Intuitionistic Rules
Admissible Rules of Intermediate Logics

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By Johan Goudsmit, around 1980

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Intuitionistic Rules
Admissible Rules of Intermediate Logics

Intuïtionistische Regels
Toelaatbare Regels van Intermediaire Logica's
(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de rector magnificus, prof.dr. G.J. van der Zwaan, ingevolge het besluit van het college voor promoties in het openbaar te verdedigen op vrijdag 29 mei 2015 des middags te 2.30 uur door

Jeroen Paul Goudsmit
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Contents

1 Introduction

When one is given a logic, there are two fundamentally distinct ways in which one can proceed. First, one might study its inner workings. Matters to be studied in this regard might be the efficiency with which one can generate theorems, or the structure of transformations between this logic's proofs. Second, one could study the body of theorems this logic yields, all the while abstracting away from its intricate inner workings. One could be concerned with describing the theorems of this logic, for instance by means of an algorithm or some sort of models. Moreover, one could consider the largest possible collection of rules which would give rise to this body of theorems. These rules are known as *admissible rules*. Admissible rules are the central object of study in this thesis.

The admissible rules of a logic are those rules under which the set of this logic's theorems is closed. The totality of all admissible rules can be seen as a meta-property of the logic, defined only in terms of the theorems present, irrespective of the rules that happen to be chosen. This in sharp contrast to *derivable rules*, those rules where the conclusion can be shown to follow from the premises using only the specific rules of inference available. Derived rules are strongly bound to an axiomatisation, present by convention, whereas admissible rules are true invariants.

Derived rules are chosen, but this does not make them any less important. To the contrary; rules matter. Let us spend a few words on this. First, when one aims to describe a certain body of theorems by means of axioms and rules, there is a choice. In general, there is no straightforward or unique way in which one must proceed, as explicated by Porte (1981, p. 409):

> "[...] it is not apparent at once which rules are or are not 'natural'. Thus, the literature of the subject contains, for the same sets of theses, several axiomatizations, the [derivable rules] of which are different."

Second, this choice of rules may influence the properties of the thus constructed logic, as already argued by Harrop (1965, p. 227):

> "[...] reasonable care must be taken before properties proved to hold for one formulation of a calculus are used in connection with some other formulation of the 'same' calculus."

Indeed, a sizeable portion of logic is devoted to constructing convenient calculi to a certain body of theorems. Proof Theory is concerned with such matters, see Buss (1998), Prawitz (1971), and Troelstra and Schwichtenberg (1996). In this thesis, we abstract away from the underlying logic, and only take notice of the admissible rules it yields.

We now return our focus to admissible rules, in particular, to the development of such rules as an object of study. The currently prevalent nomenclature around admissibility derives from Lorenzen (1955, p. 19), who wrote:

> "Wir fragen jetzt nach Regeln, deren Hinzufügung ebenfalls die Klasse der ableitbaren Aussagen nicht echt erweitert. Elche solche Regel wollen wir zulässig nennen."

That is to say, a rule is "zulässig" whenever adding this rule to the logic does not enlarge the set of this logic's theorems. In his review of Lorenzen's book, Craig (1957a) translated "zulässig" into "admissible".[1] Moreover, he tacitly switched to the definition of admissibility as we employed above.[2] Some early works mention the term "permissible rule". This is no different than an admissible rule, and the use

[1] Interestingly, a review by Skolem (1957) of the same book makes no mention of admissibility whatsoever.
[2] The difference is quite subtle, but pointed out, for instance, by Schütte (1960, p. 40):

> "Die Schlußregel ist also genau dann zulässig, wenn sich in jedem Einzelfall der Formelschemata [...] aus der Herleit barkeit der Prämissen auf die Herleitbarkeit der entsprechenden Konklusion schließen laßt."

of this term appears to originate from Pogorzelski (1968), where “dopuszczalna” — Polish for admissible — is translated as “permissible”.

Logics abound, and although one could study admissible rules in their full generality, the scope of this thesis is more modest. We are concerned with the admissible rules of the *intuitionistic propositional calculus* (IPC) and its consistent axiomatic extensions, the so-called *intermediate logics*. In this, we restrict attention to the propositional, ruling out the study of first-order logics.[3] However limited this scope may seem, the structure of admissible rules in intermediate logics is, actually, already quite complex. We also exclude modal logics, although we do often point to them, for they relate intimately to intermediate logics.[4]

Although intuitionistic ideas go back quite a while, the core principles of what is now commonly known as *intuitionism* were first formulated by Luitzen Egbertus Jan Brouwer (1907).[5] These principles have since been developed into a formal logic by Kolmogorov (1925), Glivenko (1929) andHeyting (1930), now known as the intuitionistic propositional calculus. Extensions of this system were already considered by Gödel (1932), but the first systematic study of the totality of consistent extensions of IPC, the intermediate logics, was performed by Umezawa (1959a).

When thinking about admissible rules, there are two questions that immediately come to mind:

(i) Can the admissible rules of a logic be described?

(ii) Can the admissible rules describe the logic?

Naturally, these questions need much refinement before answering them becomes a worthwhile endeavour. In this thesis, all the matters we study concern such refinements. We spend Sections 1.1 and 1.2 on indicating several, more concrete, instances of the above two questions respectively. We give some of their background, and we indicate where each of them are covered in this thesis.

The final section of this introduction, Section 1.3, gives a brief overview of the contents of this thesis, with direct pointers to our results. An even more succinct list of results can be found in Table 1.1. At the end of this thesis, in Chapter 8, we look to the future, and provide some perspectives on the study of admissibility.

We refer to Iemhoff (2013) for more details on this.

[3] Admissible rules have been studied in the context of first-order logics, see for instance Rybakov (1999b) and Visser (1999, 2002).

[4] This connection is made explicit through the Gödel–McKinsey–Tarski translation, see McKinsey and Tarski (1948, Theorem 5.1) and Rybakov (1997, Section 2.7) for more details.

[5] There are numerous works available on the life and work of Brouwer, we mention but van Dalen (2013b).

1.1. Describing the admissible rules of a logic

Can the admissible rules of a logic be described? Yes they can, but the degree to which a description is satisfying may vary quite a bit. We enumerate four refinements of this question below, and go over them in order.

(i) In which logics are the admissible rules equal to the derivable rules?

(ii) In which logics is the set of admissible rules decidable?

(iii) For which rules can one characterise admissibility in a semantic manner?

(iv) For which logics can one give a nicely described set of rules, from which all other admissible rules follow?

Naturally, these questions are not incomparable. For instance, the answer to question (i) certainly sheds some light on question (iv). Depending on what you consider nice, an answer to question (ii) might constitute an answer to question (iv) as well. Moreover, the techniques involved in answering the one question may be quite helpful in answering another. We make an effort to point out such connections.

Before we proceed, we spend some words on expectation management. The above questions are quite grand, and it would be misleading to suggest this thesis answers all of them. In fact, we answer none. We see these questions as dividing the field, and cast our results in this light.

In the remainder of this section, we give a brief overview of the literature on admissibility, arranged according to the above questions. We list several problems and indicate answers given in the literature or this thesis. Three of this thesis' chapters are directly related to the latter three questions above, to wit: (ii) is treated in Chapter 4, (iii) is covered by Chapter 3, and (iv) is considered in Chapter 5.

1.1.1. Structural completeness

The *derivable rules* and the *admissible* rules of a logic, although the latter always contain the former, need not be equal in general. Intuitively, one can feel the difference when thinking of extensions of the logic at hand. Whatever one may add to the logic, derivable rules remain derivable. The same cannot be said for admissible rules. This difference was already noted by Kleene (1952, p. 94) and, more explicitly, by Moh (1957).[6]

[6] See also the review of Moh (1957) by Wang (1960), and his subsequent paper Wang (1965). The latter, in turn, was reviewed by Church (1975).

One may wonder whether the admissible rules and the derivable rules could be the same. Indeed, this may certainly happen, and the logics where this is the case are said to be *structurally complete*. Pogorzelski (1971) was the first to study this concept, and coined its name.[7] Quite early on, it was shown that the admissible rules of the *classical propositional calculus* (CPC) are all derivable, as discussed by Belnap and Thomason (1963).[8] Their method of proof is quite syntactic; in proving structural completeness, they really reason about proofs directly.

Mints (1972) used a similarly syntactic method when he proved that the fragments of IPC which do not include at least one of the connectives $\to$ and $\vee$ are all structurally complete. A quite extreme example of a syntactic approach can be found in the work of Dekkers (1995). He proved a statement about type-inhabitation in the simply-typed λ-calculus, which, through the Curry–Howard correspondence, can be construed as a proof of structural completeness for the $[\to]$-fragment of IPC.[9]

A more semantic approach was taken by Wroński (1986, Corollary 2), who proved the structural completeness of all intermediate logics in the fragments $[\to]$, $[\wedge, \to]$, $[\wedge, \to, \bot]$. See Latocha (1983) for a brief overview of results similar to the above. An algebraic characterisation of the structurally complete logics was given by Prucnal and Wroński (1974). Pogorzelski and Prucnal (1974) showed that the structural completeness of CPC is equivalent to the Stone (1936) representation theorem of Boolean algebras. Several intermediate logics are structurally complete, most notably Medvedev's logic of finite problems, as proven by Prucnal (1976).[10] Another example is the intermediate logic LC of Dummett (1959), as proven by Dzik and Wroński (1973).[11] A characterisation of structurally complete intermediate logics has been given by Citkin (1978), and Rybakov (1995b) later generalised this approach to modal logics.

To re-iterate: in a structurally complete logic, the admissible rules and the derivable

[7] This definition has two subtly differing interpretations, depending on whether one thinks of a rule as being either finitary or potentially infinitary in nature. For a discussion on this distinction we refer to Makinson (1976). In this thesis, we consider rules to be finitary. Structural completeness in the infinitary sense has been treated by Tokarz (1973).

[8] See also Belnap, Leblanc, et al. (1963), and note they use the term "admissible rule" with an explicit attribution to Lorenzen (1955).

[9] For details on the Curry–Howard correspondence, see Sørensen and Urzyczyn (2006). Dekkers (1995) arose out of a question posed by Arnon Avron and Furio Honsell at the "Workshop on the Typed Lambda Calculus", Turin, Italy, 15—20 May 1988, as confirmed through private communication with Avron and Dekkers. The question was posed purely in the language of λ-calculus, and no connection with the established literature on structural completeness was made.

[10] See Wojtylak (2004) for a modern exposition of this proof, and Skvortsov (1999) for a different approach. Medvedev's logic ML is treated in Section 2.4, and is revisited in Sections 7.2 and 7.4.2.

[11] Note that von Plato (2003) argues that LC was already studied by Skolem in 1913.

rule coincide. It is clear that this gives a full description of the admissible rules in the logic at hand. In this thesis, we are not concerned with such logics. We do, however, use the structurally complete logic LC as a running example to illustrate our techniques.

1.1.2. Decidability

When given a class of mathematical objects, one may wonder whether it can be algorithmically described. This applies to the theorems of a logic, as well as it does to the set of a logic's admissible rules. It is thus natural to ask: "In which logics is the set of admissible rules decidable?" Before looking intermediate logics in general, let us first focus in the weakest of them all: IPC itself.

Since Gentzen (1935), it has been known that the set of theorems of IPC is decidable. Later, much machinery was developed to settle the matter of decidability for other intermediate logics, see for instance Harrop (1958, Lemma 4.1) and McKay (1968). In Chapter 4, we give an answer to the following problem.

1 **Problem (Friedman, 1975, Problem 40)**
Is the set of admissible rules of IPC decidable?

A partial, positive answer to the above was already announced by Citkin (1979b). It was not until Rybakov (1984a) that Problem 1 was answered in the affirmative. In Chapter 4, we prove that the set of admissible rules of IPC is decidable, while roughly following the idea as given in Rybakov (1984a). Our treatment differs in that we connect the argument to the many semantic notions that have since arisen, pointing out the relation to the concepts described in Chapter 3.

A wholly different approach to the same problem was taken by Ghilardi (1999). He employed insights obtained from a sheaf representation of finitely presented Heyting algebras, as described by Ghilardi and Zawadowski (1995, 2002). His proof combines the thus obtained knowledge of the models of IPC with a novel perspective on unification, as presented in Ghilardi (1997). We make liberal use of these insights, and treat them in Chapters 5 and 6. Moreover, in this introduction, this approach is mentioned again in Section 1.1.4 and 1.2.2.

One can pose a question analogous to Problem 1 for any intermediate logic, under the proviso that some sensibly chosen constraints are satisfied. We spend the remainder of this subsection on such questions, varying the involved constraints roughly in decreasing order of restrictiveness. That is, we start with very strong logics, and move to weaker ones.

We first treat the case of the so-called *tabular logics*, those logics that are specified by a single finite frame.[12] Examples of such logics include CPC and GSc, which are specified by the singleton frame and the two-fork respectively. McKay (1967b) and Hosoi (1967b, Theorem 3.8) showed that such a logic is always finitely axiomatizable, see also Bezhanishvili (2008, Theorem 3.24) for a modern treatment of this observation. General theory thus ensures the decidability of the set of theorems of any such logic.

From the perspective of Universal Algebra, an intermediate logic corresponds to a certain variety of Heyting algebras. Within said variety, one can consider the sub-quasi-variety generated by the free algebra on countably infinitely many generators. The admissible rules correspond to the quasi-equations that hold there. Another way to look at this, is to see that the admissible rules are encompassed by the universal theory of the free algebra on countably infinitely many generators. This theory is contained within the elementary theory of the same algebra, which was proven to be undecidable by Rybakov (1985b) and Idziak (1989) independently.[13]

More concretely, a tabular logic corresponds to a variety generated by a single finite algebra. It is known since Birkhoff (1935, Corollary 2) that the finitely generated free algebras of such a variety are finite. Rybakov (1997, Lemma 4.1.10) proved that the free algebra on countably infinitely many generators yield the same quasi-equations as a particular finitely generated free algebra. This observation was already employed by Rybakov (1981, Lemma 2) to prove that the admissible rules of a tabular logic are decidable.[14] The decision procedure one obtains in this manner is quite inefficient. Practically feasible algorithms are studied by Metcalfe and Röthlisberger (2013) and Röthlisberger (2013).

Pretabular logics are those intermediate logics that are maximal among the non-tabular logics. The earliest known example of such a pretabular logic is Gödel–Dummett logic LC, as proven by Dunn and Meyer (1971, Theorem 7). It was shown by Maksimova (1972) that there exist only two other pretabular logics: BD_2 and $\mathsf{BD}_3 + \mathsf{KC}$.[15] Subsequently, Rybakov (1981) proved that the admissible rules in all of these logics are decidable. Moreover, detailed descriptions of the admissible rules of the former two logics LC and BD_2 are given in Theorems 5.14 and 5.34 respectively.

[12] We formally specify tabular logics in Definition 2.91, and provide several examples in Lemma 2.92.

[13] Rybakov (1996) proved analogous results for the modal case. More precisely, he showed that the free algebras of the varieties corresponding to the model logics K, K4, T, S4 and GL have undecidable first-order theories.

[14] A logical formulation of this observation has been given by Citkin (1977b, Theorem 12).

[15] These logics go by the names $\mathcal{L}_3$ and $\mathcal{L}_2$ in the referenced paper. The logics BD_2, BD_3 and KC are treated in Section 2.4. Note that the results of Ghilardi (2004, p. 108) entail the decidability of KC's admissible rules.

The logics of a *finite slice*, as introduced by Hosoi (1967a, Section 4), include the above examples of BD_2 and $\mathsf{BD}_3 + \mathsf{KC}$.[16] Furthermore, all tabular logics are of the finite slice. In the above, we argued that all of these logics have a decidable set of admissible rules. From this perspective, the following problem is quite reasonable.

2 **Problem (Rybakov, 1984, Appendix A)**
Is the problem of admissibility of inference rules decidable in each finitely axiomatizable intermediate logic of a finite slice?

Maksimova (1975) considered modal logics of the finite slice, in analogy to the logics of the finite slice in the intermediate setting as mentioned above. Rybakov (1984c, Corollary 6) proved that, in this modal setting, the admissible rules of the weakest logic of each finite slice are decidable. Note that this does not solve the problem, neither in the modal nor in the intermediate setting. Indeed, the problem is still mentioned as open in Rybakov (1989, Question 7).

One may wonder whether the mere decidability of an intermediate logic is sufficient to ensure the decidability of its admissible rules. This is precisely the statement of Problem 3 below.

3 **Problem (Rybakov, 1989, Question 1)**
Is admissibility decidable in every decidable intermediate logic or modal logic?

Unfortunately, this problem has been settled in the negative by Chagrov (1992). An exposition of this proof is also given by Rybakov (1997, Section 3.8). Later, Wolter and Zakharyaschev (2008, Theorem 2.10) showed that admissibility is undecidable in the decidable multi-modal logics K_u and $\mathsf{K4}_u$, which arise naturally by augmenting K and $\mathsf{K4}$ with a universal modality.

After one knows that a problem can be decided, it makes good sense to ask how well one can decide it. Complexity theory deals with such matters. It is known since Statman (1979) that the problem of determining whether a formula is a theorem in IPC is PSPACE-complete.[17] This gives a lower bound on the complexity of testing whether a rule is admissible in IPC, as the problem of testing for admissibility encompasses that of testing whether a formula is a theorem. Problems of this nature are not treated in this thesis. This is not to say that such problems are ignored in

[16] Hosoi (1967a, Lemma 4.1) proved that to each intermediate logic Λ there is a unique $n = 1, 2, \ldots, \omega$ such that $\mathsf{G}_n = \Lambda + \mathsf{LC}$, in the understanding that $\mathsf{G}_\omega := \mathsf{LC}$. All logics Λ that satisfy $\mathsf{G}_n = \Lambda + \mathsf{LC}$ are said to be of the n^{th} slice, and this slice is a *finite slice* whenever n is finite. In Hosoi (1967a), the logics G_n and LC are denoted by S_n and Z respectively. We discuss these intermediate logics in Section 2.4.
[17] See Švejdar (2003) for a particularly elegant semantical proof of this theorem.

the literature. To the contrary, we refer to Cintula and Metcalfe (2010) and Jeřábek (2007, 2013) for more details on this topic.

1.1.3. Semantics

It has been known since Gödel (1932) that IPC enjoys the disjunction property: a disjunction is a theorem only if at least one of its disjuncts is a theorem, too. This property is shared by many an intermediate logic. An early example includes the logic KP of Kreisel and Putnam (1957). Eventually, Wroński (1973, 1974) proved there to be continuum many intermediate logics with the disjunction property.[18]

When given an intermediate logic, there are several ways through which one may learn whether it enjoys the disjunction property. One could proceed syntactically, like Gödel (1932) and Harrop (1956).[19] When the intermediate logic at hand is given in a semantic way, for instance by means of a class of Kripke frames satisfying a certain property, then proceeding semantically might be more convenient. Gabbay and de Jongh (1974, Lemma 14) showed that, within such a setting, a logic has the disjunction property whenever the class of models satisfies a certain closure condition. In the setting of Heyting algebras, a similar description has been given by Maksimova (1986, Theorem 1). This latter formulation is amendable to great generalisation, see for instance Galatos et al. (2007, Theorem 5.21).

The disjunction property can be construed as the admissibility of a certain multi-conclusion rule. Such multi-conclusion admissible rules are an important object of study in the field of admissibility, studied explicitly since Jeřábek (2005) after so suggested by Kracht (1999). Seen in this light, the study of the disjunction property is encompassed by the study of admissibility in general.

In Chapter 3, we study the semantics of rules. We show how the validity of certain rules in a particular model can be expressed in terms of simple semantic properties of said model. This description can be applied to the *universal model* of an intermediate logic, giving simple criteria under which a given rule is admissible. These criteria are similar to those of Iemhoff (2005, 2006). Moreover, we discuss the so-called *exact models*, of which the universal model is an example, and show that admissible rules are sound and complete with respect to these models.

Iemhoff (2001b) described the so-called "AR-models", which amount to models of a particular set of rules. She subsequently proved these models to be sound and

[18] We refer to Chagrov and Zakharyaschev (1991) for a survey of the disjunction property up until the early nineties.

[19] See also de Jongh (2009).

complete with respect to the admissible rules of IPC. Qua results, this relates to what we discuss in Section 1.1.4. This approach has been adapted to the modal setting by Jeřábek (2005).

1.1.4. Basis

Out of the many ways in which one can present a logic, among the most satisfying is presenting it by means of a *finite axiomatisation*. A finite axiomatisation consists of a finite collection of axioms and rules from which all the theorems of a logic can be generated. It seems quite reasonable to expect that similar such descriptions exist for the admissible rules of intermediate logics. The first problem on this goes back to Alexander Vladimirovich Kuznetsov.[20]

4 **Problem (Kuznetsov, 1973)**
Is there a finite basis of the admissible rules of IPC?

In the above, a *basis* is a set of rules that generates all rules under consideration. Rybakov (1985a) answered Problem 4 in the negative. In many modal logics, a finite basis is impossible as well, see Rybakov (1985a, 1987a, 1991a) for results on S4, Grz, GL and S.[21]

One may wonder whether a stronger logic allows for a finite basis of axiomatizability. On the modal side, many results point in this direction. We mention Rybakov (1981, Proposition 6) and Rybakov (1984b, Theorem 5), who proved that S5 and S4.3 have a finite basis of admissible rules respectively. From this perspective, the following conjecture seems quite reasonable.

5 **Problem (Rybakov, 1984, Appendix A)**
Does each tabular intermediate or modal logic have a finite basis of admissible rules?

From the algebraic perspective, a basis of admissible rules corresponds to an axiomatization of the quasi-variety generated by the free algebras corresponding to the logic at hand. There exist tabular quasi-varieties that are not finitely axiomatizable, as shown by Dziobiak (1982, Theorem 2.1) and Rybakov (1982, Theorem 2). This does not at all rule out that Problem 5 might be answered in the affirmative.

[20] We refer to Muravitsky (2008) and Citkin (2008) for descriptions of Kuznetsov's life and work, and to Kushner (1994, pp. 175–178) for more anecdotes.

[21] We do not treat these modal logics in this thesis. For a brief description of them, we refer to Chagrov and Zakharyaschev (1997). The latter two logics, GL and S, are known as provability logics, for a survey of the development of such logics we refer to Artemov and Beklemishev (2005).

Rybakov (1995a, Theorem 4), on the other hand, conclusively proves the conjecture to be false in both the modal case and the intuitionistic case.

Recall that question (iv) asked: "For which logics can one give a nicely described set of rules, from which all other admissible rules follow?" In the above, we showed that one is set up for failure when "nice" is taken to mean finite. Indeed, the weakest intermediate logic, IPC itself, does not satisfy this demand, nor is it satisfied by the strongest intermediate logics imaginable: even tabular logics do not necessarily have finite bases. On the other hand, the deck is stacked in one's favour when "nice" means decidable, as already argued in Section 1.1.2.

In the literature, one has often been interested in an *explicit* basis. Contrary to a finite basis or decidable basis, this term does not have a formally defined meaning. As such, the question whether a certain basis is explicit is a matter of taste. There are some natural side conditions; it should, for instance, certainly be algorithmically describable. In Chapter 5, we are concerned with such explicit bases.

The earliest conjecture known on explicit bases is given by Citkin (1979b). He explicitly described an infinite sequence of rules that he conjectured to be a basis of admissibility for IPC. These rules were independently considered by Visser and de Jongh, who made the same conjecture. Again independently, as treated in more detail in Section 1.2.1, these rules occurred in the work of Skura (1989a,b). Eventually, the conjecture was settled by Iemhoff (2001b) and Rozière (1992, 1993) independently. The former called these rules *Visser rules*, which is how they got their currently prevalent name.

On the modal side, Jeřábek (2005) considered modal analogous of these rules, and employed techniques similar to those of Iemhoff (2001b) to describe bases for, among others, K4, S4, and GL. Using an approach vaguely reminiscent of Citkin (1977a), Jeřábek (2009) employed a novel technique to provide bases of admissibility for a large variety of intermediate and modal logics, including all the ones mentioned in this paragraph.

The rules we consider in Chapter 5 are variants of these so-called Visser rules. We show how one can prove such rules to be a basis of admissibility for several quite distinct intermediate logics, including IPC, BB_n, BD_2 and GSc.[22] Chapter 5 heavily relies on the semantic information we obtain in Chapter 3, following the techniques explored by Goudsmit (2013) and Goudsmit and Iemhoff (2014).

Other kinds of explicit bases have been considered in the literature. We do not treat these in this thesis, but for completeness' sake, let us mention them here. Explicit

[22] Descriptions of these logics are given in Section 2.4.

bases for IPC have been described by Rybakov, Terziler, and Rimatskij (2000a,b). On the modal side, an explicit basis for S4 is described by Rybakov (1999a, 2001), and Fedorishin (2007) gave an explicit basis for the modal logic GL. Numerous modal logics have been given explicit basis by Rimatskij (2008, 2009a,b, 2011). Cintula and Metcalfe (2010) described a basis of the admissible rules within the $[\neg, \rightarrow]$-fragment of IPC, both in single-conclusion and multi-conclusion form.

Outside the realm of modal and intermediate logics, admissibility has been studied, too. Although such logics fall outside the scope of this thesis, we spend a few words on one prominent example: the non-classical, many-valued logic Ł, which was originally introduced by Łukasiewicz and Tarski (1930).[23] Jeřábek (2010a,b) constructed an explicit basis of the admissible rules of this logic. The techniques he employed are based on the same ideas as used by Iemhoff (2001b), although, naturally, some generalisations were required. We treat some of the general concepts used in his approach in Chapter 5, as they can also be applied to study the admissible rules of intermediate logics.

At the start of this subsection, we chose to interpret question (iv) as asking for a basis of the admissible rules. One might also provide a proof system to generate all admissible rules. This is the approach taken by Iemhoff (2003), which lead to the proof system of Iemhoff and Metcalfe (2009). This system is largely inspired by the rules treated in Iemhoff (2001b). Yet another approach would be to provide a modal logic to reason about admissible rules. This idea goes back to Citkin (1979a), which he finalised in Citkin (2010).

Let us close this subsection with a few problems. The explicitly described bases we considered above contain some redundancy. Indeed, as is apparent when we describe the Visser rules in Section 3.5.2, the rules in this sequence are increasing in strength, such that a later rule entails all the ones before it. This does not occur in an *independent basis*: a basis where no rule is expressible in terms of the others. It is not straightforward that such a basis can always be constructed. It has been shown by Rybakov (1995a) that such a basis need not exist, not even in the setting of tabular modal logics. With this in mind, the following question is quite the challenge.

6 Problem (Rybakov, 1989, Question 8)
Do the logics S4, Grz and IPC have independent bases of admissible rules?

This question was answered by Jeřábek (2008, Theorem 3.1), who provided an independent basis for many intermediate and modal logics. Rybakov, Kiyatkin, and

[23] See Cignoli (2007) for an historical survey and more details on this logic itself.

Terziler (1999, 2000) gave explicit bases for all *pretabular* modal and intermediate logics, and Rimatskij and Kiyatkin (2013) considered extensions of these logics.

Iemhoff (2005, 2006) studied the Visser rules in detail. She posed the two questions below, all pertaining to the power of such rules. The Visser rules have a semantic interpretation, as is treated in Section 3.5.2. Moreover, the admissibility of these rules naturally stratify along the sequence of intermediate logics of bounded branching, BB_n.[24] It is thus natural to ask whether the rules corresponding to a certain stratum are a basis of admissibility for the corresponding logic of bounded branching.

7 **Problem (Iemhoff, 2006)**
Is the rule V_n a basis for the admissible rules of BB_n?

This problem was addressed by Goudsmit and Iemhoff (2014), and essentially answered in the positive. We cover a slight variation of this problem in Chapter 5, replacing the rule V_n by one of the variants we discuss in Section 3.5.2. These variants are all quite similar, although there are subtle differences. For instance, in Section 4.3.1, we treat a variant that is particularly convenient when one wishes to restrict attention to formulae of a bounded implication degree. The following problem is concerned with a proper difference between two variants of the Visser rules.[25]

8 **Problem (Iemhoff, 2006)**
Do there exist intermediate logics for which the restricted Visser rules are non-derivable admissible rules and the Visser rules are not?

Citkin (2012a, Corollary 2) was the first to answer this problem in the affirmative, proving that the intermediate logic BD_2 satisfied this constraint. Goudsmit (2013) subsequently gave a basis of admissibility of this logic, making use of the admissible rules mentioned by Skura (1992a). We discuss this result in Theorem 5.34.

1.2. Describing logics by their admissible rules

Can the admissible rules describe the logic? Of course they can, as the admissible rules include all the logic's theorems, they characterise logics up to differences in

[24] The logic BB_n is discussed in Section 2.4, and plays quite the central role in this thesis. See Lemma 3.72, Corollaries 3.89 and 6.20, and Theorems 4.7, 4.73, 5.36, 7.33 and 7.42 for some highlights.

[25] We do not go into this problem in much detail. Let us but briefly state that the restricted Visser rules correspond to the scheme D_ω^-, whereas the Visser rules correspond to D_ω. It is an immediate corollary of Theorem 3.77 and Lemma 2.96 that D_ω^- is admissible in BD_2, yet Corollary 3.90 proves that D_ω is not admissible in BD_2 . This yields a negative answer to Problem 8.

axiomatisations. This kind of description is, however, wholly unsatisfactory, as the admissible rules add nothing new to the picture. In this section, we look at the insights admissibility can yield to give more intrinsic descriptions of logics.

As in Section 1.1, we give intentionally vaguely phrased questions. In this thesis, neither of the following two questions is answered in full. However, as before, we use these questions to guide us trough the field, leading to two different ways in which admissibility can help to describe logics. Question (i) is covered in Chapter 7, and question (ii) is treated in Chapter 6.

(i) Which intermediate logics can be described as the greatest intermediate logic that admits a certain set of rules?

(ii) What do the admissible rules of an intermediate logic say about its unification type?

1.2.1. Refutation

Franz Brentano considered the two fundamental logical forms of judgement to be affirmations and denials of existence.[26] Łukasiewicz (1951, Section 27) employed this distinction in his formalisation of logic, and gave a formal system, a so-called *refutation system*, in which one can formally reason about both the "rejection" and "assertion" of a propositional statement. In particular, he described CPC by means of such a refutation system.

Recall the disjunction property for IPC, as originally described by Gödel (1932). In an attempt to provide a *refutation system of* IPC, Łukasiewicz (1952, p. 209) wrote the following:

> "If we add to these general rules a special rule of rejection which is valid according to Gödel in the intuitionistic system:
>
> (g) If α and β are rejected, then $[\alpha \vee \beta]$ must be rejected,
>
> we get, as far as I see, a categorical system in which all the classical theses not accepted by the intuitionists can easily be disproved."

Kreisel and Putnam (1957) observed that this statement amounts to the conjecture that IPC is the maximal intermediate logic with the disjunction property. Recall from Section 1.1.3 that the disjunction property corresponds to the admissibility of

[26] We refer to Simons (2004) for more details on the philosophy of Brentano and its relation to logic.

a certain multi-conclusion rule. The above can thus be construed as a characterisation of an intermediate logic as being the maximal logic in which a certain, explicitly described set of rules is admissible. These are the types of characterisations we consider in Chapter 7.

As has been known since Kreisel and Putnam (1957) introduced the intermediate logic KP, IPC is not the maximal intermediate logic with the disjunction property. Indeed, KP is a proper extension which enjoys the disjunction property, disproving Łukasiewicz' conjecture. Later, a countable infinity of decidable, finitely axiomatizable logics with the disjunction property were given by Gabbay and de Jongh (1974): the so-called Gabbay–de Jongh logics, now also known under the name BB_n.

Refutation systems in themselves have become quite the object of study. Słupecki et al. (1971, 1972) developed a great deal of general theory on such systems. A concrete example includes Słupecki and Bryll (1973), who gave a refutation system describing the modal logic $\mathsf{S5}$. Throughout the years, several fixes have been proposed to Łukasiewicz's conjecture, in terms of such refutation systems.

Scott (1957) gave a *refutation system* which is both sound and complete for IPC. The system he proposed, however, contains a schema of rules. This schema has side-conditions that makes it inherently non-structural, in that it contains rules with invalid substitution instances. Another refutation system was given by Dutkiewicz (1989), based on the semantic tableaux of Evert Willem Beth.[27] This system is not structural either.

A most satisfying solution was given by Skura (1989a,b). He added an infinite series of explicitly described rules to the general rules as described by Łukasiewicz (1951), and thus obtained a refutation system characterising IPC. These rules are structural, and are, in fact, a variant of the Visser rules we mentioned earlier in Section 1.1.4. This thus gives a description of IPC as the maximal intermediate logic that admits a particular, explicitly described, infinite set of rules. This connection between refutation and admissibility remained unexplored until Goudsmit (2014a).

From the perspective of admissibility, Iemhoff (2001a) gave what amounts to essentially the same characterisation. Much like Skura (1995b, 1999), Iemhoff (2001a, Proposition 5.1) made the connection to a different, syntactic description of IPC given by de Jongh (1968, Chapter IV).[28] This syntactic description uses the so-called

[27] We refer to Fitting (1969b) for details on semantic tableaux, and to Troelstra and van Ulsen (1999) for a historical reconstruction of the invention of these tableaux.
[28] See also de Jongh (1970).

Kleene slash, as described by Kleene (1962) and modified slightly by Aczel (1968).[29]

In Chapter 7, we describe a characterisation of the intermediate logics IPC, BB_n, and ML, similar in nature to the characterisation of Iemhoff (2001a). Moreover, we show how these characterisations give rise to refutation systems similar to those of Skura (1989b). We emphasise how several concepts, central to the study of admissibility, play a role in developing such characterisations. The description of IPC and ML already occur in Skura (1989b, 1992a). A description of BB_n in terms of a non-structural refutation system was given by Skura (2004). Our approach yields a structural refutation system, based on admissible rules, and is thus quite different.

When given a base logic, one can consider all logics in which all admissible rules of the base logic are admissible, too. Naturally, such logics must be extensions of the base logic. The above characterisations can be cast in this light when one takes rule to mean multi-conclusion rules. Indeed, the logics that extend IPC while preserving all admissible multi-conclusion rules are few in number: IPC is the only one. The very same observation holds for the logics BB_n. We prove these observations in Chapter 7.

The same question can be considered for single-conclusion rules. That is to say: which intermediate logics admit all single-conclusion rules that are admissible in IPC? Partial answers to this question are known. Most prominently, Rybakov (1993) described all intermediate logics with the finite model property whose admissible rules extend the admissible rules of IPC. On the modal side, Rybakov (1994b, 1995c) and Rybakov, Gencer, et al. (1999) gave a similar description of modal logics with the finite model property whose admissible rules extend the admissible rules of $\mathsf{S4}$. Gencer (2002) and Rutskii and Fedorishin (2002) did analogous work, replacing $\mathsf{S4}$ by $\mathsf{K4}$.

Let us close this section with some pointers to the literature on refutation systems. Refutation systems can be used to provide decidability results, as argued by Skura (1990a, 1991). In this, they provide a role similar to the finite model property. A good example of this is given by Skura (1994, Section 4), who gave a refutation system for a modal logic lacking the finite model property, introduced by Makinson (1969).

Skura (1992a, Theorem 2.3) and Skura (1990b) show how tabular logics can be endowed with finitely described refutation systems. These results have been generalised considerably by Citkin (2013). Furthermore, Skura (1992a) provides *refutation systems* for the intermediate logics LC, BD_2 and ML. More syntactically presented refutation systems, whose rules are not necessarily structural, were considered by

[29] See also Bezhanishvili (2004) for a more modern proof of this syntactic description, who uses some of the techniques we describe in Sections 3.1 and 3.2.

Skura (1995b, 1996, 2002, 2011), giving more effective refutation systems describing the modal logics GL, $\mathsf{S4}$, and $\mathsf{K4}$. An overview of many results on syntactic refutation systems can be found in the book by Skura (2013).

Finally, the refutation systems for $\mathsf{S4}$, Grz and $\mathsf{S5}$ given by Skura (1992b) contain rules that bear great semblance to the explicit bases given by Jeřábek (2005) of the former two, indicating yet again the interplay between admissibility and refutation. Do note that only the system for Grz presented there is structural. Another refutation system for Grz was given earlier by Goranko (1991).

1.2.2. Unification

Unification theory is concerned with solving equations modulo a certain theory.[30] There are many facets to unification theory. One particular instance is solving equations within a certain variety, in the sense of Universal Algebra. Yet more concretely, one could be concerned with solving equations in the free algebras associated to certain intermediate logics. This is the perspective we take in this subsection, and it is from this angle that we approach admissibility in Chapter 6.

Admissible rules describe the connection between formulae and the substitutions under which they become theorems. To each formula one can associate an equation, such that a substitution that makes the formula into a theorem is exactly the same thing as a unifier of this equation. Seen from this angle, admissible rules describe relations between the unifiers associated to formulae. When one is interested in unifiers, formulae need only be considered modulo admissible rules. We treat unification from this perspective in Chapter 6.

There are numerous ways in which one could describe the set of unifiers associated to a formula. Typically, one orders unifiers via factorisation, and considers a unifier less general than another unifier if the latter factors through the former. A complete set of unifiers is such that each unifier is less general than some element in said set. Redundancy is avoided by considering only minimal such sets. Whenever they exist, such sets specify all there is to know about the unifiers of the fixed formula at hand. One compares logics by their unification type, determined by the size of such minimal complete sets to all the unifiable formulae of the logic.

Questions concerning unification in IPC go back quite a while. We mention the following question, posed by Grigori Efroimovich Mints. When one considers the

[30] We refer to Siekmann (1989) for a historical survey of this subject, and to Baader (1992) for a brief introduction.

variables $a_1, \dots, a_m$ as additional constants added to the language of intermediate logics, in the understanding that substitutions ought to fix these constants, then it becomes apparent that this is a unification problem in the resulting logic. The first statement is concerned with finding a unifier, and the second with finding all of them.

9 Problem (Mints, 1984, Appendix A)
For any propositional formula $\phi(x_1, \dots, x_n, a_1, \dots, a_m)$ find out whether there exist formulas $\psi_1, \dots, \psi_n$ such that $\phi(\psi_1, \dots, \psi_n, a_1, \dots, a_m)$ is a theorem of IPC. Find all such solutions $\psi_1, \dots, \psi_n$.

This problem was solved by Rybakov (1990b, Theorem 36), see also Rybakov (1991b, 1992). Note that his solution shows that the set of unifiers to a formula is decidable, yet it does describe the set of unifiers through a minimal compete set of unifiers. Such descriptions were later given by Rybakov (2011, 2013a) for several modal logics. Odintsov and Rybakov (2013) answered an analogous question for the minimal logic of Johansson (1937), and their approach does provide an explicit description of the unifiers. The techniques employed by these authors are similar to those described in Section 4.4, although we do not illustrate how they can be utilised to settle unification types.

A quite different approach was taken by Ghilardi (1997). He described how one could utilise projective algebras, well-established within the literature on both Universal Algebra and Category Theory, in the study of unification. In Ghilardi (1999), it is shown that these projective algebras can be used to obtain minimal complete sets of unifiers in the intermediate logics IPC and KC.[31] Unifiers for intermediate logics with the finite model property are described by Ghilardi (2004), including BD_n.

In Chapter 6, we deal with the fruits of this approach. In particular, we cover the unification problem for the intermediate logics BB_n, as presented in Goudsmit and Iemhoff (2014). We also treat the unification problem for several strong intermediate logics, such as BD_2 and GSc, even though they already are covered by Ghilardi (2004). Our description follows quite naturally from their treatment in Chapters 3 and 5, which is why we choose to include it.

Many results have been obtained through the method of Ghilardi. The modal analogue of Ghilardi (1999) is presented by Ghilardi (2000), proving the unification type of $\mathsf{K4}$, $\mathsf{S4}$, GL, and Grz to be finitary. The decidability of admissible rules in these logics follow as a matter of course. A similar approach is employed by Dzik and Wojtylak (2012) to describe unification above $\mathsf{S4.3}$, and by Ghilardi and Sacchetti (2004)

[31] The latter goes by the name of DM in said paper, abbreviating "De Morgan Logic".

to describe those modal logics above K4 where all unifiers of the same formula are comparable. Moreover, this approach sheds a new light on old problems. Consider for instance the following, rather technical question.

10 Problem (Citkin, 1984, Appendix A)
Is there a function $f(n)$ such that a rule expressed in n variables is admissible precisely if it holds on the *free Heyting algebra* on $f(n)$ generators?

Recall that a rule is admissible in IPC precisely if it is valid on the free Heyting algebra on countably infinitely many generators. It is not hard to see that the implication from right to left in Problem 10 must hold, but the converse is more challenging. In Chapter 6, we show how one can utilise the approach of Ghilardi to obtain this result.

1.3. Contribution of this thesis

Through the above, we intended to give an overview of the subject of admissibility. Refinements of six different questions have been covered in Sections 1.1.1 to 1.1.4, 1.2.1 and 1.2.2. These sections guided the reader through the literature, all the while indicating which topics are treated by this thesis, and where this treatment could be found. The purpose of this section is to indicate the contributions of this thesis, ordered along the page-axis. Most of all, this overview is meant to be brief. A succinct, schematic survey of the most significant results of this thesis can be found in Table 1.1.

In Chapter 2, we cover the majority of our preliminaries. We introduce intermediate logics and the notion of admissibility, and develop some machinery to work with them. All of the material here is present elsewhere in either the literature or folklore, some more scattered than others. In particular, traces of *concrete models*, *refined models*, and *order-defined models* can be found in many different works; we chose to treat them in a unified manner in Sections 2.2.2, 2.2.3 and 3.1.

In Chapter 3, we treat the semantics of rules in general, and the semantics of *admissible rules* in particular. Much of our arguments use the *universal model*, which we treat in some detail in Section 3.2. We are somewhat ambivalent in this matter; although we do not cover it in full, we do aim to provide sufficient background for the unacquainted reader, providing ample pointers to the literature at large. We discuss *exact models*, which we show to offer sound and complete semantics for the admissible rules of any intermediate logic with the *finite model property* in Theorem 3.38. This is an example of semantics for admissible rules. In Section 3.5, we characterise the models where specific rules are valid. Most importantly, we characterise the rules DP, $\mathsf{DP}_n^{\neg\neg}$, and D_n^- in Theorems 3.54, 3.63 and 3.70 respectively. These results are original and first presented in Goudsmit (2013, 2014a), although such descriptions could be distilled out of work by others, such as Iemhoff (2005, 2006).

The framework we set up so far allows for several fairly convenient proofs. In particular, Theorems 3.77 and 3.79, which state that subframe logics and stable logics admit D_n^-, follow as a matter of course. We use our semantic description of D_n to prove in Theorem 3.88 that the admissible rules of many an intermediate logic do not enjoy the finite model property. This argument appears in Goudsmit (2014c).

In Chapter 4, we show that the admissible rules of IPC are decidable. To this end, we first treat *extendible models* in Section 4.1. These models are characterised in the context of the intermediate logics IPC, LC, BB_n and $\mathsf{BD}_2 + \mathsf{BW}_n$ in Theorems 4.4 and 4.6 to 4.8 respectively. In the case of BB_n, this description can be made effective, which we show in Theorem 4.73. This description is original and the result of joint work, first published as Goudsmit and Iemhoff (2014). We relate extendible models to the exact models treated earlier and to *projective formulae* introduced by Ghilardi (1999) in Theorem 4.24.

Finally, we prove that the admissible rules of IPC are decidable in Section 4.4. This work, albeit an adaptation of that of Rybakov (1984a), is original, and first presented in Goudsmit (2014b). We first generalise exact models to *adequately exact models* in Theorem 4.75, and show them to be sound and complete with respect to the admissible rules of IPC. We subsequently provide a more effective description of such models in Theorem 4.78 as *adequately extendible models*, generalising extendible models in the context of IPC. This leads to Theorem 4.79, proving the desired decidability of IPC's admissible rules.

In Chapter 5, we treat bases of admissible rules for several intermediate logics. The work in this section is based on Goudsmit (2013) and the joint work in Goudsmit and Iemhoff (2014). The results given in those papers are presented in a uniform framework, depending heavily upon the characterisations of extendible models of the previous chapter. We show that the rules Con, $\mathsf{DP}_n^{\neg\neg}$, D_n^- are a bases of $\mathsf{BD}_2 + \mathsf{BW}_n$ and that Con, DP, D_n^- yields a basis of BB_n, respectively in Theorems 5.34 and 5.36, making use of what we call *admissible approximations*. To indicate the ease with which one can derive a bases through a description of extendible models, we give a rather trivial proof in Theorem 5.14 that shows Con to be a basis of LC.

In Chapter 6, we see that the methodology of Chapter 5 can help one to settle the *unification type* of intermediate logics. This work, too, is original and based on Goudsmit

(2013) and the joint work in Goudsmit and Iemhoff (2014). We show that BB_n and $\mathsf{BD}_2 + \mathsf{BW}_n$ have *finitary unification type* in Corollary 6.20 and Theorem 6.22. Moreover, we show that the unifiers of a formula in these logics can be quite readily extracted from said formula's admissible approximation. This technique also applies to LC, which we show to have *unary unification type* in Theorem 6.14.

In Chapter 7, we show how certain intermediate logics can be characterised as the greatest intermediate logic that admits some particular set of rules. The arguments presented here are original, and based on Goudsmit (2014a). Most interestingly, in Theorem 7.33 we show that BB_n is the greatest intermediate logic that admits $\overline{\mathsf{D}}_n$. The key component to this proof is the observation that whenever a formula is not a theorem of BB_n, there is some syntactic witness to this fact. In our terminology: BB_n is *admissibly expressible* in CPC through $\overline{\mathsf{D}}_n$. To illustrate the versatility and intuitiveness of our machinery, we re-prove an observation by Maksimova (1986) in Theorem 7.33, showing that ML is the greatest intermediate logic above KP that admits DP. Refutation systems have been used to give characterisations of intermediate logics. We show that our characterisations naturally lead to refutation systems for both BB_n and ML in Theorems 7.41 and 7.42 respectively.

The contribution of this thesis lies not only in its results, but also in its composition. The author sincerely hopes that the exposition of the several different approaches to the study of admissibility in the framework presented here clarifies their interconnectedness, leading to further unification and generalisation. There appears to be no recent, comprehensive overview of the field of admissibility.[32] Although this thesis is not comprehensive either, a serious attempt is made to point at as many relevant angles as sensible. In particular, reference is made (and due deference paid) to several older works that have gone unnoticed in the recent literature.

[32] Rybakov (1997) treats a vast portion of the field. However, this book appeared almost two decades ago, thus predating the influential works by Ghilardi (1999), Iemhoff (2001b), Jeřábek (2009), and Rybakov (2013b). The two most recents works on this list, admittedly, are also not treated in this thesis, whereas the others most certainly are.

Page	Ref.	Description
93	3.38	Admissible rules are sound and complete w.r.t. exact models.
102	3.54	Semantics of DP.
106	3.63	Semantics of $\mathsf{DP}_n^{\neg\neg}$.
109	3.70	Semantics of D_n^-.
112	3.77	All subframe logics admit D_n^-.
113	3.79	All stable logics admit D_n^-.
117	3.88	Failure of finite model property for admissible rules.
126	4.4	Description of IPC-extendible models.
127	4.7	Description of BB_n-extendible models.
128	4.8	Description of $(\mathsf{BD}_2 + \mathsf{BW}_n)$-extendible models.
134	4.19	Equivalence of exact models and exact formulae.
137	4.24	Description of projective formulae.
167	4.73	Effective description of BB_n-extendible formulae.
169	4.75	Admissible rules sound and complete w.r.t. adequately exact models.
171	4.78	Equivalence of adequately extendible and adequately exact.
172	4.79	Decidability of admissibility for IPC.
188	5.14	Basis of LC is Con.
196	5.34	Basis of $\mathsf{BD}_2 + \mathsf{BW}_n$ is Con, $\mathsf{DP}_n^{\neg\neg}$, D_n^-.
197	5.36	Basis of BB_n is Con, DP, D_n^-.
206	6.14	Unification type of LC is unary.
209	6.20	Unification type of BB_n is finitary.
210	6.22	Unification type of $\mathsf{BD}_2 + \mathsf{BW}_n$ is finitary.
226	7.24	ML is the greatest intermediate logic above KP that admits DP.
232	7.33	BB_n is the greatest intermediate logic that admits $\overline{\mathsf{D}}_n$.
239	7.41	Structural refutation system of ML.
239	7.42	Structural refutation system of BB_n.

Table 1.1.: Overview of this thesis' main results.

2 Logic

Intuitionistic logic was first formalised by Kolmogorov (1925), Glivenko (1929), and, in its present form, by Heyting (1930), based on the principles of *intuitionism* put forth by Luitzen Egbertus Jan Brouwer (1907). More background on the historical developments can be found in the works of van Dalen (1973, 2013a) and Kreisel and Newman (1969).

In this chapter, we briefly describe the basic definitions that are used in the remainder of this thesis. Little originality is claimed here. Indeed, the majority of the contents of this chapter can be found elsewhere in the literature. We chose to include them all here for the sake of both completeness and notational consistency. Some of the topics we treat lack systematic treatment elsewhere, we indicate these topics clearly in the sections to follow.

Section 2.1 covers consequence relations. Using this formalism, we give a definition of both the intuitionistic propositional calculus (IPC) and arbitrary intermediate logics in Section 2.1.1. Subsequently, we treat multi-conclusion consequence relations in Section 2.1.2. Finally, in Section 2.1.3, we introduce admissible rules. The contents of this section are mostly based on the text in Goudsmit and Iemhoff (2014, Section 2) and Goudsmit (2014a, Section 1).

In Section 2.2, we discuss relational semantics for IPC. We introduce the notion of a *cover* in Section 2.2.1. Sections 2.2.2 and 2.2.3 respectively treat *refined* and *concrete* models. These notions are of particular interest in Chapter 3. Much of the contents of this section is taken from Goudsmit (2013, Section 3).

We close this chapter with Section 2.4, in which we describe the intermediate logics of interest in this thesis. First, we describe their semantics in Section 2.4.1, and then we discuss some general properties in Section 2.4.2. The contents of this section are based on the descriptions of the intermediate logics treated in Goudsmit (2013, 2014a,c) and Goudsmit and Iemhoff (2014), scattered throughout these papers.

2.1. Axiomatisation

We study propositional logics, expressed within the same propositional language. In the following, it is often quite important to keep proper track of the variables involved. To this end, we always specify the set of variables under consideration. Such sets of variables are denoted by X, Y, Z, and their elements, typically, are to be denoted by their lower-case variants. The propositional language with variables X is given by the following *Backus–Nauer form*:

$$\mathcal{L}(X) ::= \top \mid \bot \mid X \mid \mathcal{L}(X) \wedge \mathcal{L}(X) \mid \mathcal{L}(X) \vee \mathcal{L}(X) \mid \mathcal{L}(X) \to \mathcal{L}(X).$$

Formulae are elements of $\mathcal{L}(X)$ for some X, and they are denoted by ϕ, ψ, χ , and θ. Sets of formulae are denoted by Γ, Π, Δ, Θ, and, unless stated otherwise, such sets are assumed to be finite. We employ several abbreviations. We write $\neg\phi$ to abbreviate $\phi \to \bot$. Furthermore, we write $\phi \equiv \psi$ to mean $(\phi \to \psi) \wedge (\psi \to \phi)$. Finally, given a finite set of formulae Δ we write $\bigwedge \Delta$ and $\bigvee \Delta$ for the iterated conjunction and disjunction of Δ, in the understanding that $\bigwedge \varnothing = \top$ and $\bigvee \varnothing = \bot$.

In an effort to reduce syntactic burden, we adopt common conventions on the placement of parenthesis. Implication has *right associativity*, in that we write $\phi \to \psi \to \chi$ to mean $\phi \to (\psi \to \psi)$.[1] Furthermore, conjunction and disjunction take precedence over implication, and are of equal precedence. This means that $\phi \wedge \psi \to \chi$ denotes $(\phi \wedge \psi) \to \chi$, and an analogous convention holds for $\vee$.

[1]Both conjunction and disjunction are interpreted as having right associativity as well. In all the logics we consider, however, this does not matter much. Indeed, the formula $\big((\phi \wedge \psi) \wedge \chi\big) \equiv \big(\phi \wedge (\psi \wedge \chi)\big)$ is a theorem in each of these logics.

A logic is given by means of rules and axioms. As they form half of this thesis' title, rules surely merit a proper definition.

2.1 Definition (Rule)
A *rule* is a pair of finite sets of formulae $\langle \Gamma, \Delta \rangle$, denoted Γ/Δ. Such a rule is said to be *single-conclusion* whenever Δ is a singleton set, and it is *multi-conclusion* otherwise.

Consequence relations, originally introduced in the works of Tarski (1931), provide us with a comfortable formalism for dealing with logics.[2] We restrict ourselves to what are commonly known as *finitary* consequence relations, as described, for instance, by Suszko (1961). Such a finitary consequence relation $\vdash$ amounts to a set of single-conclusion rules, where we write $\Gamma \vdash \chi$ to mean that the rule Γ/χ is a member of this set.

We refer to such consequence relations as being single-conclusion, in contrast to the multi-conclusion consequence relations we consider in Definition 2.7. Note that we dropped finitary from the nomenclature, which causes no confusion as all consequence relations we consider are finitary in nature.[3] Let us introduce just a bit more notation. We write Γ, Π to mean $\Gamma \cup \Pi$. Moreover, we often denote a singleton set $\{\chi\}$ simply as χ.

2.2 Definition (Single-Conclusion Consequence Relation)
A set of single-conclusion rules $\vdash$ is said to be a *single-conclusion consequence relation* whenever the following hold for all finite sets of formulae Γ, Π, Δ, and formulae χ and θ:[4]

Reflexivity $\chi \vdash \chi$;

Weakening if $\Gamma \subseteq \Pi$ and $\Gamma \vdash \chi$ then $\Pi \vdash \chi$;

Transitivity if $\Gamma \vdash \chi$ and $\Pi, \chi \vdash \theta$ then $\Gamma, \Pi \vdash \theta$.

Substitutions play an essential role in this thesis. Although the following definition is fairly standard, we give it for completeness' sake.

[2] See Wójcicki (1988) for a systematic treatment of the subject, and Pogorzelski and Surma (1969) a review of Tarski's original works on consequence relations.

[3] The difference between infinitary and finitary rules is not at all trivial. We refer to Makinson (1976) for a treatment of this difference (who uses the word "sequential" instead of "finitary"), and to de Jongh and Visser (1996, Example 2.2) for an amusing example of an infinitary rule. Łoś and Suszko (1958) show that every infinitary consequence relation is a disjoint union of finitary consequence relations.

[4] Many consider a single-conclusion consequence relation to be a relation between finite sets of formulae and formulae. Our formulation in Definition 2.2 is subtly different, in that we consider it to be a relation between finite sets of formulae and singleton sets of formulae. We take this stance merely for more convenience in switching to multi-conclusion consequence relations later on.

2.3 Definition (Substitution)
A *substitution* $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ is a function satisfying the following, for all $\phi, \psi \in \mathcal{L}(X)$.

$$\begin{aligned} \sigma(\phi) &= \phi && \text{for } \phi = \top, \bot \\ \sigma(\phi \oplus \psi) &= \sigma(\phi) \oplus \sigma(\psi) && \text{for } \oplus = \wedge, \vee, \to \end{aligned}$$

A single-conclusion consequence relation is said to be *structural* whenever the following holds, for all finite sets of formulae Γ, all formulae χ, and substitutions σ.

$$\text{if } \Gamma \vdash \chi \text{ then } \sigma(\Gamma) \vdash \sigma(\chi) \tag{2.1}$$

We think of consequence relations as describing derivability, so if $\Gamma \vdash \chi$ we say that χ is *derivable* from Γ. When we say that a formula χ is *derivable*, we mean that $\{\chi\}$ is derivable from $\varnothing$. Such formulae we also call *theorems*.

The rule below is among the most basic rules of logic; all the logics we consider in this thesis contain it. It is known as *modus ponens*, or the *rule of detachment* for conditionals.

$$\{x, x \to y\} / \{y\} \tag{\textit{modus ponens}}$$

In the following, we often omit the braces around the sets of formulae involved in rules, as the syntax allows for but one reading. Let us briefly remark that, although we only consider logics that include this rule, there certainly are logics that do not. For an example and discussion of this, we refer to Hiż (1959) and Wasserman (1974) respectively.

2.1.1. Intermediate logics

We now have sufficient machinery available to give a formulation of the intuitionistic propositional calculus. The definition we employ here can be found in many text books, see for instance Sørensen and Urzyczyn (2006, Chapter 5), Chagrov and Zakharyaschev (1997, Chapter 2), Troelstra and van Dalen (1988), and Troelstra (1973).

2.4 Definition (Intuitionistic Propositional Calculus)
The *intuitionistic propositional calculus* (IPC) is the least structural, single-conclusion consequence relation $\vdash$ containing the rule given in (*modus ponens*), such that the formulae mentioned in Table 2.1 are theorems. We denote this consequence relation as $\vdash_{\mathsf{IPC}}$, and we conflate the name IPC with the set of *theorems* of IPC.

$\phi \to \psi \to \phi$	$(\phi \to \psi \to \chi) \to (\phi \to \psi) \to \phi \to \chi$
$\phi \wedge \psi \to \phi$	$\phi \to \phi \vee \psi$
$\phi \wedge \psi \to \psi$	$\psi \to \phi \vee \psi$
$\phi \to \psi \to \phi \wedge \psi$	$(\phi \to \chi) \to (\psi \to \chi) \to (\phi \vee \psi) \to \chi$
$\psi \to \top$	$\bot \to \phi$

Table 2.1.: The axioms of IPC.

After the introduction of IPC, many extension were considered. Such extensions are commonly known as *intermediate logics* or *superintuitionistic logic*. The former are logics intermediate between IPC and CPC, the latter can be any extension of IPC.

2.5 Definition (Superintuitionistic and Intermediate Logics)
A set of formulae Λ is said to be a *superintuitionistic logic* whenever it contains IPC and it satisfies the following two conditions:

(i) if $\phi \in \Lambda$ and σ is a substitution, then $\sigma(\phi) \in \Lambda$;

(ii) if $\phi, \phi \to \psi \in \Lambda$, then $\psi \in \Lambda$.

When Λ does not contain $\bot$, we say that Λ is an *intermediate logic*.

The two conditions of Definition 2.5 are typically referred to as "closure under substitution" and "closure under *modus ponens*". It is easy to see that IPC itself is an intermediate logic. When one is given a potentially infinite set of formulae Γ, one can always construct the least superintuitionistic logic extending Γ. We denote this logic by IPC + Γ, and say that Γ is its *axiomatisation*. A special case of the above occurs when $\Gamma := \Lambda \cup \{\phi\}$, where Λ is an intermediate logic and ϕ is a formula. We denote this logic by $\Lambda + \phi$, and say that ϕ is a *finite axiomatisation over* Λ of $\Lambda + \phi$.

Recall that, as mentioned in the introduction, one typically has a consequence relation in mind when thinking about a logic. To fulfil this need, we associate a single-conclusion consequence relation $\vdash_\Lambda$ to any intermediate Λ as follows:

$$\Gamma \vdash_\Lambda \chi \text{ if and only if } \bigwedge \Gamma \to \chi \in \Lambda \tag{2.2}$$

It is easy to see that this consequence relation is *structural*, and that it contains the rule (*modus ponens*). In the case of IPC, the above introduces ambiguity in notation. Indeed, one has $\vdash_{\mathsf{IPC}}$ as introduced by Definition 2.4, and $\vdash_{\mathsf{IPC}}$ as defined by (2.2). One can readily prove that these two consequence relations are equal, thus resolving the potential ambiguity.

2.6 Example (Classical Propositional Calculus)
The *classical propositional calculus* (CPC) is defined as:

$$\mathsf{CPC} := \mathsf{IPC} + (x \vee \neg x)\,.$$

It is well-known that CPC is consistent, so it is an intermediate logic.

From Definition 2.5, it is not immediately clear that the name "intermediate logic" is justified. Indeed, we do not show that the thus defined logics are intermediate in any way. It is easy to see that any consistent superintuitionistic logic Λ satisfies:

$$\mathsf{IPC} \subseteq \Lambda \subseteq \mathsf{CPC}, \tag{2.3}$$

and as such, intermediate logics are truly *intermediate* between the logics IPC and CPC. We refer to Chagrov and Zakharyaschev (1997, Theorem 4.1) for more details.

2.1.2. Multi-Conclusion consequence relations

Below we consider a generalisation of Definition 2.2 to multi-conclusion consequence relations.[5] In the literature, one can find many different approaches to multi-conclusion rules, see for instance Scott (1971, 1974) and Shoesmith and Smiley (1978). The applicability of such consequence relations to the study of admissibility was suggested by Kracht (1999) in his review of Rybakov (1997).

This suggestion was taken up by, among others, Jeřábek (2005) and Cintula and Metcalfe (2010). Our Definition 2.7 is quite similar to theirs, except that ours lacks structurality. This property is absent with good reason; although the majority of this thesis is only concerned with structural consequence relations, we do need this additional generality in Chapter 7.

2.7 Definition (Multi-Conclusion Consequence Relation)
A set of rules $\vdash$ is said to be a *multi-conclusion consequence relation* whenever it is closed under the following for all finite sets of formulae $\Gamma, \Pi, \Delta, \Theta$, and formula ψ:

Reflexivity if $\Gamma \cap \Delta \neq \varnothing$ then $\Gamma \vdash \Delta$;

Weakening if $\Pi \subseteq \Gamma$, $\Theta \subseteq \Delta$ and $\Pi \vdash \Theta$ then $\Gamma \vdash \Delta$;

Transitivity if $\Gamma \vdash \psi, \Delta$ and $\Pi, \psi \vdash \Theta$ then $\Gamma, \Pi \vdash \Delta, \Theta$.

[5] Gabbay (1981, p. 6) calls a multi-conclusion consequence relation a Scott consequence relation, and refers to the single-conclusion consequence relations we treated in Section 2.1 as Tarski consequence relations.

Do note that the rules within a multi-conclusion consequence relation need not all be proper multi-conclusion rules. Indeed, the rule ϕ/ϕ occurs in any multi-conclusion consequence relation, and it is most certainly not a proper multi-conclusion rule. In analogy to Definition 2.2, we refer to the above three conditions as *reflexivity*, *weakening* and *transitivity*, too. Whenever $\Gamma \vdash \Delta$ entails $\sigma(\Gamma) \vdash \sigma(\Delta)$ for all substitutions σ, we say that the consequence relation $\vdash$ is *structural*. Note that this is analogous to the definition of structurally for single-conclusion consequence relations.

Given any set of rules $\mathcal{R}$, one can consider the least, structural consequence relation containing these rules, which we denote by $\vdash^{\mathcal{R}}$. When one considers a single-conclusion consequence relation $\vdash$, there are several ways in which one could extend this relation into a multi-conclusion consequence relation. Let us mention but two.

2.8 Definition (Multi-Conclusion Extensions)
Let $\vdash$ be a single-conclusion consequence relation. We define the multi-conclusion consequence relations $\vdash^{\mathsf{min}}$ and $\vdash^{\mathsf{max}}$ by means of the following two equivalences.

$$\Gamma \vdash^{\mathsf{min}} \Delta \text{ iff there is a } \chi \in \Delta \text{ such that } \Gamma \vdash \chi \tag{2.4}$$

$$\begin{aligned}\Gamma \vdash^{\mathsf{max}} \Delta \text{ iff } & \text{for all } \Pi \text{ and } \theta \text{ such that } \Pi \vdash \phi \text{ for all } \phi \in \Gamma \\ & \text{and } \chi \vdash \theta \text{ for all } \chi \in \Delta \\ & \text{it follows that } \Pi \vdash \theta\end{aligned} \tag{2.5}$$

We omit the proof that relations $\vdash^{\mathsf{min}}$ and $\vdash^{\mathsf{max}}$ as introduced in Definition 2.8 in fact are multi-conclusion consequence relations. The following lemma shows that these two extensions are edge cases.

2.9 Lemma (Došen, 1999)
Let $\vdash$ be a single-conclusion consequence relation, and let $\mathcal{R}$ be any multi-conclusion relation satisfying $\Gamma \mathrel{\mathcal{R}} \chi$ iff $\Gamma \vdash \chi$ for all Γ and χ. We now have that:

$$\vdash^{\mathsf{min}} \subseteq \mathcal{R} \subseteq \vdash^{\mathsf{max}}.$$

Proof. The first inclusion is immediate, for if $\Gamma \vdash^{\mathsf{min}} \Delta$ then there is some $\chi \in \Delta$ such that $\Gamma \vdash \chi$. Hence $\Gamma \mathrel{\mathcal{R}} \chi$ holds by assumption, yielding $\Gamma \mathrel{\mathcal{R}} \Delta$ by weakening. To prove the other inclusion, suppose $\Gamma \mathrel{\mathcal{R}} \Delta$ and assume $\Pi \vdash \phi$ holds for all $\phi \in \Gamma$ and $\chi \vdash \theta$ holds for all $\chi \in \Delta$. We want to prove that $\Pi \vdash \theta$, which follows immediately when we can prove $\Pi \mathrel{\mathcal{R}} \theta$. Through repeated use of transitivity and weakening we obtain $\Gamma \mathrel{\mathcal{R}} \theta$. Similar reasoning allows us to conclude $\Pi \mathrel{\mathcal{R}} \theta$, as desired. $\square$

Take care to note that $\vdash^{\mathsf{min}}$ and $\vdash^{\mathsf{max}}$ are structural whenever $\vdash$ is, as follows from spelling out the definitions. Given an intermediate logic Λ and a set of rules $\mathcal{R}$, we can consider the least, structural, multi-conclusion consequence relation extending $\vdash_{\Lambda}^{\mathsf{min}}$ and $\mathcal{R}$. This is denoted by $\vdash_{\Lambda}^{\mathcal{R}}$. In Lemma 5.2, we describe $\vdash_{\Lambda}^{\mathsf{max}}$ as $\vdash_{\Lambda}^{\mathcal{R}}$, where $\mathcal{R}$ consists of the rules DP and Con introduced in Examples 2.12 and 2.13 below.

2.1.3. Admissible rules

We close this section with the introduction of admissible rules. Extensive treatment of admissible rules is deferred until the next chapter, Chapter 3. At this point, we merely give the abstract definition. In the next section, in Example 2.23, we show that there exist non-derivable admissible rules in IPC. Naturally, derivability is to be taken with respect to the consequence relation $\vdash_{\mathsf{IPC}}$ we defined earlier.

2.10 Definition (Admissible Rule)
Let $\vdash$ be a consequence relation, and let $\Gamma, \Delta \subseteq \mathcal{L}(X)$ be finite sets of formulae. A rule Γ/Δ is said to be *admissible* with respect to $\vdash$ whenever the following holds for each substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ and each finite set of variables Y.

$$\text{if} \vdash \sigma(\phi) \text{ holds for all } \phi \in \Gamma\text{, then} \vdash \sigma(\chi) \text{ holds for some } \chi \in \Delta \tag{2.6}$$

The collection of all admissible rules with respect to $\vdash$ is denoted $\mathrel{|\!\sim}$.

The above definition is given with sufficient generality to be applicable in both the single-conclusion and multi-conclusion case. We study both kinds of admissible rules in the following, and it is always be clear from context which kind of rules are under consideration.

Definitions for single-conclusion admissible rules can be found throughout the literature, we mention Kleene (1952, p. 94), Lorenzen (1955, p. 19), Moh (1957), Schütte (1960, p. 40), Wasserman (1974, p. 270), and Rybakov (1997, Section 1.7). The above Definition 2.10 is based on to the multi-conclusion formulation of Jeřábek (2005, p. 413) and Cintula and Metcalfe (2010, Section 2.2).

We make sure that the relation between $\vdash$ and $\mathrel{|\!\sim}$ is clear in every context. For instance, if $\vdash$ is $\vdash_{\mathsf{IPC}}$, then we refer to $\mathrel{|\!\sim}$ as $\mathrel{|\!\sim}_{\mathsf{IPC}}$. Finally, note that $\mathrel{|\!\sim}$ is defined only with respect to the *theorems* of $\vdash$. With this in mind, let us look at some examples

2.11 Example (Harrop's Rule)
The rule below is known as *Harrop's rule* or the *Kreisel–Putnam rule*. It is an example of a single-conclusion rule that is *admissible* in IPC.

$$\neg z \to x \lor y \big/ (\neg z \to x) \lor (\neg z \to y) \tag{H}$$

Its admissibility is proven by syntactic means in Harrop (1960, Theorem 3.1), and Kreisel and Putnam (1957) showed it to not be derivable.[6] In Example 2.23, we show that this rule is admissible by semantic means. Much is known about this rule. Prucnal (1979, Theorem 1) proved it to be admissible in every intermediate logic.[7] Later, Minari and Wroński (1988) showed that a generalised form of this rule is admissible in every intermediate logic, too.

2.12 Example (Disjunction Property)
A logic admits the rule below precisely if it enjoys the *disjunction property*.

$$x \lor y / x, y \tag{DP}$$

It is known since Gödel (1932) that IPC enjoys this property, hence the above rule is certainly *admissible* for IPC.

2.13 Example (Consistency)
Admissibility of the following rule in $\vdash$ merely means that $\vdash \bot$ does not hold, as can be easily verified by spelling out the definitions.

$$\bot / \varnothing \tag{Con}$$

This rule is admissible for vacuous reasons; there simply is not any way to turn its antecedent into a theorem. The rule Con is an example of what Rybakov, Terziler, and Gencer (1999, Definition 3.1) call a *passive rule*.

Typically, one would say that a logic is consistent precisely if there is a non-derivable formula. As $\bot \vdash_{\mathsf{IPC}} \phi$ can be readily seen to hold, this condition, in the case of IPC, is

[6] When one replaces the "/" in this rule by an $\to$, it equals the axiomatising formula of the intermediate logic KP, introduced originally by Kreisel and Putnam (1957). The rule in (H) has been connected to Harrop at least since Mints (1968), and is called Harrop's rule by several authors, e.g. Citkin (1977a, p. 280), Rybakov (1989, p. 124) and de Jongh (2009). On the other hand, references to this rule as the Kreisel–Putnam rule go back to at least Prucnal (1979). We follow the former practice, and call this rule *Harrop's rule*.

[7] In this paper, Prucnal also solved Problem 41 of Friedman (1975). This problem, rephrased in modern language, conjectured that there exists an intermediate logic with the disjunction property for which all single-conclusion admissible rules are derivable. The logic Prucnal used was Medvedev's logic, known as ML. See Wojtylak (2004) for an exposition of this result, and see Grigolia (1995) for an algebraic proof. We revisit ML in Definition 2.90 and Sections 7.2 and 7.4.2.

equivalent to the statement that $\bot$ is not derivable. Consequently, one could interpret the admissibility of the rule above as the *consistency* of IPC.

2.14 Lemma
Let $\vdash$ be a multi-conclusion consequence relation, and let $\mid\!\sim$ be the corresponding set of *admissible rules*. Now, $\mid\!\sim$ is a structural, multi-conclusion consequence relation.

Proof. This follows directly from spelling out the definitions. $\square$

2.2. Relational semantics

Many different types of semantics exist for intuitionistic logic. In this section, we consider the relational semantics as originally introduced by Kripke (1965). The following is by no means meant as a general introduction, for this we refer to Chagrov and Zakharyaschev (1997), Fitting (1969a), Gabbay (1981), Smoryński (1973), and Troelstra and van Dalen (1988).

Before we continue, let us fix some notation and nomenclature. By a *partial order* (*poset* for short) we mean a set P endowed with a binary relation $\leq$, such that this relation is reflexive, transitive and anti-symmetric. We denote such posets by P and Q, and their elements are referred to by p and q. Arbitrary subsets of a poset are denoted by W and S, and their elements by w and s. A function $f : P \to Q$ is said to be a *monotonic map* (or a *map of posets*) whenever $p_1 \leq p_2$ entails $f(p_1) \leq f(p_2)$ for all $p_1, p_2 \in P$.

A subset $W \subseteq P$ is said to be an *anti-chain* whenever:

$$p \leq q \text{ implies } p = q \text{ for all } p, q \in W.$$

Note that the empty set and every singleton set are examples of anti-chains. We do not save a letter to anti-chains, although in most settings, the subsets of posets we consider can without loss of generality be assumed anti-chains.

A subset $W \subseteq P$ such that the inequality $w \leq p$ entails $p \in W$ for all $w \in W$ and $p \in P$ is said to be an *upset*. Similarly, a subset $W \subseteq P$ such that for all $w \in W$ and $p \in P$ the inequality $w \geq p$ entails $p \in W$ is said to be a *downset*. Upsets play an important role, they are be denoted by U and V.[8]

[8]Downsets matter as well, yet they do not merit their own letter.

Given an arbitrary subset $W \subseteq P$, one can consider the *upset generated by* W, the *strict upset generated by* W, and the *downset generated by* W. These are, respectively, defined as follows, in the understanding that $q < p$ means that $q \leq p$ and $p \neq q$.

$$\begin{aligned} \uparrow W &:= \{p \in P \mid \text{there is a } w \in W \text{ such that } w \leq p\} \\ \upharpoonleft W &:= \{p \in P \mid \text{there is a } w \in W \text{ such that } w < p\} \\ \downarrow W &:= \{p \in P \mid \text{there is a } w \in W \text{ such that } w \geq p\} \end{aligned}$$

If $W \subseteq P$ is an *anti-chain*, observe that $\upharpoonleft W = \uparrow W - W$. Whenever an upset U is equal to the upset generated by a singleton set, we say that U is a *principal upset*. We simply write $\uparrow p$ to mean $\uparrow\{p\}$, and similarly, we write $\upharpoonleft p$ for $\upharpoonleft\{p\}$.

2.15 Definition (Kripke Frame and Kripke Model)
A *Kripke model* is given by a triple $\langle v, P, X\rangle$, consisting of a partial order P, called the *Kripke frame*, and a *monotonic map* $v : P \to \mathcal{P}(X)$, called the *valuation*.[9] We inductively define when a formula $\chi \in \mathcal{L}(X)$ is *valid at a point* $p \in P$ of the Kripke model, denoted $v, p \Vdash \chi$, as follows.

$$\begin{array}{lll} v, p \Vdash x & := & x \in v(p) \text{ for variables } x \in X \\ v, p \Vdash \top & := & \text{true} \\ v, p \Vdash \bot & := & \text{false} \\ v, p \Vdash \phi \wedge \psi & := & p \Vdash \phi \text{ and } p \Vdash \psi \\ v, p \Vdash \phi \vee \psi & := & p \Vdash \psi \text{ or } p \Vdash \psi \\ v, p \Vdash \phi \to \psi & := & q \Vdash \psi \text{ whenever } q \Vdash \phi, \text{ for all } q \geq p \end{array}$$

We say that χ is *valid in* v, denoted $v \Vdash \chi$, to mean that $v, p \Vdash \chi$ holds for all $p \in P$. Moreover, we write $P \Vdash \phi$ to mean that $v \Vdash \phi$ for all valuations $v : P \to \mathcal{P}(X)$; that is to say, ϕ is *valid on the frame* P. Finally, we write $v, p \Vdash \Gamma$, where $\Gamma \subseteq \mathcal{L}(X)$ is a potentially infinite set of formulae, to mean that $v, p \Vdash \phi$ for all $\phi \in \Gamma$. The notations $v \Vdash \Gamma$ and $P \Vdash \Gamma$ are defined analogously.

We often refer to a Kripke model by merely mentioning the valuation, as all necessary information is encoded within this mapping. When $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(X)$ are models such that $Q \subseteq P$ is an upset and $u = v \upharpoonright Q$, then u is said to be a *generated submodel* of v.[10] A frame is said to be *rooted* whenever P is a principal upset, and the point generating this upset is said to be its *root*. Many

[9] In this particular instance, a map $v : P \to \mathcal{P}(X)$ is monotonic whenever for all $p_1, p_2 \in P$ we know that $p_1 \leq p_2$ entails $v(p_1) \subseteq v(p_2)$.
[10] As is usual, $f \upharpoonright B$ denotes the restriction of the function $f : A \to C$ to the subset $B \subseteq A$.

of the frames we consider are such that principal upsets are finite. Such frames are known as *image-finite* frames, as formally specified in Definition 2.16. We refer to Bezhanishvili and Bezhanishvili (2008) for more background on this.

2.16 Definition (Image-Finite)
A Kripke frame P is said to be *image-finite* whenever the set $\uparrow p$ is finite, for each $p \in P$. We say that a Kripke model $v : P \to \mathcal{P}(X)$ is image-finite whenever P is.

Given a model, one can be interested in the formulae that are valid in that model. We give three definitions, each concerned with describing this structure.

2.17 Definition (Theory of a Point)
Let $v : P \to \mathcal{P}(X)$ be a model. The *theory of a point* $p \in P$, denoted $\mathrm{Th}_v(p)$ is defined as:

$$\mathrm{Th}_v(p) := \{\phi \in \mathcal{L}(X) \mid v, p \Vdash \phi\} .$$

The *theory of a model* $v : P \to \mathcal{P}(X)$ and the *theory of a frame* P, respectively, are given by:

$$\begin{aligned} \mathrm{Th}(v) &:= \bigcap \{\mathrm{Th}_v(p) \mid p \in P\} , \\ \mathrm{Th}(P) &:= \bigcap \{\mathrm{Th}(v) \mid \text{for all variables } X \text{ and models } v : P \to \mathcal{P}(X)\} . \end{aligned}$$

Kripke models are *sound* with respect to IPC, as stated in Lemma 2.18 below. We omit the proof, as this is wholly standard.

2.18 Lemma (Soundness of Kripke Models)
Let $v : P \to \mathcal{P}(X)$ be a model, let $p \in P$ be a point, let $\Gamma \subseteq \mathcal{L}(X)$ be a set of formulae, and let $\chi \in \mathcal{L}(X)$ be a formula. Suppose that $\Gamma \vdash_{\mathsf{IPC}} \chi$. If $v, p \Vdash \phi$ for all $\phi \in \Gamma$, then $v, p \Vdash \chi$. In particular, all the theorems of IPC are included in the theory of any model.

2.19 Example (Non-Derivability of Harrop's Rule)
Recall Example 2.11, where we considered *Harrop's rule*. To prove that this rule is not derivable, it suffices to provide a single Kripke model on which it does not hold. Such a model is given in Fig. 2.1. See that $\neg z \to x \vee y$ is valid at the root of this model, yet neither $\neg z \to x$ nor $\neg z \to y$ are. Hence, Lemma 2.18 ensures that Harrop's rule is *not* derivable.

Looking at the same type of information from another perspective, one might wonder at which points in a model a certain formula is valid. The following definition captures this.

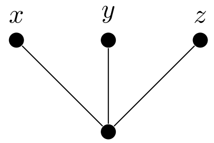


Figure 2.1.: Model $v : P \to \mathcal{P}(X)$, indicating the non-derivability of H rule.

2.20 Definition (Interpretation)
Let $v : P \to \mathcal{P}(X)$ be a model. The *interpretation of a formula* $\phi \in \mathcal{L}(X)$ *in* v, denoted $[\![\phi]\!]_v$, is defined as:

$$[\![\phi]\!]_v := \{p \in P \mid v, p \Vdash \phi\}\,.$$

The following Lemma 2.21 illustrated that all such interpretations are upsets. We use this lemma tacitly throughout this thesis.

2.21 Lemma (Preservation of Truth)
Let $v : P \to \mathcal{P}(X)$ be a Kripke model, let $p, q \in P$ be points satisfying $p \leq q$, and let $\phi \in \mathcal{L}(X)$ be a formula. If $v, p \Vdash \phi$, then $v, q \Vdash \phi$.

Proof. Immediate by induction on the structure of ϕ. □

Kripke models are complete with respect to IPC, which can be shown through the construction of the so-called *canonical model*, as introduced in the following theorem. We spend more words on this in Example 2.68, and yet more in Section 4.2.1. For now, let it suffice to say that the following theorem is wholly standard, see for instance Sørensen and Urzyczyn (2006, Theorem 2.5.9) or Chagrov and Zakharyaschev (1997, Section 5.1).

2.22 Theorem (Completeness of IPC)
Let X be a set of variables. There exists a model $\mathsf{c} : \mathsf{C}(X) \to \mathcal{P}(X)$ (called the *canonical model of* IPC) such that the following equivalence holds for all $\Gamma \subseteq \mathcal{L}(X)$ and all $\chi \in \mathcal{L}(X)$.

$$\Gamma \vdash_{\mathsf{IPC}} \chi \text{ if and only if } \bigcap_{\phi \in \Gamma} [\![\phi]\!]_{\mathsf{c}} \subseteq [\![\chi]\!]_{\mathsf{c}}$$

2.23 Example (Admissibility of Harrop's Rule)
Recall Harrop's rule from Examples 2.11 and 2.19. To prove that this rule is *admissible*, it suffices to show that if $\nvdash_{\mathsf{IPC}} \neg\chi \to \phi$ and $\nvdash_{\mathsf{IPC}} \neg\chi \to \psi$, then $\nvdash_{\mathsf{IPC}} \neg\chi \to \phi \vee \psi$

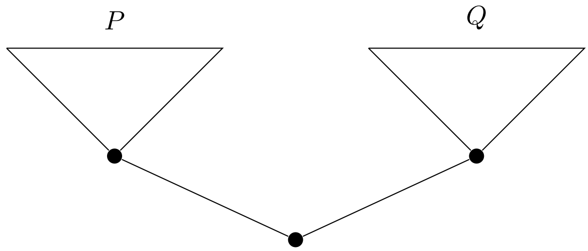


Figure 2.2.: Model used to prove the admissibility of Harrop's rule.

for all $\phi, \psi, \chi \in \mathcal{L}(X)$. Assume the former two hold. Through the completeness of IPC given in Theorem 2.22, this ensures rooted models $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(X)$ such that v and u both make $\neg\chi$ valid, yet $v \nVdash \phi$ and $u \nVdash \psi$.

Consider the model as depicted in Fig. 2.2, obtained by putting v and u disjointly next to one another, followed by adjoining a root. It is an easy matter to verify that $\neg\chi$ is valid at this model, but neither ϕ nor ψ are. This proves that Harrop's rule is indeed admissible in IPC.

