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Stochastic Discount Factor Approach to International Risk-Sharing: A Robustness Check of the Bilateral Setting

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Abstract
This paper presents a robustness check of the stochastic discount factor approach to international (bilateral) risk-sharing given in Brandt, Cochrane, and Santa-Clara (2006). We demonstrate two main inherent limitations of the bilateral SDF approach to international risk-sharing. First, the discount factors are not uniquely determined in the bilateral framework and crucially depend on the partner country included in the calculations. Second, the deviations between the discount factors obtained in this way (the imprecision in the measurement of marginal utility growth) are larger for countries whose stock market excess return shocks are relatively less important. In order to account for some of these criticisms, we extend the bilateral into a three-country setting. Although the trilateral framework demonstrates that the (final) results for the international risk-sharing index are quite robust to the number of countries used in their calculation, it does not resolve the inherent incoherence found in the bilateral SDF approach.

Keywords: International Risk-Sharing, Stochastic Discount Factor, Exchange Rate Volatility

JEL Classification: F31, G12, G15

1 Introduction

Depending on the data sources and the theoretical framework used in order to quantify the degree of international risk-sharing, one arrives at very different conclusions. For example, methods that use consumption data and are based on specific underlying utility functions imply that there is not much risk to be shared (consumption growth is not very volatile) and that countries share a very small portion of this risk because cross-country consumption growth correlations are very low (Backus, Kehoe, and Kydland, 1992; Backus and Smith, 1993; Lewis, 1999, 2000; Obstfeld, 1994; Van Wincoop, 1994, 1999). Portfolio calculations based on empirical risk-return profiles and certain specification(s) for the utility function find higher potential gains from international risk-sharing (more risk to be shared), but also very low degrees of actual risk diversification (Lewis, 1999, 2000). On the contrary, stochastic discount factor-based measures imply that there is a lot of risk to be shared (high volatility of the discount factors) and that a large portion of this risk is actually shared across countries.

In the latter approach, Brandt, Cochrane, and Santa-Clara (2006) calculate domestic and foreign marginal utility growth rates through stochastic discount factors derived from asset markets data[1]. Subsequently, they compare the volatility of these stochastic discount factors with the volatility of the real exchange rate. Their main finding is that real exchange rates (difference between marginal utility growth rates) are much less volatile than what the stochastic discount factors (proxies for marginal utility growth) of the corresponding countries would imply. Therefore, they conclude that marginal utility growth rates must be very highly correlated across countries,

[1]Following Hansen and Jagannathan (1991), this approach is based on excess returns of the stock market index above the risk-free rate.

i.e. a large portion of macroeconomic risk is shared internationally.

This paper presents a robustness check of the (bilateral) stochastic discount factor approach to measuring international risk-sharing given in Brandt, Cochrane, and Santa-Clara (2006). We demonstrate that there are two main limitations of the bilateral SDF approach to international risk-sharing. First, the discount factors in the bilateral framework are not uniquely determined and crucially depend on the partner country included in the calculation. Second, the deviations between the discount factors obtained in this way (the imprecision in the measurement of marginal utility growth) are larger for countries whose stock market excess return shocks are relatively less important (Sharpe ratios are lower).

In order to account for some of these criticisms about the bilateral SDF approach, we extend the bilateral framework into a three-country (trilateral) setting. However, although the trilateral framework demonstrates that the (final) results for the international risk-sharing index are quite robust to the number of countries used in their calculation, it does not resolve the inherent incoherence found in the bilateral SDF model. In fact, it only shifts the problem with the internal incoherence of the SDF approach by one country ahead.

The rest of this paper is organized as follows: section 2 develops the theoretical framework and presents the calculations of the stochastic discount factors and the risk-sharing index. Section 3 describes the data, replicates the bilateral results obtained by Brandt, Cochrane, and Santa-Clara (2006), and shows some limitations of the bilateral approach. Section 4 extends this approach to a three-country setting. We discuss the relevance of our findings in section 5. Section 6 concludes the paper.

2 Theoretical Framework

2.1 Pricing Kernels

In this section we derive the theoretical framework linking the change in the real exchange-rate with the domestic and foreign marginal utility growth rates (stochastic discount factors). Following the approach taken in Backus, Foresi, and Telmer (1996) and Backus, Foresi, and Telmer (2001), we model asset prices with pricing kernels, i.e. stochastic processes that govern the prices of state-contingent securities[2].

Let v_t represent the domestic currency value at time t of an uncertain, stochastic cash flow of d_{t+1} domestic currency units one period in the future. Then, the basic asset pricing relation relates v_t and d_{t+1} in the following way:

$$v_t = E_t(m_{t+1}d_{t+1}) \tag{1}$$

by dividing both sides of equation 1 by the initial investment v_t at time t, i.e. the value of the uncertain cash flow at time t, we get an expression in terms of returns:

$$1 = E_t(m_{t+1}R_{t+1}) \tag{2}$$

where $R_{t+1} = d_{t+1}/v_t$ is the gross return on this asset/investment between time t and $t+1$, and m_{t+1} is the domestic currency pricing kernel. The kernel m_{t+1} occupies a central place since it gives the "gross rate" at which economic agents discount the uncertain payment d_{t+1} one period in

[2]Several conditions should be satisfied in order to derive a relationship between the (real) exchange rate and the stochastic discount factors in the two currencies. First, there should be free trade in assets denominated in each currency as well as free trade in each of the corresponding currencies. Second, no pure (zero initial investment) arbitrage opportunities should exist on any of the markets.

the future, i.e. it represents the (nominal) intertemporal marginal rate of substitution between time t and $t+1$ for all assets traded in the domestic economy[3].

Similar relations should hold for assets denominated in foreign currency and traded in the foreign economy. In fact, there are two equivalent ways to show these relations for foreign assets. First, through substitution of all domestic variables from equations 1 and 2 with their foreign counterparts we get the following equations for foreign assets:

$$v_t^* = E_t(m_{t+1}^* d_{t+1}^*) \tag{3}$$

and, in terms of gross returns:

$$1 = E_t(m_{t+1}^* R_{t+1}^*) \tag{4}$$

Second, the cash flows (or gross returns) received in foreign currency can be converted into domestic currency units at the expected future spot exchange rate, and then discounted using the domestic pricing kernel or domestic discount factor, just as in the case of domestic assets. According to this approach, we get the following relations:

$$v_t^* = E_t\Big[m_{t+1}(S_{t+1}/S_t)d_{t+1}^*\Big] \tag{5}$$

and, in terms of gross returns:

$$1 = E_t\Big[m_{t+1}(S_{t+1}/S_t)R_{t+1}^*\Big] \tag{6}$$

[3] m_{t+1} will be a unique solution of equations 1 and 2 only if the domestic economy has a complete set of state-contingent securities that can be freely traded. Otherwise, there are multiple solutions for m_{t+1}.

where S_t stands for the current spot nominal exchange rate (the price of foreign currency in domestic currency units) at time t, and S_{t+1}/S_t represents its gross rate of change between time t and $t+1$.

Because these two approaches must give equivalent results, we can equate 3 with 5:

$$E_t(m^*_{t+1}d^*_{t+1}) = E_t\Big[m_{t+1}(S_{t+1}/S_t)d^*_{t+1}\Big] \tag{7}$$

or 4 with 6, respectively:

$$E_t(m^*_{t+1}R^*_{t+1}) = E_t\Big[m_{t+1}(S_{t+1}/S_t)R^*_{t+1}\Big] \tag{8}$$

If no pure arbitrage opportunities exist and markets in both countries are complete, then the following should hold[4]:

$$m^*_{t+1} = m_{t+1}(S_{t+1}/S_t) \tag{9}$$

which, in turn, gives the relation between the change of the exchange rate and the nominal discount factors in the two countries. Hence, the (nominal) exchange rate should move (depreciate/appreciate) exactly by the difference between the discount factors in the respective countries. More specifically, equation 9 implies that domestic currency depreciates when the domestic nominal discount factor is lower than the foreign nominal discount factor in the corresponding period.

Although the discussion in this section focused on *nominal* variables, a similar condition can be stated in terms of *real* variables. Thus, taking

[4]This relation holds in the case of complete markets in both countries (for currencies and risky assets). In incomplete markets, m^*_{t+1} and m_{t+1} will not be uniquely determined - combinations of the discount factors with some random disturbances ϵ^*_{t+1} and ϵ_{t+1} that are orthogonal to the underlying shocks will also price all assets.

the logarithm of both sides of equation 9 and changing all nominal variables (exchange rates, gross returns, discount factors) into their real counterparts, we arrive at a condition that equates the real exchange rate to the difference between changes in foreign and domestic intertemporal marginal rates of substitution between time t and $t+1$:

$$\ln \frac{e_{t+1}}{e_t} = \ln \frac{\lambda^*_{t+1}}{\lambda_{t+1}} = \ln \lambda^*_{t+1} - \ln \lambda_{t+1} \tag{10}$$

where e_t is the real exchange rate - the relative price of foreign in terms of domestic goods[5], λ_{t+1} is the gross rate of change in domestic marginal utility between time t and $t+1$, λ^*_{t+1} is the gross rate of change in foreign marginal utility between time t and $t+1$ (both measured in units of real, consumption goods)[6]. Rearranged in real terms, this condition states that in equilibrium the change in the relative price of foreign in terms of domestic goods (given by gross rate of change in the real exchange rate) should equal the ratio between foreign and domestic marginal utility changes (stochastic discount factors or pricing kernels). Derived through this simple asset pricing framework, equation 10 is of central importance for the stochastic discount factor approach to measuring international risk-sharing, elaborated in this study[7].

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[5]The real exchange rate is defined as the price of foreign goods over the price of domestic goods. Therefore, an increase in the real exchange rate implies a real appreciation (depreciation) of foreign (domestic) goods.

[6]The stochastic discount factors λ_{t+1} and λ^*_{t+1} represent gross real returns in the corresponding markets. They can be defined through in traditional consumption-based models as $\lambda_{t+1} = \beta(u'(c_{t+1}/u'(c_t))$, where β is the reciprocal of the gross rate of time preference and $(u'(c_{t+1}/u'(c_t))$ is the gross rate of change in marginal utility growth between time t and $t+1$. Therefore, the values for the discount factors will be always positive in this framework, typically in the vicinity of 1.

[7]For more extensive discussion on the application of this equation see Backus et al. (2001) and Brandt and Santa-Clara (2002) for example.

2.2 Risk-Sharing Index

The perfect international risk-sharing hypothesis implies complete equalization of marginal utility growth rates across countries. In our framework, given by equation 10, it means equality between λ_{t+1} and λ^*_{t+1} at any point in time. Thus, if this asset pricing condition holds *and* all country-specific risks are shared internationally, then the left-hand side of this equation should always be zero. Put differently, the departures from this perfect situation can be measured by the deviations on the left-hand side, i.e. the fluctuations of the real exchange rate.

Brandt et al. (2006) use this intuition to propose a measure of international risk-sharing based on asset markets. First, they take variances of both sides of equation 10:

$$\begin{aligned}\sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big) &= \sigma^2\Big(\ln\lambda^*_{t+1} - \ln\lambda_{t+1}\Big) = \\ &= \sigma^2\Big(\ln\lambda^*_{t+1}\Big) + \sigma^2\Big(\ln\lambda_{t+1}\Big) - 2\rho\sigma\Big(\ln\lambda^*_{t+1}\Big)\sigma\Big(\ln\lambda_{t+1}\Big) \qquad (11)\end{aligned}$$

where σ^2 symbolizes a variance, σ a standard deviation, and ρ is the coefficient of correlation between the two discount factors λ_{t+1} and λ^*_{t+1}. Therefore, if the following two conditions hold: i) assets and currencies are priced according to equation 10 at any point in time; and ii) all risks are shared internationally, then: $\rho = 1$, $\lambda_{t+1} = \lambda^*_{t+1}$ and $\sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big) = 0$. In general, the correlation between marginal utility growth rates will be given by:

$$\rho = \frac{\Big[\sigma^2\Big(\ln\lambda^*_{t+1}\Big) + \sigma^2\Big(\ln\lambda_{t+1}\Big) - \sigma^2\Big(\ln\frac{e_{t+1}}{e_t}\Big)\Big]}{2\sigma\Big(\ln\lambda^*_{t+1}\Big)\sigma\Big(\ln\lambda_{t+1}\Big)} \qquad (12)$$

indicating that risk-sharing across countries decreases in the variability of the real exchange rate. Based on this idea, Brandt et al. (2006) construct the following risk-sharing index

$$RSI = 1 - \frac{\sigma^2\Big(\ln \frac{e_{t+1}}{e_t}\Big)}{\sigma^2\Big(\ln \lambda^*_{t+1}\Big) + \sigma^2\Big(\ln \lambda_{t+1}\Big)} \tag{13}$$

where the numerator of the second term captures the variability in the real exchange rate (which, according to the argumentation above, measures the deviations from perfect risk-sharing), and the denominator is the sum of the variabilities in marginal utility growth in the two countries (the total risk that exists and can be shared across countries). Hence, this term gives a ratio between risk still not shared and total risk that can be shared between the two countries. Brandt et al. (2006) indicate that this index gives the portion of total (diversifiable) risk that is already shared by the two countries[8].

2.3 Basic Calculations

In order to calculate the risk-sharing index given in the previous section, first we have to recover the log discount factors (or marginal utility growth rates) from asset markets data in the corresponding countries[9]. For this purpose, we closely follow the exposition given in Brandt et al. (2006). We start by assuming that the following assets are traded in a two-country setting:

$$\frac{dB^d}{B^d} = r^d dt \tag{14}$$

[8]In this way, the framework presented by Brandt et al. (2006) can be viewed as an extension of the Hansen-Jagannathan (1991) volatility bounds to the international setting.

