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Reprint of: Bounding the locus of the center of mass for a part with shape variation

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ABSTRACT

The shape and center of mass of a part are crucial parameters to algorithms for planning automated manufacturing tasks. As industrial parts are generally manufactured to tolerances, the shape is subject to variations, which, in turn, also cause variations in the location of the center of mass. Planning algorithms should take into account both types of variation to prevent failure when the resulting plans are applied to manufactured incarnations of a model part.

We study the relation between variation in part shape and variation in the location of the center of mass for a part with uniform mass distribution. We consider a general model for shape variation that only assumes that every valid instance contains a shape P_I while it is contained in another shape P_E. We characterize the worst-case displacement of the center of mass in a given direction in terms of P_I and P_E. The characterization allows us to determine an adequate polytopic approximation of the locus of the center of mass. We also show that the worst-case displacement is small if P_I is convex and fat and the distance between the boundary of P_E and P_I is bounded.

1. Introduction

Many automated part manufacturing tasks involve manipulators that perform physical actions—such as pushing, squeezing [1], or pulling [2]—on the parts. Over the past two decades, researchers in robotics in general and algorithmic automation in particular have thoroughly studied the effect of physical actions as well as their potential role in accomplishing high-level tasks like orienting or sorting. It is evident that shape and—in many cases (see e.g. [1,3–7])—location of the center of mass are important parameters in determining the effect of a physical action on a part.

Industrial parts are always manufactured to tolerances as no production process is capable of delivering parts that are perfectly identical. Tolerance models [8,9] are therefore used to specify the admitted variations with respect to the CAD model. A consequence of these variations [10,11] is that actions that are computed on the basis of a CAD model of a part may easily lead to different behavior when executed on a manufactured incarnation of that part, and thus to failure to accomplish the higher-level task. It is important to note that the shape variations not only directly affect the behavior of the part but indirectly as well because they also cause a displacement of the center of mass of the part.

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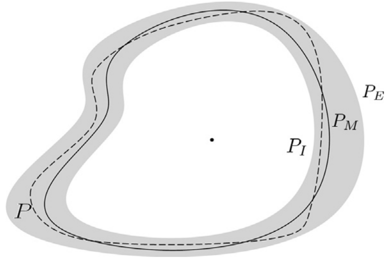

Fig. 1. A family of shapes specified by a subshape P_I and a supershape P_E of a model part P_M, along with a valid instance $P \in S(P_I, P_E)$.

To extend the planning algorithms to imperfect manufactured incarnations, it is important to understand the effects of variations and take them into account during planning. Larger variations in part shape and center-of-mass location inevitably result in a larger range of possible part behaviors, which reduces the likeliness that a manufacturing task can be accomplished. Therefore we will study how variations in part shape influence the location of the center of mass. (Note that variations in shape and center of mass are not the only sources of uncertainty in robotics. Additional uncertainty can result from the inaccuracy of the actuators and manipulators [12] and sensors [13].)

Several geometric approaches have been proposed to overcome the problems occurring in the presence of uncertainty and to smooth the effects of errors. Among the existing approaches are the model of ϵ-geometry [14], tolerance and interval geometry [15,16] and region-based models [17]. Generally, in all these models an uncertain point is represented by a region in which it may vary. The model of ϵ-geometry assumes that a point can vary within a disk of radius ϵ. Tolerance and interval-geometry take into account coordinate errors which results in an axis-aligned rectangular region in which a point can vary. In general, region-based models represent a point by any convex region. After modeling uncertainty as a point surrounded by a region, it is possible to study worst (and best) cases for a problem under the specific uncertainty model.

As observed before, variation of the shape causes variation of the center of mass of a part. The locus of the centroid of a set of points with approximate weights has been studied by Bern et al. [18]. Akella et al. [19] estimated the locus for a polygon under the ϵ-geometry model [19]. The problem of finding the locus of the center of mass of a part with shape variation and uniformly distributed mass has been mentioned as an open problem [11,19]. Akella et al. [19] studied rotating a convex polygon whose vertices and center of mass lie inside predefined circles centered at their nominal locations. The problem of orienting a part by fences has been studied by Chen et al. [11]. They define disk and square regions for the vertices of a part and proposed a method for computing the maximum allowable uncertainty radius for each vertex. They also discussed in a more general way the key role of the center of mass and the successfulness of part feeding (or orienting) algorithms in a setting of shape variation. Chen et al. [20] presented algorithms for squeezing and pushing problems. Kehoe et al. [21] explored cloud computing in a context of grasping and push-grasping under shape variation.

All the previous models for shape variation only allow the vertices to vary. In this paper we use a more general model for shape variation. For given shapes P_I and P_E such that $P_I \subseteq P_E$ we consider the family of shapes P satisfying $P_I \subseteq P \subseteq P_E$. In the practical setting of toleranced parts the shapes P_I and P_E will be fairly similar. We will show in Section 3 that the valid instance that yields the largest displacement of the center of mass in a given direction is a shape that combines a part of P_I with a part of P_E. The corresponding displacement is computable in $O(n \log n)$ time where n is the complexity of P_I and P_E; it can be used to obtain a k-facet outer approximation of the set of all possible loci of the center of mass in $O(kn \log n)$ steps.

In Section 4, we will study the size of the set of possible center-of-mass loci. Fatness of the objects under consideration has led to lower combinatorial complexities and more efficient algorithms for various problems, including union complexities [22], motion planning [23], hidden surface removal [24], and range searching [25]. Here we show that fatness and convexity of P_I together with the assumption that no point in P_E has a distance larger than ϵ to some point in P_I leads to a bound on the distance between the centers of mass of any two valid instances of a part which is proportional to ϵ and the fatness of P_I.