Let us now focus on the maps between the structures we defined below, namely Kripke frames and Kripke models. As is clear, a Kripke frame is nothing but a partial order by a different name. However, the conditions imposed upon maps between Kripke frames are stricter than those between partial orders. In the literature, some call a map of Kripke frames a pseudo-epimorphism (or p-morphism, for short), and some call it a bounded morphism.

2.24 Definition (Map of Kripke Frames)
Let P and Q be Kripke frames, and let $f : P \to Q$ be a *monotonic map*. If, in addition, the function f is such that for all $p_1 \in P$ and $q \in Q$ the inequality $f(p_1) \leq q$ ensures the existence of a $p_2 \in P$ such that $p_1 \leq p_2$ and $f(p_2) = q$, then we call f a *map of Kripke frames*. We write $\mathbf{KF}$ for the category of Kripke frames and their maps.

2.25 Definition (Map of Kripke Models)
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ be Kripke models. A map of Kripke frames $f : P \to Q$ which satisfies $v = u \circ f$ is said to be a *map of Kripke models* $f : v \to u$.

We, very briefly, remark that any partial order can be endowed with the *Alexandrov* (or Alexandroff) topology, in which open sets are upsets.[11] Endowed with such a topology, continuous maps are precisely the monotonic maps, and open maps are maps of Kripke frames.

Lemma 2.26 below is quite standard. We include for the sake of reference, as it compares nicely to the more general Lemma 4.38.

2.26 Lemma
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ be Kripke models, and let $f : v \to u$ be a map of Kripke models. If $v, p \Vdash \chi$, then $v, f(p) \Vdash \chi$ for all $\chi \in \mathcal{L}(X)$. In particular, $\mathrm{Th}(u) \subseteq \mathrm{Th}(v)$, and this becomes an equality when f is surjective.

Proof. The second statement readily follows from the first. Indeed, if $u \Vdash \phi$ yet $v \nVdash \phi$ then there would be some $p \in P$ such that $v, p \Vdash \phi$, hence $u, f(p) \nVdash \phi$, a contradiction. If f is surjective, then the converse also holds.

We now prove the first statement by structural induction along $\chi \in \mathcal{L}(X)$. In the base case, we know $\chi = x \in X$. By definition, we know $v, p \Vdash x$ to be equivalent to:

$$x \in v(p) = (u \circ f)(p) = u(f(p)).$$

Again through definition, this is equivalent to $u, f(p) \Vdash x$, resolving this case.

Both the conjunctive and disjunctive cases hold immediately by induction. Now, consider the case where $\chi = \phi \to \psi$ for $\phi, \psi \in \mathcal{L}(X)$. Suppose $v, p \Vdash \phi \to \psi$, and let $q \geq f(p)$ be such that $u, q \Vdash \phi$. There exists a $k \geq p$ such that $f(k) = q$. Induction ensures us that $v, k \Vdash \phi$. By definition we deduce $v, k \Vdash \psi$, whence $u, q \Vdash \psi$ follows by induction.

Finally, suppose $u, f(p) \Vdash \phi \to \psi$. Let $k \geq p$ be given, and suppose $v, k \Vdash \phi$. As $f(k) \geq f(p)$, we thus infer, by induction and by definition, that $u, f(k) \Vdash \psi$. Induction finishes the proof. □

2.27 Lemma
Let $f : P \to Q$ be a surjective map of Kripke frames. If the Kripke frame P is *image-finite*, then so is Q.

Proof. Suppose P is image-finite, and let $q \in Q$ be arbitrary. By surjectivity, there exists a $p \in P$ such that $f(p) = q$. Now, we compute:

$$f(\uparrow p) = \uparrow f(p) = \uparrow q,$$

hence as $\uparrow p$ is finite, so is $\uparrow q$. This proves the desired. □

[11] Such spaces were originally introduced by Alexandrov (1937), see Johnstone (1982) for more details.

We note that substitutions yield models in a straightforward manner. This observation plays a crucial role in our reasoning, in particular in Theorem 3.38. A partial converse to this is given in Lemma 3.8.

2.28 Lemma (Ghilardi, 1999, Proposition 2)
Let $\sigma : \mathcal{L}(Y) \to \mathcal{L}(X)$ be a substitution, and let $v : P \to \mathcal{P}(X)$ be a model. Define the model $\sigma^*(v) : P \to \mathcal{P}(Y)$ as

$$\sigma^*(v)(p) = \{y \in Y \mid v \Vdash \sigma(y)\} \tag{2.7}$$

This model is such that the following holds.

$$v, p \Vdash \sigma(\chi) \text{ if and only if } \sigma^*(v), p \Vdash \chi \text{ for all } p \in P \text{ and } \chi \in \mathcal{L}(Y). \tag{2.8}$$

Proof. The proof is immediate by induction and appropriately unfolding the definitions. □

2.2.1. Covers

In Example 2.23, we considered a point which was put directly below two known points. Based on this information, we could easily extract information on the theory of this new point out of the theories of the pre-existing points. We come across a similar type of construct quite often in this thesis. It thus merits a proper investigation.

First, we specify this situation in some generality. Later, in Section 4.2.2, we generalise this notion even further, but this need not concern us now. The definition below is equivalent to that of Ghilardi (2004).

2.29 Definition (Cover)
Let P be a Kripke frame, let $W \subseteq P$ be an arbitrary subset, and let $p \in P$ be a point. We say that W *covers* p, denoted $W \kappa\ p$, whenever the following equivalence holds:

$$p \leq q \text{ iff } p = q \text{ or } q \in \uparrow W \tag{2.9}$$

This means that W covers p if and only if $\uparrow W$ equals the upset generated by p or the strict upset generated by p.

2.30 Example
Consider the Kripke frame as depicted in Fig. 2.3. We give some examples and non-examples of the cover relation. Note that $\{2, 3\}\ \kappa\ 1$ and $\{6, 9, 15\}\ \kappa\ 3$ hold, yet $\{2\}\ \kappa\ 1$ and $\{6, 15\}\ \kappa\ 3$ do not. Moreover, $\{1\}\ \kappa\ 1$ and $\varnothing\ \kappa\ 15$ hold, $\varnothing\ \kappa\ 3$ does not.

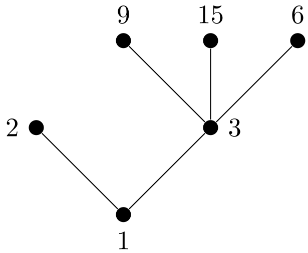


Figure 2.3.: Examples of covers in a Kripke frame.

2.31 Example (Immediate Successor)
Let P be a finite Kripke frame, and let $p \in P$ be a point. We say that $q \in P$ is an *immediate successor* of p whenever:

(i) $p \leq q$;

(ii) there exists no $k \in P$ such that $p < k$ and $k < q$.

Clearly, if W is the set of all immediate successors of p, then $W \ltimes p$. This covers the cases $\{2, 3\} \ltimes 1$ and $\{6, 9, 15\} \ltimes 3$ of Example 2.30.

2.32 Example (Maximal Points)
Let P be a Kripke frame. We define the set of *maximal points* as:

$$\mathsf{max}(P) := \{p \in P \mid \text{ for all } q \in P, \text{ if } p \leq q \text{ then } p = q\}.$$

The maximal points of Fig. 2.3 are 2, 6, 9, and 15. It is easy to see that $\varnothing \ltimes p$ precisely if $p \in \mathsf{max}(P)$. This covers the case $\varnothing \ltimes 15$, and simultaneously shows why $\varnothing \ltimes 3$ does not hold, both in Example 2.30.

2.33 Example (Reflexivity)
Any point is covered by itself, in the sense that $\{p\} \ltimes p$ holds, regardless of the choice of $p \in P$. This covers the case $\{1\} \ltimes 1$ of Example 2.30. As per the above, one can think of the covering-relation as being reflexive. This in contrast to the notion of a so-called "*total cover*", as employed by Grigolia (1995) and Bezhanishvili (2006). Using our definition, W is a *total cover* of p if $W \ltimes p$ and $p \notin W$. Jeřábek (2005) would call p a "*tight predecessor*" of W in precisely the same situation, following Iemhoff (2001b).

Within the scope of this thesis, the motivation for considering the notion of a cover comes largely from Lemma 2.34 below. Later, in Lemma 2.38, we give a partial converse to this lemma. One can think of this lemma as specifying the theory of a point in terms of its valuation and the points that cover it. The observation is obvious, and perhaps for this reason does not often appear in the literature. Without going into further detail, we mention the semblance to the definition of the *Kleene-slash*, as discussed briefly in Section 7.4.

2.34 Lemma
Let $v : P \to \mathcal{P}(X)$ be model, let $W \subseteq P$ be a set of points, and let $p \in P$ be such that $W \ \kappa \ p$. The following equivalence holds for all $\phi, \psi \in \mathcal{L}(X)$.

$$v, p \Vdash \phi \to \psi \text{ iff } v, W \Vdash \phi \to \psi \text{ and either } v, p \nVdash \phi \text{ or } v, p \Vdash \psi \text{ hold} \tag{2.10}$$

Proof. By definition we know $v, p \Vdash \phi \to \psi$ if and only if $v, q \Vdash \phi$ implies $v, q \Vdash \psi$ for all $q \geq p$. Now, because $W \ \kappa \ p$ the latter is equivalent to the statement that $v, q \nVdash \phi$ or $v, q \Vdash \psi$ holds for $q \in P$ satisfying $p = q$ or $q \in \uparrow W$, from whence the desired is immediate. □

Recall that the cover-relation is reflexive, as discussed in Example 2.33. There is good reason to allow this reflexivity in the notion of covering. The following lemma shows that covers are preserved by maps, which would not be the case were we to impose irreflexivity. Here, a Kripke frame P is said to be *conversely well-founded* whenever $\geq$ is *well-founded*. It is easy to see that every image-finite Kripke frame is conversely well-founded.

2.35 Lemma (Ghilardi, 2004)
Let P and Q be Kripke frames, and let $f : P \to Q$ be a map of partial orders. The statement (i) entails (ii), and the converse holds whenever P is *conversely well-founded*.

(i) The map f is a *map of Kripke frames.*

(ii) For all $p \in P$ and $W \subseteq P$ such that $W \ \kappa \ p$ we have $f(W) \ \kappa \ f(p)$.

Proof. The implication from (i) to (ii) follows from straightforward computation. Indeed, if $W \ \kappa \ p$ then $f(W) \ \kappa \ f(p)$ follows from the equation below.

$$\uparrow f(p) = f(\uparrow p) = f(\uparrow W \cup \{p\}) = f(\uparrow W) \cup \{f(p)\} = \uparrow f(W) \cup \{f(p)\}$$

Suppose (ii) holds. We prove, by well-founded induction, that for all $p \in P$ we have $\uparrow f(p) = f(\uparrow p)$. Consider p and $W := \mathrel{\rlap{\uparrow}{\,\cdot}}p$, and assume that $\uparrow f(w) = f(\uparrow w)$ for all $w \in$

W. It follows that $\uparrow f(W) = f(\uparrow W)$. We know that $W \; \kappa \; p$, and thus $f(W) \; \kappa \; f(p)$ holds by assumption. From here, we compute:

$$f(\uparrow p) = f(\{p\}) \cup f(\uparrow W) = f(\{p\}) \cup \uparrow f(W) = \uparrow f(p),$$

proving (i) as desired. □

2.2.2. Refined models

In the canonical model, the order is fully determined by the theories of its points. This can be the case in many more models, in particular in generated submodels of the canonical model. Many consequences can be drawn from this definability of order alone, so let us give it a name. This definition occurs elsewhere in the literature, usually in the context of *general frames*, see for instance Chagrov and Zakharyaschev (1997, pp. 133–134.) and Jeřábek (2008, p. 252). Note that the left-to-right implication in Definition 2.36 below always holds.

2.36 **Definition (Refined Model)**
A model $v : P \to \mathcal{P}(X)$ is said to be *refined* when for all $p, q \in P$ we know that $p \leq q$ holds if and only if $\mathrm{Th}_v(p) \subseteq \mathrm{Th}_v(q)$.

Refined models are hardly exotic. Indeed, most models one would come up with by hand are, in fact, refined. Moreover, any Kripke frame can be endowed with a valuation which makes it refined, as we prove below. The following lemma illustrates that one can endow any Kripke model with a refined valuation, the reasoning of which is well-known in folklore. By using the elements of the order as variables, all elements can easily be distinguished, making the proof of concreteness straightforward.

2.37 **Lemma**
Let P be a Kripke frame. The model d_P defined as below is *refined*.

$$\mathsf{d}_P : P \to \mathcal{P}(P), \quad p \mapsto \downarrow p.$$

Proof. First, note that d_P is a monotonic map, hence it is a bonafide model. Let $p, q \in P$ be arbitrary, and suppose that $\mathrm{Th}_{\mathsf{d}_P}(p) \subseteq \mathrm{Th}_{\mathsf{d}_P}(q)$. We see that:

$$\mathsf{d}_P(p) \subseteq \mathrm{Th}_{\mathsf{d}_P}(p) \subseteq \mathrm{Th}_{\mathsf{d}_P}(q),$$

which proves that $p \in \mathsf{d}_P(q) = \mathsf{d}_P(q)$. This shows $p \leq q$, as desired. □

Recall Lemma 2.34, in which we, roughly speaking, described the theory of a point through the theories of its cover. In Lemma 2.38 below, we provide a partial converse. Note that the equivalence in the statement of this lemma is simply (2.10).

2.38 Lemma
Let $v : P \to \mathcal{P}(X)$ be a *refined model*, and let $W \subseteq P$ be a finite set of points. If $p \in P$ satisfies $\uparrow W \subseteq \uparrow p$ and is such that:

$$v, p \Vdash \phi \to \psi \text{ iff } v, W \Vdash \phi \to \psi \text{ and either } v, p \nVdash \phi \text{ or } v, p \Vdash \psi \text{ hold,}$$

then $W \; \kappa \; p$.

Proof. We need to prove that $\uparrow p = \uparrow W \cup \{p\}$. The inclusion from right to left holds by assumption, so we need but focus on the other direction.

We proceed by contradiction, so assume the existence of a point $q \in P$ that satisfies both $p < q$ and $q \notin \uparrow W$. The former, combined with the refinedness of v, ensures that there is a $\phi \in \mathcal{L}(X)$ such that $v, p \nVdash \phi$ and $v, q \Vdash \phi$. Through the latter, combined with refinedness, we obtain $\psi_w \in \mathcal{L}(X)$ such that $v, w \Vdash \psi_w$ and $v, q \nVdash \psi_w$ for each $w \in W$.

We note that $\psi := \bigvee_{w \in W} \psi$ is such that $v, W \Vdash \psi$, and thus $v, W \Vdash \phi \to \psi$. By the equivalence (2.10), we know that $v, p \Vdash \phi \to \psi$, and so $v, q \Vdash \phi \to \psi$ follows by the preservation of truth of Lemma 2.21. But $v, q \Vdash \phi$, so this ensures $q \Vdash \psi$. By definition, this gives a $w \in W$ such that $q \Vdash \psi_w$, a contradiction, as desired. □

In refined models, the order between two points is fully determined by their respective theories. This makes these models quite rigid, as shown by Lemma 2.39 below.

2.39 Lemma
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ be models, and let $f, g : v \to u$ be arbitrary maps between Kripke models. If u is *refined*, then $f = g$.

Proof. First, observe that if $q_1, q_2 \in Q$ are such that $\mathrm{Th}_u(q_1) = \mathrm{Th}_u(q_2)$ then $q_1 = q_2$, as readily follows from the refinedness of u. Pick any $p \in P$ and see that the following holds through Lemma 2.26.

$$\mathrm{Th}_v(f(p)) = \mathrm{Th}_v(p) = \mathrm{Th}_v(g(p))$$

Consequently, $f(p) = g(p)$ holds for all $p \in P$, proving the desired. □

Every image-finite model on a set of variables has a unique map to the canonical model on the same set of variables. Below we show this, making use of the existence criterion given by Lemma 2.38.

2.40 Lemma
Let $v : P \to \mathcal{P}(X)$ be an *image-finite* model. There exists a unique map of Kripke models $\mathrm{Th}_v(-) : v \to \mathsf{C}(X)$.

Proof. Refinedness of the canonical model is obvious, hence, due to Lemma 2.39, we only need to prove existence. Define the map $\mathrm{Th}_v(-)$ as follows:

$$\mathrm{Th}_v(-) : v \to \mathsf{C}(X), \quad p \in P \mapsto \mathrm{Th}_v(p)$$

Monotonicity of $\mathrm{Th}_v(-)$ is clear by the preservation of truth. Let $W \subseteq P$ be arbitrary, and $p \in P$ such that $W \ \kappa \ P$. By Lemma 2.35, we are done when we can prove that $\mathrm{Th}_v(W) \ \kappa \ \mathrm{Th}_v(p)$. First, note that $W \subseteq \uparrow p$, and so W is finite as v is image-finite. Now, also observe that $\mathrm{Th}_v(W)$ and $\mathrm{Th}_v(W)$ satisfy the equivalence (2.10). The proof is now immediate through Lemma 2.38. □

2.2.3. Concrete models

How does one know that a particular model is refined? Definition 2.36 is not easily testable, and expressed in a mixture of syntax and semantics. The purpose of this subsection is to explore the notion of *concreteness*, which we show to be a special case of being refined. As notions similar to concreteness have been considered throughout the literature, we deem it worthwhile to devote this subsection to its systematic study, consolidating folklore into one coherent exposition. Note that concrete models coincide precisely with refined models in the image-finite case. However, we do not prove this until Corollary 3.12.

Before we formally introduce concrete models, let us first describe what this notion ought to entail. In a refined model, any two distinct points can be discerned by their theories. Consider a generic Kripke model $v : P \to \mathcal{P}(X)$, and suppose that there are points $p, q \in P$ such that $v(p) = v(q)$ holds, and both $W \ \kappa \ p$ and $W \ \kappa \ q$ hold for a given *anti-chain* $W \subseteq P$. Through Lemma 2.34, it is immediate that p and q must have equal theories.

In fact, there exists an obvious map of Kripke models from v to another model where these two points are conflated. Hence, if v were to be refined, then p and q ought to be equal. Concrete models are such that this is always the case.

The idea behind this goes back to de Jongh and Troelstra (1966, Definition 4.4), whose roots can be found in de Jongh (1964, pp. 14–18). In the case where p and q are comparable, they called the resulting map of Kripke models an α-reduction. In the case where p and q are incomparable, the resulting map is said to be a β-reduction.

Let us, for convenience, refer to such pairs of points p and q as α-redexes and β-redexes, respectively.[12]

These redexes appear throughout the literature. Indeed, they occur in the work of Anderson (1969, Section 4), who called the resulting maps operation 1 and operation 2.[13] Bellissima (1986), too, considers this, and speaks of α-degenerate and α-duplicate points respectively. Rybakov (1987b) calls such α-redexes "doubles", and speaks of "duplicates" in Rybakov (1987b, 1990b). These same notions occur in Odintsov and Rybakov (2013, p. 773), under the names duplicates and twins respectively.

In Definition 2.41 below, we forego the distinction between α-redexes and β-redexes, and speak of analogous points in both cases. It is easy to see that comparable analogous points form an α-redex, and incomparable analogous points form a β-redex.

2.41 Definition (Analogous Points, Concrete Model)
Let $v : P \to \mathcal{P}(X)$ be a model, and let $p, q \in P$ be points. We say that p and q are *analogous*, written $p \equiv q$, whenever $v(p) = v(q)$ and:

$$p \leq k \text{ if and only if } q \leq k \text{ for all } k \in P - \{p, q\}.$$

The model v is said to be *concrete* when $p \equiv q$ holds if and only if $p = q$, for all $p, q \in P$.

Observe that the relation $\equiv$ is reflexive and symmetric, but it need not be transitive. As a consequence, the right-to-left portion of the equivalence that is guaranteed to hold in concrete models, holds in any model.

2.42 Example
In Fig. 2.4, three models are depicted, each of which have non-trivial instances of points being analogous. There is an obvious map of Kripke models from the leftmost model to the inner most model, and there is yet another such map from the innermost model to the rightmost model.

We make the connection with our motivating example in Lemma 2.45 below. Before we proceed, let us first fix a definition.

[12] This nomenclature is, of course, not entirely proper, as "redex" is an abbreviation for "reducible expression". Still, the name suits the intuitive feel of the situation quite well.

[13] This similarity was already noted by Troelstra in his review of Anderson (1969) for Mathematical Reviews.

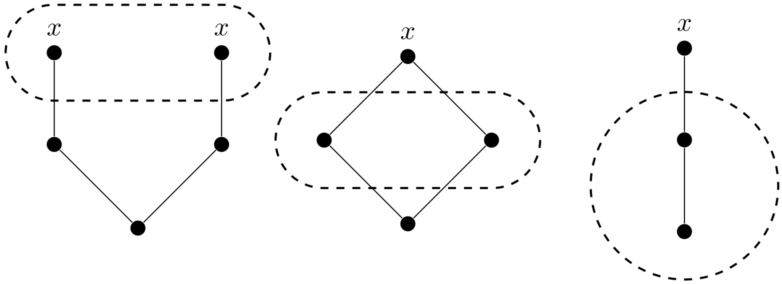


Figure 2.4.: Models in which some analogous points are indicated.

2.43 Definition (Reduction)
Let $f : P \to Q$ be a map of Kripke frames. This map is said to be a *reduction* when there exist $p, q \in P$ satisfying both $p \equiv q$ and $p \neq q$, such that the following are equivalent for all $k_1, k_2 \in P$:

(i) the equality $f(k_1) = f(k_2)$ holds;

(ii) either $k_1 = k_2$ or $\{k_1, k_2\} = \{p, q\}$ hold.

2.44 Example
Consider any Kripke frame P, and suppose that $p, q \in P$ are such that $p \equiv q$. The smallest equivalence relation R on P such that $p\ R\ q$ holds is a congruence relation with respect to the order on P. That is to say, if $k_1 \leq k_2$ and $k_1\ R\ k_1'$ and $k_2\ R\ k_2'$ then $k_1' \leq k_2'$ holds, too. Consequently, there is a natural order on P/R, the equivalence classes of R. The quotient map $P \to P/R$ is a reduction. The maps alluded to in Example 2.42 are all instances of this, and as such, they are all reductions.

The following lemma can also be found in Bezhanishvili (2006, Lemma 3.1.7). Its proof is comparable in nature to the one given below, yet there is a difference lies in the concepts used. Because we use the notion of being analogous in the definition of a reduction, instead of α-redexes and β-redexes, the proof below proceeds with less case-distinctions than elsewhere.

2.45 Lemma (de Jongh and Troelstra, 1966)
Let P be a finite model. For every map of Kripke frames $f : P \to Q$ that is not injective, there exists a chain of *reductions* $f_1, \ldots, f_n$ such that $f_n \circ f_{n-1} \circ \ldots \circ f_1 = f$.

Proof. We proceed by induction on the size of the model P. Let $f : P \to Q$ be given

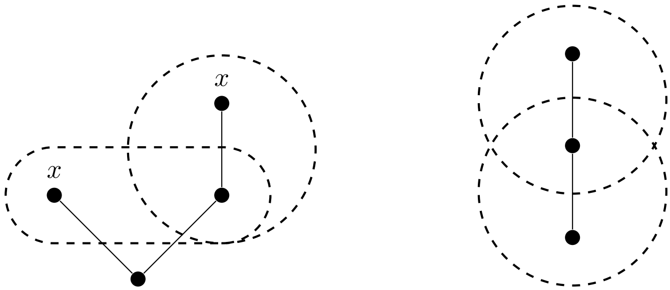


Figure 2.5.: Example of two Kripke models in which the relation of being analogous is *not* transitive.

and consider the set:

$$E := \left\{ \langle p, q \rangle \in P \times P \;\middle|\; p \neq q \text{ and } f(p) = f(q) \right\}.$$

Order E by $\langle p_1, q_1 \rangle \leq \langle p_2, q_2 \rangle$ iff $p_1 \leq p_2$ and $q_1 \leq q_2$. Because f is not injective, we know E to be non-empty, and as P is finite we can pick a maximal $\langle p, q \rangle \in E$. We claim that $p \equiv q$.

Indeed, if $k \in P - \{p, q\}$ is given and $p \leq k$ then $f(q) = f(p) \leq f(k)$, and so there must be a $k' \geq q$ such that $f(k) = f(k')$. Now $k = k'$ must hold, for otherwise $\langle k', k \rangle > \langle p, q \rangle$, contradicting the maximality of $\langle p, q \rangle$. This proves that $q \leq k' = k$, as desired. The other direction can be proven in a similar manner.

Now, consider the smallest equivalence relation R on P such that $p \; R \; q$. Define the map $f_1 : P \to P/R$ to be the quotient map, and let $f' : P/R \to Q$ be defined on representatives by f. It follows that f' is a well-defined map and that $f' f_1 = f$. Moreover, note that the size of P/R is smaller than that of P. Induction yields maps $f_2, \ldots, f_n$ such that $f_n \ldots f_2 = f'$. This proves that $f_n \ldots f_1 = f$, as desired. □

Note that the relation $\equiv$ is reflexive and symmetric, but Fig. 2.5 shows that, in general, it is *not* transitive. One can extend the notion of being analogous away from the binary into the finitary, as below.

2.46 Definition (Analogous Set)
Let $v : P \to \mathcal{P}(X)$ be a Kripke model. A set of points $W \subseteq P$ is said to be *analogous* whenever:

(i) the equality $v(w_1) = v(w_2)$ holds for all $w_1, w_2 \in W$;

(ii) for all $p \in P - W$, the inequality $w \leq p$ holds for some $w \in W$ precisely if the inequality $w \leq p$ holds for all $w \in W$.

Note that, in both the left-hand model and the right-hand model of Fig. 2.5, the set of all encircled points is an analogous set. Moreover, it is easy to see that a doubleton set is analogous precisely when its constituents are analogous points. We entertain this digression for a bit more, and define a generalisation of analogous based on the above notion.

2.47 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model. Define the relations $\cong$ and $\sqsubseteq$ on P as follows:

$$p \cong q \text{ if and only if there is an } \textit{analogous set } W \subseteq P \text{ with } p, q \in W$$
$$p \sqsubseteq q \text{ if and only if there are } p', q' \in P \text{ such that } p \cong p' \leq q' \cong q$$

The relation $\cong$ is an equivalence relation congruent with $\leq$. The relation $\sqsubseteq$ is the least reflexive, transitive relation extending both $\cong$ and $\leq$ such that $p \sqsubseteq q$ and $q \sqsubseteq p$ entail $p \equiv q$ for all $p, q \in P$.

Proof. Reflexivity and symmetry of $\cong$ are both evident. To prove transitivity, assume $a \cong b \cong c$ holds for $a, b, c \in P$. This gives us analogous sets $W_{ab} \ni a, b$ and $W_{bc} \ni b, c$. We see that $W_{ab} \cup W_{bc}$ is an analogous set, whence the transitivity follows.

It is clear that $\sqsubseteq$ extends $\leq$ and $\equiv$. There are three matters left to prove: reflexivity, transitivity and that $p \sqsubseteq q$ and $p \sqsubseteq q$ entail $p \equiv q$ for all $p, q \in P$. The former is immediate from reflexivity of $\leq$ and $\equiv$.

To prove transitivity, assume $a \sqsubseteq b \sqsubseteq c$. This yields $p_{ab}, p_{ba}, p_{bc}, p_{cb} \in P$ such that:

$$a \cong p_{ab} \leq p_{ba} \cong b \cong p_{bc} \leq p_{cb} \cong c.$$

Let W be an analogous set such that $p_{ba}, b, p_{bc} \in W$. If $p_{cb} \in W$ then $p_{ba} \cong p_{cb}$ whence the desired is immediate. Assume the contrary, then we know from $p_{bc} \leq p_{cb}$ that $p_{ba} \leq p_{cb}$. But now $a \cong p_{ab} \leq p_{cb} \cong c$, as desired.

We now turn to the third property, so assume $a \sqsubseteq b$ and $b \sqsubseteq a$. This yields us points a_{ab}, b_{ab}, b_{ba}, a_{ba} in P satisfying:

$$a \cong a_{ab} \leq b_{ab} \cong b \text{ and } b \cong b_{ba} \leq a_{ba} \cong a.$$

Consider analogous sets W_a and W_b such that $a, a_{ab}, a_{ba} \in W_a$ and $b, b_{ab}, b_{ba} \in W_b$. If these sets intersect, then we are done, so assume the contrary. It follows that $a_{ba} \leq b_{ab}$ because $a_{ab} \leq b_{ab}$ and $a_{ba}, a_{ba} \in W_a$. Similarly, $b_{ab} \leq a_{ba}$ because $b_{ba} \leq a_{ba}$

and $b_{ba}, b_{ab} \in W$. We now have, through anti-symmetry of $\leq$, that $a_{ba} = b_{ba}$, *quod non.* Minimality we leave to the reader, the proof technique is similar to the above.□

When given a model, one can conflate all points that are equivalent under $\cong$. This results in a new model, which satisfies a certain universal property. The following Definition 2.48 specifies this model, and Lemma 2.49 specifies the universal property. We show that the construction $\mathsf{Con}\,(-)$ can be construed as a functor in Corollary 2.50.

2.48 Definition (Contraction of a Model)
Let $v : P \to \mathcal{P}(X)$ be a model, and let $\cong$ and $\sqsubseteq$ be the relations of Lemma 2.47. Define $\mathsf{Con}\,P$ to be the set of $\cong$-equivalence classes, ordered by $\sqsubseteq$ on representatives, and define the model $\mathsf{Con}\,v : \mathsf{Con}\,P \to \mathcal{P}(X)$ on representatives.

2.49 Lemma
Let $v : P \to \mathcal{P}(X)$, and let $\pi : v \to \mathsf{Con}\,v$ be the canonical quotient function. The function π is a surjective map of Kripke models. Moreover, for each map of Kripke models $f : v \to u$ such that $f(p) = f(q)$ when $p \cong q$, there is a unique map $g : \mathsf{Con}\,v \to u$ such that the diagram below commutes.

$$\begin{tikzcd} v \arrow[r, "\pi"] \arrow[dr, "f"'] & \mathsf{Con}\,v \arrow[d, dashed, "g"] \\ & u \end{tikzcd}$$

Proof. By definition, we know that $\mathsf{Con}\,v \circ \pi = v$, and π is clearly monotonic. Let us prove that π is a map of Kripke models. If $\pi(p) \leq q \bmod \cong$, then this means that there are $p', q' \in K$ such that $p \cong p' \leq q' \cong q$.[14] When $p' = q'$ then clearly $\pi(p) = q$, so we are done. If $p' < q'$ then $p < q'$, so $\pi(q') = q' \bmod \cong = q \bmod \cong$ as desired. The subjectivity of π is clear.

Now, let $f : v \to u$ be a map such that $p \cong q$ entails $f(p) = f(q)$ for all $p, q \in P$. Define the map g on representatives as below:

$$g : \mathsf{Con}\,v \to u, \quad p \bmod \cong \mapsto f(p).$$

[14] As $\pi(p)$ is an element of $\mathsf{Con}\,P$, it is to be understood as an $\cong$-equivalence class. These equivalence classes are ordered by $\sqsubseteq$ on representatives, as indicated in Definition 2.48. We simply write $\leq$ for this order, in the same way as we deal with the order in any other Kripke frame.

The map g is well-defined by the assumption on f. Commutativity of the diagram is clear, as $g(\pi(p)) = f(p)$ holds by construction, whence uniqueness automatically follows. We need only verify that g indeed is a map of models. Note that:

$$\mathsf{Con}\, v \circ \pi = v = u \circ f = u \circ (g \circ \pi) = (v \circ g) \circ \pi,$$

hence $\mathsf{Con}\, v = u \circ g$ follows from the subjectivity of π. By an argument similar to the above one can show that g is a map of Kripke frames, finishing the proof. □

2.50 Corollary
Let $f : v \to u$ be a map of Kripke models. There is a map $\mathsf{Con}\, f : \mathsf{Con}\, v \to \mathsf{Con}\, u$ such that the diagram below commutes.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "\pi_v"'] & u \arrow[d, "\pi_u"] \\ \mathsf{Con}\, v \arrow[r, "\mathsf{Con}\, f"] & \mathsf{Con}\, u \end{tikzcd}$$

When we apply Lemma 2.45 to the map $\pi : v \to \mathsf{Con}\, v$ described in Lemma 2.49, it becomes apparent that $\cong$ is, intuitively, like a transitive closure of $\equiv$. More concretely, if the relation $\cong$ on a model v is not equal to the identity relation, then there must be two non-equal analogous points.

As we argue in Corollary 2.54, analogous points have equal theories. This shows that for any image-finite model $v : P \to \mathcal{P}(X)$, we know the unique map into the canonical model $f : v \to \mathsf{C}(X)$, as guaranteed to exist by Lemma 2.39, to factor through $\pi : v \to \mathsf{Con}\, v$. If P is finite, then after finitely many applications of $\mathsf{Con}\,(-)$ to the model v, one obtains a model isomorphic to $\mathsf{c} \restriction \pi(P)$.

2.51 Theorem
Let $v : P \to \mathcal{P}(X)$ be a finite model. There exists a model:

$$\mathsf{Con}_\infty\, v : \mathsf{Con}_\infty\, P \to \mathcal{P}(X)\,,$$

and a surjective map of Kripke models $\pi_\infty : v \to \mathsf{Con}_\infty\, v$, such that for every *concrete* model $u : Q \to \mathcal{P}(X)$ and every map of Kripke models $f : v \to u$ there is a unique map $g : \mathsf{Con}_\infty\, v \to u$ satisfying $f = g \circ \pi_\infty$.

Proof. We proceed by well-founded induction along $n := |P|$. Suppose the desired has been achieved for all models $u : Q \to \mathcal{P}(X)$ with $|Q| < n$. Consider the model

$\mathsf{Con}\, v : \mathsf{Con}\, P \to \mathcal{P}(X)$ as given by Definition 2.48, and take $\pi : v \to \mathsf{Con}\, v$ to be the surjective map of Kripke models guaranteed by Lemma 2.49.

Suppose that π is injective. It immediately follows that v is concrete. Hence we can let π_∞ be the identity map $\mathrm{id}_v : v \to v$, and all constraints are satisfied.

Finally, suppose that π is not injective. This ensures that $|\mathsf{Con}\, P| < |P| = n$. By induction, we obtain a map $\pi^+_\infty : \mathsf{Con}\, v \to \mathsf{Con}_\infty \mathsf{Con}\, v$. We define a map of Kripke models $\pi_\infty := \pi^+_\infty \circ \pi$ and let the Kripke model $\mathsf{Con}_\infty\, v$ be given by $\mathsf{Con}_\infty \mathsf{Con}\, v$. Clearly, π_∞ is a surjective map of Kripke models from v to $\mathsf{Con}_\infty\, v$.

Let $u : Q \to \mathcal{P}(X)$ be a concrete model, and suppose there is a map of Kripke models $f : v \to u$. By Lemma 2.49, there must be a map $h : \mathsf{Con}\, v \to u$ such that $f = h \circ \pi$. Similarly, by induction, we know there to be a map $g : \mathsf{Con}_\infty \mathsf{Con}\, v \to u$ such that $h = g \circ \pi^+_\infty$. It is easy to see that:

$$f = h \circ \pi = (g \circ \pi^+_\infty)\pi = g \circ \pi_\infty,$$

proving the desired. □

We do not explore this generalised notion of analogousness any further, and return to the binary case. Let us first tie the concept to that of coverings. Note again that the "non-strictness" of the covering relation is quite essential.

2.52 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model. The following are equivalent, for all $p_1, p_2 \in P$:

(i) the points p_1 and p_2 are *analogous*;

(ii) there is a $W \subseteq P$ such that $W \ \kappa\ p_1, p_2$ and $v(p_1) = v(p_2)$.

Proof. Assume that (ii) holds, and let $p_1, p_2 \in P$ and $W \subseteq P$ be such that $v(p_1) = v(p_2)$ and $W \ \kappa\ p_i$ holds for both $i = 1, 2$. If $p \in P - \{p_1, p_2\}$ is such that $p_1 \leq p$, then $p \in W$ because $W \ \kappa\ p_1$. As $W \ \kappa\ p_2$, this proves $p_2 \leq p$. We can prove the converse through a similar argument, showing (i) to hold.

Conversely, suppose (i) holds. We distinguish two cases, either p_1 and p_2 are comparable or they are not. In the latter case, we define $W_i := {\uparrow} p_i - \{p_1, p_2\}$ for $i = 1, 2$. Observe that $W_1 = W_2$ because $p_1 \equiv p_2$. It is easy to see that $W_i \ \kappa\ p_i$ through the incomparability of p_1 and p_2, proving the desired.

In the former case, we assume that $p_1 \leq p_2$ without loss of generality. Now, define $W := {\uparrow} p_2$ and see that $W \ \kappa\ p_2$ and $W \ \kappa\ p_1$. The first statement is trivial, and the second holds because if $p \in P$ is such that $p_1 < p$ then $p_2 < p$ or $p_2 = p$. In both cases we derived (ii). □

The following can be shown by a straightforward computation, but is also an immediate corollary of Lemma 2.35 and Lemma 2.52.

2.53 Corollary
Let $v : P \to \mathcal{P}(X)$ be a model, and let $f : v \to u$ be a map of Kripke models. If $p \equiv q$ then $f(p) \equiv f(q)$ for all $p, q \in P$. In particular, if f is bijective and the model u is *concrete*, then the model v is concrete too.

Corollary 2.55 follows immediately from Corollary 2.54, and the former is a direct consequence of Lemma 2.52 and Lemma 2.34. This shows, as promised, that concreteness is a special case of refinedness.

2.54 Corollary
Let $v : P \to \mathcal{P}(X)$ be a model, and let $p, q \in P$ be points. If p and q are *analogous*, then $\mathrm{Th}_v(p) = \mathrm{Th}_v(q)$ follows.

2.55 Corollary
Any *refined* model is *concrete*.

2.3. Algebraic semantics

Universal algebra is the study of finitary operations on a set, see Burris and Sankappanavar (2012), Cohn (1965), Grätzer (1979), and Mal'tsev (1973) for an extensive treatment of this field in general. Blok and Pigozzi (1989) indicated that many logics can be endowed with semantics through universal algebra. In this section, we explain how Heyting algebras can provide semantics for IPC. We do not delve into the details of this topic, we merely provide the basic definitions.

Definition 2.56 below encodes the structure of a partially ordered set with a greatest and least element, in which greatest lower bounds and least upper bounds exists of all pairs of elements. A proof verifying that this encoding is correct can be found, for instance, in Balbes and Dwinger (1974, p. 44). We tacitly use this characterisation throughout this thesis.

2.56 Definition (Bounded Lattice)
A tuple $\mathfrak{A} = \langle \mathfrak{A}, \vee, \wedge, 0, 1 \rangle$, where $\mathfrak{A}$ is a nonempty set, $\vee$ and $\wedge$ are binary operations on $\mathfrak{A}$, and $0, 1$ are elements of $\mathfrak{A}$, is said to be a *bounded lattice* whenever the equations of Table 2.2 hold for all $a, b, c \in \mathfrak{A}$.

$$
\begin{aligned}
(a \vee b) \vee c &= a \vee (b \vee c) & (a \wedge b) \wedge c &= a \wedge (b \wedge c) \\
a \vee b &= b \vee a & a \wedge b &= b \wedge a \\
a \vee a &= a & a \wedge a &= a \\
a \vee (a \wedge b) &= a & a \wedge (a \vee b) &= a \\
a \vee 0 &= a & a \wedge 1 &= a
\end{aligned}
$$

Table 2.2.: The axioms of a bounded lattice.

When speaking of a bounded lattice, we always use the same symbols for the operations involved, and trust that the appropriate meaning can be inferred from the context. Any partial order can be viewed as a category in which any pair of objects has at most one arrow between them. In this perspective, a bounded lattice is a partial order which has finite products and finite co-products when it is seen as a category.

2.57 Example (Lattice of Upsets)
Consider an arbitrary Kripke frame P. We define $\mathsf{ups}(P)$ to be the the poset of all upsets of P ordered by inclusion. It is easy to verify that $\langle \mathsf{ups}(P), \cup, \cap, \varnothing, P\rangle$ is a bounded lattice.

Definition 2.56 above defined a bounded lattice to be a model of a specific equational theory. The class of all such models is known as a *variety*, see Cohn (1965, Chapter 4), Mal'tsev (1973, Chapter 6), Grätzer (1979, Section 23) and Burris and Sankappanavar (2012, Section 2.9) for background on this. Maps between such structures simply are set-based functions that preserve all available operations. In the following definition, we spell this out for the case of bounded distributive lattices. Note that a variety can be construed as a category with some additional structure. We refer to Adámek (2004) for more details on this perspective, which we do not revisit in this thesis.

2.58 Definition (Map of Bounded Distributive Lattices)
Let $\mathfrak{A}$ and $\mathfrak{B}$ be bounded lattices. A function $f : \mathfrak{A} \to \mathfrak{B}$ is said to be a *map of bounded lattices* whenever it satisfies the equations below for all $a, b \in \mathfrak{A}$.

$$
\begin{aligned}
f(a \vee b) &= f(a) \vee f(b) & f(a \wedge b) &= f(a) \wedge f(b) \\
f(0) &= 0 & f(1) &= 1
\end{aligned}
$$

$a \wedge (a \Rightarrow b) = a \wedge b$	$a \Rightarrow a = \top$
$b \wedge (a \Rightarrow b) = b$	$a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$

Table 2.3.: The axioms of a Heyting algebra.

We are now ready to introduce Heyting algebras, also known as *pseudo-boolean algebras*. The following definition is taken from Johnstone (1982, p. 8), cf. Balbes and Dwinger (1974, p. 177). The notion originates from Heyting (1930), we refer to Birkhoff (1948, Sections 9.12 and 12.7) for more context. Many textbooks contain information on Heyting algebras, let us but mention Rasiowa and Sikorski (1963, Chapter 4), Rybakov (1997, Section 2.1), Chagrov and Zakharyaschev (1997, Chapter 7), and Sørensen and Urzyczyn (2006, Section 2.4).

2.59 Definition (Heyting Algebra)
A tuple $\mathfrak{A} = \langle \mathfrak{A}, \vee, \wedge, \Rightarrow, 0, 1 \rangle$ is said to be a *Heyting algebra* whenever it satisfies the equations of Table 2.3 for all $a, b, c \in \mathfrak{A}$ and the tuple $\langle \mathfrak{A}, \vee, \wedge, 0, 1 \rangle$ is a *bounded lattice*.

2.60 Lemma (Johnstone, 1982, Section 1.11)
Every Heyting algebra satisfies the following equation, for all $a, b, c \in \mathfrak{A}$.[15]

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \tag{2.11}$$

2.61 Example
Let us revisit Example 2.57, and again consider the bounded distributive lattices of all upsets $\mathsf{ups}(P)$ within a given Kripke frame P. Consider the operation $\Rightarrow$ on $\uparrow P$ defined as:

$$U \Rightarrow V := \{p \in P \mid \text{ for all } q \geq p \text{, if } q \in U \text{ then } q \in V\}.$$

One can readily verify that the thus resulting structure $\langle \mathsf{ups}(P), \cup, \cap, \Rightarrow, \varnothing, P \rangle$ is a Heyting algebra. The above might seem to suggest we had a choice in defining $\Rightarrow$, but this is most certainly not the case; $\Rightarrow$, whenever it can be defined, is uniquely specified by the bounded distributive lattice structure at hand. In fact, $a \Rightarrow b$ equals the least c such that $a \wedge c \leq b$.

[15] Any *bounded lattice* that satisfies this equation is said to be *distributive*. We refer to Birkhoff (1935, Section 13), Birkhoff (1948, Chapter 9), and Balbes and Dwinger (1974) for more details

2.62 Definition (Map of Heyting Algebras)
Let $\mathfrak{A}$ and $\mathfrak{B}$ be Heyting algebras. A function $f : \mathfrak{A} \to \mathfrak{B}$ is said to be a *map of Heyting algebras* if it is a map of bounded lattices, and in addition, it satisfies the following equation for all $a, b \in \mathfrak{A}$.

$$f(a \Rightarrow b) = f(a) \Rightarrow f(b) \tag{2.12}$$

We denote the category of Heyting algebras and their maps by $\mathbf{HA}$.

2.63 Example
Given a map $f : P \to Q$ of Kripke frames, one can construct a map between the corresponding Heyting algebras $\mathsf{ups}(Q) \to \mathsf{ups}(P)$. One can thus construe the mapping $\mathsf{ups}(-)$ as a functor $\mathsf{ups}(-) : \mathbf{KF} \to \mathbf{HA}^{\mathrm{op}}$ by defining:

$$\mathsf{ups}(f : P \to Q) := \mathsf{ups}(f) : \mathsf{ups}(Q) \to \mathsf{ups}(P)\,, \quad U \mapsto f^{-1}(U).$$

It is not hard to verify that all required equations holds. Crucially, (2.12) holds because f is a map of Kripke frames, rather than a map of posets.

There exists an evident functor $\mathsf{U} : \mathbf{HA} \to \mathbf{Set}$ which maps a Heyting algebra to its underlying set. One can wonder whether there exists a functor $\mathsf{F} : \mathbf{Set} \to \mathbf{HA}$ that is left-adjoint to U. Such a functor would satisfy the following, for all sets X and Heyting algebras $\mathfrak{A}$.

$$\mathbf{HA}(\mathsf{F}(X), \mathfrak{A}) \cong \mathbf{Set}(X, \mathsf{U}) \tag{2.13}$$

More generally speaking, a functor $\mathsf{U} : \mathcal{C} \to \mathbf{Set}$ can be defined for any variety $\mathcal{C}$. It is well-known that this functor has a left adjoint, see for instance Grätzer (1979, Corollary 25.5), Cohn (1965, Corollary 5.2), and Burris and Sankappanavar (2012, Theorem 10.2).

Let us return to the specific case of Heyting algebras. In this context, the algebra $\mathsf{F}(X)$ is referred to as the *free Heyting algebra generated by* X. In Section 3.2, we describe the structure of $\mathsf{F}(X)$ for finite sets X. This description is given by means of a particular Kripke model, known as the *universal model*.

The free Heyting algebra generated by X can also be constructed by more concrete means. Indeed, consider the set of all propositional formulae in variables X, that is, the set $\mathcal{L}(X)$. The Heyting algebra $\mathsf{F}(X)$ can be considered as the algebra whose underlying set is $\mathcal{L}(X)$ modulo provable equivalence, and where the operations are interpreted as the obvious corresponding connectives. It is not hard to verify that the resulting indeed satisfies all the necessary equations to be a proper Heyting algebra, and that it in fact satisfies the defining equivalence of (2.13).

Given a formula $\phi \in \mathcal{L}(X)$, there is an obvious interpretation of this element in the free Heyting algebra generated by X. Indeed, one can interpret ϕ as a representative of one of the elements of $\mathsf{F}(X)$.

2.64 Definition (Semantics)
Let $\mathfrak{A}$ be a Heyting algebra, and let X be a set of variables. A *valuation on* X is a map of Heyting algebras $f : \mathsf{F}(X) \to \mathfrak{A}$. We say that a formula $\phi \in \mathcal{L}(X)$ is *valid with respect to* f whenever $f(\phi) = 1$. A formula ϕ is *valid on* $\mathfrak{A}$ if it is valid for all valuations $f : \mathsf{F}(X) \to \mathfrak{A}$.

2.65 Lemma
Let $v : P \to \mathcal{P}(X)$ be a Kripke model. The map defined by:

$$[\![-]\!]_v : \mathsf{F}(X) \to \mathsf{ups}(P)$$

is a valuation.

Proof. We know that $[\![x]\!]_v$ is an upset by Lemma 2.21. One needs but verify that the equalities below are satisfied. Their validity is immediate from Definitions 2.15 and 2.20.

$$\begin{aligned}
[\![\top]\!]_v &= P && = 1\\
[\![\bot]\!]_v &= \varnothing && = 0\\
[\![\phi \wedge \psi]\!]_v &= \{p \in P \mid p \in [\![\phi]\!]_v \text{ and } p \in [\![\psi]\!]_v\} && = [\![\phi]\!]_v \wedge [\![\psi]\!]_v\\
[\![\phi \vee \psi]\!]_v &= \{p \in P \mid p \in [\![\phi]\!]_v \text{ or } p \in [\![\psi]\!]_v\} && = [\![\phi]\!]_v \vee [\![\psi]\!]_v\\
[\![\phi \wedge \psi]\!]_v &= \{p \in P \mid \text{for all } q \geq p,\ q \in [\![\phi]\!]_v \text{ implies } q \in [\![\psi]\!]_v\} && = [\![\phi]\!]_v \Rightarrow [\![\psi]\!]_v
\end{aligned}$$ □

Heyting algebras are sound and complete with respect to IPC, see for instance Rybakov (1997, Theorem 2.1.10), Chagrov and Zakharyaschev (1997, Theorem 7.21) and Sørensen and Urzyczyn (2006, Theorem 2.4.7). This can be generalised to arbitrary intermediate logics, see Chagrov and Zakharyaschev (1997, Theorem 7.73).

2.66 Theorem (Sound and Completeness)
Let Λ be an intermediate logic. The variety $\mathcal{V}_\Lambda$ of all Heyting algebras on which the theorems of Λ are valid is sound and complete with respect to Λ. That is to say, for any formula $\phi \in \mathcal{L}(X)$, one has $\phi \in \Lambda$ if and only if ϕ is valid on all algebras of $\mathcal{V}_\Lambda$. The variety $\mathcal{V}_{\mathsf{IPC}}$ is the variety of all Heyting algebras.

Note that, as explained above, each variety can be endowed with a free algebra functor. When Λ is an intermediate logic, we denote this functor by $\mathsf{F}_\Lambda(-) : \mathbf{Set} \to \mathcal{V}_\Lambda$.

Above, we described how to translate a Kripke frame P into a Heyting algebra $\mathsf{ups}(P)$. Conversely, one may translate a Heyting algebra into a Kripke frame, which we describe below. To this end, we give the following definition. This definition is standard, and can be found in many text books, see Johnstone (1982, Section 1.2) and Chagrov and Zakharyaschev (1997, Section 7.4).

2.67 Definition (Filter)
Let $\mathfrak{A}$ be a Heyting algebra. A subset $F \subseteq \mathfrak{A}$ is said to be a *filter* whenever it satisfies the following two conditions:

(i) if $a, b \in F$ then $a \wedge b \in F$;

(ii) if $a \in F$, $b \in \mathfrak{A}$, and $a \leq b$ then $b \in F$.

A filter F is said to be *prime* whenever we have $a \in F$ or $b \in F$ holds precisely if $a \vee b \in F$ for all $a, b \in \mathfrak{A}$. We denote the poset of all prime filters in $\mathfrak{A}$ by $\mathsf{spec}(\mathfrak{A})$.

In analogy with Example 2.63, one can construe the mapping $\mathsf{spec}(-)$ as a functor $\mathsf{spec}(-) : \mathbf{HA} \to \mathbf{KF}^{\mathrm{op}}$, when defining:

$$\mathsf{spec}(f : \mathfrak{A} \to \mathfrak{B}) := \mathsf{spec}(f) : \mathsf{spec}(\mathfrak{B}) \to \mathsf{spec}(\mathfrak{A}), \quad F \mapsto f^{-1}(F).$$

The above described is a well-defined map of Kripke frames.[16] Indeed, it is clear that the above maps prime filters of $\mathfrak{B}$ to prime filters of $\mathfrak{A}$ in a monotonic manner. Suppose that $\mathsf{spec}(f)(\mathfrak{p}) \subseteq \mathfrak{q}$ for $\mathfrak{p} \in \mathsf{spec}(\mathfrak{B})$ and $\mathfrak{q} \in \mathsf{spec}(\mathfrak{A})$. We claim that $f(\mathfrak{q})$ is a prime filter extending $\mathfrak{p}$, satisfying $\mathsf{spec}(f)(f(\mathfrak{q})) = \mathfrak{q}$. All of this is clear when we show that latter equality, so let us focus on this. Naturally, $\mathfrak{q} \subseteq f^{-1}\big(f(\mathfrak{q})\big)$, hence only the converse requires justification. Let $b \in f^{-1}\big(f(\mathfrak{q})\big)$ be arbitrary, and note that there must be some $a \in \mathfrak{q}$ such that $f(a) = f(b)$. We derive that:

$$f(a \Rightarrow b) = f(a) \Rightarrow (b) = 1,$$

hence $a \Rightarrow b \in \mathfrak{q}$. This yields $a \in \mathfrak{q}$, proving the desired.

2.68 Example (Canonical Model)
Consider the free Heyting algebra $\mathsf{F}(X)$ generated by a set of variables X. Through the above, this gives rise to a Kripke frame $\mathsf{spec}(\mathsf{F}(X))$. There is an obvious valuation $\mathsf{c} : \mathsf{spec}(\mathsf{F}(X)) \to \mathcal{P}(X)$ defined by the equivalence $x \in \mathsf{c}(\mathfrak{p})$ if and only if $x \in \mathfrak{p}$ for all $x \in X$ and $\mathfrak{p} \in \mathsf{spec}(\mathsf{F}(X))$. We denote $\mathsf{spec}(\mathsf{F}(X))$ by $\mathsf{C}(X)$, and call $\mathsf{c} : \mathsf{C}(X) \to \mathcal{P}(X)$ the *canonical model*. In Definition 4.48, we give a specification of the canonical model that is both more concrete and more general.

[16] An argument to this end can be reconstructed from Chagrov and Zakharyaschev (1997, Theorem 8.59).

The functors $\mathsf{spec}\,(-) : \mathbf{HA} \to \mathbf{KF}^{\mathrm{op}}$ and $\mathsf{ups}(-) : \mathbf{KF} \to \mathbf{HA}^{\mathrm{op}}$ do not give rise to an equivalence of categories. When one enriches the objects of category $\mathbf{KF}$ with a topology compatible with the order present in the Kripke frames, such a duality is possible. We refer to Esakia (1974) and Priestley (1972) for a description of this, more details can also be found in Bezhanishvili (2006, Section 2.3), Chagrov and Zakharyaschev (1997, Section 8.4) and Gehrke (2014).

Below, we treat the finite case, in which an equivalence does hold. Results of this type have been known for a long time, going back to de Jongh and Troelstra (1966), cf. Chagrov and Zakharyaschev (1997, Theorem 7.30).

2.69 **Theorem (de Jongh and Troelstra, 1966)**
When restricting the maps $\mathsf{spec}\,(-) : \mathbf{HA} \to \mathbf{KF}^{\mathrm{op}}$ and $\mathsf{ups}(-) : \mathbf{KF} \to \mathbf{HA}^{\mathrm{op}}$ to finite Heyting algebras and finite Kripke models, respectively, one obtains an equivalence of categories.

2.4. Some intermediate logics

The purpose of this section is to give an overview of the intermediate logics we treat in this thesis. First, we give an exhaustive list of the intermediate logics we consider. We go over each of the logics in Table 2.4, and describe them semantically. Some references to their origins are be given, but no effort is made to be exhaustive. Second, we discuss some properties one typically considers in the study of intermediate logics. Although much of this information is available elsewhere, we do consider it appropriate to include it here, mainly because of the convenience it induces when referencing these logics and their properties later on.

Intermediate logics were first studied by Gödel (1932), who introduced the logics G_n. Systematic studies were performed by, among others, Umezawa (1959a,b), Kuznetsov (1974) and Zakharyaschev (1989). Jankov (1968) proved that there are uncountably many intermediate logics. Many of the logics we consider in this section are finitely axiomatizable, yet the aforementioned result ensures that not all intermediate logics are of this type.

2.4.1. Descriptions

In this subsection, we describe several intermediate logics as the sets of formulae valid on particular classes of frames. Such a description always leads to an interme-

Name	Semantics	Finite Axiomatization
IPC	Definition 2.4	IPC
CPC	Corollary 2.76	$\mathsf{IPC} + (x \vee \neg x)$
BD_n	Corollary 2.75	$\mathsf{IPC} + \mathbf{bd}_n$
BW_n	Theorem 2.79	$\mathsf{IPC} + \left(\bigvee_{i=0}^{n} \left(x_i \to \bigvee_{j \neq i} x_j\right)\right)$
GSc	Corollary 2.80	$\mathsf{BD}_2 + ((p \to q) \vee (p \to q) \vee ((p \equiv \neg q))$
KC	Lemma 2.81	$\mathsf{IPC} + (\neg\neg x \vee \neg x)$
LC	Lemma 2.82	$\mathsf{IPC} + (x \to y) \vee (y \to x)$
Sm	Lemma 2.83	$\mathsf{LC} + \mathsf{BD}_2$
G_{n+1}	—	$\mathsf{LC} + \mathsf{BD}_n$
KP	—	$\mathsf{IPC} + ((\neg z \to x \vee y) \to (\neg z \to x) \vee (\neg z \to y))$
ND_n	Corollary 2.86	$\mathsf{IPC} + ((\neg z \to \bigvee_{i=1}^{n} \neg x_i) \to \bigvee_{i=1}^{n} (\neg z \to \neg x_i))$
BB_n	Lemma 2.89	$\mathsf{IPC} + \left(\bigwedge_{i=1}^{n+2} \left(\left(x_i \to \bigvee_{j \neq i} x_j\right) \to \bigvee_{j \neq i} x_j\right) \to \bigvee_{i=1}^{n+2} x_i\right)$
ML	Definition 2.90	does not exist

Table 2.4.: List of important intermediate logics.

diate logic, as justified by the following Lemma 2.70.

2.70 Lemma
Let $\mathcal{K}$ be a class of Kripke frames. The intersection of all $\mathrm{Th}(P)$ for $P \in \mathcal{K}$ forms an intermediate logic.

Proof. The only difficulty is in proving that the resulting set of formulae is closed under substitutions, which follows from Lemma 2.28. □

The first intermediate logics we consider, are the so-called superintuitionistic logics of bounded depth, denoted BD_n. This sequence contains many interesting logics. Indeed, the logic BD_1 is also known as CPC, the classical propositional calculus, as already defined in Example 2.6.

The logic BD_2 is also quite interesting; it was among the first intermediate logics to be studied. It was introduced by Jankov (1963a) under the name M. He proved this logic to be complete with respect to a particular class of Heyting algebras.[17] McKay (1967c) proved that, when restricting attention to those formulae that are built up out of only variables and implications, BD_2 has the same theorems as IPC.

More generally, when Hosoi (1967a) introduced the *finite slices*, he considered the logics BD_n. In this paper, BD_2 appeared in the guise of LP_2, the weakest intermediate logic of the second slice.[18] Finally, we note that BD_2 also appears as one of the seven intermediate logics with interpolation, as proven by Maksimova (1979).[19]

2.71 Definition (Logic of Bounded Depth)
We recursively define the formula $\mathbf{bd}_n \in \mathcal{L}(x_1, \dots, x_n)$ as:

$$\begin{aligned} \mathbf{bd}_0 &\coloneqq \bot, \\ \mathbf{bd}_{n+1} &\coloneqq x_{n+1} \vee (x_{n+1} \to \mathbf{bd}_n). \end{aligned}$$

The logic BD_n is defined as $\mathsf{BD}_n \coloneqq \mathsf{IPC} + \mathbf{bd}_n$.

One can describe the logics BD_n semantically as those intermediate logics that are complete with respect to finite Kripke models of height at most n, as for instance proven by Maksimova (1972, Assertion 4.1). We first define precisely what we mean by height. It is easy to see that any frame of height at most n is a frame of BD_n.

[17] See Rose (1970) for a more detailed description of the results by Jankov (1963a).

[18] See Hosoi (1967a, Corollary 4.7) for this statement. Hosoi and Ono (1970) fully described the second slice. This description shows, in particular, that BD_2 is among the only three *pretabular intermediate logics* as proven by Maksimova (1972).

[19] We discuss interpolation in Section 4.1.2, and give an overview of all intermediate logics with interpolation in Fig. 4.2.

2.72 Definition (Chains and Height)
Let P be a partial order. A subset $W \subseteq P$ is said to be a *chain* if for all $w_1, w_2 \in W$ we have $w_1 \leq w_2$ or $w_2 \leq w_1$. We say that P is of *height at most* n if each chain $W \subseteq P$ satisfies $|W| \leq n$.

In Lemma 2.73 below, we show how any Kripke frame that satisfies a certain semantic property must also be a frame of the logic BD_n. This proof is fairly standard. We give such an argument once to illustrate the technique, but we omit such arguments in the following.

2.73 Lemma
Let P be any Kripke frame. If P is of *height at most* n, then $P \Vdash \mathsf{BD}_n$.

Proof. By induction along n. In the base case, note that any singleton set is automatically a chain. This means that P can only be of height at most 0 if it is empty, whence the desired is immediate.

In the inductive case, we suppose that there is some model $v : P \to \mathcal{P}(Y)$ of height at most $n + 1$, some substitution $\sigma : \mathcal{L}(x_0, \dots, x_{n+1}) \to \mathcal{L}(Y)$ and some $p \in P$ such that:

$$v, p \nVdash \sigma(\mathbf{bd}_{n+1}) = \sigma(x_{n+1}) \vee (\sigma(x_{n+1}) \to \sigma(\mathbf{bd}_n)).$$

This immediately shows that $v, p \nVdash \sigma(x_{n+1})$, and it yields a $q \geq p$ such that $v, q \Vdash \sigma(x_{n+1})$ and $v, q \nVdash \sigma(\mathbf{bd}_n)$. From this, it follows $p < q$, hence $\uparrow q$ is of height at most n. By induction, we know that $u \restriction (\uparrow q) \Vdash \mathsf{BD}_n$, a contradiction with $u, q \nVdash \sigma(\mathbf{bd}_n)$. This proves the desired. □

The converse to Lemma 2.73 would be to prove that any Kripke frame that satisfies BD_n must also be of height at most n. We show something ostensibly stronger, namely that if a refined model satisfies BD_n, then this is already enough to ensure that the semantic property holds. That the latter type of argument is at least as strong as the former is immediate from Lemma 2.37. Later, in Corollary 3.13, we show that if the frame at hand is finite, there is no difference in strength at all.

2.74 Lemma
Let $v : P \to \mathcal{P}(X)$ be a refined model. If v is a model of BD_n, then P is of *height at most* n.

Proof. Suppose that there exists a chain $w_n < w_{n-1} < \dots < w_0 \in W$. Through refinedness, we know of $\phi_i \in \mathcal{L}(X)$ such that $w_i \Vdash \phi_i$ but $w_{i+1} \nVdash \phi_i$ per $0 \leq i < n$. Define a substitution:

$$\sigma : \mathcal{L}(x_1, \dots, x_n) \to \mathcal{L}(X), \quad x_i \mapsto \phi_{i-1}.$$

We prove, by induction along m, that $w_m \nVdash \sigma(\mathbf{bd}_m)$. The base case is clear, because $w_0 \nVdash \bot$. Now suppose $w_m \nVdash \sigma(\mathbf{bd}_m)$ and:

$$w_{m+1} \Vdash \sigma(\mathbf{bd}_{m+1}) = \sigma(x_{m+1} \vee (x_{m+1} \to \mathbf{bd}_m)) = \phi_m \vee (\phi_m \to \sigma(\mathbf{bd}_m)) .$$

As a consequence, at least one of $w_{m+1} \Vdash \phi_m$ and $w_{m+1} \Vdash \phi_m \to \sigma(\mathbf{bd}_m)$ must hold. The former case contradicts the choice of ϕ_m. In the latter case, because $w_m \Vdash \phi_m$, we know $w_m \Vdash \sigma(\mathbf{bd}_m)$, which is false by induction. This proves the desired. □

2.75 **Corollary**
The logic BD_n is the logic of finite frames of *height at most* n.

Proof. This is immediate from Lemmas 2.37, 2.73 and 2.74. □

Recall that, given a frame P, we write $\mathrm{Th}(P)$ for the theory of this frame. In the following Corollary 2.76, we make use of this notation, writing P in situ. The Kripke frame we describe there is the sole one-point frame. This type of notation is also used used in Lemma 2.83 and Corollary 2.80.

2.76 **Corollary**
The classical propositional calculus satisfies $\mathsf{CPC} = \mathrm{Th}(\bullet)$.

Proof. This follows directly from Corollary 2.75, as $\mathsf{CPC} = \mathsf{BD}_1$. □

The following Lemma 2.77 shows that a single-conclusion rule is *derivable* in CPC whenever it is *admissible* in some intermediate logic. This observation is folklore, see for instance Iemhoff (2005, Fact 3.12).

2.77 **Lemma**
Let Λ be an intermediate logic, and let $\Gamma \subseteq \mathcal{L}(X)$ and $\chi \in \mathcal{L}(X)$ be given. If $\Gamma \mathrel{|\!\sim}_\Lambda \chi$ then $\Gamma \vdash_{\mathsf{CPC}} \chi$.

Proof. Suppose that Γ/χ is an *admissible rule* of Λ. By Corollary 2.76, we know that $\Gamma \vdash_{\mathsf{CPC}} \chi$ iff $P \Vdash \bigwedge\Gamma \to \chi$, where P is the one-point poset. Consider an arbitrary valuation $v : P \to \mathcal{P}(X)$ on the Kripke frame P, and define:

$$\sigma : \mathcal{L}(X) \to \mathcal{L}(\varnothing), \quad x \mapsto \begin{cases} \top & \text{if } v \Vdash x; \\ \bot & \text{otherwise.} \end{cases}$$

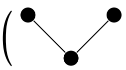
It is easy to verify that $v \Vdash \psi$ if and only if $\vdash_\Lambda \sigma(\psi)$ for each $\psi \in \mathcal{L}(X)$. Suppose that $v \Vdash \bigwedge\Gamma$. This entails $\vdash_\Lambda \sigma(\phi)$ for all $\phi \in \Gamma$, hence $\vdash_\Lambda \sigma(\chi)$ must hold. We thus derive $v \Vdash \chi$, proving $v \Vdash \bigwedge\Gamma \to \chi$. This yields $\Gamma \vdash_{\mathsf{CPC}} \chi$, as desired. □

Next to the logics of bounded height, one can consider the logics of bounded width. Note that a rooted frame is of width at most n precisely when each *anti-chain* within said frame is of size at most n. Our definition of "width" differs slightly from that of Chagrov and Zakharyaschev (1997, p. 43), in order to ensure that "width" stands to Theorem 2.79 in the same way as "depth" stands to both Lemmas 2.73 and 2.74.

2.78 Definition (Width)
A Kripke frame P is said to be of *width at most* n whenever for all $p, q_0, q_1, \ldots, q_n \in P$ satisfying $q_0, q_1, \ldots, q_n \geq p$ there exist some $i \neq j$ such that $q_i \leq q_j$.

2.79 Theorem (Chagrov and Zakharyaschev, 1997, Proposition 2.39)
A Kripke frame is of *width at most* n exactly if it satisfies BW_n.

One can combine the logics BD_2 and BW_2 to yield the intermediate logic GSc. The name comes from Avellone et al. (1999, Section 6), based on the observation of Ferrari and Miglioli (1993, p. 1371) that it is the largest *semiconstructive intermediate logic*.[20]

2.80 Corollary
The intermediate logic GSc satisfies $\mathsf{GSc} = \mathrm{Th}\Big($  $\Big)$.

The logic KC, introduced by Jankov (1963a, p. 1103) as "the calculus of the weak law of excluded middle'," is also known as Jankov's logic and de Morgan logic.

2.81 Lemma
Any *rooted frame* with a single maximal point satisfies KC. Moreover, if a rooted, refined model is a model of KC, then it has only one maximal point.

Proof. Both statements can be proven with similar ease, we focus on the latter. Let $v : P \to \mathcal{P}(X)$ be a refined and rooted model, and suppose that there exists $p, q \in$

[20]Ferrari and Miglioli (1993, Section 3) say that an intermediate logic Λ is *semiconstructive* whenever $\vdash_\Lambda \phi \vee \psi$ implies $\vdash_{\mathsf{CPC}} \phi$ or $\vdash_{\mathsf{CPC}} \psi$ for all formulae ϕ and ψ. Through *Glivenko's theorem*,[21] we know $\vdash_{\mathsf{CPC}} \phi$ to hold precisely if $\vdash_\Lambda \neg\neg\phi$. Consequently, a logic is semiconstructive precisely if it admits the rule:

$$x \vee y / \{\neg\neg x, \neg\neg y\}\,. \qquad (\mathsf{DP}_2^{\neg\neg})$$

We consider this rule in detail in Section 3.5.1.

[21]This theorem was first formulated by Glivenko (1929, p. 185) as:

> "Si une certaine expression de la logique de propositions est démonstrable dans la logique classique, c'est la fausseté de la fausseté de cette expression qui est démonstrable dans la logique brouwerienne."

Its formalised version states that $\vdash_{\mathsf{CPC}} \phi$ implies $\vdash_{\mathsf{IPC}} \neg\neg\phi$. Naturally, the implication in the other direction holds as well.

$\mathsf{max}(P)$ with $p \neq q$. Without loss of generality we can pick a formula $\phi \in \mathcal{L}(X)$ such that $v, p \Vdash \phi$ yet $v, q \nVdash \phi$. Now consider the root of P, which we denote by k. Observe that $v, k \nVdash \neg\neg\phi \vee \neg\phi$, a direct contradiction with the assumption that $v \Vdash \mathsf{KC}$. ☐

Dummett (1959) introduced the "linear calculus", the intermediate logic LC. When combining LC and BD_n, one obtains the logics G_n, as already studied by Gödel (1932) and axiomatised by Hosoi (1966a,b). For a hint at a proof of the following, see Chagrov and Zakharyaschev (1997, Example 4.15).

2.82 Lemma
A frame satisfies LC if it is is a chain. Moreover, if a refined model satisfies LC, then it is a chain.

We now consider Sm, known as Smetanič's logic. It first appeared in Smetanič (1960), and was given a semantic description in Jankov (1963a). This logic is known to be the greatest intermediate logic that is not equal to CPC.[22]

2.83 Lemma
The logic Sm satisfies $\mathsf{Sm} = \mathrm{Th}\Big($  $\Big)$.

Kreisel and Putnam (1957) introduced the intermediate logic KP, in order to show that there exist other intermediate logics with the disjunction property besides IPC. A semantic description of this logic was later given by Gabbay (1970). We do not include this description here, as it is not relevant to the remainder of this thesis.

The logics ND_n were introduced by Maksimova (1986, Lemma 7), as a means of proving the non-finite axiomatizability of the logic:

$$\mathsf{ND}_\omega := \mathsf{IPC} + \bigcup_{n\in\mathbb{N}} \mathsf{ND}_n. \tag{2.14}$$

The semantic description of ND_n is much more intricate than the types of descriptions we have seen so far. We first give the definition of a semantic property, and then prove that is characterises frames of ND_n in Lemma 2.85. We do include the full proof, as it is not that straightforward. Moreover, we later employ Lemma 2.85 in proving Lemma 7.15, a crucial step towards Theorem 7.41, in which we present a *refutation system* for Medvedev's logic.

[22]See, for instance, Wolter and Zakharyaschev (2007, p. 479).

2.84 Definition
Let P be a Kripke frame. We say that P is of *divergence* n if for all $p \in P$ and all $W \subseteq \mathsf{max}(P)$ satisfying $|W| \leq n$ and $W \subseteq \uparrow p$, there exists a $q \geq p$ such that $\mathsf{max}(\uparrow q) = W$.

2.85 Lemma
Let P be an image-finite Kripke frame. The following are equivalent:

(i) P is of *divergence* n;

(ii) $P \Vdash \mathsf{ND}_n$.

Proof. Suppose that (i) holds, and let $v : P \to \mathcal{P}(X)$ be a valuation such that $v \nVdash \mathsf{ND}_n$. As a consequence, we know of formulae $\Delta, \phi \in \mathcal{L}(X)$ with $|\Delta| \leq n$ and a point $p \in P$ such that

$$v, p \Vdash \neg\phi \to \bigvee_{\chi \in \Delta} \neg\chi \text{ yet } v, p \nVdash \bigvee_{\chi \in \Delta} \neg\phi \to \neg\chi. \tag{2.15}$$

The latter ensures the existence of $q_\chi \geq p$ such that $v, q_\chi \Vdash \neg\phi$ and $v, q_\chi \nVdash \neg\chi$ per $\chi \in \Delta$. Given $\chi \in \Delta$, we see that there must be a $w_\chi \in \mathsf{max}(\uparrow q_\chi)$ with $v, w_\chi \Vdash \chi$. Define $W := \{w_\chi \mid \chi \in \Delta\}$ and note that $W \subseteq \mathsf{max}(p)$ and $|w| \leq n$.

It is clear that $W \subseteq \max \uparrow p$, and so, by assumption, we know of a $q \geq p$ such that $W = \mathsf{max}(\uparrow q)$. From here, it is clear that $v, q \Vdash \neg\phi$, so from (2.15) we learn that $v, q \Vdash \bigvee_{\chi \in \Delta} \neg\chi$. This gives us some $\chi \in \Delta$ such that $v, q \Vdash \neg\chi$. As a consequence $v, w_\chi \Vdash \neg\chi$, which is blatantly false, proving (ii).

To prove the converse, suppose that (ii) does hold, yet (i) does not. The latter yields some $W \subseteq \mathsf{max}(P)$ with $|W| \leq n$ and a $p \in P$ with $W \subseteq \uparrow p$ such that, for all $q \geq p$, we have $\mathsf{max}(\uparrow q) \neq W$. As a consequence, we know that for all $q \geq p$ that whenever $q \nleq k$ for all $k \in \max p - W$ then $W \nsubseteq \max \uparrow q$. Define a model $v : P \to \mathcal{P}(\{z, x_1, \ldots, x_n\})$ as below, in the understanding that we enumerate $W = \{w_1, \ldots, w_n\}$.

$$\begin{aligned} v, k \Vdash z \quad &\text{iff} \quad k \in \mathsf{max}(p) - W \text{ for all } k \in P \\ v, k \Vdash x_i \quad &\text{iff} \quad k \leq w_i \text{ for all } k \in P \text{ and } i = 1, \ldots, n \end{aligned}$$

Suppose that $q \geq p$ and $v, q \Vdash \neg z$. This means that $q \nleq k$ for all $k \in \mathsf{max}(p) - W$. As a consequence, we know that $W \nsubseteq \mathsf{max}(\uparrow q)$. This yields some $i = 1, \ldots, n$ such that $v, q \Vdash \neg w_i$. We can thus conclude that $v, p \nVdash \neg z \to \bigvee_{i=1}^n w_i$.

By (ii), it now follows that $v, p \nVdash \bigvee_{i=1}^n (\neg z \to \neg w_i)$. This results in an $i = 1, \ldots, n$ such that $v, p \nVdash \neg z \to \neg w_i$. Yet $p \leq w_i$ and $w_i \Vdash \neg z$, hence $w_i \Vdash x_i$, *quod non.* This proves that (i) holds, as desired. $\square$

2.86 Corollary
The logic ND_n is the intermediate logic of finite frames of divergence n.

Proof. Immediate from the above Lemma 2.85 and Rybakov (1978, Corollary 7). □

We now turn to describe the logics BB_n, the logics of bounded branching. The logic BB_{n+1} is also known as the n^{th} *Gabbay–de Jongh logic*, as originally introduced by Gabbay and de Jongh (1969, 1974).[23]

For convenience we write BB_ω for the logic IPC. Intuitively, this makes sense, as IPC allows arbitrarily high degrees of branching.

2.87 Definition (Branching)
Let P be a Kripke frame. We say that P is of *branching degree at most* n whenever for each *anti-chain* $W \subseteq P$ and $p \in P$ with $W \ \kappa \ p$ we have $|W| \leq n$.

Proof of the following can be found in Gabbay and de Jongh (1974, p. 19) and Chagrov and Zakharyaschev (1997, Proposition 2.41).

2.88 Lemma
Every finite Kripke frame of BB_n is of *branching degree at most* n, for all $n \in \mathbb{N}$.

2.89 Lemma
The intermediate logic BB_n is sound and complete with respect to finite Kripke frames of branching degree at most n.

The final intermediate logic we consider is known as the logic of finite problems, or Medvedev's logic, introduced by Medvedev (1962). This intermediate logic is different from all the others above, in that it cannot be given a *finite axiomatisation*, as proven by Maksimova et al. (1979).

2.90 Definition (Medvedev Frames and Medvedev's Logic)
Let X be a set. The *Medvedev frame*, denoted $\mathsf{B}(X)$, is defined as the partial order with the underlying set $\mathcal{P}(X) - \varnothing$, ordered by $\supseteq$. *Medvedev's logic* ML is the intermediate logic defined by:

$$\mathsf{ML} := \bigcap \mathrm{Th}\Big(\big\{ \mathsf{B}\big(\{0, 1, \ldots, n\}\big) \bigm| n \in \mathbb{N} \big\} \Big).$$

[23] These logics were introduced as an example of an infinite series of decidable, *finitely axiomatizable* intermediate logics with the *disjunction property*. The results first appeared in a technical rapport, Gabbay and de Jongh (1969), which already garnered some attention, as attested by Segerberg (1973). Eventually, these results were published in Gabbay and de Jongh (1974).

2.4.2. Properties

There are numerous properties of intermediate logics one could study. The following is certainly not an exhaustive analysis of all such properties. We merely mention those properties that are of direct use in this thesis. Virtually no proofs are given, as all of this is well-established within the literature.

The first property we consider is that of tabularity. Tabular logics are among the strongest intermediate logics, as their full behaviour can be described by a single, finite frame. Note that this frame need not be rooted, although the examples we give all fit that mould. We refer to Chagrov and Zakharyaschev (1997, Chapter 12) for more details on this topic.

2.91 Definition (Tabular)
An intermediate logic Λ is said to be *tabular* if there is a finite Kripke frame P such $\mathrm{Th}(P) = \Lambda$.

Some of the intermediate logics under consideration are tabular. The lemma below collates the results of Corollaries 2.76 and 2.80 and Lemma 2.83.

2.92 Lemma
The intermediate logics CPC, Sm and GSc are tabular.

All tabular logics can be finitely axiomatised, as has been proven by Hosoi (1967b, Theorem 3.8) and McKay (1967b, Theorem 3), cf. Chagrov and Zakharyaschev (1997, Theorem 12.4).

2.93 Theorem
Every tabular logic is finitely axiomatizable.

The finite model property has been playing a central role in the study of intermediate logics for quite some time. It is well-known, see Harrop (1958) and Urquhart (1981), that there is a connection between decidability and the finite model property. In Definition 2.94 below, we consider finite *models*, whereas we might also have considered finite *frames*. As already noted by Hansson and Gärdenfors (1975), these two notions coincide.[24]

2.94 Definition (Finite Model Property)
An intermediate logic Λ is said to have the *finite model property* (FMP for short) if for each formula $\phi \in \mathcal{L}(X)$ such that $\nvdash_\Lambda \phi$ there is a finite model $v : P \to \mathcal{P}(X)$ such that $v \Vdash \Lambda$ yet $v \nVdash \phi$.

[24] A proof to this end can also be reconstructed by combining Corollary 3.13 and Theorem 2.51.

There exist intermediate logics without the finite model property, a first such example was given by Jankov (1968, p. 807).[25] Komori and Furuya (1975, Theorem 2.2) proved that all intermediate logics of the *finite slice* enjoy the *finite model property*.[26] The finite axiomatisability of a logic does not guarantee it to enjoy the finite model property, as has been proven by Kuznetsov and Gerchiu (1970, Theorem 4).

Now, let us introduce two classes of intermediate logics that are guaranteed to enjoy the finite model property. The first class under consideration is that of the *subframe logics*, defined formally in Definition 2.95. This type of logic goes back to Fine (1985), who introduced a modal counterpart of this. These logics were subsequently studied by Zakharyaschev (1992, 1996). Much work has been done in this direction, we but mention Bezhanishvili and Bezhanishvili (2009), Bezhanishvili and Ghilardi (2007), and Yang (2008). There are many equivalent definitions one could take, the formulation below is adapted from Zakharyaschev (1996, Theorem 5.1).

2.95 Definition (Subframe Logic)
An intermedia logic Λ is said to be a *subframe logic* when for each Kripke frame P with $P \Vdash \Lambda$ and each arbitrary subset $W \subseteq P$ one has $P \restriction W \Vdash \Lambda$.

Subframe logics are by no means scarce. In fact, Zakharyaschev (1996, Theorem 3.3) proved that there are continuum many. Let us give some concrete examples below.

2.96 Lemma
The intermediate logics IPC, CPC, BD_n, BW_n, LC and G_n are subframe logics.

Proof. Immediate from Chagrov and Zakharyaschev (1997, Table 9.7). □

2.97 Theorem (Zakharyaschev, 1996, Theorem 4.1)
Every *subframe logic* has the *finite model property*.

We now turn to another particularly interesting class of intermediate logics; the so-called *stable logics*. These logics were introduced by Bezhanishvili and Bezhanishvili (2013). Our definition below follows their Definition 6.6.

2.98 Definition (Stable Logic)
An intermedia logic Λ is said to be a *stable logic* when for each Kripke frame P with $P \Vdash \Lambda$ and each arbitrary map of posets $f : P \to Q$ one has $f(P) \Vdash \Lambda$.

[25] Jankov's example, in particular, disproved Theorem 3.4 of Troelstra (1965), as also indicated in the opening paragraph of McKay (1967b). As an aside, we note that there even exist propositional logics with *no finite models at all*, as proven by McKay (1985).

[26] Clearly, this includes the intermediate logics BD_n defined above. We do not use this observation, and instead proceed via Lemma 2.96 and Theorem 2.97 below.

Stable logics, too, are continuum in number, as proved by Bezhanishvili and Bezhanishvili (2013, Theorem 6.6). We give some examples below.

2.99 Lemma (Bezhanishvili and Bezhanishvili, 2013, Theorem 7.5)
The intermediate logics IPC, CPC, BW_n, KC, and LC are stable logics.

2.100 Theorem (Bezhanishvili and Bezhanishvili, 2013, Theorem 6.8)
Every *stable logic* has the *finite model property*.

We close this section with pointers to proofs of the observation that all intermediate logics under consideration enjoy the finite model property.

2.101 Lemma
The intermediate logics IPC, CPC, BD_n, BW_n, BB_n, GSc, KC, LC, Sm, G_n, KP, ND_n and ML all enjoy the *finite model property*.

