to allow students to come to regard the knowledge they acquire as their own knowledge.

Some Preliminaries About an Alternative Approach

In order to realize our problem-posing ideal, we looked for problems that students would recognize as relevant and real, and that would evoke solution strategies that have the potential of being mathematized towards the desired concepts and skills. Our emphasis was on students developing a thorough understanding of basic principles rather than on the training of techniques. In particular, we aimed at genuine understanding of the relationship between taking differences and adding them up, and of the difference quotient as a means for grasping and quantifying changing quantities.

Historically, the basic principles of calculus originated from thought experiments about falling objects and from grasping the relationship between velocity and distance traveled (Sawyer, 1961). In addition, graphs and other mathematical symbols such as tables and algebraic notations play key roles. Traditionally, these are presented as ready-made symbols to students. However, for students it is not at all obvious how to interpret graphs. Terms such as "high," "steep," "quick," and "constant," which have specific meanings in interpreting graphs, are very quickly mingled with the situations that are represented by the graphs, especially in the case of motion (Doorman & Gravemeijer, 2009).

It seems that learning calculus and learning kinematics are intertwined, and it is difficult, maybe even impossible, to say what must be taught first. In the historical development of calculus (starting before Leibniz and Newton), clues can be found for how graphical representations of motion emerged and supported the understanding of the relation between velocity and distance traveled (Doorman & van Maanen, 2008).

A starting point for reasoning about changing quantities is students' commonsense understanding that when you travel at high speed, you will cover more distance in equal time intervals than when you travel slower. Intervals of distances traveled have proven to be basic structuring elements for reasoning about motion (Boyd & Rubin, 1996). Often this reasoning with intervals is sufficient, but it does not always lead to precise predictions. In order to meet a demand for more precision, our idea is to connect reasoning with intervals to reasoning about change with two-dimensional graphs that represent motion. This connection has the potential to be mathematized into reasoning with difference quotients.

Sketch of an Alternative Approach

The principal theme of our alternative approach to calculus is grasping change in order to make predictions. The structure is outlined in Figure 10.7.

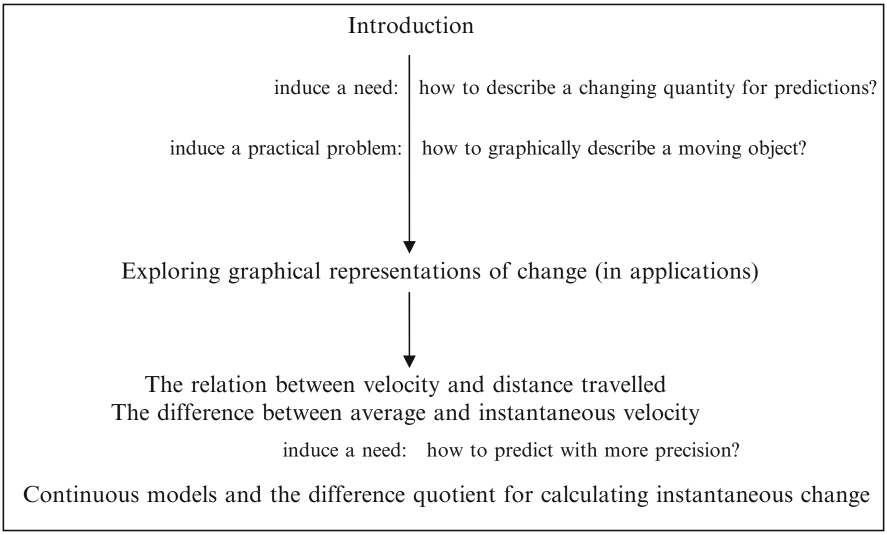

Figure 10.7. A didactical structure with coherence between modeling motion and grasping change.

Inducing a need: How to describe a changing quantity for predictions? The overarching question of the sequence is how to describe changing quantities in order to better predict. This question is initially posed in the context of motion by considering weather forecasts (moving clouds and hurricanes). Change and predictions are well-known notions in this context and we expected that this context would provide students with content-specific reasons to make predictions. During the sequence, the perspective on this overarching question changes from situation specific, to generalizing over different kinds of quantities in various contexts, and finally to context-independent concepts and skills expressed in a formal mathematical language. The overarching question supports coherence between the successive lessons by evoking contributions from students to the problems that have to be solved in order to improve conceptual understanding and tackle the global overarching question.