2. Preliminaries

In this section, we first present a general model for shape variations, then review the notion of a center of mass, and finally introduce a few notions that allow us to characterize the shapes that maximize the displacement of the center of mass. Let $P_M \subset \mathbb{R}^d$ be the model part, with $d = 2$ or $d = 3$. The part P_M has a uniform mass distribution.

No production process ever delivers parts that are perfectly identical to the model part P_M and therefore industrial parts are manufactured to tolerances. We use a very general model for permitted shape variations that only requires that any manufactured instance of P_M contains a given subshape P_I of P_M while it is contained in a supershape P_E of P_M. As a result, the set of acceptable instances of P_M is a family of shapes $S(P_I, P_E) = \{P \subset \mathbb{R}^d \mid P_I \subseteq P \subseteq P_E\}$ for given P_I and P_E satisfying $P_I \subseteq P_M \subseteq P_E$. In other words, the boundary ∂P of an instance $P \in S(P_I, P_E)$ should be entirely contained in $Q = P_E - int(P_I)$ where $int(P)$ denotes the interior of the set P. The region Q is referred to as the *tolerance zone*. The objects P_I and P_E are assumed to be closed semi-algebraic sets with a total of n boundary features. (Fig. 1 shows an

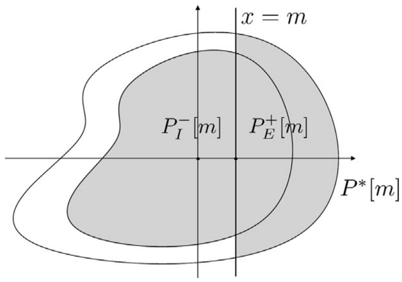

Fig. 2. A minmax object.

example of a model part P_M, shapes P_I and P_E, and a valid instance $P \in S(P_I, P_E)$.) We denote by $COM(P_I, P_E)$ the set of all centers of mass of instances $P \in S(P_I, P_E)$.

We let $X_c(P)$ denote the x-coordinate of the center of mass and $V(P)$ be the volume of the object P, with the understanding that the volume of a two-dimensional object is its area. The x-coordinate of the center of mass of an object with uniform mass distribution satisfies

$$X_c(P) = \frac{1}{V} \int_V x dV, \tag{1}$$

where V is the volume of the object. A similar equality holds for the other coordinate(s) of the center of mass. In the case of uniform mass distribution the center of mass corresponds to the centroid of the object. We will often decompose an object P into sub-objects P_i ($1 \le i \le n$) and then express its center of mass as a function of the centers of mass of its constituents, through the equation

$$X_c(P) = \frac{\sum_{i=1}^n X_c(P_i) V(P_i)}{\sum_{i=1}^n V(P_i)}. \tag{2}$$

We conclude this section by defining several useful objects. Balls play a prominent role in Section 4 of this paper. We denote by $B(p, r)$ the closed d-dimensional ball with radius r centered at p, and use the abbreviation $B(r) = B(O, r)$ where O is the origin.

For an object P and a value m we define its right portion $P^+[m]$ with respect to m by $P^+[m] = \{(x, y) \in P \subset \mathbb{R}^d \mid x \ge m\}$. Similarly, we define its left portion $P^-[m]$ with respect to m by $P^-[m] = \{(x, y) \in P \subset \mathbb{R}^d \mid x \le m\}$. With these portions we can define *minmax* objects, which allow us to capture the intuition that the largest displacement of the center of mass in a given direction is achieved by the object from $S(P_I, P_E)$ that 'maximizes mass' in that direction and 'minimizes mass' in the opposite direction. The minmax object $P^*[m]$ consists of a left portion of P_I and a right portion of P_E with respect to the same m, so $P^*[m] = P_I^-[m] \cup P_E^+[m] = P_I^- \cup P_E^+[m]$ (see Fig. 2). Note that an alternative way to describe $P^*[m]$ is by the equation $P^*[m] = P_I^-[m] \cup Q^+[m] = P_I \cup Q^+[m]$.

3. Displacement of the center of mass

In this section, we find an upper bound on the displacement of the center of mass in a given direction. The resulting bound allows us to determine a good polytopic outer approximation of the set $COM(P_I, P_E)$ of possible loci of the center of mass.

3.1. Bounding the displacement in one direction

Without loss of generality we assume that P_I and P_E are positioned and oriented in such a way that the center of mass of P_I coincides with the origin (so $X_c(P_I) = 0$) and the direction in which we want to bound the displacement aligns with the positive x-axis. Although we will bound the displacement with respect to the center of mass of P_I we observe that the result also induces a bound with respect to the center of mass of P_M as $P_M \in S(P_I, P_E)$ by definition. We let $X_r = \max_{(x,y) \in P_E} x$.

Our first lemma establishes a connection between the minmax objects $P^*[x]$ for $0 \le x \le X_r$ and the location of their centers of mass.

Lemma 1. *There is exactly one minmax object $P^*[m]$ ($0 \le m \le X_r$) that satisfies $X_c(P^*[m]) = m$. Moreover $x < X_c(P^*[x]) \le m$ for all $0 \le x < m$ and $X_c(P^*[x]) < m$ for all $m < x \le X_r$.*

Proof. From $X_c(P_I) = 0$ and $X_c(Q^+[0]) \ge 0$ and the fact that $P^*[0] = P_I \cup Q^+[0]$ it follows that $X_c(P^*[0]) \ge 0$; moreover, it is clear that $X_c(P^*[X_r]) \le X_r$. As the center of mass of $P^*[x]$ moves continuously as x increases from 0 to X_r there must be at least one x such that $X_c(P^*[x]) = x$. It remains to show that there is also at most one such x. Let m be such that $X_c(P^*[m]) = m$. We consider a minmax object $P^*[x]$ for $x \neq m$ and distinguish two cases: (i) $0 \le x < m$ and (ii) $m < x \le X_r$.