Proof. Through Lemma 2.96 and Theorem 2.97, we know that the desired needs but be proven for the logics BB_n, GSc, KC, Sm, KP, ND_n and ML. We apply Lemma 2.99 and Theorem 2.100 in order to prove that KC has the FMP. Lemma 2.92 shows the desired for GSc and Sm. Now, see that the FMP of BB_n and ND_n follows from Lemma 2.89 and Corollary 2.86 respectively. Finally, the finite model property for KP is proven by Gabbay (1970, Section 3), and the FMP holds for ML by its very definition. □

3 Semantics

Rules, like formulae, are syntactic objects. Think of an intermediate logic. With a logic in mind, there is a natural way to split the set of formulae in two. Indeed, there are those formulae that are *theorems*, and those that are not. Similarly, the set of rules is divided into those that are *admissible*, and those that are not. Given a syntactic object — be it a theorem or a rule — how does one know into which category it falls?

When discerning between theorems and non-theorems, models play a central role. As we have seen in Chapter 2, the theorems of a logic are those formulae that are valid on models of a certain special form. In this chapter, we take an analogous approach to describing the admissible rules of a logic.

We define what it means for a rule to be valid on a model. We subsequently describe, in various levels of concreteness, those models on which certain sets of rules are valid. Models of all admissible rules, and models of a particular rule are the extreme cases of this more general pattern. Later, in Chapter 4, we see cases where these two extremes coincide.

Let us give a brief overview of the structure of this chapter. Section 3.1 serves to bridge the gap between syntax and semantics. We introduce the notion of *definable upsets*, and show a tight correspondence between the *concrete* and *refined* models of Sections 2.2.2 and 2.2.3 through *order-defined* models.

In Section 3.2, we describe the so-called *universal models*. These models are such that their definable upsets correspond to the elements of the *free Heyting algebra* on a given set of generators, as we show in Theorem 3.22. In fact, we show that this holds for arbitrary intermediate logics with the *finite model property*, in the understanding that one takes the free algebra in the variety corresponding to the intermediate logic at hand. The point of this section is that it provides us with a particular model that reflects the totality of theorems on a given set of variables. Said model an is oft-used tool in the study of admissibility at large, and this thesis in particular.

After the above groundwork has been laid, we move to the semantics of rules. In Section 3.3, we define what it means for a rule to be *valid* on a Kripke model. We then introduce what we call the *exact models* of an intermediate logic, and in Theorem 3.38, we prove these to be both sound and complete with respect to the admissible rules of the logic at hand. This is the first of the above mentioned two extreme cases.

As alluded to in Section 2.2.1, covers play a central role in the study of admissibility. In Section 3.4, we provide precise criteria characterising when a given set of points must have a cover. The main characterisation is given in Theorem 3.45, and in Lemma 3.44 we show that one can *internalise* this description. That is to say, we show that the validity and non-validity of certain formulae is enough to guarantee the existence of a point covered by a given set of points. This description is used repeatedly in the subsequent section.

In Section 3.5, we turn to the second of the extreme cases: a description of those order-refined models in which a particular rule is valid. We cover several rules, and endow them with a very concrete semantic description. This description can then be applied to the universal model, leading to characterisation of the admissibility of said rules in terms of the structure of their universal models.

We close this chapter with Section 3.6, in which we treat the limits of the approach sketched in the other sections. We show that the exact models of Section 3.3 are infinite whenever the logic admits the rule D_2, making extensive use of their semantic description in Section 3.5.2. In particular, this entails that many of the admissible rules of many of the logics we consider do not enjoy the finite model property, in sharp contrast to their theorems. Said succinctly: the finite model property fails for admissible rules, even in IPC.

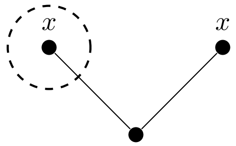


Figure 3.1.: A finite Kripke Model with an undefinable upset.

3.1. Definability

Through Theorem 2.69, we know that there is an equivalence between the categories of finite Heyting algebras and their maps, and the category of finite Kripke frames and their maps. As per this equivalence, a finite Kripke frame corresponds to the partial order of its upsets, and this partial order can be endowed with a Heyting algebra structure in a unique manner. In the following, we are concerned with a slightly different connection.

3.1 Definition (Definable Upset)
Let $v : P \to \mathcal{P}(X)$ be a model. An upset $U \subseteq \mathcal{P}(X)$ is said to be *definable* when there exists a formula $\phi \in \mathcal{L}(X)$ such that $U = [\![\phi]\!]_v$. The formula ϕ is said to be a *defining formula* of U, and is denoted by $\operatorname{def} U$.

Note that, in the above definition, $\operatorname{def} U$ does not uniquely define a formula in $\mathcal{L}(X)$. Indeed, if $\psi \in \mathcal{L}(X)$ is such that $v \Vdash \phi \equiv \psi$ then ϕ would be just as good a candidate for $\operatorname{def} U$ as ψ. We make sure to only use the notation "def −" when this difference is immaterial. Although it introduces some ambiguity, the convenience it affords us compensates this by a decent margin.

When constructing a finite Kripke model by hand, typically, all upsets are definable. Indeed, when one thinks of a Kripke model, one typically thinks of a *concrete model*. In Lemma 3.11 below, we show that all upsets in a concrete, finite model are definable. Hence, in order to find a finite example of a Kripke model with an undefinable upset, one has to consider an inconcrete model.

3.2 Example
Consider the model as depicted in Fig. 3.1. The top points are clearly *analogous*, hence the model is *not concrete*. There can be no formula ϕ defining the indicated upset, as any formula that is valid on the top left point, is also valid on the top right point.

Given a model $v : P \to \mathcal{P}(X)$, one can consider $\mathsf{defs}(v)$, the poset of its definable upsets. The underlying set of this poset is given by the following, and its elements are ordered by inclusion.

$$\mathsf{defs}(v) := \left\{ U \in \mathsf{ups}(P) \;\middle|\; U \text{ is definable} \right\}$$

It is an easy matter to verify that this partial order is a bounded distributive lattice. We claim that it is a Heyting algebra as well.

3.3 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model. The partial order $\mathsf{defs}(v)$ is a *Heyting algebra*.

Proof. This follows immediately from Lemma 2.65, coupled with the observation that $\mathsf{defs}(v)$ is the direct image of $\mathsf{F}(X)$ under the mapping $[\![-]\!]_v : \mathsf{F}(X) \to \mathsf{ups}(P)$. □

Consider a map between Kripke models $f : v \to u$. It immediately follows from Lemma 2.26 that $f^{-1}(U)$ is definable by ϕ whenever U is, for any upset U within u. The following definition is concerned with capturing this behaviour.

3.4 Definition (Definable Map)
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ Kripke models, and let $f : P \to Q$ be a *map of Kripke frames*. We say that f is a *definable map* $f : v \to u$ if:

$$f^{-1}(U) \in \mathsf{defs}(v), \text{ for all } U \in \mathsf{defs}(u).$$

The above definition could lead one to the faulty conclusion that a definable map is some special type of map of Kripke models. This is not true; definable maps are more general than maps of Kripke models, as we show in Corollary 3.6. The confusion disappears when one considers "definable map" as a primitive term on par with "map". Later, in Definition 4.58, we consider yet a different kind of map, namely *adequate maps*. These maps, in general, need not even be maps of Kripke frames, though they are defined with respect to models.

It can be helpful to think of this in terms of categories. There is a category of posets and their maps, and a distinct category of Kripke frames and their maps. Although the objects are the same, the arrows are quite different. Similarly, there is a category of Kripke models and *maps of Kripke models*. There are two other, distinct categories with exactly the same objects. The first is that of Kripke models and their *definable maps*. Finally, in Definition 4.58, we introduce the category of Kripke models and their *adequate maps*.

All maps of Kripke models are definable maps, and all definable maps are adequate maps. Maps of Kripke models and definable maps give rise to maps of Kripke frames between their underlying frame structure. Adequate maps, in general, do not.

A definable map $f : v \to u$ yields a map between the associated Heyting algebras $\mathsf{defs}(u)$ and $\mathsf{defs}(v)$. Indeed, one can define:

$$\mathsf{defs}(f) : \mathsf{defs}(u) \to \mathsf{defs}(v)\,, \quad U \mapsto f^{-1}(U).$$

It is easy to verify that the equation (2.12) is satisfied. In order to verify whether a map of Kripke frames can be construed as a definable map between chosen models, one can restrict attention to the upsets defined by variables, as illustrated below.

3.5 Lemma
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ be Kripke models, and let $f : P \to Q$ be a map of Kripke frames. Suppose that the upset $f^{-1}(\llbracket y \rrbracket_u)$ is definable for all $y \in Y$. Now, $f : v \to u$ is a *definable map.*

Proof. Suppose that f satisfies the constraint given above, and let $U \subseteq Q$ be a definable upset. This gives a formula $\chi \in \mathcal{L}(Y)$ such that $U = \llbracket \chi \rrbracket_u$. We prove, by induction along the structure of χ, that $f^{-1}(U)$ is definable as well. In the base case, this holds by assumption. Consider the implicative case, where $\chi = \phi \to \psi$. We compute:

$$f^{-1}\left(\llbracket \phi \to \psi \rrbracket_u\right) = f^{-1}\left(\llbracket \phi \rrbracket_u\right) \Rightarrow f^{-1}\left(\llbracket \psi \rrbracket_u\right),$$

which demonstrates that if $f^{-1}(\llbracket \phi \rrbracket_u)$ is defined by ϕ', and $f^{-1}(\llbracket \psi \rrbracket_u)$ is defined by ψ', then $f^{-1}(U)$ is defined by $\phi' \to \psi'$. Thus, this case is resolved by induction. All other cases can be treated similarly. □

3.6 Corollary
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(X)$ be Kripke models, and let $f : P \to Q$ be a map of Kripke frames. If f is a map of Kripke models, then f is a *definable map.*

Proof. By Lemma 2.26, we know that:

$$v, p \Vdash x \text{ if and only if } u, f(p) \Vdash x, \text{ for all } p \in P \text{ and } x \in X.$$

This ensures that $f^{-1}(\llbracket x \rrbracket_u) = \llbracket x \rrbracket_v$, hence Lemma 3.5 guarantees the desired. □

Suppose that one has two models, v and u, on the same underlying Kripke frames, P say. It may happen that the identity mapping $\mathrm{id}_P : P \to P$ can be construed as a definable map $f : v \to u$. This means that all upsets that are definable with respect to u, are definable with respect to v, too. In this case, we say that the model u is *definable with respect to* v.

3.7 Example

Let $v : P \to \mathcal{P}(X)$ be a model, and let $\sigma : \mathcal{L}(Y) \to \mathcal{L}(X)$ be a substitution. Consider the model $\sigma^*(v) : P \to \mathcal{P}(Y)$. Through Lemma 2.28 we know that:

$$v, p \Vdash \sigma(\chi) \text{ if and only if } \sigma^*(v), p \Vdash \chi \text{ for all } p \in P \text{ and } \chi \in \mathcal{L}(Y).$$

The identity map $f = \mathrm{id}_P : P \to P$ surely is a map of Kripke frames. This map carries the structure of a *definable map* $v \to \sigma^*(v)$, too. Indeed, suppose $U \subseteq P$ is definable in $\sigma^*(v)$ by ϕ. It immediately follows that $\sigma(\phi)$ defines this upset in v.

3.8 Lemma

Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ be Kripke models, and let $f : v \to u$ be a definable map. There exists a substitution $\sigma_f : \mathcal{L}(Y) \to \mathcal{L}(X)$ such that the following holds for all $p \in P$ and $\chi \in \mathcal{L}(Y)$

$$v, p \Vdash \sigma_f(\chi) \text{ iff } u, f(p) \Vdash \chi \tag{3.1}$$

Proof. The equation (3.1) suggests a definition for σ_f, namely:

$$\sigma_f(x) = \mathsf{def}\; f^{-1}\left(\llbracket x \rrbracket_u\right) \text{ for all } x \in X.$$

Let us now prove (3.1) through structural induction along χ. The base case $\chi = y \in Y$ follows from the following equivalences.

$$\begin{aligned} v, p \Vdash \sigma_f(y) &\text{ iff } p \in f^{-1}\left(\llbracket y \rrbracket_u\right) \\ &\text{ iff } u, f(p) \Vdash y. \end{aligned}$$

All inductive cases hold automatically. □

Recall the notions of *concreteness* and *refinedness*, given in Definition 2.41 and 2.36 respectively. In Corollary 2.55, we proved that every refined model is concrete. The remainder of this subsection is devoted to proving a partial converse of this statement. Before we continue, let us introduce an ostensibly stronger notion. Later, in Corollary 3.12, we show that the three notions coincide in the *image-finite* case.

3.9 Definition (Order-Defined)

Let $v : P \to \mathcal{P}(X)$ be a model. We say that v is *order-defined* when for each point $p \in P$ there exist formulae $\mathsf{nd}_v p, \mathsf{up}_v p \in \mathcal{L}(X)$ satisfying:

$$v, q \Vdash \mathsf{up}_v p \text{ if and only if } p \leq q \tag{3.2}$$

$$v, q \Vdash \mathsf{nd}_v p \text{ if and only if } q \nleq p \tag{3.3}$$

Whenever one can infer the model v from context, we abbreviate $\mathsf{up}\, p := \mathsf{up}_v\, p$ and $\mathsf{nd}\, p := \mathsf{nd}_v\, p$. Moreover, if P has a root ρ, then we write $\mathsf{up}\, P$ and $\mathsf{nd}\, P$ to abbreviate $\mathsf{up}\, \rho$ and $\mathsf{nd}\, \rho$ respectively.

The symbols introduced by Definition 3.9 are meant to be suggestive: $\mathsf{up}_v\, p$ defines the *upset* generated by p, and $\mathsf{nd}_v\, p$ defines those points that are *not* in the *downset* of p. Indeed, note that the above equations (3.2) and (3.3) amount exactly to the equations:

$$\begin{aligned} [\![\mathsf{up}_v\, p]\!]_v &= \uparrow p, \\ [\![\mathsf{nd}_v\, p]\!]_v &= P - \downarrow p, \end{aligned}$$

stating that there are formulae defining each principal upset and the complement of each principal downset. From here, it is easy to see that every order-defined model is refined. Below, we show how to obtain a partial converse to this. The idea behind this proof goes back to Jankov (1963b) and de Jongh (1968, Chapter 4) independently.[1] As such, the formulae $\mathsf{nd}\, p$ are known as *Jankov–de Jongh formulae*.

3.10 Theorem
Let $v : P \to \mathcal{P}(X)$ be an *image-finite*, *concrete* model. Now, v is *order-defined*.

Proof. We show, using well-founded induction, that the following equivalences hold for any $p \in P$. For convenience, we write W for the set of immediate successors of p as in Example 2.31, and recall that $W \;\kappa\; p$.[2]

$$\begin{aligned} p \leq q \quad & \text{iff } v(p) \subseteq v(q) \text{ and} && (3.4) \\ & \text{for all } k \geq q, \text{ if } v(p) \subset v(k) \text{ or } k \nleq w \text{ for some } w \in W \\ & \text{then } k \in \uparrow W. \\ q \nleq p \quad & \text{iff for all } k \geq q, \text{ if } k \in \uparrow p \text{ then } k \in \uparrow W && (3.5) \end{aligned}$$

Indeed, assuming this equivalence, the existence of the formulae $\mathsf{up}\, p$ and $\mathsf{nd}\, p$ is easy to see. Each atomic part of the right-hand side of these equivalences corresponds to an upset, and each of these upsets is definable, be it by induction or on their own

[1] See, in particular, de Jongh (1970, pp. 213–215), traces of which can already be seen in de Jongh (1964). A detailed proof can also be found in Bezhanishvili (2006, Theoem 3.2.2).

[2] We note that only two things really matter in the choice of W: we should have $W \;\kappa\; p$ and $p \notin W$. Instead of choosing the immediate successors of W, we might as well set $W := \updownarrow p$, and the same argument goes through.

right.[3] More explicitly, we can define these formulae as follows.

$$\mathsf{up}\,p := \bigwedge \mathsf{props}\,p \wedge \left(\left(\bigvee \mathsf{news}\,p \vee \bigvee_{w \in W} \mathsf{nd}\,w \right) \to \bigvee_{w \in W} \mathsf{up}\,w \right) \tag{3.6}$$

$$\mathsf{nd}\,p := \mathsf{up}\,p \to \bigvee_{w \in W} \mathsf{up}\,w \tag{3.7}$$

making use of the following auxiliary definitions:

$$\begin{aligned} \mathsf{props}\,p &:= \{x \in X \mid v, p \Vdash x\} \\ \mathsf{news}\,p &:= \{x \in X \mid v, \updownarrow p \Vdash x \text{ and } p \nVdash x\} \end{aligned} \tag{3.8}$$

Let us first focus on (3.4). The implication from left to right is immediate. From right to left, suppose that $p \nleq q$ yet the right-hand side does hold. Suppose that $q \nleq w$ for some $w \in W$. This immediately entails that $q \in \uparrow w \supseteq \uparrow p$, a contradiction.

We may thus assume that q is the maximal point such that $W \subseteq \uparrow q$. The right-hand side of (3.4) still holds for q, as it is upwards closed. We prove that $q \equiv p$, proving $p = q$ by concreteness, *quod non.*

To this end, take $k \in P - \{p, q\}$ to be such that $q < k$. By maximality, we know $W \nsubseteq \uparrow k$. It now follows through (3.4) that $k \in \uparrow W$, as desired. Conversely, if $p < k$, then $k \in \uparrow W \subseteq \uparrow q$. To finish the argument, we aim to prove that $v(p) = v(q)$. We know that $v(p) \subseteq v(q)$, so we need but exclude $v(p) \subset v(q)$. If this were the case, then $q \in \uparrow W \subseteq \uparrow p$, a contradiction. This finishes the proof of (3.4). As the equivalence (3.5) is clear, we are done. □

When reading the above definition (3.6) of $\mathsf{up}\,p$, recall that we understand that an empty disjunction to abbreviate for falsity ($\bot$), and an empty conjunction stands for truth ($\top$). The definition (3.6) encompasses the edge-case where $W = \varnothing$. In this case, Example 2.32 shows that p is a maximal point, and the defining equations (3.6) and (3.7) instantiate to:

$$\mathsf{up}\,p = \bigwedge_{x \in X} (\text{if } p \Vdash x \text{ then } x \text{ else } \neg x) \text{ and } \mathsf{nd}\,p = \neg \mathsf{up}\,p \tag{3.9}$$

When $U \subseteq P$ is a finite upset in the model $v : P \to \mathcal{P}(X)$, we write $\mathsf{up}\,U$ to denote the formula $\bigvee_{p \in U} \mathsf{up}\,p$. It is an easy matter to verify that:

$$U = \bigcup_{p \in U} \uparrow p = \bigcup_{p \in U} [\![\mathsf{up}\,p]\!]_v = \left[\!\!\left[\bigvee_{p \in U} \mathsf{up}\,p \right]\!\!\right]_v = [\![\mathsf{up}\,U]\!]_v.$$

[3] The idea that any such formula can be internalised has been worked out formally by McCullough (1971), which can also be found in de Jongh (1968, Theorem 3.5). In our specific case, note that the finiteness of X is crucial to the definability of $v(p) \subseteq v(q)$ and $v(p) \subset v(q)$.

3.11 Lemma
Let $v : P \to \mathcal{P}(X)$ be a finite, *order-defined model*. Every model $u : P \to \mathcal{P}(X)$ is *definable* with respect to v.

Proof. Immediate from Theorem 3.10, when one notes that each upset U is finite, and thus can be defined by $\mathsf{up}\, U$. Indeed, this shows that every upset $U \subseteq P$, regardless whether it is definable in u, is known to be definable in v. □

3.12 Corollary
An *image-finite* model is *concrete* if and only if it is *order-defined*, if and only if it is *refined*.

Proof. The implication from concrete to order-defined is given by Theorem 3.10. Any order-defined model clearly is refined. In turn, every refined model is concrete due to Corollary 2.55. □

A finite, refined model behaves in a similar manner as a frame when it comes to a logic. That is to say, all formulae of a logic are valid on a finite, refined model precisely if they are valid on the underlying Kripke frame. This follows immediately from Corollary 3.13 below.

3.13 Corollary
Let $v : P \to \mathcal{P}(X)$ be a finite, *refined* model, and let $\phi \in \mathcal{L}(Y)$ be a formula. The following are equivalent:

(i) $v \Vdash \sigma(\phi)$ for all $\sigma : \mathcal{L}(Y) \to \mathcal{L}(X)$;

(ii) $P \Vdash \phi$.

Proof. Suppose (i) holds, and let $u : P \to \mathcal{P}(Y)$ be a valuation on P. By Corollary 3.12, we know v to be *order-defined*, hence through Lemma 3.11 we know that u is definable with respect to v. Through Lemma 3.8, we thus obtain a substitution $\sigma : \mathcal{L}(Y) \to \mathcal{L}(X)$ such that $\sigma^*(v) = u$. Because $v \Vdash \phi$, this shows $u \Vdash \phi$, proving (ii). The implication from (ii) to (i) is immediate. □

3.2. Universal model

The purpose of this section is to give a description of the *free Heyting algebra* on a finite number of generators. We describe, given a finite set of variables, the so-called *universal model* alluded to in the title of this section. This is an *image-finite*

Kripke model, whose definable upsets correspond to the elements of a free Heyting algebra. This model is surprisingly manageable, and plays a crucial role in our search for semantics of admissible rules.

Each intermediate logic Λ corresponds to a variety of Heyting algebras $\mathcal{V}_\Lambda$, as described in Theorem 2.66. The free algebra generated by X in this variety, $\mathsf{F}_\Lambda(X)$, can be described in a manner similar to $\mathsf{F}(X) = \mathsf{F}_{\mathsf{IPC}}(X)$, whenever the intermediate logic Λ enjoys the *finite model property*. Below, we explain how one can define a universal model per intermediate logic, and show when this model corresponds to a free algebra.

3.2.1. History

Before we continue with the technical contents of this section, let us, for but a moment, reflect upon the history of the universal model. Its origins lie in the work of Rieger (1949), and independently, Nishimura (1960), who gave a full description of the free Heyting algebra on one generator.[4] This algebra is surprisingly simple, and can be described by the Kripke model depicted in Fig. 3.2. We refer to Wroński and Zygmunt (1974) for an historical survey of this special case.

The structure of free Heyting algebras with more than one generator remained unknown for quite some time. Urquhart (1973) was the first to give a description of its structure. Balbes and Dwinger (1974, p. 182) were aware of this result,[5] yet still write:

> "[...] very little is known about the structure of free Heyting algebras with more than one generator."

Closure algebras stand to the modal logic $\mathsf{S4}$ as Heyting algebras stand to the intermediate logic IPC. The structure of the free closure algebra remained mysterious for quite some time. Rieger (1957) proved the free closure algebra on one generator to be infinite.[6] When Horn (1978, p. 189) described the structure of the Lindenbaum algebras for $\mathsf{S5}$, he stated:

> "The free closure algebra with one generator is already so complicated that its structure is unknown."

[4] We refer to Thompson (1952) and Prior (1967) respectively for reviews of these works.

[5] Interestingly, they were not aware of the work of Rieger (1949), as pointed out in the review of their book by Urquhart (1977).

[6] See Wroński and Zygmunt (1974, Section 2) for more details on the history surrounding this discovery.

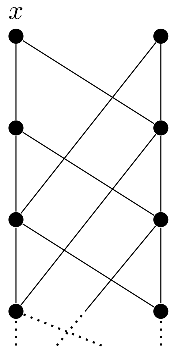


Figure 3.2.: The Rieger–Nishimura ladder.

Blok (1977, p. 362) wrote, in a paper on the structure of the open elements in free closure algebras:

> "Except for the free Heyting algebra on one generator, [...] the structure of these algebras is extremely complex and has remained rather obscure. [...] Here, even a description of the free object on one generator seems to be beyond reach, as yet."

An answer was eventually given by Shehtman (1978b), who built on the work of Esakia and Grigolia (1975). See also Grigolia (1987) for a different description of a similar technique. An independent description was given, later, by Bellissima (1985).

These types of descriptions have been used throughout the study of admissibility. Rieger and Nishimura's description of the free Heyting algebra on one generator was used by de Jongh (1982) to study the admissible rules of IPC in one variable. Most famously, Rybakov (1984a) used descriptions of free Heyting algebras and free closure algebras to prove the decidability of admissibility in IPC and S4. He called such models *characterising models.*

Over time, many descriptions of free Heyting algebras have arisen. The following is not an exhaustive enumeration of the literature; undoubtably, many works are omitted. In particular, no reference is made to the recent work on describing free algebras in a "step-by-step" manner. Let us mention Bezhanishvili and Gehrke (2009), Bezhanishvili and Ghilardi (2013), Bezhanishvili, Ghilardi, and Jibladze (2014), and

Coumans and Gool (2013) as examples of this approach, which we do not pursue in this thesis. Moreover, we do not treat the work by Ghilardi and Zawadowski (1995, 1997, 2002), who give a categorical description of the duals of finitely presented Heyting algebras.[7]

Although the descriptions are similar — they are, after all, concerned with the same object — they each have their own flavour. In particular, one can discern the approach taken by de Jongh (1968), Urquhart (1973), Rybakov (1984a), and Bellissima (1986). The approach taken by de Jongh (1968) is reflected in the work of Bezhanishvili (2006) and de Jongh and Yang (2011). Hendriks (1996, Section 2.5) is similar in background, but his construction proceeds via the the notion of *semantic types*. The description of Rybakov (1984a) is used by Gencer (2002) and Odintsov and Rybakov (2013), to name but a few. Finally, the method of Bellissima (1986) is employed by Darnière and Junker (2010) and Elageili and Truss (2012) and Elageili (2011).

3.2.2. Specification

Let us give a more precise specification of the object we are after. We first restrict attention to IPC, later on we widen the scope to encompass arbitrary intermediate logics that enjoy the FMP. Per finite set X, we seek an image-finite model $v : P \to \mathcal{P}(X)$, such that its Heyting algebra of definable upsets $\mathsf{defs}(v)$ is isomorphic to the free Heyting algebra $\mathsf{F}(X)$. This model is the *universal model on X for* IPC. The point of this subsection is to provide intrinsic conditions that guarantee that the above can be attained.

By Lemma 2.101, we know IPC to enjoy the *finite model property*: a formula is a theorem of IPC precisely if it holds in all finite, rooted models. Picture a model on a fixed set of variables, and assume that it contains a copy of every finite, rooted model on that very set of variables. This model must be complete with respect to all formulae on said variables. Each element of the free Heyting algebra generated by this fixed set of variables defines an upset in this model, and two distinct elements yield distinct upsets by this model's completeness.

The above reasoning suggests to define the universal model as being a particular model that contains a copy of each finite, rooted model. Indeed, as can be seen in Theorem 3.22, this is sufficient to prove that its definable upsets correspond in a one-to-one fashion to the elements of a free Heyting algebra. We use precisely this to define the universal model of any intermediate logic.

[7] We do appeal to some of the consequences of this work, in particular in Theorem 4.18 and Section 4.1.3.

3.14 Definition (Universal Model)
Let Λ be an intermediate logic, and let X be a finite set of variables. The *universal model of* Λ *on* X, denoted $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$, is the terminal object in the category of image-finite Kripke models on X and their maps.

The above definition, naturally, only makes sense if such a terminal object actually exists. In the case of IPC, this has been proven many times over, see for instance Bezhanishvili (2006, Theorem 3.2.2) and Yang (2008, Theorem 3.2.3). As terminal objects are isomorphic, speaking of "the universal model" is justified.[8]

Recall that earlier, when introducing Kripke models, we mentioned that the valuation encodes all information pertaining to the model. In the definition above, this is not reflected in the notation. Indeed, when reading u, one can infer neither the relevant set of variables, nor the logic under consideration. We make sure to disambiguate the notation in each context of usage, by at least once fully specifying $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. We partake in this abuse of notation for the sake of brevity.

In the definition below, we introduce an intrinsic property of a model which ensures that it contains a copy of each image-finite Kripke models on the same set of variables. Both the necessity and the sufficiency of this property is proven in Theorem 3.17. The name of this property is meant to be suggestive. Do note that the statements "v is a universal model" and "v is a model that is universal", in the terminology of Definitions 3.14 and 3.15 respectively, are not *a priori* equivalent. However, under the proviso that v is image-finite, said equivalence does follow through Theorem 3.17.

3.15 Definition (Universality)
A model $v : P \to \mathcal{P}(X)$ is said to be *universal* if for all finite *anti-chains* $W \subseteq P$ and all $Y \subseteq X$ with $Y \subseteq v(w)$ for all $w \in W$, there exists a unique $p \in P$ such that $v(p) = Y$ and $W \; \kappa \; p$.

Given a not necessarily rooted Kripke model, we can adjoin a new root to it. At this new root, we have to choose a valuation; any choice that ensures monotonicity of the valuation suffices. The operation of adjoining a root and selecting a suitable valuation plays an important role, so let us define it here.

3.16 Definition (Extension)
Let $v : P \to \mathcal{P}(X)$ be a model, and let $Y \subseteq X$ be a set of variables such that $v, P \Vdash Y$. Write P^+ for the partial order of P adjoined with a smallest element, denoted ρ. We

[8] We refer to Mac Lane (1978) for more details on such statements pertaining to category theory.

define the *extension of* v *with* Y, denoted v/Y to be the model:

$$v/Y : P^+ \to \mathcal{P}(X), \quad p \in P^+ \mapsto \begin{cases} v(p) & \text{if } p \in P, \\ Y & \text{otherwise.} \end{cases}$$

It is easy to see that $P \kappa \rho$ holds in P^+. That is to say, the set P covers the point ρ in the Kripke frame P^+. Note that $-/\varnothing$, in the notation of Definition 3.16, is the same as the Smoryński-operator $(-)'$ of Smoryński (1973).[9]

3.17 Theorem
Let X be a finite set of variables, and let $u : Q \to \mathcal{P}(X)$ be a model. The following are equivalent:

(i) for any image-finite model $v : P \to \mathcal{P}(X)$ there exists a unique map of Kripke models $f : v \to u$;

(ii) the model u is *universal*.

Proof. Suppose that (i) holds, let $W \subseteq Q$ be a finite *anti-chain*, and let $Y \subseteq X$ be such that $Y \subseteq u(w)$ for all $w \in W$. We define $U := \uparrow W$, and note that $v \restriction U$ is a finite model, because U is a finite union of principle upsets in Q, all of which are known to be finite. There is a unique map of Kripke models $i : (v \restriction U) \to v$ induced by the inclusion $U \subseteq P$. Moreover, there must be a map of Kripke models $f : v/Y \to u$. Now, $g \restriction U = i$ follows through the uniqueness of i. Because $W \kappa \rho$, Lemma 2.35 immediately yields $W = i(W) \kappa \rho$, proving (ii).

To prove the converse, suppose that (ii) holds. We prove (i) with the proviso that v is finite, from whence the image-finite case is immediate. Indeed, existence of such a map in the image-finite case follows from taking the union of all such maps in the finite, rooted case. This map is well-defined and unique, anything else would contradict the unicity in the finite, rooted case.

We proceed by induction on the number of elements in P. In the base case, we know P to be empty, and so the desired surely holds. Now, suppose we know the desired for all finite rooted models v where P is of size at most n. Write p for the root of P, and consider the upset $U = \updownarrow p$. Induction yields a map Kripke models $f_k : (v \restriction (\uparrow k)) \to u$ per $k \in U$. We know that the desired $f : v \to \mathsf{U}(X)$ must satisfy $f \restriction (\uparrow k) = f_k$ for all $k \in U$.

[9]Through the duality between finite Kripke frames and finite Heyting algebras as explained in Theorem 2.69, this operation is known as the Troelstra sum (Troelstra, 1965), star sum (Balbes and Horn, 1970a), vertical sum (Bezhanishvili, 2006), concatenation (Citkin, 2012b), and glued sum (Grätzer, 2011).

Consider the function defined by $f_U = \bigcup_{k \in U} f_k : (v \restriction U) \to u$. This function is well-defined due to the uniqueness that is ensured by induction. Moreover, it is a map of Kripke models. We see that $V = f_U(U)$ is an upset, and note that there must be a finite anti-chain $W \subseteq V$ such that $\uparrow W = V$. Hence, there exists a unique $q \in \mathsf{U}(X)$ such that both $W \kappa q$ and $\mathsf{u}(q) = v(p)$. It is clear that $V \kappa q$ holds.

By Lemma 2.35, we know that the map f ought to send p to q. Define $f = f_U \cup \langle p, q \rangle$, and the desired follows. $\square$

The above Theorem 3.17 thus shows that a model on X that is universal, whenever it exists, must be the terminal object in the category of Kripke models on X and their maps. From this perspective, the following Corollary 3.18 is immediate, explicitly stating the unicity of universal models.

3.18 Corollary
Up to isomorphism of Kripke models, there is at most one *image-finite* model that is *universal* on any given finite set of variables.

3.19 Lemma
The universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$ is *order-defined* for each finite X.

Proof. The desired result follows immediately from Theorem 3.10 when we can prove u to be concrete. Let $p, q \in \mathsf{U}_{\mathsf{IPC}}(X)$ be given, and suppose that $p \equiv q$. We claim that there exists a finite $W \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$ such that $W \kappa p$ and $W \kappa q$. By Theorem 3.17 and the observation that $\mathsf{u}(p) = \mathsf{u}(q)$, it readily follows that $p = q$, as desired.

We now prove the claim. To this end, and without loss of generality, we distinguish between whether $p \leq q$ holds. If it does, then $W := \uparrow q$ does the trick. Otherwise, we note that:

$$\updownarrow p = \updownarrow p - \{q\} = \updownarrow q - \{p\} = \updownarrow q,$$

whence the desired is immediate. $\square$

We now have sufficient machinery to prove the main theorem of this section, giving substance to Definition 3.14.

3.20 Theorem
Let Λ be an intermediate logic, and let X be a finite set of variables. There exists an image-finite, order-defined model $\mathsf{u} : \mathsf{U}_{\Lambda}(X) \to \mathcal{P}(X)$ such that for each image-finite model $v : P \to \mathcal{P}(X)$ with $v \Vdash \Lambda$, there exists a unique map of Kripke models $f : v \to \mathsf{u}$.

Proof. First, recall that we know of the universal model $\mathsf{u}_{\mathsf{IPC}} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$. By Lemma 3.19, we know this model to be order-defined. We now define the upset U as:

$$U := \left\{ p \in \mathsf{U}_{\mathsf{IPC}}(X) \;\middle|\; \mathsf{u}_{\mathsf{IPC}}, p \Vdash \Lambda \right\}.$$

We define the model $\mathsf{u} := \mathsf{u}_{\mathsf{IPC}} \upharpoonright U$. As this is a generated submodel of $\mathsf{u}_{\mathsf{IPC}}$, it is clearly both order-defined and image-finite. If $v : P \to \mathcal{P}(X)$ is any image-finite model such that $v \Vdash \Lambda$, then there exists a unique map $f : v \to \mathsf{u}_{\mathsf{IPC}}$. It is easy to see that for each $p \in P$ we have $f(p) \Vdash \Lambda$, as follows through Lemma 2.26. Hence $f(P) \subseteq U$, so the map f can be restricted to a map $v \to \mathsf{u}$. Uniqueness is clear, hence the desired is proven. □

3.21 Corollary
The underlying Kripke frame of every finite submodel of $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ is a frame of Λ.

Proof. Immediate from Corollary 3.13 and Theorem 3.20. □

The following Theorem 3.22 has been proven many times over. A proof in the intuitionistic case can be found in Urquhart (1973, Theorem 3), Bellissima (1986, Corollary 2.5), and Bezhanishvili (2006, Theorem 3.2.20). We refer to Chagrov and Zakharyaschev (1997, Theorem 8.86) and Shehtman (1978b, Theorem 6) for modal counterparts of the same theorem.

3.22 Theorem
Let Λ be an intermediate logic with the *finite model property*, and let X be a finite set of variables. The Heyting algebra of definable upset of $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ is isomorphic to $\mathsf{F}_\Lambda(X)$ via the mappings:

$$[\![-]\!]_{\mathsf{u}} : \mathsf{F}_\Lambda(X) \to \mathsf{defs}(\mathsf{u}) \text{ and } \mathsf{def} : \mathsf{defs}(\mathsf{u}) \to \mathsf{F}_\Lambda(X).$$

Proof. It is easy to verify that both functions mentioned in the theorem are maps of Heyting algebras. Let us argue that these maps are mutually inverse.

The one direction is straightforward enough. Indeed, $[\![\mathsf{def}\, U]\!]_{\mathsf{u}} = U$ follows immediately when writing out the definitions. We focus on the other direction. Know that $\mathsf{def}\, [\![\phi]\!]_{\mathsf{u}}$ is the formula ψ such that $[\![\phi]\!]_{\mathsf{u}} = [\![\psi]\!]_{\mathsf{u}}$. We wish to show that $\phi = \psi$ holds in $\mathsf{F}_\Lambda(X)$, that is to say, that $\vdash_\Lambda \phi \equiv \psi$. We reason by contradiction, and assume, without loss of generality, that $\phi \nvdash_\Lambda \psi$. By the finite model property, this gives us a finite, rooted Kripke model $v : P \to \mathcal{P}(X)$ such that $v \Vdash \Lambda$, $v \Vdash \phi$ and $v \nVdash \psi$.

We know there to be a map of Kripke models $f : v \to \mathsf{u}$. Consequently, $f(p) \Vdash \phi$ yet $f(p) \nVdash \psi$. This shows that $f(p) \in [\![\phi]\!]_{\mathsf{u}}$ and $f(p) \notin [\![\psi]\!]_{\mathsf{u}}$, proving $[\![\phi]\!]_{\mathsf{u}} \neq [\![\psi]\!]_{\mathsf{u}}$ as desired. □

Corollary 3.23 below is an immediate consequence of Theorem 3.22. This, too, has been proven many times in the past, let us point to Rybakov (1997, Theorem 3.3.6) in particular. The first appearance of a statement of this nature in the literature on admissibility appears to be Rybakov (1984a, Theorem 2), where it is formulated with respect to the modal logic $\mathsf{S4}$.

3.23 Corollary (Completeness of the Universal Model)
Let Λ be an intermediate logic with the *finite model property*, let X be a finite set of variables, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. For all formulae $\phi \in \mathcal{L}(X)$, one has $\mathsf{u} \Vdash \phi$ if and only if $\vdash_\Lambda \phi$.

Combining Corollary 3.23 with the *order-definedness* of the universal model, we obtain the Corollary 3.24 below. It plays quite an important role in Chapter 7, in particular in the form of Lemma 7.5.

3.24 Corollary
Let Λ be an intermediate logic with the *finite model property*, and let X be a finite set of variables. For all $\phi \in \mathcal{L}(X)$ and $p \in \mathsf{U}_\Lambda(X)$, we have that:

$$\mathsf{u}, p \nVdash \phi \text{ iff } \phi \vdash_\Lambda \mathsf{nd}\, p. \tag{3.10}$$

Proof. By Corollary 3.23, we need only prove that $\mathsf{u}, p \nVdash \phi$ holds if and only if $\mathsf{u} \Vdash \phi \to \mathsf{nd}\, p$. From right to left, the desired is immediate, as $\mathsf{u}, p \nVdash \mathsf{nd}\, p$. In the other direction, suppose $\mathsf{u}, p \nVdash \phi$ and assume that $\mathsf{u} \nVdash \phi \to \mathsf{nd}\, p$. The latter yields a point $q \in \mathsf{U}_\Lambda(X)$ such that $\mathsf{u}, q \Vdash \phi$ yet $\mathsf{u}, q \nVdash \mathsf{nd}\, p$. We thus derive $q \leq p$, hence $\mathsf{u}, p \Vdash \phi$ follows, a contradiction. This proves the desired. □

We close this section with the following lemma, which indicates that image-finite, order-defined models are isomorphic to generated submodels of the universal model. In the following, we make extensive use of this observation.

3.25 Lemma
Let Λ be an intermediate logic, let X be a finite set of variables. Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$, and let $v : P \to \mathcal{P}(X)$ be an image-finite, order-defined model of Λ. There exists an upset $U \subseteq \mathsf{U}_\Lambda(X)$ such that v and $\mathsf{u} \restriction U$ are isomorphic as Kripke models.

Proof. We know of a unique map of Kripke models $f : v \to \mathsf{u}$. We first show that f is injective. Indeed, suppose $p_1, p_2 \in P$ are given such that $f(p_1) = f(p_2)$. Note that $\uparrow p_1$ is definable in v by $\mathsf{up}\, p_1$. We observe that $v, p_1 \Vdash \mathsf{up}\, p_1$, and hence $\mathsf{u}, f(p_1) \Vdash \mathsf{up}\, p_1$, leading to $v, p_2 \Vdash \mathsf{up}\, p_1$. We can thus conclude $p_1 \leq p_2$, and the converse holds for a similar reason. This proves $p_1 = p_2$, as desired.

To finish the argument, we define $U := f(P)$. The existence of a map of Kripke models $g : (\mathsf{u} \restriction U) \to v$ satisfying $f \circ g = \mathrm{id}_v$ and $g \circ f = \mathrm{id}_{\mathsf{u}}$ readily follows. □

3.2.3. Construction

Through Corollary 3.18, it is clear that the universal model of IPC is *universal* in the technical sense of Definition 3.15. This immediately suggests a particular construction. Below we indicate how one might perform such a construction. This exposition is by no means meant as a solid proof, it is merely an intuitive explanation of the idea behind the explicit constructions present in the literature. We refer to Bezhanishvili (2006, Theorem 3.2.2), Odintsov and Rybakov (2013, Section 3), de Jongh and Yang (2011) and Bezhanishvili and de Jongh (2012, Definition 3.3) for more details on the actual construction.

Intuitively, one starts with an empty model, and iteratively adds points such that to each finite subset and to each set of variables that holds at this subset one has some point covered by this subset, at which precisely these variables hold. This yields a sequence of Kripke models $\mathsf{u}_n : P_n \to \mathcal{P}(X)$, such that

$$\mathsf{u} = \bigcup_{n \in \mathbb{N}} \mathsf{u}_n, \quad \mathsf{U}(X) = \bigcup_{n \in \mathbb{N}} P_n.$$

At the zeroth stage, we consider the empty Kripke model. Hence u_0 is the unique function $P_0 = \varnothing \to \mathcal{P}(X)$. The only anti-chain in P_0 thus is $\varnothing$. One would have to expand P_0 into P_1 to accommodate more points, and one would extend the valuation u_0 appropriately into u_1. There would be points $p_Y \in P_1$ per $Y \subseteq X$ satisfying $\varnothing \kappa\, p_Y$ and $\mathsf{u}_1(p_Y) = Y$. This means that the universal model must have a maximal point per subset of X.

After the $(n+1)^{\text{th}}$ stage, one inspects each subset $W \subseteq P_{n+1}$. If W would be contained within P_n, and one would add a new point p into P_{n+2} such that $W \kappa\, p$, then the uniqueness demanded by universality would be broken. Indeed, a point q satisfying $W \kappa\, q$ with $\mathsf{u}_{n+1}(q) = \mathsf{u}_{n+2}(q)$ already exists, as it was added in a previous step. One thus only considers sets W where the intersection with $P_{n+1} - P_n$ is non-empty. Moreover, if W would be the singleton set $\{p\}$, and one would add a point such that $W \kappa q$ and $\mathsf{u}_{n+2}(q) = \mathsf{u}_{n+1}(p)$, then uniqueness would be violated, too. Indeed, $W \kappa p$ also holds, and so the point q would be superfluous.

The model described by Definition 3.14 is the union of all the models obtained from the above construction. Although we do not consider the concrete details of the above construction in the following, let us spend a few words on the attention we paid to avoiding violating the uniqueness demanded by universality. In Theorem 3.22, we proved that any two distinct points in the universal model can be discerned by their theories. As a consequence of Corollary 3.12, the universal model thus must be concrete. Note that the precautions taken against constructing two points covered by the same set, in our loose description of the construction of $\mathsf{U}(X)$ above, correspond to preventing the introduction of non-equal *analogous points*, as discussed in Section 2.2.3.

There are many different ways in which one could describe the universal model of IPC. The above sketches the construction by layers, in which one "grows" the universal models by progressively adding those points that need to exist in order to satisfy universality. Another way to describe the universal model is as all those points in the canonical model of finite height, see for instance Bezhanishvili (2006, Theorem 3.2.9) and Shehtman (1978b, Theorem 4). Yet another perspective would be to consider the disjoint union of all finite, concrete Kripke models on a given set of variables X; the universal model on X is what remains when one divides this model by its largest bisimulation. Alternatively, one can iteratively apply contraction as given in Definition 2.48. Roughly speaking, the n^{th} stage in the above described process amounts to taking the disjoint union of all concrete Kripke frames of height at most n, and applying contradiction n times.

3.3. Exact models

In this section, we explore potential notions of semantics for admissible rules.[10] First, we define what it means for a rule to be *valid* on a Kripke model. Second, we introduce the so-called *exact* Kripke models. We then prove that the class of exact models corresponding to a particular intermediate logic is both sound and complete with respect to the admissible rules of said logic, under the proviso that this logic enjoys the *finite model property*.

[10] When one speaks of "semantics for a logic", one typically means semantics for the *theorems* of a logic. That is to say, one gives a class of models such that a *formula* is a *theorem* precisely if it is valid on all models in said class. By analogy, providing semantics for admissible rules amounts to defining a class of models such that a *rule* is *admissible* precisely if it is valid on all models in said class.

Later, in Section 4.4, we generalise the concept of exact models to *adequately exact models*. Moreover, exact models return in Section 4.1.3, where we give a more intrinsic description of these models. The main point we wish to make about exact models at this stage, is that they can serve as semantics for admissible rules in many an intermediate logic. Of secondary importance is the intuition they provide for the more intricate notion of adequately exact models, which play a crucial rule in proving the decidability of admissibility in IPC. Before we continue along this vein, we first define the appropriate notion of validity.

3.26 Definition (Valid)
Let $v : P \to \mathcal{P}(X)$ be a model, and let $\Gamma, \Delta \subseteq \mathcal{L}(X)$ be sets of formulae. We say that the rule Γ/Δ is *valid on* v whenever:

$$\text{if } v \Vdash \phi \text{ for all } \phi \in \Gamma, \text{ then } v \Vdash \chi \text{ for some } \chi \in \Delta. \tag{3.11}$$

Given a set of rules $\mathcal{R}$, we say that $\mathcal{R}$ is valid on v precisely if Γ/Δ is valid on v, for each $\Gamma/\Delta \in \mathcal{R}$ satisfying $\Gamma, \Delta \subseteq \mathcal{L}(X)$.

Recall that the validity of a formula on a universal model corresponds to the derivability of said formula in IPC, as expressed in Corollary 3.23. Because admissible rules are concerned with a connection between the derivability in IPC of two formulae, it makes sense to use universal models as a notion of semantics. However, the definition of admissibility refers to an unbounded totality of substitutions, which may make use of arbitrarily many variables. As such, in general, it does not suffice to consider only validity in universal models itself. To remedy this, we provide the following definition.[11]

3.27 Definition (Exact Model)
Let Λ be an intermediate logic, let $v : P \to \mathcal{P}(X)$ be a model. We say that v is *exact with respect to* Λ (or Λ-exact, for short) whenever there is a surjective, definable map $f : \mathsf{u} \to v$, where $\mathsf{u} : \mathsf{U}_\Lambda(Y) \to \mathcal{P}(Y)$ is the universal model on some finite set of variables Y.

Note that every Λ-exact model must be image finite. Indeed, the universal model is image-finite by its definition, hence Lemma 2.27 ensures any exact model to be image-finite, too.

[11] The name stems from "exactly provable", as introduced by de Jongh (1982, p. 52), and it is not related to the notion of an "exact model" as introduced by de Bruijn (1975). An exact model in the sense of de Bruijn (1975) is a finite model in which *all* upward closed sets (in contrast to merely the principally generated ones) are definable by means of formulae from a chosen fragment. It is purely coincidental that these notions occasionally coincide. We refer to de Jongh, Hendriks, et al. (1991) for further background on this matter.

3.28 Example
The identity map $\mathrm{id}_{\mathsf{u}} : \mathsf{u} \to \mathsf{u}$ for each $\mathsf{u} : \mathsf{U}_\Lambda(X)$ is clearly a surjective, definable map. Consequently, the universal model u is Λ-exact whenever X is finite.

When the intermediate logic at hand is IPC, we drop the relativisation, and simply speak of exact models instead of IPC-exact models.

3.29 Example
Let $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ be a substitution, and consider $\mathsf{u}_Y : \mathsf{U}_\Lambda(Y) \to \mathcal{P}(Y)$. Through Example 3.7, we know that this substitution gives rise to a definable map $f : \mathsf{u}_Y \to \sigma^*(\mathsf{u}_Y)$. Clearly, this map f is surjective. As a consequence, the model $\sigma^*(\mathsf{u}_Y)$ is exact.

Moreover, as $\sigma^*(\mathsf{u}_Y)$ is an image-finite model on X, it fits into the universal model $\mathsf{u}_X : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. More precisely, there is a unique map of Kripke models $g : \sigma^*(\mathsf{u}_Y) \to \mathsf{u}_X$. We thus obtained a definable map $g \circ f : \mathsf{u}_X \to \mathsf{u}_X$ such that:

$$\mathsf{u}_X, p \Vdash \sigma(\phi) \text{ iff } \mathsf{u}_Y, (g \circ f)(p) \Vdash \phi. \text{ for all } \phi \in \mathcal{L}(X). \tag{3.12}$$

3.30 Example
Consider the setting in which X is taken to be the singleton set $\{x\}$, and think of the model $v : P \to \mathcal{P}(X)$ as depicted on the right-hand side of Fig. 3.3. The required definable map $f : \mathsf{U}_{\mathsf{IPC}}(X) \to v$ is depicted by the dashed lines, whose behaviour is partially described by Lemma 2.35. Observe that the following equalities hold. Definability already follows from the first equation, the other two are merely given for reference.

$$f^{-1}\left(\llbracket x \rrbracket_v\right) = \llbracket \neg\neg x \rrbracket_{\mathsf{u}}$$
$$f^{-1}\left(\llbracket \neg x \rrbracket_v\right) = \llbracket \neg\neg\neg x \rrbracket_{\mathsf{u}} = \llbracket \neg x \rrbracket_{\mathsf{u}}$$
$$f^{-1}\left(\llbracket \neg\neg x \to x \rrbracket_v\right) = \llbracket \neg\neg\neg\neg x \to \neg\neg x \rrbracket_{\mathsf{u}} = \llbracket \top \rrbracket_{\mathsf{u}}$$

The above Example 3.30 is special in two ways. First, the model given in Fig. 3.3 is finite. Second, the model corresponds to a definable upset of the universal model, namely the upset $\llbracket \neg\neg x \to x \rrbracket_{\mathsf{u}} \subseteq \mathsf{U}(\{x\})$. To get a better understanding of exact models, let us look into these edge cases in detail. Do note that the former case is certainly encompassed by the latter, as can be seen via Lemma 3.25.

In Lemma 3.31 below, we show that an exact model gives rise to a substitution. Were v to be a definable submodel of the universal model, then it is easy to see that a defining formula must satisfy the properties as specified in Definition 3.32 below. This is a precise characterisation, as we prove in Theorem 3.33.

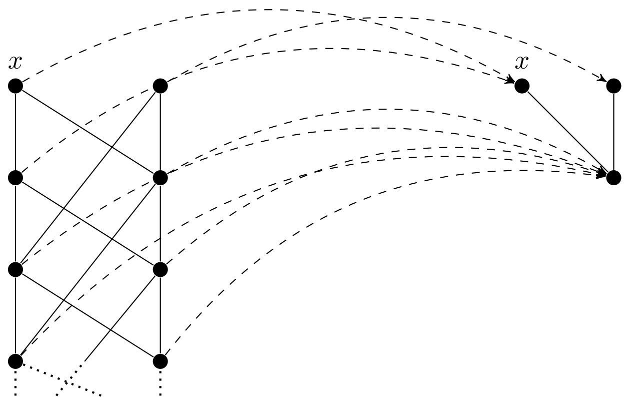


Figure 3.3.: An example of an IPC-exact model, together with a definable, surjective map from the universal model.

3.31 Lemma
Let Λ be an intermediate logic with the *finite model property*, and let $v : P \to \mathcal{P}(X)$ be an exact model. There exists a finite set of variables Y, and a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that:

$$\vdash_\Lambda \sigma(\phi) \text{ iff } v \Vdash \phi \text{ for each } \phi \in \mathcal{L}(X). \tag{3.13}$$

Proof. By definition, we know of a finite set of variables Y and a surjective, definable map $f : \mathsf{u} \to v$, where $\mathsf{u} : \mathsf{U}_\Lambda(Y) \to \mathcal{P}(Y)$ is the universal model. Through Lemma 3.8 we obtain a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that:

$$\mathsf{u}, p \Vdash \sigma(\phi) \text{ iff } v, f(p) \Vdash \phi \text{ for all } \phi \in \mathcal{L}(X).$$

Now, note that the map f is surjective, hence we know that $v \Vdash \phi$ precisely if $\mathsf{u} \Vdash \sigma(\phi)$. The latter statement is equivalent to $\vdash_\Lambda \sigma(\phi)$ through Corollary 3.23, proving the desired. □

The notion of an exact formula derives from de Jongh (1982), and it was further developed by de Jongh and Visser (1996, Section 2).

3.32 Definition (Exact Formula)
Let Λ be an intermediate logic with the *finite model property*. A formula $\phi \in \mathcal{L}(X)$

is said to be *exact* whenever there exist a finite set of variables Y and a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that:

(i) the formula $\sigma(\phi)$ is a theorem of Λ;

(ii) for all formulae $\psi \in \mathcal{L}(X)$, $\vdash_\Lambda \phi \to \psi$ holds if and only if $\vdash_\Lambda \sigma(\psi)$.

We now have two different kinds of objects we call exact, as introduced in Definitions 3.27 and 3.32. In Theorem 3.33, we show that the latter is a special case of the former. More precisely speaking, we prove that, when restricting attention to models that are definable upsets in the universal model, these notions coincide. Immediately following this theorem, we provide some examples of exact formulae, in the understanding that these constitute examples of exact models as well.

3.33 Theorem (Bezhanishvili and de Jongh, 2012, Corollary 4.6)
Let Λ be an intermediate logic with the *finite model property*, let X be a finite set of variables, and let $\phi \in \mathcal{L}(X)$ be a formula. Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$, and define the generated submodel $v := \mathsf{u} \restriction [\![\phi]\!]_{\mathsf{u}}$. The model v is *exact* if and only if the formula ϕ is *exact*.

Proof. Suppose that v is exact. By Lemma 3.31, there is a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ satisfying (3.13). As $v \Vdash \phi$ holds, it follows that $\vdash_\Lambda \sigma(\phi)$. Similarly, if $\vdash_\Lambda \sigma(\psi)$ holds, then $v \Vdash \psi$. The latter is equivalent to the statement that $\mathsf{u} \Vdash \phi \to \psi$, which, in turn, entails $\vdash_\Lambda \phi \to \psi$ through Corollary 3.23. We have thus proven that ϕ is exact.

Conversely, suppose that ϕ is exact. By definition, this provides us with a finite set of variables Y and a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ satisfying Definition 3.32. Consider the universal models $\mathsf{u}_Y : \mathsf{U}_\Lambda(Y) \to \mathcal{P}(Y)$. We obtain a definable map $f : \mathsf{u}_Y \to \mathsf{u}$ as in Example 3.29, such that $\mathsf{u}_Y, p \Vdash \sigma(\chi)$ holds precisely if $\mathsf{u}, f(p) \Vdash \chi$ for each $\chi \in \mathcal{L}(X)$ and $p \in \mathsf{U}_\Lambda(Y)$. When we can prove that $f(\mathsf{U}_\Lambda(Y)) = [\![\phi]\!]_{\mathsf{u}}$, we are done.

Consider an arbitrary $p \in \mathsf{U}_\Lambda(Y)$. By exactness, we know $\mathsf{u}_Y, p \Vdash \sigma(\phi)$. Hence $f(p) \Vdash \phi$, and so $f(p) \in [\![\phi]\!]_{\mathsf{u}}$. Conversely, suppose $f(p) \nVdash \phi$. It follows from Corollary 3.24 that $\vdash_\Lambda \phi \to \mathsf{nd}\, f(p)$. Hence, exactness entails that $\vdash_\Lambda \sigma(\mathsf{nd}\, f(p))$. By Corollary 3.23, this means that $\mathsf{u}_Y, p \Vdash \sigma(\mathsf{nd}\, f(p))$, which, in turn, is equivalent to the statement that $\mathsf{u}, f(p) \Vdash \mathsf{nd}\, f(p)$. Yet this is a blatant falsehood, proving the desired. □

3.34 Example (Variables)
Let X be a set of variables. In every intermediate logic, the formula $y \in X$ is exact. Indeed, we define the substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ as below, from whence the

desired is immediate.

$$x \in X \mapsto \begin{cases} \top & \text{if } x = y \\ x & \text{otherwise} \end{cases}$$

3.35 Example (Prucnal's Trick)
We consider a trick introduced by Prucnal (1976), using the exposition of Wojtylak (2004, Lemma 2). Consider an arbitrary formula $\phi \in \mathcal{L}(X)$, and define the substitution:

$$\sigma : \mathcal{L}(X) \to \mathcal{L}(X),\ x \mapsto \phi \to x.$$

It is easy to see that $\phi \vdash_{\mathsf{IPC}} (\chi \equiv \sigma(\chi))$ for all $\chi \in \mathcal{L}(X)$. In the case where ϕ is an element of the $[\top, \to, \wedge]$-fragment, then $\vdash_{\mathsf{IPC}} (\phi \to \chi) \equiv \sigma(\chi)$ holds as well.[12] This entails that any formula of the thus described form must be exact. Indeed, because $\vdash_{\mathsf{IPC}} (\phi \to \phi) \equiv \sigma(\phi)$, we know that $\vdash_{\mathsf{IPC}} \phi$. The other requirement also readily follows.

3.36 Example (Finite Exact Models)
Let $v : P \to \mathcal{P}(X)$ be a finite exact model. There must be a finite upset $U \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$ such that there is a map of Kripke models $v \to (\mathsf{u} \upharpoonright U)$. Moreover, as P is finite, the upset U is finite, and hence definable. All finite exact models can thus be described by means of exact formulae that induce finite upsets in the universal model.

When we restrict attention to but one variable, it is not very hard to enumerate all such exact formulae. Indeed, up to provable equivalence, they amount to the list $\top$, x, $\neg x$, $\neg\neg x$ and $\neg\neg x \to x$. For more details on this, and for a complete description of all finite exact models, we refer to Arevadze (2001).

Now that we have given numerous examples of exact models, let us prove that they can be utilised as semantics for admissible rules. First, we define what it means for a set of rules to be *sound* and *complete* with respect to a class of Kripke models. This notion is used in other parts of this thesis as well, in particular in Theorem 4.75.

3.37 Definition (Soundness and Completeness for Rules)
Let $\mathcal{K}$ be a class of Kripke models, and let $\mathcal{R}$ be a set of rules. We say that $\mathcal{R}$ is *sound* with respect to $\mathcal{K}$ if every rule $\Gamma/\Delta \in \mathcal{R}$ is valid in each model $v \in \mathcal{K}$. We say that $\mathcal{R}$ is *complete* with respect to $\mathcal{K}$ if $\Gamma/\Delta \in \mathcal{R}$ holds whenever Γ/Δ is valid in each model $v \in \mathcal{K}$ for all rules Γ/Δ.

[12] This fragment is the largest fragment excluding disjunction, where $\bot$ can is interpreted as the empty disjunction.

The following theorem, in essence, comes down to Rybakov (1997, Theorem 3.3.10). It amounts to a reformulation of stating that a rule is admissible, precisely if it is valid on the corresponding free algebra. Such statements can be found in many places in the literature, let us but mention Rybakov (1990b, Lemma 2), Rybakov (1985a, Lemma 3), and Cabrer and Metcalfe (2012, Theorem 2).

Similar theorems can be formulated for other classes of models. In this thesis, we do this in Corollary 4.20 and Theorem 4.75. Let us also mention Iemhoff (2001b, Corollary 3.14), where the admissible rules of IPC are shown to be both sound complete with respect to "AR-models". We treat such models in Section 4.1 under the name of IPC-extendible models.

3.38 Theorem (Soundness and Completeness of Admissible Rules w.r.t. Exact Models) Let Λ be an intermediate logic with the *finite model property*. The following are equivalent for all finite sets of formulae $\Gamma, \Delta \in \mathcal{L}(X)$:

(i) the rule Γ/Δ is admissible in Λ;

(ii) the rule Γ/Δ is valid on every model $v : P \to \mathcal{P}(X)$ that is Λ-exact.

One may wonder whether it suffices to restrict attention to *finite exact models*. This is not possible in general. Indeed, in Section 3.6 we see that this fails in many intermediate logics, in particular for IPC and BB_n with $n \geq 2$ in particular.

Proof. Suppose (i) holds, and let $v : P \to \mathcal{P}(X)$ be a Λ-exact model. Assume, furthermore, that $v \Vdash \Gamma$. By Lemma 3.31, we know of a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that Eq. (3.13) holds. This ensures us that $\vdash_\Lambda \sigma(\phi)$ holds for all $\phi \in \Gamma$. Through (i), we know that $\vdash_\Lambda \sigma(\chi)$ holds for some $\chi \in \Delta$. The desired statement $v \Vdash \chi$ follows immediately via (3.13), proving (ii).

Conversely, suppose (i) does not hold. By definition, this yields a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$, where Y is a finite set of variables, satisfying $\vdash_\Lambda \sigma(\phi)$ for all $\phi \in \Gamma$, yet $\nvdash_\Lambda \sigma(\chi)$ holds for all $\chi \in \Delta$. Corollary 3.23 ensures that $\mathsf{u} : \mathsf{U}_\Lambda(Y) \to \mathcal{P}(Y)$ now satisfies:

$$\mathsf{u} \Vdash \sigma(\phi) \text{ for all } \phi \in \Gamma \text{ and } \mathsf{u} \nVdash \sigma(\chi) \text{ for all } \chi \in \Delta.$$

Now, through Example 3.29, we know that the model $\sigma^*(X)$ is exact, and that Γ/Δ is not valid on it. This disproves (ii), as desired. □

Completeness with respect to exact models is not very satisfactory. The notion of exactness is formulated in a fairly extrinsic fashion, in that it is hard to look at a model, and at a glance observe that it must be exact. We provide a more intrinsic notion in Section 4.1, and we provide conditions under which these two notions coincide in Section 4.1.3. Moreover, in Section 4.4, we generalise the notion of an exact model to an *adequately exact model*. This provides us with decidable semantics for the admissible rules of IPC.

3.4. Characterising covers

In this section, we delve into the details of describing when certain subsets of models cover a point. That is to say, we give a criterion that holds for a finite subset of a model precisely if there is a point covered by this finite subset. Later, in Section 4.3, we generalise this criterion to suit the generalised notion of a cover given by Definition 4.54. The reasons for treating covers in the usual and their generalised sense separately are twofold.

First, the methods of proof involved are fairly different. The machinery developed in Sections 3.1 and 3.2 lends itself beautifully to describing covers in image-finite, order-defined models. Moreover, the resulting criterion given in Theorem 3.45 can be used quite smoothly in Section 3.5, ultimately leading to a rather straightforward construction of *bases of admissibility* in Section 5.3. On the other hand, the machinery involved in treating the more generalised form is less semantic in nature. Indeed, these arguments fit more naturally into the treatise of decidability of Chapter 4.

Second, many of the intermediate logics we are concerned with enjoy the finite model property, from which perspective the greater generality is of little value. In fact, there is a certain beauty in being able to give a description of the admissible rules of a logic in terms of the structure of its universal model. This is the type of description we give here. By contrast, one can also give descriptions in terms of a class of models with respect to which the logic at hand is sound and complete. This approach has been pursued by Iemhoff (2001b, 2005, 2006).

Recall that Lemma 2.40 proved that for each image-finite model, there is a unique map into the canonical model. By Lemma 2.35, such a map must preserve covers. This suggests a close relation between the points covered by the theory of a model and the possible *extensions* of this model. The following Lemma 3.39 expresses this observation. More precisely, it shows that finding an extension of a model amounts to finding a cover of the points representing this model's theory. Do note that all statements in Lemma 3.39 still hold after replacing the canonical model, by the universal model.

3.39 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model, consider the *canonical model* $\mathsf{c} : \mathsf{C}(X) \to \mathcal{P}(X)$, and let $W \subseteq \mathsf{C}(X)$ be a set of points such that $\mathrm{Th}_{\mathsf{c}}(W) = \mathrm{Th}(v)$. For all $Y \subseteq X$ with $v \Vdash Y$ we have that $W \ \kappa \ \mathrm{Th}(v/Y)$. Moreover, if $p \in \mathsf{C}(X)$ is such that $W \ \kappa \ p$ then $\mathrm{Th}(v/Y) = p$ for $Y := \mathrm{Th}_{\mathsf{c}}(p) \cap X$.

Proof. The first statement is immediate from Lemma 2.38. Let $p \in \mathsf{C}(X)$ be such that $W \ \kappa \ p$. From the first statement we gather that $W \ \kappa \ \mathrm{Th}(v/Y)$. It is quite clear that $\mathrm{Th}(v/Y)$ and p make the same variables true. So Lemma 2.52 shows these points to be analogous. But, as the model is *concrete* through Corollary 2.55, we know these points to be equal, hence they have equal theories. □

Let us now introduce some notions that approximate the existence of covers. In Lemma 3.43, these properties are all related to one another. The following definition is a generalisation of the set Δ of Iemhoff (2001b, page 288), as already investigated in Goudsmit and Iemhoff (2014).

3.40 Definition (Vacuous Implications)
Let $v : P \to \mathcal{P}(X)$ be a model. The set of *vacuous implications of* v is defined as:

$$\mathsf{I}(v) := \{\phi \to \psi \in \mathcal{L}(X) \mid v \Vdash \phi \to \psi \text{ and } v \nVdash \phi\}.$$

The following Definition 3.41 is convenient in semantically describing points that are just shy of being covered by a given subset, as one can see in Lemma 3.42 below.

3.41 Definition (Comparable Above)
Let $v : P \to \mathcal{P}(X)$ be a model, let $W \subseteq P$ be a subset, and let $p \in P$ be a point. We say that W is *comparable above* p when the following holds, for all $q \geq p$.

$$\uparrow q \subseteq \uparrow W \text{ or } \uparrow W \subseteq \uparrow q$$

3.42 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model, let $W \subseteq P$ be a subset and $p \in P$ be a point. If W is comparable above p, and p is maximal with respect to $\uparrow W \subseteq \uparrow p$, then $W \ \kappa \ p$.

Proof. We need to prove that $\uparrow p = \uparrow W \cup \{p\}$. The inclusion from right to left holds by assumption. To prove the converse, let $q \geq p$ be given. If $p = q$ we are done, so assume $p < q$. The maximality of p ensures us that $\uparrow W \nsubseteq \uparrow q$. But we also know that $\uparrow W \subseteq \uparrow q$ or $\uparrow q \subseteq \uparrow W$, so $\uparrow q \subseteq \uparrow W$ must follow. This proves that $q \in \uparrow W$, as desired. □

The following Lemma 3.43 illustrates the partial internalisability of being comparable above. That is to say, W is comparable above some point in $v : P \to \mathcal{P}(X)$

precisely when the sub-theory $\mathsf{I}(v \upharpoonright W)$ of $\mathrm{Th}(v \upharpoonright W)$ holds on v. We thus capture a property of the model in propositional language. We speak of partial internalisation because the theory need not be finite in general, so the property is not fully expressed in one propositional statement. This can, however, be done when the model v is assumed to be image-finite. We prove this in Lemma 3.44.

3.43 Lemma
Let $v : p \to \mathcal{P}(X)$ be a refined model, let $W \subseteq P$ be finite, and let $p \in P$ be a point such that $\uparrow W \subseteq \uparrow p$. The following are equivalent:

(i) $v, p \Vdash \mathsf{I}(v \upharpoonright W)$;

(ii) W is *comparable above* p.

Proof. Assume (i) holds. We proceed by contradiction, so we assume there is some $q \geq p$ such that $\uparrow q \nsubseteq \uparrow W$ and $\uparrow W \nsubseteq \uparrow q$. The former ensures that for all $w \in W$ we know $w \nleq q$, and the latter proves that $q \nleq w$ for some $w \in W$. By refinedness, we thus know of $\psi_w \in \mathcal{L}(X)$ such that $v, w \Vdash \psi_w$ yet $v, q \nVdash \psi_w$ per $w \in W$. Again through refinedness, we know of a $\phi \in \mathcal{L}(X)$ such that $v, q \Vdash \phi$ and $w \nVdash \phi$ for some $w \in W$.

Note that $\psi := \bigvee_{w \in W} \psi_w$ is a bonafide formula, because W is finite. It follows that $v, W \Vdash \psi$ and $v, p \nVdash \psi$. Moreover, $v, W \nVdash \phi$ and $p \Vdash \phi$. As a consequence, we derive that $\phi \to \psi \in \mathsf{I}(v \upharpoonright W)$. Hence $v, q \Vdash \phi \to \psi$ holds as well. But now $v, q \Vdash \psi$ follows, a clear contradiction. This proves (ii).

To prove the other direction, assume that (ii) holds. Suppose that $v, p \nVdash \phi \to \psi$ for some $\phi \to \psi \in \mathsf{I}(W)$. This gives us a $q \geq p$ such that $v, q \Vdash \phi$ yet $v, q \nVdash \psi$. We distinguish two cases, either $\uparrow q \subseteq \uparrow W$ or $\uparrow W \subseteq \uparrow q$. In both cases, we immediately arrive at a contradiction through upwards persistency, proving (i). $\square$

3.44 Lemma
Let $v : P \to \mathcal{P}(X)$ be order-defined, and let $W \subseteq P$ be finite. The following are equivalent for any $p \in P$:

(i) W is comparable above p and $W \subseteq \uparrow p$;

(ii) $v, p \Vdash \bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W$ and $v, p \nVdash \bigvee_{w \in W} \mathsf{nd}\, w$.

Proof. It is easy to see that $v, q \nVdash \bigvee_{w \in W} \mathsf{nd}\, w$ holds precisely if $v, q \nVdash \mathsf{nd}\, w$ for all $w \in W$ and $q \in P$. Recall that by Definition 3.9, this is equivalent to $W \subseteq \uparrow q$. This proves that the second conjuncts of (i) and (ii) are equivalent.

By definition, (ii) holds if and only if for all $q \geq p$ one has $v, q \Vdash \mathsf{up}\, W$ whenever $v, q \Vdash \bigvee_{w \in W} \mathsf{nd}\, w$. This, in turn, is equivalent to the statement that for all $q \geq p$ one has $p \nVdash \mathsf{nd}\, w$ for all $w \in W$ or $q \Vdash \mathsf{up}\, W$. Through Definition 3.9, we see the former disjunct to be equivalent to $\uparrow W \subseteq \uparrow q$, whereas the latter is equivalent to $\uparrow q \subseteq \uparrow W$. The compound statement can now readily be seen equivalent to (i), finishing the proof. ☐

In Theorem 3.45 below, we pinpoint precisely when a finite set of points covers some point. This point is extensionally quantified in (i), (ii), and (iii), with the express reason that the points referred to in each of these statements may differ. Indeed, although the argument from (i) to (ii), leading to (iii), carries the same point over, the same cannot be said for the implication from (iii) to (i). In general, there may exist many points covered by the very same set, and there is no guarantee that we end up with the same point we started from. This matters not, as this guarantee would add little to the type of arguments we use this theorem for.

Observe that, through Lemma 3.44, one can internalise (iii) in the case that v is order-defined. The theorem, as stated below, applies to both the *canonical model* and the *universal model*, yet only the latter is guaranteed to be order-defined. In Section 3.5, we only apply it to *order-defined* models, as indicated at the start of the present section. It can also be applied to the *canonical model*, to semantically characterise the admissibility of a rule even in the case that the intermediate logic at hand does not enjoy the finite model property. We do not pursue this line of reasoning.

3.45 Theorem

Let $v : P \to \mathcal{P}(X)$ be a refined model, and let $W \subseteq P$ be finite. If P is *conversely well-founded*, then the following are equivalent:

(i) there exists a point $p \in P$ such that $W \ \kappa \ p$;

(ii) there exists a point $p \in P$ with $v, p \Vdash \mathsf{I}(v \restriction W)$ and $\uparrow W \subseteq \uparrow p$;

(iii) there exists a point $p \in P$ such that $\uparrow W \subseteq \uparrow p$ and W is comparable above p.

Proof. Suppose (i) holds, and let $\phi \to \psi \in \mathsf{I}(v \restriction W)$ be given. First, observe that $v, W \Vdash \phi \to \psi$. Second, note that $v, W \nVdash \phi$, hence $v, p \nVdash \phi$. Through Lemma 2.34, it is immediate that $v, p \Vdash \phi \to \psi$. This proves (ii).

The implication from (ii) to (iii) is immediate via Lemma 3.43. Finally, suppose (iii) holds. We know of a $p \in P$ such that both $\uparrow W \subseteq \uparrow p$ holds, and W is comparable above p. Pick a $q \geq p$ that is maximal with respect to the former. This is possible, as P was assumed to be conversely well-founded. Through 3.42, we obtain $W \ \kappa \ q$, proving (i). ☐

3.5. Semantics of rules

Think of an intermediate logic. Can one describe all its potential models? In the case of IPC, the answer is clear. Several such descriptions of intermediate logics were given in Section 2.4.1. Good examples include the logics BD_n, described by Lemmas 2.73 and 2.74, or the logics ND_n, described by Lemma 2.85.

The purpose of this section is to give analogous descriptions, not merely for logics, but for rules. The study of rules encompasses that of logics, as each theorem ϕ corresponds to a rule $\varnothing/\phi$. As there are many rules one could consider, we restrict attention to those given in Table 3.1. Much like when studying the frames on which a given, finitely axiomatised, intermediate logic is valid, one considers all substitution instances of its axiomatising formulae, we are concerned with all substitution instances of a given rule, or set thereof.

We already have a formalism for dealing with the body of rules generated by a particular set of rules. Indeed, the least structural consequence relation containing the given set of rules, be it single-conclusion or multi-conclusion, does the trick. This might seem an excessively heavy formalism for dealing with something as plain as all substitution instances of a set of rules. Our reason for using consequence relations is in circumventing some inconvenient edge-cases in the proof below. We indicate the first occurrence of this phenomenon as it arises.

First, we give a definition for closing the rules under substitutions as in Definition 3.46. Second, in Lemma 3.47, we show that the validity of a set of rules amounts to the validity of the least multi-consequence relation generated by these rules. Naturally, an analogous statement holds when one would restrict attention to single-conclusion rules and the least single-conclusion consequence relation they generate.

3.46 Definition (Closure under Substitutions)
Let $\mathcal{R}$ be a set of rules. The *closure under substitutions* of $\mathcal{R}$, denoted $\mathcal{R}^*$, is given by:

$$\mathcal{R}^* := \left\{\sigma(\Gamma)/\sigma(\Delta) \;\middle|\; \Gamma, \Delta \subseteq \mathcal{L}(X), \sigma : \mathcal{L}(X) \to \mathcal{L}(Y) \text{ and } \Gamma/\Delta \in \mathcal{R} \right\}.$$

3.47 Lemma
Let $\mathcal{R}$ be a set of rules, and let $\vdash$ be the least consequence relation containing $\mathcal{R}$. The following are equivalent for any model $v : P \to \mathcal{P}(X)$:

(i) all rules in $\vdash$ are valid on v;

(ii) all rules in $\mathcal{R}^*$ are valid on v.

Name	Rule
DP	$x \vee y \,/\, \{x, y\}$
Con	$\bot \,/\, \varnothing$
$\mathsf{DP}_n^{\neg\neg}$	$\bigvee_{i=1}^n x_i \,/\, \{\neg\neg x_i\}_{i=1}^n$
H	$\neg z \to (x \vee y) \,/\, \bigvee \{\neg z \to x, \neg z \to y\}$
M	$(x \to z) \to (x \vee y) \,/\, \big((x \to z) \to x\big) \vee \big((x \to z) \to y\big)$
V_n	$\bigwedge_{i=1}^n (x_i \to z_i) \to x_{n+1} \vee x_{n+2} \,/\, \bigvee_{j=1}^{n+2} (\bigwedge_{i=1}^n (x_i \to z_i) \to x_j)$
S_n	$\bigwedge_{i=1}^n (x_i \to z_i) \to \bigvee_{j=1}^n x_j \,/\, \bigvee_{j=1}^n (\bigwedge_{i=1}^n (x_i \to z_i) \to x_j)$
D_n^-	$(\bigvee_{i=1}^n x_i \to z) \to \bigvee_{j=1}^n x_j \,/\, \bigvee_{j=1}^n ((\bigvee_{i=1}^n x_i \to z) \to x_j)$
D_n	$y \vee \big((\bigvee_{i=1}^n x_i \to z) \to \bigvee_{j=1}^n x_j\big) \,/\, y \vee \bigvee_{j=1}^n ((\bigvee_{i=1}^n x_i \to z) \to x_j) \vee y$
$\overline{\mathsf{D}}_n$	$(\bigvee_{i=1}^n x_i \to z) \to \bigvee_{j=1}^n x_j \,/\, \big\{ (\bigvee_{i=1}^n x_i \to z) \to x_j \,\big|\, j = 1, \dots, n\big\}$

Table 3.1.: Several schemes of rules, both single conclusion and multi-conclusion, all of which are admissible in IPC.

Proof. The implication from (i) to (ii) is immediate, so we focus on the converse. First, know that $\vdash$ can be inductively described as:

(a) if $\Gamma/\Delta \in \mathcal{R}$ then $\sigma(\Gamma) \vdash \sigma(\Delta)$ for all substitutions σ;

(b) if $\Gamma \cap \Delta \neq \varnothing$ then $\Gamma \vdash \Delta$;

(c) if $\Pi \subseteq \Gamma$, $\Theta \subseteq \Delta$ and $\Pi \vdash \Theta$ then $\Gamma \vdash \Delta$;

(d) if $\Gamma \vdash \psi, \Delta$ and $\Pi, \psi \vdash \Theta$ then $\Gamma, \Pi \vdash \Delta, \Theta$.

We proceed by induction on the number of steps (a), (b), (c), (d) taken in the derivation of $\Gamma \vdash \Delta$. There are two base cases, (a) and (b). In the former, validity follows from (ii). In the latter, validity trivially holds.

The case (c) is also immediate, as Π/Θ is known to be valid on v by induction. Finally, we treat (d), so suppose $\Gamma \vdash \psi, \Delta$ and $\Pi, \psi \vdash \Theta$ both hold. By induction, both $\Gamma/\psi, \Delta$ and $\Pi, \psi/\Theta$ are valid on v. Suppose $v \Vdash \phi$ for all $\phi \in \Gamma \cup \Pi$. This yields a $\chi \in \Delta$ or $\chi = \Delta$ by the validity of the rule $\Gamma/\psi, \Delta$. In the case that $\chi \in \Delta$, we are done. In the other case, the validity of the rule $\Pi, \psi/\Theta$ finishes the argument. This proves (i), as desired. $\square$

We state the following Corollary 3.48 without proof, as it quite readily follows from the above. Recall that $\vdash_\Lambda^\mathcal{R}$ is the least structural consequence relation extending

$\mathcal{R}$ and $\vdash_{\Lambda}^{\mathsf{min}}$, the least multi-conclusion consequence relation extending the single-conclusion consequence relation associated to the intermediate logic Λ.

3.48 Corollary
Let Λ be an intermediate logic, and let $\mathcal{R}$ be a set of rules. The following are equivalent, for any model $v : P \to \mathcal{P}(X)$:

(i) the rules of $\vdash_{\Lambda}^{\mathcal{R}}$ are all valid on v;

(ii) the theorems of Λ and the rules of $\mathcal{R}^*$ are valid on v.

3.5.1. Variants of the disjunction property

The first rule we consider is DP, as given below. This rule is admissible precisely if the logic at hand enjoys the *disjunction property*. In Examples 3.56 and 3.57 we give examples of logics that do and that do not admit the rule DP, making use of the semantic description given in Theorem 3.54.

$$x \vee y / x, y$$

It is not hard to see that $\vdash^{\mathsf{DP}}$, the least structural multi-conclusion consequence relation containing DP, contains the following rule for each finite, non-empty set of formulae Δ:

$$\bigvee \Delta / \Delta \tag{3.14}$$

Indeed, one can readily infer this by an argument proceeding inductively along the size of Δ. The case $\Delta = \varnothing$ is intentionally excluded, as in this case, the rule amounts to $\bot/\varnothing$, treated in Example 2.13 under the name Con. Let this be the first rule we semantically characterise.

3.49 Lemma
Let $v : P \to \mathcal{P}(X)$ be a Kripke model. The rule Con is valid on v precisely if $P \neq \varnothing$.

Proof. See that $v \Vdash \bot$ precisely if $P = \varnothing$, from whence the desired is clear. □

3.50 Corollary
The least structural multi-consequence relation extending DP does not include the rule Con.

Proof. Suppose that it did. By Lemma 3.47, we know that every model on which DP in valid, also validates $\bot/\varnothing$. But DP is valid on the empty model, a contradiction with Lemma 3.49. □

As mentioned in Section 1.1.3, semantic descriptions of logics that admit DP have long since been known. In Definition 3.51, we provide a simple semantic property, and in Theorem 3.54 we show this property to hold in an order-defined model exactly if DP^* is valid on said model. For the sake of convenience, we occasionally say that a model has a semantic property, only to mean that its underlying frame has the property.

This description is applied to the universal model in Corollary 3.55, which gives a characterisation of those intermediate logics with the finite model property that admit DP. The appeal of this characterisation is that one need not be concerned with the totality of all models, as in Maksimova (1986, Theorem 1 and Proposition 2) and Galatos et al. (2007, Theorem 5.21), or with the intricate structure of the free algebra $\mathsf{F}_\Lambda(X)$, as in Example 3.53. Indeed, it suffices to know the universal model, which is a much more manageable, mathematical structure.

