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Kris de Jaegher

Universiteit Utrecht

Utrecht School of Economics

Tjalling C. Koopmans Research Institute
Utrecht School of Economics
Utrecht University

Janskerkhof 12
3512 BL Utrecht
The Netherlands
telephone +31 30 253 9800
fax +31 30 253 7373
website www.koopmansinstitute.uu.nl

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How to reach the authors

Please direct all correspondence to the first author.

Kris de Jaegher
Utrecht University
Utrecht School of Economics
Janskerkhof 12
3512 BL Utrecht
The Netherlands
E-mail: k.dejaegher@econ.uu.nl

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Kris de Jaegher[a]

[a]Utrecht School of Economics
Utrecht University

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Abstract
This paper shows that Rubinstein's results on the two-player electronic mail game do not extend to the N-player electronic mail game.

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1. Introduction

In the two-player electronic mail game (Rubinstein, 1989), an informed player 1 knows whether or not there is an opportunity for coordinated action. If there is, an automatic communication network sends a message to an uninformed player 2, who then automatically sends a confirmation of receipt to 1, after which 1 sends a confirmation of the confirmation to 2, etc. In one variant of the game, this process continues indefinitely, as long as no message gets lost. In this case, coordinated action never takes place. In a second variant of the game, which is the focus of our paper, the process stops when a message gets lost or when a final stage is reached. In this case, coordinated action only takes place when the maximum possible number of messages is received.[1] The current paper shows that the latter result does not extend to three versions of an *N*-player electronic mail game, differentiated by the network assumed, and by the manner in which the players' strategies are defined.[2]

Our *N*-player electronic mail game takes the following form. There are two states of nature, state *a* and state *b*. State *a* occurs with probability $(1-p)>1/2$. The *N* players can choose from two actions, namely actions *A* and *B*. If all *N* players choose action *A* (*B*) in state *a* (*b*), then each player obtains payoff *M*. If all *N* players choose action *A* (*B*) in state *b* (*a*), then each player obtains payoff 0. If players choose different actions then those who choose action *A* obtain 0, and those who choose action *B* incur a loss of *L*. It is assumed that $L > M > 0$.[3] Only player 1 knows the state of nature.[4]

2. Maximal communication network, strategies in terms of knowledge

In the spirit of Rubinstein (1989), the following automatic communication network allows players to achieve t^{th}-order knowledge at stage *t*.[5] At stage 1, when observing state *b*, player 1 automatically sends an e-mail to each uninformed player. At all further stages up to the final stage *z*, each player automatically forwards each message received to each other player. Each e-mail gets lost with small probability ε. Thus, at any particular stage *t*, up to $(N-1)^t$ e-mails are sent.

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1 Rubinstein's (1989) treats the case without a final stage for the following reason. For small probability that the message gets lost, the communication network lets players approach a situation of common knowledge (approximate common knowledge). However, as long as the probability that a message gets lost is positive, coordinated action never takes place. This shows that approximate common knowledge is different from common knowledge. Our focus is not on common knowledge, which is why we treat the more realistic case of a finite number of stages.

2 The paper can also be interpreted as a comparison of exogenously given networks. In this sense, a related exercise is shortly analysed by Chwe (1995) for a particular example, namely a three-player three-action electronic mail game with one informed player. Chwe compares a communication network where the informed player informs each of the uninformed players separately to one where the informed player informs one uninformed player, who then informs the other uninformed player.

3 Together with the assumption that $(1-p)>1/2$, this implies that, if all other players would play *A* in state *a* and *B* in state *b*, an individual player plays *A*.

4 Morris (2001) treats an *N*-player electronic mail game where only *n* players, with $n < N$, need to act to make coordination beneficial. In Morris's communication network, a random *m* players find out the state of nature at stage 0. With probability $(1-\varepsilon)$, at stage 1, another random *m* players find out what the random *m* players found out at stage 0. And so on, ad infinitum. Morris shows that under certain conditions, coordinated action is never achieved in this game. See also Footnote 7.

5 Everybody knows that state *b* occurs by state 1, everybody knows that everybody knows that state *b* occurs by stage 2...