[9]For ease of exposition and manipulation in the further calculations (translating between levels and logarithms), the demonstration here uses continuous time formulation. Empirically, all variables are calculated using the corresponding discrete time approximations, see the section on data issues.

$$\frac{dS^d}{S^d} = \theta^d dt + dz^d \tag{15}$$

$$\frac{de}{e} = \theta^e dt + dz^e \tag{16}$$

$$\frac{dB^f}{B^f} = r^f dt \tag{17}$$

$$\frac{dS^f}{S^f} = \theta^f dt + dz^f \tag{18}$$

where B^d is the domestic risk-free bond (with expected return r^d), S^d is the domestic risky asset (expected return θ^d), e is the real exchange rate, i.e. the relative price of foreign in terms of domestic goods (expected return θ^e), B^f is the foreign risk-free bond, and S^f is the foreign risky asset (expected return θ^f). There are three sources of uncertainty in this setting, related to the domestic asset, the real exchange rate, and the foreign asset. These shocks can be collected into a vector of shocks dz:

$$dz = \begin{bmatrix} dz^d \\ dz^e \\ dz^f \end{bmatrix}$$

with a corresponding variance-covariance matrix given by[10]:

$$\Sigma = \frac{1}{dt} E(dzdz') = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de} & \Sigma^{df'} \\ \Sigma^{ed'} & \Sigma^{ee} & \Sigma^{ef'} \\ \Sigma^{fd'} & \Sigma^{fe} & \Sigma^{ff'} \end{bmatrix}$$

Furthermore, the calculation of the discount factor(s) from asset markets depends primarily on the variability of the excess returns on risky assets,

[10]This variance-covariance matrix is the same for domestic and foreign investors because they face the same vector of shocks in this symmetric, bilateral setting.

driven by the shocks in vector dz[11]. We derive all excess return equations in the appendix, and here present only their expected values. Thus, the domestic investor faces the following set of expected excess returns:

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ \theta^e + r^f - r^d \\ \theta^f - r^f + \Sigma^{ef} \end{bmatrix}$$

The first term in this vector gives the excess return that a domestic resident expects to get by investing on the domestic stock market. It equals the difference between the average real return on the domestic stock market index (θ^d) and the average real risk-free rate in the domestic economy (r^d) during the entire investment period. The expected excess return on the foreign exchange market is given by the second term in vector μ^d. It represents the average deviation from (uncovered) interest parity, calculated as borrowing in the domestic currency, converting the borrowed amount into the foreign currency, lending at the ongoing one-month foreign interest rate, and converting the proceeds back into domestic currency after one month. The last term in vector μ^d gives the expected excess return that a domestic investor expects to get by investing in the foreign stock market. Therefore, it represents a difference between the average return on the foreign stock market and the domestic one-month risk-free interest rate. The last part of this term Σ^{ef} results from the continuous-time formulation and gives the (average) co-movement between the returns on the foreign stock market and the exchange rate. Therefore, by correcting for the movements of the nominal exchange rate, this term facilitates the translation of excess returns obtained on the foreign market[12].

[11] Since we work with (expected) excess returns in this analysis, we do not make a real/nominal returns distinction.

[12] For example, Σ^{ef} is added to the excess return on the foreign market for the domestic

A similar vector of expected excess returns applies to the foreign investor:

$$\mu^f = \begin{bmatrix} \theta^d - r^d - \Sigma^{ed} \\ -(\theta^e + r^f - r^d - \Sigma^{ee}) \\ \theta^f - r^f \end{bmatrix}$$

The interpretation of the terms is analogous to that given for the domestic investor. The expected excess return on the foreign exchange market is exactly the opposite of the one for the domestic investor (corrected for the continuous-time term Σ^{ee}).

Then, the following discount factors price all assets according to the basic pricing conditions[13]:

$$\frac{d\Lambda^i}{\Lambda^i} = -r^i dt - \mu^{i\prime}\Sigma^{-1}dz, i = d, f \tag{19}$$

where $\frac{d\Lambda^i}{\Lambda^i}$ is the growth rate of the discount factor, r^i is the risk-free return, and μ^i is the vector of excess returns for risky assets in country i. In order to calculate the change in the log discount factor $\ln \lambda^i$ required in equation 10, we use Ito's lemma and get the following expression:

$$d\ln\Lambda^i = \frac{d\Lambda^i}{\Lambda^i} - \frac{1}{2}\frac{d\Lambda^{i2}}{\Lambda^{i2}} = -\Big(r^i + \frac{1}{2}\mu^{i\prime}\Sigma^{-1}\mu^i\Big)dt - \mu^{i\prime}\Sigma^{-1}dz \tag{20}$$

and for its standard deviation:

$$\frac{1}{dt}\sigma^2(d\ln\Lambda^i) = \mu^{i\prime}\Sigma^{-1}\mu^i, i = d, f \tag{21}$$

The change in the log discount factor $d\ln\Lambda$ corresponds to $\ln\lambda_{t+1}$ in the basic asset pricing condition 10. Therefore, the risk-sharing index given

investor, suggesting that foreign expected excess returns are amplified when associated with appreciation of the foreign currency.

[13]For more details on finding the discount factor in this setting see Brandt et al. (2006, p.675-677) or Chapter 4 in Cochrane (2004).

by 13 can be calculated directly from the second moments according to the following expression:

$$RSI = 1 - \frac{\sigma^2(d\ln\Lambda^d - d\ln\Lambda^f)}{\sigma^2(d\ln\Lambda^d) + \sigma^(d\ln\Lambda^f)} = 1 - \frac{\Sigma^{ee}}{\mu^{d\prime}\Sigma^{-1}\mu^d + \mu^{f\prime}\Sigma^{-1}\mu^f} \tag{22}$$

In order to show the symmetric structure of our framework, we relate the shocks facing the domestic with those facing the foreign investor. The expected excess returns vectors μ^d and μ^f differ only by the exchange rate changes[14]:

$$\mu^d - \mu^f = \begin{bmatrix} \theta^d - r^d \\ \theta^e + r^f - r^d \\ \theta^f - r^f + \Sigma^{ef} \end{bmatrix} - \begin{bmatrix} \theta^d - r^d - \Sigma^{ed} \\ \theta^e + r^f - r^d - \Sigma^{ee} \\ \theta^f - r^f \end{bmatrix} = \begin{bmatrix} \Sigma^{ed} \\ \Sigma^{ee} \\ \Sigma^{ef} \end{bmatrix} \tag{23}$$

From these formulae, it is clear that the expected excess return vectors differ exactly by the middle column of the common variance covariance matrix Σ^e:

$$\mu^d - \mu^f = \begin{bmatrix} \Sigma^{ed} \\ \Sigma^{ee} \\ \Sigma^{ef} \end{bmatrix} = \Sigma^e \tag{24}$$

In turn, we can derive a relationship between the domestic and foreign discount factor loadings (given by the last term of equation 20):

$$\mu^d\Sigma^{-1} = (\mu^f + \Sigma^e)\Sigma^{-1} = \mu^f\Sigma^{-1} + \Sigma^e\Sigma^{-1} = \mu^f\Sigma^{-1} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{25}$$

[14]In order to derive this relation, we disregard the change in sign before the foreign exchange excess returns when moving from domestic to foreign investor perspective.

Equation 25 shows that domestic and foreign discount factors load equally on domestic and foreign stock market shocks, while their loadings on the foreign exchange shocks differ by exactly 1. Therefore, this implies that the only difference between the two discount factors comes from fluctuations in the real exchange rate.

3 Data and Replication of Results

3.1 Data Description

In this section we replicate the results for the bilateral setting presented in Brandt et al. (2006). For that purpose, we construct a dataset that is as close as possible to the one used in the original study. In particular, we employ three types of time-series: for the risk-free rate we use interest rates on one-month Eurocurrency deposits, while for the return on the risky asset we use total returns on the stock market index for the corresponding country. We calculate inflation rates from the changes in the consumer price indices (CPI). The nominal exchange rates are expressed in terms of domestic currency per unit of foreign currency.

Our analysis includes three economies: USA, UK, and Japan. We use monthly data from January 1975 till June 1998 for the USA and the UK. For Japan interest rates on Eurocurrency deposits are not available before August 1978. Therefore, all data series for Japan start in August 1978 and go through June 1998. The series on Eurocurrency deposit interest rates, nominal exchange rates and total stock market index returns are measured at the beginning of the month, while the CPI series refer to mid-month values[15]. All data come from Datastream[16].

For stock market returns, we use the same indices employed in the original study[17]: S&P 500 for the USA, FTSE ALL for the UK, and NIKKEI 225 for Japan.

3.2 Summary Statistics

We use discrete time approximations of the continuous time formulae derived in section 2.3. The following sample counterparts are used in the calculation:

$$
\begin{aligned}
&\theta^d - r^d = \tfrac{1}{\Delta} E_T R^d_{t+\Delta} \\
&\theta^f - r^f = \tfrac{1}{\Delta} E_T R^f_{t+\Delta} \\
&\theta^e + r^f - r^d = \tfrac{1}{\Delta} E_T \left(\tfrac{e_{t+\Delta} - e_t}{e_t} + r^f_{t+\Delta} - r^d_{t+\Delta} \right) \\
&dz^d = \tfrac{1}{\Delta} (R^d_{t+\Delta} - E_T R^d_{t+\Delta}) \\
&dz^f = \tfrac{1}{\Delta} (R^f - E_T R^f_{t+\Delta}) \\
&dz^e = \tfrac{1}{\Delta} \left(\tfrac{e_{t+\Delta} - e_t}{e_t} \right) - \tfrac{1}{\Delta} E_T \left(\tfrac{e_{t+\Delta} - e_t}{e_t} \right) \\
&\Sigma = E_T (dz dz')
\end{aligned}
$$

In these sample moments T is the sample size (281 monthly observations), E_T denotes the sample mean for the entire time period, $\Delta = \frac{1}{12}$years, $R^d_{t+\Delta}$ and $R^f_{t+\Delta}$ correspond to the domestic and foreign excess stock returns, and $r^d_{t+\Delta}$ and $r^f_{t+\Delta}$ refer to the domestic and foreign risk-free (Eurocurrency deposits) interest rates, respectively[18].

[15] The results are very robust with respect to the use of lag or lead values for the inflation rate

[16] CPI data is retrieved from Datastream and comes from the IMF International Financial Statistics (IFS) database.

[17] For the UK we do the same calculations using FTSE 100 index. The results change only slightly.

[18] The formulae for the expected excess returns and the shocks on domestic and foreign

In accordance with the approach taken before, we use real variables: real (excess) stock returns, real risk-free interest rates and real exchange rates. Hence, we correct all data series by the inflation rate (measured by changes in the mid-month CPI)[19]. Moreover, we calculate stock market returns in two ways: i) assuming continuous-time specification and ii) with discrete time specification. Since the results are very similar, in the rest of the analysis we only present stock market returns calculated using the discrete time framework.

The summary statistics are presented in Table 1. Its upper panel shows means and standard deviations for excess stock market returns (Stock) and for excess foreign exchange returns (X-rate). The former are derived as returns on the stock market index above the one-month Eurocurrency interest rate, while the latter are derived as deviations from the uncovered interest parity (UIP), calculated as excess returns from borrowing in the domestic currency (dollar), investing in one-month Eurocurrency deposits in the foreign country (pounds sterling or yen), and translating these yields back to the domestic currency at the end of the period. All entries in the table are annualized and reported in percentages.

The statistics in Table 1 are very similar to and convey the same message as the ones presented by Brandt et al. (2006)[20]. In fact, the mean excess returns given in the first row illustrate the high equity premium found in

stock markets and the foreign exchange market are annualized through division by $\Delta = \frac{1}{12}$years.

[19]Our main results are based on excess market returns. Therefore, they are not sensitive to whether nominal or real variables are used in the calculations.

[20]The first moments are similar and normally keep the same ranking between different countries, but are not identical. On the other hand, the second moments are almost identical as the ones presented by Brandt et al. (2006). This is to be expected as the second moments are usually much less sensitive to the exact procedure used in the calculation.

stock markets data. They range from 4.29 percent in Japan, 9.97 percent in the USA, to 10.21 percent in the UK. All of them are statistically different from zero. Moreover, their associated standard errors, reported in the row beneath, are typically very high. Thus, they result in values for the Sharpe ratio between 0.22 for Japan, 0.62 for the UK, to 0.72 for the USA. Therefore, these results suggest that investors in the USA got the highest excess returns per unit of risk taken, while investors in Japan got the lowest. On the other hand, mean excess returns for foreign exchange are much smaller and not statistically different from zero [21]. Furthermore, the annualized standard deviations for foreign exchange excess returns are about half the values for excess stock market returns (11.56 percent for the first, 12.67 percent for the second, and 12.16 for the third exchange rate).

Finally, the lower panel of this table presents a returns correlation matrix. Three conclusions are evident from this table. First, foreign exchange excess returns are very weakly correlated with excess returns on stock markets. Second, foreign exchange excess returns on one currency pair are highly correlated with excess return on the other currency pair (correlations of 0.507. 0.551 and −0.439). Third, excess returns for different stock markets are highly correlated among themselves (correlations ranging from 0.32 between USA and Japan to 0.58 between USA and UK).

3.3 Replication of the Results for the Bilateral Setting

3.3.1 Results for the Risk-Sharing Index

Using the dataset described in the previous section, here we present a replication of the results obtained by Brandt et al. (2006) for the bilateral

[21] In fact, all mean excess returns on the foreign exchange market are within the range 1-2 percent.