3.51 Definition (Downwards Directed)
A Kripke frame P is said to be *downwards directed* if for each finite, non-empty $W \subseteq P$ there is a $p \in P$ such that $W \subseteq {\uparrow}p$.

3.52 Example (Rooted Models)
Any rooted model is downwards directed. The converse is not necessarily true, though it is easy to see that it does hold in the finite case. That is to say, a finite model is downwards directed precisely if it is rooted.

3.53 Example (Canonical Model, Revisited)
Let Λ be an intermediate logic. Recall the *canonical model*, as introduced loosely in Theorem 2.22, and treated in more detail in Example 2.68. In addition, recall the free algebra functor $\mathsf{F}_\Lambda(-) : \mathbf{Set} \to \mathcal{V}_\Lambda$ introduced in Section 3.2, where $\mathcal{V}_\Lambda$ is as in Theorem 2.66. We define the *canonical model of* Λ to be:

$$\mathsf{c} : \mathsf{C}_\Lambda(X) := \mathsf{spec}(\mathsf{F}_\Lambda(X)) \to \mathcal{P}(X), \quad \mathfrak{p} \mapsto \{x \in X \mid x \in \mathfrak{p}\}.$$

The set of all theorems of Λ is a *filter*. Moreover, this filter is *prime* whenever Λ admits DP. It is easy to see that this prime filter is the root of $\mathsf{C}_\Lambda(X)$, making $\mathsf{C}_\Lambda(X)$ downwards directed via Example 3.52. This connection is tight: an intermediate logic Λ admits DP precisely if the canonical model $\mathsf{C}_\Lambda(X)$ is rooted for each finite set of variables X.

Now that we have seen several examples of models that are downwards directed, we are ready to use this property to semantically characterise the validity of $\vdash^{\mathsf{DP}}$ in order-defined models.

3.54 Theorem
Let $v : P \to \mathcal{P}(Z)$ be an order-defined model. The rules $\vdash^{\mathsf{DP}}$ are valid on v precisely if v is downwards directed.

Proof. Suppose that v is downwards directed, and let $\sigma : \mathcal{L}(\{x, y\}) \to \mathcal{L}(Z)$ be a substitution. We need to show that the rule:

$$\sigma(x \vee y) \Big/ \{\sigma(x), \sigma(y)\}$$

is valid on v. We proceed by contraposition, so assume $v \nVdash \sigma(x)$ and $v \nVdash \sigma(y)$. This yields w_z such that $v, w_z \nVdash \sigma(z)$ for each $z = x, y$.

Define the set $W := \{w_x, w_y\}$. By assumption, there exists a point $p \in P$ such that $W \subseteq {\uparrow}p$. Suppose that $v, p \Vdash \sigma(x \vee y)$. This yields $v, p \Vdash \sigma(z)$ and hence $w_z \Vdash \sigma(z)$ for some $z = x, y$, a contradiction. This proves that the rules DP^* are valid on v.

Conversely, suppose that the rules DP^* are valid on v, and let $W \subseteq P$ be a finite, non-empty subset. Recall Definition 3.9, and note that $v, w \nVdash \mathsf{nd}\, w$ for each $w \in W$. Know that the rule:

$$\bigvee_{w \in W} \mathsf{nd}\, w \Big/ \{\mathsf{nd}\, w \mid w \in W\}$$

is contained in $\vdash^{\mathsf{DP}}$, so we know it to be valid on v.[13] This proves $v \nVdash \bigvee_{w \in W} \mathsf{nd}\, w$, yielding a $p \in P$ such that $p \nVdash \mathsf{nd}\, w$ for each $w \in W$. The desired is immediate by Definition 3.9. □

In the above proof, we have been exceedingly precise in spelling out some of the details. We quantified over substitutions that take formulae using the variables occurring within the rule DP to the set of variables Z. In the following, we proceed a bit more loosely. Instead of considering substitutions explicitly, we quantify over formulae in the appropriate set of variables, and compose them in such a way that, in effect, we quantify over all substitution instances of the rule at hand.

The following Corollary 3.55 follows quite readily from Theorem 3.54 through the exactness and completeness of the universal model. We thus obtain a very simple characterisation of logics that admit the rule DP, through a straightforward argument.

[13]In the edge-case where $|W| = 2$, this rule is a direct substitution instance of DP. However, in every other case, we know it to be included in $\vdash^{\mathsf{DP}}$ due to the reasoning at the start of Section 3.5.1. Had we merely assumed DP^* to be valid on v, we would have had to reason to the effect of Corollary 3.48. It is important to note that the case $|W| = 0$ is explicitly excluded.

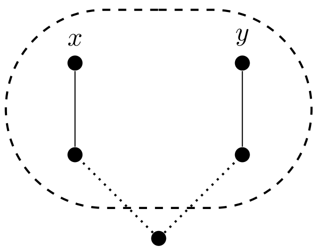


Figure 3.4.: Counterexample to the admissibility of DP in BD_2.

3.55 Corollary

Let Λ be an intermediate logic with the finite model property. The following are equivalent:

(i) the rule DP is admissible in Λ;

(ii) $\mathsf{U}_\Lambda(X)$ is downwards directed for each finite set X.

Proof. Suppose (i) holds. Because the universal model is exact, as shown in Example 3.28, Theorem 3.38 proves that every admissible rule is valid on the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ for each finite set of variables X. As $\mathsf{DP} \in \mathrel{\sim}_\Lambda$, we know $\mathsf{DP}^* \subseteq \mathrel{\sim}_\Lambda$. Theorem 3.38 thus readily proves (ii) through the above Theorem 3.54.

Conversely, suppose (i) does not hold. This provides us with a finite, non-empty set of formulae $\Delta \subseteq \mathcal{L}(X)$ such that $\vdash_\Lambda \bigvee \Delta$, yet $\nvdash_\Lambda \chi$ for each $\chi \in \Delta$. By Theorem 3.38, this means that the rule $\bigvee \Delta / \Delta$ is not valid on $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. Consequently, Theorem 3.54 shows that (ii) is false, proving the desired. $\square$

3.56 Example (A Logic that does not Admit DP)

Consider the intermediate logic BD_2, as semantically described by Corollary 2.75. Fix the set of variables $Z = \{x, y\}$, and consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{BD}_2}(Z) \to \mathcal{P}(Z)$. The marked four points of Fig. 3.4 constitute a generated submodel of this model.

Suppose BD_2 did admit DP. By Lemma 2.101 we know BD_2 to have the finite model property, hence Corollary 3.55 shows that u must be downwards directed. This ensures the existence of the bottom, fifth point in Fig. 3.4. Lemma 2.74 proves that this root must be of height at most 2, yet it clearly is not. In conclusion, we have thus proven that the intermediate logic BD_2 *does not* admit DP.

3.57 Example (Some Logics That Admit DP)
We know the intermediate logics IPC, BB_n, ML and KP to all have the finite model property by Lemma 2.101. The logics IPC, KP and BB_n have been shown to admit DP respectively by Gödel (1932), Kreisel and Putnam (1957, p. 75) and Gabbay and de Jongh (1974, Lemma 14).[14] A nice, syntactic proof that KP and ND_n admit DP for all $n \geq 2$ is given by de Jongh (2009). The intermediate logic ML has been proven to admit DP by Szatkowski (1981, Lemma 4.3). Through Corollary 3.55, this means that their universal models are all downwards directed.

Fix an intermediate logic Λ with the finite model property, and suppose that Λ admits DP. From Theorem 3.38, it immediately follows that DP is valid on every Λ-exact model. A slightly stronger statement in fact holds, as we show in Lemma 3.58 below.

3.58 Lemma
Let $f : P \to Q$ be a surjective map of Kripke frames. If P is downwards directed, then so is Q.

Proof. Suppose P is downwards directed, and let $S \subseteq Q$ be a finite set of points. Through the surjectivity of f, there exists a w_s such that $f(w_s) = s$ per $s \in S$. Consider the set :

$$W := \{w_s \mid s \in S\},$$

and know there exists a p such that $W \subseteq \uparrow p$. Now, see that:

$$S = f(W) \subseteq f(\uparrow p) = \uparrow f(p),$$

proving the desired. □

3.59 Corollary
Let Λ be an intermediate logic which admits DP. Now, every Λ-exact model is downwards directed.

Proof. This follows immediately from Lemma 3.58 and Corollary 3.55. □

3.60 Example
Consider an Λ-exact formula $\phi \in \mathcal{L}(X)$, in the sense of Definition 3.32. Through Theorem 3.33, we know that the following generated submodel of the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ is exact:

$$\mathsf{u} \upharpoonright [\![\phi]\!]_{\mathsf{u}} \to \mathcal{P}(X). \tag{3.15}$$

[14]See also Gabbay (1981, p. 84) and Gabbay (1981, p. 101) for proofs that BB_n and KP enjoy the DP respectively.

By Theorem 3.38, DP^* is valid on this model. This means that if $\phi \vdash_\Lambda \bigvee \Delta$, then $\phi \vdash_\Lambda \chi$ for some $\chi \in \Delta$ for any non-empty $\Delta \subseteq \mathcal{L}(X)$.

In Example 3.60 above, we showed that DP^* holds on the model as given in (3.60). When reformulating this, in the last sentence of the example, our conclusion would not have been at fault were to omit the constraint of non-emptiness on Δ. This can easily be derived through the following. In short, exact models are always non-empty, and always validate the rule Con.

3.61 Lemma
For every intermediate logic Λ, the Kripke frame $\mathsf{U}_\Lambda(X)$ is non-empty. Moreover, the rule Con is valid on every Λ-exact model.

Proof. This is immediate through Example 2.13, Lemma 3.49, and Theorem 3.38. □

The intermediate logic BD_2, as we saw in Example 3.56, does not admit the rule DP. We consider the rule $\mathsf{DP}_n^{\neg\neg}$ shown below, a weakening of DP, inspired by Skura (1992a, Theorem 4.2).

$$\bigvee_{1=1}^{n} x_i \Big/ \{\neg\neg x_i\}_{i=1}^{n}$$

When considering the rule DP, we noted that its binary formulation is not essential, in that it entails a formulation of every positive arity as in (3.14). This is most definitely *not* the case with the rule $\mathsf{DP}_n^{\neg\neg}$, which we prove in Lemma 3.67. Eventually, in Theorem 5.34, we employ the rule $\mathsf{DP}_n^{\neg\neg}$ to obtain a basis of admissibility for BD_2 and $\mathsf{BD}_2 + \mathsf{BW}_n$ with $n \geq 2$. First, though, we start with a semantic description of the validity of this rule, analogous to Definition 3.51 and Theorem 3.54. It is clear that a non-empty model is classically n-ary downwards directed whenever it is downwards directed, although the converse need not hold, which we indicate in Example 3.66.

3.62 Definition (Classically n-ary Downwards Directed)
A Kripke frame P is said to be *classically n-ary downwards directed* if for each finite $W \subseteq \max(P)$ with $|W| \leq n$ there is a $p \in P$ such that $W \subseteq {\uparrow}p$.

The above definition carries quite a long name; this is not without a good reason. Note that maximal points of any universal model are models of CPC, thus explaining the prefix "classically". The remainder of the name is by analogy with Definition 3.51. We, in passing, point out that $\max(P)$ consists precisely of those points that generate a submodel whose frame is a frame of the logic BD_0. One could generalise Definition 3.62 by replacing BD_0 with BD_m for a given m. We do not pursue this line of reasoning.

3.63 Theorem
Let $v : P \to \mathcal{P}(X)$ be an order-defined model. The rules $\vdash^{\mathsf{DP}_n^{\neg\neg}}$ are valid on v precisely if P is classically n-ary downwards directed.

Proof. Suppose $\mathsf{DP}_n^{\neg\neg}$ is valid on v, and let $W \subseteq \max(P)$ be such that $|W| \leq n$. Observe that $v, w \Vdash \phi \equiv \neg\neg\phi$ for all $\phi \in \mathcal{L}(X)$ and $w \in W$. Moreover, $v, w \nVdash \mathsf{nd}\, w$ for each $w \in W$. By the validity of $\mathsf{DP}_n^{\neg\neg}$, we know that $v \nVdash \bigvee_{w \in W} \mathsf{nd}\, w$, yielding a point $p \in P$ such that $W \subseteq {\uparrow}p$. The converse can be proven with similar ease. □

The following Corollary 3.64 is analogous to Corollary 3.55. As the proofs are virtually identical, we omit a proof here.

3.64 Corollary
Let Λ be an intermediate logic with the finite model property. The following are equivalent for any $n \in \mathbb{N}$:

(i) the rule $\mathsf{DP}_n^{\neg\neg}$ is admissible in Λ;

(ii) $\mathsf{U}_\Lambda(X)$ is classically n-ary downwards directed for each finite set X.

3.65 Example
Consider the intermediate logic BD_n for some $n \geq 2$, and fix an $m \geq 2$. We know BD_n to have the finite model property by Lemma 2.101. Moreover, we know BD_n to admit $\mathsf{DP}_m^{\neg\neg}$ whenever $\mathsf{u} : \mathsf{U}_{\mathsf{BD_n}}(X) \to \mathcal{P}(X)$ is classically m-ary downwards directed, as per Corollary 3.64. We can demonstrate that the latter indeed is the case.

To this end, fix some $W \subseteq \max(\mathsf{U}_{\mathsf{BD_n}}(X))$ satisfying $|W| \leq m$. We define the model:

$$v := (\mathsf{u} \restriction W)/\varnothing : \mathsf{U}_{\mathsf{BD_n}}(X)^+ \to \mathcal{P}(X),$$

using Definition 3.16. Clearly, this model is of *height at most* 2, hence Lemma 2.73 proves $v \Vdash \mathsf{BD}_n$ for each $n \geq 2$. There exists a unique map $f : v \to \mathsf{u}$ by Theorem 3.20, and the point $p := f(\rho)$ is such that ${\uparrow}p \supseteq W$. We thus have proven that each universal model of BD_n is classically m-ary downwards directed, hence the rule $\mathsf{DP}_m^{\neg\neg}$ is admissible for BD_n, for each $n \geq 2$ and $m \geq 2$.

3.66 Example
Recall Example 3.56, where we proved that the model

$$\mathsf{u} : \mathsf{U}_{\mathsf{BD_2}}(Z) \to \mathcal{P}(Z)$$

does not validate DP, where Z is a doubleton set. By Example 3.65 and Corollary 3.64, we know u to be a model on which $\mathsf{DP}_m^{\neg\neg}$ is valid for each $m \geq 2$. Hence, via Lemma 3.47, we know $\vdash^{\mathsf{DP}_m^{\neg\neg}}$ to *not* include DP.

3.67 Lemma
The rule $\mathsf{DP}_n^{\neg\neg}$ is admissible in $\mathsf{BD}_2 + \mathsf{BW}_m$ if and only if $m \geq n$.

Proof. A proof of the implication from right to left can be found in Example 3.65, after observing that the therein constructed model is also of *width at most* m, hence a model of BW_m by Theorem 2.79. The converse is also easy enough to prove.

Indeed, suppose $\mathsf{DP}_n^{\neg\neg}$ is admissible in $\Lambda := \mathsf{BD}_2 + \mathsf{BW}_m$ for some $m < n$. Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(Z) \to \mathcal{P}(Z)$, where $Z = \{z_1, \ldots, z_n\}$. There exist points $w_i \in \max(\mathsf{U}_\Lambda(Z))$ such that $w_i \Vdash z_j$ if and only if $i = j$. By Corollary 3.64, there must be a $p \in \mathsf{U}_\Lambda(Z)$ such that $w_1, \ldots, w_n \in {\uparrow}p$. Hence $\mathsf{u} \restriction {\uparrow}p$ is *not* of width m, a contradiction via Corollary 3.13 and Theorem 2.79. This proves the desired. ☐

3.5.2. Visser rules

In this subsection, we treat the true heroes of our story, the Visser rules. Visser rules come in many flavours, and spring from many different sources. They are surprisingly powerful, yet they can be quite simple. We meet them in many places throughout this thesis, most notably in Chapter 5. Here, we introduce them, and describe their semantics. Before we get to this, let us first give a brief overview of the variants we consider, in order of origination.

The following rule was introduced by Mints (1972). Through reasoning similar to that of Citkin (2012a, Proposition 1), one can readily derive that this rule is a special case of V_n below with n instantiated to 1.

$$(x \to z) \to (x \vee y) \Big/ \big((x \to z) \to x\big) \vee \big((x \to z) \to y\big) \tag{M}$$

Citkin (1977a) studied the rule M. Based on this work, he introduced a generalisation of Mints' rule in Citkin (1979b), given below. This rule was independently rediscovered by Rozière (1992, Section 7.1) and Iemhoff (2001b, p. 285), who used it to give a basis of admissible rules for IPC.

$$\bigwedge_{i=1}^{n} (x_i \to z_i) \to x_{n+1} \vee x_{n+2} \Bigg/ \bigvee_{j=1}^{n+2} \left(\bigwedge_{i=1}^{n} (x_i \to z_i) \to x_j \right) \tag{V_n}$$

During the eighties, Albert Visser and Dick de Jongh studied the admissible rules of Heyting Arithmetic HA.[15] Eventually, based on the pioneering work of Ghilardi (1999), Visser (1999) proved that the admissible rules of HA are equal to those of IPC. Iemhoff (2001b,c) provided a basis of the admissible rules of IPC, and thus of HA. She employed the rules V_n described above, which De Jongh and Visser throughout their studies believed to be a basis of the admissible rules for HA. Unfortunately, this conjecture was never explicitly written down. One can, however, find traces of this rule through their works, see for instance Visser (1994, p. 8) and Visser (1984, Theorem 1.1.11).[16]

Completely independently from all of the above, Skura (1989b, p. 75) introduced the rules shown in (S_n) below. He used these rules to characterise IPC by way of a Łukasiewicz-style refutation system, remedying a faulty conjecture by Łukasiewicz (1952). Iemhoff (2001a), independently, provided a kind of characterisation that intuitively feels quite similar to that of Skura. In Chapter 7, we treat these kinds of characterisations in detail.

$$\bigwedge_{i=1}^{n}(x_i \to z_i) \to \bigvee_{j=1}^{n} x_j \Bigg/ \bigvee_{j=1}^{n}\left(\bigwedge_{i=1}^{n}(x_i \to z_i) \to x_j\right) \tag{S_n}$$

We are interested in the rule scheme shown in (D_n^-) below. It naturally occurs as a restriction of S_n, and was already studied by Rozière (1992, Section 7.1).[17]

$$\left(\bigvee_{i=1}^{n} x_i \to z\right) \to \bigvee_{j=1}^{n} x_j \Bigg/ \bigvee_{j=1}^{n}\left(\left(\bigvee_{i=1}^{n} x_i \to z\right) \to x_j\right) \tag{D_n^-}$$

Do note that the rules as introduced have gotten progressively more specific; in the presence of IPC, any consequence relation that contains V_n must also contain S_n, which in turn encompasses D_n^-. Now that the rules have been introduced, let us get down to the semantics. The first property we consider is given in Definition 3.68. This property is based on what Iemhoff (2006) calls the *weak extension property*.

[15] As examples of this, see de Jongh (1982), de Jongh and Chagrova (1995), de Jongh and Visser (1996), Visser (1982, 1984), and Visser et al. (1994).

[16] We refer to Visser (2002) for the Visser rules in the context of HA (his Theorem 9.1) and more historical information (his Section 1.4).

[17] The indices Rozière (1992) employs are slightly different, but the difference is so minuscule this hardly merits a reformulation. Indeed, instead of limiting to n as we do in (D_n^-), he limits to $n + 1$.

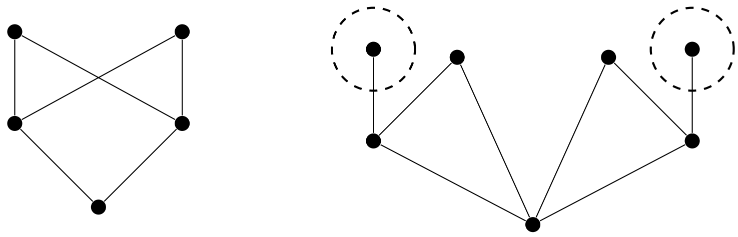


Figure 3.5.: A Kripke frame that is weakly 2-ary covered, and an example of a model that is *not* 2-ary covered.

3.68 Definition (Weakly n-ary Covered)
A Kripke frame P said to be *weakly n-ary covered* if for each finite set W and each $q \in P$ satisfying $W \subseteq {\uparrow}q$ and $|W| \leq n$ there exists some $p \in P$ such that $W \ \kappa \ p$. We say that P is weakly covered whenever it is weakly n-ary covered for all $n \in \mathbb{N}$.

3.69 Example
Consider the models depicted in Fig. 3.5. The left-hand model is weakly binary covered. Also note that the model constructed by taking two disjoint copies of this left-hand model is weakly binary covered, too. The right-hand model in Fig. 3.5, however, is *not* weakly binary covered. Indeed, the two marked points do not cover anything.

3.70 Theorem
Let $v : P \to \mathcal{P}(X)$ be an order-defined model, and let $n \in \mathbb{N}$. The rules $\vdash^{\mathsf{D}^-_n}$ are valid on v precisely if P is weakly n-ary covered.

Proof. Suppose that the rules D_n^{-*} are valid on v. Let $W \subseteq P$ and $q \in P$ be such that $W \subseteq {\uparrow}q$ and $|W| \leq n$. Consider an arbitrary $w \in W$, and note that $v, w \Vdash \mathsf{up}\, w$ holds, yet $v, w \Vdash \mathsf{nd}\, w$ does not hold. From here, it is easy to see that:

$$v \nVdash \bigvee_{s \in W} \left(\left(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W \right) \to \mathsf{nd}\, s \right).$$

The validity of D_n^{-*} on v ensures us that the following must hold as well, for some point $k \in P$.

$$v, k \Vdash \bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W \text{ and } v, k \nVdash \bigvee_{s \in W} \mathsf{nd}\, s$$

By Lemma 3.44, this proves that W is comparable above k, and $W \subseteq {\uparrow}k$. We thus obtain a $p \in P$ such that $W \ \kappa \ p$ through Theorem 3.45, proving the desired.

Conversely, suppose P is weakly n-ary covered, and let $\Delta \subseteq \mathcal{L}(X)$ and $\phi \in \mathcal{L}(X)$ be such that $|\Delta| \leq n$ and:

$$v \nVdash \bigvee_{\chi \in \Delta} (\bigvee \Delta \to \phi) \to \chi.$$

This yields a point $q \in P$ and points $w_\chi \geq q$ per $\chi \in \Delta$ such that:

$$v, w_\chi \Vdash \bigvee \Delta \to \phi \text{ and } v, w_\chi \nVdash \chi \text{ per } \chi \in \Delta.$$

Define $W := \{w_\chi \mid \chi \in \Delta\}$, and note that $|W| \leq |\Delta| \leq n$. As v is weakly n-ary covered, there must be a $p \in P$ such that $W \; \kappa \; p$. Persistence ensures that $v, p \nVdash \chi$ for each $\chi \in \Delta$. By Lemma 2.34, it automatically follows that:

$$v, p \Vdash \bigvee \Delta \to \phi \text{ and } v, p \nVdash \bigvee \Delta \,.$$

We have thus shown D_n^{-*} to be valid on v, as desired. □

3.71 Corollary

Let Λ be an intermediate logic with the finite model property. The following are equivalent for any $n \in \mathbb{N}$:

(i) the rule D_n^- is admissible in Λ;

(ii) $\mathsf{U}_\Lambda(X)$ is weakly n-ary covered for each finite set X.

We first give an example of a logic that admits D_n^- per $n \in \mathbb{N}$, immediately followed by a logic that does not admit D_2^-. As the admissibility of D_m^- entails the admissibility of D_n^- whenever $m \geq n$, this means that there are logics in which D_n^- is not admissible for all $n \geq 2$.

3.72 Lemma

The rule D_n^- is admissible in BB_n.

Proof. We can proceed via Corollary 3.71, as BB_n was shown to have the finite model property in Lemma 2.101. Let X be a finite set, and consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{BB_n}}(X) \to \mathcal{P}(X)$. Suppose that $W \subseteq \mathsf{U}_{\mathsf{BB_n}}(X)$ and $q \in \mathsf{U}_{\mathsf{BB_n}}(X)$ are such that $W \subseteq \uparrow q$ and $|W| \leq n$. We claim that the model $u := v/\varnothing \to \mathcal{P}(X)$ is a model of BB_n, where $v := \mathsf{u} \restriction (\uparrow W)$.

The desired readily follows through the claim. Indeed, note the model v is concrete, and hence *order-defined* by Corollary 3.12. We obtain a map $f : v- > \mathsf{u}$ by Theorem 3.20, and observe that the same theorem ensures that $f(W) = W$. Now, as $W \; \kappa \; \rho$, we know $W \; \kappa \; f(\rho)$ by Lemma 2.35, as desired. All that thus remains is

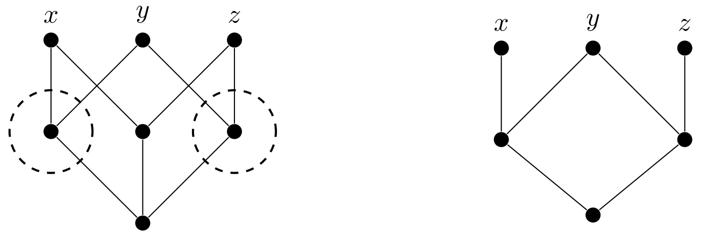


Figure 3.6.: Models to show that ND_2 does not admit D_2^-

proving the claim. Clearly, the model v is a rooted, finite, refined model, and its underlying frame is of branching degree at most n. We thus know v to be a model of BB_n by Lemma 2.89, finishing the argument. □

3.73 Example (D_2^- is not Admissible in ND_2)
Consider the intermediate logic ND_2, which we semantically described using Definition 2.84 in Lemma 2.85.[18] We know ND_2 to have the finite model property by Lemma 2.101, so Corollary 3.71 shows that ND_2 admits D_2^- precisely if $\mathsf{U}_{\mathsf{ND}_2}(X)$ is weakly binary covered for each finite set X. Let us prove that this is *not* the case.

First, consider the left-hand Kripke model $v : P \to \mathcal{P}(X)$ depicted in Fig. 3.6, where $X = \{x, y, z\}$. There are precisely three elements in $\mathsf{max}(P)$, and to each subset of the set of these three points there is exactly one point that sees precisely this subset. Consequently, through the semantic description of ND_2 given in Lemma 2.85, we know v to be a model of ND_2. Moreover, this model is clearly *concrete*, hence it occurs as a generated submodel of the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{ND}_2}(X) \to \mathcal{P}(X)$ via Lemma 3.25.

Suppose u were weakly binary covered. There are two marked points in the left-hand model of Fig. 3.6, call the set of these points W. We know that there must be a $q \in \mathsf{U}_{\mathsf{ND}_2}(X)$ such that $W \kappa q$. The resulting model $\mathsf{u} \restriction (\uparrow q)$ is the right-hand model in Fig. 3.6.

Note that $\mathsf{u} \restriction (\uparrow q)$ could not possibly be a model of ND_2. Indeed, fix the doubleton $S \subseteq \mathsf{max}(P)$ consisting of those points on which x and z is valid, and see that although $S \subseteq \uparrow q$, there is no $k \geq q$ such that $\mathsf{max}(\uparrow k) = W$. This violates the semantic description of an ND_2 model, finishing the argument.

[18] One may wonder why we do not apply this reasoning to ND_1. The reason is simple: ND_1 equals IPC, and D_2^- surely is admissible there.

The line of reasoning portrayed in Example 3.73 can readily be generalised to prove Lemma 3.74 below, which we state without proof.

3.74 Lemma
The rule D_n^- is not admissible in ND_m for each $n \geq 2$ and $m \geq 2$.

Through Lemma 3.58, we know that being downwards directed is preserved by surjective maps of Kripke frames. One can wonder whether the same holds for being weakly n-ary covered. It does, as we state in Lemma 3.75 show that this is the case.

3.75 Lemma
Let $f : P \to Q$ be a surjective map of Kripke frames, and suppose that P is weakly n-ary covered for a fixed $n \in \mathbb{N}$. Now, Q is weakly n-ary covered as well.

Proof. Let $S \subseteq Q$ and $q \in Q$ be such that $S \subseteq {\uparrow} q$ and $|S| \leq n$. First, pick some $p \in P$ such that $f(p) = q$. Because $f(p) \leq s$, we can choose a $w_s \geq p$ per $s \in S$. Define $W := \{w_s \mid s \in S\}$, and know that there exists $k \in P$ such that $W \ \kappa \ k$. Through Lemma 2.35, we know that $f(W) \ \kappa \ f(k)$. Now, as $f(W) = S$, the desired is proven. □

Through the semantic description of Theorem 3.70, it is easy to see that D_n^- must be valid on any finite, non-empty chain. Something even stronger holds. Indeed, the formula (3.16) below is valid on every chain. Lemma 2.82 shows us that any chain is a model of LC, and Lemma 3.76 below shows that the rules D_n^- are derivable (and thus admissible) in all intermediate logics that extend LC. We state this lemma without proof, as the reader can readily construct the required derivations.

3.76 Lemma (Rozière, 1992, Section 8.2)
Let Λ be any intermediate logic that extends LC. The formula below is a theorem of Λ for each $n \in \mathbb{N}$. In particular, the rule D_n^- is both admissible and derivable in Λ.

$$\left(\bigvee_{i=1}^{n} x_i \to z\right) \to \bigvee_{j=1}^{n} x_j \to \bigvee_{j=1}^{n} \left(\left(\bigvee_{i=1}^{n} x_i \to z\right) \to x_j\right) \tag{3.16}$$

Recall the *subframe logics*, as treated in Section 2.4.2. Through the above semantic description of D_n^-, one can readily prove the admissibility of this rule in all subframe logics. As there are continuum many subframe logics, this also means that the rule D_n^- is admissible in continuum many intermediate logics.

3.77 Theorem
The rule D_n^- is admissible in all intermediate *subframe logics*.

Proof. Let Λ be a subframe logic. Theorem 2.97 shows us that Λ has the finite model property, hence we can proceed via Corollary 3.71. Let X be a finite set of variables, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. Take a subset $W \subseteq \mathsf{U}_\Lambda(X)$ and a point $q \in \mathsf{U}_\Lambda(X)$ such that $W \subseteq \uparrow q$. Our goal is to find a point $p \in P$ such that $W \mathrel{\kappa} p$.

We define $U := \uparrow q$, and consider the model $\mathsf{u} \upharpoonright U : U \to \mathcal{P}(X)$. First, note that this model is finite, as the universal model is image-finite by definition. Second, we know this model to be refined by Theorem 3.20 and Corollary 3.12. Combining these two observations with Corollary 3.13 allows us to deduce that U, when considered as a Kripke frame, is a frame of Λ.

The set $W \cup \{q\} \subseteq U$ is a subset of the frame U of Λ. Because Λ is a subframe logic, the Kripke frame $Q := U \upharpoonright \big(W \cup \{q\}\big)$ is a frame of Λ. Hence, the model $u := \mathsf{u} \upharpoonright Q : Q \to \mathcal{P}(X)$ is a model of Λ. There exists a unique map of Kripke models $f : u \to \mathsf{u}$. Note that $q \mathrel{\kappa} W$ holds in Q, hence Lemma 2.35 ensures us that $f(W) \mathrel{\kappa} f(q)$. But $f(w) = w$ clearly holds for each $w \in W$, proving $W \mathrel{\kappa} f(q)$, as desired. ☐

3.78 Corollary
The rule D_n^- is admissible in IPC, CPC, BD_n, BW_n, LC and G_n.

Proof. Immediate by Lemma 2.96 and Theorem 3.77. ☐

In Section 2.4.2, we also discussed *stable logics*. One can perform the same kind of construction as in Theorem 3.77 to prove that stable logics, too, admit D_n^-. We make this explicit in Theorem 3.79 below.

3.79 Theorem
The rule D_n^- is admissible in all *stable logics*.

Proof. Let Λ be a stable logic, let X be a finite set of variables, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. Take a subset $W \subseteq \mathsf{U}_\Lambda(X)$ and a point $q \in \mathsf{U}_\Lambda(X)$ such that $W \subseteq \uparrow q$. The frame $\uparrow q := U$ is a frame of Λ, via Corollary 3.13.

Consider the Kripke models $v := \mathsf{u} \upharpoonright U$ and $u = \mathsf{u} \upharpoonright Q$, where $Q := \uparrow W$. It readily follows from Corollary 3.13 that the underlying frame of v is a frame of Λ. We now define the map:

$$g : U \to Q^+, \quad p \in P \mapsto \begin{cases} \rho & \text{if } p \notin U, \\ p & \text{otherwise.} \end{cases}$$

The map g is monotonic. This proves that Q^+ is a frame of Λ, so $u/\varnothing$ is a model of Λ. Note that $u/\varnothing$ is both finite and refined, hence it is order-defined by Corollary 3.12. Due to Lemma 3.25, we know of a unique map of Kripke models $f : u/\varnothing \to \mathsf{u}$. One readily infers $f(W) = W \ltimes f(\rho)$ from Lemma 2.35, proving the desired. □

3.80 Corollary
The rule D_n^- is admissible for IPC, CPC, BW_n, KC, and LC.

Proof. This follows immediately through Lemma 2.99 and Theorem 3.79. □

It is often convenient to consider the rules DP, Con and D_n^- at the same time. The rule scheme $\overline{\mathsf{D}}_n$, described below, fulfils precisely this task.

$$\left(\bigvee_{i=1}^{n} x_i \to z\right) \to \bigvee_{j=1}^{n} x_j \Big/ \left\{\left(\bigvee_{i=1}^{n} x_i \to z\right) \to x_j\right\}_{j=1}^{n} \tag{$\overline{\mathsf{D}}_n$}$$

It is easy to see that $\overline{\mathsf{D}}_0$ trivialises to the following rule. Any multi-conclusion consequence relation that contains $\vdash_{\mathsf{IPC}}^{\mathsf{min}}$ contains this rule precisely if it contains Con. From this, it is easy to see that $\overline{\mathsf{D}}_0$ is admissible precisely if Con is admissible.

$$(\bot \to z) \to \bot \big/ \varnothing$$

The rule $\overline{\mathsf{D}}_1$ trivial, in that it is contained in every multi-conclusion consequence relation. From $n \geq 2$ onwards, however, the rule $\overline{\mathsf{D}}_n$ becomes more interesting. In this case, it is easy to see that $\overline{\mathsf{D}}_n$ is contained in a multi-conclusion consequence relation whenever both DP and D_n^- are.

3.81 Example
The rule $\overline{\mathsf{D}}_n$ is admissible in IPC and BB_n for each $n \in \mathbb{N}$. Indeed, these logics admit DP due to Example 3.57, and they admit D_n^- by Corollary 3.78 and Lemma 3.72 respectively. Later, in Chapter 7, we prove that IPC is the sole *subframe logic* (and also the sole *stable logic*) which admits $\overline{\mathsf{D}}_n$ for any $n \geq 2$.

We close this section with the rule given below, which can be seen as a generalisation of M.

$$y \vee \left(\left(\bigvee_{i=1}^{n} x_i \to z\right) \to \bigvee_{j=1}^{n} x_j\right) \Big/ y \vee \bigvee_{j=1}^{n} \left(\left(\bigvee_{i=1}^{n} x_i \to z\right) \to x_j\right) \vee y \tag{D_n}$$

Semantics for the rule D_n can be given in much the same way as semantics for D_n^-. In Definition 3.82 below, we suitably adapt Definition 3.68 to fit D_n. The precise correspondence between validity and semantics is stated in Theorem 3.83. We omit the proof, as it is fairly similar to that of Theorem 3.70.

3.82 Definition (Strongly n-ary Covered)
A Kripke frame P is said to be *strongly n-ary covered* if for each finite set W and each $q \in P$ satisfying $W \subseteq \uparrow q$ and $|W| \leq n$ there exist $p, p_+ \in P$ such that $W \; \kappa \; p$ and $p_+ \leq p, q$. We say that P is strongly covered whenever it is strongly n-ary covered for all $n \in \mathbb{N}$.

3.83 Theorem
Let $v : P \to \mathcal{P}(X)$ be an order-defined model, and let $n \in \mathbb{N}$ be arbitrary. The rules $\vdash^{\mathsf{D}_n}$ are valid on v precisely if v is strongly n-ary covered.

3.84 Corollary
Let Λ be an intermediate logic with the finite model property. The following are equivalent for any $n \in \mathbb{N}$:

(i) the rule D_n is admissible in Λ;

(ii) $\mathsf{U}_\Lambda(X)$ is *strongly n-ary covered* for each finite set X.

3.85 Lemma
The rule D_n is admissible in BB_n and IPC for each $n \geq 2$.

3.6. Failure of the finite model property

All the intermediate logics we have considered enjoy the finite model property. That is to say, a formula is false precisely if it is falsified on a finite frame. One might wonder whether a similar property could hold for the admissible rules of an intermediate logic. In this section, we show that this is often not the case. In fact, not even IPC has the finite model property for admissible rules.

Before we continue, let us first formally define what we mean by the finite model property. This definition makes use of Definition 3.37, which pins down what we mean by soundness and completeness. Note that we might as well replace finite Kripke frame with finite, refined model, as per Corollary 3.13. As any model can be made refined while preserving the theory, we might even drop this proviso.

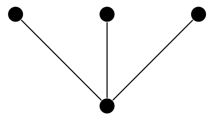


Figure 3.7.: The three-fork.

3.86 Definition (Finite Model Property for Rules)
Let $\mathcal{R}$ be a set of rules. We say that $\mathcal{R}$ has the *finite model property* if there exists a class of finite Kripke frames $\mathcal{K}$ such that $\mathcal{R}$ is both sound and complete with respect to $\mathcal{K}$.

Consider a frame that is not of *width at most 2*. By the very definition of width, we now know that the three-fork, as depicted in Fig. 3.7 must occur within this frame. The following lemma shows that the presence of the three-fork yields an infinity of points, whenever the model has the binary offspring property. Pictorially, the lemma is extremely straightforward; Fig. 3.8 basically says it all. To be a bit more precise, we do spell out the details below.

3.87 Lemma
Let P be a frame that is not of *width at most 2*, and, in addition, assume that P is strongly binary covered. Now, P must be infinite.

Proof. We prove that for each natural number n there exists a sequence of subsets $W_0 \subseteq W_1, \ldots \subseteq W_n \subseteq P$ satisfying the following three properties:

(i) the elements in W_i are pairwise incomparable;

(ii) there exists a $p \in P$ such that $W_i \subseteq {\uparrow}p$ for each $i = 0, 1, \ldots, n$;

(iii) $W_j \subset {\uparrow}W_i$ for all $j < i \leq n$.

Assuming such a sequence, it is immediate that $\bigcup_i W_i$ is infinite, proving the desired.

We need but show that one can construct said sequence, which we readily do by induction along its length. In the base case $n = 0$ there is little to do. As P is not of width at most 2, there must be points $p, w_0, w_1, w_2 \in P$ such that $p \leq w_0, w_1, w_2$, satisfying the additional condition that $W_0 := \{w_0, w_1, w_2\}$ be an *anti-chain*. Properties (i) and (ii) follow immediately from our assumption, and property (iii) holds vacuously.

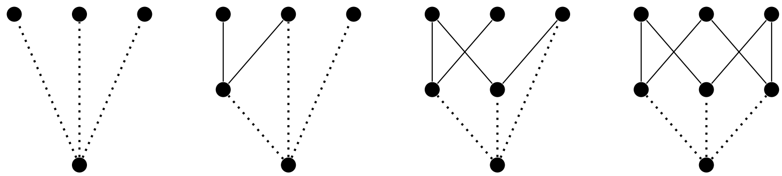


Figure 3.8.: Extending one layer in Lemma 3.87.

Now, suppose we have such a chain $W_0 \subseteq \ldots \subseteq W_n$, satisfying the three conditions. By property (ii), there exists a point $p_0 \leq w$ for all $w \in W_n$. Note that there are precisely 3 subsets of size 2 within W_n, let us call these S_1, S_2 and S_3. Because P is strongly binary covered, we know of six points p_j, p_j^+ with $j = 1, 2, 3$, satisfying $p_{i+1} \leq p_i, p_{1+1}^+$ and $W_i \; \kappa \; p_i$ for $i = 0, 1, 2$. We claim that $W_{n+1} = \{p_1, p_2, p_3\}$ does the trick.

Property (i) holds through immediate verification. Indeed, suppose that $p_i < p_j$ holds for some $j \neq i$. Recall that $S_i \; \kappa \; p_i$, so there must be a $w \in W_i$ such that $w \leq p_j$. This entails that two elements in W_n are comparable, a contradiction by induction.

To show property (ii), simply observe that $p = p_3^+$ does the trick. Finally, property (iii) holds immediately by construction. □

We now have sufficient machinery to prove this section's main theorem. Its proof is but a simple composition of the elementary results gathered above. The implications of this Theorem 3.88 are, however, quite far reaching. One can apply this theorem to many of the intermediate logics we considered above; see Corollary 3.89 for some examples.

Rybakov, Kiyatkin, and Oner (1999) proved the failure of the finite model property for admissible rules in a great variety of modal logics, including K4, S4 and GL. Moreover, they describe conditions under which the finite model property for admissible rules does hold. Building on these techniques, Fedorishin and Ivanov (2003) proved the failure of the finite model property for admissibility in many intermediate logics. Their argument requires a fair amount of machinery, which we are able to hide away in Theorem 3.88 below.

3.88 Theorem

Let Λ be an intermediate logic that admits the rule D_2, and furthermore assume that Λ does not extend BW_2. Now, the set of single-conclusion admissible rules of Λ does not have the *finite model property*.

Proof. We proceed by contradiction, that is to say, suppose there does exist a class of finite frames $\mathcal{K}$ such that the following holds for all formulae ϕ and ψ.

$$P \Vdash \phi/\psi \text{ for all } P \in \mathcal{K} \text{ iff } \phi/\psi \text{ is admissible in } \Lambda. \tag{3.17}$$

We consider the rule below, and note that this rule is admissible precisely if $\mathsf{BW}_2 \in \Lambda$.

$$\top \Big/ \bigvee_{i=0}^{2} \left(x_i \to \bigvee_{j \neq i} x_j \right)$$

By assumption, this is not the case.

We thus obtain a $P \in \mathcal{K}$ and a $v : P \to \mathcal{P}(X)$ such that $v, P \Vdash \top$ yet $v, P \nVdash \mathsf{BW}_2$. As a consequence, $P \nVdash \mathsf{BW}_2$, hence Theorem 2.79 allows us to conclude that P is *not* of *width at most* 2.

As the rule D_2 was assumed to be admissible in Λ, we know that this rule is valid on P. By Theorem 3.83, it follows that P has the binary offspring property. This proves that P is infinite through Lemma 3.87, *quod non.* □

Note that, in the above (3.17), one could weaken the constraint of being complete with respect to admissible rules of Λ to that of being complete with respect to the theorems of Λ. That is to say, we need only know that each model is *sound* with respect to all admissible rules (or indeed, a very particular subset thereof), and furthermore impose that said models collectively are complete for the theorems of Λ.

3.89 Corollary
Neither IPC nor any of the logics BB_n for $n \geq 2$ have the finite model property for their admissible rules

Proof. Immediate by Theorem 3.88 and Lemma 3.85 □

As a minor application of Lemma 3.87, we provide the following Corollary 3.90. In particular, this shows that BD_2 does not admit the rule D_2. Do recall that BD_2 admits D_2^-, so we can answer Problem 8 in the positive. It goes without saying that BD_1 does admit D_2, as BD_1 equals CPC.

3.90 Corollary
The rule D_2 is not admissible in BD_n for all $n = 2, 3, \ldots$.

Proof. Suppose that BD_n admits D_2. Let $Z = \{x, y\}$, and consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{BD}_n}(Z) \to \mathcal{P}(Z)$. First, note that $\mathsf{U}_{\mathsf{BD}_n}(Z)$ is finite, as follows through a straightforward inductive argument. Second, the frame $\mathsf{U}_{\mathsf{BD_n}}(Z)$ can readily be seen to not be of width at most 2. We immediately obtain a contradiction with Lemma 3.87 through Theorem 3.83, proving the desired. □

4

Decidability

Can one know the rules of a logic? More precisely: given an intermediate logic, does there exist an algorithm that decides whether a rule is admissible in said logic? This is most certainly not possible in general. Indeed, there are plenty of logics in which even the set of theorems cannot be algorithmically decided, not even when the logic at hand is finitely axiomatizable.[1] In this chapter, we consider the special case of IPC, and prove that this logic's admissible rules are decidable.

What methods could one employ towards proving decidability of admissible rules? One could try to repurpose some of the familiar machinery commonly used to prove the decidability of a logic's theorems. Since Harrop (1958), it is known that any finitely axiomatizable intermediate logic with the finite model property must have a decidable set of theorems. A first try would be to consider the finite model property

[1] Harrop (1965, pp. 280–283) and Friedman (1975, Problem 39) asked whether the finite axiomatizability of a logic ensures its decidability. This was answered in the negative by Shehtman (1978a), see also the review by Urquhart (1985).

for admissible rules. Unfortunately, as shown in Section 3.6, the finite model property fails for many intermediate logics in general, and IPC in particular. One thus needs a more sophisticated argument to prove the decidability of admissible rules.

The well-known technique of filtration, as pioneered by Lemmon and Scott (1977, Section 3), is one of the methods by which one can prove a logic to enjoy the finite model property. This technique has been considered through many lenses, see for instance Lemmon (1966a,b) for an algebraic perspective and Bezhanishvili, Bezhanishvili, and Iemhoff (2014), Conradie et al. (2013), and Ghilardi (2010) for recent developments. One can also apply filtration techniques to the study of admissible rules, an approach which goes back to Rybakov (1981). This eventually lead to the original proof for the decidability of IPC's admissible rules, as presented by Rybakov (1984a). His proof proceeds via the Gödel–McKinsey–Tarski translation.[2] A direct argument, albeit in the setting of the minimal logic of Johansson (1937), was presented in Odintsov and Rybakov (2013).

Let us paint a picture of this proof in broad brushstrokes. After this messy painting, we indicate how this argument is presented in this chapter, and how it fits into the totality of this thesis. First, recall the exact models as discussed in Section 3.3. By virtue of being models of admissibility, IPC-exact models enjoy certain semantic properties. When one takes the maximal filtration of an exact model through an adequate set of formulae, one is left with a model that still bears some semblance to its original; the semantic features did not fade away entirely. These remnants are enough to intrinsically characterise those models that have arisen in such a way, and this description is *effective*. Moreover, there are but finitely many such models when one filtrates through a finite set. To check whether a rule is admissible, it suffices to verify its validity in the thus obtained finite set of finite models obtained via filtrating through the finitely many subformulae of the rules at hand. This is, in essence, what the argument of Rybakov amounts to.

At the time of writing, three techniques towards proving the decidability of admissible rules can be found in the literature: those of Rybakov (1984a), Ghilardi (1999) and Rozière (1992). We restrict attention to the former two.[3] The material presented in this chapter roughly corresponds to said approaches. First, spanning Sections 4.1 to 4.3, we treat several alternative notions of semantics for admissibility. In particular, we consider *projective formulae* and *extendible formulae*, which play a central

[2] As modal logics fall outside the scope of this thesis, we do not expand on this. See McKinsey and Tarski (1948, Theorem 5.1), Dummett and Lemmon (1959), and Rybakov (1997, Section 2.7) for more details.

[3] Rozière (1992, Proposition 5.6.2) proved the admissible rules of IPC to be decidable using syntactic means. As attested by Rozière (1993), this happened after having seen the approach by Dekkers (1995) to a problem solved earlier by Mints (1972). We do not go deeper into this approach, mostly for lack of space. We do refer to Iemhoff and Rozière (2012) for recent work along this vein.

role in constructing a basis of admissibility in Chapter 5. These notions are derived from the technique developed by Ghilardi (1997, 1999, 2004). Moreover, we introduce generalisations of these notions, considering a weakening of the constraints imposed upon maps. Intuitively, one can think of these as the "filtrated remnants" alluded to in the painting above. Second, in Section 4.4, we show how to interrelate these notions, proving the admissible rules of IPC to be algorithmically decidable. We now go over the sections of this chapter in some more detail.

The chapter opens with Section 4.1, in which we introduce *extendible models*. Exact models were defined *extrinsically*; they are those models that occur as the image of the universal model under a definable map. As such, they inherit some semantic properties of the universal model, as many such properties were shown to be preserved under maps of Kripke frames, see for instance Lemmas 3.58 and 3.75. Moreover, it is rather straightforward to prove that such models are both sound and complete with respect to admissible rules, as we showed in Theorem 3.38. Extendible models are their *intrinsically* described counterparts; they are those models that enjoy the semantic properties we have come to expect from exact models.

The notion of extendible models is not ours. We consider a straightforward generalisation that is quite common in the literature. Indeed, our definition of IPC-extendible amounts to the co-cover property of Rybakov (1993), it corresponds closely to the extension property of Ghilardi (1999, p. 866) and Iemhoff (2001b, p. 284), and coincides precisely with being extendible as defined by Bezhanishvili and de Jongh (2012, Definition 4.7).

We restrict attention to those extendible models that arise as definable upsets of universal models in Section 4.1.1. In Section 4.1.2, we show that this is no real restriction in seven intermediate logics, including IPC and BD_2, using the technique of *uniform interpolation*. Finally, we introduce *projective formulae* in Section 4.1.3, and prove that a formula is projective precisely if the corresponding upset in the universal model is extendible. This argument is but a straightforward adaptation of Ghilardi (1999, Theorem 5), but as it plays quite the crucial role in Chapters 5 and 6, we include it in full.

In Section 4.2, we introduce all the necessary notions for performing filtration. We weaken the restriction imposed upon a map of Kripke frames to satisfying the *closed domain condition*, as studied by Zakharyaschev (1992, p. 1383). This notion has already been employed very fruitfully in the study of admissibility by Jeřábek (2009), see also Bezhanishvili and Bezhanishvili (2013), Bezhanishvili, Bezhanishvili, and Iemhoff (2014), and Bezhanishvili and de Jongh (2014) for other recent developments around this notion. In Section 4.2.2, we weaken the notion of *cover* correspondingly, in such a way that analogues of Lemmas 2.34 and 2.35 can be shown to hold. Finally, Section 4.2.3 suitably generalises the notion of definable maps to this new setting. Substitutions correspond to these new-found *adequate maps* in a manner encompassing the correspondence seen between substitutions and exact models.

A quite different perspective is taken in Section 4.2.1. Here, we consider the syntactic side of the story. The reasoning presented here is a generalisation of some of the argument spread throughout the work of Iemhoff (2001b, 2005, 2006). These arguments play a crucial role in providing a basis for the admissible rules of BB_n as given in Chapter 5. In Section 4.3, we provide necessary and sufficient conditions for a formula to be extendible using the machinery of Section 4.2.1. These conditions can readily be seen to be effective. The material mentioned in this paragraph appeared earlier, albeit in slightly different form, in Goudsmit and Iemhoff (2014).

This chapter closes with Section 4.4, wherein we give an exposition of Rybakov's approach to proving the decidability of IPC's admissible rules. The main theorem is Theorem 4.79, which proves the admissible rules of IPC to be decidable. The work presented here, together with the work of Sections 4.2.2 and 4.2.3, appeared in Goudsmit (2014b). Rybakov's approach has proven to be both powerful and of great applicability, and it has given rise to numerous results over the past three decades. We claim no originality in the results presented in Section 4.4; instead we offer originality in composition and presentation. The purpose of this exercise is to clarify and celebrate a central result in the study of admissibility, in the hope that connecting it to known concepts in novel ways leads to alternative avenues of generalisation.

4.1. Extendible models

Exact models, as treated in Section 3.3, provide us with sound and complete semantics for IPC. On the one hand, their definition is quite natural. Indeed, Λ-exact models are given in such a way that they are guaranteed to encode the theory of a substitution. The soundness and completeness of the admissible rules of Λ with respect to Λ-exact models, as covered in Theorem 3.38, can thus be proven quite smoothly. On the other hand, their definition is not at all *intrinsic*; it refers to a definable map that must exist somewhere outside the model. In this section, we lay the groundwork towards giving a more *intrinsic* definition.

Such an intrinsic description is of use for roughly two different purposes. First, it plays a central role in providing bases of admissibility, as we describe in Section 5.1. Second, it yields a motivating example for the type of models we introduce towards proving the decidability of IPC's admissible rules. This latter purpose is what we pursue here.

The definition of Λ-exact models was purposely formulated in a general manner. Nonetheless, it immediately follows from the definition that every Λ-exact model is image-finite. Through Theorem 3.20, we thus know that it occurs as a generated submodel of the universal model. Here, we restrict ourselves immediately to this setting, as it allows for a more convenient formulation of the aspired intrinsic definition. Let us start with an example.

4.1 Example
Consider IPC, and fix an upset $U \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$. This upset generates a submodel $\mathsf{u}_U : \mathsf{u} \restriction U \to \mathcal{P}(X)$ of the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$. Suppose that the model u_U is IPC-exact. As we know exact models to be both sound and complete with respect to admissible rules, for this was proven in Theorem 3.38, it follows that all admissible rules of IPC are valid on u_U. We mention but three consequences.

First, the Kripke frame U must be *non-empty* due to Lemma 3.49 and the admissibility of the rule Con. Second, the Kripke frame U is *downwards directed* by Theorem 3.54, as DP is admissible in IPC. Third, Theorem 3.70 proves that U is *weakly n-ary covered* as D_n^- is admissible for all $n \in \mathbb{N}$.

The three observations made in Example 4.1 can be conveniently encoded by the following general definition. In Theorem 4.4, we show that in the specific case of IPC, these facts coincide precisely with Definition 4.2.

4.2 Definition (Extendible Model)
Let Λ be an intermediate logic, let X be a finite set of variables, and let $U \subseteq \mathsf{U}_\Lambda(X)$ be an upset of the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. We say that U is *Λ-extendible* if for each $p \in \mathsf{U}_\Lambda(X)$ and each $W \subseteq U$ satisfying $W \;\kappa\; p$ there exists a $q \in U$ such that $W \;\kappa\; q$. The generated submodel $\mathsf{u}_U := \mathsf{u} \restriction U : U \to \mathcal{P}(X)$ is said to be a *Λ-extendible model*.

4.3 Example (Consistency)
Consider a Λ-extendible upset $U \subseteq \mathsf{U}_\Lambda(X)$. It follows that $U \neq \varnothing$. Indeed, we know $\mathsf{U}_\Lambda(X)$ to be non-empty by Lemma 3.61. Hence, as $\mathsf{U}_\Lambda(X)$ is image-finite, we can pick a $p \in \mathsf{U}_\Lambda(X)$ such that $\varnothing \;\kappa\; p$. It is clearly the case that $\varnothing \subseteq U$, so Definition 4.2 ensures the existence of a $q \in U$ such that $\varnothing \;\kappa\; q$. This proves that U is *non-empty*. In particular, this means that Con is valid on $\mathsf{u} \restriction U$. Note that we *do not* need to assume that Λ enjoys the finite model property in order to make the above inferences.

The following Theorems 4.4 and 4.6 to 4.8 characterise extendible models in several intermediate logics. In Corollary 4.5, we employ the insights of Section 3.6 to show that finite IPC-extendible models are models of BW_2.

4.4 Theorem
Let X be a finite set of variables, and let $U \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$ be given. Consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$, and define $\mathsf{u}_U := \mathsf{u} \upharpoonright U$. The following are equivalent:

(i) the Kripke frame U is IPC*-extendible*;

(ii) the rules Con, DP, and D_n^- are valid in model u_U for all $n \in \mathbb{N}$;

(iii) the Kripke frame U is *non-empty*, *downwards directed*, and *weakly covered*.

Proof. Suppose (i) holds. We first prove that U is weakly covered. Let $p \in U$ and $W \subseteq {\uparrow}p$ be given. As W is finite, we know there to be a point $p^- \in \mathsf{U}_{\mathsf{IPC}}(X)$ such that $W \;\kappa\; p^-$. Because U is IPC-extendible, there must be a $p^+ \in \mathsf{U}_{\mathsf{IPC}}(X)$ such that $W \;\kappa\; p^+$, as desired.

Let us now prove that U is downwards directed. Take some non-empty, finite subset $W \subseteq U$. The universality of $\mathsf{U}_{\mathsf{IPC}}(X)$ ensures the existence of a $p \in \mathsf{U}_{\wedge}(X)$ such that $W \;\kappa\; p$. By the IPC-extendibility of U, we thus know of a $q \in U$ such that $W \;\kappa\; q$. In particular, this proves $W \subseteq {\uparrow}q$, as desired. Finally, note that Example 4.3 already argued for the non-emptiness of U. This proves (iii).

The equivalence between (iii) and (ii) is immediate through Lemma 3.49 and Theorems 3.54 and 3.70. Finally, suppose (iii) holds. Let $p \in \mathsf{U}_{\mathsf{IPC}}(X)$ and $W \subseteq U$ be such that $W \;\kappa\; p$. As U is non-empty and downwards directed, we know there to be a point $q^- \in U$ such that $W \subseteq {\uparrow}q^-$. Now, as U is weakly covered, there must be a $q \in U$ satisfying $W \;\kappa\; q$. This proves (i), as desired. $\square$

The following follows quite naturally when combining the above Theorem 4.4 with the results of Section 3.6. An analogous result holds for the logic BB_n, as one can prove by replacing the appeal to Theorem 4.4 in Corollary 4.5 by Theorem 4.7.

4.5 Corollary
Every IPC*-extendible model* that is finite, is also a model of BW_2.

Proof. Let $U \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$ be an arbitrary upset, and consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$. Suppose that the generated submodel $v := \mathsf{u} \upharpoonright U$ is IPC-extendible. By Theorem 4.4, it is immediate that $\overline{\mathsf{D}}_2$ is valid on v. As per Theorem 3.83, we know U to be a strongly binary covered frame. The desired is immediate through Lemma 3.87. □

Recall Lemma 3.76, which proved that D_n^- is derivable in each intermediate logic above LC. This trivialises the question as to whether D_n^- is valid on a model of LC; it simply always is. Similarly, the notion of extendibility trivialises in logics above LC, as shown by Theorem 4.6 below.

4.6 Theorem
Let Λ be any intermediate logic extending LC. Every *non-empty* generated submodel of $\mathsf{U}_\Lambda(X)$ is Λ-*extendible*.

Proof. Let $U \subseteq \mathsf{U}_\Lambda(X)$ be an upset, and let $p \in \mathsf{U}_\Lambda(X)$ and $W \subseteq U$ be such that $W \ \kappa \ p$. Due to Corollary 3.13, we know the Kripke frame $\uparrow p$ to satisfy Λ. Hence, in particular, it satisfies LC. Lemma 2.82 proves that $\uparrow p$ is a chain. We distinguish between whether $W = \varnothing$. In the case that this equality holds, note that U is non-empty by assumption, hence we are done. In the other case, there exists a point $q \in W$ such that $\uparrow W = \uparrow q$. As in Example 2.33, we clearly have $\{q\} \ \kappa \ q$, so $W \ \kappa \ q$. Because $q \in W \subseteq U$, we are done. □

4.7 Theorem
Let X be a finite set of variables, let $n \geq 1$ be a natural number, and let $U \subseteq \mathsf{U}_{\mathsf{BB_n}}(X)$ be an upset. Consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{BB_n}}(X) \to \mathcal{P}(X)$, and fix the generated submodel $\mathsf{u}_U := \mathsf{u} \restriction U$. The following are equivalent:

(i) the Kripke frame U is BB_n-*extendible*;

(ii) the rules Con, DP and D_n^- are valid on the model u_U;

(iii) the Kripke frame U is *non-empty*, *downwards directed* and *weakly* n-*ary covered*.

Proof. The proof is similar to that of Theorem 4.4; the reasoning therein can be applied *mutatis mutandis* to prove this theorem. The most substantial difference is in the proof from (iii) to (i). Suppose $p \in \mathsf{U}_{\mathsf{BB_n}}(X)$ and $W \subseteq U$ are such that $W \ \kappa \ p$, without loss of generality we may assume W to be an *anti-chain*. First, we proceed as earlier, and through downwards directness we obtain a $q^- \in U$ such that $W \subseteq \uparrow q^-$. Consider the Kripke model $\mathsf{u} \restriction \uparrow q^-$, which we know to be both finite and order-defined. Corollary 3.13 ensures the Kripke frame $\uparrow p$ to be a frame of BB_n. Now, Lemma 2.88 guarantees any such frame to be of *branching degree at most* n. Because W is an anti-chain, this proves $|W| \leq n$. As U is weakly n-ary covered, this yields a point $q^+ \in U$ such that $W \ \kappa \ q^+$, proving the desired. □

The following Theorem 4.8 is particularly interesting. Observe that $\mathsf{BD}_2 + \mathsf{BW}_n$ is a subframe logic, and hence admits D_n^- for all $n \in \mathbb{N}$ by Theorem 3.77. One might expect it to be necessary to include all of these rules in (ii) below, much like in Theorem 4.4. However, this is not the case. The restricted form of the models of the intermediate logic $\mathsf{BD}_2 + \mathsf{BW}_n$ ensure that this additional strength, although it is present, is not necessary.

4.8 Theorem

Let $n \geq 2$ be a natural number and fix the intermediate logic $\Lambda := \mathsf{BD}_2 + \mathsf{BW}_n$. Let X be a finite set of variables, and let $U \subseteq \mathsf{U}_\Lambda(X)$ be an upset. Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$, and define $\mathsf{u}_U := \mathsf{u} \upharpoonright U$. The following are equivalent:

(i) the Kripke frame U is Λ-*extendible*;

(ii) the rules Con, $\mathsf{DP}_n^{\neg\neg}$ and D_n^- are valid on the model u_U;

(iii) the Kripke frame U is non-empty, *classically n-ary downwards directed*, and *n-ary weakly covered*.

Proof. The equivalence between (iii) and (ii) follows from Lemma 3.49 and Theorems 3.63 and 3.70. Suppose (i) holds. Non-emptiness of U follows from Example 4.3. To prove that U is classically n-ary downwards directed, let $W \subseteq \mathsf{max}(U)$ be such that $|W| \leq n$. Consider the model $v := \mathsf{u} \upharpoonright W$. It is easy to see that $u := v/\varnothing$ is a model of *height at most two* and of *width at most* n.

This proves $u \Vdash \mathsf{BD}_2 + \mathsf{BW}_n = \Lambda$, hence there must be a unique map of Kripke models $f : u \to \mathsf{u}$. As $W \;\kappa\; \rho$, we know $f(W) \;\kappa\; f(\rho)$ by Lemma 2.35. We readily see that $f(W) = W$, hence $W \;\kappa\; f(\rho)$. The Λ-extendibility of U now yields a $p \in U$ such that $W \;\kappa\; p$, proving that $W \subseteq {\uparrow} p$ holds in particular. We thus know that U is classically n-ary downwards directed.

Finally, we show that U is weakly covered. Take some $p \in U$ and let $W \subseteq {\uparrow} p$ be non-empty. Without loss of generality, we may assume W to be an *anti-chain*. Because $\mathsf{u}, p \Vdash \mathsf{BW}_n$, we know that $|W| \leq n$ through Theorem 2.79 and Corollary 3.13. Let us proceed by cases. First, suppose W is a singleton set. It immediately follows that this set covers it sole inhabitant. Second, suppose that W is not a singleton set. It is easy to see that $W \subseteq \mathsf{max}(U)$ must follow. By reasoning similar to the above, we now obtain a $p \in U$ such that $W \;\kappa\; p$. This proves (iii).

Conversely, suppose (iii) holds. Let $p \in \mathsf{U}_\Lambda(X)$ and $W \subseteq U$ be such that $W \;\kappa\; p$. Without loss of generality, we may assume W to be an *anti-chain*. From Theorem 2.79 and Corollary 3.13 and $p \Vdash \mathsf{BW}_n$ it follows that $|W| \leq n$. If W is empty, we are done by the non-emptiness of U. In the case that W is a singleton set, the desired is also immediate. In the case that $|W| \geq 2$, we know $W \subseteq \mathsf{max}(U)$.

As U is classically n-ary downwards directed, we obtain a $q^- \in U$ such that $W \subseteq \uparrow q^-$. Finally, because U is weakly n-ary downwards directed, this yields a point $q^+ \in U$ satisfying $W \ \kappa \ q^+$. We have thus proven (i). □

4.9 Corollary
Let Λ be among IPC, LC, BB_n, BD_2, and let X be a finite set of variables. Every generated submodel of $\mathsf{U}_\Lambda(X)$ that is Λ-*exact* is Λ-*extendible*.

Proof. Immediate through the characterisations of extendible models Theorems 4.4 and 4.6 to 4.8, coupled with Theorem 3.38 and the observation that the rules mentioned therein are admissible in the respective logics. □

The above Corollary 4.9 is rather concrete. Indeed, we used a fair bit of information specific to the logics at hand in order to draw the above inference. Below, in Definition 4.10, we give a more general condition one could impose upon a model to ensure that it be extendible. This formulation can be found elsewhere in the literature, it is based on Ghilardi (2004, pp. 105–106) and Bezhanishvili and de Jongh (2012, Theorem 4.17). In Section 4.1.3 we show that this condition is equivalent to being extendible, and that it can be re-formulated in a much more syntactic manner.

4.10 Definition (Injective Model)
Let Λ be an intermediate logic, let X be a finite set of variables, and let $U \subseteq \mathsf{U}_\Lambda(X)$ be an upset of the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. Consider the generated submodel $\mathsf{u}_U := \mathsf{u} \restriction U : U \to \mathcal{P}(X)$. We say that u_U is a Λ-*injective Kripke model* whenever there exists a definable map $f : \mathsf{u} \to \mathsf{u}_U$ such that the equality $f \restriction U = \mathrm{id}_{\mathsf{u}_U}$ holds.

The specification given in Definition 4.10 can best be summarised by means of the diagram in Fig. 4.1. Note that the definable map f is a *section* of the inclusion $\mathsf{u}_U \to \mathsf{u}$. We do not require that this latter map be definable, and postpone discussing this special case until Theorem 4.22.

In the following Lemma 4.11, we relate Definition 4.10 to Λ-exact and Λ-extendible models Later, in Theorem 4.24, we show that a definable upset is Λ-extendible precisely if it is Λ-injective. The lemma is phrased more concretely than necessary. Indeed, any section of a map into a universal model yields a model that is both exact and extendible.

$$\begin{tikzcd}
U \arrow[rr, "\subseteq"] \arrow[dr, "id_v"'] & & \mathsf{U}_\Lambda(X) \arrow[dl, "f"] \\
& U &
\end{tikzcd}$$

Figure 4.1.: Diagrammatic description of Definition 4.10.

4.11 Lemma
Every Λ-injective model is Λ-exact and Λ-extendible.

Proof. Let X be a finite set of variables, and let $v : U \to \mathcal{P}(X)$ be a generated submodel of the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. Suppose that v is Λ-injective. We thus know of a definable map $f : \mathsf{u} \to v$ satisfying $f \upharpoonright U = \mathrm{id}_v$.

First, we prove that v is Λ-exact. To this end, we need but prove that f is surjective, which readily follows from $f \upharpoonright U = \mathrm{id}_v$. Second, we prove that v is Λ-extendible. Let $p \in \mathsf{U}_\Lambda(X)$ and $W \subseteq U$ be such that $W \;\kappa\; p$. By Lemma 2.35 we know that $f(W) \;\kappa\; f(p)$. As $f(W) = W$ and $f(p) \in U$, this proves that v is Λ-extendible. $\square$

4.1.1. Extendible formulae

We have just seen Λ-extendible models, and considered some of their properties. By definition, a Λ-extendible model arises as the restriction of a universal model to some arbitrary upset. In this section, we limit attention to *definable* upsets. Recall the Λ-exact formulae, as treated in Definition 3.32, which we proved to correspond to definable Λ-exact models in Theorem 3.33. We proceed analogously in this section, in that we give a description of those formulae that give rise to extendible models. Limiting to attention to definable upsets might seem arbitrary and overly restrictive. Looks can be deceiving, though, as Section 4.1.2 proves that in the case of seven particular intermediate logics including IPC, nothing is lost when considering exact *formulae* rather than exact *models*. In these cases, there is no restriction at all.

One may wonder why we then bother with restricting to extendible formulae at all. Indeed, without risk of loss, what stands to be gained? The answer rests in Chapters 5 and 6. An intrinsic, semantic description of those formulae that are extendible amounts, via Theorem 4.24, to a description of the *projective formulae*. Such projective formulae play a key role in both finding a basis of admissibility and in solving the unification problem of an intermediate logic. This restriction does not, however, play any role at all in our proof of the decidability of IPC.

The first step we take is in translating Definition 3.26, the validity of a rule, to a statement pertaining to formulae. Definition 4.12 below will play this role, as we prove in Lemma 4.14. Note that its formulation is more general than strictly necessary. Indeed, we could have specialised the definition, instantiating $\mathcal{R}$ as $\mathrel{|\!\sim}_\Lambda$. We do not proceed in this manner in order to allow this definition to be of use in Section 4.3 as well.

4.12 Definition (Closed under Rules)
Let Λ be an intermediate logic, and let $\Pi \in \mathcal{L}(X)$ be a set of formulae, and let $\mathcal{R}$ be a set of rules. We say that Π *is closed under* $\mathcal{R}$ *with respect to* Λ whenever:

$$\text{for all } \Gamma/\Delta \in \mathcal{R}, \text{ if } \Pi \vdash_\Lambda \phi \text{ for all } \phi \in \Gamma \text{ then } \Gamma \vdash_\Lambda \chi \text{ for some } \chi \in \Delta.$$

We often omit reference to Λ, as it should be clear from context which intermediate logic is meant.

4.13 Example
Let us treat one example in almost obnoxious detail. Recall the rule as given in (*modus ponens*) on Page 26, whose substitution instances are of the form:

$$\phi, \phi \to \psi/\psi \ .$$

Every formula χ is closed under this rule. Indeed, suppose that both $\chi \vdash_\Lambda \phi$ and $\chi \vdash_\Lambda \phi \to \psi$ hold. By definition, this means that

$$(\chi \to \phi)\,, (\chi \to \phi \to \psi) \in \Lambda.$$

The axioms of IPC, as given in Table 2.1, have the following formula as a substitution instance.

$$(\chi \to \phi) \to (\chi \to \phi \to \psi) \to (\chi \to \psi)$$

Hence the above formula is an element of $\Lambda \supseteq \mathsf{IPC}$. Two applications of the second condition of Definition 2.5 now prove that $\chi \vdash_\Lambda \psi$.

Above, we characterised Λ-extendible models in terms of intrinsic semantic properties for several intermediate logics. Such semantic descriptions were readily transformed into a syntactic one via the characterisations of the validity of rules treated in Section 4.3. Through Lemma 4.14, we move to the syntactic in full, relating the validity of a rule in a definable generated submodel of the universal model to the closure of the corresponding defining formulae under said rule. Corollaries 4.15 to 4.17 will play a crucial role in Chapter 5.

4.14 Lemma
Let Λ be an intermediate logic with the finite model property, let X be a finite set of variables, and let $\phi \in \mathcal{L}(X)$ be a formula. Consider the submodel $v : U \to \mathcal{P}(X)$ of $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ generated by $[\![\phi]\!]_\mathsf{u}$. A rule on $\mathcal{L}(X)$ is valid on v precisely if ϕ is closed under this rule.

Proof. Recall Corollary 3.23, which shows that $\vdash_\Lambda \chi$ is equivalent to $\mathsf{u} \Vdash \chi$ for any $\chi \in \mathcal{L}(X)$. In particular, the model $v := \mathsf{u} \upharpoonright [\![\phi]\!]_\mathsf{u}$ satisfies the equivalence:

$$v \Vdash \chi \text{ if and only if } \phi \vdash_\Lambda \chi \text{ for all } \chi \in \mathcal{L}(X). \tag{4.1}$$

To prove this equivalence, suppose $v \Vdash \chi$ holds for some $\chi \in \mathcal{L}(X)$. If $\phi \nvdash_\Lambda \chi$ holds, then there exists some $p \in \mathsf{U}_\Lambda(X)$ such that $\mathsf{u}, p \Vdash \phi$ and $\mathsf{u}, p \nVdash \chi$. The former ensures $p \in [\![\phi]\!]_\mathsf{u}$, whereas the latter yields $v \nVdash \chi$, a contradiction. The converse is immediate, because $\phi \vdash_\Lambda \chi$ entails $\vdash_\Lambda \phi \to \chi$, and $v \Vdash \phi$ holds.

Now, let Γ/Δ be a rule satisfying $\Gamma, \Delta \subseteq \mathcal{L}(X)$. Suppose that Γ/Δ is valid on v, and assume $\Gamma \vdash_\Lambda \phi$ for all $\phi \in \Gamma$. By (4.1), the latter ensures that $v \Vdash \Gamma$. We thus obtain a $\chi \in \Delta$ such that $v \Vdash \chi$. Again, through (4.1), we derive $\vdash_\Lambda \chi$. The converse is similar, proving the desired. □

4.15 Corollary (IPC-Extendible Formulae)
A formula is IPC-extendible precisely if it is closed under Con, DP and D_n^- for all $n \in \mathbb{N}$.

Proof. Immediate via Lemma 4.14 and Theorem 4.4. □

4.16 Corollary (BB_n-Extendible Formulae)
Let $n \geq 2$ be a natural number. A formula is BB_n*-extendible* precisely if it is closed under Con, DP and D_n^-.

Proof. Immediate via Lemma 4.14 and Theorem 4.7. □

4.17 Corollary (BD_2-Extendible Formulae)
Let $n = 2, 3, \ldots, \omega$ be given. A formula is $(\mathsf{BD}_2 + \mathsf{BW}_n)$*-extendible* precisely if it is closed under Con, $\mathsf{DP}_n^{\neg\neg}$ and D_n^-.

Proof. Immediate via Lemma 4.14 and Theorem 4.8. □

4.1.2. Uniform interpolation and exactness

Exact formulae, via Theorem 3.33, can be construed as a special case of exact models. One might wonder how "special" this special case really is. The purpose of this subsection is to provide sufficient conditions on an intermediate logic under which all exact models arise from exact formulae. The tool we use towards this end is *uniform interpolation.* None of the results obtained in this section are essential to the remainder of this thesis, although they do provide an interesting perspective.

Craig (1957b) proved that to every pair of formulae $\phi \in \mathcal{L}(X)$ and $\psi \in \mathcal{L}(Y)$ such that $\phi \vdash_{\mathsf{CPC}} \psi$ there is a formula $\chi \in \mathcal{L}(X \cap Y)$ such that $\phi \vdash_{\mathsf{CPC}} \chi$ and $\chi \vdash_{\mathsf{CPC}} \psi$. In this statement, one could replace CPC by an arbitrary logic Λ and wonder whether it still holds. If a logic satisfies said statement, then it is said to enjoy the *interpolation property*. This property has been studied intensively. We mention but a few relevant results, see D'Agostino (2008) and Hoogland (2001) for a more extensive overview.

Maksimova (1977, 1979) proved that there are precisely seven intermediate logics with the interpolation property: CPC, Sm, GSc, KC, LC, BD_2 and IPC, depicted as ordered by inclusion in Fig. 4.2. In the formulation of the interpolation property, the interpolant χ depends on both ϕ and ψ. Renardel de Lavalette (1989, p. 1429) asked whether it is possible to choose χ uniformly for all ψ in IPC, after observing that this is the case for CPC. This was answered in the affirmative by Pitts (1992, Theorem 13), by means of a syntactic proof. Semantic proofs were later given by Ono (1986, Corollary 4.7), Ghilardi and Zawadowski (1995, Corollary 4.4) and Visser (1996, Theorem 5.1). See Shavrukov (1993) for a proof of uniform interpolation in GL.

In Theorem 4.18 below, we state that all the intermediate logics with interpolation in fact have *uniform interpolation.* We do not include the proofs, and refer to Ghilardi and Zawadowski (2002, p. 101) and Ghilardi and Zawadowski (1997, pp. 44–46) for further detail.

4.18 Theorem (Uniform Interpolation)
Let Λ be one of the intermediate logics CPC, G_2, GSc, KC, LC, BD_2, or IPC, and let X and Y be finite sets of variables. For every substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ and every formula $\phi \in \mathcal{L}(Y)$ there exists formulae $\exists_\sigma \phi, \forall_\sigma \phi \in \mathcal{L}(X)$ such that for all $\psi \in \mathcal{L}(X)$ we have:

$$\begin{aligned} \phi \vdash_\Lambda \sigma(\psi) \quad &\text{iff} \quad \exists_\sigma \phi \vdash_\Lambda \psi, \\ \sigma(\psi) \vdash_\Lambda \phi \quad &\text{iff} \quad \psi \vdash_\Lambda \forall_\sigma \phi. \end{aligned}$$

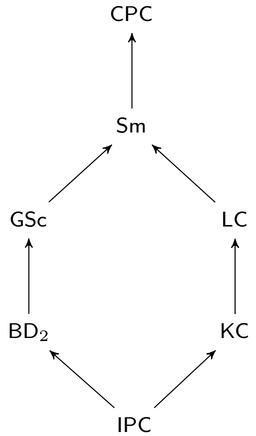


Figure 4.2.: The intermediate logics with the *interpolation property* ordered by inclusion, as illustrated by Rothenberg (2010, Figure 3.1).

The following Theorem 4.19 was first given by de Jongh and Visser (1996, Theorem 2.3). The same connection has been pointed out elsewhere in the literature, see for instance de Jongh and Chagrova (1995, p. 500) and Bezhanishvili and de Jongh (2012, Corollary 4.4). As we spell out in Corollary 4.20, this readily entails that the admissible rules of IPC are complete with respect to the set of exact formulae in some precise, technical sense. This observation can be used to prove the set of admissible rules of IPC to be decidable, an approach taken by Ghilardi (1999, p. 874).

4.19 Theorem (de Jongh and Visser, 1996, Theorem 2.3)
Let Λ be one of the intermediate logics CPC, G_2, GSc, KC, LC, BD_2, or IPC, and let $v : P \to \mathcal{P}(X)$ be a Λ-exact model. There exists a formula $\phi \in \mathcal{L}(X)$ such that

$$\phi \vdash_\Lambda \chi \text{ iff } v \Vdash \chi \text{ for all } \chi \in \mathcal{L}(X).$$

Proof. By Lemma 3.31, there exists a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ for some finite set of variables Y such that $v \Vdash \chi$ precisely if $\vdash_\Lambda \sigma(\chi)$. Now, consider the formula $\exists_\sigma \top \in \mathcal{L}(X)$, which is guaranteed to exist by Theorem 4.18. We define $\phi := \exists_\sigma \top$, and observe that:

$$\phi \vdash_\Lambda \chi \text{ iff } \top \vdash_\Lambda \sigma(\chi) \text{ iff } v \Vdash \chi \text{ for all } \chi \in \mathcal{L}(X).$$

This proves the desired. $\square$

4.20 Corollary
Let Λ be one of the intermediate logics CPC, G_2, GSc, KC, LC, BD_2, or IPC. The following are equivalent for all finite sets of formulae $\Gamma, \Delta \subseteq \mathcal{L}(X)$:

(i) the rule Γ/Δ is admissible in Λ;

(ii) each Λ-exact formula $\phi \in \mathcal{L}(X)$ is closed under the rule Γ/Δ.

Proof. This follows readily from Theorems 3.38 and 4.19. □

We close this section with a brief enumeration of the results known about the admissibility of the intermediate logics depicted in Fig. 4.2. It is well-known that CPC is structurally complete, that is to say, $\mathrel{|\!\sim}_{\mathsf{CPC}} = \vdash_{\mathsf{CPC}}$. The structural completeness of LC and G_2 was proven by Dzik and Wroński (1973), and Citkin (1978) showed that these logics are hereditarily structurally complete.[4] Both IPC and KC have non-trivial admissible rules, and all admissible rules follow from the Visser rules by Iemhoff (2005, Theorem 5.1) and Iemhoff (2001a). We revisit some of these results in Section 5.3.

Through Skura (1992a), it can be seen that BD_2 admits some non-trivial rules, yet no basis of admissibility of these logics was known until Goudsmit (2013). We have shown, in Example 3.65 and Corollary 3.78, that BD_2 admits the rules $\mathsf{DP}_n^{\neg\neg}$ and $\mathsf{D}_n^{\neg}$ for all $n \in \mathbb{N}$. By the same results, we know that GSc admits $\mathsf{DP}_2^{\neg\neg}$ and $\mathsf{D}_2^{\neg}$. We employ these rules to give a basis of admissibility for these logics in Section 5.3. As a consequence, the admissible rules of all intermediate logics with interpolation have been described in full.

4.1.3. Projective formulae

In Section 4.1.1, we described those formulae that give rise to Λ-extendible models. A similar kind of description was given for Λ-exact models through Definition 3.32 and Theorem 3.33. The purpose of this section is to provide a similar type of description for injective models. First, we treat Λ-projective formulae, and show that this class captures precisely those formulae that happen to give rise to Λ-injective models. Second, we prove that Λ-extendible formulae are the same thing as Λ-projective formulae.

Let us start with a brief explanation of the involved nomenclature. Projective algebras have been introduced by Everett and Ulam (1946, Section 2). In category theory, projective objects arose as a generalisation of projective modules, see for instance

[4] For more on structural completeness from the perspective of admissibility, we refer to Rybakov (1997, Chapter 5).

Cartan and Eilenberg (1956).[5] Projective objects occur as objects of study in Universal Algebra, and in Lattice Theory in particular, let us but mention Balbes (1967), Balbes and Horn (1970a,b), and Halmos (1961) as examples.

In the context of admissible rules of IPC, projective algebras play an important role. Citkin (1977a, Theorem 2) gave a description of the finite Heyting algebras on which the rule M (see Page 107) is valid. Through Balbes and Horn (1970a, Theorem 4.10), it immediately follows that these are precisely the *projective Heyting algebras*.[6] Ghilardi (1997) used the notion of projective algebras in the context of unification, which we expand upon in Chapter 6.

