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Asymptotic series of Poincaré and Stieltjes

At the end of the nineteenth century both Henri Poincaré and the Dutch mathematician Thomas Stieltjes had their first publication on asymptotic series. The question arises whether they had contact with each other at that time on this new subject. Hasse van Boven, Rob Wesselink and Steven Wepster analyse both publications and investigate these simultaneous developments.

In 1886, two seemingly independent papers appeared that introduced the new subject of asymptotic series: one written by Henri Poincaré [1] in Paris appeared in the December issue of *Acta Mathematica*; the other consisted of the dissertation of the young Dutch mathematician Thomas Jan Stieltjes [6], produced under the supervision of Charles Hermite and defended in June, also in Paris.

At first sight, both texts have quite different goals. Stieltjes uses asymptotic series to find practical approximations for various functions and integrals, whereas Poincaré searches for formal, analytic properties of those series. Many modern textbooks on perturbation theory and asymptotic series mention both authors side by side (e.g., [9]). But how is it possible that two mathematicians introduce a new research area at nearly the same time and place? How similar are their ideas, and were they actually conceived independently? First, we analyse both papers; next we investigate possible forms of contact between the authors.

Divergent series

Poincaré discusses in his paper first a concrete example, i.e. Stirling's series development of the gamma function

$$\log\Gamma(x+1) = \tfrac{1}{2}\log(2\pi) + (x+\tfrac{1}{2})\log(x) - x + \frac{B_2}{1\cdot 2\cdot x} + \frac{B_4}{3\cdot 4\cdot x^3} + \frac{B_6}{5\cdot 6\cdot x^5} + \cdots,$$

where the B_k denote Bernoulli numbers. He soon directs his attention to the more general properties of asymptotic series, which he introduces as follows. Let $m_0 + \frac{m_1}{x} + \cdots$ be a not necessarily convergent series and denote by $S_n = S_n(x)$ the partial sum of the first $n+1$ terms. In Poincaré's terminology, such a series represents a function $F(x)$ asymptotically if

$$\lim_{x\to\infty} x^n(F(x) - S_n(x)) = 0 \quad \text{for fixed } n$$

and

$$\lim_{n\to\infty} x^n(F(x) - S_n(x)) = \infty \quad \text{for fixed } x.$$

The second condition excludes convergent series from being asymptotic representations.

In contrast, Stieltjes sets out first to discuss various difficulties encountered when dealing with divergent series; then he turns to the function $F(x) = m_0 + \frac{m_1}{x} + \frac{m_2}{x^2} + \cdots$ (where the series need not be convergent) subject to the condition that

$$\lim_{x\to\infty}\left(F(x) - \sum_{i=0}^{n-1}\frac{m_i}{x_i}\right)x^n = m_0.$$

He calls the series *semi-convergent* when the condition is fulfilled.

Poincaré's asymptotic representations and Stieltjes' semi-convergent series cover identical concepts which we now address as asymptotic series. Before considering the question whether they had a common source of inspiration, let us first see how these authors used these concepts in rather different ways.

Stieltjes' method

Stieltjes illustrates his method of approximating functions with a well-known example. He considers the logarithmic integral

Photo: Gauthier-Villars [8]

Thomas Stieltjes

$$\mathrm{li}(a) = \int_0^a \frac{du}{\log u} = \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon} \frac{du}{\log u} + \int_{1+\epsilon}^a \frac{du}{\log u}\right)$$

for $a > 1$. Using the substitution $u = e^{a(1-v)}$ and the first n terms of the geometric series expansion for $\frac{1}{1-v}$ he gets

$$\begin{aligned}\mathrm{li}(e^a) &= e^a \lim_{\epsilon\to 0}\left(\int_0^{1-\epsilon} \frac{e^{-av}}{1-v}\,dv + \int_{1+\epsilon}^{\infty} \frac{e^{-av}}{1-v}\,dv\right)\\ &= e^a\left(\sum_{k=1}^{n} \frac{(k-1)!}{a^k} + R_n\right)\end{aligned}$$

where (putting $a = n+\eta$, with $0 \le \eta < 1$) the remainder term is

$$R_n = \lim_{\epsilon\to 0}\int_0^{1-\epsilon} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv + \int_{1+\epsilon}^{\infty} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv. \qquad (1)$$