Table 1: Summary Statistics (Annualized)

	USA	UK		Japan		
	Stock	Stock	X-Rate ($\$/£$)	Stock	X-Rate ($\$/Y$)	X-Rate ($£/Y$)
				Returns (%)		
Mean	9.97	10.21	0.98	4.29	2.06	1.08
Std Dev	13.80	16.49	11.77	19.52	12.67	12.16
Sharpe ratio	0.72	0.62	0.08	0.22	0.16	0.09
	USA	UK		Japan		
	Stock	Stock	X-Rate ($\$/£$)	Stock	X-Rate ($\$/Y$)	X-Rate ($£/Y$)
				Return Correlations		
USA Stock	1					
UK Stock	0.583	1				
X-Rate ($\$/£$)	0.010	-0.050	1			
Japan Stock	0.324	0.342	0.077	1		
X-Rate ($\$/Y$)	-0.023	-0.063	0.507	0.101	1	
X-Rate ($£/Y$)	-0.037	0.065	-0.439	0.030	0.551	1

Note: The table contains summary statistics and correlations for real excess returns on stock and foreign exchange markets. All figures are calculated over the time period January 1975-June 1998 (for USA and UK) or over the period August 1978-June 1998 (for Japan). The upper panel figures for the means, standard deviations and Sharpe ratios of all shocks. The lower panel contains figures for the coefficient of correlation between the corresponding returns. Stock market excess returns are calculated as returns on the stock market indices over the one-month Eurocurrency deposit rate for the corresponding country/currency. Excess returns on the foreign exchange market are calculated as (real) deviations from uncovered interest rate parity ($\theta^e + r^f - r^d$): borrowing at the US interest rate, converting to the foreign currency, investing on the foreign interest rate, and converting the proceeds back to US dollars. All data-series are retrieved from Datastream. The summary statistics presented in the upper panel are annualized and expressed in percentage terms (rounded to two decimal places).

setting. The most important result is presented in the first row of Table 2. The risk-sharing index obtains values higher than 0.98, which indicates that an extremely large portion of total macroeconomic risks faced by investors in different countries is shared internationally. This is the central result and the most important message from Brandt et al. (2006). In order to understand these high values for the risk-sharing index, we present its two components in the lower part of Table 2. The volatility of the real exchange rate (numerator in the second term of the risk-sharing index) is several times lower than the volatility of the stochastic discount factors, i.e. the volatility of the intertemporal marginal utility growth rates (denominator in the second term of the risk-sharing index). In fact, the discount factors calculated from asset markets are very volatile, implying that marginal utility varies by about $65 - 75$ percent per year[22]. In turn, this implies low values for the second term in 13 and high value for the overall risk-sharing index.

3.3.2 Discount Factor Loadings

The volatility of the stochastic discount factor (marginal utility growth rate) comes from three sources: domestic and foreign stock market excess return shocks and the foreign exchange excess return shock. The loadings on each of these shocks enter the equations for the discount factors with a negative sign, meaning that a positive shock leads to a decrease in the discount factor (equation 19). For example, a positive (negative) shock on the US stock market (dz^d) leads to a decrease (increase) in domestic and foreign marginal

[22] The volatility of the stochastic discount factor crucially depends on the (average) excess returns earned by the asset markets (equation 21). Therefore, high values for the discount factor volatility reflect the (abnormally) high equity premium earned by investors (Mehra and Prescott, 1985; Kocherlakota, 1996).

Table 2: Risk Sharing Index

	USA vs. UK	USA vs. Japan	UK vs. Japan
Risk Sharing Index	0.9878	0.9857	0.9821
Real X-Rate Volatility	11.75	12.47	12.05
Volatility of Marginal Utility Growth:			
Domestic	75.49	74.83	62.21
Foreign	75.11	73.09	65.06

Note: The table presents results for the bilateral risk-sharing index. The first row gives figures for the overall risk-sharing index calculated according to the following formula: $RSI = 1 - \frac{\Sigma^{ee}}{\mu^{d\prime}\Sigma^{-1}\mu^{d} + \mu^{f\prime}\Sigma^{-1}\mu^{f}}$. The second row refers to the volatility of the real exchange rate found in the numerator of the risk-sharing index, while the last two rows refer to the volatility of the stochastic discount factors found in the denominator of the risk-sharing index. Domestic refers to the first country, while foreign refers to the second country mentioned in the country-pair. The volatilities of the real exchange rate and the marginal utility growth are measured as annualized standard deviations and are expressed in percentage terms (rounded to two decimal places).

utility growth rates (discount factor levels)[23]. Table 3 presents figures for the discount factors loadings ($\mu^{d}\Sigma^{-1}$ and $\mu^{f}\Sigma^{-1}$) on each of these underlying shocks.

In line with equation 25, domestic and foreign discount factors are restricted to load equally on each of the stock market shocks, and the domestic discount factor loads on the exchange rate shocks by one more than the foreign discount factor. The last point implies that the difference between the two discount factors at each point in time equals the fluctuations in the real exchange rate[24]. Furthermore, these foreign exchange loadings are of similar magnitude in all three country-pairs (in absolute value terms) and are always lower than the dominant stock market loadings.

[23] A favorable stock market shock leads to lower marginal utility growth rate as shown by the negative sign in front of the disturbance term in equation 20. Moreover, this shock is “scaled” by the loading coefficient $\mu'\Sigma^{-1}$.

[24] This reflects the symmetric nature of the foreign exchange excess return shocks given by equation 25.

There are large differences between stock markets discount factor loadings for each of the three bilateral country-pairs. For example, the loadings on the domestic (USA) stock market (3.76 and 5.36, respectively) are much higher than the loadings on the other two stock markets (1.95 for UK and -0.13 for Japan) in the first two country-pairs. This suggests that the USA stock market represents the dominant source of variability for both discount factors (domestic and foreign) for these pairs (USA vs. UK and USA vs. Japan). In fact, this finding reflects the superior return compensation per unit of risk undertaken that investors get in the USA compared to the other two stock markets given by the Sharpe ratios in Table 1. Since investors' utility directly depends on the Sharpe ratio, i.e. the compensation they get per unit of risk, excess return shocks on markets/assets with the highest Sharpe ratio matter more for the stochastic discount factor (marginal utility growth). Therefore, excess return shocks on the USA stock market matter most, while shocks on the Japanese stock market matter the least for investors' utility changes.

Furthermore, the discount factors load negatively (and load much less in absolute value) on the Japanese excess return shocks in the second country-pair (USA vs. Japan). This finding (partially) reflects the low price of risk on the Japanese relative to the American stock market (Sharpe ratio of 0.22 for Japan compared to 0.72 for the USA). In fact, since the Japanese stock market is clearly dominated by the American stock market, holding any non-negative investment position on the Japanese market implies that investors forego better investment opportunities on the American market. Hence, this sub-optimal behavior explains the anomalous loadings on the Japanese stock market reported in the middle columns of Table 3.

Table 3: Discount Factor Loadings (Bilateral)

	USA vs. UK		USA vs. Japan		UK vs. Japan	
	Domestic	Foreign	Domestic	Foreign	Domestic	Foreign
dz^d	3.76	3.76	5.36	5.36	3.69	3.69
dz^e	-1.02	-2.02	1.51	0.51	-0.41	-1.41
dz^f	1.95	1.95	-0.13	-0.13	0.12	0.12

Note: The table presents figures for the discount factor loadings in the bilateral setting. The loadings for the domestic discount factor are given by $\mu^{d\prime}\Sigma^{-1}$ and the corresponding loadings for the foreign discount factor are given by $\mu^{f\prime}\Sigma^{-1}$. For each of the three bilateral country-pairs domestic refers to the first country and foreign refers to the second country mentioned in the country-pair. The row marked dz^d contains figures for discount factor loadings on the domestic stock market shocks, row dz^e refers to discount factor loadings on the foreign exchange market shocks, and row dz^f refers to discount factor loadings on the foreign stock market shock for the corresponding country-pair.

3.3.3 Visual Evidence

In order to give a visual representation of the main result in our study, we present several plots for the discount factors. First, in Figure 1 we show time paths for the log discount factors in the three country pairs. We calculate the log level of the discount factor in line with equation 20. It contains two components: a trend component given by the expected value of equation 20 (the term in brackets) and a disturbance component given by the loadings on the underlying excess return shocks. The development of the log level discount factors can be best understood through the contribution of each of its components.

There are several interesting issues in this figure. First, the log level discount factors typically slope downward as a result of the trend component. In fact, as long as the sum of the average real risk-free rate and the discount factor volatility (the expected value of equation 20 given by the term in brackets) is positive (as normally observed), the log level discount factors will follow a downward trend. The easiest way to understand why this is usually the case is by looking at an economy with one only risk-free bond. If this economy experiences real growth over an extended period of time, then its average real risk-free interest rate will be positive (and the trend component will be negative). That is, a downward trend in the log level discount factor corresponds with a decreasing trend in marginal utility growth rates or continual improvement in overall economic conditions. Second, it is clear from the figure that both discount factors follow a similar pattern and move very closely together. In fact, the only difference between them comes from the real exchange rate fluctuations (see equations 10 and 25). Based on this observation, we can conclude that marginal utility growth rates across countries follow very similar time paths, just as implied by the perfect risk-sharing condition.

Moreover, in Figure 2 we present scatterplots for the discount factor growth rates. We calculate these monthly growth rates according to equation 19. This figure just strengthens our conclusion from Figure 1 : there is a very high positive correlation between the discount factor growth rates for each country pair. Most observations/points are literally lying on the 45 degree line, thereby indicating that the stochastic discount factor approach implies nearly perfect levels of (bilateral) international risk-sharing.

3.4 Discussion about the Results from the Bilateral Setting

Section 3.3 demonstrated that measures based on the stochastic discount factor approach imply very high levels of international risk-sharing among three different country-pairs: USA-UK, USA-Japan, and UK-Japan. In fact, we showed that discount factors for each country in the bilateral pair display very similar levels of volatility (Table 2), follow similar time paths (Figure 1), and have almost identical growth rates (Figure 2). However, all these

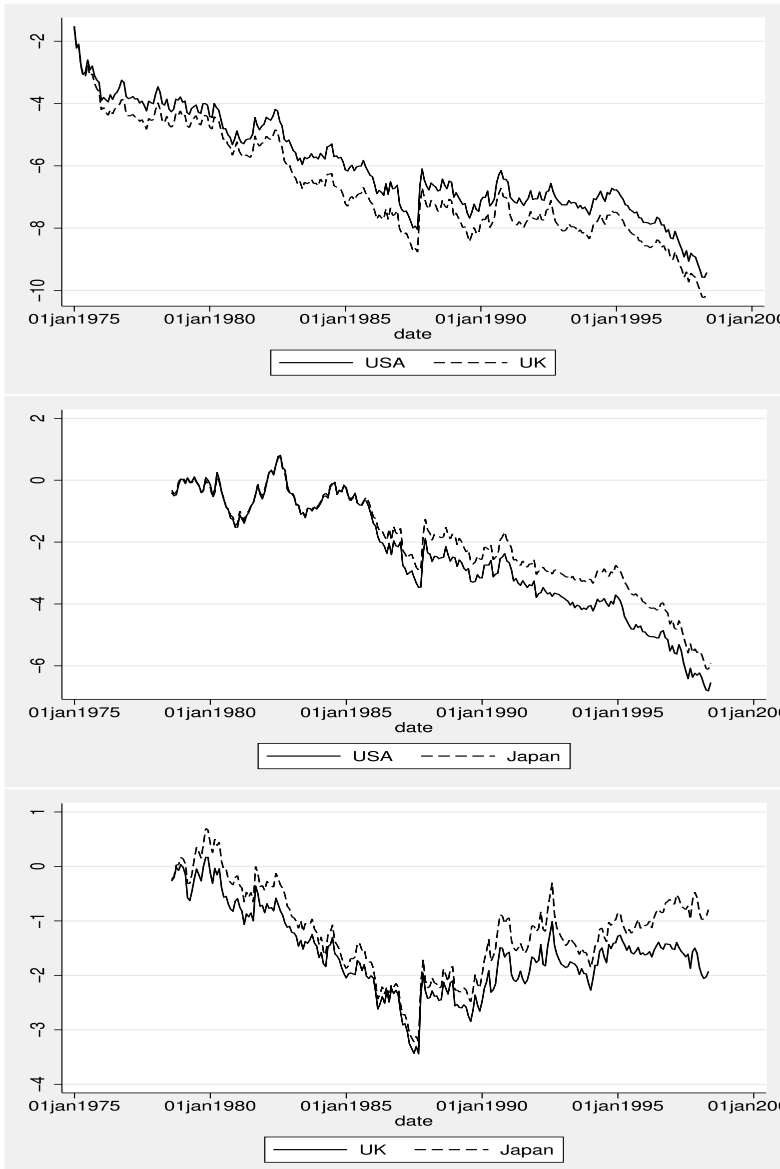

Figure 1: Log Levels of Discount Factors (Bilateral)

Note: The figure presents time lines of the log levels of the discount factors calculated in the bilateral setting. Each plot refers to separate country-pair. The log levels of the discount factors are calculated through accumulation of the changes in the log discount factors given in equation 20.