Consider case (i). Using the notation $Q' = Q^+[m]$ and $Q'' = P^*[x] - P^*[m] = Q^+[x] - Q^+[m]$ we have that $P^*[m] = P_I \cup Q'$ and $P^*[x] = P_I \cup Q' \cup Q''$. Note that $Q'' \subset [x, m] \times \mathbb{R}^{d-1}$ and thus

$$x \le X_c\big(Q''\big) \le m.$$

As $x < X_c(P^*[m]) = X_c(P_I \cup Q') = m$ it follows from applying Eq. (2) to $P^*[m] = P_I \cup Q'$ that

$$x\big(V(P_I) + V\big(Q'\big)\big) < X_c\big(Q'\big)V\big(Q'\big) = m\big(V(P_I) + V\big(Q'\big)\big).$$

If we then apply Eq. (2) to $P^*[x] = P_I \cup Q' \cup Q''$ and use the aforementioned equation and inequalities we obtain

$$\begin{aligned} X_c\big(P^*[x]\big) &= \frac{X_c(Q')V(Q') + X_c(Q'')V(Q'')}{V(P_I) + V(Q') + V(Q'')} \\ &> \frac{x(V(P_I) + V(Q')) + xV(Q'')}{V(P_I) + V(Q') + V(Q'')} = x \end{aligned}$$

and

$$\begin{aligned} X_c\big(P^*[x]\big) &= \frac{X_c(Q')V(Q') + X_c(Q'')V(Q'')}{V(P_I) + V(Q') + V(Q'')} \\ &\le \frac{m(V(P_I) + V(Q')) + mV(Q'')}{V(P_I) + V(Q') + V(Q'')} = m. \end{aligned}$$

Consider case (ii). Using the notation $Q' = Q^+[x]$ and $Q'' = P^*[m] - P^*[x] = Q^+[m] - Q^+[x]$ we have that $P^*[x] = P_I \cup Q'$ and $P^*[m] = P_I \cup Q' \cup Q''$. Note that $Q'' \subset [m, x] \times \mathbb{R}^{d-1}$ and thus

$$m \le X_c\big(Q''\big) \le x.$$

As $X_c(P^*[m]) = X_c(P_I \cup Q' \cup Q'') = m$ it follows from applying Eq. (2) to $P^*[m] = P_I \cup Q' \cup Q''$ that

$$X_c\big(Q'\big)V\big(Q'\big) = m\big(V(P_I) + V\big(Q'\big)\big) + \big(m - X_c\big(Q''\big)\big)V\big(Q''\big).$$

If we then apply Eq. (2) to $P^*[x] = P_I \cup Q'$ and use the above equations and inequality we obtain

$$\begin{aligned} X_c\big(P^*[x]\big) &= \frac{X_c(Q')V(Q')}{V(P_I) + V(Q')} \\ &\le \frac{m(V(P_I) + V(Q')) - (X_c(Q'') - m)}{V(P_I) + V(Q')} \\ &\le \frac{m(V(P_I) + V(Q'))}{V(P_I) + V(Q')} = m < x. \end{aligned}$$

Combining both cases we find that there is no $x \neq m$ that satisfies $X_c(P^*[x]) = x$. □

In addition to the fact that there is only one minmax object $P^*[m]$ that satisfies $X_c(P^*[m]) = m$, Lemma 1 also reveals that $X_c(P^*[x]) > x$ for $x < m$ and $X_c(P^*[x]) < x$ for $x > m$. Moreover, it shows that $X_c(P^*[x]) < m$ for all $x \neq m$ which means that the minmax object $P^*[m]$ with $X_c(P^*[m]) = m$ achieves larger displacement of the center of mass in the direction of the positive x-axis than any other minmax object $P^*[x]$ with $x \neq m$. The following theorem shows that $P^*[m]$ in fact achieves the largest displacement of the center of mass among *all* objects in $S(P_I, P_E)$.

Theorem 2. *Let $P^*[m]$ $(0 \le m \le X_r)$ be the unique minmax object that satisfies $X_c(P^*[m]) = m$. Then $X_c(P) < X_c(P^*[m])$ for all $P \in S(P_I, P_E)$, $P \neq P^*[m]$.*

Proof. Let $P \in S(P_I, P_E)$, $P \neq P^*[m]$ be the object that yields the largest displacement $m' \ge m$ of the center of mass, so $X_c(P) = m'$. If $P = P^*[m']$ then it follows immediately from Lemma 1 that $m' = m$. Now assume for a contradiction that $P \neq P^*[m'] = P_I^-[m'] \cup P_E^+[m']$ which implies that (i) $P_E^+[m'] - P^+[m'] \neq \emptyset$ or (ii) $P^-[m'] - P_I^-[m'] \neq \emptyset$.

Consider case (i) and let R be a closed connected subset with $V(R) > 0$ of $P_E^+[m'] - P^+[m']$. Observe that $P \cup R \in S(P_I, P_E)$. Note that $R \subset (m', \infty) \times \mathbb{R}^{d-1}$ and thus $X_c(R) > m'$. We get

$$\begin{aligned} X_c(P \cup R) &= \frac{X_c(P)V(P) + X_c(R)V(R)}{V(P) + V(R)} \\ &> \frac{m'V(P) + m'V(R)}{V(P) + V(R)} = m' \end{aligned}$$

which contradicts the assumption that P is the object in $S(P_I, P_E)$ that achieves the largest displacement of the center of mass.