Definition 4.21 below is a straightforward generalisation of Ghilardi (1999, p. 863). The formulae described by this definition are said to be *projective* because dividing the free algebra $\mathsf{F}_\Lambda(X)$ of the variety $\mathcal{V}_\Lambda$ by the filter generated by any such formula yields an algebra that is *projective* in in the category $\mathcal{V}_\Lambda$.[7] Our Theorem 4.22 intuitively amounts to a dual description phrased in terms of Kripke models rather than Heyting algebras. This is why we speak of *injective models* and *projective formulae*, the name refers to their respective role in the corresponding category.

4.21 Definition (Projective Formula)
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$ be a formula. We say that ϕ is a Λ-*projective formula* if there exists a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ such that:

(i) the formula $\sigma(\phi)$ is a theorem of Λ;

(ii) for all formulae $\psi \in \mathcal{L}(X)$, the statement $\phi \vdash_\Lambda \sigma(\psi) \equiv \psi$ holds.

Recall Examples 3.34 and 3.35, and note that the formulae mentioned therein are not only IPC-*exact*, but also IPC-*projective*. The following shows that projective formulae are precisely those formulae that give rise to injective models. Our proof below is similar to that of Bezhanishvili and de Jongh (2012, Theorem 4.17).

4.22 Theorem
Let X be a finite set of variables, let $\phi \in \mathcal{L}(X)$ be a formula, and consider the submodel $v : U \to \mathcal{P}(X)$ of the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ generated by the upset $[\![\phi]\!]_\mathsf{u}$. The following are equivalent:

[5] The definition of a projective module can be found in any text book on modern algebra, see for instance Lang (2002, p. 137).

[6] Citkin was not aware of this particular result at the time. Grigolia, who worked on finitely presented projective Heyting algebras, see for instance Grigolia (1987, 1995), pointed him to the work of Balbes and Horn (1970a).

[7] We spend but a few words on this, more details are given by Arevadze (2001, Section 4).

(i) the model v is injective;

(ii) the formula ϕ is projective.

Proof. Suppose (i) holds. Fix a definable map $f : \mathsf{u} \to v$ such that $f \restriction U = \mathrm{id}_v$, which we know to exist. By Lemma 3.8, this definable map f gives rise to a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ such that:

$$\mathsf{u}, p \Vdash \sigma(\chi) \text{ iff } v, f(p) \Vdash \chi, \text{ for all } p \in \mathsf{U}_\Lambda(X) \text{ and } \chi \in \mathcal{L}(X). \tag{4.2}$$

Because $v \Vdash \phi$ holds, it follows from the above equivalence that $\mathsf{u} \Vdash \sigma(\phi)$. This, through Corollary 3.23, ensures (i) of Definition 4.21 to hold. To prove (ii) of the same definition, we proceed via Corollary 3.23. Suppose $p \in \mathsf{U}_\Lambda(X)$ is such that $\mathsf{u}, p \Vdash \phi$. We know that $p \in U$. For any $q \geq p$ we know $f(q) = q$, so $\mathsf{u}, q \Vdash \psi$ is equivalent to $\mathsf{u}, q \Vdash \sigma(\psi)$ for any $\psi \in \mathcal{L}(X)$ by (4.2). We have thus proven (ii).

Conversely, suppose (ii) holds. The substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ gives rise to a definable map $f : \mathsf{u} \to \mathsf{u}$ as per Example 3.29, again satisfying (4.2). Through (i) of Definition 4.21, it follows that $f(\mathsf{U}_\Lambda(X)) \subseteq [\![\phi]\!]_\mathsf{u}$. We thus see that f can be restricted to a map $f : \mathsf{u} \to v$. We now need but prove that $f \restriction U = \mathrm{id}_v$. As u is order-refined, it suffices to prove that $f(p)$ and p have equal theories for all $p \in U$. But this follows immediately from (4.2) and (ii) of Definition 4.21, proving (i) as desired. □

The proof of the following is easy to obtain by spelling out the definitions. Similarly, one might proceed via Theorems 3.33 and 4.22, from whence the proof is also immediate.

4.23 Corollary
Every Λ-*projective* formula is Λ-*exact*.

The remainder of this subsection is spent on proving Theorem 4.24 below. We present the reasoning as given by Ghilardi (1999), adapted to our setting. The present setting is more general, as we are concerned with arbitrary intermediate logics Λ with the finite model property instead of IPC alone. Admittedly, the reasoning applies practically verbatim to this more general setting. We do deviate a bit from the original argument, in that we present it couched in terms of definable maps and the universal model.

4.24 Theorem (Ghilardi, 1999, Theorem 5)
Let Λ be an intermediate logic with the finite model property. A formula ϕ is Λ-*projective* if and only if it is Λ-*extendible*.

We already know the implication from left to right to hold. Indeed, this follows readily from Theorem 4.22 and Lemma 4.11. In order to prove the other direction, we first introduce a particular kind of substitution in Definition 4.25. This substitution is reminiscent of the trick by Prucnal we discussed in Example 3.35. In particular, this substitution will be such that it almost automatically satisfies (ii) of Definition 4.21, yet (i) need not hold in general.

4.25 Definition
Let X be a finite set of variables, let $Y \subseteq X$ be a subset, and let $\phi \in \mathcal{L}(X)$ be a formula. We define the substitution $\sigma_\phi^Y : \mathcal{L}(X) \to \mathcal{L}(X)$ by:

$$\sigma_\phi^Y : \mathcal{L}(X) \to \mathcal{L}(X), \quad x \in X \mapsto \begin{cases} \phi \to x & \text{if } x \in Y, \\ \phi \wedge x & \text{otherwise.} \end{cases} \tag{4.3}$$

Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$, and think of the generated submodel $v : [\![\phi]\!]_\mathsf{u} \to \mathcal{P}(X)$. Through Example 3.29, it is clear that the above substitution $\sigma_\phi^Y : \mathcal{L}(X) \to \mathcal{L}(X)$ gives rise to a definable map of Kripke frames $f_\phi^Y : \mathsf{u} \to \mathsf{u}$. Instantiating (3.12) to this specific case, we know:

$$\mathsf{u}, f_\phi^Y(p) \Vdash \chi \text{ iff } \mathsf{u}, p \Vdash \sigma_\phi^Y(\chi) \text{ for all } \chi \in \mathcal{L}(X). \tag{4.4}$$

Lemmas 4.26 to 4.28 below collect some observations about these substitutions and their corresponding definable maps. These observations have already been made by Ghilardi (1999, Lemma 2 and 3).

4.26 Lemma
For any $Y \subseteq X$, we have that $f_\phi^Y \restriction [\![\phi]\!]_\mathsf{u} = \mathrm{id}_v$.

Proof. Let $p \in [\![\phi]\!]_\mathsf{u}$ be given. We proceed by well-founded induction, so assume $f_\phi^Y(w) = w$ for all $w \in W$, where $W = \mathrel{\updownarrow}p$. By Lemma 2.35 and $W \kappa p$ we know that:

$$W = f_\phi^Y(W) \ \kappa \ f_\phi^Y(p).$$

As the universal model is concrete, Lemma 2.52 shows that it suffices to prove that:

$$\mathsf{u}\left(f_\phi^Y(p)\right) = \mathsf{u}(p).$$

We proceed via (4.4). The desired is immediate by Corollary 3.23, because we know that $\phi \vdash_\Lambda \sigma_\phi^Y(x) \equiv x$ for all $x \in X$. □

4.27 Lemma
For any $Y \subseteq X$, if $p \in \mathsf{U}_\wedge(X)$ is such that $\mathsf{u}, p \nVdash \phi$, then $\mathsf{u}\big(f^Y_\phi(p)\big) \subseteq \mathsf{u}\big(p\big)$.

Proof. Let $x \notin Y$ be given, suppose that $\mathsf{u}, f^Y_\phi(p) \Vdash x$. We thus infer $\mathsf{u}, p \Vdash \phi \wedge x$ via (4.4), a contradiction. □

4.28 Lemma
Let $W \subseteq [\![\phi]\!]_\mathsf{u}$ and p^+, p^- be such that $W \ \kappa\ p^\pm$. If $p^+ \in [\![\phi]\!]_\mathsf{u}$ and $p^- \notin [\![\phi]\!]_\mathsf{u}$, then $f^Y_\phi(p^-) = p^+$ for $Y = \mathsf{u}(p^+)$.

Proof. To prove the desired, it suffices to show that:

$$\mathsf{u}\big(f^Y_\phi(p^-)\big) = \mathsf{u}\big(p^+\big) = Y. \tag{4.5}$$

Indeed, we know the universal model to be concrete, hence the above equation implies the desired via Lemma 2.52. First, suppose $x \in Y$ holds. Through Lemma 2.34 we see that $\mathsf{u}, p^- \Vdash \phi \to x$, because $\mathsf{u}, W \Vdash x$ and $\mathsf{u}, p^- \nVdash \phi$. By (4.4), we thus know $\mathsf{u}, f^Y_\phi(p^-) \Vdash x$. The other direction follows by Lemma 4.27. This proves the desired. □

We wish to prove Theorem 4.24. To this end, it suffices to show that the generated submodel $v : [\![\phi]\!]_\mathsf{u} \to \mathcal{P}(X)$ of $\mathsf{u} : \mathsf{U}_\wedge(X)$ is injective, as per Theorem 4.22. To reiterate, we but need to find a definable map $f : \mathsf{u} \to v$. We obtain such a map by composing the definable maps f^Y_ϕ in a suitable order.

First, let us enumerate the elements of $\mathcal{P}(X)$ as $Y_1, \ldots, Y_n$, where $n := 2^{|X|}$. One can construct the enumeration in such a way that we adhere to:

$$Y_i \subseteq Y_j \text{ implies } i \leq j \text{ for all } i, j \leq n. \tag{4.6}$$

Second, with this order at hand, we construct a sequence of definable map by:

$$f^i := f^{Y_i}_\phi \circ \ldots \circ f^{Y_n}_\phi : \mathsf{u} \to \mathsf{u},$$

where $i = 1, \ldots, n$. It is clear that each f^i satisfies $f^i \restriction [\![\phi]\!]_\mathsf{u} = \mathrm{id}_v$. Indeed, it is a composition of maps satisfying this property, and this property is closed under composition.

We claim that the map $f := f^1$ can be restricted to a map $\mathsf{u} \to v$. The desired readily follows from this claim, as already argued above, so we but need to prove it. Before we continue, let us first prove the following lemma.

4.29 Lemma (Ghilardi, 1999, Lemma 4)
Let $k \in \mathsf{U}_\wedge(X)$ and $i \le n$ be given. If $f(k) \in [\![\phi]\!]_\mathsf{u}$ and $f(k) \Vdash Y_i$, then $f^i(k) \in [\![\phi]\!]_\mathsf{u}$.

Proof. Suppose the desired does not hold. Let $1 \le j < i$ be the maximal such that $f^j(k) \in [\![\phi]\!]_\mathsf{u}$. Subsequently, through Lemma 4.27 and then via iterative applications of Lemma 4.26, we gather that:

$$Y_j \supseteq \mathsf{u}\big(f^j(k)\big) = \mathsf{u}\big(f^1(k)\big) = \mathsf{u}\big(f(k)\big) \supseteq Y_i.$$

Yet this readily entails $Y_i \subseteq Y_j$, which contradicts $j < i$ through our choice of enumeration (4.6). This proves the desired. $\square$

We reason by contradiction, so suppose that $f(p) \notin [\![\phi]\!]_\mathsf{u}$ holds for some $p \in \mathsf{U}_\wedge(X)$. Without loss of generality we may assume p to be the maximal such point, as follows from the image-finiteness of $\mathsf{U}_\wedge(X)$. For convenience, we define $V := \updownarrow p$, and note that $V \;\kappa\; p$. The maximality of p ensures that $f(V) \subseteq [\![\phi]\!]_\mathsf{u}$. Moreover, Lemma 2.35 proves that $f(V) \;\kappa\; f(p)$.

Because ϕ is assumed to be extendible, we now know that there must be a point $q \in [\![\phi]\!]_\mathsf{u}$ such that $f(V) \;\kappa\; q$. Lemma 4.28 shows that $f_\phi^Y f(p) = q$ for $Y := \mathsf{u}(q)$. Every $k \in \updownarrow p$ is such that $\mathsf{u}, k \Vdash Y$. Hence Lemma 4.29 ensures us that there is some $i \le n$ such that:

$$f^i(k) \in [\![\chi]\!]_\mathsf{u} \text{ for all } k \in \updownarrow p. \tag{4.7}$$

We now show, for all $m \le i$, that:

$$f^i(k) = f^{i-m}(k) \text{ for all } k \in \uparrow p.$$

The base case $m = 0$ is immediate. Suppose $f^i(p) = f^{i-m+1}(p)$ is known, and consider a point $k \in \uparrow p$. In the case that $k \neq p$, we know that:

$$f^{i-m}(k) = f_\phi^{Y_{i-m}} \circ f^{i-m+1} = f^i(k) \in [\![\phi]\!]_\mathsf{u},$$

as follows through (4.7) and Lemma 4.26. In the case where $k = p$, we know that $f^{m-i}(p) \notin [\![\phi]\!]_\mathsf{u}$, for otherwise repeated application of Lemma 4.26 would prove $f(p) \in [\![\phi]\!]_\mathsf{u}$. This thus proves $f^{i-m}(k) = f^i(k)$, as desired.

Summarising the above, we have shown that $f^i(p) = f(p)$. It is thus easy to infer that

$$f^i(p) = f_\phi^{Y_i} \circ f^i(p) = f_\phi^{Y_i} \circ f(p) = q$$

for we know that $f_\phi^{Y_i} \circ f_\phi^{Y_i} = f_\phi^{Y_i}$. Repeated applications of Lemma 4.26 now prove that $f(p) \in [\![\phi]\!]_\mathsf{u}$. But this is a clear contradiction with the assumption $f(p) \notin [\![\phi]\!]_\mathsf{u}$. This completes the proof of Theorem 4.24.

4.30 Theorem
A formula is IPC-*exact* iff it is IPC-*extendible*, iff it is IPC-*projective*.

Proof. Every IPC-exact formula is known to be IPC-extendible by Corollary 4.9 and Theorem 3.33. Through Theorem 4.24, we know every IPC-extendible formula to be IPC-projective. From here, Lemma 4.11 and Theorem 4.22 finish the argument.□

4.2. Adequate notions

Filtration is one of the classic techniques used to prove the finite model property for logics, both modal and intuitionistic. The key observation is that, when trying to determine the validity of a given formula, it suffices to distinguish but finitely many truth values within any model. To be a tad more precise, one can restrict attention to a finite set of formulae, and only observe a model up to the equivalence relation that identifies nodes which behave identically with respect to that chosen set of formulae. We employ the same type of observation to the study of admissibility.

We have considered several notions of models for admissibility, in particular we considered Λ-*exact models*. Such a Λ-exact model comes equipped with a surjective map of Kripke frames from a universal model of Λ, preserving the validity of all formulae in the language. In Definition 4.58 below, we generalise our notion of maps to a form which only guarantees the validity of a specific set of formulae to be preserved. For most of our practical applications, Theorem 4.78 in particular, this set will be finite.

Before we continue, let us first give a definition of the types of sets of formulae we wish to consider. We then formulate desiderata on a suitably generalised notion of semantics for admissible rules in Theorem 4.32, and prove the decidability of admissible rules, assuming said conditions can be met.

4.31 Definition (Subformulae and Adequate Sets)
Let $\chi \in \mathcal{L}(X)$ be a formula, and let $\Gamma \subseteq \mathcal{L}(X)$ be a set of formulae. We define the *subformulae of* χ , denoted $\mathsf{Sub}\,(\chi)$, inductively by:

$$\begin{aligned} \mathsf{Sub}\,(\chi) &:= \chi && \text{if } \chi \in X \text{ or } \chi = \top, \bot, \\ \mathsf{Sub}\,(\phi \oplus \psi) &:= \mathsf{Sub}\,(\phi) \cup \mathsf{Sub}\,(\psi) \cup \{\phi \oplus \psi\} && \text{for } \oplus = \wedge, \vee, \to. \end{aligned}$$

The set Γ is said to be *adequate* whenever $\mathsf{Sub}\,(\phi) \subseteq \Gamma$ for all $\phi \in \Gamma$.

4.32 Theorem
Suppose that there exists an algorithm that produces a set of Kripke models $\mathcal{K}_\Sigma$ subject to the following conditions whenever one inputs a finite adequate set of formulae $\Sigma \subseteq \mathcal{L}(X)$.

Condition 1 The rule Γ/Δ is admissible in IPC if and only if it is valid on all members of $\mathcal{K}_\Sigma$, for all finite $\Gamma, \Delta \subseteq \Sigma$.

Condition 2 The set of models $\mathcal{K}_\Sigma$ is finite.

Condition 3 Given a model in $\mathcal{K}_\Sigma$, it is decidable whether a rule is valid on it.

Now, the set of admissible rules for IPC is decidable.

Proof. We provide an algorithm that decides whether a given rule is decidable. On input Γ/Δ we construct the adequate set $\Sigma := \mathsf{Sub}(\Gamma) \cup \mathsf{Sub}(\Delta)$. By assumption, we can effectively produce a class of Kripke models $\mathcal{K}_\Sigma$ satisfying the three conditions above. Verify whether the rule Γ/Δ is valid on all $v \in \mathcal{K}_\Sigma$. The rule is admissible precisely if the above holds, due to Condition 1. Condition 2 ensures that we can effectively run through all models in $\mathcal{K}$, and Condition 3 guarantees we can effectively test validity. This proves the desired. □

In the following, we often restrict attention to a specific set of adequate formulae. As continuously explicating this throughout all equations would induce quite some overhead, we abbreviate:

$$\begin{aligned} \Gamma &=_\Sigma \Pi \quad \text{iff} \quad \Gamma \cap \Sigma = \Pi \cap \Sigma, \\ \Gamma &\subseteq_\Sigma \Pi \quad \text{iff} \quad \Gamma \cap \Sigma \subseteq \Pi \cap \Sigma. \end{aligned}$$

First, let us reconsider the requirements we impose upon maps. A map $f : P \to Q$ between Kripke models $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(X)$ was defined in such a way that the equivalence below holds for all formulae. In general, this is much more than we need. Indeed, when one is only interested in the validity of formulae in a given adequate set Σ, we need only know the behaviour of maps with respect to the formulae from Σ. In symbols, it suffices to know that:

$$v, p \Vdash \chi \text{ if and only if } u, f(p) \Vdash \chi \text{ for all } \chi \in \Sigma \text{ and } p \in P.$$

To define maps in such a manner would mix syntax and semantics where no such collusion is necessary. Instead, we make use of maps satisfying the "*closed domain condition*" of Zakharyaschev (1992), or rather, the intuitionistic variant as described by Bezhanishvili and Bezhanishvili (2013, Section 4).[8] Lemma 4.38 shows that this semantic condition is sufficient to retrieve the desired syntactic information. Take care to note that any map of Kripke frames satisfies this condition.

4.33 Definition (Closed Domain Condition)
Let $f : P \to Q$ be a monotonic map between posets, and let D be a subset of Q. We say that f *satisfies the closed domain condition for* D (in short: f has the CDC for D) whenever the following holds.

$$\text{if } {\uparrow} f(p) \cap D \neq \varnothing \text{ then } f({\uparrow} p) \cap D \neq \varnothing \text{ for all } p \in P. \tag{4.8}$$

When the above holds for all $D \in \mathcal{D}$, and $\mathcal{D}$ is a set of subsets of Q, then f is said to have the CDC for $\mathcal{D}$.

In the above context, we call D a *domain*, and refer to $\mathcal{D}$ as a *set of domains*. A domain should always be understood as a subset of a given Kripke frame. Maps that satisfy the CDC are closed under composition in the technical sense of Lemma 4.36.

4.34 Example
Consider the models depicted in Fig. 4.3. We call the left-hand model $v : P \to \mathcal{P}(X)$ and the right-hand model $u : Q \to \mathcal{P}(X)$, where $X = \{x\}$. The arrows indicate a monotonic map $f : P \to Q$. It is clear that this map if *not* a map of Kripke frames. Indeed, the three-point chain in P is not mapped to an upset under f. As a consequence, this map does not satisfy the CDC for the indicated subset of Q.

4.35 Example
Consider the models as in Fig. 4.4. As in Example 4.34 above, we let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$ denote the left-hand and right-hand model respectively. The indicated arrows constitute a monotonic map $f : P \to Q$. This map f is *not* a map of Kripke frames. Indeed, the upset generated by the middle point in P does not get mapped to an upset in Q. We claim that this map satisfies the CDC for the indicated domains. We prove the claim for the right-hand domain, which, for convenience, we denote by $D = \{m\}$. Let $p \in P$ be such that ${\uparrow} f(p) \cap D \neq \varnothing$. This simply means that $f(p) \leq m$. It is easy to see that there exists a $m^* \in P$ such that $f(m^*) = m$ and $m^* \geq p$. This proves the desired.

4.36 Lemma
Let $f : P \to Q$ and $g : Q \to K$ be monotonic maps, and let $\mathcal{D} \subseteq \mathcal{P}(K)$ be a set of

[8] Bezhanishvili and Bezhanishvili (2013) formulate the closed domain condition in terms of Esakia spaces, as introduced by Esakia (1974). As we work with finite models, this additional topological data is not necessary, and we choose to side-step it entirely.

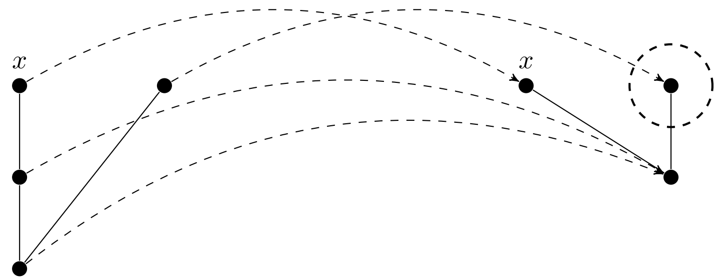


Figure 4.3.: The indicated arrows constitute a monotonic map that does not satisfy the CDC for the indicated domain.

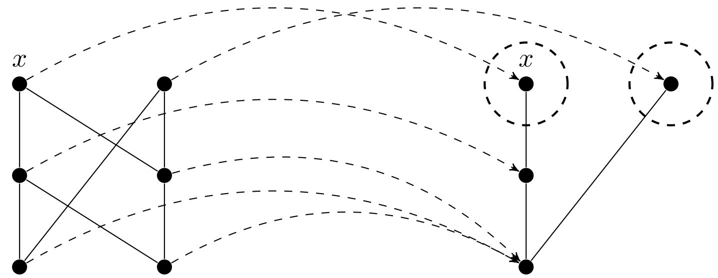


Figure 4.4.: The indicated arrows constitute a monotonic map that does satisfy the CDC for all the indicated domains.

domains. Suppose that g has the CDC for D and f has the CDC for $g^{-1}(D)$. Now, $g \circ f$ has the CDC for D.

Proof. Suppose that $\uparrow(g \circ f)(p) \cap D \neq \varnothing$. By assumption, we get $g\left(\uparrow f(p)\right) \cap D \neq \varnothing$. We can thus readily deduce that $\uparrow f(p) \cap g^{-1}(D) \neq \varnothing$, proving $f(\uparrow p) \cap g^{-1}(D)$ to be non-empty. We thus obtain $(g \circ f)(\uparrow p) \cap D \neq \varnothing$, as desired. $\square$

Out of all the potential domains one could define on a model, we are particularly interested in those domains that arise syntactically as in Definition 4.37. These domains are precisely the sets of points where certain implications fail to hold. Note that such domains most certainly need not be upsets, as illustrated by Fig. 4.5.

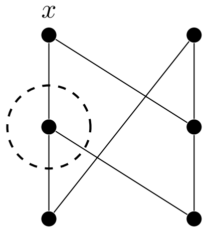


Figure 4.5.: A model on the variables $X = \{x\}$, where the marked subset is the domain on which the implication $\neg\neg x \to x$ is not valid.

4.37 Definition
Let $v : P \to \mathcal{P}(X)$ be a model, and let $\Sigma \subseteq \mathcal{L}(X)$ be an *adequate set*. We define the *domains specified by* Σ as $\mathcal{D}_v^\Sigma := \left\{ [\![\phi]\!]_v - [\![\psi]\!]_v \;\middle|\; \phi \to \psi \in \Sigma \right\}$.

Lemma 4.38 shows that a monotonic map respects the validity of Σ precisely if it satisfies the CDC for $\mathcal{D}_v^\Sigma$, much like a map of Kripke models was shown to respect the validity of all formulae in Lemma 2.26. Moreover, monotonic maps that satisfy the CDC are a generalisation of maps of Kripke frames, which we illustrate in Lemma 4.39 below. Intuitively speaking, a monotonic map into an order-defined Kripke model is a map of Kripke frames precisely whenever it satisfies the CDC for all domains that can be specified in the sense of Definition 4.37.

4.38 Lemma
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set of formulae, let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(X)$ be models, and let $f : P \to Q$ be a monotonic map satisfying $v = u \circ f$.[9] The following are equivalent:

(i) the function f has the CDC for $\mathcal{D}_u^\Sigma$;

(ii) the equivalence (4.9) holds.

$$v, p \Vdash \chi \text{ if and only if } u, f(p) \Vdash \chi \text{ for all } \chi \in \Sigma \text{ and } p \in P. \tag{4.9}$$

Proof. Suppose that (i) holds. We prove (4.9) for all $p \in P$ by structural induction along $\chi \in \Sigma$. In the base case, the desired is immediate by the requirement that $v = u \circ f$. Both the conjunctive and disjunctive case follow straightforwardly by

[9] Note that the requirement that $v = u \circ f$ is included in Definition 2.25, the definition of a map of Kripke models. As f is merely assumed to be a monotonic map, a map between posets, we need to require this.

induction. Now, suppose $\chi = \phi \to \psi$, and note that $\phi, \psi \in \Sigma$ holds as Σ was assumed to be adequate. Consider $p \in P$ and $q \in Q$ such that $v, p \Vdash \phi \to \psi$, $f(p) \leq q$, and $u, q \Vdash \phi$. If $u, q \nVdash \psi$ then we know:

$$\uparrow f(p) \cap \left(\llbracket\phi\rrbracket_u - \llbracket\psi\rrbracket_u\right) \neq \varnothing.$$

Through the CDC for $\mathcal{D}_u^\Sigma$, we have some $k \geq p$ such that $u, f(k) \Vdash \phi$ and $v, f(k) \nVdash \psi$. By induction, (4.9) allows us to deduce that $v, k \nVdash \phi \to \psi$, a contradiction. This proves the implication from left to right in (4.9); the other direction is immediate.

Conversely, suppose (ii) holds. We assume that $\uparrow f(p) \cap D \neq \varnothing$ for some $D \in \mathcal{D}_u^\Sigma$. This gives us some $\phi \to \psi \in \Sigma$ such that $D = \llbracket\phi\rrbracket_u - \llbracket\psi\rrbracket_u$. As a consequence, we immediately know that $u, f(p) \nVdash \phi \to \psi$. It follows through (4.9) that $v, p \nVdash \phi \to \psi$, so there is some $q \geq p$ such that $v, q \Vdash \phi$ and $v, q \nVdash \psi$. Using (4.9) again, we obtain $u, f(q) \Vdash \phi$ and $u, f(q) \nVdash \psi$. This, in turn, yields $f(\uparrow p) \cap D \neq \varnothing$, proving (i) as desired. □

4.39 Lemma
Let P be a poset, and let $u : Q \to \mathcal{P}(X)$ be an *order-defined* model. Suppose the monotonic map $f : P \to Q$ satisfies the CDC for $\mathcal{D}_u^\Sigma$, where $\Sigma := \mathcal{L}(X)$. Then f is a map of Kripke frames.

Proof. Take $p \in P$ and $q \in Q$ to be such that $f(p) \leq q$. Now consider the formulae:

$$\phi := \mathsf{up}\, q \text{ and } \psi := \mathsf{nd}\, q.$$

It is clear that $q \Vdash \phi$ and $q \nVdash \psi$, hence $\uparrow f(q) \cap \llbracket\phi\rrbracket_u - \llbracket\psi\rrbracket_u$ is non-empty. By assumption, this yields us some $k \geq p$ such that $u, f(k) \Vdash \phi$ and $u, f(k) \nVdash \psi$. The former proves $q \leq f(k)$, whereas the latter proves $f(k) \leq q$. We thus derive $f(k) = q$, as desired. □

In order to provide some examples of maps that satisfy the CDC, we recall the technique of *filtration*, first described by Lemmon and Scott (1977). The following Definition 4.40 and Lemma 4.42 are folklore, although the lemma is phrased using the terminology given above, cf. Chagrov and Zakharyaschev (1997, Chapter 5) and Gabbay (1981, Chapter 4). We refer to Bezhanishvili, Bezhanishvili, and Iemhoff (2014, Section 4) for a very recent perspective on these matters.

Given an adequate set Σ and a model $v : P \to \mathcal{P}(X)$, one can consider points to be equivalent whenever their theories restricted to Σ coincide. This induces an equivalence relation on P.

$$p_1 \sim p_2 \text{ precisely if } \mathrm{Th}_v(p_1) =_\Sigma \mathrm{Th}_v(p_2) \tag{4.10}$$

We write $[p]$ for the equivalence class of p under $\sim$, and denote the set of all such equivalence classes by $P_{\sim}$. Using these definition, one defines *filtration* as in Definition 4.40 below.

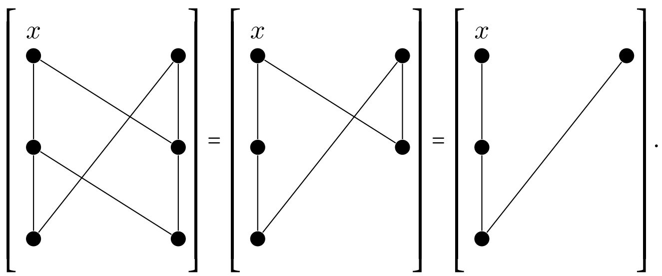
4.40 Definition (Filtration)
Let $v : P \to \mathcal{P}(X)$ be a model, and let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set. A Kripke model $u : P_{\sim} \to \mathcal{P}(X)$ is said to be a *filtration of* v *through* Σ whenever:

(i) if $p_1 \leq p_2$ then $[p_1] \leq [p_2]$ for all $p_1, p_2 \in P$;

(ii) $u([p]) = v(p)$ for each $p \in P$;

(iii) if $[p_1] \leq [p_2]$ and $p_1 \Vdash \phi$ then $p_2 \Vdash \phi$ for all $\phi \in \Sigma$.

4.41 Example
Consider the adequate set $\Sigma = \{\bot, x, \neg x, \neg\neg x\}$, and recall Fig. 4.4 as discussed in Example 4.35. We compute that:



All other points have singleton equivalence classes. One can construe the right-hand model of Example 4.35 as filtration through identifying a point $q \in Q$ with $f^{-1}(q)$. This readily follows when we can show $\mathrm{Th}_v(p) =_{\Sigma} \mathrm{Th}_u(f(p))$. Because $v \circ f = u$, we know this to hold by Lemma 4.38 whenever f has the CDC for $\mathcal{D}_u^{\Sigma}$. Now, note that $\mathcal{D}_u^{\Sigma}$ consists of the two singleton sets indicated in Fig. 4.4. Yet we already argued that f satisfies the CDC for these two sets in Example 4.35, proving the claim.

The above Example 4.41 showed a very concrete instance of a filtration in which the canonical quotient map is a map that satisfies the CDC for the appropriately chosen set of domains. This property holds true in general, which we explicate in Lemma 4.42 below.

4.42 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model, let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and let the Kripke model $u : Q \to \mathcal{P}(X)$ be a *filtration* of v through Σ. There is a surjective, monotonic map $f : v \to u$ that satisfies both the CDC for $\mathcal{D}_u^{\Sigma}$ and $v = u \circ f$.

Proof. Define the map f by:

$$f : v \to u, \quad p \in P \mapsto [p] = \{q \in P \mid p \sim q\}.$$

First, the map f is monotonic by condition (i) of Definition 4.40. Second, observe that $v = u \circ f$, as follows from condition (ii). Finally, we verify whether f satisfies the CDC for $\mathcal{D}_u^\Sigma$. This follows from Lemma 4.38 whenever:

$$v, p \Vdash \chi \text{ iff } u, f(p) \Vdash \chi \text{ for all } p \in P \text{ and } \chi \in \Sigma.$$

We proceed by structural induction along $\chi \in \Sigma$. The base case is immediate. The disjunctive and conjunctive cases follow readily by induction, so we focus solely on the implicative case. We know that $\chi = \phi \to \psi$ for some $\phi, \psi \in \Sigma$. Suppose $v, p \Vdash \phi \to \psi$ and $q \geq f(p)$ such that $v, q \Vdash \phi$. By construction, there is some $k \in p$ such that $f(k) = q$. Induction ensures $v, k \Vdash \phi$. Moreover, the condition (iii) of Definition 4.40 ensures $v, k \Vdash \phi \to \psi$. Hence $v, k \Vdash \psi$, which proves $u, q \Vdash \psi$ by induction. The other direction is immediate. $\square$

In the case of IPC, it is obvious that any filtration of a model of IPC is again a model of IPC. This need not hold in general, as will become apparent in Example 4.49. Stable logics, as given in Definition 2.98, do always allow for filtrations. The technique of filtration has been extended throughout the years, to allow it to be applied to a greater variety of logics. Think, for example, of *selective filtration* as pioneered by Gabbay (1972) and employed by Gabbay and de Jongh (1974) to prove the finite model property for BB_n. We do not expand on this, nor do we employ such techniques in the following.

4.2.1. Syntactic constructions

The canonical model has been mentioned several times in the above. Regardless of the properties of the intermediate logic at hand, the canonical model on a given number of variables contains all information about this logic's theorems. In this section, we define *adequately canonical models*. As the name suggests, they play the role of canonical models, but then only with respect to a given adequate set. Naturally, the full canonical model is but a special case. This endeavour has been undertaken by others, see for instance Visser (1984, pp. 12-14), Visser (1994, pp. 5-6), Visser (1996, Section 4) and de Jongh and Visser (1996, pp. 196-197).

A warning is in order. Within the canonical model, one can safely interchange validity and derivability; they mean precisely the same thing. This is not necessarily the case in adequately canonical models. Phrased differently: adequately canonical models need not be models of the logic at hand. One has to be careful to not have one's intuition be contaminated with the idea validity and derivability can be safely interchanged, as this quite possibly erroneous assumption could lead one astray. Theorem 4.53 regains some of the former safety, proving adequately canonical models to be models in the setting of stable logics. The machinery we devise here returns in Sections 4.3 and 5.3, where it is put to good use.

Throughout this section, Λ denotes an arbitrary intermediate logic. It is explicitly *not* required that Λ enjoys the finite model property, contrary to most other parts of this thesis. Moreover, again contrary to elsewhere in this thesis, the symbols Γ, Π, Θ, Σ can denote infinite sets of formulae. Note that they need not be infinite, it simply should not and need not be excluded as a possibility. On the one hand, practical purposes such as proving the decidability of admissible rules for a given logic could lead one to restrict to finite sets Σ.[10] On the other hand, when characterising the admissibility of a rule it makes good sense to let Σ be the set of all formulae, which is clearly not finite.

Recall that $\vdash_\Lambda$, being a single-conclusion consequence relation, is a relation between finite sets of formulae and formulae. For notational convenience, we extend this relation to potentially infinite sets of formulae by defining:

$$\Gamma \vdash_\Lambda \chi \text{ iff there is a finite } \Pi \subseteq \Gamma \text{ such that } \Pi \vdash_\Lambda \chi. \tag{4.11}$$

It is an easy matter to verify that the resulting relation satisfies the conditions of Definition 2.2 when interpreting them as pertaining to the above relation, and it coincides with the earlier definition on finite sets.

We start with a generalisation of being a *saturated set of formulae*, which in the propositional case essentially amounts to being closed under the rule DP. Lemmas 4.45 and 4.46 provide means of constructing specific sets satisfying the definition below.

4.43 Definition (Saturation)
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and let $\Gamma, \Pi \subseteq \mathcal{L}(X)$ be sets of formulae. We say that Γ *is* Σ*-saturated in* Π, denoted $\Gamma \preccurlyeq_\Sigma \Pi$, whenever the following two properties hold:

(i) Γ is a subset of Π;

(ii) if $\Gamma \vdash_\Lambda \bigvee \Delta$ then $\Pi \cap \Delta \neq \varnothing$ for all finite $\Delta \subseteq \Sigma$.

[10] We phrase this cautiously for a reason. Theorem 4.53 indicates that one can even obtain a decidability result while keeping Σ infinite, as long as Σ is finite up to provable equivalence with respect to the intermediate logic at hand.

A set of formulae Γ that is Σ-saturated in Γ is simply said to be Σ-*saturated*. Whenever Σ is taken to be the set of all formulae, we drop all reference to it from the nomenclature. As an example, we easily see that the theory of a rooted model is saturated. Throughout the literature, there are several synonyms for saturated sets of formulae, amongst them are prime theories and theories with the disjunction property. In this setting, a theory would be a set of formulae Γ where $\Gamma \vdash_\Lambda \chi$ entails $\chi \in \Gamma$ for all $\chi \in \Sigma$. One can readily see this to hold for any Σ-saturated set.

Recall Theorem 3.45, in which we describe the existence of covers. The proof from (iii) to (i), at some point, picks a point p that is maximal with respect to lying below a chosen set of points W. In this setting, the theory of p is saturated in the theory of W. Definition 4.44 below generalises this to the adequate context.

There is some subtlety in choosing the appropriate generalisation of the situation sketched above. Indeed, simply demanding maximality with respect to the order $\subseteq_\Sigma$ would be too weak. Summarising Definition 4.44 informally, it states that there can be no strict extension of Γ along Σ all the while remaining Σ-saturated in Π.

4.44 Definition (Maximally Saturated)
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and let $\Gamma, \Pi \subseteq \mathcal{L}(X)$ be sets of formulae. We say that Γ is Σ-*maximally saturated in* Π when the following two hold:

(i) Γ is Σ-saturated in Π;

(ii) if $\Gamma \subseteq \Theta \preccurlyeq_\Sigma \Pi$ then $\Gamma =_\Sigma \Theta$ for every set $\Theta \subseteq \mathcal{L}(X)$.

One may then wonder whether there exists a set that is Σ-maximally saturated in some set Π whenever there exists a set that is Σ-saturated in Π. This turns out to be the case. Moreover, such sets are always Σ-saturated in themselves. We prove these two assertions in Lemmas 4.45 and 4.46 below. They arise as a natural generalisation of a lemma well-established in folklore, see for instance Iemhoff (2001b, Lemma 3.4). The final statement in Lemma 4.45 is void when $\Sigma = \mathcal{L}(X)$, yet quite important otherwise.

4.45 Lemma
Let $\Gamma, \Pi, \Sigma \subseteq \mathcal{L}(X)$ be sets of formulae, and suppose that Σ is adequate. If Γ is Σ-saturated in Π, then there exists a set of formulae $\Gamma' \subseteq \mathcal{L}(X)$ that is Σ-maximally saturated in Π, satisfying both $\Gamma \subseteq \Gamma'$ and $\Gamma' - \Gamma \subseteq \Sigma$.

Proof. We construct a sequence of sets:

$$\Gamma =: \Gamma_0 \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_\omega := \bigcup_{n<\omega} \Gamma_n,$$

such that Γ_n is Σ-saturated within Π for each $n \leq \omega$. We claim that the set Γ_ω satisfies all required constraints.

The sequence is constructed in the following manner. Consider any enumeration of Σ, say $(\phi_n)_{n\in\mathbb{N}}$. We inductively define the following. Note that this construction clearly ensures that $\Gamma_\omega - \Gamma \subseteq \Sigma$.

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\phi_n\} & \text{if } \Gamma_n \cup \{\phi_n\} \preccurlyeq_\Sigma \Pi \\ \Gamma_n & \text{otherwise.} \end{cases}$$

First, we prove that $\Gamma_\omega \preccurlyeq_\Sigma \Pi$. It is clear from the construction that $\Gamma_\omega \subseteq \Pi$. Take some finite $\Delta \subseteq \Sigma$, and assume that $\Gamma_\omega \vdash_\Lambda \bigvee \Delta$. By definition, this ensures the existence of a finite set $\Theta \subseteq \Gamma_\omega$ such that $\Theta \vdash_\Lambda \bigvee \Delta$. There must be some $n \in \mathbb{N}$ such that $\Theta \subseteq \Gamma_n$. As a consequence, $\Gamma_n \vdash_\Lambda \bigvee \Delta$, and as $\Gamma_n \preccurlyeq_\Sigma \Pi$ holds by construction, we now know $\Delta \cap \Pi \neq \varnothing$, proving the desired.

Second, we prove maximality. Let $\Theta \subseteq \mathcal{L}(X)$ be a set of formulae satisfying $\Theta \supseteq \Gamma_\omega$ and $\Theta \preccurlyeq_\Sigma \Pi$. We aim to show that $\Theta \subseteq_\Sigma \Gamma_\omega$. To this end, consider any $\phi \in \Theta \cap \Sigma$. We know that $\phi = \phi_n$ for some $n \in \mathbb{N}$, fix such an n. Note that:

$$\Gamma_n \cup \{\phi\} \subseteq \Gamma_\omega \cup \{\phi\} \subseteq \Theta \preccurlyeq_\Sigma \Pi,$$

so $\Gamma_n \cup \{\phi\} \preccurlyeq_\Sigma \Pi$. The construction of $(\Gamma_n)_{n\geq 0}$ now ensures that $\phi \in \Gamma_{n+1} \subseteq \Gamma_\omega$. This proves $\Theta \subseteq_\Sigma \Gamma_\omega$, as desired. □

4.46 Lemma
Let $\Gamma, \Pi, \Sigma \subseteq \mathcal{L}(X)$ be sets of formulae, and suppose that Σ is adequate. If Γ is Σ-*maximally saturated in* Π, then Γ is Σ-*saturated*.

Proof. Let $\Delta \subseteq \Sigma$ be a finite set of formulae, and suppose that $\Gamma \vdash_\Lambda \bigvee \Delta$. If there is a $\chi \in \Delta$ such that $\Gamma \cup \{\chi\}$ is Σ-saturated within Π, then $\Gamma \cup \{\chi\} =_\Sigma \Gamma$ follows from the maximality of Γ. This proves that Δ intersects Γ. We reason by contradiction, so we assume that this is not the case.

As such, we can pick a finite set of formulae $\mathsf{cx}(\chi) \subseteq \Delta$ for each $\chi \in \Delta$ such that $\Gamma \cup \{\chi\} \vdash_\Lambda \bigvee \mathsf{cx}(\chi)$ but $\mathsf{cx}\,\chi \cap \Pi = \varnothing$. This set $\mathsf{cx}(\chi)$ can be thought of as a ***counterexample*** to $\Gamma \cup \{\chi\} \preccurlyeq_\Sigma \Pi$. We consequently know that:

$$\Gamma \vdash_\Lambda \bigvee_{\chi\in\Delta} \bigvee \mathsf{cx}(\chi).$$

Because $\Gamma \preccurlyeq_\Sigma \Pi$, this must entail that $\mathsf{cx}(\chi)$ intersects Π for some $\chi \in \Delta$. But this is excluded by design, a contradiction, proving the desired. □

We can readily prove Corollary 4.47 below using the machinery developed above. The case for $\Sigma = \mathcal{L}(X)$ is quite old, with proofs going back to at least Aczel (1968, Theorem 3.1).

4.47 Corollary
Let $\Gamma, \Sigma \subseteq \mathcal{L}(X)$ be sets of formulae, and suppose that Σ is adequate. If $\Gamma \nvdash_\Lambda \phi \to \psi$ for some $\phi, \psi \in \Sigma$, then there exists a Σ-saturated set $\Pi \subseteq \mathcal{L}(X)$ such that $\Gamma \subseteq \Pi$, $\phi \in \Pi$, $\psi \notin \Pi$ and $\Pi - \Gamma \subseteq \Sigma$.

Proof. Define the set $\Theta := (\Gamma \cup \Sigma) - \{\psi\}$. We claim that $\Gamma \cup \{\phi\}$ is Σ-saturated in Θ. First, note that $\Gamma \subseteq \Theta$ clearly holds. Second, suppose:

$$\Gamma, \phi \vdash_\Lambda \bigvee \Delta$$

holds for some $\Delta \subseteq \Sigma$. Clearly, $\psi \in \Delta$ cannot hold, as this would entail $\Gamma \vdash_\Lambda \phi \to \psi$. Hence we know $\Delta \cap \Theta \neq \varnothing$. We now apply Lemmas 4.45 and 4.46 to obtain the desired Π, finishing the argument. □

We now provide the promised generalisation of the canonical model. Definition 4.48 is much more concrete than our description in Example 2.68. Note that this model need not be sound for the intermediate logic at hand, as we show in Example 4.49.

4.48 Definition (Adequately Canonical Model)
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and let Λ be an intermediate logic. We define the Σ*-adequately canonical model of* Λ by:

$$\mathsf{c} : \mathsf{can}^{\Sigma}_{\Lambda}(X) := \{\Gamma \mid \Gamma \subseteq \Sigma \text{ is } \Sigma\text{-saturated}\} \to \mathcal{P}(X), \quad \Gamma \mapsto \Gamma \cap X.$$

4.49 Example
Consider the intermediate logic $\Lambda = \mathsf{BD}_2$, and fix $\Sigma = X = \{x, y\}$. It is easy to see that each subset of Σ is Σ-adequate and Σ-saturated. The Σ-adequately canonical model $\mathsf{c} : \mathsf{can}^{\Sigma}_{\mathsf{BD}_2}(X) \to \mathcal{P}(X)$ is depicted in Fig. 4.6. It is easy to see that this model is concrete, yet it is not of *height at most two*. Consequently, Lemma 2.74 and Corollary 3.13 ensure that c could not possibly be a model of BD_2.

4.50 Theorem
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and consider the model $\mathsf{c} : \mathsf{can}^{\Sigma}_{\Lambda}(X) \to \mathcal{P}(X)$. For each $\Gamma \in \mathsf{can}^{\Sigma}_{\Lambda}(X)$ one has $\mathrm{Th}_{\mathsf{c}}(\Gamma) =_\Sigma \Gamma$.

Proof. We prove, by structural induction along $\chi \in \Sigma$, that for each $\Gamma \in \mathsf{can}^{\Sigma}_{\Lambda}(X)$ we have:

$$\mathsf{c}, \Gamma \Vdash \chi \text{ if and only if } \Gamma \vdash_\Lambda \chi.$$

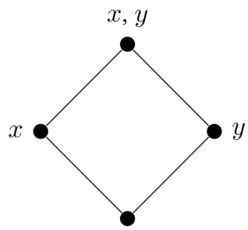


Figure 4.6.: An example of the Σ-adequately canonical model, where Σ consists solely of variables.

The base case follows directly by Definition 4.48. Indeed, $x \in \mathsf{c}(\Gamma)$ holds precisely if $x \in \Gamma$, which is equivalent to $\Gamma \vdash_\Lambda x$ for any $x \in X$. The conjunctive and disjunctive cases readily follow via induction.

Let us treat the implicative case, where $\chi = \phi \to \psi$, in some detail. From left to right, suppose $\mathsf{c}, \Gamma \Vdash \phi \to \psi$ yet $\Gamma \nvdash_\Lambda \phi \to \psi$. By Corollary 4.47 and induction, we know there to be a $\Pi \subseteq \mathcal{L}(X)$ that is Σ-saturated, satisfying $\phi \in \Pi$, $\psi \notin \Pi$ and $\Pi - \Gamma \subseteq \Sigma$. The above first and final property ensure that Π is a point $\Pi \geq \Gamma$ in $\mathsf{can}^\Sigma_\Lambda(X)$. Through induction, we know $\mathsf{c}, \Pi \Vdash \phi$ and $\mathsf{c}, \Pi \nVdash \psi$. Yet this immediately proves $\mathsf{c}, \Gamma \nVdash \phi \to \psi$, a contradiction.

Conversely, suppose $\mathsf{c}, \Gamma \nVdash \phi \to \psi$. We obtain a Σ-saturated set Π such that $\Pi \Vdash \phi$ and $\Pi \nVdash \psi$. Again, though induction, we see $\Pi \vdash_\Lambda \phi$ and $\Pi \nvdash_\Lambda \psi$. If $\Gamma \vdash_\Lambda \phi \to \psi$ then surely $\Pi \vdash_\Lambda \phi \to \psi$. Yet *modus ponens* now guarantees $\Pi \vdash_\Lambda \psi$, a contradiction. This proves the desired. $\square$

The canonical model clearly is refined in the sense of Definition 2.36. The same observation holds for adequately canonical models. In fact, an even stronger property can be seen to hold, which we specify in Definition 4.51 below. Often, especially when Σ is finite, there is a clear upper bound on the amount of different Σ-adequately refined models.

4.51 Definition (Adequately Refined)
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and let $v : P \to \mathcal{P}(X)$ be a model. We say that v is Σ-*adequately refined* whenever:

$$p \leq q \text{ iff } \mathrm{Th}_v(p) \subseteq_\Sigma \mathrm{Th}_v(q) \text{ for all } p, q \in P.$$

It goes without saying that the implication from left to right in the above always holds.

4.52 Lemma
Let Λ be an intermediate logic, let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set of formulae, and let $v : P \to \mathcal{P}(X)$ be a model. The model $\mathsf{c} : \mathsf{can}_\Lambda^\Sigma(X) \to \mathcal{P}(X)$ is Σ-adequately refined.

Proof. Suppose $\Gamma, \Pi \in \mathsf{can}_\Lambda^\Sigma(X)$ are such that $\Gamma \nleq \Pi$. By definition, there is a $\phi \in \Sigma$ such that $\phi \in \Gamma$ and $\phi \notin \Pi$. Through Theorem 4.50 we know $\mathsf{c}, \Gamma \Vdash \phi$ and $\mathsf{c}, \Pi \nVdash \phi$, proving the desired. □

Theorem 4.53 below shows that there is a correspondence between the size of Σ-refined models and the size of Σ.

4.53 Theorem
Let Λ be an intermediate logic, and let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set of formulae. Suppose that there are at most finitely many formulae $\phi_1, \ldots, \phi_n \in \Sigma$ such that:

$$\vdash_\Lambda \phi_i \equiv \phi_j \text{ iff } i = j \text{ for all } i, j \leq n. \tag{4.12}$$

Every Σ-adequately refined model $v : P \to \mathcal{P}(X)$ is of size at most 2^n.

Proof. Let $\phi_1, \ldots, \phi_n \in \Sigma$ be the longest list of formulae satisfying (4.12). We claim that for any Σ-saturated $\Gamma, \Pi \subseteq \Sigma$ we have that $\Gamma = \Pi$ precisely if $\phi_i \in \Gamma$ iff $\phi_i \in \Pi$ for all $i \leq n$. The size-bound on $v : P \to \mathcal{P}(X)$ follows immediately from the above claim.

To prove the claim, first note that the implication from left to right is immediate. To see the converse, suppose $\Gamma \neq \Pi$. Without loss of generality, there is a $\phi \in \Sigma$ such that $\phi \in \Gamma$ and $\phi \notin \Pi$. See that $\vdash_\Lambda \phi \equiv \phi_i$ holds for some $i \leq n$. Indeed, if not, then this would violate the maximality of n. By the Σ-saturation of Γ, we know $\phi_i \in \Gamma$. For the same reason, $\phi_i \notin \Pi$ follows. This proves the claim.

4.2.2. Covers revisited

Recall Lemma 2.40, which states that there exists a unique map of Kripke models between any image-finite model and the corresponding canonical model. We proved this via a syntactic description of covers, and the guarantee that a monotonic map leaving an image-finite model is a map of Kripke frames precisely when all covers are preserved, respectively given by Lemmas 2.35 and 2.38. In this subsection, we describe similar machinery for maps satisfying the CDC. This leads to a generalisation of Lemma 2.40, provided by Corollary 4.57.

4.54 Definition (Adequate Cover)
Let P be a poset, let $\mathcal{D}$ be a set of subsets, let $p \in P$, and let $W \subseteq P$. We say that W *is a $\mathcal{D}$-adequate cover of* p, denoted $W \,\kappa_{\mathcal{D}}\, p$, whenever:

$$\text{if } {\uparrow}p \cap D \neq \varnothing \text{ then } p \in D \text{ or } {\uparrow}W \cap D \neq \varnothing, \text{ for all } D \in \mathcal{D}. \tag{4.13}$$

In the following, we write $W \,\kappa_{\Sigma}\, p$ to mean that $W \,\kappa_{\mathcal{D}}\, p$ for $\mathcal{D} = \mathcal{D}_v^{\Sigma}$ in the context where $v : P \to \mathcal{P}(X)$. Recall Definition 2.29, where we defined $W \,\kappa\, p$. The above definition is a generalisation of this concept. Indeed, if the ambient model is order-defined, then $W \,\kappa\, p$ is equivalent to $W \,\kappa_{\Sigma}\, p$ for $\Sigma := \mathcal{L}(X)$.[11]

4.55 Lemma
Let $v : P \to \mathcal{P}(X)$ be a model, let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and let $\mathcal{D} = \mathcal{D}_v^{\Sigma}$. The following are equivalent for all $p \in P$ and finite $W \subseteq P$:

(i) $W \,\kappa_{\Sigma}\, p$;

(ii) the equivalence (4.14) holds for all $\phi \to \psi \in \Sigma$ and $p \in P$.

$$v, p \Vdash \phi \to \psi \text{ iff } v, W \Vdash \phi \to \psi \text{ and } (v, p \Vdash \phi \text{ implies } v, p \Vdash \psi) \tag{4.14}$$

Proof. Suppose (i) holds, and let $\phi \to \psi \in \Sigma$ be arbitrary. The implication from left to right in (4.14) is immediate, as $W \subseteq {\uparrow}p$. In order to prove the other direction, we assume that $v, p \nVdash \phi \to \psi$. It is clear that ${\uparrow}p \cap [\![\phi]\!]_v - [\![\psi]\!]_v \neq \varnothing$, hence we know that either $p \in [\![\phi]\!]_v - [\![\psi]\!]_v$ or ${\uparrow}W \cap [\![\phi]\!]_v - [\![\psi]\!]_v \neq \varnothing$. The former entails $v, p \Vdash \phi$ and $v, p \nVdash \psi$, whereas the latter ensures $v, W \nVdash \phi \to \psi$. This proves (ii).

Conversely, suppose (ii) holds. Take an arbitrary $D \in \mathcal{D}_v^{\Sigma}$, a point $p \in P$, and suppose that ${\uparrow}p \cap D \neq \varnothing$. We know that $D = [\![\phi]\!]_v - [\![\psi]\!]_v$ for some $\phi \to \psi \in \Sigma$. It thus follows that $v, p \nVdash \phi \to \psi$. By (4.14), we know that $v, W \nVdash \phi \to \psi$ or $v, p \Vdash \phi$ and $v, p \nVdash \psi$. The latter disjunct entails $p \in D$, whereas the former entails ${\uparrow}W \cap D \neq \varnothing$. This proves (i), as desired. $\square$

[11] A more granular equivalence can be given as well. Definition 3.9, the specification of what it means to be order-defined, can easily be generalised to the adequate case. Say that a model $v : P \to \mathcal{P}(X)$ is *order-defined by* Σ whenever each principal upset and the complement of each principal downset can be defined by means of a formula from Σ. With this in mind, $W \,\kappa\, p$ is equivalent to $W \,\kappa_{\Sigma}\, p$ whenever the ambient model is order-defined by Σ.

Recall that we motivated the usefulness of the notion of covers via Lemma 2.35. The following Lemma 4.56 plays an analogous role in justifying the purpose of adequate covers.

4.56 Lemma
Let P and Q be Kripke frames, let $\mathcal{D}$ be a set of subsets of Q, and let $f : P \to Q$ be a monotonic map. Suppose that P is image finite. The following are equivalent:

(i) the function f satisfies the CDC for $\mathcal{D}$;

(ii) for every $p \in P$ and for every finite $W \subseteq P$, if $W \ \kappa\ p$, then $f(W) \ \kappa_{\mathcal{D}}\ f(p)$.

Proof. Suppose (i) holds. Let $p \in P$ and $W \subseteq P$ be such that W is finite and $W \ \kappa\ p$. Via monotonicity, it follows that $f(W) \subseteq \uparrow f(p)$. Fix a $D \in \mathcal{D}$, and suppose that $\uparrow f(p) \cap D \neq \varnothing$. As f is assumed to have the CDC for D, we know that $f(\uparrow p) \cap D \neq \varnothing$, so we can take some $q \geq p$ such that $f(q) \in D$. Because $W \ \kappa\ p$, we now know that either $p = q$ or $q \in \uparrow W$. In the former case, we know that $f(p) = f(q) \in D$. In the latter case, there is a $w \in W$ such that $w \leq q$. Consequently, $f(q) \geq f(w)$, and so $f(q) \in \uparrow f(W) \cap D$, as desired.

Conversely, suppose (ii) holds. Let $p \in P$ and $D \in \mathcal{D}$ be such that $\uparrow f(p) \cap D \neq \varnothing$. We proceed by well-founded induction along $n := |\uparrow p|$. Suppose that we know:

$$\uparrow f(k) \cap D \neq \varnothing \text{ implies } f(\uparrow k) \cap D \neq \varnothing \text{ for all } k \in P \text{ with } |\uparrow k| < n. \tag{4.15}$$

Our goal is to prove that $f(\uparrow p) \cap D \neq \varnothing$. First, note that $W := \mathop{\uparrow}\limits^{\circ} p$ is finite and satisfies $W \ \kappa\ p$. By (ii), we thus know $f(W) \ \kappa_{\mathcal{D}}\ f(p)$. We gather that $f(p) \in D$ or $\uparrow f(W) \cap D \neq \varnothing$. In the former case, we are done, so assume we are in the latter case. This yields some $w \in W$ such that $\uparrow f(w) \cap D \neq \varnothing$. We know that:

$$\bigcup_{s \in W} f(\uparrow s) \subseteq f(\uparrow p),$$

hence (4.15) finishes the argument when instantiating $k = w$. We have thus proven (i), as desired. □

4.57 Corollary
Let $v : P \to \mathcal{P}(X)$ be an image-finite Kripke model, let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set of formulae, and consider the *adequately canonical model* $\mathsf{c} : \mathsf{can}^{\Sigma}_{\Lambda}(X) \to \mathcal{P}(X)$. There exists a unique monotonic map $f : v \to \mathsf{c}$ satisfying the CDC for $\mathcal{D} := \mathcal{D}^{\Sigma}_{\mathsf{c}}$ and $v \circ f = \mathsf{c}$.

Proof. Let us first prove uniqueness. Suppose there are monotonic maps $f, g : v \to \mathsf{c}$ satisfying the CDC for $\mathcal{D}^{\Sigma}_{\mathsf{c}}$ and $v \circ f = \mathsf{c} = v \circ g$. Let $p \in P$ be arbitrary. By Lemma 4.38, we know that:

$$\mathsf{c}, f(p) \Vdash \chi \text{ iff } v, p \Vdash \chi \text{ iff } \mathsf{c}, g(p) \Vdash \chi \text{ for all } \chi \in \Sigma.$$

Combining this with Theorem 4.50 we derive:

$$f(p) = \mathrm{Th}_{\mathsf{c}}(f(p)) \cap \Sigma = \mathrm{Th}_v(p) \cap \Sigma = \mathrm{Th}_{\mathsf{c}}(g(p)) \cap \Sigma = g(p),$$

proving the desired uniqueness.

The above suggests the definition:

$$f : P \to \mathsf{can}^{\Sigma}_{\Lambda}(X),\ p \mapsto \mathrm{Th}_v(p) \cap \Sigma.$$

We proceed via Lemma 4.56, so let $W \subseteq P$ and $p \in P$ be such that $W \ \kappa \ p$. The desired follows when we can prove $f(W) \ \kappa_{\mathcal{D}} \ f(p)$. By Lemma 2.34, for all $\phi \to \psi \in \mathcal{L}(X)$ we know that:

$$v, p \Vdash \phi \to \psi \text{ iff } v, W \Vdash \phi \to \psi \text{ and } v, p \Vdash \phi \text{ implies } v, p \Vdash \psi.$$

Consequently, we are done by Theorem 4.50 followed by Lemma 4.55. □

4.2.3. Adequate maps

We introduced definable maps in Section 3.1, and upon this notion we built semantics for admissibility in Section 3.3. We close this section with a suitably adjusted notion of definability of monotonic maps that satisfy the CDC. Consider models $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(Y)$. When we merely know a monotonic map $f : P \to Q$ to have the CDC for $\mathcal{D}^{\Sigma}_{u}$, it is not reasonable to require that the pre-image of every definable set is definable. The most one could reasonably expect is the preservation of definability under pre-images of f for upsets defined by formulae from Σ. It is both sufficient and necessary to require this for the variables in Σ, which is how we will define it.

4.58 **Definition (Adequate Map)**
Let $v : P \to \mathcal{P}(Y)$ and $u : Q \to \mathcal{P}(X)$ be Kripke models, and let $\mathcal{D}$ be a set of subsets of Q. We say that a function $f : P \to Q$ is a *$\mathcal{D}$-adequate map* $f : v \to u$ whenever f is monotonic, f has the CDC for $\mathcal{D}$, and the set $f^{-1}(\llbracket x \rrbracket_u)$ is definable for all $x \in X$. Moreover, a function $f : P \to Q$ is said to be a *Σ-adequate map* $f : v \to u$ whenever it is an $\mathcal{D}^{\Sigma}_{u}$-adequate map $f : v \to u$.

It is easy to see that $\mathcal{D}$-adequate maps are a special case of definable maps. Indeed, every definable map is $\mathcal{D}$-adequate, no matter the choice of $\mathcal{D}$. A partial converse can be obtained through Lemmas 3.5 and 4.39. Indeed, suppose that v and $u : P \to \mathcal{P}(X)$ are Kripke models, and assume that u is order-defined. It readily follows that any $\mathcal{L}(X)$-adequate map $f : v \to u$ is a definable map as well.

4.59 Example
Consider again the setting of Corollary 4.57, and think of the monotonic map $f : v \to \mathsf{c}$ satisfying the CDC for $\mathcal{D} := \mathcal{D}^{\Sigma}_{\mathsf{c}}$ described therein. This map is a Σ-adequate map. Do recall that c need not be a model of Λ.

4.60 Example
Recall Fig. 4.4 and Example 4.35, and think of the map $f : P \to Q$. We can construe this as a Σ-adequate map $f : v \to u$. Consider the three statements below:

$$f^{-1}\left(\llbracket x \rrbracket_u\right) = \llbracket x \rrbracket_v,$$
$$f^{-1}\left(\llbracket \neg x \rrbracket_u\right) = \llbracket \neg x \rrbracket_v,$$
$$f^{-1}\left(\llbracket \neg\neg x \to x \rrbracket_u\right) \neq \llbracket \neg\neg x \to x \rrbracket_v.$$

Whereas the former two equations hold true, the latter most certainly *does not hold true*. Would f have been a definable map, then the third equation would have necessarily followed from the first, as in Example 3.30. This illustrates that Σ-adequate are strictly more general than definable maps.

There is a close connection between *exact models* and *substitutions*, as explored in Example 3.29 and Lemma 3.31. To be a bit more precise, we have shown that a potentially infinite set of formulae Γ encodes the theory of a substitution precisely if it is the theory of an exact model. The following Theorem 4.61 shows the precise correspondence between substitutions and adequate maps. When instantiating Σ to $\mathcal{L}(X)$, this theorem can be seen to encompass both Example 3.29 and Lemma 3.31 through Lemma 4.39. It is important to bear in mind that the model v referred to in (i) below need not be a model of the logic at hand.

4.61 Theorem
Let Λ be a intermediate logic with the finite model property, let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set with $X \subseteq \Sigma$, and let $\Gamma \subseteq \Sigma$ be a set of formulae. The following are equivalent for all finite sets of variables Y:

(i) there exists a Σ-adequately refined model $v : P \to \mathcal{P}(X)$ and a surjective Σ-adequate map $f : \mathsf{u} \to v$ such that $\mathrm{Th}(v) =_{\Sigma} \Gamma$, where $\mathsf{u} : \mathsf{U}_{\Lambda}(Y) \to \mathcal{P}(Y)$;

(ii) there exists a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that $\vdash_\Lambda \sigma(\phi)$ if and only if $\phi \in \Gamma$ for all $\phi \in \Sigma$.

Proof. Suppose (i) holds. We define the substitution $\sigma_f : \mathcal{L}(X) \to \mathcal{L}(Y)$ by setting, for each $x \in X$:

$$\sigma_f(x) = \mathsf{def}\, f^{-1}\left([\![x]\!]_u\right).$$

We claim that $\mathsf{u}, p \Vdash \sigma(\chi)$ holds iff $v, f(p) \Vdash \chi$ for all $p \in \mathsf{U}_\Lambda(y)$ and $\chi \in \Sigma$. Indeed, this follows immediately through Lemma 4.38. Observe that for all $\chi \in \Sigma$ we know $\mathsf{u} \Vdash \sigma(\chi)$ to be equivalent to $\vdash_\Lambda \sigma(\chi)$ through Corollary 3.23. Moreover, note that f is assumed to be surjective, and see that $v \Vdash \chi$ is equivalent to $\chi \in \Gamma$ for all $\chi \in \Sigma$. We have thus proven (ii).

Conversely, suppose (ii) holds. By Lemma 3.8, we know the identity mapping to be a definable map $g : \mathsf{u} \to \sigma^*(\mathsf{u})$. Consider the adequately canonical model $\mathsf{c} : \mathsf{can}^\Sigma_\Lambda(X) \to \mathcal{P}(X)$. We thus obtain a unique Σ-adequate map $h : \sigma^*(\mathsf{u}) \to \mathsf{c}$ through Corollary 4.57, satisfying $\sigma^*(\mathsf{u}) = \mathsf{c} \circ h$. We know $h \circ g : \mathsf{u} \to \mathsf{c}$ to be a Σ-adequate map by Lemma 4.36.

Define the model $v := \mathsf{c} \restriction (h \circ g)(\mathsf{U}_\Lambda(X))$, and note that v is Σ-adequately refined by Lemma 4.52. Observe that:

$$\mathsf{c}, (h \circ g)(p) \Vdash \phi \text{ iff } \mathsf{u}, p \Vdash \sigma(\phi) \text{ for all } p \in \mathsf{U}_\Lambda(X).$$

From here, (i) is immediate by Corollary 3.23. □

4.3. Semantics of rules

How can one know that a formula is IPC-*extendible*? Although the definition is given in an intrinsic manner, it is not at all apparent that one can verify extendibility in an algorithmic fashion. In this section, we investigate this problem.

First, in Theorem 4.67 we give a criterion under which a finite subset of the universal model is known to have a cover satisfying a certain formula. This criterion only makes use of formulae of a bounded implication degree, in the technical sense as given in Definition 4.62. Subsequently, we provide convenient rule scheme which stratifies nicely over implication degrees. Finally, the answer is provided by Theorem 4.73 in terms of closure under a finite set of rules. As closure under a finite set of rules, in the sense of Definition 4.12, in essence comes down to verifying the validity of some formulae, we have thus provided an algorithm.

A solution to this problem is known in the literature. Indeed, Ghilardi (2002) showed how one can algorithmically verify projectivity for IPC. Because the IPC-extendible formulae coincide with IPC-projective formulae, as shown in Theorem 4.24, this immediately solves the problem. Our approach shows that IPC-extendible formulae are precisely those formulae that are closed under a finite number of instances of rules quite similar to $\overline{\mathsf{D}}_n$. As such, this characterisation can be applied quite readily to providing a basis of admissibility, as we see in Section 5.3.

Let us first formally define what we mean by the implication degree in Definition 4.62 below. Alternatively, we could have defined the third clause in the above definition as:

$$\mathsf{d}(\phi \to \psi) = \max(\mathsf{d}(\phi) + 1, \mathsf{d}(\psi)),$$

the major difference being that in this latter definition the formulae $\phi \to \psi \to \chi$ and $(\phi \wedge \psi) \to \chi$ would have equal degree. We do not pursue this line of reasoning any further.

4.62 Definition (Implication Degree)
Let $\chi \in \mathcal{L}(X)$ be a formula. We inductively define the *implication degree* of χ, denoted $\mathsf{d}(\chi)$, by:

$$\begin{aligned} \mathsf{d}(\chi) &:= 0 && \text{if } \chi = \top, \bot \text{ or } \chi \in X, \\ \mathsf{d}(\phi \oplus \psi) &:= \max(\mathsf{d}(\phi), \mathsf{d}(\psi)) && \text{for } \oplus = \wedge, \vee \text{ and } \phi, \psi \in \mathcal{L}(X), \\ \mathsf{d}(\phi \to \psi) &:= \max(\mathsf{d}(\phi), \mathsf{d}(\psi)) + 1 && \text{for } \phi, \psi \in \mathcal{L}(X). \end{aligned}$$

We define the set of *formulae with implication degree at most* n in a given set of variables X, denoted $\mathcal{L}_n(X)$, by $\mathcal{L}_n(X) := \{\phi \in \mathcal{L}(X) \mid \mathsf{d}(\phi) \le n\}$. For convenience, we write $\mathcal{L}_\omega(X)$ to denote $\mathcal{L}(X)$.

The following Theorem 4.63 follows by straightforward induction along n, as already pointed out by Visser (1996, Theorem 4.5). In the case where $n = 0$, this amounts to the observation that the variety of distributive lattices is locally finite.

4.63 Theorem
Let Λ be an intermediate logic, let $n \ge 1$ be a natural number, and let X be a finite set of variables There exists a finite set of formulae $\Sigma \subseteq \mathcal{L}_n(X)$ such that:

(i) for all $\phi \in \mathcal{L}_n(X)$ there exists a $\psi \in \Sigma$ such that $\vdash_\Lambda \phi \equiv \psi$;

(ii) for all $\phi, \psi \in \Sigma$ we have $\vdash_\Lambda \phi \equiv \psi$ iff $\phi = \psi$.

Recall Definition 3.40, wherein we defined the so-called set of *vacuous implications*. This definition can be readily relativized to a fixed implication degree, by defining:

$$\mathsf{I}_n(\Gamma) := \{\phi \to \psi \mid \phi, \psi \in \mathcal{L}_n(X), \Gamma \vdash_\Lambda \phi \to \psi \text{ and } \Gamma \nvdash_\Lambda \phi\},$$

where Γ is a set of formulae. This is a deviation from Definition 3.40, which was defined with respect to a model.

Lemma 4.64 below is similar to Lemma 3.42, although the former is formulated in the context of sets of formulae, whereas the latter is specified in terms of models. Nonetheless, the reasoning below nicely shows that the role of being "maximally comparable below" is taken up by being "Σ-maximally saturated" in this more syntactic context.

4.64 Lemma
Let Λ be an intermediate logic, let $n, m \in \mathbb{N}$, and let $\Theta_1, \dots, \Theta_m \subseteq \mathcal{L}(X)$ be $\mathcal{L}_n(X)$-saturated sets. Suppose that Γ is $\mathcal{L}_n(X)$-maximally saturated in $\Theta := \bigcup_{i=1}^m \Theta_i$, and that $\mathsf{I}_n(\Theta) \subseteq \Gamma$. The following are equivalent for all $\phi, \psi \in \mathcal{L}_n(X)$:

(i) $\Gamma \vdash_\Lambda \phi \to \psi$;

(ii) $\Theta \vdash_\Lambda \phi \to \psi$, and $\Gamma \vdash_\Lambda \phi$ implies $\Gamma \vdash_\Lambda \psi$.

Proof. The implication from (i) to (ii) is immediate. To prove the converse, suppose (ii) holds yet (i) does not. By Corollary 4.47, we know of a $\mathcal{L}_n(X)$-saturated set $\Pi \subseteq \mathcal{L}(X)$ such that $\Gamma \subseteq \Pi$, $\phi \in \Pi$, $\psi \notin \Pi$ and $\Pi - \Gamma \subseteq \mathcal{L}_n(X)$. First, we observe that $\Gamma \neq \Pi$. Indeed, otherwise $\Gamma \vdash_\Lambda \phi$ yet $\Gamma \nvdash_\Lambda \psi$, a contradiction. The maximality ensures that $\Pi \nsubseteq \Theta_i$ for some $i = 1, \dots, m$. We thus obtain a $\chi \in \Pi \cap \mathcal{L}_n(X)$ such that $\chi \in \Pi$ yet $\chi \notin \Theta_i$. Consider $\chi \wedge \phi \to \psi$, and observe that $\chi \wedge \phi, \psi \in \mathcal{L}_n(X)$. Clearly, we have $\Theta_i \vdash_\Lambda \chi \wedge \phi \to \psi$ and $\Theta_i \nvdash_\Lambda \chi \wedge \phi$. We thus know:

$$\chi \wedge \phi \to \psi \in \mathsf{I}_n(\Theta) \subseteq \Gamma \subseteq \Pi.$$

As a consequence, $\Pi \vdash_\Lambda \psi$, a contradiction. This proves the desired. □

See that the equivalence proven in Lemma 4.64 is quantified over formulae of implication degree $n+1$ of a particular form. We capture this form in Definition 4.65. Take care to note that if $n = \omega$, then nothing is added between $\mathcal{L}_n^+(X)$ and $\mathcal{L}_n(X)$.

4.65 Definition
Let $n \in \mathbb{N}$ be a natural number. We define:

$$\mathcal{L}_n^+(X) := \mathcal{L}_n(X) \cup \{\phi \to \psi \mid \phi, \psi \in \mathcal{L}_n(X)\} \subseteq \mathcal{L}_{n+1}(X),$$
$$\mathcal{L}_n^*(X) := \{\textstyle\bigwedge \Gamma \to \bigvee \Delta \mid \Gamma \subseteq \mathcal{L}_n^+(X) \text{ and } \Delta \subseteq \mathcal{L}_n(X)\}.$$

The formulae in $\mathcal{L}_n^*(X)$ are of a rather restricted form. This type of formulae is of interest, as will be apparent in Section 4.3.1. Moreover, formulae of this form appear naturally in Theorem 4.67.

Let us spend a few words indicating the implications of Lemma 4.64. Consider a finite subset $W \subseteq \mathsf{U}_{\mathsf{IPC}}(X)$ of the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$, take some $n \in \mathbb{N}$, and write $\Theta_w := \mathrm{Th}_{\mathsf{u}}(w) \cap \mathcal{L}_{n+1}(X)$. In this setting, it readily follows that:

$$\Theta = \bigcup_{w \in W} \Theta_w = \mathrm{Th}_{\mathsf{u}}(W) \cap \mathcal{L}_{n+1}(X).$$

Fix a set Γ as in the setting of Lemma 4.64, define the set of variables $Y = \Gamma \cap X$, write $v := \mathsf{u} \upharpoonright (\uparrow W)$ for the submodel generated by $\uparrow W$. The point of Lemma 4.64 is to ensure that:

$$v/Y \Vdash \chi \text{ iff } \chi \in \Gamma \text{ for all } \chi \in \mathcal{L}_n^+(X).$$

Through the usual argument, there exists a point $p \in \mathsf{U}_{\mathsf{IPC}}(X)$ satisfying $W \;\kappa\; p$ that has the same theory as v/Y. We now know that Γ must be a subset of said theory, giving us information on the formulae satisfied on covers of W. The precondition given in Lemma 4.64 seems strange though, in Section 4.3.1 we show that this saturation-property is related to being *closed under* a suitable set of rules.

In the above argument, we worked with points in the universal model, and hence we worked with finite models. This finiteness is not essential, hence neither is the universal model. We can even abstract away from IPC, and move to arbitrary intermediate logics. This is the approach we take in Corollary 4.66 below. We revisit the argument above in Theorem 4.67 in a more precise form.

4.66 Corollary
Let Λ be an intermediate logic, let $n, m \in \mathbb{N}$, and let $v_i : P_i \to \mathcal{P}(X)$ be a rooted model of Λ per $i = 1, \ldots, m$. Define the model $v := \coprod_{i=1}^m v_i$ and the set of variables $Y := \Gamma \cap X$. Suppose that $\Gamma \subseteq \mathcal{L}(X)$ extends $\mathsf{I}_n(\mathrm{Th}(v))$ and is $\mathcal{L}_n(X)$-maximally saturated in $\mathrm{Th}(v)$. The following holds for each $\chi \in \mathcal{L}_n^+(X)$:

$$v/Y \Vdash \chi \text{ iff } \Gamma \vdash_\Lambda \chi.$$

Proof. We prove the desired by structural induction along χ. The base case is immediate by definition. In the disjunctive case, we know that $\chi = \phi_1 \vee \phi_2$ for some $\phi_1, \phi_2 \in \mathcal{L}_n^+(X)$.[12] See that $v/Y \Vdash \chi$ iff $v/Y \Vdash \phi_i$ for $i = 1, 2$. Because Γ was assumed to be $\mathcal{L}_n(X)$-saturated, we know $\phi_1 \vee \phi_2$ to holds precisely if $\phi_i \in \Gamma$ holds for some $i = 1, 2$. Induction thus resolves this case.

[12]It is crucial that we consider $\chi \in \mathcal{L}_n^+(X)$, and not merely $\chi \in \mathcal{L}_{n+1}(X)$. The latter would not provide sufficient information to finish the argument here.

Finally, let us treat the implicative case where $\chi = \phi \to \psi$ for some $\phi, \psi \in \mathcal{L}_n(X)$. By Lemma 2.34, we know $v/Y \Vdash \phi \to \psi$ to hold precisely if $v \Vdash \phi \to \psi$ and $v/Y \Vdash \phi$ implies $v/Y \Vdash \psi$. The former conjunct is equivalent to $\mathrm{Th}(v) \vdash_\Lambda \phi \to \psi$, whereas the latter conjunct is equivalent to $\phi \in \Gamma$ implies $\psi \in \Gamma$ by induction. We apply Lemma 4.64, finishing the argument. $\square$

In Theorem 4.67 we lay the groundwork towards providing a criterion under which a formula from $\mathcal{L}^*_n(X) \subseteq \mathcal{L}_{n+2}(X)$ is known to be BB_n*-extendible*. Indeed, all that remains is appropriately formulating (i) of Theorem 4.67 in terms of being *closed under rules*, whence Theorem 4.73 follows quite readily.

The following argument relies heavily on the specific sets of formulae under consideration, and less so on the intermediate logic BB_m. Other base logics are conceivable, for instance KC, but great care has to be taken to ensure that an extension of the submodel of the universal model generated by $\uparrow W$ is again a model of the logic. We do not explore such generalisations here.

At first reading, it is convenient to let both n and m be ω. In this setting, Λ becomes IPC and Σ may as well be replaced with $\mathcal{L}(X)$, although this is not literally what the theorem states. Under this reading, (i) states that the set $\phi \cup \mathsf{I}(v)$ is saturated in $\mathrm{Th}_\mathsf{u}(W)$, and (ii) posits the existence of a point $p \in \mathsf{U}_\mathsf{IPC}(X)$ such that $W \ \kappa \ p$ and $p \Vdash \phi$. Said succinctly, the theorem provides a convenient tool towards describing IPC-extendible formulae.

4.67 Theorem
Let Λ be BB_m for some $m = 1, \ldots, \omega$, let $n = 1, \ldots, \omega$, define $\Sigma := \mathcal{L}^+_{n+1}(X)$, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. The following are equivalent for any finite subset $W \subseteq \mathsf{U}_\Lambda(X)$ satisfying $|W| \leq m$ and any formula $\phi \in \mathcal{L}^*_n(X)$:

(i) the set $\phi \cup \mathsf{I}_n(\mathrm{Th}_\mathsf{u}(W))$ is $\mathcal{L}_n(X)$-saturated in $\mathrm{Th}_\mathsf{u}(W)$;

(ii) there exists a point $p \in \mathsf{U}_\Lambda(X)$ such that $W \ \kappa \ p$ and $p \Vdash \phi$.

Proof. Suppose (i) holds. By Lemma 4.45, we know of a set of formulae Γ that is $\mathcal{L}_n(X)$-maximally saturated in$\mathrm{Th}_\mathsf{u}(W)$, satisfying both:

$$\{\phi\} \cup \mathsf{I}_n(\mathrm{Th}_\mathsf{u}(W)) \subseteq \Gamma,$$

and $\Gamma \subseteq \Sigma \cup \{\phi\}$. We now apply Corollary 4.66 to the models $v_w := \mathsf{u} \restriction (\uparrow w)$ per $w \in W$. This thus yields a $Y \subseteq X$ such that:

$$v/Y \Vdash \chi \text{ iff } \Gamma \vdash_\Lambda \chi \text{ for all } \chi \in \Sigma. \tag{4.16}$$

Indeed, this follows readily from Lemmas 2.34 and 4.64. Let us now prove that $v/Y \Vdash \phi$. As $\phi \in \mathcal{L}^*_n(X)$, we know that:

$$\phi = \bigwedge \Pi \to \bigvee \Delta,$$

for some finite $\Pi, \Delta \subseteq \mathcal{L}^+_n(X)$. Suppose $v/Y \nVdash \phi$. It follows that $v/Y \Vdash \psi$ for all $\psi \in \Pi$ yet $v/Y \nVdash \chi$ for all $\chi \in \Delta$. By (4.16), the former ensures $\Gamma \vdash_\Lambda \psi$ for all $\psi \in \Pi$. As $\phi \in \Gamma$, it thus follows that $\Gamma \vdash_\Lambda \bigvee \Delta$. We know Γ to be $\mathcal{L}_n(X)$-saturated, hence $\Gamma \vdash_\Lambda \chi$ follows for some $\chi \in \Delta$. This proves, again through (4.16), that $v/Y \Vdash \chi$, *quod non.*

Now, observe that the underlying frame of v is of *branching degree at most* m, as follows through Corollary 3.13 and Lemma 2.88. Because $|W| \leq m$, it is easy to see that v/Y is of branching degree at most m as well. Hence $v/Y \Vdash \Lambda$, proving the existence of a unique map of Kripke models $f : v/Y \to \mathsf{u}$ via Theorem 3.20. The uniqueness proves $f(W) = W$, and as $W \; \kappa \; \rho$, Lemma 2.35 shows that $W \; \kappa \; f(\rho)$.[13] When we set $p := f(\rho)$, we thus clearly see (ii) to hold.