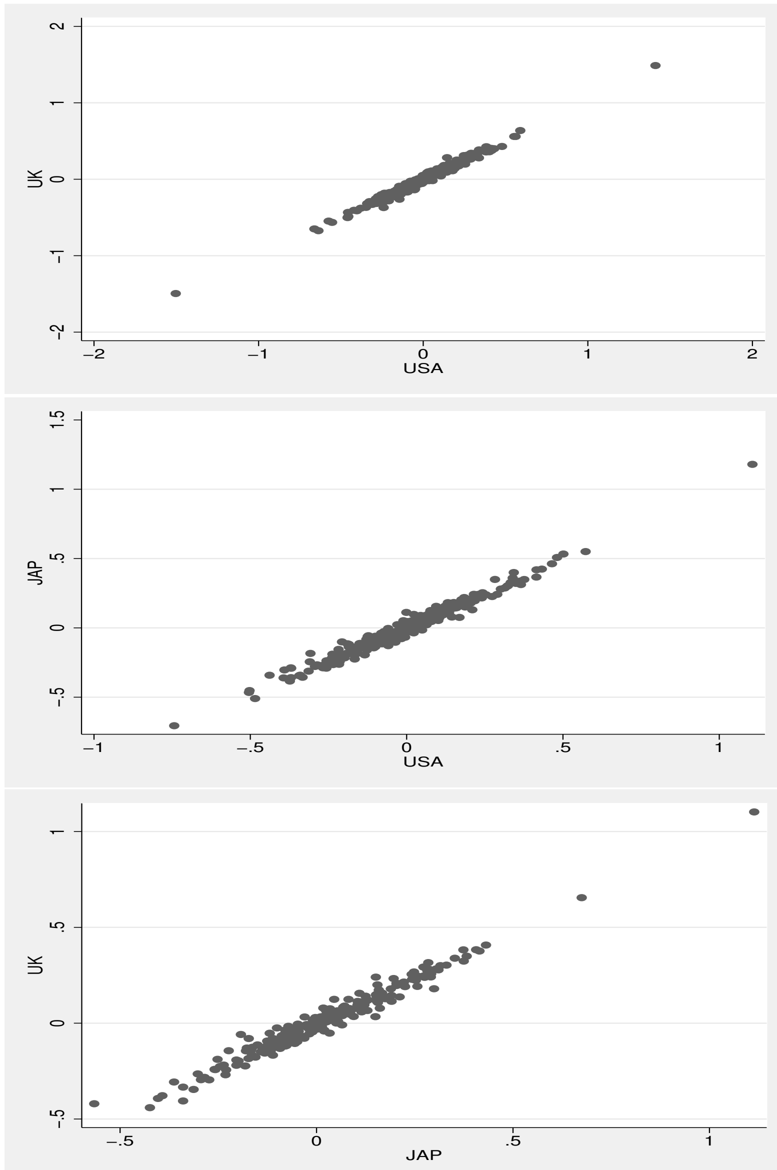

Figure 2: Growth of Discount Factors (Bilateral)

Note: The figure presents scatterplots for growth rates of the discount factors calculated in the bilateral setting. Each plot refers to separate country-pair. The growth of discount factors is calculated according to equation 19.

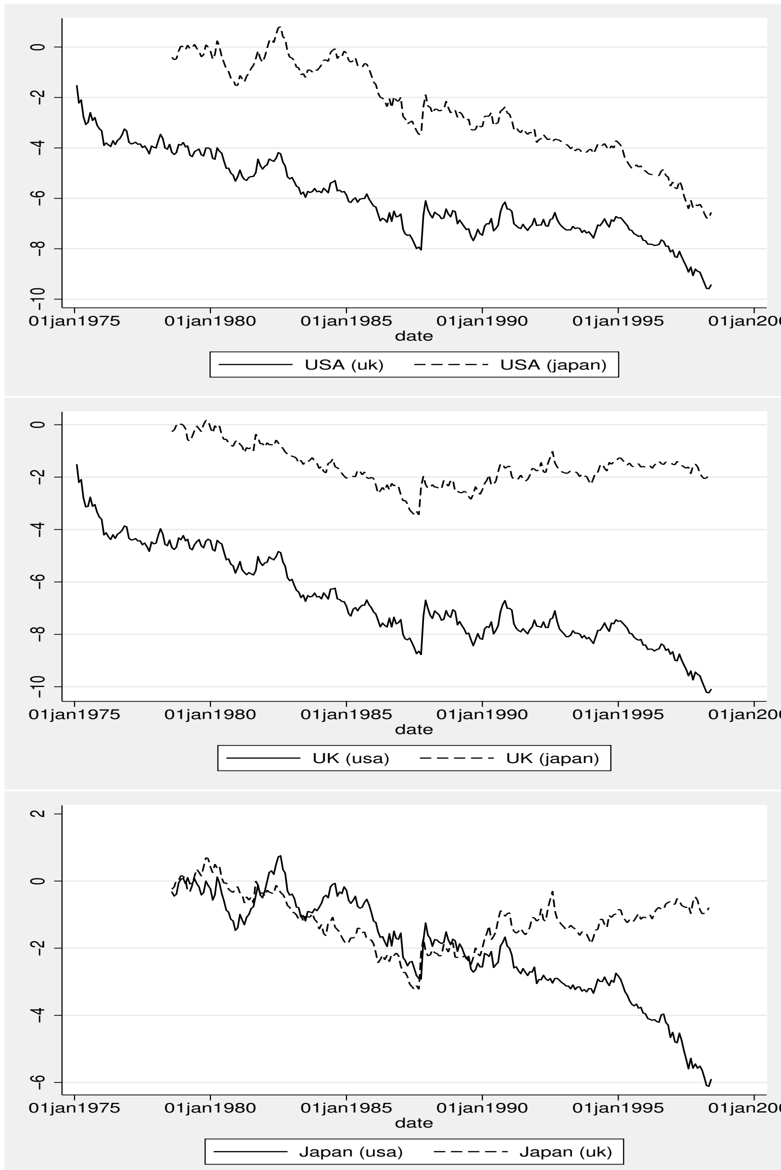

Figure 3: Comparison of Log Levels of Discount Factors (Bilateral)

Note: The figure presents time lines of the log levels of the discount factors calculated in the bilateral setting. Each plot refers to log levels for one country when alternative countries are used as partners. The log levels of the discount factors are calculated through accumulation of the changes in the log discount factors given in equation 20.

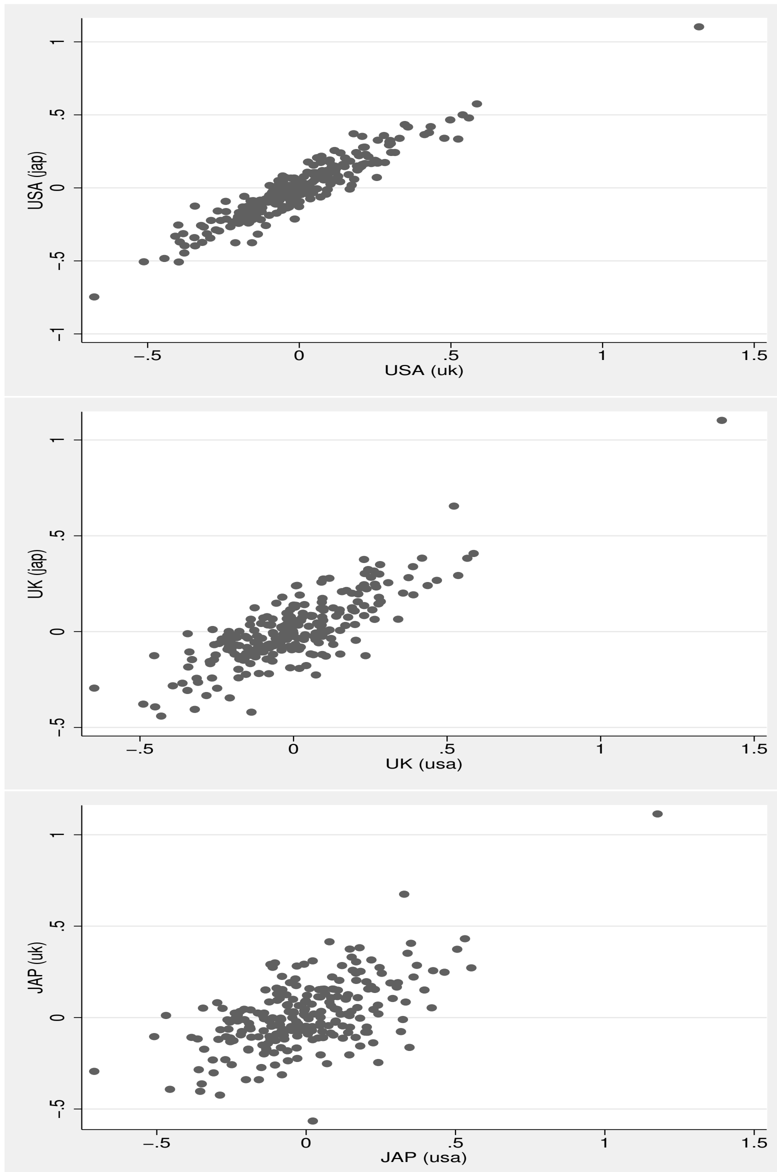

Figure 4: Comparison of Discount Factor Growth Rates (Bilateral)

Note: The figure presents scatterplots for growth rates of the discount factors calculated in the bilateral setting. Each plot refers to discount factor growth for one country when alternative countries are used as partners. The growth of discount factors is calculated according to equation 19.

calculations were conducted within a bilateral setting, i.e. treating only two countries at the time. Therefore, one possible criticism of this approach is that a country's discount factor obviously depends on the choice of the second country. In particular, the USA log discount factor displays a very similar behavior with the UK log discount factor (in the first panel of Figure 1). Similarly, in the second panel of Figure 1, the USA and Japan discount factors are much alike too. However, the USA log discount factor from the first panel is quite different from the USA log discount factor given in the second panel. Correspondingly, the difference between the two UK discount factors in the first and the third panel and between the two Japan discount factors in the second and the third panel is even larger. In other words, this shows that the discount factors in this framework are chosen in such a way as to satisfy the restrictions imposed by one bilateral country pair at the time.

To show this more clearly, Figure 3 compares the log levels of the discount factor for each country relative to each of the other two countries. For example, the first plot compares the time path of the log level discount factor for the USA when UK and Japan are used as partner countries, respectively. This time plot suggests that the discount factor for the USA is not uniquely determined, but clearly depends on the second country. Moreover, the differences between discount factors for the same country are the smallest for the USA and the largest for Japan, reflecting the relative importance of each country's excess return shocks on the log level of the discount factor.

Figure 4 presents scatterplots for the growth rates of the discount factors for each country when the other two countries are used as partners. The evidence in these scatterplots gives additional support to the findings from Figure 3. First, the measures for marginal utility growth (discount factor growth) for the same country are far from perfect[25]. Second, this imprecision in the measurement of discount factor growth increases with the "marginalization" of certain country's stock market shocks in the discount factor calculation. Hence, these measures are the least precise for Japan because it is the country with the lowest Sharpe ratio, and therefore, with the lowest discount factor loading (see Table 3). On the contrary, the imprecision is the lowest for the USA because this is the dominant country (highest Sharpe ratio and discount factor loading) in both country-pairs.

There is an intuitive interpretation of these findings as well. If an investor holds a portfolio of three risky assets with different risk-return profiles, then the asset that makes up the largest part of his total utility/well-being (highest Sharpe ratio) is the most important one for (the change in) his utility (represented by the stochastic discount factor). Following this argument, the contribution of the inferior asset (Japanese stock in this case) for investor's utility is very limited. Therefore, assets with relatively low Sharpe ratios represent residual assets for the investor. In turn, their contribution for his overall utility is quantified in a less precise manner.

Overall, the results suggest two main limitations of the bilateral SDF approach to international risk-sharing. First, the discount factors in the bilateral setting are not uniquely determined and show high sensitivity to the choice of particular partner country. Second, this sensitivity is especially important for countries with relatively low Sharpe ratios (on their stock markets), since their discount factors change substantially from one bilateral setting to another.

[25]Uniquely determined discount factors imply perfect relationships in all scatterplots, i.e. all points should lie along the 45 degrees line.

4 Trilateral Setting

In general, the discount factor for a certain country should be uniquely determined and incorporate all (direct) investment opportunities available to its residents (and therefore, should price all these assets). In order to investigate to what extent the results from section 3 depend on the specific, bilateral structure, we extend it into a three-country (trilateral) setting[26]. Therefore, the discount factors calculated in this trilateral setting are *unique* for each country and *simultaneously* price all assets available to its residents (all risky assets in each of the three countries)[27].

4.1 Results from the Trilateral Setting

Table 4 presents figures for the real exchange rate and discount factor volatilities in the trilateral setting. Similar as in the bilateral case, marginal utility growth volatility is several times larger (about $70 - 80$ percent, measured by the discount factor volatility) than real exchange rate volatility (about 12 percent), suggesting that a lot of risk-sharing takes place among them.

We modify the risk-sharing index given by equation 13 in order to adapt it to our trilateral framework. Hence, we include all three countries in its calculation. For example, for the domestic country (USA), we include both real exchange rates (with respect to the UK and with respect to Japan) and all three discount factor volatilities. Moreover, we allow for differences between partner countries by assigning them specific weights α and $(1-\alpha)$, respectively. In this way, all foreign partner weights for a certain country must sum up to 1. The easiest way to think about this approach is as an "effective, trade-weighted" combination of foreign partners.

[26] All calculations for the trilateral setting can be found in the appendix

[27] The extension to an n-country (n-assets) setting follows the same lines.