Consider case (ii) and let R be a closed connected subset with $V(R) > 0$ of $P^-[m'] - P_I^-[m']$. Observe that $P - R \in S(P_I, P_E)$. Note that $R \subset (-\infty, m') \times \mathbb{R}^{d-1}$ and thus $X_c(R) < m'$. We get

$$X_c(P - R) = \frac{X_c(P)V(P) - X_c(R)V(R)}{V(P) - V(R)} > \frac{m'V(P) - m'V(R)}{V(P) - V(R)} = m$$

which again contradicts the assumption that P is the object in $S(P_I, P_E)$ that achieves the largest displacement of the center of mass. As a result we find that $P^*[m]$ with $X_c(P^*[m]) = m$ is the unique object in $S(P_I, P_E)$ that achieves the largest displacement of the center of mass. □

The theorem shows that the set $COM(P_I, P_E)$ does not extend beyond the plane or line $x = m$ where m is such that $X_c(P^*[m]) = m$. The bound is tight because $P^*[m] \in S(P_I, P_E)$. In fact, the theorem shows that $P^*[m]$ is the only instance in $S(P_I, P_E)$ that has its center of mass on that plane or line. Since the result holds in any direction, this implies that the boundary of $COM(P_I, P_E)$ bounds a convex set.

3.2. A k-facet approximation for $COM(P_I, P_E)$

The results in the previous subsection suggest an easy approach to determine an outer approximation of the set $COM(P_I, P_E)$ of possible centers of mass of instances in $S(P_I, P_E)$. If we select k different directions that positively span the d-dimensional space ($d = 2, 3$) and apply Theorem 2 in each of these directions then we obtain a bounded polytope with k facets enclosing $COM(P_I, P_E)$. Every facet of the polytope contains a point of the convex set $COM(P_I, P_E)$.

Our method to efficiently compute the largest displacement of the center of mass in the positive x-direction relies on a covering P_I and P_E by a set of signed cones, following an idea by Lien and Kajiya [26], using that an integral (such as the center of mass) over an object is the sum of appropriately signed integrals over the cones. To get the required covering of P_I (and P_E) we pick a point v on its boundary and decompose its facets into $O(n)$ subfacets of constant complexity such that every half-line emanating from v intersects the subfacet at most once. The cones are then obtained by connecting v to every single subfacet.

To find the largest displacement of the center of mass in the positive x-direction, we sort the features of P_I and P_E by x-coordinate and perform a binary search. For each x considered during this search we compute $X_c(P^*[x]) = X_c(P_I^-[x] \cup P_E^+[x])$ by using the intersections of the constant-complexity cones covering P_I and P_E with the respective half-spaces. This requires the computation of $O(n)$ integrals of a bounded-degree polynomial over a constant-complexity domain. By comparing the resulting $X_c(P^*[x])$ to x it can be determined how to continue the search. Once the x satisfying $X_c(P^*[x]) = x$ is found to lie between two consecutive features, we can find it by solving a polynomial equation of bounded degree. Applying the same procedure in each of the k selected directions yields that we can find a k-facet outer approximation of $COM(P_I, P_E)$ in $O(kn \log n)$ steps, where each step requires the computation of a bounded-degree polynomial over a constant-complexity domain or the solution of a polynomial equation of bounded degree.

Fig. 3 shows a two-dimensional P_I and P_E and 4-, 8-, 16-, and 64-edge outer approximations of $COM(P_I, P_E)$. Recall that every edge of the polygonal approximation contains one point of the convex set $COM(P_I, P_E)$, so $COM(P_I, P_E)$ strongly resembles its approximation.

The examples in Fig. 3 seem to suggest that the displacement of the center of mass is proportional to the distance between the boundaries of P_I and P_E and does not depend on the sizes of P_I and P_E themselves. In the next section we will see that this is not true in general. We will derive a bound on the size of $COM(P_I, P_E)$ for a convex P_I that depends on the distance between the boundaries of P_I and P_E and the fatness of P_I.

4. Bounding the size of $COM(P_I, P_E)$

The admitted shape variation for a manufactured part is usually small compared to the dimensions of the part itself. As a result, the enclosed shape P_I and enclosing shape P_E do not deviate much from the model shape P_M, and therefore also not from each other. To capture this similarity we will assume that $P_I \subseteq P_E \subseteq P_I \oplus B(\epsilon)$, where $\oplus$ denotes the Minkowski sum. Note that this means that every point in P_E lies within a distance of at most ϵ from some point in P_I.

We must also assume that P_I is convex and fat to obtain a bound on the diameter of $COM(P_I, P_E)$ that depends on ϵ and the fatness. There are many different definitions of fatness and we will use the one by De Berg et al. [27], which is based on a similar definition presented in the thesis of van der Stappen [23].

Definition 1. Let $P \subseteq \mathbb{R}^d$ be an object and let β be a constant with $0 < \beta \leq 1$. Define $U(P)$ as the set of all balls centered inside P whose boundary intersects P. We say that the object P is *β-fat* if for all balls $B \in U(P)$ we have $V(P \cap B) \geq \beta \cdot V(B)$. The *fatness* of P is defined as the maximal β for which P is β-fat.