Inducing a practical problem: How to describe a moving object graphically? The sequence started with two satellite photos taken with 3 hours between them. The aim was to predict whether the clouds, which clearly changed position, would reach the Netherlands in the next 6 hours. This was important to know for the organizers of a pop concert that evening. The context was expected to provide a need-to-know for students and to offer opportunities for an initial orientation on the main theme. As expected students measured displacements, and extrapolated these in making predictions. Next students were shown successive positions of an accelerating hurricane on a map. They were asked to predict when and where it would hit the coastline. These questions led to opportunities for discussing patterns and for using changes in successive positions as a basis for predictions. As Boyd and Rubin (1996) have found, students naturally think of intervals as a measure of change of velocity. They were therefore expected to realize that it made sense to display the measurements graphically for investigating and extrapolating patterns in intervals.

Exploring graphical representations of change. After working with the hurricane and the stroboscopic photographs, two types of two-dimensional graphs emerged: discrete graphs of intervals between successive positions, and discrete graphs of total distances traveled. The classroom discussion led to consensus about the use of, and the relationship between, these two-dimensional graphs for describing and predicting motional phenomena. It also became clear that drawing such graphs was a sensible way to proceed.

In the graphs distances are represented, not as the height of a dot, but as lengths of vertical bars. The discrete case of the main theorem of calculus was implicitly touched on in this kinematic context. The sum of intervals was equal to the total distance traveled, and the difference between two successive values of the distance traveled was equal to the interval (see Figure 10.8).

Inducing a need: How to predict with more precision? The newly developed tools were evaluated with respect to the overarching question: do the tools enable us

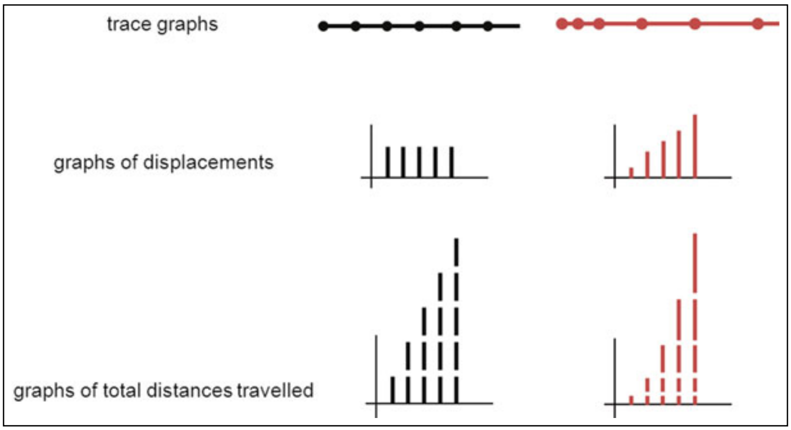

Figure 10.8. From trace graphs to discrete two-dimensional graphs of motion.

to make better predictions? We expected students to suggest measuring the successive positions of a hurricane at shorter time intervals in order to gain a better view of the pattern in the displacements. Furthermore, they were expected to differentiate between changes in average velocities based upon the measurements and the actual velocity after the last measurement. Subsequently, this was to be used by the teacher to induce a content-directed motive for introducing hypothetical continuous models for predictions. However, the next lesson dealt with a new (historical) context, which turned out to hinder rather than facilitate the teacher in tapping, emphasizing, and using the required conceptual connections.

Introducing a hypothesis: A continuous model for free fall? The transition to continuous models was introduced in the context of a narrative about Galileo's work. Students were asked to interpret Galileo's hypothesis that the velocity of a falling object increases in proportion to the time it falls, and to compare this hypothesis with other ideas in that period. We chose the story about Galileo because we thought that it would be a relevant problem for the students. Moreover, it offered opportunities for students to connect discrete approximations and discrete graphs with continuous models. Finally, it gave students a view on a milestone in science.

With intervals of distances traveled in specific time intervals students could calculate constant average velocities for the chosen time intervals. The graph of the average velocities will also increase linearly. The multiplication of a time interval and a constant velocity resulted in a displacement in the corresponding time interval. From there on, as expected, students saw the connection with the discrete case. Adding intervals traveled (areas in the velocity graph) resulted in total distances traveled (inspired by Kindt, 1996; Polya, 1963). These procedures used an informal limit concept.

By approximating changing velocities with bars (representing constant velocities in specific time intervals), the first step was made towards creating an experiential base for the process of describing motion leading up to integrating functions.

Improving prediction by using continuous models. A situation about a Dutch comic character who drove his car through a village (inspired by Kindt, 1979) was presented together with a continuous time graph of his distance traveled. The question was: Do you think he broke the speed limit? We expected students to reason about velocity with discrete approximations of time and distance (Δt and Δs) in this graph. Students managed to reach consensus on how to calculate instantaneous velocity approximately. After discussing the activity, students used a computer program for drawing a difference quotient on a graph as a chord, and for zooming in on part of the graph (inspired by Tall, 1996). As a result of this exploration, students developed a strong graphic and dynamic image to support the formalization in mathematical language of the relationship between the slope of a chord and the approximation of instantaneous change. During subsequent lessons, the teacher and the students regularly referred to this dynamic image.