Table 4: Real X-Rate and Discount Factor Volatility (Annualized)

Real X-Rate		Discount Factor	
e_1 (USA/UK)	11.58	USA	77.56
e_2 (USA/JAP)	12.47	UK	79.19
e_3 (JAP/UK)	12.05	JAP	76.05

Note: The table presents results for the components of the risk-sharing index in the trilateral setting. The first column gives figures for the the volatility of the real exchange rate, while the second column refers to the volatility of the stochastic discount factors over the time period August 1978-June 1998. Both volatilities (of the real exchange rate and the marginal utility growth) are measured as annualized standard deviations and are expressed in percentage terms (rounded to two decimal places). Real exchange rate e_1 is defined as the price of UK goods in terms of USA goods, i.e. the ratio of prices in the UK over prices in the USA ($e_1 = S^{\$/£}(P^{UK}/P^{USA})$). Similarly, e_2 is ratio of Japanese over USA prices and e_3 is ratio of UK over Japanese prices.

$$RSI = 1 - \frac{\alpha\Sigma^{e_1e_1} + (1-\alpha)\Sigma^{e_2e_2}}{\mu^{d\prime}\Sigma_d^{-1}\mu^d + \alpha\mu^{f_1^\prime}\Sigma_{f_1}^{-1}\mu^{f_1} + (1-\alpha)\mu^{f_2^\prime}\Sigma_{f_2}^{-1}\mu^{f_2}} \tag{26}$$

In fact, these weights should correspond to the relative importance of specific partner countries for international risk-sharing. Hence, there is no specific theoretical way to derive them[28]. Rather, in this study we allow the value for α to fluctuate anywhere between 0 and 1. Figure 5 shows results for the risk-sharing index for each country when different weights are assigned to its other two partners. In fact, the value for α, indicated on the horizontal axis, goes from one extreme (0) to the other (1) (where at each extreme only one of the partner countries matters for risk-sharing) and covers all possible intermediate cases.

For example, the line for the USA represents different values for the USA risk-sharing index going from $\alpha = 0$ (all risk-sharing is done with Japan) to $\alpha = 1$ (all risk-sharing takes place with the UK). The upward slope of this line with respect to α suggests that USA achieves a higher level of

[28]For example, they can be calculated according to the share of trade or the portion of a country's assets portfolio invested in each country.

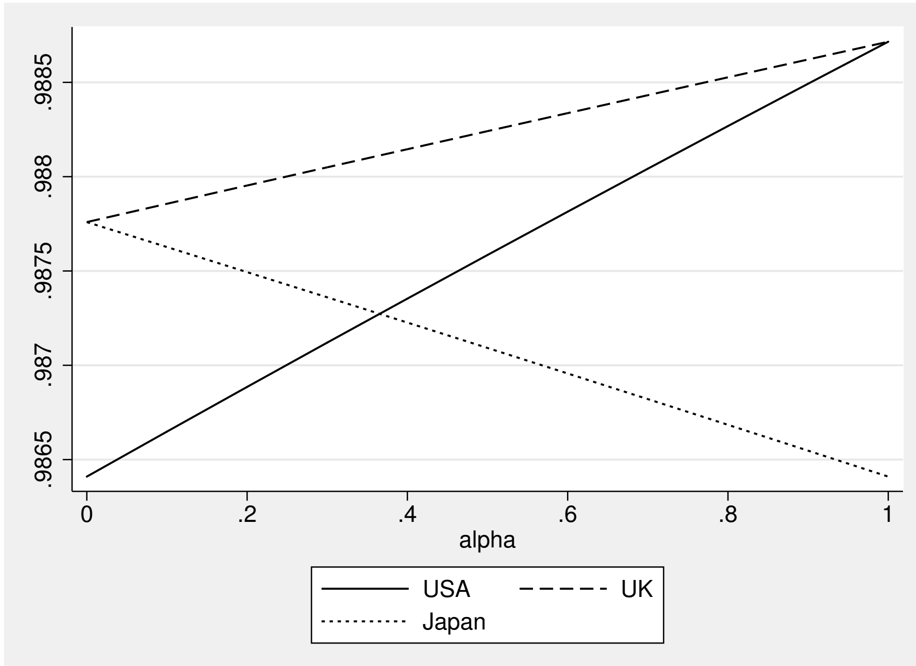

Figure 5: Risk-Sharing Index (Trilateral)

Note: The figure presents values for the risk-sharing index obtained from the trilateral setting given in equation 26: $RSI = 1 - \frac{\alpha\Sigma^{e_1 e_1} + (1-\alpha)\Sigma^{e_2 e_2}}{\mu^{d'}\Sigma_d^{-1}\mu^d + \alpha\mu^{f_1'}\Sigma_{f_1}^{-1}\mu^{f_1} + (1-\alpha)\mu^{f_2'}\Sigma_{f_2}^{-1}\mu^{f_2}}$. Each line refers to values of the risk-sharing index for one country when different weights (α) are assigned to its other two partners. For $\alpha = 0$ and $\alpha = 1$, the index measures risk-sharing between two countries. For $\alpha = 0$ the index refers to the following pairs: USA-Japan, UK-Japan, and Japan-UK, while for $\alpha = 1$ the index refers to the following pairs: USA-UK, UK-USA, and Japan-USA.

international risk-sharing when UK becomes the relatively more important partner. The similar logic applies to the calculations for the other two countries: the upward line for the UK indicates increasing risk-sharing levels when USA becomes relatively more important partner (compared to Japan), and the downward sloping line for Japan indicates decreasing risk-sharing levels when USA becomes relatively more important partner (compared to the UK).

We can derive two conclusions from this figure. First, though differences exist, the risk-sharing index does not vary a lot with respect to the specific combination of partner countries[29]. Second, irrespective of the relative importance of different partner countries, the risk-sharing index for each country-pair is higher than the corresponding index in the bilateral setting. This is the central result from our trilateral setting: measures of risk-sharing based on the stochastic discount factor approach are not very sensitive to the number of countries used in their calculation. If anything, then this trilateral framework suggests somewhat higher risk-sharing compared to the bilateral setting.

4.2 Visual Evidence

Figure 6 depicts the development of log discount factors through time. In this setting, all three discount factors are *simultaneously and uniquely* determined. As can be seen from the figure, their behavior closely resembles that for the bilateral country pairs. In fact, all three log discount factors move very closely together, the only difference being assigned to the fluctuations in the real exchange rates.

Finally, we complete our visual inspection with a 3-dimensional scatterplot of the discount factor growth rates given in Figure 7. In fact, this plot visualizes the joint correlation among the discount factor growth rates for all three countries. The figure shows that almost all points (observations) lie along the spatial diagonal, suggesting quasi-equalization of all three discount factor growth rates. Thus, the evidence from this 3-dimensional scatterplot just strengthens the conclusion that the stochastic discount factor approach implies somewhat higher international risk-sharing in the trilateral than it does in the bilateral setting.

[29] The index fluctuates within the range 0.9865-0.9885.

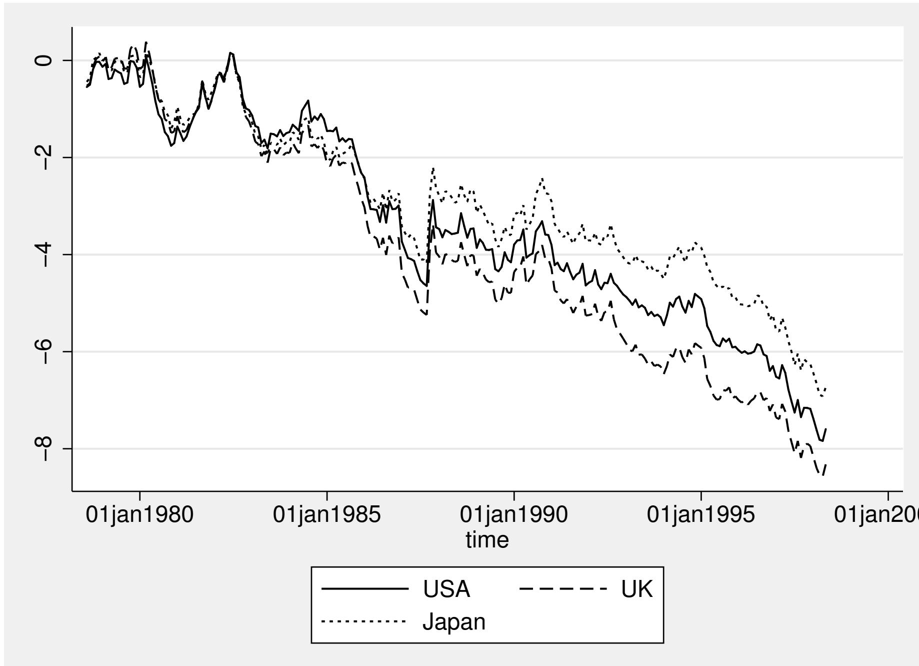

Figure 6: Log Level of Discount Factors

Note: The figure presents time lines of the log levels of the discount factors calculated in the trilateral setting. The log levels of the discount factors are calculated through accumulation of the changes in the log discount factors given in equation 20. All log levels are uniquely determined and price all assets in each of the three countries.

5 Discussion: Limitations of the SDF Approach

In this section we discuss two main (possible) limitations of the SDF approach: the first refers to the internal incoherence of the bilateral approach, while the second refers to the discrepancy of its results with the macroeconomic evidence.

5.1 Inherent Incoherence

The trilateral framework, presented in the previous section, tries to account for some of the (possible) criticisms about the inherent incoherence of the bi-

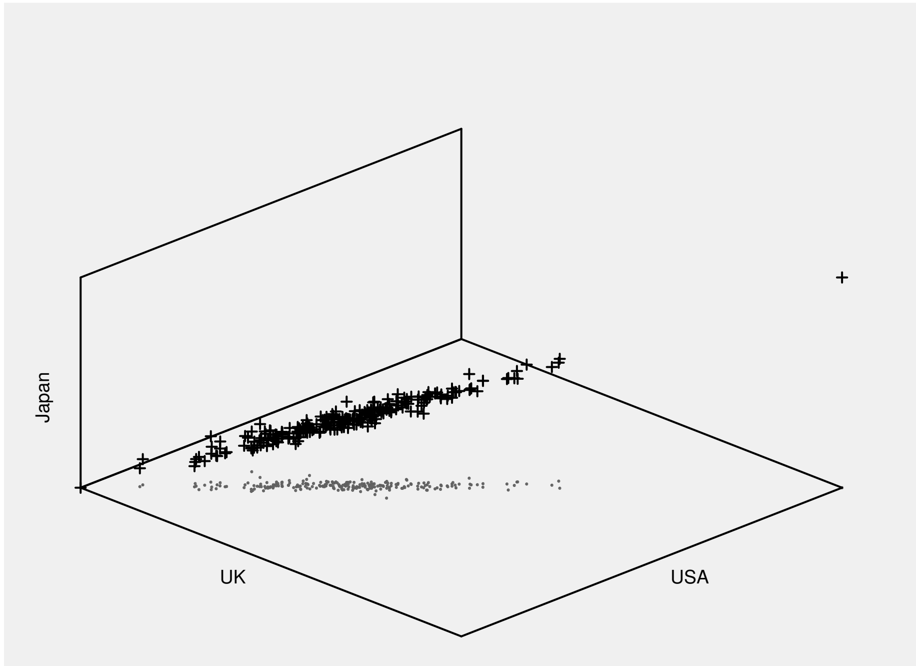

Figure 7: Correlation of Discount Factor Growth Rates (3-D)

Note: The figure presents a threedimensional scatterplot for growth rates of the discount factors calculated in the trilateral setting. The growth of discount factors is calculated according to equation 19.

lateral SDF approach identified in section 3.4. However, although it demonstrates that the (final) results for the international risk-sharing index are quite robust to the number of countries used in their calculation, the trilateral framework does not (completely) resolve the inherent incoherence found in the bilateral SDF model. In fact, it only (temporarily) fixes the problems from the bilateral setting, and therefore, faces the same type of (incoherence) criticisms with the addition of new countries.

Figure 6 shows that the discount factors calculated using the trilateral framework will be uniquely determined and price all assets in a three-country world. However, as soon as a fourth country enters this world, the trilateral framework will face the same problem(s) as the bilateral one[30]. In that case, the stochastic discount factors calculated from the trilateral setting will not be uniquely determined and will not price all assets simultaneously anymore. In fact, the addition of a fourth country brings two additional (independent) shocks into the system that cannot be (uniquely) priced by the discount factors computed from the trilateral setting[31]. Moreover, the time paths of the three discount factors in Figure 6 are very similar with the time paths of the discount factors for the bilateral pairs that include USA (the first and second panel in Figure 1). This observation suggests that the discount factors for all three countries crucially depend on the shock with the highest Sharpe ratio, i.e. the USA stock market shock again[32]. Hence, including a fourth country with a Sharpe ratio (for its stock market) even higher than the one for the USA might lead to dramatic changes in all discount factors from the trilateral setting. Following the same argument as for the bilateral setting in section 3.4, the discount factors computed from one trilateral setting (group of three countries) will (in general) be quite different from the discount factors computed from another trilateral setting in that case (another combination of three countries)[33]. Therefore, the trilateral setting only shifts the problem with the internal incoherence of the SDF approach to international risk-sharing by one country ahead, but

[30] The trilateral framework, elaborated on in section B, offers a simple extension to calculate risk-sharing among a group of several countries.

[31] Each country adds two additional (independent) shocks: one related to its stock market, and the other related to its foreign exchange market.

[32] Table 5 in the appendix shows that all three discount factors load much more on the USA stock market shock than on the other shocks.

[33] There will be a total of four different three- country groups/combinations in this four-country world. Three of these groups will be strongly influenced by the highest Sharpe ratio country, and therefore, will be quite different from the last group/combination that excludes this country.

does not resolve it.