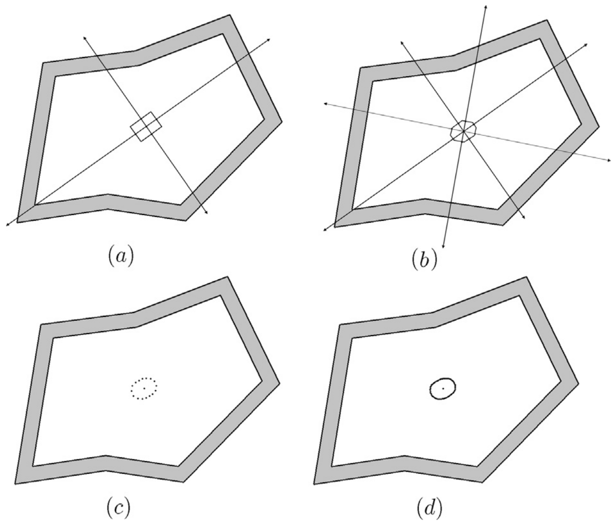

Fig. 3. Outer approximations of $COM(P_I, P_E)$ with (a) 4, (b) 8, (c) 16, and (d) 64 vertices.

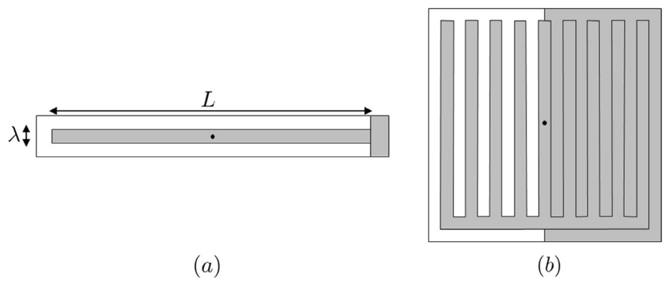

Fig. 4. (a) The part $P^*[L/2] = P_I \cup Q^+[L/2]$ (in gray) has its center of mass at least $L/4$ to the right of the center of mass of thin convex part P_I. (b) The part $P^*[0] = P_I \cup Q^+[0]$ (in gray) has its center of mass at least $L/16$ to the right of the center of mass of the fat non-convex part P_I.

For bounded objects the value of β is at most $1/2^d$; larger values only occur for unbounded objects [23].

Two planar polygonal examples in Sections 4.1 and 4.2 show that both fatness and convexity of P_I are necessary for a bound that is independent of the size of P_I (and P_E). In Section 4.3 we derive a bound for the case that both assumptions hold.

4.1. A thin convex part

When P_I is a sufficiently long and narrow box the set $S(P_I, P_E)$ contains shapes whose centers of mass are a distance proportional to the diameter of P_I apart. Let $L \gg \epsilon$ and pick λ such that $0 < \lambda < 2\epsilon^2/(L - \epsilon)$. We define $P_I = [-L/2, L/2] \times [-\lambda/2, \lambda/2]$ and $P_E = [-(L+\epsilon)/2, (L+\epsilon)/2] \times [-(\lambda+\epsilon)/2, (\lambda+\epsilon)/2]$, and note that $P_E \subseteq P_I \oplus B(\epsilon)$. See Fig. 4(a).

Now consider the object $P^*[L/2] = P_I^-[L/2] \cup P_E^+[L/2] = P_I \cup Q^+[L/2]$. We observe that $V(P_I) = \lambda L$, $X_c(P_I) = 0$, $V(Q^+[L/2]) = \epsilon(\epsilon + \lambda)/2$, and $X_c(Q^+[L/2]) = L/2 + \epsilon/4 > L/2$. The upper bound on λ implies that $V(Q^+[L/2]) > V(P_I)$. From Eq. (2) it follows that $X_c(P^*[L/2]) > L/4$, showing that the diameter of $COM(P_I, P_E)$ is *not* proportional to ϵ in this case.

4.2. A fat non-convex part

We define an auxiliary box $A = [-L/2, L/2]^2$ that will contain P_I and let $P_E = [-(L+\epsilon)/2, (L+\epsilon)/2]^2$. We subdivide A both horizontally and vertically into an odd number (≥ 5) of strips of width just smaller than $\epsilon/2$. See Fig. 4(b). We construct P_I by taking the union of every second vertical strip, starting with the first and ending with the last, and the bottommost horizontal strip. The resulting object P_I is a comb-shaped object that is known to be at least $1/4\pi$-fat [23]. Note that $P_E \subseteq P_I \oplus B(\epsilon)$. From the symmetry of P_I it is immediately clear that $X_c(P_I) = 0$.

Now consider the object $P^*[0] = P_I^-[0] \cup P_E^+[0] = P_I \cup Q^+[0]$. It is clear from the construction that $V(Q^+[0]) > V(P_I)/2$ and that $V(Q^+[0]) < V(P_I)$. It is also easy to verify that $X_c(Q^+[0]) > L/4$. Combining the inequalities with Eq. (2) yields that $X_c(P^*[0]) > L/16$ showing that the diameter of $COM(P_I, P_E)$ is also *not* proportional to ϵ in this case.

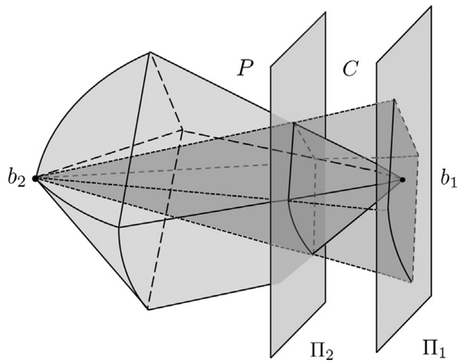

Fig. 5. Illustration of Lemma 4.

4.3. Fat convex parts

We turn our attention to the case that P_I is both convex and β-fat ($0 < \beta \leq 1$), and recall that $P_E \subseteq P_I \oplus B(\epsilon)$. The Steiner formula for convex bodies (see e.g. [28,29]) establishes a useful connection between the properties of a convex object P and a value ϵ on the one hand and the volume of $P \oplus B(\epsilon)$ on the other hand. Lemma 3 summarizes the formula for two- and three-dimensional objects.