The unit closed with a reflection on the successive steps that had been taken, from the perspective of the overarching question: To what extent are we now capable of describing and predicting change? The connection between the successive representations in the context of modeling motion supported students in reconstructing the meaning of the difference quotient, its power (you can be precise), and its limitations (you need a function, a continuous model, which is not always at hand).

Reflection on Problem-Posing Features

When we look back at our teaching sequence from the point of view of our problem-posing ideal, we have mixed feelings. Throughout, our aim has been to introduce situations that evoke the need for new tools or concepts by problematizing students' understandings and experiences in the context of the overarching goal of grasping change in order to make more appropriate predictions. We provided the teacher with information on students' reasoning about changing quantities, and on how this reasoning could be used to elicit productive questions and suggestions. In the first part of the teaching sequence, this worked out rather well. The teacher was indeed able to regulate classroom discussions in such a way that students understood that displaying and investigating patterns in displacements was a sensible way to proceed for describing and predicting motion. Suggestions by students for using two-dimensional graphs were welcomed by the teacher as a valuable way of reasoning. Moreover, this way of reasoning was accepted by all students, as we concluded from their contributions and questions while discussing the graphs.

The transition from discrete motion graphs to continuous models, however, did not proceed so smoothly. Although in the end students managed to reason adequately with continuous models, we did not succeed in providing students with advance content-specific motives for the transition itself. Above we indicated that the historical context of Galileo's work somehow hindered adequate scaffolding for students in building their reasoning with formula-based graphs upon their reasoning with data-based graphs. In retrospect, we now have a clearer view of the cause of the observed "friction." The transition to continuous models simply was not functional for students in view of the overarching goal of making better predictions. Up to the transition, students had made predictions on the basis of available data by linear extrapolation. In order to improve the predictions, there was a sudden switch to making predictions on the basis of imagined data or hypothesized models. But instead, it may have been more "logical" for students to use readily available data and to improve their predictions by extending the method of extrapolation beyond *linear* continuation. Furthermore, it was possible to do so, even if no hypothetical continuous models were available. In retrospect, this reinterpretation of the friction we observed during the transition was so obvious that one may wonder why we did not see it before, when we designed the relevant teaching–learning activities. We will not address this question here. Our point merely is to illustrate how our problem-posing ideal at least in retrospect has guided us to understand more fully what may have caused the observed friction.

We conclude that, as designers, we are faced with a choice. Either we retain our original aim of reasoning with continuous models, in which case there still is the need to provide students with a motive for making the transition. This seems to demand a change of overarching goal. Or, we retain the original overarching goal, and then the natural course rather seems to be towards the idea of Taylor-expansion as a controlled step-by-step improvement of prediction. We will not argue here for either option, and there may be more. Our point merely is to indicate how our problem-posing ideal has oriented us towards possible resolutions.

Reflection and Extension

Our problem-posing approach is not a general theory of learning or teaching, but a programmatic view of the possibilities for improving educational practice at a content-specific level which can be further explored and empirically realized by educational research. It is not easy to achieve the goal that all along students know, on content-specific grounds, what they are doing and why. It is not just a matter of asking students what they would want to learn. In order to appreciate the difficulty, it is useful to distinguish between the content-specific purposes of students (their goals) and the aims of the course designer (the attainment targets). The student goals should become worthwhile to them in advance of comprehending the attainment targets. Students should also come to experience the work they are going to do as instrumental to reaching their goals. From the perspective of the course designer, moreover, students' work should bring them closer to their attainment targets. It is a difficult challenge to meet all of these requirements at the same time. Hence, the reason why the problem pointed out by Gunstone (1992)—students not knowing the purpose(s) of what they are doing—is difficult to avoid. But to the extent that one manages to meet these requirements, it will contribute to having students regard the knowledge they acquire as their own. First, because the knowledge is then acquired on a need-to-know basis. Second, because the knowledge is then acquired by continually tapping their own conceptual resources, thus helping to avoid alienation and compartmentalization.

From the two cases discussed above, it should be apparent that meeting our problem-posing ideal involves a detailed analysis of students' existing knowledge and abilities, as well as a careful and detailed outlining of teaching–learning activities that support and build on each other. There are no general procedures for how to achieve this. It is a matter of finding local solutions to local problems, and in many cases critical details such as the actual wording of tasks are of vital importance (Viennot, 2003). It typically takes several cycles of design, testing, and redesign, before the ideal is just beginning to come in sight. In this respect, we feel we have made more progress in the case of radioactivity than in the case of calculus. In part, this will have to do with the nature and complexity of the topics at hand. It is much easier to involve students in the practical concerns associated with radioactivity than to set and keep them in the right kind of theoretical mood that is required for calculus.