5.2 Reconciliation with Macroeconomic Evidence

Unambiguously, the results from the trilateral framework just strengthen the evidence about the discrepancy that exists between the measures of international risk-sharing derived from asset markets data following the stochastic discount factor approach and those derived with macroeconomic data and specific utility function. One possible reason for these differences is the absence of complete capital markets. In fact, if asset markets account for only a small portion of total macroeconomic risks, then the low values for international risk-sharing implied by macroeconomic data can co-exist with the high risk-sharing measures presented here. However, these additional, non-marketable/non-insurable shocks not spanned by assets markets should be very large, negatively correlated across countries, and even more variable than the ones already observed in asset markets. In fact, Brandt et al. (2006) demonstrate that it is extremely difficult to justify the existence of such shocks[34]. Subsequently, shocks must be even larger and more variable to rationalize the results from the trilateral setting presented in this study. Therefore, it is very unlikely that the reconciliation between these two approaches to measuring international risk-sharing would go along these lines.

The arguments above suggest that equation 10 cannot hold if the two different approaches are to be reconciled[35]. In fact, by assuming that equa-

[34]Brandt et al. (2006) show that these additional, non-insurable shocks should be very volatile (adding 50-100 percent volatility in marginal utility growth per year) and poorly (or negatively) correlated in order to reconcile the risk-sharing figures from the SDF approach with those found in the macroeconomic studies.

[35]Equation 10 need not always hold in the presence of incomplete markets. However, there are (infinitely many) combinations of non-insurable shocks in the two countries, for

tion 10 holds at any point in time, i.e. that real exchange rates depreciate/appreciate exactly by the difference between domestic and foreign marginal utility growth rates, this approach implicitly "imposes" (almost) perfect risk-sharing. Compared to the macroeconomic literature, this condition is equivalent to the risk-sharing condition in the presence of non-traded goods proposed by Backus and Smith (1993): marginal utility growth (usually measured through consumption growth) can differ across countries as long as real exchange rates do not stay constant[36]. In that case, the Backus-Smith condition suggests that real exchange rates appreciate for countries that experience relatively higher marginal utility growth rates (relatively lower consumption growth rates).

It is important to realize that the basic asset pricing equation 10 gives an equilibrium, no-arbitrage condition between three macroeconomic variables. Nonetheless, none of these variables refers to asset(s) that is continually traded on the asset markets[37]. Instead, all variables correspond to abstract concepts about aggregate macroeconomic behavior, which prevents direct empirical testing of this condition. Therefore, it might be interesting to test not only whether this condition holds as parity (as assumed here), but rather to see whether it has the correct sign (+). If this is not the case, then the reconciliation of the two approaches might be very closely related to the solution(s) of other puzzles in international macroeconomics and finance: the uncovered interest parity (UIP) anomaly and the Backus-Smith puzzle

which equation 10 still holds in an incomplete markets setting.

[36]For exposition of this risk-sharing condition see Backus and Smith (1993), Kollmann (1995), Ravn (2001), or Corsetti, Dedola, and Leduc (2007).

[37]Condition 10 was derived under the assumptions that there is free trade in all assets and there are no pure (zero initial investment) arbitrage opportunities. Therefore, the nature of the three macroeconomic variables used in condition 10 seriously questions both of these assumptions.

(consumption-real exchange rate correlation puzzle).

6 Concluding Remarks

In this study we present an extension of the stochastic discount factor approach to international risk-sharing. At the beginning, we present the theoretical framework that links the minimum-variance discount factors in two countries with the corresponding real exchange rate. We elaborate on the calculation of the discount factors, the construction of the risk-sharing index and the replication of the results for the bilateral setting given in Brandt et al. (2006). There are two possible criticisms about the inherent inconsistency of the bilateral approach. First, the discount factors are not uniquely determined in the bilateral framework and crucially depend on the partner country included in the calculation. Second, the deviations between the discount factors obtained in this way (the imprecision in the measurement of marginal utility growth) are larger for countries whose stock market excess return shocks are relatively less important. Both of these criticisms suggest that the (bilateral) SDF approach to international risk-sharing is very sensitive to the choice of particular partner countries.

In order to account for some of these shortcomings of the bilateral framework, we propose an extension to a three-country (trilateral) setting. However, although the trilateral framework demonstrates that the (final) results for the international risk-sharing index are quite robust to the number of countries used in their calculation, it does not resolve the inherent incoherence found in the bilateral SDF model. In fact, as soon as a fourth country enters this world, the trilateral framework will face the same problem(s) as the bilateral one: the discount factors will not be uniquely determined and their behavior will crucially depend on the shock with the highest Sharpe ratio. Therefore, we conclude that the trilateral setting only shifts the problem with the internal incoherence of the SDF approach to international risk-sharing by one country ahead, but does not resolve it.

Finally, we give a note of caution on the interpretation of the results in this study. The stochastic discount factor approach to international risk-sharing is derived under the assumption that equation 10 always holds. Moreover, the replication of the results for the bilateral setting, but also the extension to a trilateral setting are performed retaining the assumption that equation 10 prices all assets at any point in time. However, if this is not the case, i.e. if the economies are far-away from what is implied by the first principles, then this approach cannot give valid measures of international risk-sharing in the first place.

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A Excess Returns in Bilateral Setting

This section presents formulae for excess returns in the bilateral framework. First, we present a general derivation for excess return formulae for each asset. Second, we derive vectors of expected excess returns for each country. A general distinction is made between formulae for domestic country (with superscript d) and foreign country (with superscript f) assets. USA is the domestic country in the first two country-pairs, and UK is the domestic country in the last country-pair.

A.1 Excess Return Processes for Domestic Investor

The investors in the domestic country face the following three types of excess return shocks: domestic stock, foreign bond, and foreign stock.

The excess returns on domestic stock are calculated difference between returns on the domestic stock market and the risk-free rate on domestic bond:

$$\frac{dS^d}{S^d} - \frac{dB^d}{B^d} = (\theta^d - r^d)dt + dz^d \tag{27}$$

The corresponding excess return on the foreign bond for the domestic investor is given as the difference between foreign bond return expressed in domestic currency and domestic bond return. Hence, although foreign bond is risk-free for the foreign investor, it is risky asset from the perspective of the domestic investor due to the currency risk it contains.

$$\frac{d(eB^f)}{eB^f} - \frac{dB^d}{B^d} = \frac{de}{e} + r^f dt - r^d dt = (\theta^e + r^f - r^d)dt + dz^e \tag{28}$$

Finally, excess returns on foreign stock for the domestic investor are calculated as the difference between returns on foreign stock and returns on foreign bonds when both are expressed in domestic currency units. Excess returns on the foreign stock for the domestic investor is calculated as follows:

$$
\begin{aligned}
\frac{d(eS^f)}{eS^f} - \frac{d(eB^f)}{eB^f} &= \frac{dS^f}{S^f} + \frac{de}{e}\frac{dS^f}{S^f} - \frac{dB^f}{B^f} - \frac{de}{e}\frac{dB^f}{B^f} \\
&= \Big(1 + \frac{de}{e}\Big)\Big(\frac{dS^f}{S^f} - \frac{dB^f}{B^f}\Big) \\
&= (1 + \theta^e dt + dz^e)(\theta^f dt + dz^f - r^f dt) \\
&= \theta^f dt + dz^f - r^f dt \\
&+ \theta^e dt\theta^f dt + \theta^e dt dz^f - \theta^e dt r^f dt + dz^e\theta^f dt + dz^e dz^f - dz^e r^f dt \\
&= (\theta^f - r^f)dt + dz^e dz^f + dz^f \\
&= (\theta^f - r^f + \Sigma^{ef})dt + dz^f \qquad (29)
\end{aligned}
$$

A.2 Excess Return Processes for Foreign Investor

The investors in the foreign country face the following three types of excess return shocks: domestic bond, domestic stock, and foreign stock.

The excess return on the domestic bond for the foreign investor is given as the difference between domestic bond return expressed in domestic currency and foreign bond return. Hence, although domestic bond is risk-free for the domestic investor, it is a risky asset from the perspective of the foreign investor due to the currency risk it contains. These excess returns are given as follows:

$$
\begin{aligned}
\frac{d\Big(\frac{B^d}{e}\Big)}{\Big(\frac{B^d}{e}\Big)} - \frac{dB^f}{B_f} &= \Big(\frac{dB^d}{B^d} - \frac{de}{e} + \frac{de_1^2}{e_1^2} - \frac{de}{e}\frac{dB^d}{B^d}\Big) - \frac{dB^f}{B_f} \\
&= r^d dt - \theta^e dt - dz^e + \Sigma^{ee} dt - \theta^d dt r^d dt - r^f dt \\
&= (r^d - r^f - \theta^e + \Sigma^{ee})dt - dz^e \\
&= -[(\theta^e + r^f - r^d - \Sigma^{ee})dt + dz^e] \qquad (30)
\end{aligned}
$$

The excess returns on domestic stock from the perspective of foreign investor are calculated as difference between domestic stock and domestic bond returns, both translated into foreign currency:

$$
\begin{aligned}
\frac{d\left(\frac{S^d}{e}\right)}{\frac{S^d}{e}} - \frac{d\left(\frac{B^d}{e}\right)}{\frac{B^d}{e}} &= \Big(\frac{dS^d}{S^d} - \frac{de}{e} + \frac{de_1^2}{e_1^2} - \frac{de}{e}\frac{dS^d}{S^d}\Big) \\
&- \Big(\frac{dB^d}{B^d} - \frac{de}{e} + \frac{de_1^2}{e_1^2} - \frac{de}{e}\frac{dB^d}{B^d}\Big) \\
&= \frac{dS^d}{S^d} - \frac{dB^d}{B^d} - \frac{de}{e}\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \frac{de}{e})\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \theta^e dt - dz^e)(\theta^d dt + dz^d - r^d dt) \\
&= \theta^d dt + dz^d - r^d dt - \theta^e dt\theta^d dt - \theta^e dt dz^d \\
&+ \theta^e dt r^d dt - dz^e\theta^d dt + dz^e dz^d - dz^e r^d dt \\
&= (\theta^d - r^d - \Sigma^{ed})dt + dz^d \qquad (31)
\end{aligned}
$$

Finally, the excess returns that foreign investors get by investing on the foreign stock market are given as the difference between returns on foreign stock market and returns on foreign bond. Since the latter is a risk-free asset from the perspective of foreign investors. Hence, the foreign stock market excess returns are given by the following equation:

$$
\frac{dS^f}{S^f} - \frac{dB^f}{B^f} = (\theta^f - r^f)dt + dz^f \qquad (32)
$$

A.3 Expected Excess Returns

This section presents the expected values for the excess return processes calculated in the previous two sections. The term in front of the dt term refers to the expected values in the continuous-time formulation employed here. Therefore, domestic investor faces the following set of expected excess returns:

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ \theta^e + r^f - f^d \\ \theta^f - r^f + \Sigma^{ef} \end{bmatrix}$$

This vector stacks the expected values of the expected return processes given by equations 27 (domestic stock), 28 (foreign bond), and 32 (foreign stock).

The foreign investor faces a similar set of expected excess returns. The following vector stack the expected values of the expected return processes given by equations 30 (domestic bond), 31 (domestic stock), and 32 (foreign stock):

$$\mu^f = \begin{bmatrix} \theta^d - r^d - \Sigma^{ed} \\ -(\theta^e + r^f - r^d - \Sigma^{ee}) \\ \theta^f - r^f \end{bmatrix}$$

B Calculations for the Trilateral Setting

This section presents calculations for the trilateral framework. The discount factors in this trilateral setting can be calculated according to equations 19 and 20:

$$\frac{d\Lambda^i}{\Lambda^i} = -r^i dt - \mu^{i\prime}\Sigma_i^{-1} dz_i, i = d, f_1, f_2$$

$$d \ln \Lambda = \frac{d\Lambda}{\Lambda} - \frac{1}{2}\frac{d\Lambda^2}{\Lambda^2} = -\Big(r + \frac{1}{2}\mu'\Sigma_i^{-1}\mu\Big)dt - \mu'\Sigma_i^{-1} dz_i$$

and their volatility according to equation 21:

$$\frac{1}{dt}\sigma^2(d \ln \Lambda^i) = \mu'\Sigma_i^{-1}\mu, i = d, f_1, f_2$$

where d refers to the domestic country, f_1 to the first foreign country, and f_2 to the second foreign country. In the calculations below, d stands for the USA, f_1 for the UK, and f_2 for Japan. In the trilateral setting, residents in each country are faced with five (instead of three) sources of uncertainty. Apart from shocks to domestic risky assets, they face two exchange rate shocks, and two foreign risky assets shocks. Thus, all these sources of uncertainty can be summarized in the following three vectors, each referring to residents of the corresponding country:

$$dz_d = \begin{bmatrix} dz^d \\ dz^{e_1} \\ dz^{e_2} \\ dz^{f_1} \\ dz^{f_2} \end{bmatrix}$$

$$dz_{f_1} = \begin{bmatrix} dz^d \\ dz^{e_1} \\ dz^{e_3} \\ dz^{f_1} \\ dz^{f_2} \end{bmatrix}$$

$$dz_{f_2} = \begin{bmatrix} dz^d \\ dz^{e_3} \\ dz^{e_2} \\ dz^{f_1} \\ dz^{f_2} \end{bmatrix}$$

with the following set of three variance-covariance matrices:

$$\Sigma_d = \frac{1}{dt} E(dz_d dz_d') = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de_1} & \Sigma^{de_2} & \Sigma^{df_1'} & \Sigma^{df_2'} \\ \Sigma^{e_1 d'} & \Sigma^{e_1' e_1} & \Sigma^{e_1' e_2} & \Sigma^{e_1 f_1'} & \Sigma^{e_1 f_2'} \\ \Sigma^{e_2 d'} & \Sigma^{e_2' e_1} & \Sigma^{e_2' e_2} & \Sigma^{e_2 f_1'} & \Sigma^{e_2 f_2'} \\ \Sigma^{f_1' d} & \Sigma^{f_1' e_1} & \Sigma^{f_1' e_2} & \Sigma^{f_1' f_1} & \Sigma^{f_1' f_2} \\ \Sigma^{f_2' d} & \Sigma^{f_2' e_1} & \Sigma^{f_2' e_2} & \Sigma^{f_2' f_1} & \Sigma^{f_2' f_2} \end{bmatrix}$$

$$\Sigma_{f_1} = \frac{1}{dt} E(dz_{f_1} dz_{f_1}') = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de_1} & \Sigma^{de_3} & \Sigma^{df_1'} & \Sigma^{df_2'} \\ \Sigma^{e_1 d'} & \Sigma^{e_1' e_1} & \Sigma^{e_1' e_3} & \Sigma^{e_1 f_1'} & \Sigma^{e_1 f_2'} \\ \Sigma^{e_3 d'} & \Sigma^{e_3' e_1} & \Sigma^{e_3' e_3} & \Sigma^{e_3 f_1'} & \Sigma^{e_3 f_2'} \\ \Sigma^{f_1' d} & \Sigma^{f_1' e_1} & \Sigma^{f_1' e_3} & \Sigma^{f_1' f_1} & \Sigma^{f_1' f_2} \\ \Sigma^{f_2' d} & \Sigma^{f_2' e_1} & \Sigma^{f_2' e_3} & \Sigma^{f_2' f_1} & \Sigma^{f_2' f_2} \end{bmatrix}$$

$$\Sigma_{f_2} = \frac{1}{dt}E(dz_{f_2}dz'_{f_2}) = \begin{bmatrix} \Sigma^{dd'} & \Sigma^{de_3} & \Sigma^{de_2} & \Sigma^{df'_1} & \Sigma^{df'_2} \\ \Sigma^{e_3d'} & \Sigma^{e'_3e_3} & \Sigma^{e'_3e_2} & \Sigma^{e_3f'_1} & \Sigma^{e_3f'_2} \\ \Sigma^{e_2d'} & \Sigma^{e'_2e_3} & \Sigma^{e'_2e_2} & \Sigma^{e_2f'_1} & \Sigma^{e_2f'_2} \\ \Sigma^{f'_1d} & \Sigma^{f'_1e_3} & \Sigma^{f'_1e_2} & \Sigma^{f'_1f_1} & \Sigma^{f'_1f_2} \\ \Sigma^{f'_2d} & \Sigma^{f'_2e_3} & \Sigma^{f'_2e_2} & \Sigma^{f'_2f_1} & \Sigma^{f'_2f_2} \end{bmatrix}$$

Moreover, we must impose an additional restriction in the calculation. Namely, we have to exclude the possibilities for triangular (cross-currency) arbitrage. In particular, if the exchange rate returns are given by:

$$\frac{de_1}{e_1} = \theta_1^e dt + dz_1^e, \frac{de_2}{e_2} = \theta_2^e dt + dz_2^e, \frac{de_3}{e_3} = \theta_3^e dt + dz_3^e \tag{33}$$

then the following cross-currency condition must hold:

$$\theta_3^e dt + dz_3^e = \theta_2^e dt + dz_2^e + \theta_1^e dt + dz_1^e \tag{34}$$

The excess return vectors can be related using the restrictions imposed by the cross-currency condition 34 (no triangular arbitrage possibilities). For example, the excess returns for a domestic resident can be related with the excess returns for a resident in the first foreign country (f_1) as follows[38]:

$$\mu^{f_1} = A\mu^d \tag{35}$$

where the matrix A is defined as:

[38]For reasons of symmetry we use directly the discrete-time equivalents of the continuous-time formulae, just as implemented in the calculations. Thus, we disregard the continuous-time terms in the excess return vectors.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{36}$$

Equation 35 shows that residents in both countries face the same expected excess returns on all three stock markets, while their foreign exchange excess returns form a linear combination. In turn, the variance covariance matrix with shocks facing the residents in the first foreign country is given by:

$$\Sigma_{f_1} = A\Sigma_d A' \tag{37}$$

and its inverse:

$$\Sigma_{f_1}^{-1} = (A\Sigma_d A')^{-1} = (A')^{-1}\Sigma_d^{-1}A^{-1} \tag{38}$$

Therefore the domestic and first foreign (f_1) discount factor loadings will be related as follows:

$$\mu^{f_1}\Sigma_{f_1}^{-1} = \mu^d A'(A')^{-1}\Sigma_d^{-1}A^{-1} = \mu^d\Sigma_d^{-1}A^{-1} \tag{39}$$

Equation 39 indicates that the only difference between domestic and foreign discount factors is given by A^{-1}. It means that all discount factors load equally on all three stock market shocks, while their foreign exchange loadings differ by a linear combination of the exchange rate shocks.

B.1 Excess Returns in Trilateral Setting

This section presents formulae for excess returns in the trilateral framework. Similar as in the bilateral case, we present a derivation for (expected) excess returns on each asset. A general distinction is made between formulae for domestic country (with superscript d) and two foreign countries (with superscript f_1 and f_2). USA is the domestic country, UK refers to foreign country f_1 and Japan refers to foreign country f_2.

B.1.1 Domestic - USA

A USA-based resident gets the following excess returns on the domestic stock market:

$$\frac{dS^d}{S^d} - \frac{dB^d}{B^d} = (\theta^d - r^d)dt + dz^d \tag{40}$$

similarly, he gets the following excess returns on the foreign bond in country f_1 (UK):

$$\begin{aligned}
\frac{d\left(\frac{B^d}{e_1}\right)}{\left(\frac{B^d}{e_1}\right)} - \frac{dB^{f_1}}{B_{f_1}} &= \Big(\frac{dB^d}{B^d} - \frac{de_1}{e_1} + \frac{de_1^2}{e_1^2} - \frac{de_1}{e_1}\frac{dB^d}{B^d}\Big) - \frac{dB^{f_1}}{B_{f_1}} \\
&= r^d dt - \theta^{e_1}dt - dz^{e_1} + \Sigma^{e_1 e_1}dt - \theta^d dt r^d dt - r^{f_1}dt \\
&= (r^d - r^{f_1} - \theta^{e_1} + \Sigma^{e_1 e_1})dt - dz^{e_1} \\
&= -[(\theta^{e_1} + r^{f_1} - r^d - \Sigma^{e_1 e_1})dt + dz^{e_1}]
\end{aligned} \tag{41}$$

on the foreign bond in Japan (country f_2):

$$\frac{d(e_2 B^{f_2})}{e_2 B^{f_2}} - \frac{dB^d}{B^d} = \frac{de_2}{e_2} + r^{f_2}dt - r^d dt = (\theta^{e_2} + r^{f_2} - r^d)dt + dz^{e_2} \tag{42}$$

on the stock market in the UK (country f_1):

$$
\begin{aligned}
\frac{d(e_1 S^{f_1})}{e_1 S^{f_1}} - \frac{d(e_1 B^{f_1})}{e_1 B^{f_1}} &= \frac{dS^{f_1}}{S^{f_1}} + \frac{de_1}{e_1}\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}} - \frac{de_1}{e_1}\frac{dB^{f_1}}{B^{f_1}} \\
&= \Big(1 + \frac{de_1}{e_1}\Big)\Big(\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}}\Big) \\
&= (1 + \theta^{e_1} dt + dz^{e_1})(\theta^{f_1} dt + dz^{f_1} - r^{f_1} dt) \\
&= \theta^{f_1} dt + dz^{f_1} - r^{f_1} dt \\
&+ \theta^{e_1} dt \theta^{f_1} dt + \theta^{e_1} dt dz^{f_1} - \theta^{e_1} dt r^{f_1} dt + dz^{e_1} \theta^{f_1} dt + dz^{e_1} dz^{f_1} - dz^{e_1} r^{f_1} dt \\
&= (\theta^{f_1} - r^{f_1}) dt + dz^{e_1} dz^{f_1} + dz^{f_1} \\
&= (\theta^{f_1} - r^{f_1} + \Sigma^{e_1 f_1}) dt + dz^{f_1} \qquad (43)
\end{aligned}
$$

and on the stock market in Japan (country f_2):

$$
\begin{aligned}
\frac{d(e_2 S^{f_2})}{e_2 S^{f_2}} - \frac{d(e_2 B^{f_2})}{e_2 B^{f_2}} &= \frac{dS^{f_2}}{S^{f_2}} + \frac{de_2}{e_2}\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}} - \frac{de_2}{e_2}\frac{dB^{f_2}}{B^{f_2}} \\
&= \Big(1 + \frac{de_2}{e_2}\Big)\Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= (1 + \theta^{e_2} dt + dz^{e_2})(\theta^{f_2} dt + dz^{f_2} - r^{f_2} dt) \\
&= \theta^{f_2} dt + dz^{f_2} - r^{f_2} dt \\
&+ \theta^{e_2} dt \theta^{f_2} dt + \theta^{e_2} dt dz^{f_2} - \theta^{e_2} dt r^{f_2} dt + dz^{e_2} \theta^{f_2} dt + dz^{e_2} dz^{f_2} - dz^{e_2} r^{f_2} dt \\
&= (\theta^{f_2} - r^{f_2}) dt + dz^{e_2} dz^{f_2} + dz^{f_2} \\
&= (\theta^{f_2} - r^{f_2} + \Sigma^{e_2 f_2}) dt + dz^{f_2} \qquad (44)
\end{aligned}
$$

B.1.2 Foreign 1 - UK

UK-based investor gets the following excess return on the USA (domestic) stock market:

$$
\frac{d\Big(\frac{S^d}{e_1}\Big)}{\frac{S^d}{e_1}} - \frac{d\Big(\frac{B^d}{e_1}\Big)}{\frac{B^d}{e_1}} = \Big(\frac{dS^d}{S^d} - \frac{de_1}{e_1} + \frac{de_1^2}{e_1^2} - \frac{de_1}{e_1}\frac{dS^d}{S^d}\Big)
$$

$$
\begin{aligned}
&-\Big(\frac{dB^d}{B^d}-\frac{de_1}{e_1}+\frac{de_1^2}{e_1^2}-\frac{de_1}{e_1}\frac{dB^d}{B^d}\Big)\\
&=\frac{dS^d}{S^d}-\frac{dB^d}{B^d}-\frac{de_1}{e_1}\Big(\frac{dS^d}{S^d}-\frac{dB^d}{B^d}\Big)\\
&=(1-\frac{de_1}{e_1})\Big(\frac{dS^d}{S^d}-\frac{dB^d}{B^d}\Big)\\
&=(1-\theta^{e_1}dt-dz^{e_1})(\theta^d dt+dz^d-r^d dt)\\
&=\theta^d dt+dz^d-r^d dt-\theta^{e_1}dt\theta^d dt-\theta^{e_1}dtdz^d+\\
&\theta^{e_1}dtr^d dt-dz^{e_1}\theta^d dt+dz^{e_1}dz^d-dz^{e_1}r^d dt\\
&=(\theta^d-r^d-\Sigma^{e_1d})dt+dz^d \qquad (45)
\end{aligned}
$$

and the following excess return on the USA (domestic) bond:

$$
\frac{d(e_1B^{f_1})}{e_1B^{f_1}}-\frac{dB^d}{B^d}=\frac{de_1}{e_1}+r^{f_1}dt-r^d dt=(\theta^{e_1}+r^{f_1}-r^d)dt+dz^{e_1} \qquad (46)
$$

while investment in the UK (foreign f_1) stock market brings him the following excess return:

$$
\begin{aligned}
\frac{d\Big(\frac{B^{f_2}}{e_3}\Big)}{\Big(\frac{B^{f_2}}{e_3}\Big)}-\frac{dB^{f_1}}{B_{f_1}}&=\Big(\frac{dB^{f_2}}{B^{f_2}}-\frac{de_3}{e_3}+\frac{de_3^2}{e_3^2}-\frac{de_3}{e_3}\frac{dB^{f_2}}{B^{f_2}}\Big)-\frac{dB^{f_1}}{B_{f_1}}\\
&=r^{f_2}dt-\theta^{e_3}dt-dz^{e_3}+\Sigma^{e_3e_3}dt-\theta^{f_2}dtr^{f_2}dt-r^{f_1}dt\\
&=(r^{f_2}-r^{f_1}-\theta^{e_3}+\Sigma^{e_3e_3})dt-dz^{e_3}\\
&=-[(\theta^{e_3}+r^{f_1}-r^{f_2}-\Sigma^{e_3e_3})dt+dz^{e_3}] \qquad (47)
\end{aligned}
$$

similar calculation can be made for Japanese bonds:

$$
\frac{dS^{f_1}}{S^{f_1}}-\frac{dB^{f_1}}{B^{f_1}}=(\theta^{f_1}-r^{f_1})dt+dz^{f_1} \qquad (48)
$$

and investment on the Japanese stock market:

$$
\frac{d\Big(\frac{S^{f_2}}{e_3}\Big)}{\frac{S^{f_2}}{e_3}}-\frac{d\Big(\frac{B^{f_2}}{e_3}\Big)}{\frac{B^{f_2}}{e_3}}=\Big(\frac{dS^{f_2}}{S^{f_2}}-\frac{de_3}{e_3}+\frac{de_3^2}{e_3^2}-\frac{de_3}{e_3}\frac{dS^{f_2}}{S^{f_2}}\Big)
$$

$$
\begin{aligned}
&- \Big(\frac{dB^{f_2}}{B^{f_2}} - \frac{de_3}{e_3} + \frac{de_3^2}{e_3^2} - \frac{de_3}{e_3}\frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= \frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}} - \frac{de_3}{e_3}\Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= (1 - \frac{de_3}{e_3})\Big(\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}}\Big) \\
&= (1 - \theta^{e_3}dt - dz^{e_3})(\theta^{f_2}dt + dz^{f_2} - r^{f_2}dt) \\
&= \theta^{f_2}dt + dz^{f_2} - r^{f_2}dt - \theta^{e_3}dt\theta^{f_2}dt - \theta^{e_3}dtdz^{f_2} + \theta^{e_3}dtr^{f_2}dt \\
&- dz^{e_3}\theta^{f_2}dt + dz^{e_3}dz^{f_2} - dz^{e_3}r^{f_2}dt \\
&= (\theta^{f_2} - r^{f_2} - \Sigma^{e_3 f_2})dt + dz^{f_2} \qquad (49)
\end{aligned}
$$