Since he is dealing with divergent series, the problem is to find an optimal number of terms that gives the best possible approximation, i.e., the number n for which $|R_n|$ is minimal. Stieltjes evaluates (1) by putting $ve^{-v} = e^{-1-x^2}$ and $1 - v = t = a_1x + a_2x^2 + a_3x^3 + \cdots$; by way of differentiation and recurrent relations he finds the coefficients $a_1 = \sqrt{2}, a_2 = -\frac{2}{3}, a_3 = \frac{\sqrt{2}}{18}, \ldots$ Consequently,

$$-\frac{dv}{1-v} = \left(1 - \frac{\sqrt{2}}{3}x - \frac{1}{9}x^2 + \frac{\sqrt{2}}{270}x^3 + \cdots\right)\frac{dx}{x}$$

and also

$$e^{-\eta v} = e^{-\eta}e^{\eta t} = e^{-\eta}\left(1 + \eta t + \frac{\eta^2 t^2}{2} + \cdots\right).$$

Combining these, he gets

$$\begin{aligned}&\int_{1-h}^{1-\epsilon} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv\\ &\quad = e^{-a}\int_{\epsilon\sqrt{\frac{1}{2}}}^{L} e^{-nx^2}\left(1 + A_1x + A_2x^2 + A_3x^3 + \cdots\right)\frac{dx}{x},\end{aligned}$$

where the A_j represent polynomials in η, L depends on h, and h is chosen so that the series in brackets is convergent on the integration interval. Attacking the second integral in (1) in the same way, Stieltjes finds

$$\begin{aligned}&\int_{1+\epsilon}^{1+k} \frac{(ve^{-v})^n}{1-v} e^{-\eta v}\,dv\\ &\quad = -e^{-a}\int_{\epsilon\sqrt{\frac{1}{2}}}^{L} e^{-nx^2}\left(1 - A_1x + A_2x^2 - A_3x^3 + \cdots\right)\frac{dx}{x},\end{aligned}$$

where k is chosen so that the integral extends to the same L as before. Hence the constant term between brackets disappears in the sum of the two integrands; the division by x can be carried through and the limit $\epsilon \to 0$ can be taken; and finally the remaining parts $\int_L^\infty$ are argued to converge to 0 when $n \to \infty$. Termwise integration and working out of the polynomials A_j then gives him the remainder term as

$$\begin{aligned}R_n = e^{-a}\sqrt{\frac{2\pi}{n}}&\left(\eta - \frac{1}{3} + \left(\frac{\eta^3}{6} - \frac{\eta^2}{2} + \frac{\eta}{12} + \frac{1}{540}\right)\frac{1}{n}\right.\\ &\left. + (\cdots)\frac{1}{n^2} + \cdots\right).\end{aligned}$$

Now, using $a = n+\eta$ and expressing η as a power series with unknown coefficients $\sum \frac{\beta_k}{n^k}$, he also reduces this expression for R_n to a power series in $\frac{1}{n}$; equating the coefficients of this series to zero and solving for the β_i leads finally to the identity

$$n = a - \frac{1}{3} - \frac{8}{405a} + \frac{16}{25515a^2} - \cdots \qquad (2)$$

for the supposedly optimal number of terms n in the divergent series approximation

$$\mathrm{li}(e^a) \approx e^a\left(\sum_{k=1}^{\lfloor n\rfloor} \frac{(k-1)!}{a^k} + \frac{\lfloor n\rfloor!}{a^{\lfloor n\rfloor+1}}(n - \lfloor n\rfloor)\right),$$

where $\lfloor n\rfloor$ denotes the largest integer not bigger than n; the last term is derived from a heuristic reasoning about R_n. In practice it is sufficient to compute (2) up to the quadratic term. Stieltjes adds that the 'order of approximation' is $\sqrt{2\pi/a}$, however, this is likely to be a very large estimate of the error (as he observes himself) and his concept of approximation order is not well defined. Moreover, Stieltjes shows a numerical example to demonstrate that his asymptotic series approximates the logarithmic integral much more rapidly than the classical result $\mathrm{li}(e^a) = \gamma + \log a + \sum_1^\infty \frac{a^n}{n\,n!}$.