Of course this does not imply that the ideal must be abandoned for the case of calculus, though it may make one wonder if one values the ideal strongly enough to go through the amount of trouble that apparently is needed to attain it. As far as we are concerned, we have not yet reached the stage that we would rather leave our teaching sequence on calculus as it is. Instead, our tendency is to analyze the weak points of our approach and to try harder, perhaps by exploring other avenues. In our discussion, we indicated what we see as weak points, in particular that the guiding theme of how to describe change for predictions was not always functional for the students. Could the weak points be addressed by changing some of the examples, or is a more drastic modification needed such as a replacement of the guiding theme itself? An alternative avenue may be to explore the educational usefulness of one of Zeno's paradoxes. Achilles and a turtle are involved in a running contest. The turtle has a head start on Achilles. Zeno reasons that Achilles will never overtake the turtle because when Achilles reaches the spot where the turtle started, the turtle will already have moved on, and so on ad infinitum. It will be obvious for students that Zeno's conclusion is false (of course Achilles will overtake the turtle), but it will not be obvious at all for them to pinpoint the flaw in Zeno's reasoning. The potentially useful element of this example is that it naturally sets students to think about change *within a theoretical context*, that is, within the context of sound reasoning. We have not sufficiently worked out this line of thought though. Clearly, clever ideas are needed here. Of course, it cannot be enforced that one gets good ideas, but at least we are more receptive now. We do hope that some readers, after having been sensitized to our ideal, will come up with useful suggestions. In our opinion, it is an essential aspect of educational research to thus engage the broader research community.

Several other attempts have been made, with more or less success, to realize the ideal that all along students know what they are doing and why. Vollebregt (1998) designed a teaching sequence on particle models, in which conceptual progress on particle models drives and is driven by issues of a metaphysical, ontological, and epistemological nature (e.g., What does it mean to explain something? Do particles really exist? How do we know which properties they have?). Kortland (2001) designed a teaching sequence in environmental education. In a process structured by students' existing decision-making skills and basic knowledge about life cycles of materials, students eventually arrive at well-argued decisions in the context of dealing with household package waste. Another attempt concerns an introductory mechanics course. By tapping core causal knowledge and epistemic resources, students eventually arrive at theoretical insights in explanations of motion and a justified preference of Newton's to Kepler's theory of planetary motion (Emmett, Klaassen, & Eijkelhof, 2009; Klaassen, Westra, Emmett, Eijkelhof, & Lijnse, 2008). Other attempts have been based on the idea of adapting an established professional practice, e.g., the chemistry-related practice of monitoring water quality (Westbroek, Klaassen, Bulte, & Pilot, 2010).). A professional practice can be thought of as an organized system of activities, the coordinated execution of which leads to the attainment of some goal. The basic idea is to "transform" this hierarchy of means-to-end relations in the context of professional practice into a hierarchy of content-specific motives for students to engage in learning activities.

Mathematical Problem Posing from the Point of View of Our Problem-Posing Approach

When we think about mathematical problem posing from the perspective of our problem-posing approach, our main message is *not* to view mathematical problem posing as an optional activity alongside, or over and above, students' learning about some mathematical topic (long division, calculus, statistics, or whatever). If one thinks about organizing a teaching sequence about a particular topic in such a way that all along students know on content-specific grounds what they are doing and why, one cannot but think about appropriate contexts to make students raise the right sort of problems. What makes the problems of the right sort is that they are clearly connected to a worthwhile goal (for students) and also suggest a direction for a solution. Following that direction, moreover, is to lead students eventually to the attainment targets, perhaps via some redirections engendered by newly raised problems or reformulated old problems, and so on. Just like in Vollebregt's (1998) approach, students' learning about the nature of science is not something added on to their learning of science, but naturally integrated within their learning of science, so we think of mathematical problem posing as something to be naturally integrated within students' learning of mathematics, and the same goes for mathematical modeling.

We have one final reflection. We have argued against the tendency of structuring a teaching sequence along the lines of a rational reconstruction. But this does not rule out the possibility, within a problem-posing approach, of inviting students to make a rational reconstruction, namely towards the end of the teaching sequence, in order to summarize what they have learned. Such a rational reconstruction may also concern the role played by mathematical problem posing in the teaching sequence. It may even be given a useful point within an educational setting, as a preparation for the test. That is, in order to prepare well for the test students can be challenged to design good test items for each other and to reflect on why they think these are good problems. The aim for students would then be to make explicit the sorts of elements that were also central to the programs of English (1997a, 1997b, 1998) discussed earlier in the chapter, by classifying the types of problems that have been treated, separating problem structures from contextual features, and so on.

Acknowledgments We would like to thank Carolina Kühn, two anonymous reviewers, and the editors of this book, for their help in improving previous versions of this chapter.

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