Lemma 3. *Let $P \subset \mathbb{R}^d$ ($d = 2, 3$) be a convex object. Then*

- $V(P \oplus B(\epsilon)) = V(P) + p\epsilon + \pi\epsilon^2$, *where p is the perimeter of P, when $d = 2$, and*
- $V(P \oplus B(\epsilon)) = V(P) + a\epsilon + 2\pi w\epsilon^2 + \frac{4}{3}\pi\epsilon^3$, *where a is the surface area and w is the mean width of P, when $d = 3$.*

The results will be used in the proof of Theorem 5 to bound the additional volume that any instance of $S(P_I, P_E)$ can have in comparison to P_I.

Lemma 4 is not strictly necessary yet it leads to a better bound in our main theorem. The two-dimensional version of the lemma is based on a result by Hammer [30], which we will generalize to three-dimensional objects.

Lemma 4. *Let $P \subset \mathbb{R}^d$ ($d = 2, 3$) be a convex object with diameter δ. Then no point in P has distance larger than $\frac{\delta d}{d+1}$ to the center of mass of P.*

Proof. Hammer's result [30] says that the center of mass of every planar convex body divides every chord through it in a ratio less than or equal to 2/3, meaning that the part of the chord on one side of the center of mass cannot be more than twice as long as the part on the other side of the center of mass. The result then follows from the observation that no chord is longer than the diameter δ.

We will extend Hammer's construction to show that the center of mass of every three-dimensional convex body divides every chord through it in a ratio less than or equal to 3/4, which will then immediately imply the given result. Let b_1 and b_2 be the endpoints of a chord through the center of mass of P. See Fig. 5. Let Π_1 be a plane tangent to P at b_1, and let Π_2 be the plane parallel to Π_1 at 3/4 the distance between b_2 and Π_1 from b_2. The intersection of P with the plane Π_2 is a convex two-dimensional shape I. We create a generalized cone C by taking the union of all half-lines emanating from b_2 and passing through I and clipping the resulting shape with the plane Π_1. Application of Eq. (1) to C reveals that its center of mass lies on Π_2.

The plane Π_2 cuts the objects P and C into two parts each. Let P_1 and C_1 be the parts of P and C respectively between Π_1 and Π_2; let P_2 and C_2 be the parts of P and C respectively in the half-space bounded by Π_2 and containing b_2. The convexity of P implies that $P_1 \subseteq C_1$ and $P_2 \supseteq C_2$; in other words, the object P has less or equal mass than C beyond Π_2, while it has more or equal mass than C in front of Π_2 (when viewed from b_2). As a result, the object P must have its center of mass on the part of the chord between b_1 and b_2 in front of Π_2, which proves the claim. $\square$

Lemma 4 shows that any convex object with diameter δ (and uniform mass distribution) fits completely inside a ball with radius $\frac{\delta d}{d+1}$ centered at its center of mass, which is a slightly stronger result than the obvious claim that it fits inside a ball with radius δ.

We now have all the ingredients to prove our upper bound on the diameter of $COM(P_I, P_E)$.

Theorem 5. *Let $P_I \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded convex β-fat object ($0 < \beta \leq 1$) and let $P_E \subset \mathbb{R}^d$ be a bounded object satisfying $P_I \subseteq P_E \subseteq P_I \oplus B(\epsilon)$. Then the diameter of $COM(P_I, P_E)$ is bounded by $\frac{5}{2}\beta^{-1}\epsilon$ if $d = 2$ and by $3\beta^{-1}\epsilon$ if $d = 3$.*

Proof. We use δ to denote the diameter of P_I and once again assume without loss of generality that $X_c(P_I) = 0$. Theorem 2 shows that it suffices to consider objects $P^*[m]$ to bound the size of $COM(P_I, P_E)$. The assumption $X_c(P_I) = 0$ allows us to simplify Eq. (2) for $P^*[m] = P_I \cup Q^+[m]$ to

$$X_c\big(P^*[m]\big) = \frac{X_c(Q^+[m])V(Q^+[m])}{V(P_I) + V(Q^+[m])}. \tag{3}$$

Lemma 4 says that P_I lies completely inside $B(\delta d/(d+1))$. As a consequence, the object $P^*[m]$ must lie entirely inside $B(\delta d/(d+1)+\epsilon)$, which implies that $X_c(P^*[m]), X_c(Q^+[m]) \le (\delta d/(d+1)+\epsilon)$. We treat $d = 2$ and $d = 3$ separately.

Consider $d = 2$. We distinguish two cases based on the ratio of ϵ and δ.

If $\epsilon \ge \delta/6$ then $X_c(P^*[m]) \le 2\delta/3 + \epsilon \le 5\epsilon$. Since $P^*[m]$ is bounded we know that $\beta \le 1/4$ and thus $X_c(P^*[m]) \le 5\epsilon \le 5\beta^{-1}\epsilon/4$.

If $\epsilon \le \delta/6$ we use Eq. (3) to obtain an upper bound $X_c(P^*[m])$ by combining the upper bound $X_c(Q^+[m]) \le 2\delta/3 + \epsilon$ with a lower bound on $V(P_I)$ and upper and lower bounds on $V(Q^+[m])$. The lower bound on $V(P_I)$ follows from the fatness of P_I. As δ is the diameter of P_I there must be two points $p_1, p_2 \in P_I$ that are δ apart. The boundary of the ball $B(p_1, \delta)$ contains p_2 and thus belongs to the set $U(P_I)$. The β-fatness of P_I implies that $V(P_I) \ge \beta \cdot V(B(p_1,\delta)) = \beta\pi\delta^2$.