B.1.3 Foreign 2 - Japan

Japan-based investor gets the following excess return on the USA stock market:

$$
\begin{aligned}
\frac{d\Big(\frac{S^d}{e_2}\Big)}{\frac{S^d}{e_2}} - \frac{d\Big(\frac{B^d}{e_2}\Big)}{\frac{B^d}{e_2}} &= \Big(\frac{dS^d}{S^d} - \frac{de_2}{e_2} + \frac{de_2^2}{e_2^2} - \frac{de_2}{e_2}\frac{dS^d}{S^d}\Big) \\
&- \Big(\frac{dB^d}{B^d} - \frac{de_2}{e_2} + \frac{de_2^2}{e_2^2} - \frac{de_2}{e_2}\frac{dB^d}{B^d}\Big) \\
&= \frac{dS^d}{S^d} - \frac{dB^d}{B^d} - \frac{de_2}{e_2}\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \frac{de_2}{e_2})\Big(\frac{dS^d}{S^d} - \frac{dB^d}{B^d}\Big) \\
&= (1 - \theta^{e_2}dt - dz^{e_2})(\theta^d dt + dz^d - r^d dt) \\
&= \theta^d dt + dz^d - r^d dt - \theta^{e_2}dt\theta^d dt - \theta^{e_2}dtdz^d \\
&+ \theta^{e_2}dtr^d dt - dz^{e_2}\theta^d dt + dz^{e_2}dz^d - dz^{e_2}r^d dt \\
&= (\theta^d - r^d - \Sigma^{e_2 d})dt + dz^d \qquad (50)
\end{aligned}
$$

on the USA bond:

$$\frac{d\left(\frac{B^d}{e_2}\right)}{\left(\frac{B^d}{e_2}\right)} - \frac{dB^{f_2}}{B_{f_2}} = \Big(\frac{dB^d}{B^d} - \frac{de_2}{e_2} + \frac{de_2^2}{e_2^2} - \frac{de_2}{e_2}\frac{dB^d}{B^d}\Big) - \frac{dB^{f_2}}{B_{f_2}}$$
$$= r^d dt - \theta^{e_2} dt - dz^{e_2} + \Sigma^{e_2 e_2} dt - \theta^d dt r^d dt - r^{f_2} dt$$
$$= (r^d - r^{f_2} - \theta^{e_2} + \Sigma^{e_2 e_2})dt - dz^{e_2}$$
$$= -[(\theta^{e_2} + r^{f_2} - r^d - \Sigma^{e_2 e_2})dt + dz^{e_2}] \qquad (51)$$

on UK (country f_1) bond:

$$\frac{d(e_3 B^{f_1})}{e_3 B^{f_1}} - \frac{dB^{f_2}}{B^{f_2}} = \frac{de_3}{e_3} + r^{f_1} dt - r^{f_2} dt = (\theta^{e_3} + r^{f_1} - f^{f_2})dt + dz^{e_3} \qquad (52)$$

on the UK stock market:

$$\frac{d(e_3 S^{f_1})}{e_3 S^{f_1}} - \frac{d(e_3 B^{f_1})}{e_3 B^{f_1}} = \frac{dS^{f_1}}{S^{f_1}} + \frac{de_3}{e_3}\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}} - \frac{de_3}{e_3}\frac{dB^{f_1}}{B^{f_1}}$$
$$= \Big(1 + \frac{de_3}{e_3}\Big)\Big(\frac{dS^{f_1}}{S^{f_1}} - \frac{dB^{f_1}}{B^{f_1}}\Big)$$
$$= (1 + \theta^{e_3} dt + dz^{e_3})(\theta^{f_1} dt + dz^{f_1} - r^{f_1} dt)$$
$$= \theta^{f_1} dt + dz^{f_1} - r^{f_1} dt$$
$$+ \theta^{e_3} dt \theta^{f_1} dt + \theta^{e_3} dt dz^{f_1} - \theta^{e_3} dt r^{f_1} dt + dz^{e_3} \theta^{f_1} dt + dz^{e_3} dz^{f_1} - dz^{e_3} r^{f_1} dt$$
$$= (\theta^{f_1} - r^{f_1})dt + dz^{e_3} dz^{f_1} + dz^{f_1}$$
$$= (\theta^{f_1} - r^{f_1} + \Sigma^{e_3 f_1})dt + dz^{f_1} \qquad (53)$$

and on the Japanese stock market:

$$\frac{dS^{f_2}}{S^{f_2}} - \frac{dB^{f_2}}{B^{f_2}} = (\theta^{f_2} - r^{f_2})dt + dz^{f_2} \qquad (54)$$

B.2 Expected Excess Returns

The set of expected excess returns for the USA-based investor is given in the following vector:

$$\mu^d = \begin{bmatrix} \theta^d - r^d \\ -(\theta^{e_1} + r^{f_1} - r^d - \Sigma^{e_1 e_1}) \\ \theta^{e_2} + r^{f_2} - r^d \\ \theta^{f_1} - r^{f_1} + \Sigma^{e_1 f_1} \\ \theta^{f_2} - r^{f_2} + \Sigma^{e_2 f_2} \end{bmatrix}$$

similarly for the set of expected excess returns facing the UK-based investor:

$$\mu^{f_1} = \begin{bmatrix} \theta^d - r^d - \Sigma^{e_1 d} \\ \theta^{e_1} + r^{f_1} - r^d \\ -(\theta^{e_3} + r^{f_1} - r^{f_2} - \Sigma^{e_3 e_3}) \\ \theta^{f_1} - r^{f_1} \\ \theta^{f_2} - r^{f_2} - \Sigma^{e_3 f_2} \end{bmatrix}$$

and for the Japan-based investor:

$$\mu^{f_2} = \begin{bmatrix} \theta^d - r^d - \Sigma^{e_2 d} \\ -(\theta^{e_2} + r^{f_2} - r^d - \Sigma^{e_2 e_2}) \\ \theta^{e_3} + r^{f_1} - f^{f_2} \\ \theta^{f_1} - r^{f_1} + \Sigma^{e_3 f_1} \\ \theta^{f_2} - r^{f_2} \end{bmatrix}$$

The interpretation of these excess returns is analogous to the one given for the bilateral setting. In fact, the main difference is that residents can invest in two (instead of one) foreign risk-free bonds and three (instead of two) stock markets.

Table 5: Discount Factor Loadings (Trilateral)

		USA	UK	Japan
dz^{d}	(dz^{USA})	4.07	4.07	4.07
dz^{e_1}	$(dz^{USA/UK})$	-0.23	1.74	
dz^{e_2}	$(dz^{USA/JAP})$	1.51		0.23
dz^{e_3}	$(dz^{JAP/UK})$		-1.51	-1.74
dz^{f_1}	(dz^{UK})	1.86	1.86	1.86
dz^{f_2}	(dz^{JAP})	-0.45	-0.45	-0.45

Note: The table presents figures for the discount factor loadings in the trilateral setting. The loadings for the discount factor in country i are given by $\mu^{i\prime}\Sigma_i^{-1}$. There are three stock market shocks and three real exchange rate shocks in this trilateral framework. The row marked $dz^{d}(dz^{USA})$ contains figures for discount factor loadings on the USA stock market shocks, row $dz^{f_1}(dz^{UK})$ refers to discount factor loadings on the UK stock market shocks, and row $dz^{f_2}(dz^{JAP})$ refers to discount factor loadings on the Japanese stock market shock. Rows marked dz^{e_i} contain figures for discount factor loadings on shocks for real exchange rate i. $dz^{e_1}(dz^{USA/UK})$ is defined as the relative price of UK in terms of USA goods, i.e. as the ratio of UK price level of USA price level. Similar definitions apply to $dz^{e_2}(dz^{USA/JAP})$ and $dz^{e_3}(dz^{JAP/UK})$.

B.3 Discount Factor Loadings

The evolution of the stochastic discount factors in the trilateral framework depends on five excess return shocks: three associated with the stock markets in each country plus two associated with the exchange rates. Table 5 presents the discount factor loadings on these five shocks for each of the three countries. Several findings in this table deserve attention. First, in line with the results for the bilateral setting and equation 39, all discount factors load equally on the stock market excess return shocks in each country. Second, these loadings differ across stock markets, being the the strongest for the USA, and the weakest for Japan. In fact, the magnitude and the relative importance of these loadings on the stock markets in the trilateral setting closely resemble those for the two bilateral pairs in section 3. Third, as pointed out in equations 35 and 39, each exchange rate loading forms a linear combination of the other two. For example, condition 39 and the definition of matrix A given in 36 imply the following relation between the loadings on the exchange rate excess return shocks for the domestic (USA) and the first foreign country (UK): $dz_{f_1}^{e_2} = dz_d^{e_2} - dz_d^{e_1}$. The values in Table 5 confirm this linear relationship: $(1.74 = 1.51 - (-0.23))$. Similar conclusions apply to the other exchange rate shock combinations given in the second, the third, and the fourth row of Table 5.

B.4 Pairwise Comparisons of SDF Growth Rates

Figure 8 plots the discount factor growth rates for all three country pairs (bilaterally). This figure is almost identical to Figure 2, which depicted the correlation of discount factor growth rates in the bilateral setting. As in the previous case, most observations lie on or very close to the 45 degrees lines, suggesting that marginal utility growth rates are almost equalized for each bilateral country pair. This is exactly what the perfect risk-sharing condition implies.

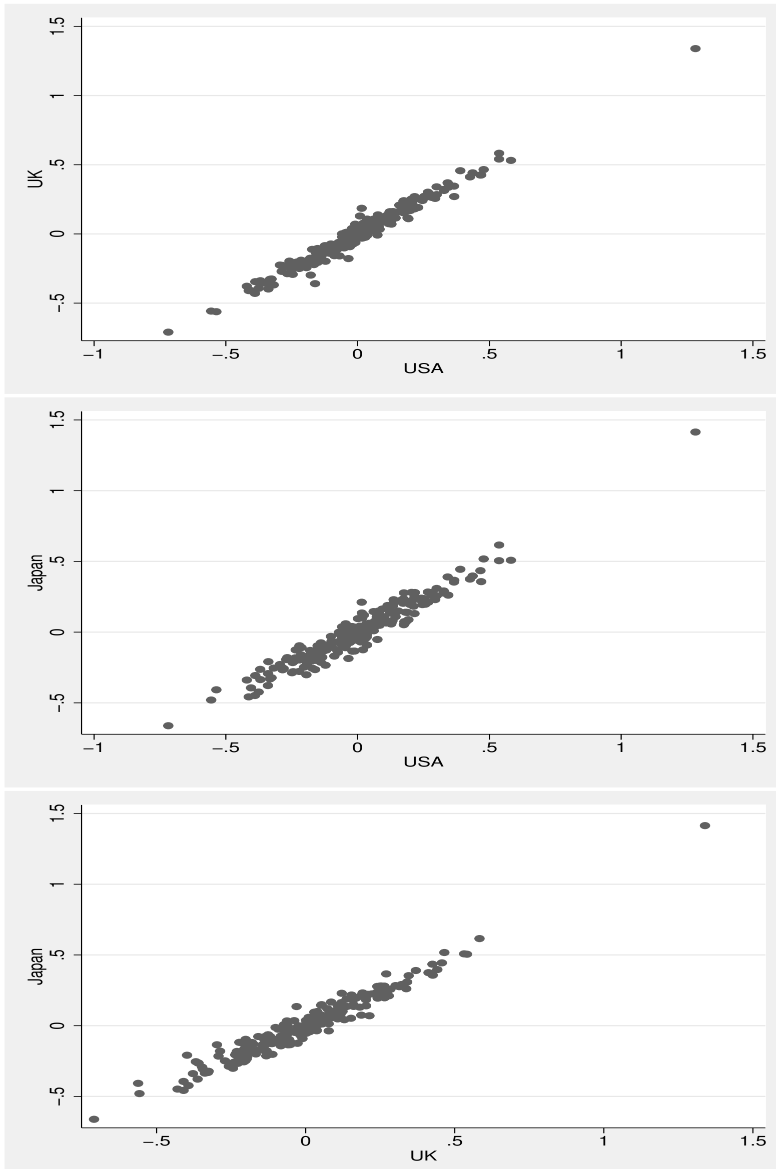

Figure 8: Discount Factor Growth Rates (Trilateral)

Note: The figure presents scatterplots for growth rates of the discount factors calculated in the trilateral setting. Each plot refers to one of the three country-pair combinations. The growth of discount factors is calculated according to equation 19.