By scrolling down an e-mail, one can observe the sequence of players through which a message was forwarded. Thus, when player i receives a particular message from player j at stage t, player i knows that j knows that k knows that l knows that … 1 knows that state b occurs, or $K_i^t K_j^{t-1} K_k^{t-2} K_l^{t-3} ... K_1^0(b)$. Superscripts refer to stages, where player 1 learns the state of nature at stage 0. This same message also implies $K_j^{t-1} K_k^{t-2} K_l^{t-3} ... K_1^0(b)$ $K_k^{t-2} K_l^{t-3} ... K_1^0(b)$, etc.

Let a pure strategy for player i consist of a *minimal amount of interactive knowledge*, in the form of a *minimal set S_i* of statements player i must know to play B. A separating equilibrium is defined as a profile of minimal sets $\left(S_i^*\right)_{i \in N}$ containing mutual best responses ("*" refers to an equilibrium). Denote by S_i^t the set of statements that player i needs to find out at stage t in order to play B, and by $\left(S_i^t\right)_{i \in N}$ a profile of such sets.

Proposition 1 shows that equilibria exist where players play B when receiving only a few messages at the stages before z. However, at stage z, each player only plays B when receiving a confirmation of each message sent at stage $(z-1)$. The intuition for this is the following. Suppose that at stage z, player i does not receive a confirmation from player j that at stage $(z-1)$, j received a message m required by j for playing B. Then the events "j did not receive m" and "j received m but her confirmation of it got lost" are about equally likely. Given that playing B by oneself is costly, i plays A. However, suppose that at stage $t < z$, i does not receive a confirmation that j received at stage $(t-1)$ a message m' required by j for playing B. Also, assume that no other player requires a confirmation that i received j's confirmation of m' at stage t. If some player now informs i at stage z that j received m' at stage $(t-1)$, then i plays B even though she did not receive a confirmation of m' from j at stage t.[6]

Proposition 1

[6] For a simple example, consider the case $N = 3$, $z = 2$. We show that an equilibrium exists where $S_1^{2*} = S_3^{2*} = \left\{K_2^1 K_1^0(b)\right\}$, $S_2^{2*} = \left\{K_1^0(b)\right\}$. *First*, let players 1 and 2 follow the candidate equilibrium, and let $\neg K_3^1 K_1^0(b)$, but let $K_3^2 K_2^1 K_1^0(b)$. Then $K_2^1 K_1^0(b)$, and for sufficiently small ε, it is extremely likely that $K_1^2 K_2^1 K_1^0(b)$; it follows that 3 plays B. If $K_3^1 K_1^0(b)$, $\neg K_3^2 K_2^1 K_1^0(b)$, then for small ε, 3 considers $\neg K_2^1 K_1^0(b)$ and $K_2^1 K_1^0(b)$ about equally likely. Given that $L > M$, 3 plays A. *Second*, let players 2 and 3 follow the candidate equilibrium, and let $\neg K_1^2 K_3^1 K_1^0(b)$, but let $K_1^2 K_2^1 K_1^0(b)$. Then $K_2^1 K_1^0(b)$, and for sufficiently small ε, it is extremely likely that $K_3^2 K_2^1 K_1^0(b)$; it follows that 1 plays B. If $\neg K_1^2 K_2^1 K_1^0(b)$, then for small ε, 3 considers $\neg K_2^1 K_1^0(b)$ and $K_2^1 K_1^0(b)$ about equally likely. Given that $L > M$, 1 plays A. *Third*, let players 1 and 3 follow the candidate equilibrium, and let $\neg K_2^2 K_3^1 K_1^0(b)$, but let $K_2^1 K_1^0(b)$. Then for sufficiently small ε, it is extremely likely that $K_1^2 K_2^1 K_1^0(b)$ and that $K_3^2 K_2^1 K_1^0(b)$; it follows that 2 plays B. If $\neg K_2^1 K_1^0(b)$, then $\neg K_1^2 K_2^1 K_1^0(b)$ and $\neg K_3^2 K_2^1 K_1^0(b)$, and players 1 and 3 play A. It follows that player 2 plays A.