It remains to bound $V(Q^+[m])$. We note that $Q^+[m] \subseteq Q \subseteq (P_I \oplus B(\epsilon)) - int(P_I)$, from which it follows that $V(Q^+[m]) \le V(P_I \oplus B(\epsilon)) - V(P_I)$. Lemma 3 says that $V(P_I \oplus B(\epsilon)) - V(P_I) = p\epsilon + \pi\epsilon^2$, where p is the perimeter of P. As P_I is contained in $B(2\delta/3)$ we know that $p \le 4\pi\delta/3$. Combining these observations with a trivial lower bound on $V(Q^+[m])$ we get $0 \le V(Q^+[m]) \le 4\pi\epsilon\delta/3 + \pi\epsilon^2$.

Plugging all the inequalities into Eq. (3) and using $\epsilon/\delta \le 1/6$ yields

$$\begin{aligned}
X_c\big(P^*[m]\big) &= \frac{X_c(Q^+[m])V(Q^+[m])}{V(P_I) + V(Q^+[m])} \\
&\le \frac{(\frac{2}{3}\delta + \epsilon)(\frac{4}{3}\pi\delta\epsilon + \pi\epsilon^2)}{\beta\pi\delta^2} \\
&= \beta^{-1}\epsilon\left(\frac{8}{9} + 2\left(\frac{\epsilon}{\delta}\right) + \left(\frac{\epsilon}{\delta}\right)^2\right) \\
&\le \frac{5}{4}\beta^{-1}\epsilon
\end{aligned}$$

which shows $COM(P_I, P_E) \subseteq B(\frac{5}{4}\beta^{-1}\epsilon)$ if $d = 2$.

Now consider $d = 3$, and again distinguish two cases based on the ratio of ϵ and δ.[2]

If $\epsilon \ge 3\delta/44$ then $X_c(P^*[m]) \le 3\delta/4 + \epsilon \le 12\epsilon$. As $P^*[m]$ is bounded we have that $\beta \le 1/8$ and thus $X_c(P^*[m]) \le 12\epsilon \le 3\beta^{-1}\epsilon/2$.

If $\epsilon \le 3\delta/44$ we again derive a lower bound on $V(P_I)$ and upper and lower bounds on $V(Q^+[m])$, and combine it with the upper bound $X_c(Q^+[m]) \le 3\delta/4 + \epsilon$. The lower bound on $V(P_I)$ follows from the fatness of P_I. If $p_1, p_2 \in P_I$ are δ apart, then the ball $B(p_1, \delta)$ again belongs to the set $U(P_I)$. The β-fatness of P_I now implies that $V(P_I) \ge \beta \cdot V(B(p_1,\delta)) = 4\beta\pi\delta^3$.

We bound $V(Q^+[m])$ using $V(Q^+[m]) \le V(P_I \oplus B(\epsilon)) - V(P_I)$. Lemma 3 says that $V(P_I \oplus B(\epsilon)) - V(P_I) = a\epsilon + 2\pi w\epsilon^2 + \frac{4}{3}\pi\epsilon^3$, where a is the surface area and w is the mean width of P. As P_I is contained in $B(3\delta/4)$ we know that $a \le 4\pi(3\delta/4)^2 = 9\pi\delta^2/4$. Moreover, the mean width w does not exceed the diameter of P_I so $w \le \delta$. Combining these observations with a trivial lower bound on $V(Q^+[m])$ we get $0 \le V(Q^+[m]) \le 9\pi\delta^2\epsilon/4 + 2\pi\delta\epsilon^2 + 4\pi\epsilon^3/3$.

Plugging all the inequalities into Eq. (3) and using $\epsilon/\delta \le 3/44$ yields

$$\begin{aligned}
X_c\big(P^*[m]\big) &= \frac{X_c(Q^+[m])V(Q^+[m])}{V(P_I) + V(Q^+[m])} \\
&\le \frac{(\frac{3}{4}\delta + \epsilon)(\frac{9}{4}\pi\delta^2\epsilon + 2\pi\delta\epsilon^2 + \frac{4}{3}\pi\epsilon^3)}{\frac{4}{3}\beta\pi\delta^3} \\
&= \beta^{-1}\epsilon\left(\frac{81}{64} + \frac{45}{16}\left(\frac{\epsilon}{\delta}\right) + \frac{9}{4}\left(\frac{\epsilon}{\delta}\right)^2 + \left(\frac{\epsilon}{\delta}\right)^3\right) < \frac{3}{2}\beta^{-1}\epsilon
\end{aligned}$$

which shows $COM(P_I, P_E) \subseteq B(\frac{3}{2}\beta^{-1}\epsilon)$ if $d = 3$. □

[2] For reasons of simplicity of the final bound we have chosen the split at $\epsilon = \delta/(44/3)$. The optimal split would be at $\epsilon = \delta/k$ where k equals the single positive real root of the equation $6k^4 - 73k^3 - 180k^2 - 144k - 64 = 0$, which yields

$$k = \sqrt[3]{\frac{977}{8} + \sqrt{654}} + \sqrt[3]{\frac{977}{8} + \sqrt{654}} + \frac{9}{2}.$$

A split at k leads to a marginally better bound on the radius of the disk containing $COM(P_I, P_E)$.

Theorem 5 confirms the intuition that the variation of the center of mass of a convex part grows if the admitted shape variation increases or the fatness decreases.

5. Conclusion

We have considered a very general model for admitted shape variations of a model part, based on enclosed shape P_I and an enclosing shape P_E. We have identified the valid instance that maximizes the displacement of the center of mass in a given direction, and used this result to find a k-facet polytopic outer approximation of the set of all possible center-of-mass loci in $O(kn \log n)$ time, where n is the number of features of P_I and P_E. If P_I is convex and β-fat and every point of P_E lies within a distance ϵ of P_I then the diameter of the set of all center-of-mass loci can be shown to be $O(\beta^{-1}\epsilon)$.