Conversely, suppose (ii) holds, and let $p \in \mathsf{U}_\Lambda(X)$ be such that $W \; \kappa \; p$ and $p \Vdash \phi$. It is easy to see that:

$$\phi \cup \mathsf{I}_n(\mathrm{Th}_\mathsf{u}(W)) \subseteq \mathrm{Th}_\mathsf{u}(p),$$

whence (i) is immediate. □

4.3.1. De Jongh rules

We know quite a bit about the structure of BB_n*-extendible models.* Indeed, through Theorem 4.7 we know that an upset in the universal model is BB_n-extendible precisely if it is *non-empty*, *downwards directed*, and *weakly* n*-ary covered*. We can summarise these three properties into the following definition.

4.68 Definition (n-Extendible)
Let P be a Kripke frame, and let $n = 0, 1, \ldots, \omega$. We say that P is n*-extendible* whenever for each $W \subseteq P$ satisfying $|W| \leq n$ there exists a $p \in P$ such that $W \; \kappa \; p$.

Through Definition 4.68, it follows that n-extendible implies m-extendible whenever $n \geq m$. Observe that any 0-extendible Kripke frame is non-empty. Moreover, any Kripke frame that is 2-extendible must be downwards directed, as readily follows through a straightforward inductive argument. Recall the rule $\overline{\mathsf{D}}_n$, as defined on Page 114, and recall that it encompasses the rules Con, DP, and D^-_n. These observations, together with Theorem 4.7, combine to prove Lemma 4.69 below.

[13] Recall that ρ is the root of v/Y, as described in Definition 3.16 on Page 81.

4.69 Lemma
Let Λ be BB_n for some $n = 2, \dots, \omega$, and let $\phi \in \mathcal{L}(X)$ be a formula. Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. The following are equivalent:

(i) the formula ϕ is Λ-*extendible*;

(ii) the formula ϕ is closed under $\overline{\mathsf{D}}_n$;

(iii) the Kripke frame $[\![\phi]\!]_\mathsf{u}$ is n-*extendible*.

Although the rule schemes D_n^-, D_n, and $\overline{\mathsf{D}}_n$ are fairly succinct and manageable, they do come with a natural disadvantage. Consider $\overline{\mathsf{D}}_n$, formulated as:

$$\left(\bigvee_{i=1}^{n} x_i \to z\right) \to \bigvee_{j=1}^{n} x_j \Bigg/ \left\{\left(\bigvee_{i=1}^{n} x_i \to z\right) \to x_j\right\}_{j=1}^{n}.$$

The antecedent of the rule is an implication, and this implication's antecedent, in turn, is but a single implication. Naturally, this latter implication is equivalent to a finite conjunction of implications all sharing the same conclusion. In order to make the connection to Theorem 4.67, it would be convenient if this rule scheme allowed for multiple, different conclusions.

A possible formulation can be sought in V_n, commonly formulated as:

$$\bigwedge_{i=1}^{n} \big((x_i \to z_i) \to x_{n+1} \vee x_{n+2}\big) \Bigg/ \bigvee_{j=1}^{n+2} \left(\bigwedge_{i=1}^{n} (x_i \to z_i) \to x_j\right).$$

As can be seen above, one stratifies this rule scheme over the number of implications in the rule's antecedent. This, albeit indirectly, influences the amount of disjuncts the conclusion has. In Definition 4.70 below, we consider a rule scheme where we directly influence the number of conclusions, all the while allowing for arbitrarily many implications in the antecedent.

4.70 Definition (De Jongh Rules)
Let $k \in \mathbb{N}$ and $\mathcal{I} \subseteq \mathcal{P}(\{1, \dots, k\})$ be a set satisfying $\bigcup \mathcal{I} = \{1, \dots, k\}$. The *de Jongh rule indexed by* $\mathcal{I}$ *on* k is given by:

$$\bigwedge_{i=1}^{k} (z_i \to x_i) \to \bigvee_{i=1}^{k} z_i \Bigg/ \left\{ \bigwedge_{i=1}^{k} (z_i \to x_i) \to \bigvee_{i \in I} z_i \;\middle|\; I \in \mathcal{I} \right\}. \tag{$\mathsf{J}_\mathcal{I}$}$$

We write J_n for the set of all de Jongh rules indexed by $\mathcal{I}$ on k satisfying $|\mathcal{I}| \leq n$.

Note that when $k = 0$, the above rule trivialises to the rule Con. The rule schemes J_n and $\overline{\mathsf{D}}_n$ are related. In fact, when adding either of these rule schemes to the minimal multi-conclusion consequence relation corresponding to an intermediate logic one obtains the very same multi-conclusion relation, as the following Lemma 4.71 shows.

4.71 Lemma
Let $n \in \mathbb{N}$, and let Λ be a intermediate logic. We have $\vdash_\Lambda^{\mathsf{J}_n} = \vdash_\Lambda^{\overline{\mathsf{D}}_n}$.

Proof. We first prove that the latter consequence relation is included in the former. This follows rather straightforwardly. Indeed, first observe that:

$$\vdash_\Lambda \left(\bigvee_{i=1}^{n} z_i\right) \to y \equiv \bigwedge_{i=1}^{n} (z_i \to y)\,. \tag{4.17}$$

Now, choose $k = n$ and set $\mathcal{I} = \big\{\{1\}, \dots, \{n\}\big\}$. It is clear that $|\mathcal{I}| \leq n$, and $\mathsf{J}_\mathcal{I}$ can be instantiated to:

$$\bigwedge_{i=1}^{k} (z_i \to x) \to \bigvee_{i=1}^{k} z_i \Big/ \left\{ \bigwedge_{i=1}^{k} (z_i \to x) \to z_j \;\middle|\; j = 1, \dots, n \right\}. \tag{$\mathsf{J}_\mathcal{I}$}$$

Through our earlier observation (4.17), this proves that $\vdash_\Lambda^{\overline{\mathsf{D}}_n} \subseteq \vdash_\Lambda^{\mathsf{J}_n}$.

Conversely, let $k \in \mathbb{N}$ be arbitrary and consider $\mathcal{I} \subseteq \mathcal{P}(\{1, \dots, k\})$ satisfying $|\mathcal{I}| \leq n$. We define formulae:

$$\chi_I := \bigvee_{i \in I} z_i \text{ per } I \in \mathcal{I} \text{ and } \phi := \bigwedge_{i=1}^{n} (z_i \to y_i).$$

We now compute as below, proving the desired through transitivity.

$$\begin{aligned}
\bigwedge_{i=1}^{k} (z_i \to x_i) \to \bigvee_{i=1}^{k} z_i &\vdash_\Lambda \left(\bigvee_{I \in \mathcal{I}} \chi_I \to \phi \right) \to \bigvee_{J \in \mathcal{I}} \chi_J \\
&\vdash_\Lambda^{\overline{\mathsf{D}}_n} \left\{ \left(\bigvee_{I \in \mathcal{I}} \chi_I \to \phi \right) \to \chi_J \;\middle|\; J \in \mathcal{I} \right\} \\
&\vdash_\Lambda^{\mathsf{min}} \left\{ \bigwedge_{i=1}^{k} (z_i \to x_i) \to \bigvee_{i \in J} z_i \;\middle|\; J \in \mathcal{I} \right\}
\end{aligned}$$ □

Fix natural numbers n and m, and consider an instance of J_m where the antecedent is an element of $\mathcal{L}_n^*(X) \subseteq \mathcal{L}_{n+2}(X)$. It automatically follows that the conclusion must be a subset of $\mathcal{L}_n^*(X)$. Indeed, the rule is of the form:

$$\bigwedge_{i=1}^{k} (\chi_i \to \phi_i) \to \bigvee_{i=1}^{k} \chi_i \Big/ \left\{ \bigwedge_{i=1}^{k} (\chi_i \to \phi_i) \to \bigvee_{i \in I} \chi_i \;\middle|\; I \in \mathcal{I} \right\}, \tag{4.18}$$

where $\chi_1, \ldots, \chi_k, \phi_1, \ldots, \phi_k \in \mathcal{L}_n(X)$. Such an instance is what we call a $\mathcal{L}_n^*(X)$-instance, as formally specified by Definition 4.72. There are but finitely many such instances up to IPC-provable equivalence, as follows through Theorem 4.63.

4.72 Definition (Instance)
Let $\Sigma \subseteq \mathcal{L}(Y)$ be an adequate set, and let $\mathcal{R}$ be a set of rules in $\mathcal{L}(X)$. We say that the rule Γ/Δ is a Σ*-instance of* $\mathcal{R}$ whenever $\Gamma, \Delta \subseteq \Sigma$ and $\Gamma/\Delta \in \mathcal{R}^*$.

Recall Definition 4.12, wherein we defined what it means for a set of formulae to be closed under a set of rules (tacitly with respect to an intermediate logic). We are now ready to state Theorem 4.73, which gives an effective criterion for being BB_m-extendible.

4.73 Theorem
Let Λ be BB_m for $m = 1, \ldots, \omega$, and let $\phi \in \mathcal{L}_n^*(X)$ be a formula. The following are equivalent:

(i) the formula ϕ is n*-extendible*;

(ii) the formula ϕ is *closed under* all $\mathcal{L}_n^*(X)$-instances of J_m.

Proof. Suppose (i) holds. By Lemma 4.71, it certainly suffices to prove that ϕ is closed under all instances of $\overline{\mathsf{D}}_n$. This holds by Lemma 4.69. We have thus proven (ii).

Conversely, suppose (ii) holds. Consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$, and let $W \subseteq [\![\phi]\!]_\mathsf{u}$ be such that $|W| \leq m$. By Theorem 4.67, it suffices to prove that:

$$\phi \cup \mathsf{I}_n(\mathrm{Th}_\mathsf{u}(W)) \preccurlyeq_{\mathcal{L}_n(X)} \mathrm{Th}_\mathsf{u}(W)\,.$$

To this end, let $\chi_1, \ldots, \chi_k \subseteq \mathcal{L}_n(X)$ be given and assume:

$$\phi \cup \mathsf{I}_n(\mathrm{Th}_\mathsf{u}(W)) \vdash_\Lambda \bigvee_{i=1}^{k} \chi_k.$$

If there is an $i = 1, \ldots, k$ such that $\mathsf{u}, W \Vdash \chi_i$ then we are done, so assume otherwise. There exists a finite $\Gamma \subseteq \mathsf{I}_n(\mathrm{Th}_\mathsf{u}(W))$ such that $\phi, \Gamma \vdash_\Lambda \bigvee_{i=1}^k \chi_k$. Without loss of generality we may assume there are $\phi_1, \ldots, \phi_k \in \mathcal{L}_n(X)$ such that:

$$\Gamma = \{\chi_1 \to \phi_1, \ldots, \chi_k \to \phi_k\}.$$

We now define $\mathcal{I} = \{I_w \mid w \in W\}$, where:

$$I_w := \{\chi_i \mid\ i = 1, \ldots, n \text{ and } \mathsf{u}, w \nVdash \chi_i\ \}\,.$$

From the assumption that $\mathsf{u}, W \nVdash \chi_i$ for all $i = 1, \ldots, k$ one can easily deduce the equality $\bigcup \mathcal{I} = \{1, \ldots, k\}$ to hold. Moreover, by definition we know that $|\mathcal{I}| \leq n$. Observe that:

$$\phi \vdash_\Lambda \bigwedge_{i=1}^{n} (\chi_i \to \phi_i) \to \bigvee_{j=1}^{k} \chi_i,$$

a clear $\mathcal{L}_n^*(X)$-instance of J_m. There thus must be some $w \in W$ such that $I = I_w$ and:

$$\phi \vdash_\Lambda \bigwedge_{i=1}^{n} (\chi_i \to \phi_i) \to \bigvee_{j \in I} \chi_j.$$

Now, as the antecedent of the right-hand implication holds on u, w and the conclusion certainly does not, this yield a contradiction. This proves (i). ☐

4.4. Decidability in IPC

In this section, we prove the admissible rules of IPC to be decidable. Much of the machinery we employ towards this goal has been set up in the sections above; but one ingredient is yet missing. Recall Theorem 4.32, which indicated a constraint that is sure to entail the decidability of the admissible rules of an intermediate logic. This theorem stated that there ought to be a set of Kripke models per adequate set of formulae that has to satisfy three conditions. First, it has to be sound and complete with respect to all admissible rules contained within said adequate set. Second, the set of models within this set should be finite and effectively enumerable. Third, validity of a rule should be decidable on each of the models in this set.

Fix an adequate set $\Sigma \subseteq \mathcal{L}(X)$. A first attempt would be to consider all *exact models* on X. We know, as per Theorem 3.38, that the set of all exact models satisfies the first condition. However, there are infinitely many IPC-exact models on X whenever $|X| \geq 2$.[14] Consequently, Theorem 4.32 is not applicable to this situation.

Definition 4.74 below generalises the exact models of Definition 3.27 to *adequately exact models* in the case of $\Lambda = \mathsf{IPC}$. Note that this in itself is not going to reduce the number of models under consideration. However, one can restrict attention to those adequately exact models that are also *adequately refined* in the sense of Definition 4.51. Whenever the adequate set at hand is finite, it follows through Theorem 4.53 that there are but finitely many such models.

[14] This readily follows through Bezhanishvili and de Jongh (2012, Section 5.2), together with the observation that IPC-exact formulae are IPC-extendible formulae, as explicated in Theorem 4.30.

4.74 Definition (Adequately Exact Model)
Let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set of formulae, and let $v : P \to \mathcal{P}(X)$ be a model. We say that v is Σ-*adequately exact* if there exists a surjective, Σ-adequate map $f : \mathsf{u} \to v$, where $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(Y)$ is the universal model on some finite set of variables Y.

Were we to replace $\mathsf{U}_{\mathsf{IPC}}(X)$ with $\mathsf{U}_{\Lambda}(X)$ in the above definition, it need not be the case that adequately exact models are models of Λ. This can cause great confusion, as consequences that hold in Λ thus need not hold on adequately exact models. As our goal is restricted to IPC, we stay within the confines of this more comfortable logic, shying away from this potential generalisation.

Theorem 3.38 showed us that exact models can serve as sound and complete semantics for arbitrary admissible rules. When restricting attention to admissible rules drawn from a particular adequate set, it suffices to consider adequately exact models instead. The major upside of this is that there is an obvious bound on the sensible size of an adequately exact model. Indeed, as we are only interested in the validity of formulae of a given adequate set, the size of all models one needs to be concerned with can be bound in terms of the size of this adequate set, as in Theorem 4.53. As such, the first and third condition of Theorem 4.32 are satisfied.

4.75 Theorem (Soundness and Completeness for Adequately Exact Models)
Let Σ be an adequate set. The following are equivalent for any $\Gamma, \Delta \subseteq \Sigma$:

(i) the rule Γ/Δ is *admissible*;

(ii) the rule Γ/Δ is valid on every model $v : P \to \mathcal{P}(X)$ that is Σ-*adequately exact*, satisfying $P \subseteq \mathsf{can}^{\Sigma}_{\mathsf{IPC}}(X)$.

Proof. Immediate by Theorem 4.61. □

Even though we now know that adequately exact models suffice to determine the admissible rules of IPC, the problem of decidability is not yet solved. The definition of adequate exactness is in no way *intrinsic*, and it is not at all apparent that one can decide whether a model is adequately exact. In the following, we provide an *intrinsic* description of those models that are Σ-adequately exact.

Roughly speaking, the notion of adequate extendibility we introduce below in Definition 4.76 stands to adequate exactness as extendibility stands to exactness. We show that a model is adequately exact precisely if it is adequately extendible. As a consequence, admissibility of IPC is decidable. Furthermore, the proofs given in this section can be used to re-prove some popular results in the literature, among which

$$\begin{tikzcd}
\mathsf{u} \arrow[rr, dashed, "f"] & & v \\
& u \arrow[ul, "\subseteq"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Figure 4.7.: A diagrammatic description of Theorem 4.77.

the characterisation of finite projective Heyting algebras, or dually, finite injective Kripke frames. We get to this in Theorem 4.80.

The following Definition 4.76 is a generalisation of Definition 4.68 in the case where $n = \omega$, which in turn is but an adaptation of Definition 4.2. When instantiating $\mathcal{D}$ to $\mathcal{D}_v^\Sigma$ for some adequate set $\Sigma \subseteq \mathcal{L}(X)$, one can see reflections of this notion in Rybakov (1990b, Theorem 15.3) and Odintsov and Rybakov (2013, Proposition 4.1.b) through the lens of Lemma 4.55.

4.76 Definition (Adequately Extendible)
Let P be a Kripke frame, and let $\mathcal{D}$ be a set of subets of P. We say that P is *adequately extendible* for $\mathcal{D}$ whenever there is a $p \in P$ to each finite $W \subseteq P$ such that $W\ \kappa_\Sigma\ p$.

A function $\mu : \mathcal{P}(P) \to P$ is said to be an *adequate choice of covers for* $\mathcal{D}$, whenever $W\ \kappa_\mathcal{D}\ \mu(W)$ for all $W \subseteq P$. Clearly, a finite poset is adequately extendible precisely if it has an adequate choice of covers. This might make the latter notion appear redundant, yet it is quite convenient in practise. We use this notion in the proof of Theorem 4.77 below, where it gives us a handle on the choices involved.

4.77 Theorem (Extension Theorem)
Let $v : P \to \mathcal{P}(X)$ be an order-defined finite model, let $\mathcal{D} \subseteq \mathcal{P}(P)$ be a set of subsets, and let $\mu : \mathcal{P}(P) \to P$ be an adequate choice of covers for $\mathcal{D}$. Consider the universal model $\mathsf{u} : \mathsf{U}(Y) \to \mathcal{P}(Y)$ on a finite set of variables Y, and the submodel $u : U \to \mathcal{P}(Y)$ generated by a definable upset $U \subseteq \mathsf{U}(Y)$. If there is a *$\mathcal{D}$-adequate* map $g : u \to v$, then there exists a $\mathcal{D}$-adequate map $f : \mathsf{u} \to v$ satisfying $f \restriction U = g$.

The above theorem can conveniently be summarised by the Fig. 4.7. We do not immediately prove this theorem, but first state a few of its corollaries. First and foremost, one can prove adequately exact models to be the same as adequately extendible models. We show this in Theorem 4.78 below. The decidability of IPC's admissible rules thus immediately follows.

4.78 Theorem
Let $v : P \to \mathcal{P}(X)$ be a finite model, let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set, and write $\mathcal{D} := \mathcal{D}_v^\Sigma$. The following are equivalent:

(i) the model v is Σ-*adequately exact*;

(ii) the frame P is *adequately extendible* for $\mathcal{D}$.

Proof. Suppose (i) holds. We know there to be a finite set of variables Y, together with a surjective $\mathcal{D}$-adequate map $f : \mathsf{u} \to v$, where $\mathsf{u} : \mathsf{U}(Y) \to \mathcal{P}(Y)$ is the universal model on Y. Let $W \subseteq P$ be a finite subset, we need to find some $p \in P$ such that $W \ \kappa_{\mathcal{D}} \ p$.

Consider the set $f^{-1}(W)$. Fix a finite *anti-chain* $S \subseteq f^{-1}(W)$ satisfying the equation $\uparrow S = \uparrow f^{-1}(W)$. We know there to be a point $q \in \mathsf{U}(Y)$ such that $S \ \kappa \ q$. It is easy to see that $f^{-1}(W) \ \kappa \ q$ holds as well. By Lemma 4.56, we know that $f\left(f^{-1}(W)\right) = W \ \kappa_{\mathcal{D}} \ f(q) =: p$, proving (ii).

Conversely, suppose (ii) holds. We define $Y := X + P$, and construct a model u^* as:

$$u^* : P \to \mathcal{P}(Y), \quad p \mapsto v(p) \cup \{q \in P \mid p \leq q\}.$$

Note that the model u^* is order-defined. Indeed, one can readily verify that the following equivalences hold for all $p, q \in P$:

$$\begin{aligned} &p \leq q \text{ iff } q \Vdash p; \\ &q \nleq p \text{ if } q \Vdash k \text{ for some } k \nleq p. \end{aligned}$$

Hence, through Lemma 3.25, there is an upset $U \subseteq \mathsf{U}(Y)$ such that the model $u := \mathsf{u} \restriction U : U \to \mathcal{P}(Y)$ is isomorphic to u^*, where $\mathsf{u} : \mathsf{U}(Y) \to \mathcal{P}(X)$. Write $g : u \to u^*$ and $g^{-1} : u^* \to u$ for the maps of Kripke models we thus know to exist.

By Theorem 4.77, there exists some $\mathcal{D}$-adequate map $f : \mathsf{u} \to u^*$ such that $f \restriction U = g$. The latter condition guarantees that the Σ-adequate map f is surjective. Finally, observe that the map $\mathrm{id}_P : P \to P$ is a definable map $h : u^* \to v$, simply because every upset in u^* is definable. We can thus construct a surjective $\mathcal{D}$-adequate map $h \circ f : \mathsf{u} \to v$ through Lemma 4.36. This shows that v is adequately exact for $\mathcal{D}$, proving (i). □

With a proof of Theorem 4.78 in hand, it is not hard to show that the admissible rules of IPC are decidable. The proof of Theorem 4.79 follows through a straightforward combination of the developed machinery, which we spell out in full.

4.79 Theorem
The admissible rules of IPC are decidable.

Proof. We proceed via Theorem 4.32. Let $\Sigma \subseteq \mathcal{L}(X)$ be finite, and let $\mathcal{K}_\Sigma$ be the set defined by:

$$\mathcal{K}_\Sigma := \left\{ v : P \to \mathcal{P}(X) \;\middle|\; v \text{ is } \Sigma\text{-adequately exact and } P \subseteq \mathsf{can}^{\Sigma}_{\mathsf{IPC}}(X) \right\}.$$

By Theorem 4.53 we know $\mathsf{can}^{\Sigma}_{\mathsf{IPC}}(X)$ to be finite, hence $\mathcal{K}_\Sigma$ is a finite set of finite models. This class is sound and complete with respect to the rules in Σ due to Theorem 4.75. Finally, by Theorem 4.78 we know that a model $v : P \to \mathcal{P}(X)$ satisfying $P \subseteq \mathsf{can}^{\Sigma}_{\mathsf{IPC}}(X)$ is Σ-exact precisely if it is adequately extensible for $\mathcal{D}^{\Sigma}_{v}$. This latter property can be checked in finite time. We have thus shown that all three prerequisites of Theorem 4.32 are satisfied, proving the desired. □

The following is a combination of Citkin (1977a, Theorem 2) and Balbes and Horn (1970a, Theorem 4.10), seen through the duality between finite Heyting algebras and finite Kripke frames we discussed in Theorem 2.69. In fact, their theorems are slightly stronger, as they only need the case $n = 2$ in (i). We can obtain a similar result through the contrapositive of Lemma 3.87. See also Rybakov (1997, Section 5.2), and in particular his Theorem 5.2.15.

4.80 Theorem
Let U be a finite upset of the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$. Consider the model $v := \mathsf{u} \upharpoonright U$. The following are equivalent:

(i) the model v admits $\overline{\mathsf{D}}_n$ for all n;

(ii) the Kripke frame U is IPC*-extendible*;

(iii) the model v is U is IPC*-injective*.

Proof. First, observe that U is definable as $\mathsf{U}_{\mathsf{IPC}}(X)$ is order-refined and U is finite. The implication from (i) to (ii) follows through Theorem 4.4. Moreover, (iii) implies (i) by 4.11.

Suppose (ii) holds. Observe that $\mathrm{id}_v : v \to v$ is a $\mathcal{D}^{\mathcal{L}(X)}_{v}$-adequate map, as it is a map of Kripke models. By Theorem 4.77, there exists a $\mathcal{L}(X)$-adequate map $g : \mathsf{u} \to v$ satisfying $g \upharpoonright U = \mathrm{id}_v$. Through Lemma 4.39, we know g to be a definable map $g : \mathsf{u} \to v$. This proves (iii), finishing the argument. □

The Extension Theorem 4.77 is the core of Rybakov's method towards obtaining decidability of admissibility. Indeed, it has been proven many times over, in many different guises. Our formulation of the proof is mostly inspired by Odintsov and Rybakov (2013, Theorem 4.2), although the presentation is quite different.

The earliest occurrence of this technique in the literature came from Rybakov (1984a, Lemma 4), where a similar statement is proven for S4.[15] It is not straightforward to recognise the statement of Theorem 4.77 in Lemma 4 of Rybakov (1984a). Indeed, this lemma makes no mention of the notion of adequate extendibility, or anything similar to it. Instead, it concretely describes six properties, some of which (property 4 and 6 to be precise) are analogous to what we encompass by adequate extendibility. A more honest description would be to say that this lemma proves the implication from (i) to (ii) of Theorem 4.78, immediately followed by the observation that *adequately exact models* are *sound* with respect to admissibility.

Before we move on to the actual proof, let us first give a rough exposition of the technique involved. To this end, we consider the edge-case where g is the identity map $\mathrm{id}_U : U \to U$. To satisfy all prerequisites, the upset $U \subseteq \mathsf{U}(X)$ of the universal model $\mathsf{u} : \mathsf{U}(X) \to \mathcal{P}(X)$ ought to be finite. The goal of the theorem thus becomes constructing a definable map $f : \mathsf{u} \to U$ such that f satisfies the CDC for $\mathcal{D}$ and f obeys the equality $f \restriction U = \mathrm{id}_U$. This goal is attainable no matter the choice of $\mathcal{D}$; indeed, it is possible to make f a *definable map of Kripke frames.*

Let us first illustrate why there exists a map of Kripke frames, and defer thoughts of definability. We have thus reduced the Extension Theorem 4.77 to the following, which is but a reformulation of Ghilardi (2004, Proposition 4).

4.81 Theorem
Let $U \subseteq \mathsf{U}(X)$ be an upset that is both finite and *extendible.*[16] Now, there exists a map of Kripke frames $f : \mathsf{U}(X) \to U$ such that $f \restriction U = \mathrm{id}_U$.

Proof by Ghilardi (2004). Observe that, if the map f were to exist, it ought to preserve covers by Lemma 2.35. We can thus define the value of the map f on $q \in \mathsf{U}(X)$ inductively along the height of q. If $q \in U$ then $f(q)$ is defined to be q. Otherwise, we know that $f\left(\uparrow q\right)$ already has been defined. This subset of U must cover at least one

[15]This technique is employed to establish decidability of admissibility in many modal and intermediate logics. To illustrate the wide applicability of this technique, let us but mention Rybakov (1986a, Lemma 3), Rybakov (1986b, Lemma 4), Rybakov (1987a, Lemma 8), Rybakov (1987b, Lemma 8), Rybakov (1990a, Proposition 5), Rybakov (1990b, Theorem 20), Rybakov (1991a, Proposition 8), Rybakov (1991b, Theorem 4), Rybakov (1991c, Theorem 7), Rybakov (1992, Theorem 4), Rybakov (1994a, Lemma 7), all of which culminate to Rybakov (1997, Theorem 3.9.6).

[16]Note that to each finite X there are only finitely many such upsets, see Arevadze (2001, Chapter 5) for details.

node, define $f(q)$ to equal one of these. The resulting function is a map of Kripke frames. $\square$

The above construction is quite elegant in its simplicity, yet it does have two major drawbacks. First, at no finite stage in the process can the map f be seen as completed or fully determined. Second, it does not show that f is definable. One can fill both of these lacunae by using the method given in the proof below.

Observe that there exists a finite number N, such that every point in U generates an upset of size at most N. In general, the number $N := |U|$ certainly does the trick. We construct a definable upset $\mathsf{A}(W) \subseteq \mathsf{U}(X)$ per subset $W \subseteq U$. These upsets will be such that their union equals the entire universal model. Moreover, the value of $f : \mathsf{U}(X) \to U$ at $q \in \mathsf{U}(X)$ is determined by the smallest $S \subseteq U$ such that $q \in \mathsf{A}(W)$, and this value will be covered by W. This also makes it clear that W must generate the same upset as $f(\uparrow q)$.

In the proof below, we take an approach similar to the above. Do note that the prerequisites are quite different; Theorem 4.77 never actually assumes that P is extendible. Indeed, the theorem requires that P is *adequately extendible for* $\mathcal{D}$, given some fixed $\mathcal{D}$. This matters not, as in the reasoning above one could replace cover by adequate cover for $\mathcal{D}$, and the argument applies *mutatis mutandis*.

For the applicability of this theorem, it is crucial that pre-images of upsets under f are definable. As illustrated above, the crux of the matter is that such pre-images are unions of $\mathsf{A}(W)$ for suitably chosen $W \subseteq P$. These sets $\mathsf{A}(W)$ are constructed inductively along the size of W, together with partial definitions of the desired map f. The majority of the work lies in making sure that these sets $\mathsf{A}(W)$ behave coherently, and that their union equals the entirety of $\mathsf{U}(X)$.

In the proof below, we construct a sequence of maps such that five conditions are satisfied. Let us make a few remarks on these conditions. The conditions of Compatibility and Closed Domain are quite straightforward; the former is a natural ingredient of a piece-wise construction, and the latter is the piece-wise formulation of the closed domain condition f ought to satisfy.

The condition Domain Growth ensures that the sequence converges to a map which has all of the universal model in its domain. Image bound, on the other hand, ensures that the division of $\mathsf{U}(X)$ into the not necessarily disjoint upsets $\mathsf{A}(W)$ for $W \subseteq P$ contains enough information to specify the behaviour of f. Finally, the condition Identity ensures that the resulting map satisfies $f \upharpoonright U = g$.

Proof of Theorem 4.77. We construct a finite sequence of monotonic maps with increasing domains, in such a way that the final map in this series is the desired $\mathcal{D}$-*adequate map* $f : \mathsf{u} \to v$. For greater notational convenience, let us write:

$$\mathcal{P}_n(P) := \{W \subseteq P \mid n = |W|\}.$$

We also define the natural number $N := |P| + 1$. We claim that for each $n \leq N$ and each $W \subseteq \mathcal{P}_n(P)$ there exists a definable upset $\mathsf{A}(W) \subseteq \mathsf{U}(X)$ and a $\mathcal{D}$-adequate map $f_n : \mathrm{dom} f_n \to P$, satisfying the following conditions for all $n \leq N$.

Compatibility For all $m \leq n$ we have that: $\mathrm{dom} f_m \subseteq \mathrm{dom} f_n \subseteq \mathsf{U}(X)$ and $f_m = f_n \upharpoonright \mathrm{dom} f_m$.

Closed Domain For all $q \in \mathrm{dom} f_n$ and $D \in \mathcal{D}$ we have that $\uparrow f_n(q) \cap D \neq \varnothing$ implies $f_n(\uparrow q) \cap D \neq \varnothing$.

Domain Growth If $|f_n(\Uparrow q)| < n$ then $q \in \mathrm{dom} f_n$.

Image Bound For all $W \in \mathcal{P}_n(P)$, $q \in \mathsf{A}(W)$ implies $f_n(\Uparrow q) \subseteq W$.

Identity The equality $f_0 = g$ holds.

Suppose that the above can be constructed. Due to Domain Growth, it is clear that $\mathrm{dom} f_N = \mathsf{U}(X)$. Indeed, any $q \in \mathsf{U}(X)$ satisfies:

$$\left|f_N(\Uparrow q)\right| \leq |P| < |P| + 1 = N,$$

so $q \in \mathrm{dom} f_N$ follows. The map f_N satisfies all constraints imposed upon f, as follows immediately from Compatibility, Closed Domain and Identity. Consequently, we know that we need but prove that these constraints can truly be satisfied.

The definitions of $\mathsf{A}(W)$ and f_n, combined with their respective proofs of correctness, will proceed by induction along n. Let us first, uniformly for all cases, define

$$\mathrm{dom} f_n := U \cup \bigcup_{i<n} \bigcup_{S \in \mathcal{P}_i(P)} \mathsf{A}(S). \tag{4.19}$$

In the case that $n = 0$, we simply define $f_0 = g$. We also construct the set $\mathsf{A}(W)$ for $W \in \mathcal{P}_0(P)$. Know that $W = \varnothing$, so it suffices to define:

$$\mathsf{A}(\varnothing) := \{q \in \mathsf{U}(X) \mid q \text{ is maximal}\}.$$

This upset is finite, and as such, definable. Let us now verify that all conditions are satisfied. Indeed, Compatibility holds trivially, Closed Domain is valid by assumption, and Identity holds by construction. See that, if $q \Vdash \mathsf{A}(\varnothing)$, then q is maximal and so $\Uparrow q = \varnothing$, proving Image Bound. Moreover, $\left|f_0(\uparrow q)\right| < 0$ is never satisfied, hence Domain Growth holds vacuously. We have thus verified all conditions.

We now turn to the case where $n = m + 1$, and define the map f_{m+1}. First note that, through (4.19), we know

$$\mathrm{dom} f_{m+1} = \mathrm{dom} f_m \cup \bigcup_{S \in \mathcal{P}_m(P)} \mathsf{A}(S) \,. \tag{4.20}$$

Recall that, for any $S \in \mathcal{P}_m(P)$, the upset $\mathsf{A}(S)$ is known to be definable by induction. Using this, we define f_{m+1} by cases:

$$f_{m+1}(q) := f_m(q) \qquad \text{if } q \in \mathrm{dom} f_m, \tag{4.21}$$

$$f_{m+1}(q) := \mu\left(f_m\left(\Uparrow q\right)\right) \qquad \text{if } q \in \mathrm{dom} f_{m+1} \text{ and } q \notin \mathrm{dom} f_m. \tag{4.22}$$

Before we continue, we first prove the following.

$$\begin{array}{l}\text{if } q \in \mathrm{dom} f_{m+1} \text{ and } q \notin \mathrm{dom} f_m, \\ \text{then } f_m(\Uparrow q) \text{ is the unique } S \in \mathcal{P}_m(P) \text{ with } q \in \mathsf{A}(S).\end{array} \tag{4.23}$$

We know that there exists some $S \in \mathcal{P}_m(P)$ such that $q \in \mathsf{A}(S)$, due to (4.20). From Image Bound, we gather that $f_m(\Uparrow q) \subseteq S$. If this inclusion were strict, then:

$$\left|f_m(\Uparrow q)\right| < |S| = m,$$

so Domain Growth would yield $q \in \mathrm{dom} f_m$. Yet we explicitly assumed this not to be the case, a contradiction. This entails $W = f_m(\Uparrow q)$, proving (4.23).

Let us now prove that the map f_{m+1} is monotonic. Suppose $q, k \in \mathrm{dom} f_{m+1}$ are given such that $q \leq k$ holds. We distinguish three cases below, these are both exhaustive and mutually exclusive.

(i) Both q and k are in $\mathrm{dom} f_m$.

(ii) $q \notin \mathrm{dom} f_m$ and $k \in \mathrm{dom} f_m$.

(iii) Neither q nor k are in $\mathrm{dom} f_m$.

In the case (i), the desired is immediate, as f_m is monotonic by induction. In the case (ii), observe that $f_m(k) \in f_m(\Uparrow q)$. By definition (4.22) and the assumption on μ, we know that $f_{m+1}(q) = \mu\left(f_m(\Uparrow k)\right) \leq f_m(k)$, resolving this case.

Finally, we treat the case (iii). Because $q \in \mathrm{dom} f_{m+1} - \mathrm{dom} f_m$, we know that $q \in \mathsf{A}(f_m(\Uparrow q))$ by (4.23). As $q \leq k$, it also follows that $k \in \mathsf{A}(f_m(\Uparrow q))$. Another application of (4.23) now yields $f_m(\Uparrow q) = f_m(\Uparrow k)$, proving:

$$f_{m+1}(q) = \mu\left(f_m(\Uparrow q)\right) = \mu\left(f_m(\Uparrow k)\right) = f_{m+1}(k).$$

We have thus shown that $f_{m+1}(q) \leq f_{m+1}(k)$ in all cases (i), (ii), (iii), hence f_{m+1} is monotonic.

We now proceed to prove that f_{m+1} is definable. To this end, let $U \subseteq P$ be a definable upset in $v : P \to \mathcal{P}(X)$. We claim that $f^{-1}_{m+1}(U)$ can be expressed as:

$$f^{-1}_{m+1}(U) = f^{-1}_{m}(U) \cup \bigcup \left\{ \mathsf{A}(W) \;\middle|\; W \in \mathcal{P}_m(P) \text{ and } \mu(W) \in U \right\}. \tag{4.24}$$

Once we know (4.24) to hold, the definability is immediate. Indeed, all constituents are known to be definable by induction, and the connectives can all readily be internalised.

To prove the inclusion from left to right, suppose that $q \in \operatorname{dom} f_{m+1}$ is such that $f_{m+1}(q) \in U$. We distinguish between whether $q \in \operatorname{dom} f_m$ does or does not hold. If it does, then $f_{m+1}(q) = f_m(q)$ by (4.21), and hence $q \in f^{-1}_m(U)$. In the case that it does not, we know that $q \in \mathsf{A}(f_m(\updownarrow q))$ by (4.23). By definition (4.22), we know that:

$$\mu(f_m(\updownarrow q)) = f_{m+1}(q) \in U,$$

proving the desired.

To prove the other direction, we suppose that $q \in \mathsf{U}(X)$ is either such that $q \in \operatorname{dom} f_m$, or $q \notin \operatorname{dom} f_m$ and $q \in \mathsf{A}(W)$ for some $W \in \mathcal{P}_m(P)$ with $\mu(W) \in U$. The former case is immediate. In the latter case, fix this W and note that (4.23) ensures $W = f_m(\updownarrow q)$. Because $\mu(W) \in U$ and $f_{m+1}(q) = \mu(W)$ holds by definition (4.22), the desired follows.

Now, let us prove that the conditions are satisfied. It is clear that Compatibility holds. To show that Closed Domain holds, take some $q \in \operatorname{dom} f_{m+1}$ and $D \in \mathcal{D}$ to be such that:

$$\uparrow f_{m+1}(q) \cap D \neq \varnothing.$$

We distinguish two cases, either $q \in \operatorname{dom} f_m$ holds or it does not. If it does, then Compatibility and induction ensure that:

$$f_{m+1}(\uparrow q) \cap D = f_m(\uparrow q) \cap D \neq \varnothing.$$

In the other case, definition (4.22) yields $f_{m+1}(q) = \mu(f_m(\updownarrow q))$. Observe that the inequality $\uparrow f_{m+1}(q) \cap D \neq \varnothing$ holds, hence we know of some $p \in P$ such that both $f_{m+1}(q) \leq p$ and $p \in D$ hold. By the assumption on μ, it follows that $f_{m+1}(q) \in D$ or $f_m(\updownarrow q) \cap D \neq \varnothing$. One can easily check that both disjuncts ensure:

$$f_{m+1}(\uparrow q) \cap D \neq \varnothing,$$

proving that the Closed Domain condition holds.

Finally, we construct the sets $\mathsf{A}(W)$ for $W \in \mathcal{P}_{m+1}(P)$, prove their definability, and show that both the conditions Domain Growth and Image Bound hold.

$$\mathsf{A}(W) := \left\{ q \in \mathsf{U}(X) \;\middle|\; \begin{array}{l} \text{if } k \in f_m^{-1}(\uparrow p) \text{ then } k \in f_m^{-1}(\updownarrow p), \\ \text{for all } k \geq q \text{ and } p \in P - W \end{array} \right\} \tag{4.25}$$

Because $v : P \to \mathcal{P}(X)$ is assumed to be order-defined, we know that $\uparrow p$ is definable. We know f_m to be a definable map through induction, hence $f_m^{-1}(\uparrow p)$ is definable as well. With this information, we can give the defining formula of $\mathsf{A}(W)$ as:

$$\mathsf{def}\,\mathsf{A}(W) := \bigwedge_{p \in P - W} \mathsf{def}\, f_m^{-1}(\uparrow p) \to \mathsf{def}\, f_m^{-1}(\updownarrow p).$$

We prove Domain Growth. Let $q \in \mathsf{U}(X)$ be such that $\left|f_{m+1}(\updownarrow q)\right| < m + 1$. If $q \in \mathrm{dom} f_m$ then $q \in \mathrm{dom} f_{m+1}$, as readily follows through Compatibility. Consider now the other case, where $q \notin \mathrm{dom} f_m$. Through Compatibility, we know that $f_m(\updownarrow q) \subseteq f_{m+1}(\updownarrow q)$. We distinguish two cases, either $\left|f_m(\updownarrow q)\right| < m$ or $\left|f_m(\updownarrow q)\right| = m$. In the former case, we know $q \in \mathrm{dom} f_m$ by induction, a contradiction.

Let us focus on the latter case, that is, we assume $\left|f_m(\updownarrow q)\right| = m$. This ensures us that $f_{m+1}(\updownarrow q) = f_m(\updownarrow q)$. We argue by contradiction, and assume that $q \notin \mathrm{dom} f_{m+1}$. Our goal will be to derive that $q \in \mathsf{A}(f_{m+1}(\updownarrow q))$, which would ensure $q \in \mathrm{dom} f_{m+1}$, an immediate contradiction.

To this end, let $p \in P - f_{m+1}(\updownarrow q)$ and $k \geq q$ be given. Assume that $k \in f_{m+1}^{-1}(\uparrow p)$. If $q = k$ holds, then $q \in \mathrm{dom} f_{m+1}$ follows, a contradiction. So suppose that $k \in \updownarrow p$. If $f_{m+1}(k) = p$, then we have $p \in f_{m+1}(\updownarrow q)$, another contradiction. This proves that $k \in f_{m+1}^{-1}(\updownarrow p)$. We have thus proven that $q \in \mathsf{A}(f_{m+1}(\updownarrow q))$, as desired.

Finally, we prove that Image Bound holds. Let $W \in \mathcal{P}_{m+1}(P)$ be given. We wish to prove that if $q \in \mathsf{A}(W)$, then $f_{m+1}(\updownarrow q) \subseteq W$. Suppose the contrary, that is, suppose there is some $k > q$ such that $k \in \mathrm{dom} f_{m+1}$ yet $f_{m+1}(k) \notin W$. We see that $p \in f_{m+1}^{-1}(\uparrow p)$ for $p := f_{m+1}(p) \notin W$. Clearly, $p < f_{m+1}(k)$ does not hold, a contradiction with $k \in f_{m+1}^{-1}(\updownarrow p)$. This proves the desired, finishing the argument. □

The above argument is based on the reasoning of Rybakov (1984a), which, over the years, has been generalised to prove the decidability of many intermediate and modal logics. Note, in particular, that Rybakov (1994a, p. 222) shows BB_n to have a decidable set of admissible rules. It seems plausible that our formulation can be generalised in a similar manner.

5

Bases

Think of an intermediate logic. Given a set of formulae, one may ask whether it is an *axiomatisation* of said logic; do all theorems follow from these formulae? When the logic at hand comes equipped with a nicely defined class of models, it suffices to show that the set of theorems induced by said formulae is both *sound* and *complete* with respect to these models. Analogously, given a set of rules, one may ask whether it is a *basis*: do all admissible rules follow from these rules? In this chapter, we consider several concrete instances of this question, and tackle it via semantics.

Consider the intermediate logic IPC. We have seen that its admissible rules are sound and complete with respect to IPC-exact models, as proven in Theorem 3.38. Moreover, as we proved in Theorem 4.19, all such models arise as upsets in universal models defined by IPC-exact formulae. This already gives us a quite firm grasp on the semantics of admissibility, but the story gets even better. Indeed, through Theorem 4.30 and Corollary 4.15, such IPC-exact formulae are precisely those formulae that are closed under the rules Con, DP and D^-_ω. In summary, the admissible rules are sound and complete with respect to those models on which the rules Con, DP and D^-_ω are valid.

Albeit quite anachronistically, the above makes a good case for the conjecture that all admissible rules of IPC follow from Con, DP and D_ω^-.[1] When we prove that these rules are sound and complete with respect to the models in which they are valid, the desired is but an immediate corollary. Yet this is no straightforward task.

The purpose of this chapter is to provide a basis of the admissible rules of the intermediate logic BB_n and $\mathsf{BD}_2 + \mathsf{BW}_n$ for $n \geq 2$, which we present in Theorems 5.34 and 5.36 respectively. These results can be obtained using fairly similar methods, which we develop in Sections 5.1 and 5.2. Let us go over this machinery in virtually no detail, merely to sketch a mental image.

We start with an intermediate logic and a set of admissible rules that serves as a prospective basis. The goal is to show that each formula has what we call an *admissible approximation* anchored by the given rules. If this happens to be the case, then each admissible rule is in fact derivable in the prospective basis, which thus truly is a basis. We iteratively decompose formulae into proper parts, in such a way that the union of the admissible approximation of the components, whenever they all exist, is guaranteed to be an admissible approximation of the original. The core idea is that formulae can be decomposed no further precisely when they are their own admissible approximation. This happens precisely when the formula at hand is *projective*. We show that one can obtain such a perfectly decomposed form in finitely many steps, thus guaranteeing that admissible approximations can be obtained in the above described manner. Naturally, one cannot blindly go through the above steps in any intermediate logic; there are several technical caveats one has to keep in mind. We point out these caveats along the way.

In Section 5.1, we provide the basic definitions. Definition 5.1 specifies what we mean by a basis, and admissible approximations are described in Definition 5.3. We give motivation for both of these definitions, and give formal substance to the observations sketched in the above paragraph. The machinery described in this section is already fairly powerful, and we immediately employ it to prove that intermediate logics above LC have a very simple basis in Theorem 5.14.

We subsequently lay out the framework through which we progressively process formulae in Section 5.2. This section makes substantial use of the language of rewrite systems, as this formalism provides a convenient language for dealing with the structures at hand. The core of this section is Definition 5.19, in which we specify the

[1] The anachronism lies in that the above is far from the origin story of this conjecture as it is usually told. Said more precisely, it has been a long standing conjecture, going back to Mints, Citkin and Visser independently, that all single-conclusion admissible rules follow from the Visser rules. This conjecture naturally arises when one inspects the sequent-calculus formulation of IPC. Unfortunately, we were unable to find any written documentation of these conjectures.

admissible reduction. We show that admissible approximations are preserved by admissible reduction, and we provide sufficient conditions under which this reduction can be shown to not go on forever. Finally, we provide more stringent conditions under which formulae that cannot be reduced any further correspond with projective formulae.

We close this chapter with Lemma 5.32, in which we reap the benefits of Section 5.3. We consider the intermediate logics $\mathsf{BD}_2 + \mathsf{BW}_n$ and BB_n, and provide bases of their admissible rules.The proofs are relatively similar, yet distinct still. In the case of BD_2 and its extensions, the logic at hand is known to be *locally tabular*: there are but finitely many distinct formulae modulo derivable equivalence when taking a fixed, finite amount of variables. Using this observation, one can quite readily prove that admissible reduction takes but finitely many steps.

The intermediate logic BB_n, on the other hand, most certainly is not locally tabular when $n \geq 2$. Instead, we restrict attention to all formulae of a bounded implication degree. When considering only these formulae, BB_n is as good as locally tabular. As per Theorem 4.61, we know formulae to be BB_n-projective precisely when they are closed under rules satisfying such a restriction. This observation is crucial to our argument.

This chapter is based on original work, presented earlier in Sections 5 and 6 of Goudsmit (2013) and Goudsmit and Iemhoff (2014). The presentation here is significantly different in two ways. First, in Goudsmit and Iemhoff (2014) formulae are approximated by formulae in $\mathcal{L}_0^*(X)$. This is no longer necessary due to our more flexible machinery. Second, we provide a basis of BB_n and BD_2 using the same general machinery. It is clear from neither Goudsmit (2013) nor Goudsmit and Iemhoff (2014) that this would be possible. Furthermore, there are many stylistic differences in the definitions, in such a way that the work presented here should be more easily adaptable to other settings.

5.1. Admissible approximation

As discussed in Section 1.1.4, several techniques exist that allow one to construct a *basis*. In this section, we focus on the approach pioneered by Iemhoff (2001b, 2005, 2006). The first matter of business is specifying what it means to be a basis of the admissible rules for a given intermediately logic. We pin this down precisely in Definition 5.1, but let us first spend a few words on the intuition.

In the literature, one has often been interested in proving a basis for the *single-conclusion* version of admissibility. The definition of a basis in this setting is rather straightforward: a set of single-conclusion rules $\mathcal{R}$ is a basis of the single-conclusion admissible rules of an intermediate logic Λ if the least single-conclusion relation extending $\vdash_\Lambda$ equals the set of all single-conclusion admissible rules.

We made the conscious decision to focus our attention on *multi-conclusion rules* early on in Section 2.1.3, when we specified our definition of admissible rules. In this setting, there is a bit of freedom. Any intermediate logic Λ gives rise to a multi-conclusion consequence relation $\vdash_\Lambda^{\mathsf{min}}$, as we have specified in Definition 2.8. This is the least such consequence relation whose single-conclusion rules coincide with $\vdash_\Lambda$, as proven in Lemma 2.9. It automatically follows that $\mathrel{|\!\sim}_\Lambda$ is an extension of $\vdash_\Lambda^{\mathsf{min}}$. We consider $\vdash_\Lambda^{\mathsf{min}}$ as the "ground floor" on which to built multi-conclusion consequence relations that have the same body of theorems as Λ.

5.1 Definition (Basis)
Let Λ be an intermediate logic, and let $\mathcal{R}$ be a set of rules. We say that $\mathcal{R}$ *is a basis of the admissible rules in* Λ whenever:

$$\Gamma \mathrel{|\!\sim}_\Lambda \Delta \text{ iff } \Gamma \vdash_\Lambda^{\mathcal{R}} \Delta \text{ for all finite sets of formulae } \Gamma \text{ and } \Delta.$$

Naturally, the set of all admissible rules in Λ is a basis of the admissible rules in Λ. Hence, the question is not *whether* a logic has a basis (for it surely does), but whether a particularly nice set of rules can act as a basis.

A word of warning is in order. The set of all single-conclusion admissible rules yields the largest single-conclusion consequence relation with the same set of theorems as the logic at hand.[2] The same most certainly does not hold for multi-conclusion admissible rules. Indeed, fix an intermediate logic Λ and consider the multi-conclusion consequence relation $\vdash_\Lambda^{\mathsf{max}}$. It is not hard to see that:

$$\Gamma \vdash_\Lambda^{\mathsf{max}} \Delta \text{ iff } \bigwedge \Gamma \to \bigvee \Delta \in \Lambda \text{ for all finite } \Gamma, \Pi \subseteq \mathcal{L}(X). \tag{5.1}$$

The multi-conclusion consequence relation $\vdash_\Lambda^{\mathsf{max}}$ is not a suitable consequence relation for studying admissibility in general. Indeed, the rule DP is included in $\vdash_\Lambda^{\mathsf{max}}$, as readily follows through (5.1). Yet we know there to be many intermediate logics that do not admit this rule, think of CPC or BD_2, for instance. Consequently, $\vdash_\Lambda^{\mathsf{max}}$ need not be included within $\mathrel{|\!\sim}_\Lambda$, and is thus not as suitable as $\vdash_\Lambda^{\mathsf{min}}$ as a general ground. Through Lemma 5.2, though, it is apparent that $\vdash_\Lambda^{\mathsf{max}}$ can be safely employed whenever DP is admissible.

[2] See Iemhoff (2013) for a more details on this.

5.2 Lemma
Let Λ be an intermediate logic. The consequence relations $\vdash_{\Lambda}^{\mathsf{max}}$ and $\vdash_{\Lambda}^{\mathsf{DP},\mathsf{Con}}$ are equal.

Proof. To prove the inclusion from left to right, it suffices to show that DP and Con are contained within $\vdash_{\Lambda}^{\mathsf{max}}$. Let $\Delta \subseteq \mathcal{L}(X)$ be finite, we wish to show $\bigvee \Delta \vdash_{\Lambda}^{\mathsf{max}} \Delta$. To this end, consider (2.5), and let Π be a finite set of formulae and let θ be a formula. Suppose that $\Pi \vdash_{\Lambda} \bigvee \Delta$ and each $\chi \in \Delta$ satisfies $\chi \vdash_{\Lambda} \theta$. An easy induction on the size of Δ proves that $\Pi \vdash_{\Lambda} \theta$. We have thus proven the desired.

Suppose that $\Gamma \vdash_{\Lambda}^{\mathsf{max}} \Delta$. Reading (2.5) with $\Pi := \Gamma$ and $\theta := \bigvee \Delta$, we derive that $\Gamma \vdash_{\Lambda} \bigvee \Delta$. Indeed, $\Gamma \vdash_{\Lambda} \phi$ holds for each $\phi \in \Gamma$ through weakening and reflexivity, and $\chi \vdash_{\Lambda} \bigvee \Delta$ readily follows for each $\chi \in \Delta$ because of the axiom $x \to x \vee y \in \Lambda$ given in Table 2.1.

We distinguish two cases. First, suppose $\Delta = \varnothing$. It follows that $\bigvee \Delta = \bot$, and hence derive:

$$\Gamma \vdash_{\Lambda} \bigvee \varnothing = \bot \;\mathsf{Con}\; \varnothing.$$

Second, consider the case where Δ is not empty. By the reasoning below (3.14) we can thus derive:

$$\Gamma \vdash_{\Lambda} \bigvee \Delta \vdash_{\Lambda}^{\mathsf{DP}} \Delta.$$

These two observations combine to prove that $\Gamma \vdash_{\Lambda}^{\mathsf{DP},\mathsf{Con}} \Delta$. □

Think of the single-conclusion consequence relation $\vdash_{\Lambda}$, and consider it as a preorder. There is an obvious inclusion intro the preorder determined by the admissibility relation $\mathrel{|\!\sim}_{\Lambda}$. In symbols, we have that:

$$\phi \vdash_{\Lambda} \psi \text{ implies } \phi \mathrel{|\!\sim}_{\Lambda} \psi.$$

Whenever one is faced with an inclusion, it is natural to ask whether there exists a left adjoint.[3] Suppose such a left adjoint exists, and denote it by $\mathsf{E}(-)$. The formula $\mathsf{E}(\phi)$ is what Iemhoff (2005, Definition 3.1) calls a *maximal admissible consequence* of ϕ. Such a formula $\mathsf{E}(\phi)$ has to satisfy:

$$\mathsf{E}(\phi) \vdash_{\Lambda} \psi \text{ iff } \phi \mathrel{|\!\sim}_{\Lambda} \psi.$$

It is easy to see that the above entails both $\mathsf{E}(\phi) \vdash_{\Lambda} \phi$ and $\phi \mathrel{|\!\sim}_{\Lambda} \mathsf{E}(\phi)$. In Definition 5.3, we introduce the admissible approximation of a formula, which generalises

[3] Of course, if this inclusion is to have a left adjoint, it had better preserve products in the preorder determined by $\mathrel{|\!\sim}_{\Lambda}$. The product of ϕ and ψ in this preorder is given by $\phi \wedge \psi$. Indeed, $\phi \wedge \psi \mathrel{|\!\sim}_{\Lambda} \phi$ and $\phi \wedge \psi \mathrel{|\!\sim}_{\Lambda} \psi$ hold, and for any χ such that $\chi \mathrel{|\!\sim}_{\Lambda} \phi$ and $\chi \mathrel{|\!\sim}_{\Lambda} \psi$ we clearly have $\chi \mathrel{|\!\sim}_{\Lambda} \phi \wedge \psi$. The product in the preorder determined by $\vdash_{\Lambda}$ is surely given by conjunction, hence products are preserved.

the above to the multi-conclusion setting. In the case where $\mathcal{R}$ equals the set of all admissible rule, Definition 5.3 seems quite similar to what Jeřábek (2010b, Definition 3.6) calls an admissibly saturated approximation. Do note that his notion is *a priori* stronger than ours.[4] The origins of this definition go back to Ghilardi (1999, p. 874), in which the notion of a *projective approximation* was first introduced.[5]

5.3 Definition (Admissible Approximation)
Let Λ be an intermediate logic, let $\phi \in \mathcal{L}(X)$ be a formula, and let $\mathcal{R}$ be a set of rules satisfying $\mathcal{R} \subseteq \mathrel{\vert\!\sim}_\Lambda$. We say that a finite set of formulae Π is an *admissible approximation of ϕ anchored by $\mathcal{R}$* whenever $\phi \vdash_\Lambda^{\mathcal{R}} \Pi$ and:

$$\psi \vdash_\Lambda^{\mathsf{min}} \Delta \text{ for all } \psi \in \Pi \text{ iff } \phi \mathrel{\vert\!\sim}_\Lambda \Delta \text{ for all finite sets of formulae } \Delta. \tag{5.2}$$

The logic Λ is said to have *admissible approximation anchored by $\mathcal{R}$* when each formula has an admissible approximation anchored by $\mathcal{R}$.

It is easy to see that if Π is an admissible approximation of ϕ anchored by $\mathcal{R}$, and if $\mathcal{R} \subseteq \mathcal{R}^*$, then Π is an admissible approximation of ϕ anchored by $\mathcal{R}^*$ as well. Because $\mathcal{R}$, in the above context, has to satisfy $\mathcal{R} \subseteq \mathrel{\vert\!\sim}_\Lambda$, it follows that Π is also an admissible approximation of ϕ anchored by $\mathrel{\vert\!\sim}_\Lambda$. When this happens, we say that Π is an admissible approximation of ϕ, omitting reference to the anchoring as it offers no additional information.

The above definition does not directly guarantee that admissible approximations are uniquely defined. Suppose Δ and Θ are both admissible approximations for ϕ. One can readily derive that $\theta \vdash_\Lambda^{\mathsf{min}} \Delta$ for each $\theta \in \Theta$, and through analogous reasoning we know $\chi \vdash_\Lambda^{\mathsf{min}} \Theta$ for each $\chi \in \Delta$. If Δ is a singleton set, then each element in Θ is equivalent to this single inhabitant of Δ. In particular, we have thus proven the following.

[4] Jeřábek (2010b) calls Π an admissibly saturated approximation of ϕ whenever Π is a finite set of admissibly saturated formulae (see Jeřábek, 2010b, Definition 3.1), $\phi \mathrel{\vert\!\sim}_\Lambda \Pi$ and $\psi \vdash \phi$ for all $\psi \in \Pi$. One can easily show that this entails that Π is an admissible approximation of ϕ in the sense of Definition 5.3. Conversely, if Π is an admissible approximation of ϕ, then it does not automatically follow that each element of Π is admissibly saturated. Nonetheless, this will be the case in the approximations we compute in Theorems 5.14, 5.33 and 5.35 below, as follows by construction. A more general criterion under which these notions are equivalent follows from Corollary 5.6.

[5] Many authors have since adopted this notion and variants thereof, let us but mention Ghilardi (2000, p. 196), Ghilardi (2002), Visser (2002, Section 4.4.2), Iemhoff (2003, p. 256), Jeřábek (2005, Definition 4.2), Artemov and Iemhoff (2007, p. 442), and Baader and Ghilardi (2011, p. 717). Some immediately impose constraints on the implication complexity of the constituents of an admissible approximation. We make no use of such a restriction at this point, as it would make the machinery inapplicable to the BD_2 setting.Later, in Theorem 5.35, such a bound will again come into play.

5.4 Lemma
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$ be a formula. If Δ and Θ are both *admissible approximations* of ϕ, then $\vdash_\Lambda \bigvee \Delta \equiv \bigvee \Theta$.

It may well happen that an admissible approximation contains two formula, one of which entails the other. In this setting, one may omit the strongest of the two, as shown in Lemma 5.5 below. Consequently, admissible approximations can safely be assumed to be incomparable with respect to $\vdash_\Lambda$.

5.5 Lemma
Let Λ be an intermediate logic, and let $\mathcal{R}$ be a set of rules satisfying $\mathcal{R} \subseteq \mathrel{|\!\sim}_\Lambda$. Suppose that $\mathsf{E}(\phi)$ is an *admissible approximation* of ϕ anchored by $\mathcal{R}$, and let $\Theta \subseteq \mathsf{E}(\phi)$ be such that for all $\theta \in \mathsf{E}(\phi)$ there is a $\theta^* \in \Theta$ with $\theta \vdash_\Lambda \theta^*$. Now, Θ is an *admissible approximation* of ϕ anchored by $\mathcal{R}$.

Proof. We first show that $\phi \vdash_\Lambda^\mathcal{R} \Theta$. This is clear from transitivity and $\phi \vdash_\Lambda^\mathcal{R} \mathsf{E}(\phi)$. Let Δ be a finite set of formulae, and suppose that for all $\theta \in \Theta$ we have $\theta \vdash_\Lambda^{\mathsf{min}} \Delta$. For any $\theta \in \mathsf{E}(\phi)$ we obtain $\theta \vdash_\Lambda \theta^* \vdash_\Lambda^{\mathsf{min}} \Delta$, which proves $\phi \mathrel{|\!\sim} \Delta$ as desired. □

When combining the setting of the following Corollary 5.6 with the results of Theorems 4.4 and 4.24, it follows that the elements of $\mathsf{E}(\phi)$ must be IPC-*projective* if $\Lambda = \mathsf{IPC}$. Replacing Theorem 4.4 by Theorems 4.6 to 4.8 in this argument allows one to make similar statements for LC, BB_n and $\mathsf{BD}_2 + \mathsf{BD}_n$ respectively, in the understanding that $n \geq 2$.

5.6 Corollary
Let Λ be an intermediate logic, and let $\mathsf{E}(\phi)$ be an *admissible approximation* of ϕ. Suppose that all elements of $\mathsf{E}(\phi)$ are incomparable with respect to $\vdash_\Lambda$. Now, each element of $\mathsf{E}(\phi)$ is *closed* under all admissible rules of Λ.

Proof. Let $\theta \in \mathsf{E}(\phi)$ be arbitrary, and let Γ/Δ be an admissible rule. Suppose that $\theta \vdash_\Lambda \phi'$ for each $\phi' \in \Gamma$. We observe that:

$$\phi \mathrel{|\!\sim}_\Lambda \mathsf{E}(\phi) = \big(\mathsf{E}(\phi) - \{\phi\}\big) \cup \{\phi\},$$

hence transitivity ensures:

$$\phi \mathrel{|\!\sim}_\Lambda \big(\mathsf{E}(\phi) - \{\phi\}\big) \cup \Delta.$$

Consequently, we know of a $\chi \in \big(\mathsf{E}(\phi) - \{\phi\}\big) \cup \Delta$ such that $\theta \vdash_\Lambda \chi$. If $\chi \notin \Delta$, then $\mathsf{E}(\phi)$ becomes comparable with respect to $\vdash_\Lambda$, a contradiction. This proves that $\theta \vdash_\Lambda^{\mathsf{min}} \Delta$, as desired. □

5.7 Lemma
Let Λ be an intermediate logic, and let $\mathcal{R}$ be a set of rules satisfying $\mathcal{R} \subseteq \mathrel{|\!\sim}_\Lambda$. If the logic Λ has *admissible approximations* anchored by $\mathcal{R}$, then $\mathcal{R}$ is a *basis* of the admissible rules for Λ.

Proof. Suppose that Λ has admissible approximations anchored by $\mathcal{R}$. This means that to each formula ϕ we have an admissible approximation $\mathsf{E}(\phi)$ anchored by $\mathcal{R}$. We wish to prove that $\mathrel{|\!\sim}_\Lambda$ equals the multi-conclusion consequence relation $\vdash_\Lambda^\mathcal{R}$. To this end, let ϕ be a formula, and let Δ be a finite set of formulae. Suppose that $\phi \mathrel{|\!\sim}_\Lambda \Delta$. Consider a $\psi \in \mathsf{E}(\phi)$, and observe that:

$$\psi \vdash_\Lambda^{\mathsf{min}} \Delta,$$

by definition. Furthermore, by assumption we know that:

$$\phi \vdash_\Lambda^\mathcal{R} \mathsf{E}(\phi)\,.$$

Through the transitivity of $\vdash_\Lambda^\mathcal{R}$, it thus readily follows that $\psi \vdash_\Lambda^\mathcal{R} \Delta$, as desired. The converse holds by assumption, for if $\mathcal{R} \subseteq \mathrel{|\!\sim}_\Lambda$ then $\vdash_\Lambda^\mathcal{R} \subseteq \mathrel{|\!\sim}_\Lambda$ naturally follows. ☐

The following Lemma 5.8 hints at how admissible approximations may be computed. Indeed, it shows that when one "decomposes" a formula using admissible rules in a lossless manner, then the admissible approximation of the original can be reconstructed from the admissible approximations of its components.

5.8 Lemma
Let Λ be an intermediate logic, let $\mathcal{R}$ be a set of admissible rules, let ϕ be a formula, and let Θ be a finite set of formulae. Suppose that:

$$\phi \vdash_\Lambda^\mathcal{R} \Theta \text{ and } \theta \mathrel{|\!\sim}_\Lambda \phi \text{ for all } \theta \in \Theta,$$

and furthermore assume that an *admissible approximation* $\mathsf{E}(\theta)$ of θ anchored by $\mathcal{R}$ exists per $\theta \in \Theta$. Now, the set $\mathsf{E}(\phi) := \bigcup_{\theta\in\Theta} \mathsf{E}(\theta)$ is an *admissible approximation* of ϕ anchored by $\mathcal{R}$.

Proof. It readily follows that $\phi \vdash_\Lambda^\mathcal{R} \mathsf{E}(\phi)$ through $\phi \vdash_\Lambda^\mathcal{R} \Theta$ and $\theta \vdash_\Lambda^\mathcal{R} \mathsf{E}(\theta)$ for each $\theta \in \Theta$. Let $\Delta \subseteq \mathcal{L}(X)$ be a finite set of formulae. We are done when we can prove that:

$$\psi \vdash_\Lambda^{\mathsf{min}} \Delta \text{ for all } \psi \in \bigcup_{\theta\in\Theta} \mathsf{E}(\theta) \text{ iff } \phi \mathrel{|\!\sim}_\Lambda \Delta.$$

From left to right, suppose that for all $\theta \in \Theta$ and $\psi \in \mathsf{E}(\theta)$ we have that $\psi \vdash_\Lambda^{\mathsf{min}} \Delta$. As we know $\phi \vdash_\Lambda^\mathcal{R} \mathsf{E}(\phi)$ to entail $\phi \mathrel{|\!\sim}_\Lambda \mathsf{E}(\phi)$, we immediately derive $\phi \mathrel{|\!\sim}_\Lambda \Delta$. From right to left, suppose $\phi \mathrel{|\!\sim}_\Lambda \Delta$. Let $\theta \in \Theta$ and $\psi \in \mathsf{E}(\theta)$ be arbitrary. We see that $\theta \mathrel{|\!\sim}_\Lambda \phi$ and thus $\theta \mathrel{|\!\sim}_\Lambda \Delta$. This readily implies $\psi \vdash_\Lambda^{\mathsf{min}} \Delta$, as desired. ☐

5.9 Corollary
Let Λ be an intermediate logic, and let $\phi, \psi \in \mathcal{L}(X)$ be such that $\vdash_\Lambda \phi \equiv \psi$. Any *admissible approximation* of ϕ is also an *admissible approximation* of ψ.

5.10 Example
Consider an intermediate logic Λ which admits DP, such as IPC or BB_n for $n \geq 2$. We immediately see that:

$$\phi \vee \psi \mathrel{|\!\sim}_\Lambda \{\phi, \psi\},\ \phi \vdash_\Lambda \phi \vee \psi, \text{ and } \psi \vdash_\Lambda \phi \vee \psi.$$

The above Lemma 5.8 now shows that when looking for the admissible approximation of $\phi \vee \psi$, it would suffice to construct the union of the admissible approximations of ϕ and ψ.

5.11 Example
Let $n \geq 2$ be given, and consider an intermediate logic Λ which admits D_n^-, such as BB_n. Observe, for each $\Delta \subseteq \mathcal{L}(X)$ and $\phi \in \mathcal{L}(X)$, that:

$$(\bigvee \Delta \to \phi) \to \bigvee \Delta \ \mathrel{|\!\sim}\ \bigvee_{\chi \in \Delta} (\bigvee \Delta \to \phi) \to \chi,$$
$$\bigvee_{\chi \in \Delta} (\bigvee \Delta \to \phi) \to \chi \vdash_\Lambda (\bigvee \Delta \to \phi) \to \bigvee \Delta.$$

Through Lemma 5.8, this thus shows that the admissible approximation of the antecedent of the rule D_n^- must equal the admissible approximation of its consequent, whenever it exists. We employ this observation to good end in Theorem 7.30.

The rule Con is of paramount importance when considering a basis of admissibility. Indeed, any basis had better include this rule, as we know it to be admissible in any intermediate logic by Example 2.13. The following Lemma 5.12 shows that it plays a crucial role in determining admissible approximations as well. This lemma ensures that the admissible approximation of a formula which has a non-empty model must be non-empty.

5.12 Lemma
Let Λ be an intermediate logic, and let ϕ be a formula. The *admissible approximation* of ϕ anchored by Con is $\varnothing$ precisely if $\vdash_\Lambda \phi \equiv \bot$.

Proof. From left to right, suppose that the admissible approximation of ϕ is $\varnothing$. It follows immediately that $\phi \mathrel{|\!\sim}_\Lambda \bot$. By Lemma 2.77, we know $\vdash_{\mathsf{CPC}} \phi \equiv \bot$. *Glivenko's theorem* now shows $\vdash_{\mathsf{IPC}} \phi \equiv \bot$, as desired.

Conversely, suppose that $\vdash_\Lambda \phi \equiv \bot$ holds. By Corollary 5.9, we can assume ϕ to equal $\bot$ without loss of generality. We need to prove that $\bot \vdash_\Lambda^{\mathsf{Con}} \Delta$ holds for any $\chi \in \Delta$. This follows immediately from $\bot \vdash_\Lambda^{\mathsf{Con}} \varnothing$ and weakening. □

Projective formulae play a special role; each projective formula is its own admissible approximation, anchored by the empty set. This is the strongest possible type of approximation, as $\vdash^{\varnothing}_{\Lambda}$ by definition equals $\vdash^{\mathsf{min}}_{\Lambda}$ for each intermediate logic Λ. As we have a firm semantic grasp on projective formulae via Theorem 4.24, this is very helpful in computing the admissible approximation of a formula.

5.13 Lemma
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$. If ϕ is Λ-*projective*, then $\{\phi\}$ is the *admissible approximation* of ϕ anchored by $\varnothing$.

Proof. Suppose that ϕ is a Λ-projective formula. By assumption, there exits a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ such that $\vdash_{\Lambda} \sigma(\phi)$ and $\phi \vdash_{\Lambda} \sigma(\psi) \equiv \psi$ for all formulae ψ. We prove that:

$$\phi \vdash^{\mathsf{min}}_{\Lambda} \Delta \text{ iff } \phi \mathrel{|\!\!\sim}_{\Lambda} \Delta \text{ for all finite sets of formulae } \Delta.$$

The proof from left to right is immediate, as $\vdash^{\mathsf{min}}_{\Lambda} \subseteq \mathrel{|\!\!\sim}$. To prove the converse, suppose $\phi \mathrel{|\!\!\sim}_{\Lambda} \Delta$ for some finite set of formulae Δ. We know that $\vdash_{\Lambda} \sigma(\phi)$, hence there exists a $\chi \in \Delta$ such that $\vdash_{\Lambda} \sigma(\chi)$. Fix this $\chi \in \Delta$, and observe that $\phi \vdash_{\Lambda} \sigma(\chi) \equiv \chi$. Transitivity thus ensures $\phi \vdash_{\Lambda} \chi$. We thus derive $\phi \vdash^{\mathsf{min}}_{\Lambda} \Delta$, as desired. $\square$

We have seen many examples of projective formulae throughout this thesis. Recall, for instance, Example 3.30, which explicitly described an IPC-projective formula which corresponds to a finite upset in the universal model $\mathsf{U}_{\mathsf{IPC}}(\{x\})$. An enumeration of all such upsets is given in Example 3.36.

We close this section with an elaborate example, albeit in a trivial case. Recall the logic LC, and think back to Theorem 4.6, where we proved each non-empty generated submodel of the universal model $\mathsf{U}_{\Lambda}(X)$ to be extendible whenever $\Lambda \supseteq \mathsf{LC}$. We exploit this observation in Theorem 5.14 below, using the machinery described above, in order to derive a basis of admissibility for such logics. In particular, this shows that Con is a basis of the admissible rules for CPC.

5.14 Theorem
The rule Con is a *basis* of admissible rules in any intermediate logic extending LC.

Proof. Let Λ be an intermediate logic satisfying $\mathsf{LC} \subseteq \Lambda$. We proceed via Lemma 5.7, so fix a formula $\phi \in \mathcal{L}(X)$. It suffices to prove there is an admissible approximation $\mathsf{E}(\phi)$ of ϕ anchored by Con.

We distinguish two cases, depending on whether $\vdash_{\Lambda} \phi \equiv \bot$. In the case where this does hold, we know through Lemma 5.12 that $\mathsf{E}(\phi) := \varnothing$ is an admissible approximation of ϕ anchored by Con.

We now turn to the other case, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$. Define the upset $U := [\![\phi]\!]_\mathsf{u}$, and know it to be non-empty. We derive via Theorem 4.6 that U is Λ-extendible. Through Theorem 4.24 we learn that ϕ is a Λ-projective formula, hence Lemma 5.13 shows that $\{\phi\}$ is the admissible approximation of ϕ anchored by $\varnothing$. Having resolved both of these cases, we have proven the desired. $\square$

5.2. Reduction

When one wants to know whether a set of rules is a basis, it suffices to show that each formula has an admissible approximation anchored by said prospective basis. In Theorem 5.14, this was a rather straightforward matter, as each formula is either equivalent to $\bot$ or it equaled its own approximation. Matters are much more intricate when one moves to logics like IPC or BD_2. The purpose of this section is to provide general machinery by which one can progressively decompose a formula into its admissible approximation.

The main technical tool of this section is the notion of *admissible reduction*, as given in Definition 5.19. We formulate this notion in terms of the language of rewrite theory, namely as an *abstract rewrite system*. Such abstract rewrite systems provide a convenient language for dealing with transformations between objects of a similar type that need not necessarily proceed in a deterministic manner. Before we proceed, let us first give a few of the relevant concepts. More details on these topics can be found in standard works by Baader and Nipkow (1999) and Terese (2003).

5.15 Definition (Abstract Rewrite System)
An *abstract rewrite system* is a pair $\langle A, \to\rangle$ where A is a set and $\to\, \subseteq A \times A$ is a binary relation. We write $\to^+$ for the *transitive closure* of $\to$, and $\to^*$ for the *reflexive, transitive closure* of $\to$. An element $a \in A$ is said to be in *normal form* if $a \to b$ holds for no $b \in A$. We say that this system is *strongly normalising* if there are no $a_1, a_2, \ldots \in A$ such that: $a_1 \to a_2 \to \ldots$.

5.16 Example
Consider an intermediate logic Λ, and let Σ be a set of formulae. We define an *abstract rewrite system* $\mathsf{L}_\Lambda(\Sigma) := \langle \Sigma, \to\rangle$ where:

$$\phi \to \psi \text{ iff } \phi \vdash_\Lambda \psi \text{ and } \psi \nvdash_\Lambda \phi \text{ for all } \phi, \psi \in \Sigma.$$

This relation need not be strongly normalizing. A straightforward example can be constructed when we take $\Lambda := \mathsf{IPC}$, $X := \{x_1, x_2, \ldots\}$, and define $\Sigma = \mathcal{L}(X)$. It is easy to see that the formulae $\mathbf{bd}_n$ of Definition 2.71 satisfy:

$$\bot = \mathbf{bd}_0 \rightarrow \ldots \rightarrow \mathbf{bd}_n \rightarrow \mathbf{bd}_{n+1} = x_{n+1} \vee (x_{n+1} \rightarrow \mathbf{bd}_n) \rightarrow \ldots.$$

We recall the following well-known lemma, see Terese (2003, p. 798) for a variant phrased in terms of trees.

5.17 Lemma (Kőnig, 1927)
Every connected, infinite, finitely branching directed graph has an infinite path.

5.18 Example
Consider the setting of Example 5.16. Let us give a more semantic example of an infinite chain in $\langle \mathcal{L}(X), \rightarrow \rangle$. To this end, fix any finite non-empty set of variables X, and consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \rightarrow \mathcal{P}(X)$. It is easy to see that $\mathsf{U}_{\mathsf{IPC}}(X)$ is not finite. In fact, whenever $|X| \geq 2$ this follows through Lemma 3.87. By Lemma 5.17, there exists a sequence of points $a_1, a_2, \ldots \in \mathsf{U}_{\mathsf{IPC}}(X)$ satisfying $a_i < a_{i+1}$ for each $i \in \mathbb{N}$. We thus obtain an infinite sequence $\mathsf{up}\, a_1 \rightarrow \mathsf{up}\, a_2 \rightarrow \ldots$, as desired.

We now have ample machinery to formulate Definition 5.19. The remainder of this section is devoted to proving three properties of this relation, namely Lemmas 5.22, 5.24 and 5.29. Intuitively speaking, these properties combine to ensure that the admissible approximation of a formula ϕ is given by a normal form Π in $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R})$ such that $\{\phi\} \rightarrowtail^* \Pi$.[6]

5.19 Definition (Admissible Reduction)
Let Λ be an intermediate logic, let Σ be be a set of formulae, and let $\mathcal{R}$ be a set of rules. We define the *abstract rewrite system* of *admissible reduction* $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R}) = \langle \mathcal{P}(\Sigma), \rightarrowtail \rangle$, where $\rightarrowtail$ is defined by:

$$\begin{aligned} \Gamma \rightarrowtail \Pi :=\ & \text{there exist } \phi \in \Gamma \text{ and } \Pi^* \subseteq \Pi \text{ such that } \Pi = (\Gamma - \{\phi\}) \cup \Pi^*, \\ & \phi \vdash^{\mathcal{R}}_{\Lambda} \Pi^*, \text{ and for all } \psi \in \Pi^*, \psi \vdash_{\Lambda} \phi \text{ and } \phi \nvdash_{\Lambda} \psi \text{ hold.} \end{aligned}$$

5.20 Example
Consider the setting in Definition 5.19, and suppose that Con is an element of $\mathcal{R}$. Let ϕ be a formula satisfying $\vdash_{\Lambda} \phi \equiv \bot$. Through Lemma 5.12, we know that $\varnothing$ is an admissible approximation of ϕ, hence $\{\phi\} \rightarrowtail^* \varnothing$ had better hold. We show that $\{\phi\} \rightarrowtail \varnothing$ holds. Indeed, take $\Pi^* = \varnothing$, and see that $\varnothing = (\{\phi\} - \{\phi\}) \cup \varnothing$. Moreover, we know $\phi \vdash^{\mathcal{R}}_{\Lambda} \varnothing$. As Π^* is empty, the remaining conditions hold vacuously.

[6] Recall that $\rightarrowtail^*$ is the *reflexive, transitive closure* of $\rightarrowtail$ as defined in Definition 5.15.

5.21 Lemma
Let Λ be an intermediate logic, let Σ be be a set of formulae, and let $\mathcal{R}$ be a set of rules. The relation $\rightarrowtail$ of $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R}) = \langle \mathcal{P}(\Sigma), \rightarrowtail \rangle$ is irreflexive.

Proof. Suppose that $\Gamma \rightarrowtail \Gamma$ holds for $\Gamma \subseteq \Sigma$. This gives a $\phi \in \Gamma$ and $\Gamma^* \subseteq \Gamma$ such that $\Gamma = (\Gamma - \{\phi\}) \cup \Gamma^*$ and for all $\psi \in \Gamma^*$ both $\psi \vdash_\Lambda \phi$ and $\phi \nvdash_\Lambda \psi$ hold. The former conjunct ensures that $\phi \in \Gamma^*$, from whence the latter conjunct can be seen to entail $\phi \nvdash_\Lambda \phi$, a clear contradiction. □

In Lemma 5.22 below, we show the relation between admissible approximations of the right-hand side and left-hand side of $\rightarrowtail^*$. The name of the lemma is made by analogy with *subject expansion* as it is commonly used in rewriting theory, or type theory in particular, see for instance Pierce (2002, p. 98). This property states that when the one object is reduced to the other, and the latter object has a given type, then the former object has the same type.

5.22 Lemma (Subject Expansion)
Let Λ be an intermediate logic, let Σ be a set of formulae, let $\mathcal{R}$ be a set of rules, let $\phi \in \Sigma$, and let $\Theta \subseteq \Sigma$ be finite. Suppose that each $\theta \in \Theta$ has an *admissible approximation* $\mathsf{E}(\theta)$ anchored by $\mathcal{R}$. If $\{\phi\} \rightarrowtail^* \Theta$, then

$$\mathsf{E}(\phi) \coloneqq \bigcup_{\theta \in \Theta} \mathsf{E}(\theta)$$

is an *admissible approximation* of ϕ anchored by $\mathcal{R}$.

Proof. We proceed by induction on the number of steps in $\{\phi\} \rightarrowtail^* \Theta$. In the base case, when no steps are taken, we know that $\{\phi\} = \Theta$, hence the desired follows immediately. In the inductive case, we derive that there exist $\Gamma, \Pi \subseteq \Theta$ and $\chi \in \Sigma$ such that:

$$\{\phi\} \rightarrowtail^* \big(\Gamma \cup \{\chi\}\big) \rightarrowtail \big(\Gamma \cup \Pi\big) = \Theta,$$

under the constraints that $\chi \vdash^{\mathcal{R}}_{\Lambda} \Pi$ and for all $\psi \in \Pi$ we know that both $\psi \vdash_\Lambda \chi$ and $\chi \nvdash_\Lambda \psi$ hold. By Lemma 5.8, we know that $\mathsf{E}(\chi)$ defined by:

$$\mathsf{E}(\chi) \coloneqq \bigcup_{\psi \in \Pi} \mathsf{E}(\psi)$$

must be an admissible approximation of χ anchored by $\mathcal{R}$. We can apply the induction hypothesis to $\{\phi\} \rightarrowtail^* \Gamma \cup \{\chi\}$, and see that:

$$\mathsf{E}(\phi) = \left(\bigcup_{\theta \in \Gamma} \mathsf{E}(\theta)\right) \cup \mathsf{E}(\chi) = \left(\bigcup_{\theta \in \Gamma} \mathsf{E}(\theta)\right) \cup \left(\bigcup_{\theta \in \Pi} \mathsf{E}(\theta)\right) = \bigcup_{\theta \in \Theta} \mathsf{E}(\theta),$$

is an admissible approximation of ϕ anchored by $\mathcal{R}$, proving the desired. □

At this point, we already have enough information to show that $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R})$ can be used to compute admissible approximations of formulae that are provably equivalent to $\bot$. We spell this out in Corollary 5.23 below.