Products of series

About a year after he had obtained his doctorate, Stieltjes proved that the product of asymptotic series is again asymptotic [7]. His proof is nearly identical to that which Poincaré had given in [1], except for a geometric consideration about the boundary terms. Poincaré's proof runs as follows. Let two asymptotic series be given:

$$J(x) = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \cdots,$$
$$J'(x) = A'_0 + \frac{A'_1}{x} + \frac{A'_2}{x^2} + \cdots + \frac{A'_n}{x^n} + \cdots,$$

and let S_n, S'_n denote their partial sums. Denote the product of the series by

$$\Sigma = B_0 + \frac{B_1}{x} + \frac{B_2}{x^2} + \cdots + \frac{B_n}{x^n} + \cdots,$$

with partial sums Σ_n. Since S_n, S'_n and Σ_n are polynomials of degree n and $2n$ in $\frac{1}{x}$, we see that

$$\lim_{x\to\infty} x^n \left(S_n S'_n - \Sigma_n\right) = 0,$$

also

$$\lim_{x\to\infty} \frac{J}{S_n} = \lim_{x\to\infty} \frac{J'}{S'_n} = 1.$$

Finally, by definition we have

$$\lim_{x\to\infty} x^n \left(J - S_n\right) = \lim_{x\to\infty} x^n \left(J' - S'_n\right) = 0.$$

Writing $J = S_n + \frac{\epsilon}{x^n}$ and $J' = S'_n + \frac{\epsilon'}{x^n}$, we get

$$JJ' = S_n S'_n + \frac{S'_n\epsilon + S_n\epsilon' + \frac{\epsilon\epsilon'}{x^n}}{x^n}.$$

When $x \to \infty$, we have $S_n \to A_0$ and also $\epsilon \to 0$ and likewise for the primed magnitudes, hence

$$\lim_{x\to\infty} x^n \left(JJ' - S_n S'_n\right) = 0$$

and

$$\lim_{x\to\infty} x^n \left(JJ' - \Sigma_n\right) = 0,$$

i.e., JJ' is an asymptotic series.

This multiplicative property and its proof fit well within Poincaré's interest in their analytic properties. Stieltjes did not need such properties initially, when he limited himself primarily to concrete examples; it was only years later that he needed them. We remark that at that time he provided the proofs himself instead of referring to Poincaré's paper. Was he unaware of Poincaré's work?

Contact

We have no evidence that Poincaré and Stieltjes had been in contact prior to the publication of their initial work in asymptotic series. There is no known correspondence between them although we have many letters written to or by each individual to several other colleagues. It is not unthinkable that they met in person in Paris as they both resided there between mid 1885 and the end of the next year. However, such a meeting is likely to have left traces in letters, and we know nothing of the kind. Also, Stieltjes' own proof of the multiplicative property of asymptotic series may be an indication that he did not know of Poincaré's work: had he known it, he could simply have referred to Poincaré's proof of the same result. But on the other hand we have the similarity between their definitions of asymptotic series which indicates that there might have been at least indirect contact.