We have presented examples that show that both fatness and convexity are necessary to bound the size of the set of all possible center-of-mass loci. Since there are many definitions of fatness it is worthwhile to see if there are versions that can lead to a similar result without requiring convexity of P_I. It seems crucial to avoid long boundaries. It is also interesting to investigate under which circumstances the results can be extended to parts with non-uniform mass distribution.

References

[1] K. Goldberg, Orienting polygonal parts without sensors, Algorithmica 10 (2) (1993) 201–225.
[2] R.-P. Berretty, M.H. Overmars, A.F. van der Stappen, Orienting polyhedral parts by pushing, Comput. Geom. Theory Appl. 21 (2002) 21–38.
[3] S. Akella, M.T. Mason, Posing polygonal objects in the plane by pushing, in: Proc. IEEE Int. Conf. on Robotics and Automation, 1992, pp. 2255–2262.
[4] M.A. Erdmann, M.T. Mason, An exploration of sensorless manipulation, IEEE J. Robot. Autom. 4 (1988) 367–379.
[5] K.M. Lynch, Locally controllable manipulation by stable pushing, IEEE Trans. Robot. Autom. 15 (1999) 314–323.
[6] R.-P. Berretty, K. Goldberg, M.H. Overmars, A.F. van der Stappen, Computing fence designs for orienting parts, Comput. Geom. Theory Appl. 10 (4) (1998) 249–262.
[7] R.C. Brost, Automatic grasp planning in the presence of uncertainty, Int. J. Robot. Res. 7 (1) (1988) 3–17.
[8] A.A.G. Requicha, Toward a theory of geometric tolerancing, Int. J. Robot. Res. 2 (1983).
[9] H. Voelker, A current perspective on tolerancing and metrology, Manuf. Rev. 6 (4) (1993).
[10] B.R. Donald, Error Detection and Recovery in Robotics, Springer-Verlag, 1987.
[11] J. Chen, K. Goldberg, M.H. Overmars, D. Halperin, K.-F. Böhringer, Y. Zhuang, Shape tolerance in feeding and fixturing, in: Robotics: The Algorithmic Perspective, A.K. Peters, 1998, pp. 297–311.
[12] M. Dogar, S.S. Srinivasa, A framework for push-grasping in clutter, Robot. Sci. Syst. 2 (2011).
[13] S.M. Lavalle, Planning Algorithms, Chapter 12: Planning under sensing uncertainty, Cambridge University Press, 2006.
[14] L. Guibas, D. Salesin, J. Stolfi, Epsilon geometry: building robust algorithms for imprecise computations, in: Proc. of the 5th Annual ACM Symp. on Computational Geometry, 1989, pp. 208–217.
[15] U. Roy, C. Liu, T. Woo, Review of dimensioning and tolerancing, Comput. Aided Des. 23 (7) (1991).
[16] Y. Ostrovsky-Berman, L. Joskowicz, Tolerance envelopes of planar mechanical parts with parametric tolerances, Comput. Aided Des. (2005) 531–534.
[17] M. Löffler, Data imprecision in computational geometry, PhD thesis, Utrecht University, 2009.
[18] M. Bern, D. Eppstein, L.J. Guibas, J.E. Hershberger, S. Suri, J. Wolter, The centroid of points with approximate weights, in: Proc. 3rd Eur. Symp. Algorithms, in: Lecture Notes in Computer Science, vol. 979, 1995, pp. 460–472.
[19] S. Akella, M.T. Mason, Orienting toleranced polygonal parts, Int. J. Robot. Res. 19 (12) (2000) 1147–1170.
[20] Y.-B. Chen, D.J. Ierardi, The complexity of oblivious plans for orienting and distinguishing polygonal parts, Algorithmica 14 (1995) 367–397.
[21] B. Kehoe, D. Berenson, K. Goldberg, Toward cloud-based grasping with uncertainty in shape: estimating lower bounds on achieving force closure with zero-slip push grasps, in: Proc. IEEE Int. Conf. on Robotics and Automation, 2012.
[22] B. Aronov, M. de Berg, Unions of fat convex polytopes have short skeletons, Discrete Comput. Geom. 48 (1) (2012) 53–64.
[23] A.F. van der Stappen, Motion planning amidst fat obstacles, PhD thesis, Utrecht University, 1994.
[24] M.J. Katz, M.H. Overmars, M. Sharir, Efficient hidden surface removal for objects with small union size, Comput. Geom. Theory Appl. 2 (1992) 223–234.
[25] M.H. Overmars, A.F. van der Stappen, Range searching and point location among fat objects, J. Algorithms (1996) 626–656.
[26] S. Lien, J.T. Kajiya, A symbolic method for calculating the integral properties of arbitrary nonconvex polyhedra, IEEE Trans. Comput. Graph. Appl. 4 (1984) 35–41.
[27] M. de Berg, M.J. Katz, A.F. van der Stappen, J. Vleugels, Realistic input models for geometric algorithms, Algorithmica 34 (2002) 81–97.
[28] F. Barthe, The Brunn–Minkowski theorem and related geometric and functional inequalities, in: Proc. International Congress of Mathematicians, vol. II, 2006, pp. 1529–1546.
[29] J.-M. Morvan, Generalized Curvatures, Part VI: The Steiner formula, Springer Series in Geometry and Computing, vol. 2, Springer-Verlag, Berlin, Heidelberg, 2008.
[30] P.C. Hammer, The centroid of a convex body, Proc. Am. Math. Soc. 2 (1951) 522–525.