5.23 Corollary

Let Λ be an intermediate logic, let Σ be a set of formulae, let $\mathcal{R}$ be a set of rules, and let $\phi \in \Sigma$. If $\mathsf{Con} \in \mathcal{R}$, then the following are equivalent:

(i) $\vdash_{\Lambda} \phi \equiv \bot$;

(ii) $\varnothing$ is a admissible approximation of ϕ anchored by $\mathcal{R}$;

(iii) $\{\phi\} \rightarrowtail^{*} \varnothing$.

Proof. The items (i) and (ii) are equivalent via Lemma 5.12, and (ii) implies (iii) via Example 5.20. The converse follows by Lemma 5.22. □

5.24 Lemma

Let Λ be an intermediate logic, let $\mathcal{R}$ be a set of rules, and let $\Sigma \subseteq \mathcal{L}(X)$ be an adequate set. Suppose that Σ is closed under conjunctions, that is to say, for all $\phi, \psi \in \Sigma$ we have a $\chi \in \Sigma$ such that $\vdash_{\Lambda} \chi \equiv (\phi \wedge \psi)$. The following are equivalent for all $\Theta \subseteq \Sigma$:

(i) each $\theta \in \Theta$ is closed under Σ-instances of $\mathcal{R}$;

(ii) Θ is a *normal form*.

Proof. Suppose (i) holds. We proceed by contradiction, so assume there is a $\Pi \subseteq \Sigma$ such that $\Theta \rightarrowtail \Pi$. This provides us with a $\phi \in \Theta$ and $\Pi^{*} \subseteq \Pi$ such that $\phi \vdash^{\mathcal{R}}_{\Lambda} \Pi^{*}$, and $\phi \nvdash_{\Lambda} \psi$ holds for all $\psi \in \Pi^{*}$. As ϕ is assumed closed under Σ-instances of $\mathcal{R}$, this immediately yields a contradiction. We have thus proven (ii).

Conversely, suppose (ii) holds. We proceed by contradiction, so we pick some $\theta \in \Theta$ that is not closed under Σ-instances of $\mathcal{R}$. This yields some $\Gamma, \Delta \subseteq \Sigma$ such that $\Gamma \vdash^{\mathcal{R}}_{\Lambda} \Delta$ and $\theta \vdash_{\Lambda} \phi$ for all $\phi \in \Gamma$ yet $\theta \nvdash_{\Lambda} \chi$ for all $\chi \in \Delta$. As Σ is closed under conjunction, we know of a formula $\chi^{*} \in \Sigma$ such that:

$$\vdash_{\Lambda} \chi^{*} \equiv (\chi \wedge \theta), \tag{5.3}$$

for each $\chi \in \Delta$. We define $\Delta^{*} = \{\chi^{*} \mid \chi \in \Delta\}$, and claim that we have:

$$\Theta \rightarrowtail (\Theta - \{\theta\}) \cup \Delta^{*}. \tag{5.4}$$

There are several things to prove. First, we wish to prove that $\theta \vdash^{\mathcal{R}}_{\Lambda} \Delta^*$. Through transitivity we derive $\theta \vdash^{\mathcal{R}}_{\Lambda} \Delta$ from $\theta \vdash_{\Lambda} \phi$ for all $\phi \in \Gamma$ and $\Gamma \vdash^{\mathcal{R}}_{\Lambda} \Delta$. Moreover, through (5.3) we know that $\theta, \chi \vdash^{\mathcal{R}}_{\Lambda} \chi^*$ holds for each $\chi \in \Delta$. Combining these two observations yields $\theta \vdash^{\mathcal{R}}_{\Lambda} \Delta^*$ by transitivity (or rather, $|\Delta|$ many applications thereof), as desired.

Second, we observe that $\theta \vdash_{\Lambda} \chi^*$ holds for no $\chi \in \Delta$. Indeed, if $\theta \vdash_{\Lambda} \chi^*$ holds for some $\chi \in \Delta$, then $\theta \vdash_{\Lambda} \chi$ follows, *quod non.* Third, and finally, note that $\chi^* \vdash_{\Lambda} \theta$ holds for each $\chi \in \Delta$, as readily follows from (5.3). This completes the proof of the claim (5.4), which provides us with a contradiction, thus proving (i). □

We spend the remainder of this section on providing conditions under which the abstract rewrite system $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R})$ is *strongly normalising*. The importance of this property follows through Lemma 5.25. We omit the proof of this lemma, as it can be readily derived using well-founded induction. Through the above Lemma 5.24, we have a good grasp on normal forms in this system. We combine the existence of normal forms, their description, and subject expansion to construct admissible approximations in Section 5.3.

5.25 Lemma
Let $A = \langle A, \to \rangle$ be an *abstract rewrite system.* Suppose that A is *strongly normalising*. Now, to every $a \in A$ there exists a *normal form* $a^* \in A$ such that $a \to^* a^*$.

Dershowitz and Manna (1979) defined the *multiset order*, which lifts an abstract rewrite system $\langle A, \to \rangle$ to an abstract rewrite system on finite multisets over the set A. They showed that the one abstract rewrite system is strongly normalising exactly if the other is strongly normalising. We employ a similar order, not defined on finite multisets, but on finite sets. The plan is to show that $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R})$ is strongly normalising (under certain conditions) by mapping it into a suitably chosen finite set order.

5.26 Definition (Finite Set Order)
Let $A = \langle A, \to \rangle$ be an *abstract rewrite system.* We define the *finite set order on A*, denoted $\mathsf{S}(A)$, to be the abstract rewrite system $\langle \mathsf{S}(A), \succ \rangle$ where:

$$\begin{aligned} \mathsf{S}(A) &:= \{ W \mid W \subseteq A \text{ and } W \text{ is finite} \}, \\ W \succ S &:= \text{there exist } W^* \subseteq W \text{ and } S^* \subseteq S \text{ such that:} \\ &\qquad \varnothing \neq W^*,\ S = (W - W^*) \cup S^* \text{ and} \\ &\qquad \text{for all } s \in S^* \text{ there is a } w \in W \text{ such that } w \to^+ s. \end{aligned}$$

We state the following Lemma 5.27 and Theorem 5.28 without proof. Their respective arguments can readily be constructed, and spelling them out in full would lead us too far afield.

5.27 Lemma
Let $A = \langle A, \rightarrow \rangle$ and $B = \langle B, \rightsquigarrow \rangle$ be *abstract rewrite systems*, and let $f : A \rightarrow B$ be a map such that $a \rightarrow b$ implies $f(a) \rightsquigarrow f(b)$ for all $a, b \in A$. If B is strongly normalising, then A is *strongly normalising*.

5.28 Theorem
Let A be an abstract rewrite system. The following are equivalent:

(i) the abstract rewrite system A is *strongly normalizing*;

(ii) the *finite set order* $\mathsf{S}(A)$ is *strongly normalizing*.

Proof. Immediate through a straightforward adaptation of the proof by Dershowitz and Manna (1979, pp. 467–468). □

Recall the relation $\mathsf{L}_\Lambda(\Sigma)$ as defined in Example 5.16. When instantiating Λ to IPC and Σ to $\mathcal{L}(X)$, we know through Example 5.18 that $\mathsf{L}_\Lambda(\Sigma)$ *is not* strongly normalising. However, when one takes Σ to be $\mathcal{L}_n(X)$, it can be proven that $\mathsf{L}_\Lambda(\Sigma)$ *is* strongly normalising. This line of reasoning we pursue in Section 5.3.

5.29 Lemma (Termination)
Let Λ be an intermediate logic, let Σ be be a set of formulae, and let $\mathcal{R}$ be a set of rules. Suppose that $\mathsf{L}_\Lambda(\Sigma)$ is *strongly normalising*. Now, $\mathsf{R}^\Sigma_\Lambda(\mathcal{R})$ is *strongly normalising* as well.

Proof. The abstract rewrite system $A := \langle \Sigma, \rightarrow \rangle$ induces a *finite set order* $\mathsf{S}(A)$. We define a map:

$$f : \mathsf{R}^\Sigma_\Lambda(\mathcal{R}) \rightarrow \mathsf{S}(A), \quad \Gamma \mapsto \Gamma,$$

and claim that $\Gamma = f(\Gamma) \succ f(\Pi) = \Pi$ whenever $\Gamma \rightarrowtail \Pi$ for all $\Gamma, \Pi \in \mathsf{R}^\Sigma_\Lambda(\mathcal{R})$. The desired follows immediately from this claim via Lemma 5.27 and Theorem 5.28.

Suppose that $\Gamma \rightarrowtail \Pi$ holds for $\Gamma, \Pi \in \mathsf{R}^\Sigma_\Lambda(\mathcal{R})$. This provides us with a $\phi \in \Gamma$ and $\Pi^* \subseteq \Pi$ such that $\Pi = (\Gamma - \phi) \cup \Pi^*$ and for each $\psi \in \Pi^*$ both $\psi \vdash_\Lambda \phi$ and $\phi \nvdash_\Lambda \psi$ hold. The latter ensures that $\phi > \psi$ for all $\psi \in \Pi^*$, proving $\Pi \subseteq \uparrow\phi$. This immediately entails the desired. □

We close this section with the following Definition 5.30 and Lemma 5.31. The latter is a convenient tool in proving that $\mathsf{L}_\Lambda(\Sigma)$ is strongly normalising, reducing yet again the burden of proof towards showing that $\mathsf{R}^{\Sigma}_{\Lambda}(\mathcal{R})$ is strongly normalising. Note that we have already seen an instance of Definition 5.30 in the past, namely in Theorem 4.63.

5.30 Definition
Let Λ be an intermediate logic, and let $\Sigma^* \subseteq \Sigma \subseteq \mathcal{L}(X)$ be sets of formulae. We say that Σ^* *is* Λ*-representative for* Σ whenever the following two properties hold:

(i) for all $\phi \in \Sigma$ there exists a $\psi \in \Sigma^*$ such that $\vdash_\Lambda \phi \equiv \psi$;

(ii) for all $\phi, \psi \in \Sigma^*$ we have $\vdash_\Lambda \phi \equiv \psi$ iff $\phi = \psi$.

5.31 Lemma
Let Λ be an intermediate logic, and let Σ^* be Λ-representative for Σ^*. Suppose that Σ^* is finite. The abstract rewrite system $\mathsf{L}_\Lambda(\Sigma)$ is *strongly normalising*.

Proof. Suppose there are $\phi_1, \phi_2, \ldots \in \Sigma$ such that $\phi_i \to \phi_{i+1}$ for all $i \in \mathbb{N}$. For all $i \in \mathbb{N}$, we know of $\phi_i^* \in \Sigma^*$ such that $\vdash_\Lambda \phi_i^* \equiv \phi_i$. By the pidgeonhole principle, there must be $i \neq j$ such that $\phi_i^* = \phi_j^*$. Without loss of generality we assume $i < j$. It follows that $\phi_j \vdash_\Lambda \phi_i$, a contradiction. This proves the desired. □

5.3. Bases of some intermediate logics

The stage is set. All machinery is in place to conveniently prove Theorems 5.34 and 5.36 below. The reasoning in these proofs is similar yet distinct. Naturally, the characterisation of projective objects in these logics differs. We use Lemma 5.32 to prove that the relation of admissible reducibility is strongly normalising in the setting of BD_2 and its extensions. In the setting of BB_n, we appeal to Theorem 4.63, and need to proceed with a little bit more care.

5.32 Lemma
Let $n \in \mathbb{N}$, let $m = 1, 2, \ldots, \omega$, let X be a finite set of variables, and consider the intermediate logic $\Lambda := \mathsf{BD}_n + \mathsf{BW}_m$. There exists a finite set $\Sigma^* \subseteq \mathcal{L}(X)$ that is Λ*-representative* for $\mathcal{L}(X)$.

Proof. The desired is immediate from Theorem 3.22, after noting that the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ is finite. Indeed, as each point is of finite height, this follows by a straightforward induction. □

On a casual reading of Theorem 5.33, one might be under the impression that this proof is not specific to $\mathsf{BD}_2 + \mathsf{BW}_n$ at all. However, this is not the case. Take care to note that we make crucial use of our description of $(\mathsf{BD}_2 + \mathsf{BW}_n)$-extendible formulae in terms of the rules Con, $\mathsf{DP}_n^{\neg\neg}$ and $\mathsf{D}_n^{\neg}$, as presented in Corollary 4.17. Without this description, the proof would not go through.

5.33 Theorem
Let $n = 1, 2, \ldots, \omega$ be given. The logic $\Lambda := \mathsf{BD}_2 + \mathsf{BW}_n$ has *admissible approximations* anchored by the rules Con, $\mathsf{DP}_n^{\neg\neg}$ and $\mathsf{D}_n^{\neg}$, each of which consists solely of Λ-*projective* formulae.

Proof. For convenience, we define $\mathcal{R}$ to equal all $\mathcal{L}(X)$-instances of Con, $\mathsf{DP}_n^{\neg\neg}$ and $\mathsf{D}_n^{\neg}$. Let $\phi \in \mathcal{L}(X)$ be a formula. By Lemma 5.32, we know of a finite set Σ^* that is Λ-representative for $\mathcal{L}(X)$. The *abstract rewrite system* $\mathsf{R}_\Lambda^\Sigma(\mathcal{R})$ is *strongly normalising*. This follows readily through Lemma 5.29, for $\mathsf{L}_\Lambda(\Sigma)$ is strongly normalising by Lemma 5.31.

We consider the set $\{\phi\}$, and note that there must exist a normal form $\mathsf{E}(\phi)$ satisfying $\{\phi\} \rightarrowtail^* \mathsf{E}(\phi)$ due to Lemma 5.25. By Lemma 5.24 we know each element of $\mathsf{E}(\phi)$ to be Λ-projective. Indeed, a formula is Λ-projective precisely if it is Λ-extedible due to Theorem 4.24, which in turn is equivalent to being closed under $\mathcal{R}$ by Corollary 4.17.

Consequently, Lemma 5.13 proves $\{\theta\}$ is an admissible approximation of θ anchored by $\mathcal{R}$ for each $\theta \in \mathsf{E}(\phi)$. It thus follows through Lemma 5.22 that:

$$\bigcup_{\theta \in \mathsf{E}(\phi)} \mathsf{E}(\theta) = \bigcup_{\theta \in \mathsf{E}(\phi)} \{\theta\} = \mathsf{E}(\phi)\,.$$

is an admissible approximation of ϕ anchored by $\mathcal{R}$. This finishes the proof. □

5.34 Theorem
Let $n = 1, 2, \ldots, \omega$ be given. The rules Con, $\mathsf{DP}_n^{\neg\neg}$ and $\mathsf{D}_n^{\neg}$ form a *basis* for the admissible rules of $\mathsf{BD}_2 + \mathsf{BW}_n$.

Proof. Immediate via Lemma 5.7 and Theorem 5.33. □

5.35 Theorem
Let $m = 2, \ldots, \omega$ be given. The intermediate logic BB_m has *admissible approximations* anchored by Con, DP and $\mathsf{D}_m^{\neg}$, each of which consists solely of BB_m-*projective* formulae.

Proof. Let $\phi^- \in \mathcal{L}(X)$ be given. Define $n := \mathsf{d}(\phi^-) + 2$, and set:

$$\Sigma := \{\bigwedge \Gamma \mid \Gamma \subseteq \mathcal{L}^*_n(X) \text{ is finite }\}.$$

It is easy to see that $\phi := (\top \to \top) \to \phi^- \in \Sigma$, and moreover, $\vdash_\Lambda \phi \equiv \phi^-$. We can thus, without loss of generality, compute an admissible approximation of ϕ due to Corollary 5.9. Define $\mathcal{R}$ to equal all $\mathcal{L}^*_n(X)$-instances of Con, DP and D^-_m. By Theorem 4.63, we know of a finite set Σ^* that Λ-represents Σ. This shows that the *abstract rewrite system* $\mathsf{R}^\Sigma_\Lambda(\mathcal{R})$ is *strongly normalising*. Indeed, as $\mathsf{L}_\Lambda(\Sigma)$ is strongly normalising by Lemma 5.31, this follows through Lemma 5.29.

We consider the set $\{\phi\}$, and note that there must exist a normal form $\mathsf{E}(\phi)$ satisfying $\{\phi\} \rightarrowtail^* \mathsf{E}(\phi)$ due to Lemma 5.25. We claim that, due to Lemma 5.24, we know each element of $\mathsf{E}(\phi)$ to be Λ-projective. First, see that a formula is Λ-projective precisely if it is Λ-extendible due to Theorem 4.24. Because $\theta \in \mathsf{E}(\phi)$ satisfies $\theta \in \Sigma$, we know that θ is Λ-extendible precisely if it is closed under all $\mathcal{L}^*_n(X)$-instances of $\mathcal{R}$ by Corollary 4.16. Do note that Corollary 4.16 is only directly applicable to formulae in $\mathcal{L}^*_n(X)$, but it can easily be extended to conjunctions of such formulae.

Consequently, Lemma 5.13 proves that $\{\chi\}$ is an admissible approximation of θ anchored by $\mathcal{R}$ for each $\theta \in \mathsf{E}(\phi)$. We compute that:

$$\bigcup_{\theta \in \mathsf{E}(\phi)} \mathsf{E}(\chi) = \bigcup_{\theta \in \mathsf{E}(\phi)} \{\chi\} = \mathsf{E}(\phi),$$

and note that $\mathsf{E}(\phi)$ is an admissible approximation of ϕ anchored by $\mathcal{R}$. This finishes the proof. □

5.36 Theorem
Let $n = 2, \ldots, \omega$ be given. The rules Con, DP and D^-_n form a *basis* for the admissible rules of BB_n.

Proof. Immediate by Theorem 5.35 and Lemma 5.7. □

In closing, we note that Theorem 5.36 provides a positive answer to Problem 7.

6
Unification

Think of a formula. One can substitute its propositional variables for other, more complex formulae. This may turn the formula into a theorem of IPC, and it may well not. Can one characterise those substitutions that do turn said formula into a theorem? For instance, can they be produced in an algorithmic manner? And does there exist a small set of substitutions from which all such substitutions follow?

In this chapter, we treat these latter two questions in a more general form, replacing IPC by the intermediate logics BB_n and $\mathsf{BD}_2 + \mathsf{BW}_n$. These questions are natural refinements of Problem 9, as originally posed by Mints. The proofs presented here are based on the work described in Goudsmit (2013) and Goudsmit and Iemhoff (2014). Much of the heavy lifting has been done in the earlier chapters of this thesis, Chapter 5 in particular. All that remains is providing the appropriate language to phrase the questions, and tying together previously presented proofs in order to answer them. The main results of this chapter can be found in Theorems 6.18 and 6.22.

We proceed as follows. First, we provide some basic definitions regarding Unification Theory in Section 6.1. This language allows us to conveniently rephrase the problems sketched colloquially above in a more precise and formal manner. We introduce the term *unifier*, and order such unifiers in a natural manner.[1] Moreover, we treat *unification types* as a measure of the amount of incomparable unifiers that suffice to describe all unifiers of a given formulae. The adage “less is more” holds; the fewer incomparable unifiers suffice the better.

In Section 6.2, we employ the language of Unification Theory to give more precise formulations of the questions alluded to above. We show how *admissible approximations* play a role in unification, especially when the approximates are *projective formulae*. Using the results of Section 5.3, we prove that unifiers in BB_n and $\mathsf{BD}_2+\mathsf{BW}_n$ can be described by finitely many incomparable unifiers; the unification type of these logics is finitary. In particular, we resolve both Problems 9 and 10.

6.1. Unification theory

We provide the basics of Unification Theory, as closely following the terminology of Baader (1992), Baader and Ghilardi (2011), and Ghilardi (1997). This section contains no original results, and merely serves to fix the language. Definition 6.1 specifies a preorder on substitutions, ordering them by factorisation. One can think of a substitution as less general than another when the former factors through the latter. Pictorially, one could summarise this definition as Fig. 6.1.

6.1 Definition
Let Λ be an intermediate logic, and let $\sigma, \tau : \mathcal{L}(X) \to \mathcal{L}(X)$ be two substitutions. We say that σ *is more general than* τ, denoted $\tau \leq^X_\Lambda \sigma$, whenever there exists a substitution $\xi : \mathcal{L}(X) \to \mathcal{L}(X)$ satisfying:

$$\vdash_\Lambda \tau(x) \equiv \xi(\sigma(x)) \text{ for all } x \in X \ .$$

In its general form, Unification Theory is concerned with finding solutions to equations modulo equational theories. A *solution* to an equation is a substitution that *unifies* both sides of said equation up to the equational theory at hand; such a substitution is called a unifier. Any intermediate logic Λ gives rise to an equational theory E_Λ which simply stipulates that each member of Λ be equal to $\top$. Conversely, any equation can be converted into a formula in such a way that the resulting equation

[1]The ordering we consider is by “generality”, as in Siekmann (1989, p. 208) and Baader (1992, p. 153). Recently, Cabrer and Metcalfe (2014, Section 3) proposed a different order. In the more general setting of unification in arbitrary algebraizable logics, this order can have a more pleasant structure than the typical order. Except for the above remarks, we do not treat this approach.

$$\begin{tikzcd} \mathcal{L}(X) \arrow[rr, "\tau"] \arrow[dr, "\sigma"'] & & \mathcal{L}(X) \\ & \mathcal{L}(X) \arrow[ur, "\xi"'] & \end{tikzcd}$$

Figure 6.1.: The substitution τ is less general than σ.

is valid modulo E_Λ precisely if the original formula is a theorem of Λ. Now, we capture the above informal idea with Definition 6.2 below. From here onwards, we no longer look back to general Unification Theory, and proceed with the terminology as instantiated to the special case of unification in intermediate logics.

6.2 Definition (Unifier)
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$ be a formula. We say that a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ is a Λ-*unifier of* ϕ whenever $\vdash_\Lambda \sigma(\phi)$. In case that such a unifier exists, we say that the formula ϕ is *unifiable*. We define the preorder of all Λ-unifiers of ϕ as $\mathsf{N}_\Lambda(\phi) = \left\langle \mathsf{N}_\Lambda(\phi), \leq_\Lambda^X \right\rangle$ where:

$$\mathsf{N}_\Lambda(\phi) := \{\sigma \mid \sigma : \mathcal{L}(X) \to \mathcal{L}(X) \text{ is a } \Lambda\text{-unifier of } \phi\}.$$

In Examples 6.3 to 6.5 below, we provide some instances of the above Definition 6.2. We briefly remark that a formula is *unifiable* precisely if it has a non-empty model, as we prove in Corollary 6.7.

6.3 Example
Consider the variables $X := \{x, y, z\}$, and consider the formula:

$$\phi := (x \wedge y) \equiv z.$$

It is easy to see that $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ defined by $\sigma(z) := x \wedge y$, $\sigma(x) := x$, and $\sigma(y) = y$ is a IPC-unifier of ϕ. It is by no means the single unifier, yet is surely is the *most general* unifier. Indeed, first note that for all $\psi \in \mathcal{L}(X)$ we have:

$$(x \wedge y) \equiv z \vdash_{\mathsf{IPC}} \sigma(\psi) \equiv \psi,$$

as readily follows through a simple inductive argument. Any unifier $\tau : \mathcal{L}(X) \to \mathcal{L}(X)$ can thus be seen to satisfy:

$$\vdash_{\mathsf{IPC}} \tau(\sigma(\phi)) \equiv \tau(\phi),$$

proving that σ is *more general* than any unifier. The above is but a special case of Example 3.35. Indeed, had we taken ϕ to be any formula in the $[\wedge, \rightarrow, \top]$-fragment of IPC, the same reasoning would go through.

6.4 Example (Exact Unifier)
Consider some intermediate logic Λ, and let $v : P \rightarrow \mathcal{P}(X)$ be a Λ-*exact model*. Through Lemma 3.31, we know of a substitution $\sigma : \mathcal{L}(Y) \rightarrow \mathcal{L}(Y)$ for some finite set of variables $Y \supseteq X$.[2]

$$\vdash_\Lambda \sigma(\phi) \text{ iff } v \Vdash \phi \text{ for each } \phi \in \mathcal{L}(X).$$

As a consequence, σ is a *unifier* of every formula in $\mathrm{Th}(v)$. Naturally, if v arises from an *exact formula* ϕ, as in Theorem 3.33, then σ is a unifier of ϕ in particular. This is also reflected in Definition 3.32, the definition of exact formulae.

6.5 Example (Projective Unifier)
Let $\phi \in \mathcal{L}(X)$ be a *projective formula* . By Definition 4.21, this provides us with a substitution $\sigma : \mathcal{L}(X) \rightarrow \mathcal{L}(X)$ that, in addition to being a Λ-unifier of ϕ, satisfies:

$$\phi \vdash_\Lambda \psi \equiv \sigma(\psi) \text{ for all } \psi \in \mathcal{L}(X) \ .$$

Such a unifier is said to be a *projective unifier*. Note that Example 6.3 is a special case of this, and as can be argued through Lemma 4.11 and Theorem 4.22, all of this is a special case of Example 6.4.

The admissibility of a rule can be expressed solely in terms of an inclusion between the sets of unifiers of said rule's antecedent and its consequent, as stated formally in Lemma 6.6 below. We omit the proof, as it readily follows by simply spelling out the definitions involved.

6.6 Lemma
Let Λ be an intermediate logic. The following are equivalent for all $\Gamma, \Delta \subseteq \mathcal{L}(X)$:

(i) the rule Γ/Δ is *admissible* in Λ;

(ii) the inclusion $\bigcap_{\phi \in \Gamma} \mathsf{N}_\Lambda(\phi) \subseteq \bigcup_{\chi \in \Delta} \mathsf{N}_\Lambda(\chi)$ holds.

The reasoning in the following is reminiscent of Lemma 5.12, and in essence similar to Ghilardi (1999, Lemma 3).

6.7 Corollary
Let Λ be an intermediate logic. A formula ϕ is unifiable precisely if $\nvdash_\Lambda \phi \equiv \bot$

[2] Do note that Lemma 3.31 provides a substitution $\sigma : \mathcal{L}(X) \rightarrow \mathcal{L}(Y^*)$, but such a substitution can readily be extended to $Y := X \cup Y^*$. We make such switches tacitly throughout the following.

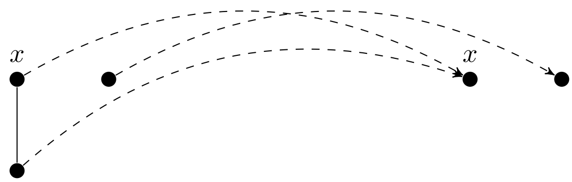


Figure 6.2.: An example of an LC-*projective model*, together with a definable, surjective map from the universal model of LC.

Proof. By Lemma 6.6, we know that $\mathsf{N}_\Lambda(\phi) = \varnothing$ precisely if $\phi/\bot$ is admissible. The latter is equivalent to $\vdash_{\mathsf{CPC}} \neg\phi$ by Lemma 2.77, proving the desired through *Glivenko's theorem.* □

Consider the formula $\phi := x \vee \neg x \in \mathcal{L}(X)$, where we take $X = \{x\}$. In CPC, this formula surely is unifiable. Even better, the formula is a theorem, and hence even the identity substitution serves as a unifier. It is easy to see that the identity substitution is the most general map, and we can describe $\mathsf{N}_{\mathsf{CPC}}(\phi)$ as the set of all those substitutions that are less general than it.

When moving to LC, the situation become slightly more interesting. It is easy to see that ϕ is not a theorem, but it certainly does have unifiers. Indeed, it is not hard to see that:

$$\sigma_\bot, \sigma_\top : \mathcal{L}(X) \to \mathcal{L}(X), \quad \sigma_\bot(x) := \bot, \quad \sigma_\top(x) := \top,$$

are both unifiers of ϕ. They are not very general, and they are incomparable.

On the other hand, consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{LC}}(X) \to \mathcal{P}(X)$ and the submodel v generated by $[\![\phi]\!]_{\mathsf{u}}$. The former comprises of the three points on the left-hand side of Fig. 6.2, and the latter are the two points on the right-hand side. The dotted lines between these two models indicate a definable map $f : \mathsf{u} \to v$. As in Example 3.30, we can spell out the substitution we know to correspond to this map through Lemma 3.8:

$$\sigma : \mathcal{L}(X) \to \mathcal{L}(X), \quad \sigma(x) := \neg\neg x.$$

Because $f \upharpoonright [\![\phi]\!]_{\mathsf{u}} = \mathrm{id}_v$, we know v to be LC-*injective*, hence σ is a projective unifier through Theorem 4.22. In Example 6.9 below, we show that any such projective unifier is a most general unifier, generalising our observations in Example 6.3. Concretely, though, it is easy to see that $\sigma_\bot$ and $\sigma_\top$ factor through σ via themselves.

The final situation we consider is that in IPC. The map f *cannot* be extended to a map starting at the universal model $\mathsf{u}^* : \mathsf{U}_{\mathsf{IPC}}(X)$. Indeed, if this were possible, then $[\![\phi]\!]_{\mathsf{u}^*}$ would have to be IPC-extendible by Lemma 4.11, which it clearly is not. As such, the substitution σ defined above *is not* a IPC-unifier of ϕ. This shows that there cannot be a most general IPC-unifier of ϕ, as noted by Ghilardi (1999, p. 859). We generalise this statement in Lemma 6.15.

As we have seen above, it may happen that a formula has a most general Λ-unifier, and it may well not. We close this section with some definitions to capture this behaviour a bit more precisely. The following all builds up to Definition 6.13, which specifies the *unification types*. First, we start with a generalisation of the notion of a most general unifier. A most general unifier is a maximal element in the preorder of unifiers. Roughly speaking, Definition 6.8 generalises this to a maximal subset that is an *anti-chain*. This definition is drawn from Ghilardi (1997, p. 734).

6.8 Definition ($\boldsymbol{\mu}$-set)
Let P be a preorder, and let $W \subseteq P$ be a subset. We say that W is a *μ-set* whenever:

(i) for all $p \in P$ there is a $w \in W$ such that $p \leq w$;

(ii) for all $w, s \in W$ we have $w \leq s$ if and only if $w = s$.

6.9 Example (Most General Unifier)
Consider an intermediate logic Λ, and let $\phi \in \mathcal{L}(X)$ be a *projective formula.* By Definition 4.21, there exists a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(X)$ that is a Λ-unifier of ϕ. Moreover, for every $\tau \in \mathsf{N}_\Lambda(\phi)$, we know that $\tau \leq^X_\Lambda \sigma$. Indeed, as in Example 6.3, one can readily derive that:

$$\vdash_\Lambda \sigma(\tau(x)) \equiv \tau(x) \text{ for all } x \in X. \tag{6.1}$$

This proves that σ is a *most general unifier*. In the just-introduced nomenclature, this means that $\{\sigma\}$ is a μ-set in $\mathsf{N}_\Lambda(\phi)$.

6.10 Example (Transparant Unifier)
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$ be a formula. Wroński (1995) defined a unifier $\sigma \in \mathsf{N}_\Lambda(\phi)$ to be *transparant* whenever:

$$\vdash_\Lambda \tau(\sigma(x)) \equiv \tau(x) \text{ for all } \tau \in \mathsf{N}_\Lambda(\phi) \text{ and } x \in X.$$

Through 6.1, it is easy to see that every *projective unifier* is transparent, too. The converse holds as well, see Słomczyńska (2012, p. 76).

All μ-sets of the same preorder are of the same cardinality, as shown in Lemma 6.11. We use this observation to note that Definition 6.12 is unambiguous.

6.11 Lemma
Let P be a preorder, and let $W, S \subseteq P$ be *μ-sets*. We have $|W| = |S|$

Proof. Pick any $w \in W$. There must be an $s \in S$ such that $w \leq s$, because S is a μ-set. Similarly, there exists a $w' \in W$ such that $s \leq w'$. This yields $w \leq s \leq w'$, hence $w' = w$ because W is a μ-set. This has shown $w \leq s$ and $w \leq s$. Consequently, there is a bijective correspondence between W and S, as desired. □

The following Definitions 6.12 and 6.13 are straightforward reformulations of Ghilardi (1997, pp. 734–735), cf. Siekmann (1989, p. 222) and Baader (1992, p. 153).

6.12 Definition (Types)
Let P be a non-empty preorder. We say that the type of P is *nullary*, *unary*, *finitary*, *infinitary* when the cardinality of every μ-set $W \subseteq P$ is zero, one, finite or infinite respectively.

The nullary type indicates an absence of μ-sets. An example of a preorder with nullary type are the integers, ordered in the usual manner. We order the types themselves, in a way such that the smaller the type, the smaller the μ-sets. Such an order would be:

$$\text{unary} < \text{finitary} < \text{infinitary} < \text{nullary}.$$

We now have ample terminology to give Definition 6.13.

6.13 Definition (Unification Type)
Let Λ be an intermediate logic. We say that Λ has *unitary*, *finitary*, *infinitary*, and *nullary* unification type respectively if the largest type of $\mathsf{N}_\Lambda(\phi)$ for any formula ϕ is unary, finitary, infinitary, and nullary respectively.

6.2. Unification in some intermediate logics

The role of projectivity in unification came to light through Ghilardi (1997), which, throughout the years, lead to many new results. Most prominently, Ghilardi (1999) established the unification type of IPC to be finitary. Building on this work, Iemhoff (2001b) provided a basis for the admissible rules of IPC. Unification types of many modal logics have been established through the method of Ghilardi (1997), let us but mention Dzik (2011), Dzik and Wojtylak (2012), Ghilardi (2000, 2004), and Ghilardi and Sacchetti (2004).

We have described *bases* for the intermediate logics BB_n and BD_2 in Section 5.3, all the while making extensive use of the description of *projective formulae* we gave in Section 4.1.3. It the following, we use some of the results we obtained towards describing bases in order to settle the unification types of these logics. Not to spoil the story, but they will turn out to be finitary.

Before we get to the actual story, we first show that the above machinery applies to the intermediate logic LC as well. This result is not at all new, see for instance Wroński (1995, Fact 3), Dzik and Wojtylak (2012, Theorem 2.13), and Ghilardi (2004, p. 108).

6.14 Theorem
Let Λ be an intermediate logic that extends LC. Now, Λ has *unary unification type.*

Proof. Consider an arbitrary formula $\phi \in \mathcal{L}(X)$. By the proof of Theorem 5.14, we know the admissible approximation $\mathsf{E}(\phi)$ to be either empty or equal to $\{\phi\}$, where ϕ is projective. In the former case, Corollary 6.7 shows that ϕ is not unifiable, so there is nothing left to prove. In the other case, we know that $\mathsf{N}_\Lambda(\phi)$ is of *unary type* due to Example 6.9. This proves the desired. □

It is not hard to show that the logics we are concerned with do not have unitary unification type. We provide a fairly general condition under which unitary unification is guaranteed to fail in Lemma 6.15. This condition is met by both BB_n and $\mathsf{BD}_2 + \mathsf{BW}_n$ for $n = 2, 3, \ldots, \omega$, as we have seen in Example 3.57 and Lemma 3.67 respectively. Naturally, it fails in LC and CPC, for otherwise a contradiction with Theorem 6.14 would readily arise.

6.15 Lemma
Let Λ be an intermediate logic that admits $\mathsf{DP}_2^{\neg\neg}$. The unification type of Λ is at least finitary.

Proof. Consider the formula $\phi := \neg x \vee \neg\neg x \in \mathcal{L}(X)$, where $X := x$. We claim that $\mathsf{N}_\Lambda(\phi)$ is not of *unary type*. First, observe that we have unifiers:

$$\sigma_{\neg x}, \sigma_{\neg\neg x} : \mathcal{L}(X) \to \mathcal{L}(X), \quad \sigma_{\neg x}(x) := \bot, \quad \sigma_{\neg\neg x}(x) := \neg\neg x \to x.$$

It is easy to see that σ_ψ is the most general unifier of ψ, for $\psi = \neg x, \neg\neg x$. We argue by contradiction, so suppose that $\mathsf{N}_\Lambda(X)$ is of unary type. This yields a $\sigma \in \mathsf{N}_\Lambda(X)$ such that, in particular, we have $\sigma_\psi \leq \sigma$ for $\psi = \neg x, \neg\neg x$. We know that the rule

$$\neg\neg x \vee x \big/ \{\neg\neg x, \neg x\}$$

is admissible, hence Lemma 6.6 shows that $\sigma \in \mathsf{N}_\Lambda(\psi)$ for some $\psi = \neg x, \neg\neg x$. Pick such a ψ, and see that $\vdash_\Lambda \sigma(z) \equiv \sigma_\psi(z)$ follows for each $\chi \in \mathcal{L}(X)$. But $\sigma_{\neg\neg x}$ is not a unifier of $\neg x$, and $\sigma_{\neg x}$ is not a unifier of $\neg\neg x$, a contradiction. This proves the desired. $\square$

As we saw in the proof of Theorem 6.14, the admissible approximation of a formula carries quite some information about the unifiers of said formula. This relation, in full generality, is given by the following lemma. Lemma 6.16 lends formal credence to the statement "when one is interested in unifiers, formulae need only be considered modulo admissible rules" we expressed in Section 1.2.2.

6.16 Lemma
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$ be a formula. Suppose $\mathsf{E}(\phi)$ is an *admissible approximation* of ϕ. We have that:

$$\mathsf{N}_\Lambda(\phi) = \bigcup_{\theta \in \mathsf{E}(\phi)} \mathsf{N}_\Lambda(\theta).$$

Proof. We know that $\phi \mathrel{|\!\sim} \mathsf{E}(\phi)$, hence the inclusion from left to right follows by Lemma 6.6. Moreover, for each $\theta \in \mathsf{E}(\phi)$ we know that $\theta \vdash_\Lambda \phi$. This proves the other inclusion. $\square$

Through Lemma 5.5, we know that admissible approximations can, without loss of generality, be assumed incomparable with respect to $\vdash_\Lambda$. Indeed, this lemma shows that any admissible approximation that violates the above posed constraint can be reduced to one that does not. Moreover, recall that admissible approximations of this form necessarily consist of formulae that are closed under all admissible rules, as we proved in Corollary 5.6. In the intermediate logics under consideration, IPC and BD_2 in particular, we know that Λ-*projective formulae* are precisely those formulae that are closed under all admissible rules.[3]

6.17 Lemma
Let Λ be an intermediate logic, and let $\phi \in \mathcal{L}(X)$ be a formula. Suppose that $\mathsf{E}(\phi)$ is an admissible approximation of ϕ, satisfying the following conditions:

(i) the elements of $\mathsf{E}(\phi)$ are incomparable with respect to $\vdash_\Lambda$;

(ii) each element $\theta \in \mathsf{E}(\phi)$ is Λ-projective, with Λ-projective unifier σ_θ.

The set $\{\sigma_\theta \mid \theta \in \mathsf{E}(\phi)\}$ is a μ-*set* for $\mathsf{N}_\Lambda(\phi)$.

[3] Recall that Λ-*projective formulae* and Λ-*extendible formulae* are the same in any intermediate logic with the *finite model property* through Theorem 4.24. In the case that Λ equals any amongst IPC, LC, BB_n and $\mathsf{BD}_2 + \mathsf{BW}_n$, we can describe the Λ-extendible formulae as those that are closed under a subset of Λ's the admissible rules of, as shown in Theorems 4.4 and 4.6 to 4.8 respectively.

Proof. By Lemma 6.16, we know a unifier of ϕ to be the same as a unifier of some element of $\mathsf{E}(\phi)$. There are two things left to prove, corresponding to (i) and (ii) of Definition 6.8. In order to prove the former, let $\sigma \in \mathsf{N}_\Lambda(\phi)$ be arbitrary. We know there is some $\theta \in \mathsf{E}(\phi)$ such that $\sigma \in \mathsf{N}_\Lambda(\theta)$. Because θ is projective, this ensures $\sigma \leq \sigma_\theta$, as argued in Example 6.9.

Now, suppose there are $\chi, \theta \in \mathsf{E}(\phi)$ such that $\sigma_\chi \leq \sigma_\theta$. In particular, this means that $\vdash_\Lambda \sigma_\chi(\theta)$. We know that $\chi \vdash_\Lambda \sigma_\chi(\theta) \equiv \theta$, and hence derive $\chi \vdash_\Lambda \theta$. By assumption, this proves $\theta = \chi$, as desired. ☐

We can recast Problem 9 in the language of Unification Theory. A full translation would deal with the so-called *parameters* $a_1, \dots, a_m$ that occur in this problem's original formulation; we merely consider the instance where $m = 0$.[4] Our reinterpretation is given in Theorem 6.18 below, formulated for the intermediate logics BB_n. Naturally, one retrieves the case IPC by setting $n = \omega$.

Note that Theorem 6.18, in particular, shows that it is decidable to check whether a formula is IPC. This property does not hold of every logic, see Wolter and Zakharyaschev (2008, Theorem 2.3) for a counterexample in the setting of modal logic. Another proof of this fact follows through the reasoning of Ghilardi (1999, p. 873), more details of which are given by Ghilardi (2002).

6.18 Theorem
Let $n = 2, 3, \dots, \omega$, and consider the intermediate logic $\Lambda := \mathsf{BB}_n$. There exists an algorithm that produces a finite *μ-set* for $\mathsf{N}_\Lambda(\phi)$ when given a formula $\phi \in \mathcal{L}(X)$ and a finite set of variables.

Proof. The algorithm is as follows. On input $\phi \in \mathcal{L}(X)$ and a finite X, we produce an *admissible approximation* $\mathsf{E}(\phi)$ of ϕ, as indicated by Theorem 5.33. By inspecting the proof of Theorem 5.33, it is not hard to see that this is an algorithmic process. Through Lemma 6.17, we know $\{\sigma_\theta \mid \theta \in \mathsf{E}(\phi)\}$ to be the desired μ-set, finishing the argument. ☐

Recall Theorem 3.38, which states that a rule in $\mathcal{L}(X)$ is admissible precisely if it is valid in every exact model on X. An exact model comes equipped with a definable map from the universal model on some finite set of variables Y. One may wonder whether there is any relation between the size of X and the size of Y. Through the correspondence between free Heyting algebras and universal models as given in Theorem 3.22, it is easy to see that this question amounts to Problem 10.

[4] We refer to Rybakov (1990b, 1991b, 1992) for a treatment of unification with parameters in the intermediate logic IPC.

To re-iterate, Problem 10 asks for a function $f : \mathbb{N} \to \mathbb{N}$ such that a rule expressed in n many variables is refutable on the free Heyting algebra on $f(n)$ many variables. More precisely, if ϕ/Δ is a rule expressed in $\mathcal{L}(\{1, \dots, n\})$, does there exist a valuation:

$$g : \mathsf{F}(\{1, \dots, n\}) \to \mathsf{F}(\{1, \dots, f(n)\}),$$

such that $g \vDash \phi$ yet $g \nvDash \chi$ for all $\chi \in \Delta$? We have ample machinery to prove that such a function exists. The function is not at all complicated; the identity map suffices. We show this in Theorem 6.19 below.

6.19 Theorem
Let X be a set of variables. The following are equivalent for all $\Gamma, \Delta \subseteq \mathcal{L}(X)$:

(i) the rule Γ/Δ is an *admissible rule* for IPC.

(ii) for every IPC-projective formula $\psi \in \mathcal{L}(X)$, we have:

$$\psi \vdash_\Lambda \phi \text{ for all } \phi \in \Gamma \text{ implies } \psi \vdash_\Lambda \chi \text{ for some } \chi \in \Delta.$$

Proof. Suppose (i) holds, and let $\psi \in \mathcal{L}(X)$ be an IPC-projective formula. Because $\psi \vdash_{\mathsf{IPC}} \phi$ for all $\phi \in \Gamma$, transitivity ensures us that $\psi \mathrel{|\!\!\sim}_{\mathsf{IPC}} \Delta$. By Lemma 5.13, we know that ψ is its own *admissible approximation*, hence the above ensures $\psi \vdash^{\mathsf{min}}_{\mathsf{IPC}} \Delta$. This proves (ii).

Conversely, suppose (ii) holds. Through Theorem 5.35, we know there to be an admissible approximation Θ of $\bigwedge \Gamma$, consisting only of projective formulae. Fix some formula $\theta \in \Theta$. Naturally, it satisfies $\theta \vdash_{\mathsf{IPC}} \phi$ for all $\phi \in \Gamma$. This yields $\theta \vdash^{\mathsf{min}}_{\mathsf{IPC}} \Delta$ by assumption. As Θ is an admissible approximation of $\bigwedge \Gamma$, this means that $\bigwedge \Gamma \mathrel{|\!\!\sim}_{\mathsf{IPC}} \Delta$. From here, the desired is immediate. □

We close this chapter by settling the unification type for some intermediate logics. The arguments we need to provide are far from long; the desired description readily follows through the results we obtained earlier.

6.20 Corollary
The unification type of BB_n is *finitary* for each $n = 2, 3, \dots, \omega$.

Proof. We know that the unification type of BB_n is at most finitary due to Theorem 6.18. Finally, Lemma 6.15 shows that the unification type cannot be *unary*, proving the desired. □

The following is but a special case of Corollary 6.20. In the above, we have settled the unification type through an adaptation on the approach of Ghilardi (1999), proceeding via projectivity. Let us note that there exists another way to settle the unification type, per the machinery of Rybakov (1984a), as explained in Section 4.4. We do not explore this line of reasoning here, but do refer to Odintsov and Rybakov (2013) and Rybakov (2013a,b) for more details. In particular, the following occurs as Rybakov (2013b, Theorem 4.4).

6.21 Corollary
The intermediate logic IPC has *finitary unification type.*

The final result in this chapter is Theorem 6.22. Our proof proceeds via the machinery described here and in Chapter 5. Alternatively, one can see this to hold through Ghilardi (2004, p. 110).

6.22 Theorem
The unification type of $\mathsf{BD}_2 + \mathsf{BW}_n$ is *finitary* for each $n \in \mathbb{N}$.

Proof. Immediate via Lemmas 6.15 and 6.17 and Theorem 5.33. $\square$

7
Characterisations

Are there intermediate logics that can be characterised as the maximal intermediate logic where a certain set of formulae are all *theorems*? Of course there are, or rather, is: CPC is the only logic that can be described in this manner. Analogously, one could ask which intermediate logics can be described as the maximal intermediate logic where certain of rules are *admissible*. In this chapter, we focus on this latter, more salient, question.

We consider several examples, most interestingly the logics BB_n. The case of BB_0, which simply amounts to CPC, is the most straightforward. Indeed, as it is the maximal intermediate logic, it can readily be described in the above manner by letting the set of rules be empty. This is yet another place where the adage "less is more" holds true; the smaller the set of rules in the logic's description, the better. In this sense, the description of any other intermediate logic is bound to be worse.

A particularly bad case can be found in BB_ω, the intermediate logic IPC. As mentioned in Section 1.2.1, and as we revisit in more detail in Section 7.4, it was conjectured by Łukasiewicz (1952, p. 209) that IPC can be described as the maximal intermediate logic in which DP is admissible.[1] This conjecture was originally disproven by Kreisel and Putnam (1957) through the introduction of KP. Later, Wroński (1973, 1974) showed there to be continuum many intermediate logics that admit DP. In fact, there does not exist any intermediate logic that could be described as the maximal intermediate logic with the disjunction property. This readily follows through Ferrari and Miglioli (1995a,b), who prove that there are continuum many such logics, evaporating any hope of unicity.[2]

One could enlarge the set of rules under consideration, in the hope that more data might characterise IPC. Skura (1989b) and Iemhoff (2001a) independently did just this. Whereas Skura gave a refutation system, Iemhoff approached the problem directly via admissibility.[3] The point of this chapter is to show how their approaches can be unified, and generalised to suit all the logics of bounded branching BB_n.

The key observation of this chapter is that in some logics, a formula is derivable precisely if all the admissible consequences of all its substitution instances are derivable in the minimal multi-conclusion extension of a much stronger logic. In this case, we say that the weaker logic is *admissibly expressible* in the stronger logic. Throughout this chapter, we only really use admissible expressibility in CPC. A precise formalisation of the above idea appears in Theorem 7.21, instantiated to the special case of ML. In Definition 7.23, we present it in full generality. It appears that this notion

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[1] Naturally, his conjecture was not phrased in this manner. It does, however, readily follow from the formulation as given by Kreisel and Putnam (1957, p. 74):

> "K sei eine Klasse von Formeln des IAK, die abgeschlossen ist in bezug auf Einsetzung und Abtrennung, alle Theoreme des IAK enthält und entweder A oder B enthält, wenn sie $A \vee B$ enthält. Dann ist K entweder genau die Klasse der Theoreme des IAK oder K ist inkonsistent d. h. enthält alle Formeln des IAK."

Here, IAK is what we today call IPC, the "intuitionistische Aussagenkalkül" of Heyting (1930). See the review of Kreisel and Putnam (1957) by Robinson (1958) for more details in English.

[2] The absence of a maximal intermediate logic with the disjunction property has been known much longer, see Kirk (1982, Theorem 2) for a proof. Let us note that this proof makes use of the intermediate logics KP and BB_2, referring to the latter as D_1.

[3] Interestingly, both made use of the semantics for IPC introduced by Jaśkowski (1936): the so-called *Jaśkowski sequence*. A description of this sequence and a proof of its completeness have been given by Rose (1953, Section 5). The Jaśkowski sequence, quite crucially, is a sequence that satisfies the precondition in the following Corollary by Smoryński (1973, p. 351):

> "Let $\{(K_n, \leq_n)\}_n$ be a sequence of finite trees with the property that every finite tree $(K, \leq)$ can be embedded as a subtree of some $(K_n, \leq_n)$. Then [IPC] is complete for the sequence $(K_n, \leq_n)$."

McKay (1967a) considered a modification of the Jaśkowski sequence that does not have the property; all his trees are binary. Consequently, each tree in his sequence is a model of BB_2, as pointed out by Segerberg (1973). We refer to Surma et al. (1974, 1975) and Smoryński (1973, pp. 348-352) for further details on the Jaśkowski sequence.

has not yet been discussed in the literature. We hope that expressing it here may lead to further results, similar in nature to Theorem 7.33, which characterises BB_n as the greatest intermediate logic that admits $\overline{\mathsf{D}}_n$ for each $n = 2, 3, \ldots$.

Let us expand on the composition of this chapter. In Section 7.1, we introduce the basic semantic notions we employ throughout this chapter. Most prominently, we show how the non-validity of a formula in a frame can be translated into a statement about derivability. We give a criterium, called *maximal separability*, under which this latter statement pertains to a formula of a particularly convenient form. The frames of the Jaśkowski sequence satisfy said criterion.

We take a brief detour through ML in Section 7.2. We re-cast a characterisation of Medvedev's logic by Levin (1969) in terms of admissible expressibility. In this, we lay the groundwork for re-proving a result by Skura (1992a), which we finalise in Theorem 7.41. The purpose of this section is to provide a non-trivial, yet simple motivating example for the notion of admissible expressibility.

In Section 7.3, we define and explore admissible expressibility. We consider maximally separable frames endowed with the valuation described in Definition 7.8. These models are *order-defined*, and as such one can consider the not-down formulae as specified in Definition 3.9. We compute the *admissible approximation* of said formulae, and show that these approximations consist of formulae that are not derivable in CPC. Through this observation, we can readily prove BB_n to be admissibly expressible in CPC for all $n \neq 1$. This readily leads to a characterisation of BB_n as the maximal intermediate logic that admits $\overline{\mathsf{D}}_n$, which we present in Theorem 7.33.

We close this chapter with Section 7.4, in which we discuss refutation systems. These systems allow one to inductively define the non-theorems of a logic. With the above described machinery, we can readily provide such refutation systems for ML and BB_n. The refutation systems we construct provide a very succinct and syntactic characterisation of these logics. In particular, we re-prove the result by Skura (1989b) and present a refutation system for IPC.

7.1. Semantics to syntax

The purpose of this section is to describe and motivate a condition on frames under which the not-downset formula of said frame's root is of a particularly convenient form. A natural reaction to the above would be to question our peculiar interest in not-downset formulae and their shapes. Indeed, why should they matter? The answer lies in the bridge they form between semantics on the one hand, and syntax on the other.

Consider the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$. Through Corollary 3.24, we know that for all $\phi \in \mathcal{L}(X)$ and $p \in \mathsf{U}_{\mathsf{IPC}}(X)$, we have that:

$$\mathsf{u}, p \nVdash \phi \text{ iff } \phi \vdash_{\mathsf{IPC}} \mathsf{nd}\, p. \tag{7.1}$$

Through Corollary 3.23, the above means that a formula is not a theorem of IPC precisely if it implies $\mathsf{nd}\, p$ for some $p \in \mathsf{U}_{\mathsf{IPC}}(X)$. Naturally, there are many such points, so this is not much of a characterisation.

The stronger the logic, the more we know about the point which falsifies the formula. In CPC, for instance, the points in the universal model are finite in number. Moreover, Lemma 7.1 shows that not-down formulae of *non-maximal points* of any universal model are theorems of CPC.

7.1 Lemma
Let Λ be an intermediate logic with the finite model property, and consider the universal model $\mathsf{u} : \mathsf{U}_{\Lambda}(X) \to \mathcal{P}(X)$. For all $p \in \mathsf{U}_{\Lambda}(X)$, we have that:

$$p \in \max(\mathsf{U}_{\Lambda}(X)) \text{ iff } \nvdash_{\mathsf{CPC}} \mathsf{nd}\, p.$$

Proof. Let $p \in \mathsf{U}_{\Lambda}(X)$ be given. Suppose that $p \in \max(\mathsf{U}_{\Lambda}(X))$. As $\mathsf{u}, p \nVdash \mathsf{nd}\, p$, this shows $\nvdash_{\mathsf{IPC}} \mathsf{nd}\, p$. Through the proof of Theorem 3.10, it follows that:

$$\vdash_{\mathsf{IPC}} \mathsf{nd}\, p \equiv \left(\neg \mathsf{up}\, p\right),$$

as explicated in (3.9). Through *Glivenko's theorem*, we thus derive $\nvdash_{\mathsf{CPC}} \mathsf{nd}\, p$.

Conversely, suppose $\vdash_{\mathsf{CPC}} \mathsf{nd}\, p$ yet $p \in \max(\mathsf{U}_{\Lambda}(X))$. We know the latter to entail $\mathsf{u}, p \Vdash \mathsf{CPC}$, as follows by Corollary 2.76. This yields $\mathsf{u}, p \Vdash \mathsf{nd}\, p$, which ensures $p \nleq p$ by Definition 3.9, the definition of $\mathsf{nd}\, p$. Yet this is a clear contradiction, proving the desired. □

Recall Lemma 2.37, which illustrates that one can endow any Kripke model with a concrete valuation. Whenever the frame at hand is finite, it readily follows through Theorem 3.10 that the resulting model is *order-defined* as well. The purpose of this section is to provide some conditions under which one can choose a set of variables that is potentially much smaller, which we see in Definition 7.7.

Consider an arbitrary image-finite, order-defined model $v : P \to \mathcal{P}(X)$. We know there to be an isomorphism of Kripke models $f : v \to u$, where $u = \mathsf{u} \restriction U$ is a restriction of the universal model $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$ to some finite upset U, as proven in Lemma 3.25. Now, there are two natural settings in which one could interpret the expression $\mathsf{up}\,p$ for any $p \in P$: either within v or within u. It stands to reason that there ought to be no discernible difference between these two formulae, as the respective ambient models are isomorphic. We present the following Lemmas 7.2 and 7.3, which together show exactly this. Note that this is the first point at which we use the more elaborate and detailed notation introduced at Definition 3.9, there simply was no prior need for this precision.

7.2 **Lemma**
Let $v : P \to \mathcal{P}(X)$ and $u : Q \to \mathcal{P}(X)$ be Kripke models, and let $f : v \to u$ and $g : u \to v$ be maps of Kripke models. Suppose that $g \circ f = \mathrm{id}_v$. Now, we have for all $p \in P$ that:

$$v \Vdash \mathsf{nd}_v\, p \equiv \mathsf{nd}_u\, f(p). \tag{7.2}$$

Proof. Let $k \in P$ be arbitrary. Consider the following equivalences:

$$\begin{aligned} v, k \nVdash \mathsf{nd}_v\, p \ \text{ iff } k \leq p \text{ iff } f(k) \leq f(p) \text{ iff } u, f(k) \nVdash \mathsf{nd}_u\, f(p) \\ \text{iff } v, (g \circ f)(k) \nVdash \mathsf{nd}_u\, f(p), \end{aligned}$$

proving the desired. □

7.3 **Lemma**
Let $v : P \to \mathcal{P}(X)$ be an image-finite concrete model, and let $U \subseteq P$ be an upset. Both v and $u := v \restriction U$ are *order-defined*, and for all $p \in U$ we have:

$$\vdash_{\mathsf{IPC}} \mathsf{up}_v\, p \equiv \mathsf{up}_u\, p \text{ and } \vdash_{\mathsf{IPC}} \mathsf{nd}_v\, p \equiv \mathsf{nd}_u\, p.$$

Proof. Straightforward through an inspection of the proof of Theorem 3.10. □

Recall the valuation d_P as introduced in Lemma 2.37. This valuation allows one to translate a negative, semantic property, the falsification of a formula on a frame, into a positive, syntactic property: a particular formula dependent only on the frame has a valid substitution instance. We treat this observation formally in Lemma 7.5. Analogues of Lemma 7.5 below are scattered throughout the literature on refutation systems. For the sake of reference, let us mention a few examples. In the setting of intermediate logics, examples include: Skura (1989b, Lemma 1), Skura (1990a), Skura (1992a, Theorem 2.2), and Skura (1999, Section 4.2). A similar lemma for modal logics is covered by: Skura (1994, Lemma 3.1), Skura (1995a, Lemma 4), and Goranko (1994, Lemma 1.3.3).

7.4 Lemma
Let $v : P \to \mathcal{P}(X)$ be a finite concrete model. To every model $u : P \to \mathcal{P}(Y)$ there is a substitution $\sigma : \mathcal{L}(Y) \to \mathcal{L}(X)$ such that:

$$v, p \Vdash \sigma(\phi) \text{ iff } u, p \Vdash \phi \text{ for all } p \in P \text{ and } \phi \in \mathcal{L}(Y).$$

Proof. By Lemma 3.11, we know the identity map $\mathrm{id}_P : P \to P$ to be construable as a definable map $v \to u$. Hence Lemma 3.8 gives rise to the desired substitution. □

7.5 Lemma
Let P be a finite rooted Kripke frame. For every formula $\phi \in \mathcal{L}(X)$, we know that the following are equivalent:

(i) $P \nVdash \phi$;

(ii) there exists a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(P)$ such that $\sigma(\phi) \vdash_{\mathsf{IPC}} \mathsf{nd}_{\mathsf{d}_P} P$.

Proof. Consider the concrete model $\mathsf{d}_P : P \to \mathcal{P}(P)$ of Lemma 2.37. We know of a $p \in \mathsf{U}_{\mathsf{IPC}}(P)$ such that d_P is isomorphic to $u := \mathsf{u} \restriction (\uparrow p)$ via Lemma 3.25. Through Lemmas 7.2 and 7.4, it follows that:

$$\mathsf{u} \Vdash \mathsf{nd}_{\mathsf{u}}\, p \equiv \mathsf{nd}_{\mathsf{d}_P} P,$$

hence Corollary 3.23 shows these formulae to be provably equivalent in IPC.

Suppose (i) holds. This yields a valuation $v : P \to \mathcal{P}(X)$ such that $v \nVdash \phi$. By Lemma 7.4, we know of a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(P)$ such that $\mathsf{d}_P \nVdash \sigma(\phi)$. This shows that $\mathsf{u}, p \nVdash \sigma(\phi)$, whence Corollary 3.24 shows (ii) through the above equivalence. Conversely, suppose (ii) holds. Suppose that $P \Vdash \phi$ holds. In particular, this implies that $\sigma^*(\mathsf{d}_P) \Vdash \phi$. We subsequently see $\mathsf{u}, p \Vdash \sigma(\phi)$, a contradiction via Corollary 3.24. We thus have proven (i), as desired. □

At this point, we point out the exceedingly convenient shape of the formulae $\mathsf{nd}\, p$ whenever p is covered by an anti-chain of size unequal to one. We do not use this lemma until Lemma 7.27.

7.6 Lemma
Let $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$ be the IPC-universal model on finitely many variables X, let $p \in P$, and let $W \subseteq P$ be an anti-chain such that $W \kappa\, p$ and $|W| \neq 1$. We now know that:

$$\vdash_{\mathsf{IPC}} \left(\mathsf{up}\, p \to \bigvee_{w \in W} \mathsf{nd}\, w \right) \equiv \mathsf{nd}\, p.$$

Proof. Through Corollary 3.23, it suffices to prove that the above formula holds in u. This amounts to proving the following equivalence for all $q \in \mathsf{U}_{\mathsf{IPC}}(X)$:

(i) for all $k \geq q$ we have that $p \leq k$ entails $k \nleq w$ for some $w \in W$;

(ii) $q \nleq p$.

Suppose (i) holds and $q \leq p$. Instantiating (i) with $k := p$ yields $p \nleq w$ for some $w \in W$, *quod non.* This proves (ii) to hold.

Conversely, suppose that (ii) holds, and suppose there is a $k \geq p$ with $q \leq k$ that satisfies $k \leq w$ for all $w \in W$. Because $k \geq p$ and $W \kappa p$, we know that either $p = k$ or $k \in \uparrow W$. In the former case, we arrive at $q \leq p$, contradicting the assumed $q \nleq p$. The latter case, there is a $w^+ \in W$ such that $w^+ \leq k$. We have $k \leq w^+$ by assumption, and so $k \in W$ follows. There must be a point $w^- \in W$ with $w^- \neq k$. Yet, on the other hand, we know $k \leq w^-$, violating the assumption that W is an anti-chain. All cases reach a contradiction, so (i) follows. □

Recall that to each partial order one can assign a valuation such that the resulting model is concrete, as shown in Lemma 2.37. The valuation there is such that all elements of the underlying order are assigned distinct variables. In Definition 7.7, we specify a kind of Kripke frame for which we can construct a more "economic" valuation, taking values only in the *maximal points* of the frame at hand. There are situations in which such a valuation suffices to distinguish between all elements, for instance when the underlying partial order is a *proper tree*, as shown in Lemma 7.13.

The advantage of such a valuation should become clear by Lemma 7.27. Roughly speaking, the formulae $\mathsf{up}\,p$ and $\mathsf{nd}\,p$ of points p in such models are of a form amendable to manipulation by admissible rules. Details on this follow later, in particular in Lemma 7.27.

7.7 **Definition (Maximally Separable)**
Let P be a partial order. We say that the Kripke frame P is *maximally separable* if for all $p, q \in P$ we have that:

$$p \leq q \text{ iff } \mathsf{max}(\uparrow q) \subseteq \mathsf{max}(\uparrow p).$$

Traces of the following definition can be found in the literature. Smoryński (1973, Lemma 5.3.9), roughly speaking, states that the frames P of the *Jaśkowski sequence* are *order-defined* whenever one endows them with the valuation m_P as defined below.

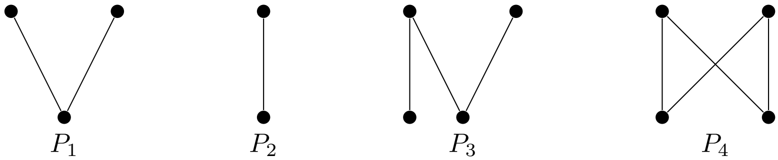

Figure 7.1.: Examples of maximal separability and distinguishability.

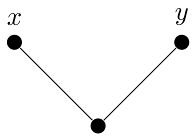

Figure 7.2.: Example of a finite, rooted, *maximally separable* frame P endowed with the valuation $\mathsf{m}_P : P \to \mathcal{P}(\mathsf{max}(P))$, in the understanding that $\mathsf{max}(P) = \{x, y\}$.

7.8 Definition (Maximally Distinguishable)
Let P be a partial order, and define the model $\mathsf{m}_P : P \to \mathcal{P}(\mathsf{max}(P))$ by:

$$\mathsf{m}_P : P \to \mathcal{P}(\mathsf{max}(P)), \quad p \mapsto \begin{cases} \{p\} & \text{if } p \in \mathsf{max}(P), \\ \varnothing & \text{otherwise.} \end{cases}$$

The Kripke frame P is said to be *maximally distinguishable* whenever m_P is *concrete*.

7.9 Example
Consider the models $P_1, \dots, P_4$ as depicted in Fig. 7.1. These are examples and non-examples of the notions introduced in Definitions 7.7 and 7.8. The frame P_1 is maximally separable and P_1, P_2 and P_3 are maximally distinguishable. Note that P_2, P_3 and P_4 are most certainly *not* maximally separable, and the latter is not maximally distinguishable.

7.10 Example
The frame P drawn in Fig. 7.2 is maximally separable. Endowing this model with the valuation m_P would result in the valuation as drawn there, in the understanding that the upper-left point is called x and the upper-right point goes by y.

Recall that IPC is complete with respect to the *Jaśkowski sequence*, a particular sequence of *proper trees*.[3] In Lemma 7.13, we prove that proper finite trees are maximally separable.[4] To be fully precise, we precisely specify what we mean by a tree and a proper frame in Definition 7.11.

7.11 Definition
A Kripke frame P is said to be a *tree* whenever it is both rooted, and for all $p, q, k \in P$ with $p, q \leq k$ we have $p \leq q$ or $q \leq p$. We say that a frame is *proper* whenever $\{p\} \kappa q$ implies $p = q$ for all $p, q \in P$.

7.12 Lemma
Let P be a *maximally separable* Kripke frame. Now, P is *proper*.

Proof. Suppose that P is maximally separable, and furthermore suppose that P is *not* proper. This yields $p, q \in P$ with $\{p\} \kappa q$ yet $p \neq q$. Consider any $m \in \mathsf{max}(P)$ and assume $q \leq m$. Because $\{p\} \kappa q$, we know that either $p \leq m$ or $q = m$. In the latter case, we obtain a contradiction because $m < p$, and m is known to be maximal. In the former case, see that maximal separability ensures $p \leq q$. As we know $q \leq p$, this yields a contradiction, proving the desired. □

7.13 Lemma
Every finite *tree* is *maximally separable* if and only if it is *proper*.

Proof. Let P be a finite tree. From right to left, suppose that P is proper. We claim that if $p, q \in P$ are such that $q < p$ then there is a $m \in \mathsf{max}(P)$ such that $q \leq m$ yet $p \nleq m$. This can readily be proven by induction along $q \in P$. Indeed, suppose that $q < p$, and let W be the set of *immediate successors* of q as in Example 2.31. Because P is proper, we know there to be a $w \in W$ with $w \neq p$. By induction, we can choose a point $m \in \mathsf{max}(P)$ such that $w \leq m$. If $p \leq m$ then w and p are comparable because P is a tree, a contradiction. This proves the claim.

Now, suppose that $p, q \in P$ are such $\mathsf{max}(\uparrow q) \subseteq \mathsf{max}(\uparrow p)$ and assume that $p \nleq q$. There must be a maximal $m \in \mathsf{max}(P)$ with $q \leq m$, and so $p, q \leq m$ holds by assumption. Because P is a tree, it follows that $p \leq q$ or $q \leq p$, which yields $q < p$. By the above paragraph, there exists a point $m^* \in \mathsf{max}(P)$ with $q \leq m^*$ and $p \nleq m^*$, a contradiction. This proves that P is maximally separable. As the implication from right to left follows via Lemma 7.12, we have proven the desired. □

We close this section with an argument showing that all *maximally separable* frames are *maximally distinguishable*.

[4] More generally speaking, any *meet-semilattice* where each element can be expressed as a meet of maximal elements is maximally separable.

7.14 Lemma
Let P be a *maximally separable* frame, and consider $\mathsf{m}_P : P \to \mathcal{P}(\mathsf{max}(P))$. For all points $p, q \in P$ we have:

$$p \leq q \text{ iff for all } m \in \mathsf{max}(P) - \uparrow p \text{ it holds that } q \Vdash \neg m. \tag{7.3}$$

In particular, the frame P is *maximally distinguishable*.

Proof. The implication from left to right in (7.3) is clear. Indeed, suppose $p \leq q$ and take some $k \geq q$ such that $k \Vdash m$ with $m \in \mathsf{max}(P) - \uparrow p$. It follows that $k = m$, and so $q \leq m$. Consequently, we know $p \leq q \leq m$, a contradiction.

In order to prove the other direction, we assume the right-hand side of (7.3). Let $m \in \mathsf{max}(P)$ to be such that $q \leq m$. When we can prove that $p \leq m$, then we are done by maximal separability. If $p \nleq m$ then $q \Vdash \neg m$ follows by the right-hand side of (7.3). We derive $m \Vdash \bot$, a contradiction. This proves the desired.

For the final statement, assume $p, q \in P$ are such that $\mathsf{m}_P(p) = \mathsf{m}_P(q)$ and for all $k \in P - \{p, q\}$ we have $p \leq k$ if and only if $q \leq k$. It is easy to verify that for each formula ϕ we have $p \Vdash \phi$ precisely if $q \Vdash \phi$. The desired is now immediate by the equivalence proven above. □

7.2. Characterising Medvedev's logic

Before we continue along our main line of reasoning, let us first spend a few words on Medvedev's logic. The purpose of this intermezzo is to motivate the notion of *admissible expressibility* we introduce in Definition 7.23. This notion is a crucial component towards characterising intermediate logics by means of their admissible rules. We recast a well-known characterisation of ML in terms that can readily be seen as a special case of admissible expressibility, making use of the machinery described in the above section. In the subsequent section, we provide similar characterisations of the intermediate logics BB_n.

The main result of this section is not at all new. In fact, it is but a reformulation of Maksimova (1986, Theorem 5) couched in terms introduced above. Our argument employs a characterisation due to Levin (1969) as described by Wojtylak (2004). Theorem 7.21 eventually leads to a *refutation system of* ML in Theorem 7.41, and this refutation system is equal to the one given by Skura (1992a, Theorem 6.1). Indeed, this section contains no novel results whatsoever, but the composition does provide intuitive motivation for our notion of *admissible expressibility*.

Before we proceed to give our characterisation, we need some additional machinery and background information. We start with the intermediate logics at hand. Recall KP, ND_n, and ND_ω as discussed to some extent in Section 2.4. In particular, we have shown ML and KP to admit the rule DP in Example 3.57. Moreover, we know there to be some inclusions between these logics. Lemma 7.15 collates these results, and adds two additional inclusions.

7.15 Lemma
For all $n \in \mathbb{N}$, we know that:

$$\mathsf{ND}_n \subseteq \mathsf{ND}_\omega \subseteq \mathsf{KP} \subseteq \mathsf{ML}.$$

Proof. The first inclusion can be readily seen to hold through the very definition of ND_ω, as it was given in (2.14). The same definition shows that the second inclusion amounts to proving that $\mathsf{ND}_n \subseteq \mathsf{KP}$ holds for all $n = 1, 2, \ldots$. One can readily prove this through induction along $n \in \mathbb{N}$. We know that $\mathsf{ND}_1 = \mathsf{IPC}$, so the base-case clearly holds. Now, suppose that $\mathsf{ND}_n \subseteq \mathsf{KP}$ holds. We can readily derive that:

$$\begin{aligned} \neg z \to \bigvee_{i=1}^{n+1} \neg x_i &\vdash_{\mathsf{KP}} \left(\neg z \to \bigvee_{i=1}^{n} \neg x_i \right) \vee (\neg z \to \neg x_{n+1}), \\ &\vdash_{\mathsf{KP}} \bigvee_{i=1}^{n+1} (\neg z \to \neg x_i), \end{aligned}$$

proving the inclusion $\mathsf{ND}_{n+1} \subseteq \mathsf{KP}$.

Finally, let us prove $\mathsf{KP} \subseteq \mathsf{ML}$. We show that each Medvedev frame, as defined in Definition 2.90, is of divergence n for all $n \in \mathbb{N}$, after which the desired follows through Lemma 2.85. To this end, let $X \neq \varnothing$ be a finite set, let $W \subseteq \max(\mathsf{B}(X))$, and let $p \in \mathsf{B}(X)$ be such that $W \subseteq \max(\uparrow p)$. We wish to show that there exists an element $q \in \mathsf{B}(X)$ satisfying both $p \leq q$ and $\max(\uparrow q) = W$. Such an element always exists, simply take $q := \bigcup W$, and see that it satisfies all requirements. $\square$

Through *Glivenko's theorem*, we know that $\vdash_{\mathsf{IPC}} \neg\phi$ holds if and only if $\vdash_{\mathsf{CPC}} \neg\phi$ for any formula ϕ. Lemma 7.16 generalises this statement to arbitrary disjunctions of negated formulae.

7.16 Lemma
Let Λ be an intermediate logic that admits DP, and let $\Delta \subseteq \mathcal{L}(X)$ be finite. We have that:

$$\vdash_\Lambda \bigvee_{\chi \in \Delta} \neg\chi \text{ if and only if } \vdash_{\mathsf{CPC}} \neg\chi \text{ for some } \chi \in \Delta. \tag{7.4}$$

Proof. Suppose $\vdash_\Lambda \bigvee_{\chi \in \Delta} \neg\chi$. It follows that $\vdash_\Lambda \neg\chi$ for some $\chi \in \Delta$, which in turn implies $\vdash_{\mathsf{CPC}} \neg\chi$. Conversely, if $\vdash_{\mathsf{CPC}} \neg\chi$, then $\vdash_{\mathsf{IPC}} \neg\chi$ follows through *Glivenko's theorem*. This proves $\vdash_\Lambda \bigvee_{\chi \in \Delta} \neg\chi$. □

Formulae that are of the form as the left-hand in (7.4) are of particular interest; we capture these with Definition 7.17. Note that our definition of disjunctive-negative formulae differs from that of Maksimova (1986). Our definition is more lax, incorporating formulae that are equivalent to a formula of the appropriate form within the ambient logic, whereas Maksimova (1986, p. 71) requires syntactic equality, and additionally demands $|\Delta| \geq 1$.

7.17 Definition (Disjunctive Negative)
Let Λ be an intermediate logic, and let ϕ be a formula. We say that ϕ is *disjunctive-negative in* Λ whenever there is some finite set of formulae Δ such that:

$$\vdash_\Lambda \phi \equiv \left(\bigvee_{\chi \in \Delta} \neg\chi \right).$$

With Definition 7.18, we capture those formulae where each variable occurs within the scope of a negation. Such *essentially negative* formulae are disjunctive-negative in ND_ω, as we show in Lemma 7.19.

7.18 Definition (Essentially Negative)
A formula $\phi \in \mathcal{L}(X)$ is said to be *essentially negative* whenever it is an element of the set $\mathcal{L}^\neg(X)$ as defined by the *Backus–Nauer form*:

$$\begin{aligned}\mathcal{L}^\neg(X) ::= \top \mid \bot \mid \neg\mathcal{L}(X) \mid \mathcal{L}^\neg(X) \wedge \mathcal{L}^\neg(X) \\ \mid \mathcal{L}^\neg(X) \vee \mathcal{L}^\neg(X) \mid \mathcal{L}^\neg(X) \to \mathcal{L}^\neg(X).\end{aligned}$$

A substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ is said to be *essentially negative* if its image is contained in $\mathcal{L}^\neg(Y)$.

7.19 Lemma
Let $\phi \in \mathcal{L}(X)$ be a formula, and let $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ be an *essentially negative* substitution. There is a finite $\Delta \subseteq \mathcal{L}(Y)$ such that:

$$\vdash_{\mathsf{ND}_\omega} \sigma(\phi) \equiv \bigvee_{\chi \in \Delta} \neg\Delta. \tag{7.5}$$

Moreover, $\vdash_{\mathsf{KP}} \sigma(\phi)$ precisely if $\vdash_{\mathsf{ML}} \sigma(\phi)$.[5]

[5] Let us remark that through Maksimova (1986, Proposition 6) we might also replace this last statement with: for all intermediate logics Λ above ND_ω with the disjunction property one has that $\vdash_\Lambda \sigma(\phi)$ holds precisely if $\vdash_{\mathsf{ND}_\omega} \sigma(\phi)$. The proof is analogous, and the original statement may be retrieved through Lemma 7.15.

Proof. The former statement can be proven by means of a straightforward inductive argument along the structure of ϕ. Let us focus on the second statement. The implication from left to right is immediate by Lemma 7.15. To prove the converse, suppose that $\vdash_{\mathsf{ML}} \sigma(\phi)$. By Lemma 7.15 we know $\mathsf{ND}_\omega \subseteq \mathsf{ML}$, hence (7.5) and Lemma 7.16 combine to prove that there is some $\chi \in \Delta$ such that $\vdash_{\mathsf{CPC}} \neg\chi$. Another application of Lemma 7.16 shows that $\vdash_{\mathsf{KP}} \bigvee_{\chi\in\Delta} \neg\chi$. We know $\mathsf{ND}_\omega \subseteq \mathsf{KP}$ through Lemma 7.15, hence combining the previous with (7.5) yields the desired. $\square$

Fix a finite *maximally separable* frame P, and endow it with the valuation m_P. Through Lemma 7.14, we know for all $p, q \in P$ that:

$$p \le q \text{ iff } q \Vdash \neg \bigvee \big(\mathsf{max}(P) - \uparrow p\big).$$

Intuitively, this amounts to saying that p lies below q precisely when every maximal element that is *not* above p is *not* above q. In maximally separable models, this naturally holds, as proven in Lemma 7.14. Through this equivalence, one can readily infer that the substitution ξ, as defined below, is such that $\mathsf{m}_P \Vdash \xi(\phi) \equiv \phi$ for all formulae ϕ.

$$\xi : \mathcal{L}(\mathsf{max}(P)) \to \mathcal{L}(\mathsf{max}(P)), \quad m \mapsto \neg \bigvee (\mathsf{max}(P) - \{m\}). \tag{7.6}$$

We employ this substitution in Theorem 7.20 below. This theorem shows that each formula that is not a theorem of ML must have a substitution instance that is not a theorem in KP. We spell this out in a bit more detail in Theorem 7.21.

7.20 Theorem (Levin, 1969)
The following are equivalent for any formula $\phi \in \mathcal{L}(X)$:

(i) $\vdash_{\mathsf{ML}} \phi$;

(ii) for all finite sets Y and all *essentially negative* substitutions $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$, we have $\vdash_{\mathsf{KP}} \sigma(\phi)$.

Proof. Suppose (i) holds, and let $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ be an essentially negative substitution. Through Lemma 7.19, we know that $\vdash_{\mathsf{ML}} \sigma(\phi)$ and $\vdash_{\mathsf{KP}} \sigma(\phi)$ are equivalent. Structurality ensures the former, hence (ii) holds.

Conversely, suppose that (i) does not hold. This gives us some finite $Z \neq \varnothing$ and a valuation $v : \mathsf{B}(Z) \to \mathcal{L}(X)$ such that $v \nVdash \phi$. We claim that the frame $P := \mathsf{B}(Z)$ is *maximally separable*, and hence the model $\mathsf{m}_P : P \to \mathcal{L}(\mathsf{max}(P))$ is *concrete* due to Lemma 7.14. Indeed, we know that $k \in \mathsf{B}(X)$ is maximal precisely if k is a singleton. Now, observe that $p \le \{x\}$ implies $q \le \{x\}$ for all $x \in X$ precisely means that $p \subseteq q$, which is equivalent to $q \le p$. This proves that K is maximally separable.

For convenience, we abbreviate $Y := \mathsf{max}(P)$. By Lemma 7.4, we know of a substitution $\tau : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that $\mathsf{m}_P \Vdash \tau(\psi)$ holds precisely if $v \Vdash \psi$. Now, consider $\xi : \mathcal{L}(Y) \to \mathcal{L}(Y)$, as given in (7.6), and define $\sigma := \xi \circ \tau : \mathcal{L}(X) \to \mathcal{L}(Y)$. It is clear that this substitution is *essentially negative*.

By Lemma 7.19, we know that $\vdash_{\mathsf{KP}} \xi(\phi)$ entails $\vdash_{\mathsf{ML}} \xi(\phi)$. But then $\mathsf{m}_P \Vdash \xi(\phi)$, which ensures $\mathsf{m}_P \Vdash \tau(\phi)$ due to the remark above this theorem. This, in turn, would yield $v \Vdash \phi$, a contradiction proving (ii) not to hold. We have thus proven the equivalence. ☐

We now have sufficient machinery to prove Theorem 7.21 below. The statement of this theorem might seem a bit peculiar. Indeed, the property it expresses does not really speak to the imagination. Its power lies in the contrapositive of the implication from right to left. Roughly speaking, it provides us with a *witness* to the statement that a formula is not a theorem. This witness is sufficiently strong to allow a smooth proof of Corollary 7.22.

Moreover, we employ this description in Theorem 7.41 to provide a *refutation system* of ML. More details follow in Section 7.4.2, but suffice to say that one can think of such a system as a syntactic description of the non-theorems of ML. The beauty is that this system can be finitely described, whereas the theorems of ML have no finite axiomatisation.

7.21 Theorem
Let $\phi \in \mathcal{L}(X)$ be given. The following are equivalent:

(i) $\vdash_{\mathsf{ML}} \phi$;

(ii) if $\sigma(\phi) \vdash^{\mathsf{DP}}_{\mathsf{KP}} \Delta$ then $\vdash^{\mathsf{min}}_{\mathsf{CPC}} \Delta$ for all $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ and $\Delta \subseteq \mathcal{L}(Y)$.[6]

Proof. Suppose (i) holds, and assume that $\sigma(\phi) \vdash^{\mathsf{DP}}_{\mathsf{KP}} \Delta$ hold, where σ is some substitution and Δ is a finite set of formulae. Lemma 7.15 proves that $\mathsf{KP} \subseteq \mathsf{ML}$ and $\mathsf{DP} \subseteq \vdash_{\mathsf{KP}}, \vdash_{\mathsf{ML}}$. These two observations combine to show that $\vdash^{\mathsf{DP}}_{\mathsf{KP}} \subseteq \vdash_{\mathsf{ML}}$. It thus follows that $\vdash_{\mathsf{ML}} \chi$ for some $\chi \in \Delta$, hence $\vdash^{\mathsf{min}}_{\mathsf{CPC}} \Delta$ holds, showing (ii)

Conversely, we suppose (i) does not hold. Via Theorem 7.20, we obtain an *essentially negative* substitution σ such that $\nvdash_{\mathsf{KP}} \sigma(\phi)$. Due to Lemma 7.19, this gives rise to a set of negated formulae Δ such that $\sigma(\phi) \vdash_{\mathsf{ND}_\omega} \bigvee \Delta$. Note that (7.4) ensures that each of the formulae in Δ is non-derivable in CPC. Finally, it follows from

[6] Recall that $\vdash^{\mathsf{DP}}_{\mathsf{KP}}$ denotes the least structural multi-conclusion consequence relation extending both $\vdash_{\mathsf{KP}}$ and DP, as defined on page 30.