It is hard to imagine that they had not met. To wit, Poincaré had been a student of Stieltjes' supervisor Hermite at the École Polytechnique, and Stieltjes had a clear interest in Poincaré's work. On 19 March 1886 he wrote to his supervisor:

"Comme seconde Thèse, je voudrais bien exposer la démonstration due à M. Poincaré de la possibilité d'une figure annulaire d'une masse fluide en rotation. C'est un sujet qui m'intéresse beaucoup et j'ai encore quelques mois de temps, certainement, avant que ma Thèse ne soit imprimée." [8, Chapter I, p. 190]

Stieltjes already expressed his interest in Poincaré's work when he was still in Leiden, when the name of Poincaré was mentioned several times in the correspondence between him and Hermite. So Stieltjes certainly had Poincaré in view. Only years later did a reverse interest take shape.

Photo: Gauthier-Villars [8]

Charles Hermite, supervisor of Thomas Stieltjes

This asymmetry in their relationship may go some way to explain why (if so) they had not met in person. In fact, Stieltjes moved to Paris in the middle of 1885. At that time he did not yet have a reputation among mathematicians, although he had published a few papers and was already a member of the Royal Dutch Academy of Sciences (KNAW). Only when in Paris did his reputation rise. Meanwhile, Poincaré was an established mathematician (though still only 31 years of age) who had many contacts with colleagues, and the academically younger Stieltjes may have missed his attention.

External infuence

As already mentioned, Hermite might have provided the link between Stieltjes and Poincaré. He was one of the most important mathematicians then and counted many students, including Mittag-Leffler. After Stieltjes obtained his doctorate from Hermite and Darboux it was Hermite who helped him to acquire a position in Toulouse. Hermite and Stieltjes remained in friendly contact and they maintained an extensive correspondence in which more than 400 letters were exchanged.

Poincaré too kept in contact with his former teacher Hermite. Therefore Hermite was able to inform Poincaré that Stieltjes worked on the generalisation of certain ideas of Lagrange and Cauchy in complex function theory. Stieltjes abstained from publishing his results because he was unable to complete the proofs; Poincaré was more successful and published his results in [2], adding that he had elaborated on the initial ideas of Stieltjes.

This proves that Hermite *did* function as an intermediary between the two, at least sometimes. Since Poincaré's paper was published near the end of 1887, Hermite and Poincaré must have discussed this matter in that summer at the latest. Probably it was not the first time that Hermite passed on ideas to Poincaré.

Another link between Stieltjes and Poincaré might have been the editor of *Acta Mathematica*, Gösta Mittag-Leffler in Stockholm. Mittag-Leffler and Poincaré have maintained an extensive correspondence of which some 258 letters have been preserved [4]. In April of 1886, Poincaré wrote him:

"Je vous adresse aujourd'hui le mémoire que vous m'aviez demandé au sujet des intégrales irrégulières des équations linéaires et de leur représentation approximative par des séries divergentes analogues à celles de Stirling." [5]

His attached article discussed the properties of asymptotic series. He added that he had been unable to work on it for two months because of his health. This implies that he had already been working on the topic for a while.

Stieltjes also had contact with Mittag-Leffler, for he had published five papers in the *Acta* of which two dated from his Leiden years and three from France. They also exchanged four letters discussing Riemann's zeta-function (these have been published as an appendix to [8]). Clearly, Mittag-Leffler had less contact with the Dutchman than with Poincaré. Although the topic of asymptotic series is discussed among Mittag-Leffler and Poincaré, the former never mentioned Stieltjes' work on asymptotic series.

Conclusion

There is a notable similarity in Stieltjes' and Poincaré's work on asymptotic series, but we have found no proof of direct contact between the two. This is remarkable because they shared academic ancestry and friendship in the person of Charles Hermite, who had often advised both men. Hermite must have been aware of his students's projects yet it seems that he did not always advise them of common factors. Sometimes he did function as an intermediary as in the case of Stieltjes getting stuck on the residues of double integrals.

Years later, Poincaré reviewed an article of Stieltjes on continued fractions and (what we now know as) the Stieltjes integral (see [3]). Poincaré praised Stieltjes: "Le travail de Stieltjes est donc un des plus remarquables Mémoires d'Analyse, qui aient été ecrits dans ces dernieres années." It is impossible to tell whether he had formed this opinion recently or already around 1886.

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