Lemma 7.15 that $\vdash_{\mathsf{ND}_\omega} \subseteq \vdash_{\mathsf{KP}}$, so the above combine to show $\sigma(\phi) \vdash^{\mathsf{DP}}_{\mathsf{KP}} \Delta$. This thus proves that (ii) does not hold, as desired. □

With the above Theorem 7.21, we have obtained a good grasp on ML. Using this property, we describe ML as the maximal intermediate logic that admits a certain set of rules. Note that Corollary 7.22 is not phrased literally in this manner, but clearly a logic Λ extends KP precisely if it admits the rules:

$$\{\varnothing/\phi \mid \phi \in \mathsf{KP}\}. \tag{7.7}$$

7.22 **Corollary (Maksimova, 1986, Theorem 5)**
The logic ML is the maximal intermediate logic which extends KP and admits DP.

Proof. Suppose Λ is an intermediate logic with $\mathsf{DP} \subseteq \mathrel{\vdash\!\!\sim}_\Lambda$ and $\mathsf{KP} \subseteq \Lambda$. We proceed by contradiction, so suppose there is a $\phi \in \Lambda - \mathsf{ML}$. Through Theorem 7.21 and the above, we know of a substitution σ and a finite set of formulae Δ satisfying:

$$\sigma(\phi) \vdash^{\mathsf{DP}}_{\mathsf{KP}} \Delta \text{ and } \nvdash_{\mathsf{CPC}} \chi \text{ for all } \chi \in \Delta.$$

As $\vdash_\Lambda \phi$, we know $\vdash_\Lambda \sigma(\phi)$. We clearly have $\vdash^{\mathsf{DP}}_{\mathsf{KP}} \subseteq \mathrel{\vdash\!\!\sim}_\Lambda$, through which the previous yields some $\chi \in \Delta$ such that $\vdash_\Lambda \chi$. We have thus shown $\vdash_{\mathsf{CPC}} \chi$ for some $\chi \in \Delta$, a contradiction proving the desired. □

7.3. Admissible expressibility

The following property is a generalisation of the above Theorem 7.21. Note that the proof of Corollary 7.22 makes no appeal to any special knowledge of ML, other than Theorem 7.21 and the observation that it extends KP and admits DP. This goes to show the strength of this property; it is a sufficient condition in showing that an intermediate logic is the maximal intermediate logic which makes a certain set of rules admissible. We repeat this statement in Theorem 7.24. Moreover, in Section 7.4, we show that any logic with this property can de endowed with a sound and complete *refutation system.*

7.23 **Definition (Admissibly Expressible)**
Let Λ and Ω be intermediate logics satisfying $\Lambda \subseteq \Omega$, and let $\mathcal{R}$ be a set of admissible rules of Λ. We say that Λ is *admissibly expressible in* Ω *through* $\mathcal{R}$ if for all formulae ϕ we have that:

$$\vdash_\Lambda \phi \text{ iff for all } \sigma \text{ and } \Delta,\ \sigma(\phi) \vdash^{\mathcal{R}}_{\mathsf{IPC}} \Delta \text{ implies } \vdash^{\min}_{\Omega} \Delta. \tag{7.8}$$

Note that the implication in (7.8) from left to right always holds. Indeed, whenever $\vdash_\Lambda \phi$ and $\sigma(\phi) \vdash^{\mathcal{R}}_{\mathsf{IPC}} \Delta$ both hold, we immediately obtain a $\chi \in \Delta$ such that $\vdash_\Lambda \chi$ because $\vdash_\Lambda \sigma(\phi)$ and $\vdash^{\mathcal{R}}_{\mathsf{IPC}} \subseteq \mathrel{\vert\!\sim}_\Lambda$. As we assumed $\Lambda \subseteq \Omega$, this readily shows $\vdash_\Omega \chi$.

7.24 Theorem
Let Λ be *admissibly expressible in* CPC through $\mathcal{R}$. Now, Λ is the maximal intermediate logic that admits $\mathcal{R}$.

Proof. Let Ω be an intermediate logic with $\mathcal{R} \subseteq \mathrel{\vert\!\sim}_\Omega$. Suppose there is a $\phi \in \Omega - \Lambda$. By definition, we know of a substitution σ and a finite set of formulae Δ satisfying:

$$\sigma(\phi) \vdash^{\mathcal{R}}_{\mathsf{IPC}} \Delta \text{ and } \nvdash_{\mathsf{CPC}} \chi \text{ for all } \chi \in \Delta.$$

As $\vdash_\Omega \phi$, we know $\vdash_\Omega \sigma(\phi)$. We clearly have $\vdash^{\mathcal{R}}_{\mathsf{IPC}} \subseteq \mathrel{\vert\!\sim}_\Omega$, through which the previous yields some $\chi \in \Delta$ such that $\vdash_\Omega \sigma(\chi)$. This proves $\vdash_{\mathsf{CPC}} \chi$ for some $\chi \in \Delta$, which gives rise to a contradiction, proving the desired. ☐

7.25 Example (Medvedev's Logic)
Through Theorem 7.21, it is easy to see that ML's logic is *admissibly expressible* in CPC through DP and the rules given in (7.7). Indeed, define $\mathcal{R}$ to consist of the rule-counterpart of the axioms of KP together with the rule DP. It is easy to see that $\vdash^{\mathcal{R}}_{\mathsf{IPC}}$, the least structural multi-conclusion relation extending both $\vdash^{\mathsf{min}}_{\mathsf{IPC}}$ and $\mathcal{R}$, must equal the multi-conclusion consequence relation $\vdash^{\mathsf{DP}}_{\mathsf{KP}}$.

We note that we could have considered a more general form of Definition 7.23. The interesting information encapsulated by Definition 7.23 is that non-theorems of Λ have substitution instances that are "decomposable" into non-theorems of Ω via rules that are admissible in Λ; this is precisely what the contrapositive of the right-to-left implication in (7.8) amounts to. Instead of letting Δ be non-theorems of Ω, we might as well have formulated this in a more positive manner. As an example of this, recall Smetanič's logic Sm, the *tabular logic* semantically characterised by Lemma 2.83. Skura (1992a, Lemma 3.3) proved that:[7]

$$\nvdash_{\mathsf{Sm}} \phi \text{ iff } \sigma(\phi) \vdash_{\mathsf{IPC}} \neg x \vee x \text{ for some } \sigma.$$

More generally, we have the following Lemma 7.26. We do not explore this line of reasoning any further.

7.26 Lemma (Skura, 1992a, Theorem 2.2)
Let $\mathsf{u} : \mathsf{U}_{\mathsf{IPC}}(X) \to \mathcal{P}(X)$ be the universal model of IPC on variables X, and let $p \in \mathsf{U}_{\mathsf{IPC}}(X)$ be any point. Consider the intermediate logic Λ defined as the logic of

[7] Note that he calls Sm the Heyting-Łukasiewicz logic.

the frame $\uparrow p$. We have:

$$\nvdash_\Lambda \phi \text{ iff there exists a substitution } \sigma \text{ such that } \sigma(\phi) \vdash_{\mathsf{IPC}} \mathsf{nd_u}\, p.$$

Proof. Immediate by Lemma 7.5. □

The remainder of this section is devoted to proving Theorem 7.32, which states that BB_n is admissibly expressible in CPC through $\overline{\mathsf{D}}_n$. Of course, Theorem 5.36 guarantees that if BB_n is admissibly expressible in CPC through some set of rules, then it must be admissibly expressible through $\overline{\mathsf{D}}_n$. Part of the beauty of our argument is in that we do not need to employ Theorem 5.36 to obtain this result. In particular, no knowledge about *projective formulae* is necessary whatsoever.

Our reasoning roughly proceeds as follows. First, we show that a formula is false precisely if it is falsified by a proper *tree* of *branching degree at most* n. As a consequence, false formulae have a substitution instance that entails the not-downset formula of the model m_P for some such tree P. Second, these not-downset formulae have a set of classically non-derivable formulae as their *admissible approximation.*

The intuitive idea here is that one can iteratively compute the admissible approximation of a formula $\mathsf{nd}\,\mathsf{m}_P$, whenever P is *maximally separable* and *proper*. One can think of this iteration as traversing the tree from the root to all of its leaves. At each stage, one considers a set of nodes and their corresponding not-downset formulae. Step-by-step we replace a point by its immediate successors, if it has any. Throughout this process, the admissible approximation of the disjunction of these formulae remains invariant.

In Lemma 7.27, we show sufficient conditions for maintaining this invariant. Theorem 7.28 provides the evidence needed to show that one is left with an admissible approximation when the computation ends. Finally, all of this is put together in Theorem 7.30. It might be helpful to remark that Lemma 7.27 and Theorem 7.30 play roles similar to Lemma 5.22 and Theorem 5.35 respectively.

We should note that, in order to obtain Theorems 7.32, 7.33 and 7.42, our eventual goals, it is not necessary to know that this computation actually delivers an admissible approximation. Indeed, it suffices to merely know that $\mathsf{nd}\,\mathsf{m}_P$ can be admissibly transformed into a *disjunctive-negative* formula. We do prove this stronger fact to stress the connection between the proof technique as employed by Skura (1989b, Theorem 2) and methods commonly used in the study of admissibility. Furthermore, we would like to drive home the idea that *bases of admissibility* and *refutation systems* can be quite closely related.

The statement of the following Lemma 7.27 contains many qualifications. At first glance, it is convenient to read it with $\Lambda := \mathsf{IPC}$ and $n := \omega$. Specialised to this specific setting, this theorem's instances are many still. Consider the model $\mathsf{m}_P : P \to \mathcal{P}(X)$ as drawn in Fig. 7.2. Through Lemma 7.6, we know that:

$$\vdash_{\mathsf{IPC}} \mathsf{nd}\, P \equiv \left(\mathsf{nd}\left(\overset{x}{\bullet}\right) \vee \mathsf{nd}\left(\overset{y}{\bullet}\right) \to \mathsf{up}\left(\overset{x}{\bullet}\right) \vee \mathsf{up}\left(\overset{x}{\bullet}\right)\right) \to \mathsf{nd}\left(\overset{x}{\bullet}\right) \vee \mathsf{nd}\left(\overset{y}{\bullet}\right).$$

It is not hard to see that this formula is *precisely* of the same form as the antecedent of $\overline{\mathsf{D}}_\omega$, and even $\overline{\mathsf{D}}_2$. The point of Lemma 7.27 is in showing that:

$$\mathsf{nd}\left(\begin{tikzcd}[column sep=small] \overset{x}{\bullet} \arrow[dr, no head] & & \overset{y}{\bullet} \arrow[dl, no head] \\ & \bullet & \end{tikzcd}\right) \vdash^{\overline{\mathsf{D}}_\omega}_{\mathsf{IPC}} \left\{\mathsf{nd}\left(\overset{x}{\bullet}\right), \mathsf{nd}\left(\overset{y}{\bullet}\right)\right\}.$$

It is in this equation that we see the not-down formula being pushed upwards along the tree at hand, as alluded to earlier. In addition to this statement, the lemma also shows that the members of the above right-hand sides entail the left-hand side. Note the similarity with the conditions in Lemma 5.8; it is for this reason that Lemma 7.27 plays an analogous role to Lemma 5.22.

7.27 **Lemma**
Let Λ be an intermediate logic with the *finite model property*, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ on a finite number of variables X. Let $n = 2, 3, \ldots, \omega$, let $W \subseteq \mathsf{U}_\Lambda(X)$ be an anti-chain satisfying $1 \neq |W| \leq n$, and let $p \in \mathsf{U}_\Lambda(X)$ satisfy $W \;\kappa\; p$. Suppose that $\mathsf{u}(w) = \mathsf{u}(s)$ iff $w = s$ for all $w, s \in W$. We now have that:

$$\mathsf{nd}\, p \vdash^{\overline{\mathsf{D}}_n}_{\Lambda} \{\mathsf{nd}\, w \mid w \in W\} \text{ and } \mathsf{nd}\, w \vdash_\Lambda \mathsf{nd}\, p \text{ for all } w \in W.$$

Proof. Before we proceed, first recall that by Lemma 7.6 we have:

$$\vdash_\Lambda \left(\mathsf{up}\, p \to \bigvee_{w \in W} \mathsf{nd}\, w\right) \equiv \mathsf{nd}\, p. \tag{7.9}$$

The second statement trivially follows from this observation. We now focus on the former statement. Let us first apply the above equivalence (7.9), and unfold the definition of $\mathsf{up}\, p$ as given in (3.6). It is crucial to note that $\mathsf{u}(w) = \mathsf{u}(s)$ iff $w = s$ for all $w, s \in W$, which guarantees that both $\mathsf{props} p$ and $\mathsf{news} p$ of (3.8) trivialise. Through these manipulations, we gather:

$$\vdash_\Lambda \mathsf{nd}\, p \equiv \left(\left(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\right) \to \bigvee_{w \in W} \mathsf{nd}\, w\right).$$

Observe that this formula matches *precisely* with the left-hand side of $\overline{\mathsf{D}}_n$. We can thus readily derive:

$$\mathsf{nd}\, p \vdash_\Lambda^{\overline{\mathsf{D}}_n} \left\{\left(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\right) \to \mathsf{nd}\, s \;\middle|\; s \in W\right\}. \tag{7.10}$$

It is clear that for all $s \in W$ we have:

$$s \nVdash \left(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\right) \to \mathsf{nd}\, s.$$

Through Corollary 3.24, we thus know that:

$$\left(\bigvee_{w \in W} \mathsf{nd}\, w \to \mathsf{up}\, W\right) \to \mathsf{nd}\, s \vdash_\Lambda \mathsf{nd}\, s.$$

Combining this observation with (7.10) finished the proof through transitivity. □

To smoothen our further reasoning, it is convenient to know that the not-down formulae of maximal points are *projective*. We prove this in Corollary 7.29, via the following, more general, Theorem 7.28. A similar theorem has been proven by Bezhanishvili and de Jongh (2012, Theorem 4.13). We re-iterate that this information about projectivity is not necessary to prove our characterisation of BB_n, although it is interesting to see that this kind of information is available at all.

7.28 Theorem
Let Λ be an intermediate logic with the finite model property, consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ on a finite set of variables X, and let $W \subseteq \mathsf{U}_\Lambda(X)$ be a non-singleton anti-chain. The following are equivalent.

(i) The formula $\mathsf{nd}\, p$ is *projective*, for all $p \in \mathsf{U}_\Lambda(X)$ such that $W \ \kappa\ p$.

(ii) There are distinct $p^+, p^- \in \mathsf{U}_\Lambda(X)$ such that $W \ \kappa\ p^\pm$.

Proof. Suppose (ii) holds, and let $p \in \mathsf{U}_\Lambda(X)$ be such that $W \ \kappa\ p$. By Theorem 4.24, it suffices to show that the formula $\mathsf{nd}\, p$ is Λ-*extendible*. To this end, let $q \in \mathsf{U}_\Lambda(X)$ and $S \subseteq [\![\mathsf{nd}\, p]\!]_{\mathsf{u}}$ be a non-singleton anti-chain such that $S \ \kappa\ q$. We seek a point

$q^* \in [\![\mathsf{nd}\, p]\!]_\mathsf{u}$ satisfying $S \ \kappa \ q^*$. Naturally, we are done whenever $q \nleq p$, so suppose the contrary. Because $S \ \kappa \ q$, this now shows us that either $p = q$ or $p \in {\uparrow} S$.

In the latter case, we obtain a contradiction with the assumption that $S \subseteq [\![\mathsf{nd}\, p]\!]_\mathsf{u}$. In the former case, see that $W \ \kappa \ p$ and $S \ \kappa \ p$. We thus know ${\uparrow} W = {\uparrow} S$, because $p \notin W, S$ and:

$${\uparrow} p = \{p\} \cup {\uparrow} W = \{q\} \cup {\uparrow} S = {\uparrow} q.$$

By assumption, we have a point q^* such that $q^* \neq p$ and $W \ \kappa \ q^*$. This yields $W \ \kappa \ q^*$. Moreover, $q^* \leq p$ would entail $q^* \in {\uparrow} S$, which is not possible as W is not a singleton. Having resolved all cases, (i) follows.

Conversely, we proceed by contraposition, so suppose (ii) does not hold. This either means that W has no covers at all, or but a unique cover. In the former case, (i) holds vacuously. In the latter case, let p be the unique cover of W. Note that $W \subseteq [\![\mathsf{nd}\, p]\!]_\mathsf{u}$. Indeed, otherwise there is some $w \in W$ such that $w \leq p$ holds. This would entail $p \in W$, violating the condition that W is an anti-chain. We obtain an immediate contradiction through Theorem 4.24, as there clearly can be no cover of W within $[\![\mathsf{nd}\, p]\!]_\mathsf{u}$. This proves (i) not to hold, as desired. □

7.29 Corollary
Let Λ be an intermediate logic with the finite model property, and consider the universal model $\mathsf{u} : \mathsf{U}_\Lambda(X) \to \mathcal{P}(X)$ on a finite, non-empty set of variables X. The formula $\mathsf{nd}\, p$ is *projective* for all $p \in \max(\mathsf{U}_\Lambda(X))$.

Proof. As in Example 2.32, we note that $\varnothing \ \kappa \ p$ holds if and only if $p \in \max(\mathsf{U}_\Lambda(X))$ for all $p \in \mathsf{U}_\Lambda(X)$. Moreover, there are at least two such points. The desired is thus immediate through Theorem 7.28. □

Roughly speaking, Theorem 7.30 below shows that the *admissible approximation* of the not-downset formula of a finite, rooted, *maximally separated* frame P endowed with the valuation m_P is given by the set of not-down formulae of its *maximal points.* Note that we do not use much of this result in our further reasoning. Indeed, all future arguments proceed via Theorem 7.32, which merely depends on Theorem 7.30 for the observation that:

$$\mathsf{nd}\, P \vdash^{\mathcal{R}}_{\mathsf{IPC}} \mathsf{E}(\mathsf{nd}\, P)\,,$$

in the terminology of the theorem below. This is what we meant when we noted earlier that no knowledge about projective formulae was necessary to our goals.

7.30 Theorem
Let $n = 2, 3, \ldots, \omega$, and let Λ be an intermediate logic with the *finite model property* that admits $\overline{\mathsf{D}}_n$. Let P be a rooted, finite, *maximally separable* frame, and consider

the model $\mathsf{m}_P : P \to \mathcal{P}(X)$. Now,

$$\mathsf{E}(\mathsf{nd}\,P) := \{\mathsf{nd}\,m \mid m \in \mathsf{max}(p)\}$$

is an *admissible approximation* of $\mathsf{nd}\,P$ anchored by $\overline{\mathsf{D}}_n$.

Proof. We claim that for all $p \in P$ we have that:

$$\mathsf{nd}\,p \vdash_{\Lambda}^{\overline{\mathsf{D}}_n} \{\mathsf{nd}\,m \mid m \in \mathsf{max}(\uparrow p)\}\,, \tag{7.11}$$

$$\mathsf{nd}\,m \vdash_{\Lambda} \mathsf{nd}\,p \text{ for all } m \in \mathsf{max}(\uparrow p). \tag{7.12}$$

Assume that the claim holds. The desired readily follows from here. Indeed, through Corollary 7.29 we know $\mathsf{nd}\,m$ to be projective, hence Lemma 5.13 shows that

$$\mathsf{E}(\mathsf{nd}\,m) := \{\mathsf{nd}\,m\}$$

is an admissible approximation of $\mathsf{nd}\,m$ anchored by $\overline{\mathsf{D}}_n$. Now, combing (7.11) and (7.12) instantiated to the root of P with Lemma 5.8, $\mathsf{E}(\mathsf{nd}\,P)$ is an admissible approximation of $\mathsf{nd}\,P$ anchored by $\overline{\mathsf{D}}_n$ to be sure.

Let us now prove the claim by well-founded induction along the order in P for all $p \in P$. For convenience, we write W for the set of *immediate successors* of $p \in P$, as in Example 2.31. Note that W cannot be a singleton set, for P is *proper* due to Lemma 7.12 In order to prove (7.11), note that:

$$\mathsf{nd}\,p \vdash_{\Lambda}^{\overline{\mathsf{D}}_n} \{\mathsf{nd}\,w \mid w \in W\}\,,$$

as follows through Lemma 7.27. From here, (7.11) readily follows through induction and transitivity. Analogous reasoning, albeit applied in reverse order, proves (7.12).□

Through the above Theorem 7.30, we can quite readily characterise BB_n as the greatest intermediate logic that admits $\overline{\mathsf{D}}_n$. Theorem 7.30 is to BB_n what Iemhoff (2001a, Theorem 3.5) is to IPC, and Maksimova (1986, Theorem 5) is to ML, namely, a characterisation of an intermediate logic as the strongest intermediate logic that admits a particular set of rules. We achieve this result in Theorem 7.33, as an immediate corollary of Theorem 7.32.

This latter theorem, in turn, is proven through the machinery set up above, with Lemma 7.31 as additional ingredient. Let us note that Iemhoff (2001a, Corollary 3.6 and 4.7) can be construed as an alternative proof to Theorem 7.33. The proofs, however, are fairly different. Whereas we proceed purely semantically, using the so-called *Jankov–de Jongh formulae* of Theorem 3.10, the proof of Iemhoff is more syntactic.[8]

7.31 Lemma
For each natural $n \geq 2$, the logic BB_n is sound and complete with respect to *proper trees* with *branching degree at most* n.

Proof. Soundness is immediate through Lemma 2.88. Let us argue for completeness. Take ϕ to be a formula such that $\mathsf{BB}_n \nvdash \phi$. By Gabbay and de Jongh (1974, Equation 34), we obtain a finite tree P of branching degree at most n such that $P \nVdash \phi$. Without loss of generality, we may assume that P is indeed proper, taking care to note that $n \geq 2$. □

7.32 Theorem
For each $n = 2, 3, \ldots, \omega$, the intermediate logic BB_n is *admissibly expressible* in CPC through the rule scheme $\overline{\mathsf{D}}_n$.

Proof. Note that $\overline{\mathsf{D}}_n$ is admissible in BB_n, as argued in Example 3.81. Moreover, as indicated in (2.3), CPC is an extension of BB_n. We thus need but prove the implication from right to left in (7.8). To this end, we argue by contraposition, so suppose that $\phi \in \mathcal{L}(X)$ is such that $\nvdash_{\mathsf{BB}_n} \phi$.

By Lemma 7.31, there exists a finite, proper tree P of *branching degree at most* n such that $P \nVdash \phi$. We know P to be maximally separable by Lemma 7.13. Through Lemma 7.5, we thus obtain a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ for some finite set of variables Y such that $\sigma(\phi) \vdash_{\mathsf{IPC}} \mathsf{nd}\, P$. Moreover, we know that:

$$\mathsf{nd}\, p \vdash^{\overline{\mathsf{D}}_n}_{\mathsf{IPC}} \mathsf{E}(\mathsf{nd}\, p) := \{\mathsf{nd}\, m \mid m \in \mathsf{max}(p)\}$$

by Theorem 7.30. Furthermore, Lemma 7.1 readily shows that $\nvdash_{\mathsf{CPC}} \chi$ holds for all $\chi \in \mathsf{E}(\mathsf{nd}\, p)$. We combine these observations into the following, proving the desired.

$$\sigma(\phi) \vdash^{\overline{\mathsf{D}}_n}_{\mathsf{IPC}} \mathsf{E}(\mathsf{nd}\, p)\,, \text{ yet not } \vdash^{\mathsf{min}}_{\mathsf{CPC}} \mathsf{E}(\mathsf{nd}\, p)$$ □

7.33 Theorem
The intermediate logic BB_n is the greatest intermediate logic that admits $\overline{\mathsf{D}}_n$ for each $n = 2, 3, \ldots, \omega$.

Proof. Immediate via Theorems 7.24 and 7.32. □

[8] See, in particular, Iemhoff (2001a, Lemma 3.4). This proof is similar in spirit to our Theorem 4.73.

7.4. Refutation systems

According to Łukasiewicz (1951), we assert true propositions, and reject false ones. He remarked that rejection had been neglected in the study of formal logic, and introduced a formal system to inductively derive rejections of false propositions. We call such systems *refutation systems*, following Scott (1957) and Skura (1990a). The general theory of such systems has been studied comprehensively by Słupecki et al. (1971, 1972). In this section, we present refutation systems for the intermediate logics BB_n and ML, making extensive use of their characterisations given above.

A refutation system can be thought of as a proof system for rejection. Instead of deriving that one can correctly assert a statement through a series of truth-preserving inferences from given axioms, as one does in a proof system of assertion, one derives the refutability of a propositional statement through a series of non-truth preserving inferences from given anti-axioms. Proofs in a refutation system will be called *refutations*, and a formula will be called *refutable* whenever a refutation exists ending in this formula.

Let us, by way of example, present a reformulation of the original *refutation system* of CPC as given by Łukasiewicz (1951). This particular presentation, and all the following, will be in the style of Skura (1992a), which goes back to Scott (1957).[9] Recall that x denotes a propositional variable, ϕ and ψ both denote propositional formulae, and σ denotes a substitution.

$$\frac{}{\dashv x}\ \mathsf{Ax} \qquad \frac{\dashv \sigma(\phi)}{\dashv \phi}\ \mathsf{Subs} \qquad \frac{\dashv \psi \qquad \phi \vdash_{\mathsf{CPC}} \psi}{\dashv \phi}\ \mathsf{MT}$$

This refutation system is both *sound* (all refutable formulae are not derivable in CPC) and *complete* (all formulae that are not derivable in CPC are refutable). Skura (1999, Section 1.2) gives a thorough proof of completeness, let us simply remark that to each classically non-derivable formula there is a substitution under which it is equivalent to falsity, from whence completeness is clear.

Since Gödel (1932), it is known that IPC enjoys the disjunction property. Quoting Chagrov and Zakharyaschev (1991, p. 190):

> "In those days only a countable set of extensions of [IPC] was known: the logics that are obtained by adding to [IPC] the formulas which Gödel used for proving that [IPC] has no finite characteristic matrix,

[9] See Słupecki and Bryll (1973) for a similar presentation and further pointers to the literature.

the logic of Kleene's realizability (see [Kleene, 1945], [Rose, 1953]), and maybe few others."

From this perspective, Łukasiewicz (1952) proposed the following refutation system for IPC, which he conjectured to be complete.

$$\frac{}{\dashv x}\,\mathsf{Ax} \qquad \frac{\dashv \sigma(\phi)}{\dashv \phi}\,\mathsf{Subs} \qquad \frac{\dashv \psi \qquad \phi \vdash_{\mathsf{IPC}} \psi}{\dashv \phi}\,\mathsf{MT} \qquad \frac{\dashv \phi \qquad \dashv \psi}{\dashv \phi \vee \psi}\,\mathsf{DP}$$

Kreisel and Putnam (1957) proved that this system is not complete when they introduced the intermediate logic KP. This logic KP has the disjunction property, and it is unequal to IPC, falsifying the conjecture.[1] To be a bit more precise, there is no possible way to derive that:

$$\dashv (\neg z \to x \vee y) \to (\neg z \to x) \vee (\neg z \to y)\,.$$

Observe that the rules DP and MT are *structural*, in the sense that every substitution instance of an instance of this rule is again an instance of this rule. This does not hold for Ax and Subs, and necessarily so, as we argue in more detail directly above Definition 7.37.

Scott (1957) gave a refutation system which is both sound and complete for IPC by replacing DP with the rule:

$$\frac{\dashv (\bigwedge_{i=1}^{n} \chi_i \to \phi_i) \to \chi_i \text{ per } i \text{ such that } \phi_i \text{ is no variable}}{\dashv (\bigwedge_{i=1}^{n} \chi_i \to \phi_i) \to x}\,\mathsf{RScott}$$

under the proviso that none of the formulae $\phi_1, \ldots, \phi_n$ equals $\top$.[10] It is not hard to see that the side-conditions on RScott make it inherently *non-structural*, in the sense that it has instances with invalid substitution instances.

Recall the Kleene—Aczel slash, as originally introduced by Kleene (1962) and modified slightly by Aczel (1968).[11] A crucial property of this slash is that if $\chi \mid \chi$, then $\chi \vdash_{\mathsf{IPC}} \phi \vee \psi$ entails $\chi \vdash_{\mathsf{IPC}} \phi$ or $\chi \vdash_{\mathsf{IPC}} \psi$. This suggests the following refutation system, obtained through replacing DP with $\mathsf{RKleene}$ as given below.

[10] The precise, original formulation of this rule can be found as (iv) in Scott (1957, p. 232). Skura (1999, Section 1.4) gives a detailed exposition of Scott's refutation system.

[11] One can find the definition of the Kleene—Aczel slash in many textbooks, for examples we refer to Gabbay (1981, p. 31) and Smoryński (1973, p. 333).

$$\frac{\dashv \psi \to \chi_i \text{ per } i = 1, \ldots, n}{\dashv \psi \to \bigvee_{i=1}^{n} \chi_i} \;\mathsf{RKleene} \text{ with } \psi \mid \psi$$

One may wonder whether this refutation system is *sound* and *complete* for IPC. It most certainly is, as readily follows from de Jongh (1968, Chapter IV).[12] Another refutation system for IPC was given by Dutkiewicz (1989) based on the semantic tableaux of Evert Willem Beth. The systems discussed in this paragraph lack *structurality* in the same way as the system discussed in the paragraph above.

In the following, we focus on the approach taken by Skura (1989b), who introduced the rules RSkura shown below. This rule scheme is *structural*, and bears great resemblance to the *Visser rules* as discussed in Section 3.5.2.

$$\frac{\dashv (\bigwedge_{i=1}^{n} \chi_i \to \phi_i) \to \chi_j \text{ per } j = 1, \ldots, n}{\dashv (\bigwedge_{i=1}^{n} \chi_i \to \phi_i) \to \bigvee_{j=1}^{n} \chi_j} \;\mathsf{RSkura}$$

To re-iterate, the point of this section is to show the close relation between admissible rules and refutation systems. We provide a refutation system for the logics of bounded branching BB_n. The method naturally extends to cover IPC, providing an intimate connection between the characterisation of IPC in terms of its multi-conclusion admissible rules, as given by Iemhoff (2001a), and the structural refutation system of Skura (1989b). Moreover, the same machinery applies to Medvedev's Logic, re-proving a result by Skura (1992a).

7.4.1. Refutation rules

Recall the refutation system we described for CPC above. This system consisted of three rules, Ax, Subs and MT. All of these rules are what we call *refutation rules*, as specified formally by Definition 7.34 below. This definition is the same as that of Skura (2009, Definition 3.2), and similar in spirit to Skura (2005, p. 456).[13]

[12]The proof given by de Jongh (1968) was eventually published as de Jongh (1970). We refer to Bezhanishvili (2004) for a proof using the universal model, and to Iemhoff (2001a, Proposition 5.1) for a proof using admissible rules. These authors do not propose a refutation system, the rule given here is taken from Skura (1999, Section 5.4).

[13]Do note that our definition differs from that of a "refutability system" by Gabbay (1981, p. 121).

7.34 Definition (Refutation Rule)
Let Λ be an intermediate logic. A rule Δ/ϕ is said to be a *refutation rule* whenever $\vdash_\Lambda \phi$ entails $\vdash_\chi$ for some $\chi \in \Delta$. A single-conclusion consequence relation $\dashv$ is said to be a *refutation system of* Λ whenever Δ/ϕ is a refutation rule for each $\Delta \vdash \phi$.[14]

Before we proceed, first note that a rule Δ/ϕ is *admissible* for Λ precisely if the rule $\sigma(\Delta)/\sigma(\phi)$ is a refutation rule of Λ for all substitutions σ. In a very real sense, admissible rules are *structural* refutation rules.

Through Definition 7.34, we endowed the term "refutation system" with a formal meaning. Let us first show that the main example we considered above, the refutation system for CPC, satisfies these formal constraints. We start by defining a relation $\mathcal{R}$ between sets of formulae and formulae as follows:

$$\Delta\ \mathcal{R}\ \phi \text{ iff } \dashv \chi \text{ for all } \chi \in \Delta \text{ entails } \dashv \phi. \tag{7.13}$$

Let us demonstrate that all the rules in $\mathcal{R}$ actually are refutation rules. Consider MT, which states that $\dashv \psi$ and $\phi \vdash_{\mathsf{CPC}} \psi$ entail $\dashv \phi$. From this observation, it naturally follows that if $\phi \vdash_{\mathsf{CPC}} \psi$ then ψ/ϕ is contained in $\mathcal{R}$, and a refutation rule to boot. In a similar manner, the rule Subs corresponds to the rule $\sigma(\phi)/\phi$. This rule, too, is an element of $\mathcal{R}$, and a refutation rule to be sure. Finally, one can also see that $\varnothing/x$ is a valid refutation rule by virtue of x, or any variable for that matter, being non-derivable in $\vdash_{\mathsf{CPC}}$. This latter rule, of course, corresponds to Ax.

From the above, we can conclude that the rules of the refutation system of CPC correspond to refutation rules in the formal sense of Definition 7.34. Moreover, see that $\mathcal{R}$ is a single-conclusion consequence relation, in the sense of Definition 2.2. When we take $\mathcal{R}$ to be the set of all refutation rules rules for a given logic, then Lemma 7.35 show that $\mathcal{R}$ itself is a single-conclusion consequence relation.

7.35 Lemma
Let Λ be an intermediate logic, let $\mathcal{R}$ be a set of refutation rules of Λ, and let $\vdash_{\mathcal{R}}$ be the least single-conclusion consequence relation extending $\mathcal{R}$. Now, $\vdash_{\mathcal{R}}$ is a refutation system.

Proof. By induction along the inference of $\Delta \vdash_{\mathcal{R}} \phi$, we prove that Δ/ϕ is a refutation rule. The base case follows by assumption. We only treat the case of transitivity. Suppose $\Delta_1, \theta \vdash_{\mathcal{R}} \phi$ and $\Delta_2 \vdash_{\mathcal{R}} \theta$ with $\Delta = \Delta_1 \cup \Delta_2$. By induction, we know that the rules:

$$\left(\Delta_1 \cup \{\theta\}\right)\Big/\phi \text{ and } \Delta_2\Big/\theta \tag{7.14}$$

[14] We simply use $\dashv$ as a dedicated symbol for refutation systems. The notation might lead one to believe that $\dashv$ is the reversal of $\vdash$, but this is not the case.

are both *refutation rules*. Assume that $\vdash \phi$ holds. As the left-hand rule of (7.14) is a refutation rule, this ensures us that either $\vdash \theta$ or $\vdash \chi$ for some $\chi \in \Delta_1$. In the former case, it follows that $\vdash \chi$ for some $\chi \in \Delta_2$ because the right-hand rule of (7.14) is a refutation rule, too. Hence $\vdash \chi$ for some $\chi \in \Delta$ holds in both cases, finishing the argument. $\square$

7.36 **Definition (Completeness)**
A refutation system $\dashv$ of Λ is said to be *complete* whenever $\nvdash \phi$ implies $\dashv \phi$ for all formulae ϕ.

One could summarise Definition 7.36 in symbols as the equation $\dashv \, = \, \nvdash$. Following Słupecki et al. (1971), we say that a logic is *Ł-decidable* when it has a complete refutation system. In Definition 7.37 below, we construct refutation systems out of a basic set of "falsehoods" (or anti-axioms) and a set of admissible rules. These are the types of structural systems we wish to consider. Do note that the consequence relation $\dashv$ defined by Definition 7.37 is not *structural* in the sense of consequence relations, and indeed, for our purposes, never could be. This follows quite readily from the observation that variables must be refutable, yet anything is a substitution instance of a variable. As such, any refutation system that would be structural in this sense has to refute everything, which violates the assumption that all its rules are refutation rules.

7.37 **Definition**
Let Λ be an intermediate logic, let Θ be a set of formulae that are not theorems in Λ, and let $\mathcal{R}$ be a set of admissible rules of Λ. The *refutation system determined by* Θ *and* $\mathcal{R}$ is the least single-conclusion consequence relation $\dashv$ such that:

(i) $\varnothing \dashv \theta$ for all $\theta \in \Theta$;

(ii) $\sigma(\phi) \dashv \phi$ for all formulae ϕ and all substitutions σ;

(iii) $\Delta \dashv \phi$ for all $\phi \, \mathcal{R} \, \Delta$.

The above definition can be reformulated in the same form as the refutation system given in the introduction, as we state in Lemma 7.38 below. We use this form throughout the following, silently sidestepping the single-conclusion consequence relation formulation of Definition 7.37. One can think of the relation $\vdash_{\mathsf{Rf}\dashv}$, defined within the proof below, as a formalised version of the relation $\vdash_{\mathcal{R}}$ of Lemma 7.35. Intuitively speaking, whereas in Lemma 7.35 we consider all refutation rules, in Lemma 7.38 we restrict ourselves to those rules where the refutation can be "witnessed formally" by the predicate $\dashv (-)$.

7.38 Lemma
Let the situation be as in Definition 7.37 and inductively define the predicate $\dashv (-)$ on formulae as below. We now have $\varnothing \dashv \phi$ precisely if $\dashv \phi$ for any formula ϕ.

$$\frac{\theta \in \Theta}{\dashv \theta}\,\mathsf{Ax} \qquad \frac{\dashv \sigma(\phi)}{\dashv \phi}\,\mathsf{Subs} \qquad \frac{\dashv \chi \text{ per } \chi \in \Delta \qquad \phi/\Delta \in \mathcal{R}}{\dashv \phi}\,\mathsf{Inv}\,\Delta/\phi$$

Proof. By structural induction along the definition of $\dashv (-)$, it immediately follows that if $\dashv \phi$ then $\varnothing \dashv \phi$. Indeed, in the base case we have $\dashv \phi$ because $\phi \in \Theta$, so $\varnothing \dashv \phi$ holds by definition. Now, suppose that $\dashv \phi$ holds because of Subs. Induction ensures us that $\varnothing \dashv \sigma(\phi)$, and we know that $\sigma(\phi) \dashv \phi$ holds as well. Through transitivity, we thus obtain $\varnothing \dashv \phi$. Finally, suppose $\dashv \phi$ holds because of $\mathsf{Inv}\,\Delta/\phi$. Induction ensures $\varnothing \dashv \chi$ for each $\chi \in \Delta$, and we know that $\Delta \dashv \phi$. Again, transitivity finishes the argument.

To prove the other direction, we first define an auxiliary relation $\vdash_{\mathsf{Rf}\dashv}$ as follows:

$$\Delta \vdash_{\mathsf{Rf}\dashv} \phi \text{ iff } \dashv \chi \text{ for all } \chi \in \Delta \text{ entails } \dashv \phi.$$

The desired is to be obtained through the following two claims. Indeed, assuming both claims it is clear that if $\varnothing \dashv \phi$ then $\varnothing \vdash_{\mathsf{Rf}\dashv} \phi$ by (i), and so $\dashv \phi$ holds by (ii). It is clear that (ii) holds, and the proof of (i) is similar to a contrapositive formulation of Lemma 7.35.

(i) $\dashv$ is a subset of $\vdash_{\mathsf{Rf}\dashv}$.

(ii) $\varnothing \vdash_{\mathsf{Rf}\dashv} \phi$ holds precisely if $\dashv \phi$ for any formula ϕ. □

7.39 Remark
When $\mathcal{R}_1$ and $\mathcal{R}_2$ are sets of admissible rules such that $\vdash_{\mathcal{R}_1} = \vdash_{\mathcal{R}_2}$ it is clear that the structural refutation system determined by Θ and $\mathcal{R}_1$ equals that determined by Θ and $\mathcal{R}_2$ for any set of formulae Θ. In particular, if $\mathcal{R}$ is a basis of admissible rules then the structural refutation system determined by Θ and $\mathcal{R}$ equals that determined by Θ and $\mathrel{|\!\!\sim}$. As such, bases of admissibility are of interest in constructing succinct structural refutation systems.

7.40 Theorem
Let Λ be an intermediate logic that is admissibly reducible to CPC through $\mathcal{R}$. The refutation system $\dashv$ determined by $\bot$ and $\vdash^{\mathcal{R}}_{\mathsf{IPC}}$ is complete, and the intermediate logic Λ is *Ł-decidable.*

Proof. In order to prove completeness, we assume $\phi \in \mathcal{L}(X)$ to be some formula such that $\nvdash \phi$. By assumption, we obtain a substitution $\sigma : \mathcal{L}(X) \to \mathcal{L}(Y)$ and a finite subset $\Delta \subseteq \mathcal{L}(Y)$ such that $\sigma(\phi)\ \mathcal{R}\ \Delta$ and $\nvdash_{\mathsf{CPC}} \chi$ for all $\chi \in \Delta$. First, we note that $\dashv \chi$ for each $\chi \in \Delta$. Indeed, because $\nvdash_{\mathsf{CPC}} \chi$ we know that there is some substitution $\tau : \mathcal{L}(X) \to \mathcal{L}(\varnothing)$ such that $\tau(\chi) \vdash_{\mathsf{IPC}} \bot$, as follows for the same reasons as in Lemma 2.77. By definition, this shows that $\bot \dashv \tau(\chi)$. When combining this with the observations that $\tau(\chi) \dashv \chi$ and $\varnothing \dashv \bot$, we can readily prove $\varnothing \dashv \chi$. Note that we have $\sigma(\phi) \dashv \phi$, so we are done when we can prove $\Delta \dashv \phi$. Yet this is clear, because we assumed $\phi \vdash^{\mathcal{R}}_{\mathsf{IPC}} \Delta$. This finishes the proof. □

7.4.2. Refutation system for some intermediate logics

Equipped with the machinery described above, we are now ready to provide the promised refutation systems. Let us start with a refutation system for the intermediate logic ML. Note that this gives a finite description of the non-theorems of ML. In sharp contrast, Maksimova et al. (1979) proved that there exists no finite axiomatisation of ML. This shows that refutation systems provide a quite different approach to describing an intermediate logic.

7.41 Theorem (Skura, 1992a, Theorem 6.1)
The refutation system below is a complete refutation system of ML.

$$\frac{}{\dashv \bot}\ \mathsf{Ax} \qquad \frac{\dashv \sigma(\phi)}{\dashv \phi}\ \mathsf{Subs} \qquad \frac{\dashv \psi \qquad \phi \vdash_{\mathsf{KP}} \psi}{\dashv \phi}\ \mathsf{Inv} \vdash_{\mathsf{KP}} \qquad \frac{\dashv \phi \qquad \dashv \psi}{\dashv \phi \vee \psi}\ \mathsf{DP}$$

Proof. Recall that ML is *admissibly expressible* in CPC through DP, as proven in Theorem 7.21. The desired is immediate through Lemma 7.38 and Theorem 7.40. □

In Theorem 7.42, we provide the promise refutation system for BB_n. This results encompasses Skura (1989b, Theorem 2). Skura (2004) also gave a refutation system for BB_n, using more syntactic arguments. This system, however, is not constructed using admissible rules.

7.42 Theorem
Let $n = 2, 3, \ldots, \omega$ be given. The refutation system below is a *complete refutation system* of BB_n. In particular, BB_n and IPC are *Ł-decidable*.

$$\frac{}{\dashv \bot}\ \mathsf{Ax} \qquad \frac{\dashv \sigma(\phi)}{\dashv \phi}\ \mathsf{Subs} \qquad \frac{\dashv \psi \qquad \phi \vdash_{\mathsf{IPC}} \psi}{\dashv \phi}\ \mathsf{MT}$$

$$\frac{\dashv (\bigvee_{i=1}^{n} \chi_i \to \phi) \to \chi_j \text{ per } j = 1, \ldots, n}{\dashv (\bigvee_{i=1}^{n} \chi_i \to \phi) \to \bigvee_{j=1}^{n} \chi_j}\ \mathsf{R}\overline{\mathsf{D}}_n$$

Proof. Recall that BB_n is *admissibly expressible* in CPC through $\overline{\mathsf{D}}_n$, as proven in Theorem 7.32. The desired is immediate through Lemma 7.38 and Theorem 7.40. □

8 Perspectives

Over the course of the past six decades, research in admissibility has blossomed into a vibrant field. A great deal of questions has been both posed and answered, using diverse methods that stem from Model Theory, Category Theory and Proof Theory. Often times, questions pertain to all of these fields. Examples of this include the characterisation of projective, finite Heyting algebras as those corresponding to Kripke models that validate Mints' rule, or the proof that the Visser rules are a basis for the admissible rules of IPC.

A community of researchers has arisen around these themes. Their work has been presented at many logic conferences, such as TACL, LATD, BLAST, and ALCOP. Since a few years, the community even has a conference to call its own: the workshop on admissible rules and unification (WARU). It seems too early to tell where the field is headed as a whole. Nonetheless, there are several lines of enquiry that seem particularly fruitful. We close this thesis with a few of these perspectives.

The higher the vantage point, the better the view. Although this thesis measures but a meagre 19 millimetres, it does hint at some interesting perspectives. The approach we have taken to admissibility in this thesis is meant to suggest the questions posed in Sections 8.1 to 8.4. We sketch these perspectives in varying degrees of detail, reflecting roughly the extent to which we believe positive answers to these questions can be justified at this moment. A slightly more lofty perspective is put forth in Section 8.5, where we signify some synergy between Topos Theory and admissibility. The perspectives put forth below are argued with similar sincerity but significantly less evidence than the work described above. Although the author believes the ideas below are to materialise, they may as well not.

8.1. Fragments

Think of the $[\to]$-fragment of IPC. Within this fragment, a single-conclusion rule is derivable precisely if it is admissible, as shown independently by Dekkers (1995) and Mints (1972, Theorem 2). By contrast, the $[\bot, \to]$-fragment has many non-derivable, admissible rules, both in single-conclusion and multi-conclusion form. These rules can, however, be described by means of a convenient basis, as proven by Cintula and Metcalfe (2010, Theorem 4.6). All these proofs are quite syntactic in nature. Can these results be obtained through more semantic means?

Any fragment of IPC gives rise to a variety. In particular, the $[\wedge, \vee]$-fragment yields the variety of distributive lattices, the $[\to]$-fragment gives rise to the variety of Hilbert algebras, and the $[\wedge, \to]$-fragment induces the variety of relatively pseudo-complemented meet semi-lattices. Wroński (1986) stated as fact that in the latter two varieties, $\mathfrak{A}/a$ is a retract of $\mathfrak{A}$ whenever $0 \neq a \in \mathfrak{A}$ for all algebras $\mathfrak{A}$. As a consequence, the single-conclusion admissible rules of these fragments equal their derivable rules. This argument is already a fair amount more semantic than the above, yet one can go further.

In Section 3.2, we illustrated how one can describe finitely generated Heyting algebras by means of Kripke models, the so-called universal models. Moreover, the order-structure in this model can be described by means of formulae, as we saw in Theorem 3.10. Similar descriptions are possible for various fragments of IPC, see for instance Hendriks (1993, 1996).[1] Analogues of the *Jankov–de Jongh formulae* can be given in these fragments, see for instance de Jongh, Hendriks, et al. (1991, p. 547). More recently, Tzimoulis and Zhao (2013) described a universal model and the corresponding Jankov–de Jongh formulae for the $[\wedge, \vee, \to, \top]$-fragment of IPC. Moreover, the decidability of this fragment has already been settled by Odintsov and Rybakov (2013, Theorem 4.7).

[1]See Renardel de Lavalette et al. (2012) for an extensive overview of the relevant literature.

Similar types of descriptions are possible, even when considering sets of formulae that one would not naturally consider as fragments. Take, for instance, NNIL: the set of formulae with *no nestings of implications to the left*, originally introduced by Visser (1984, p. 16) in the context of Heyting Arithmetic. These formulae appear in many guises, see Visser et al. (1994) for an overview. Interestingly, all subframe logics can be defined by means of NNIL-formulae.[2] Yang (2008, Section 8.5) showed that one can construct a universal model for such formulae, and defined a counterpart to the Jankov–de Jongh formulae. What are the admissible rules of IPC when restricting to this fragment? In answering this question, one will have to think carefully about what an admissible rule in the NNIL fragment really is. Indeed, all fragments we considered until now are closed under substitution, whereas NNIL most certainly is not. As such, Definition 2.10 does not seem suitable.

The formulae of bounded implication degree, as given in Definition 4.62, also constitute quite the interesting fragment. They have been of great use in this thesis, especially in constructing a basis of BB_n. One can also construct a universal model for such formulae, although this undertaking does not appear to be explicitly documented in the literature. Through Visser (1996, Sections 3 and 4) and Ghilardi (1999, pp. 868–871), one can infer that this model would amount to the totality of image-finite Kripke models modulo $\sim_n$, ordered by $\geq_n$, both in the notation of Ghilardi (1999).[3] Jankov–de Jongh formulae for this are easy to construct, see Visser (1996, Theorem 4.10), and less explicitly so, Ghilardi (1999, Proposition 1). Naturally, this universal model is finite, and thus could play the role of v in Theorem 4.77. Can one use this model to provide a more semantic proof of the decidability of IPC's admissible rules? And assuming this model as a given, how efficiently can one verify whether a point codifies an IPC-projective formula?

Taking stock, one can see that much of the machinery we employed in this thesis remains available in proper fragments of IPC. It may be very well possible to construct a basis of the $[\wedge, \vee, \to, \top]$-fragment of IPC using suitable adaptations of our machinery. A similar attempt could be made to re-prove the results by Cintula and Metcalfe (2010) on the $[\bot, \to]$-fragment of IPC.

[2] Answering the obvious question: Yes, there does exist a syntactically described class of formulae that plays the same role with respect to stable logics as NNIL plays with respect to subframe logics. This class is called ONNILLI, short for *only NNIL to the left of implications*, and it has been introduced by Bezhanishvili and de Jongh (2014).

[3] These two relations are the intuitionistic counterparts of n-equivalence and n-subsumption respectively, as introduced by Fine (1974b, p. 33). In this modal setting, one would bound the nesting of modal operators. Universal models and Jankov–de Jongh formulae for this setting can be found in Moss (2007, Section 4).

8.2. Decidability

Which logics have decidable admissible rules? We definitely know of examples. Indeed, Theorem 4.79 shows that IPC counts as such an example. *Tabular logics* and *pretabular logics*, are also examples, as proven by Rybakov (1981). Quite general conditions are known under which a logic is bound to have a decidable set of admissible rules, most prominently Rybakov (1997, Theorem 3.5.7). We believe there is still room for further results of this nature.

One potential avenue is to consider finitely axiomatizable *stable logics*. Such logics must have the finite model property, as mentioned in Theorem 2.100, and hence must be decidable due to Harrop (1958, Lemma 4.1). It seems plausible that the admissible rules of such logics are decidable. At the time of writing, this question does not appear to be settled. We have seen several examples of stable logics, including IPC, KC, LC and CPC, and they all satisfy this property.[4] Of course, this is far from sufficient information to allow such an inference. Additional data is offered by their axiomatisation and the proof of their finite model property, as described by Bezhanishvili and Bezhanishvili (2013, Theorems 6.8 and 6.11), yet conclusive evidence is not readily available.

8.3. Correspondence theory

We have seen several examples of rules, such as Con, DP, $\mathsf{DP}_n^{\neg\neg}$, and $\mathsf{D}_n^{\neg}$. These rules have been endowed with a simple semantic description in Lemma 3.49 and Theorems 3.54, 3.63 and 3.70 respectively, showing that their validity in an *order-defined model* corresponds to the validity of a simple, first-order formula in the language of partial orders. One may wonder whether these descriptions can be lifted from *models* to *frames*.

In the case of these first three rules, one can readily show this to hold true. Indeed, although the proofs mentioned above are formulated for order-defined models, they surely can be adapted to frames. Roughly speaking, one can consider the formulae $\mathsf{up}\,p$ and $\mathsf{nd}\,p$ used in these proofs as propositional variables, and define them to be valid in the obvious places. All of the proofs thus use the typical *minimal definable assignment* style of reasoning, as explained by van Benthem (2001).

[4] For proofs of this, see Theorem 4.79, Ghilardi (2004, p. 108), and Theorem 5.14 respectively, counting the latter theorem twice.

The story gets a bit more complicated when one considers D_n^-. Our argument seems to make crucial use of the *converse well-foundedness* of the ambient model. Still, in this more restricted setting, the translation readily holds true. As a consequence, the following question naturally comes to mind: which rules are such that their validity in a conversely well-founded frame can be described by means of a first-order formula?

The above brings us to the realm of *correspondence theory*, wherein a formula of the logic at hand and a first-order formula in the language of partial orders are said to *correspond* whenever they describe the same class of Kripke frames. For example, the formula $\mathbf{bd}_n$ of Definition 2.71 is known to correspond to the first-order formula expressing that a given frame is of height at most n, as shown by Lemma 2.73. It seems reasonable to ask whether this theory can be generalised from formulae to rules. We have already seen that rules allows for stronger descriptions than formulae ever could. Downwards directed models, for instance, cannot be described by formulae, whereas they can easily be described by the rule DP.

Concrete instances of correspondence theory for rules go back to Citkin (1977a, Theorem 1). Paraphrasing mildly, he showed that what he called the *characteristic rule* of a finite Heyting algebra $\mathfrak{A}$ does not hold on another Heyting algebra $\mathfrak{B}$ precisely when $\mathfrak{A}$ can be embedded in $\mathfrak{B}$. These characteristic rules are single-conclusion rules, and based on the *characteristic formulae* of Jankov (1963b), which we treated to some extent in Section 3.1.[5] Note that the non-embeddability of a given, finite Heyting algebra into a particular Heyting algebra is a first-order statement. Through the well-established duality due to Esakia (1974) and Priestley (1972), all of this can be translated into statements about descriptive frames or Esakia spaces.[6]

More recently, Jeřábek (2009, Lemma 3.3) introduced *canonical rules*. These are multi-conclusion rules, and they occur as a generalisation of the *canonical formulae* of Zakharyaschev (1992), which in turn generalise *characteristic formulae*. Summarising loosely, the canonical rule determined by a finite Kripke frame and a set of domains is not valid on a descriptive frame whenever there exists a subreduction from the latter into the former satisfying the global closed domain condition for said domains.

Subreductions arose in the work of Fine (1985, p. 421), and are used to deal with *subframe logics*. They were employed in the description of canonical formulae by Zakharyaschev (1992), and studied from an algebraic perspective by Bezhanishvili

[5]We refer to Citkin (2015) for a generalisation of this single-conclusion approach to multi-conclusion rules.

[6]More details on this duality are given by Celani and Jansana (2014) and Gehrke (2014). See e.g. Bezhanishvili (2006, Theorem 3.3.3) for a description of Jankov's characteristic formulae using descriptive frames.

and Bezhanishvili (2009). A notion that can play an analogous role for *stable logics* has been developed by Bezhanishvili and Bezhanishvili (2013). This notion was then employed to characterise the validity of *stable canonical rules* by Bezhanishvili, Bezhanishvili, and Iemhoff (2014, Theorem 5.4).

We have thus seen three particular classes of first order-formulae that can be described through the validity of rules. Can one give more general criteria under which a class of formulae is bound to satisfy this? In the context of formulae, one has the *Sahlqvist formulae*, which are guaranteed to correspond to a first-order definable class of frames.[7] It seems natural to ask: Can one provide such a class for rules?

8.4. Counting

Given a rule, how many logics admit it? Concrete instances of this question have been studied throughout the literature. The most trivial instance would be to let the set of rules be empty. In this case, the question asks for the amount of intermediate logics. This is known: there are continuum many, as proven by Jankov (1968).

One could ask this question with but a single rule in mind. For instance, how many logics admit DP? Again, this is well-known: there are continuum many, as proven by Wroński (1973, 1974). When one moves to more intricate rules, relevant literature becomes scarce. Let us consider four additional examples.

First, recall the rule H, attributed to Harrop (1960) and Kreisel and Putnam (1957). How many intermediate logics admit this rule? There are quite a few. In fact, there are none that do not admit it, as proven by Prucnal (1979, Theorem 1).

Second, think of the rule D_ω^-, the weakest variant of the Visser rules we considered. By Theorems 3.77 and 3.79, we know this rule to be admissible in all subframe logics and stable logics respectively. As there are continuum many logics in either class, as shown by Zakharyaschev (1996, Theorem 3.3) and Bezhanishvili and Bezhanishvili (2013, Theorem 6.6), there are continuum many logics that admit D_ω^-.

Third, think of the rule D_ω. This variant of the Visser rules is of equivalent power to V_ω, and encompasses the rule M. Again we ask: how many logics admit it? The answer surely is greater than one, because both IPC and KC admit it, as shown by Iemhoff (2005, Theorem 5.1). On the other hand, there definitely exist logics that do not admit this rule, for instance BB_2. Rybakov (1993, Theorem 7) settles this matter: there are continuum many intermediate logics that admit D_ω.

[7] See van Benthem et al. (2012, Section 3.1) for a recent description.

Finally, think of the rule $\overline{\mathsf{D}}_\omega$, the strongest variant of the Visser rules we consider. It encompasses both DP and D_ω^-, and as such, it also includes D_ω. How many logics admit this rule? We treated this question in Chapter 7, albeit in a slightly different formulation. The answer: there is but one logic that admits it, namely IPC.

The above illustrates that instances of this question have been studied throughout the literature. Yet, at the same time, there does not appear to be a unified approach to these questions. The arguments for DP, D_ω^- and $\overline{\mathsf{D}}_\omega$, all employ the same underlying machinery, going back to Jankov (1968). Similar machinery exists in the context of modal logics, going back to Fine (1974a), and employed to a similar purpose by Fine (1985, Corollary 3). There have been many recent developments surrounding this technical apparatus, let us but mention Bezhanishvili and Bezhanishvili (2013), Bezhanishvili, Bezhanishvili, and Iemhoff (2014), and Jeřábek (2009). It thus seems natural to ask whether one could develop general machinery through which one can settle the cardinality of the set of logics that admit a particular rule.

From this perspective, another question springs to mind. Prucnal (1979) showed that H is admissible in all intermediate logics. This result was generalised by Minari and Wroński (1988), giving sufficient conditions for a rule to be admissible in all intermediate logics. Rimatskij and Rybakov (2005, Theorem 4) described the rules that are admissible in all intermediate logics with the finite model property as those rules that hold in all tabular intermediate logics. Can one give a natural description of the set of rules that are admissible in all intermediate logics?

8.5. Beyond the propositional

Admissible rules have been studied in the context of first order intuitionistic logic. In fact, many of the notions discussed in this thesis historically stem from investigations of the rules of Heyting Arithmetic HA such as de Jongh (1982) and Visser (1984, 2008). Consider an arbitrary theory T in intuitionistic predicate logic IQC. The logic of T is the set of propositional formulae ϕ such that $T \vdash \sigma(\phi)$ for all substitutions from propositional formulae to sentences in the language of T. What does this logic look like?

In many concrete cases, answers are known. For instance, when T equals HA, then this logic equals IPC. This is what is known as *de Jongh's theorem*. The same statement holds true for many extensions of HA, see de Jongh, Verbrugge, et al. (2011, Section 2) for an overview. One can analogously define the propositional admissible rules of a first-order theory T, and aim to describe them. This endeavour has been undertaken by Visser (1999), which led to a quite general result. A crucial role in his reasoning is played by the extension property. Let us spend a few words on re-phrasing this property in a more general setting.

Consider a not necessarily rooted Kripke frame P and its extension P^+, as in Definition 3.16. The former is included in the latter as an upset, hence the inclusion induces a map of Kripke frames $i : P \to P^+$. In topological terms, this is an open map between the associated *Alexandrov spaces*, which induces an open geometric morphism $f : \mathsf{Sh}(\mathsf{P}) \to \mathsf{Sh}(\mathsf{P}^+)$, in the sense of Mac Lane and Moerdijk (1992). We say that T has the extension property when for each sheaf $F : \mathsf{ups}(P) \to \mathbf{Set}$ that is a model of T, there exists a sheaf $\overline{F} : \mathsf{ups}(P^+) \to \mathbf{Set}$ that is also a model of T satisfying the equality $f^*(\overline{F}) = F$, where f^* is the inverse image.

Examples of theories with the extension property abound. For instance, HA itself has the extension property, as can be shown using Smoryiński's trick. Other examples include the constructive theory of groups with apartness and the constructive theory of non-trivial fields with apartness, as proven by Visser (1999, Section 4.2.2). The extension property amounts to the applicability of what some call Smoryński glueing, see for instance van Dalen (1984). Many have used instances of this property to show that some first-theory or other has DP as an admissible rule. Very few have investigated other, more complicated admissible rules of such theories.

Interestingly, this machinery can be generalised to the topos theoretic setting, replacing Smoryński gluing by the more general notion of Artin glueing, and eventually by the Freyd cover of a topos, as shown by Moerdijk (1983). Ščedrov and Scott (1982) showed how the approach of Friedman (1973), which, in particular, can be used to prove that HA admits DP, falls within this more general framework. Through this method, many logics have been shown to admit DP. Moreover, Moerdijk (1982, Theorem 3.6), proved that several higher-order theories extending higher-order Heyting Arithmetic admit DP.

It would be quite interesting to see which other, familiar, rules are admissible in higher-order logics. Moreover, there are several first-order theories whose admissible rules remain elusive to this day still. Examples include HA extended with Markov's principle or Church's thesis, as discussed by Visser (1999). More elusive still is the intuitionistic Zermelo–Fraenkel set theory of Myhill (1975).

One may wonder which of the techniques described in this thesis carry over to the topos theoretic setting. For instance, is there an analogous notion of projective formulae, and can they be described similarly through closure under certain admissible rules? At least on the surface, there appear to be analogies with Lambek and Scott

(1980, Section 6) and Lambek and Scott (1981, Theorem 3.13). Moreover, the hereditary formulae of Lambek and Scott (1981, p. 203) seem analogous in nature to the IPC-extendible formulae of Section 4.1.1. Finally, Freyd et al. (1987, Theorem 2.2) seems to do for intuitionistic set theory what de Jongh (1982) does for HA, in that they both describe the kernels of Lindenbaum algebras.

These examples suggest some potential for further unification. In particular, the results of Visser (1999) could be phrased in a more topos theoretic setting, opening up new options for abstraction. At the very least, there seems little reason to restrict said paper's main theorem to first-order theories whose logic equals IPC, especially in the light of de Jongh, Verbrugge, et al. (2011). Removing this restriction would be a nice, first step.

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A

Problems from the Logic Notebook

Several problems were discussed in the introduction. In particular, we mentioned Problems 2, 5, 9 and 10. All of these problems occur in Ershov and Goncharov (1986), the so-called “Logic Notebook”. In this appendix, we provide the original Russian formulation of these problems, together with a literal translation to English.

Before we continue to the problems, first a few words on the Logic Notebook. It opens with the following (the italics are ours):

> ‘В издание вошли проблемы математической логики, предложенные для решения участниками УП Всесоюзной конференции по математической логике, проходившей в сентябре 1984 г. в г. новосибирске. Новые вопросы и комментарии направляйте по адресу: 630090, Новосибирск 90, Институт математики СО АН СССР, “*Логическая тетрадь*”.’

> 'The publication includes problems of mathematical logic, proposed for solution to the members of All-Union conference on mathematical logic, held in September 1984 in Novosibirsk. New questions and comments should be addressed to: 630090, Novosibirsk 90, Institute of Mathematics of the Academy of Sciences of the USSR, "*Logic notebook*."'

The Logic Notebook (Логическая тетрадь) contains a collection of problems posed at the final day of the seventh All-Union conference on mathematical logic, held between 5 and 7 September 1984 in honour of the 75th anniversary of Anatoly Ivanovich Mal'tsev (Анато́лий Ива́нович Ма́льцев). Several of these problems pertain to admissible rules.

Grigori Efroimovich Mints (Григорий Ефроимович Минц) proposed problems 102 through 106, of which the following is of particular interest. We quote Ershov and Goncharov (1986, p. 22):

> "103. Проблема. Разрешимость булевых уравнений в интуиционистском случае. Для любой пропозициональный формулы $A(x_1, \dots, x_n, a_1, \dots, a_m)$ узнать существуем ли формулы $X_1, \dots, X_n$ такие, что $A(X_1, ..., X_n, a_1, \dots, a_m)$ интуиционистски выводима. Найти все такие решения $X_1, ..., X_n$."

> 103. Problem. Solvability of Boolean equations in the intuitionistic case. For any propositional formula $A(x_1, \dots, x_n, a_1, \dots, a_m)$, do there exist formulae $X_1, \dots, X_n$ such that $A(X_1, ..., X_n, a_1, \dots, a_m)$, is derivable in IPC. Find all such unifiers $X_1, ..., X_n$.

We included a modest reformulation of this as Problem 9 on Page 18. V.V. Rybakov (В.В. Рыбаков) proposed problems 128 and 129, which we respectively included as Problems 2 and 5 on Pages 8 and 10.[1] We quote Ershov and Goncharov (1986, p. 27):

> "128. Во всякой ли конечно-аксиоматизируемой модальной (или суперинтуиционистской) логике конечного поля разрешим проблема допустимости правил вывода?"

> "128. In each finitely axiomatizable modal (or superintuitionistic) logic of a finite slice, is the problem of admissibility of inference rules decidable?"

> "129. Каждая ли табличная модальная (или суперинтуиционистсая) логика имеет конечный базис допустимых правил?'

[1]The translations to English have been confirmed by V.V. Rybakov on 29 August 2014.

"129. Does each tabular modal (or superintuitionistic) logic have a finite basis of admissible rules?"

Finally, A.I. Citkin (А.И. Циткин) presented problems 144 through 146. The former two were due to his doctoral supervisor, Alexander Vladimirovich Kuznetsov (А.В. Кузнецов). Problem 146 occurs as Problem 10 on Page 19. We quote Ershov and Goncharov (1986, p. 31):

"146. Существует ли функция $f(n)$ такая, что любое правило $A(p_1, \ldots, p_n)/B(p_1, \ldots, p_n)$ (A, B — пропозициональные формулы), не допустимое в интуиционистской логике, опровергается на $\mathsf{Lin}_{f(n)}(\mathsf{Int})$ — линденбаумовой алгебре интуиционистской логики с $f(n)$ образующими."

"146. Is there a function $f(n)$ such that any rule $A(p_1, \ldots, p_n)/B(p_1, \ldots, p_n)$ (A, B — propositional formulae) that is not admissible in intuitionistic logic, is refuted on $\mathsf{Lin}_{f(n)}(\mathsf{Int})$ — the Lindenbaum algebra of intuitionistic logic with $f(n)$ generators."

Samenvatting in het Nederlands

Tussen januari 2011 en januari 2015 heb ik gewerkt aan dit proefschrift. Eerlijker gezegd, in deze periode heb ik gewerkt aan mijn onderzoek, en de resultaten hiervan heb ik tussen oktober 2014 en januari 2015 omgezet in het proefschrift dat u hier voor u ziet. Wanneer u mij zou vragen wat deze resultaten dan zijn, dan hangt mijn antwoord erg af van wat u al weet. Indien u bekend bent met *intuïtionistische logica*, en misschien zelfs met het onderzoek naar *toelaatbare regels*, dan zou ik u verwijzen naar het beknopte technische overzicht gegeven in Paragraaf 1.3 van dit proefschrift. Deze samenvatting is voor eenieder die niet met deze twee begrippen bekend is.

In het onderstaande zal ik eerst wat woorden besteden aan intuïtionistische logica. Ik zal een ruw beeld schetsen van wat het is, en waarom het van belang is. Daarna licht ik toe wat toelaatbare regels zijn, en waarom ze interessant zijn om te bestuderen. Tenslotte zal ik in vogelvlucht door mijn proefschrift heengaan, en aangeven welke vragen ik met dit werk heb proberen te beantwoorden.

Intuïtionistische logica

Wiskunde vangt patronen; logica de patronen daarin. Een wiskundige ziet een patroon, en probeert dit te vangen in een stelling. Deze stelling wordt vervolgens sluitend beargumenteerd, waarmee ze als bewezen kan worden beschouwd. Vervolgens kunnen anderen vrijelijk gebruik maken van dit resultaat, in de zekerheid dat dit patroon ook daadwerkelijk bestaat.

Het bovenstaande klinkt behoorlijk utopisch. Zijn alle wiskundigen het met elkaar eens over wat een sluitend argument is? Absoluut niet! Er bestaan verscheidene stromingen, elk met een eigen blik op wat een sluitend argument is. Het intuïtionisme, dat rond het begin van de vorige eeuw ontsprongen is uit de Nederlandse wiskundige Luitzen Egbertus Jan Brouwer, is een van deze stromingen. Intuïtionisme gaat uit van de veronderstelling dat wiskunde een creatieve mentale activiteit is. De waarheid van een wiskundige uitspraak kan enkel gevat worden door middel van een mentale constructie welke laat zien dat deze uitspraak juist is. Dit legt grenzen op aan welke soort van argumenten acceptabel is. De intuïtionistische logica, ontwikkeld door Kolmogorov (1925), Glivenko (1929) en Heyting (1930), vangt de redeneerpatronen die een intuïtionistische wiskundige zou kunnen gebruiken.

Wat maakt de intuïtionistische logica zo interessant? Welnu, ze blijkt tal van toepassingen te hebben, zowel binnen de wiskunde als binnen nabijgelegen gebieden, zoals de informatica. Een intuïtionistisch argument kenmerkt zich door zijn algoritmische aard. Anders gezegd, wanneer men op een intuïtionistische wijze bewijst dat een zeker iets bestaat, dan geeft men ook daadwerkelijk aan hoe datgene te vinden is. Dit in grote tegenstelling tot klassieke bewijzen, welke een dergelijke verantwoording niet schuldig zijn, en waarvan je er dus ook niet op aan kan dat ze zo een verantwoording zullen geven. In tal van praktische situaties, wanneer men niet alleen wilt weten *dat* iets bestaat maar ook *waar* het te vinden is, hebben intuïtionische argumenten een duidelijke meerwaarde. Deze observatie heeft theoretische gevolgen, bekend als de Curry–Howard–de Bruijn correspondentie. Erg informeel gezegd: er is een nauw verband tussen intuïtionistische bewijzen en computerprogramma's.

Dusver ben ik nog niet erg precies geweest. Om u iets dieper mee te nemen in de wereld waarin ik mij de afgelopen jaren bewogen heb, zal ik kort uitweiden over wat ik hier bedoel met het woord *logica*. In het dagelijks taalgebruik gaat logica over een juiste en samenhangende manier van redeneren: een argument is al dan niet logisch. Wat men bedoelt te zeggen wanneer een argument logisch klinkt, is dat de redeneerstappen in dit argument, ongeacht hun inhoud, goed samen lijken te hangen. Logici bestuderen zulke *formele* argumenten, argumenten die kloppen enkel en alleen al bij de gratie van hun *vorm*. Een logica modelleert een manier van samenhangend redeneren.

In beginsel heeft een logica twee ingrediënten: axioma's en afleidingsregels. Een axioma is een voor bewezen aangenomen uitspraak binnen deze logica. Anders gezegd: een argument wat enkel en alleen bestaat uit een axioma is altijd juist en samenhangend. Met een afleidingsregel bewijst men een uitspraak uit al eerder bewezen uitspraken. De axioma's en afleidingsregels van een logica geven zodoende een tweedeling tussen alle uitspraken: de uitspraken die bewezen kunnen worden, en de uitspraken die niet bewezen kunnen worden.

De meest gangbare logica heet *klassieke logica.* Intuïtionitsiche logica en klassieke logica zijn aan elkaar verwant: ze delen dezelfde afleidingsregels, alleen kent klassieke logica één extra axioma. Als gevolg hiervan zijn de bewezen uitspraken in intuïtionistische logica bevat in die van klassieke logica. Er bestaan tal van logica's wiens bewezen uitspraken hier tussenin liggen, maar die ook precies dezelfde afleidingsregels kennen. Dit proefschrift richt zich op deze zogeheten *intermediare logica's.*

Toelaatbare regels

Neem een bepaalde logica in gedachten, het maakt niet uit welke. Zoals hierboven omschreven geeft dit aan elke uitspraak een label: ofwel de uitspraak kan bewezen worden, ofwel dit kan niet. Als u een axioma toe zou voegen aan de logica die u in gedachten heeft, dan kan de labeling veranderen. Immers, als dit axioma niet al een bewezen uitspraak was, is ze dat nu zeker wel. Als u een regel toe zou voegen aan de logica, dan kan de labeling eveneens veranderen. Dit proefschrift richt zich op de regels waarbij de labeling niet verandert: de zogeheten *toelaatbare regels.*

Op deze manier bekeken lijken toelaatbare regels misschien niet bijster interessant. Niets is minder waar. Hoewel toelaatbare regels precies die regels zijn die niets toevoegen *aan een logica*, voegt informatie over de toelaatbare regels *van een logica* zeker wat toe. Het geheel van toelaatbare regels is een *invariant* van de logica: elke logica die dezelfde labeling van uitspraken geeft, heeft dezelfde toelaatbare regels. Dit in scherp contrast tot de gegeven afleidingsregels, die sterk kunnen wisselen zonder enig effect op welke uitspraken bewezen kunnen worden.

Denk nogmaals aan een bepaalde logica. Onthoud welke uitspraken bewezen kunnen worden, maar vergeet alle regels. Is het mogelijk de logica te reconstrueren uit deze beperkte informatie? Een mogelijke benadering is de grootste logica te nemen die dezelfde uitspraken bewijst. De afleidingsregels van deze logica zijn precies de toelaatbare regels van de oorspronkelijke logica.

Elke afleidingsregel is een toelaatbare regel, maar het omgekeerde hoeft zeker niet waar te zijn. In klassieke logica geldt dit wel, en is er zelfs een één-op-één verband tussen toelaatbare regels en axioma's. De situatie ligt veel subtieler in intuïtionistische logica. Hier zijn er tal van toelaatbare regels die helemaal niet voorkomen als afleidingsregel, en daar ook zelfs niet van af te leiden zijn.

In dit proefschrift

In dit proefschrift staan de toelaatbare regels van intermediaire logica's centraal. Er zijn tal van zaken die men rondom toelaatbare regels zou kunnen bestuderen. Ik heb mijn onderzoek onderverdeeld in twee hoofdonderwerpen, en wel als volgt: het omschrijven van de toelaatbare regels van intermediaire logica's, en het omschrijven van intermediaire logica's aan de hand van hun toelaatbare regels.

Omschrijven van toelaatbare regels

Hoe kan men de toelaatbare regels van een logica omschrijven? Op tal van manieren, elk met hun eigen voor- en nadelen. Ik heb me gericht op een drietal verschillende omschrijvingen: door middel van modellen, door middel van een algoritme, en door middel van regels.

Het is gebruikelijk om door middel van modellen te bepalen of een zekere uitspraak al dan niet bewezen kan worden in een logica. Zo kent men waarheidstafels, waarmee de bewijsbaarheid van een uitspraak in klassieke logica getoetst kan worden. In intermediaire logica's is een fijnmaziger model nodig, maar is het niettemin nog steeds mogelijk om te toetsen of een uitspraak bewijsbaar is door middel van eindige modellen. Hoe zou men modellen kunnen maken, niet voor de *bewijsbaarheid* van *uitspraken*, maar voor de *toelaatbaarheid* van *regels*? In Hoofdstuk 3 van dit proefschrift worden modellen ontwikkeld waarmee men de toelaatbaarheid van regels kan toetsen. Het soort model lijkt op het model voor bewijsbaarheid, hoewel er extra beperkingen aan de vorm van het model opgelegd moeten worden. Verder toon ik aan dat het niet mogelijk is de toelaatbaarheid van regels in intuïtionistische logica te toetsen met enkel eindige modellen.

Het is mogelijk mechanisch te testen of een *uitspraak* al dan niet *bewezen* kan worden in klassieke logica. Hetzelfde kan voor uitspraken in intuïtionistische logica, en in tal van andere intermediaire logica's. Hiermee ligt het voor de hand te vragen of ook de *toelaatbaarheid* van *regels* in intuïtionistische logica op een dergelijke mechanische wijze getoetst kan worden. Deze vraag werd voor het eerst gesteld in de jaren zeventig. Een positief antwoord kwam in 1984, en sindsdien zijn er verschillende methoden ontwikkeld om ditzelfde feit aan te tonen. In Hoofdstuk 4 van dit proefschrift wordt de oorspronkelijke methode op een originele manier benaderd. De hierboven genoemde modellen voor toelaatbaarheid worden veralgemeniseerd, en er wordt aangetoond dat een regel toelaatbaar is precies wanneer ze geldt op al deze veralgemeniseerde modellen van een bepaalde grootte. Omdat er slechts beperkt veel van dit soort modellen bestaan, en al deze modellen eenvoudig opgesomd kunnen worden, geeft dit een algoritme om te bepalen of een regel al dan niet toelaatbaar is.

Toelaatbare regels zijn op een volkomen abstracte manier gedefinieerd. Hoe zien deze regels er uit? In Hoofdstuk 5 van dit proefschrift geven we voor een aantal intermediaire logica's basale regels waaruit alle andere toelaatbare regels volgen. Het bewijs hiervoor steunt op de hierboven genoemde modellen voor toelaatbaarheid.

Omschrijven met toelaatbare regels

Hoe kan men intermediaire logica's omschrijven door middel van toelaatbare regels? De toelaatbare regels van een logica geven informatie over de logica zelf, en deze informatie kan gebruikt worden om de logica nader te omschrijven. Ik heb me gericht op twee verschillende perspectieven: vereenzelviging en weerlegging.

Hoe zijn twee uitspraken te vereenzelvigen? Het principe hiervan is eenieder bekend. Denk bijvoorbeeld aan de vraag om "$4x + 12 = 2x + 28$" op te lossen voor x. Waar dan naar gezocht wordt is een vereenzelviging van de twee rekenkundige formules $4x+12$ en $2x+28$: een x waarvoor deze twee formules op hetzelfde uitkomen. Er is een algemene procedure om dit soort sommen op te lossen, en in dit geval is het antwoord om voor x het getal 8 in te vullen. Net zo goed als men kan vragen om de vereenzelviging van twee rekenkundige formules, kan men vragen om de vereenzelviging van twee logische uitspraken. In Hoofdstuk 6 van dit proefschrift geef ik algemene methoden waarmee dergelijke vragen opgelost kunnen worden in enkele intermediaire logica's. De oplossingsmethode leunt essentieel op een goed begrip van de toelaatbare regels in de relevante logica's.

Toen ik hierboven logica's nader toelichtte, sprak ik van de axioma's en afleidingsregels waarmee men uitspraken kan bewijzen. Dit geeft een tweedeling tussen de uitspraken, namelijk de uitspraken die wel te bewijzen zijn, en de uitspraken die dat niet zijn. Indien u zou willen laten zien dat een uitspraak te bewijzen is, hoeft u enkel een bewijs te geven: een aaneenschakeling van axioma's en afleidingsregels die eindigt in de gewenste uitspraak. Hoe zou u kunnen laten zien dat een uitspraak niet te bewijzen is? In Hoofdstuk 7 van dit proefschrift beschrijf ik weerleggingssystemen voor enkele intermediaire logica's. Een weerlegging is, net als een bewijs, een stapsgewijze afleiding. Echter, in plaats van te laten zien dat een uitspraak bewijsbaar is, laat een weerlegging zien dat een uitspraak niet bewijsbaar is. Een weerleggingssysteem staat tot een weerlegging als een logica tot een bewijs. Ik laat zien dat, in sommige intermediaire logica's, weerleggingssystemen gegeven kunnen worden door als regels omkeringen van toelaatbare regels te nemen.

Ter afsluiting

De vragen die bestudeerd worden in dit proefschrift staan hierboven omschreven. Hiermee is het meerendeel van dit proefschrift omstreken, en wel de Hoofdstuk 3 tot en met 7. Daarnaast wordt in dit proefschrift een poging gedaan een getrouw overzicht te geven van het gehele vakgebied, het onderzoek naar toelaatbare regels in intermediaire logica's, welke te vinden is in Hoofdstuk 1. Alle basisbegrippen die verder in dit werk nodig zijn, worden behandeld in Hoofdstuk 2.

Het inhoudelijke deel van dit proefschrift sluit met Hoofdstuk 8, waarin enkele toekomstperspectieven geboden worden. Deze perspectieven geven mijn blik op enkele verdere ontwikkelingsmogelijkheden van het veld. Toelaatbare regels worden al een aardige tijd bestudeerd, met wortels die tenminste teruggaan tot Kleene (1952), Lorenzen (1955) en Moh (1957). Er is sindsdien veel bereikt, maar er is nog veel onontgonnen terrein. Dit proefschrift is mijn bijdrage aan dit gebied; een paar stappen in het onbekende.

Curriculum Vitae

Jeroen Goudsmit was born on May 13th 1987 in Leidschendam, the Netherlands. He attended Alfrink College in Zoetermeer, where he graduated in 2005. In the same year, he enrolled in the Bachelor's programs of Mathematics and Cognitive Artificial Intelligence at Utrecht University. Mainly due to a growing interest in software technology, he enrolled in the Bachelor's program Computer Science in 2006 at the same university. He completed these programs between November 2008 and August 2009, the latter two of which with the distinction *cum laude*.

Subsequently, he continued his studies with a Master's in Mathematical Sciences, focusing mainly on constructive logic and algebraic number theory. He completed this program in November 2010. In January 2011, he started as a PhD candidate under the supervision of dr. Rosalie Iemhoff and prof.dr. Albert Visser on the research program "the power of constructive proofs" funded by the Netherlands Organisation for Scientific Research (NWO). The results he obtained while a PhD candidate are contained within this thesis. Between September 2013 and May 2015, Jeroen was a member of the University Council of Utrecht University.

Quaestiones Infinitae

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