(i) Let $K_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b) \in S_j^{t*}$, with $t < z$. Then $\exists h \in N$: $K_h^{t+1}K_j^tK_k^{t-1}K_l^{t-2}...\ K_1^0(b) \in S_h^{(t+1)*}$.[7]

(ii) Let $K_k^{z-2}K_l^{z-3}K_m^{z-4}...\ K_1^0(b) \in S_j^{(z-1)*}$. Then $\forall h \in N, h \neq j$: $K_j^{z-1}K_k^{z-2}K_l^{z-3}K_m^{z-4}...\ K_1^0(b) \in S_h^{z*}$.

(iii) Let $K_j^{z-1}K_k^{z-2}K_l^{z-3}K_m^{z-4}...\ K_1^0(b) \in S_h^{z*}$. Then $K_k^{z-2}K_l^{z-3}K_m^{z-4}...\ K_1^0(b) \in S_j^{(z-1)*}$, $K_l^{z-3}K_m^{z-4}...K_1^0(b) \in S_k^{(z-2)*}$, $K_m^{z-4}...\ K_1^0(b) \in S_l^{(z-3)*}$, etc.

(iv) For sufficiently small ε, every possible $\left(S_i^{z-1}\right)_{i\in N}$ fully characterises a unique separating equilibrium.

Proof:

(i)-(ii) Let $K_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b) \in S_j^{t*}$ with $t < z$, but let $\forall \sigma > t: \neg K_h^\sigma ...K_j^tK_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$. At best[8] then, $K_h^tK_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$. Player h then calculates the probability that $K_j^tK_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$ to be $(1-\varepsilon)\varepsilon/\left[\varepsilon+(1-\varepsilon)\varepsilon\right] < 1/2$, and the probability that $\neg K_j^tK_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$ to be $\varepsilon/\left[\varepsilon+(1-\varepsilon)\varepsilon\right] > 1/2$. Since $L > M$, playing *A* is a best response for *h*. This proves both claims (i) and (ii).

(iii) $K_k^{z-1}K_l^{z-2}K_m^{z-3}K_n^{z-4}...\ K_1^0(b)$ can only occur if $K_l^{z-2}K_m^{z-3}K_n^{z-4}...\ K_1^0(b)$, $K_m^{z-3}K_n^{z-4}...\ K_1^0(b)$, etc.

(iv) Claims (ii) and (iii) together imply that a given profile of sets $\left(S_i^{z-1}\right)_{i\in N}$ determines a range of statements that must *at least* be included in the sets S_i^* for all $i \in N$. It remains to be shown that *no other* statements are included in these sets. We show that, if all players $f \neq h$ follow the candidate equilibrium, then it is a best response for player *h* to play *B* when receiving *only* the messages implied through (iii) by a given $\left(S_i^{z-1}\right)_{i\in N}$.

Let (iii) imply for $\left(S_i^{z-1}\right)_{i\in N}$ that $K_l^{t-1}K_m^{t-2}K_n^{t-3}...\ K_1^0(b) \in S_k^{t*}$. Then by (ii), $\exists j: K_j^{z-1}...K_k^tK_l^{t-1}K_m^{t-2}K_n^{t-3}...\ K_1^0(b) \in S_h^{z*}$. Consider, for any $t < \sigma \leq (z-1)$, statements of the type $K_g^\sigma....K_k^tK_l^{t-1}K_m^{t-2}K_n^{t-3}...\ K_1^0(b)$ of which $\left(S_i^{z-1}\right)_{i\in N}$ through (iii) does *not* imply that they must be an element of $S_h^{(\sigma+1)*}$. Let $\neg K_h^{\sigma+1}K_g^\sigma....K_k^tK_l^{t-1}K_m^{t-2}K_n^{t-3}...\ K_1^0(b)$; at the same time, let $K_h^zK_j^{z-1}...K_k^tK_l^{t-1}K_m^{t-2}K_n^{t-3}...\ K_1^0(b)$. As each other player *f* is assumed to follow the candidate equilibrium, for any $\sigma < \tau \leq z$ and for any $f \neq h$, it is the case that

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[7] It should be noted that this part of the proposition shows that, if $z = \infty$, then Rubinstein's (1989) result that coordinated action never takes place in state *b* is confirmed. An extra message is always requested, but for an infinite number of stages the probability that all the requested messages are received is zero. This result is similar to the result obtained by Morris (2001) in his N-player electronic mail game (see Footnote 4).

[8] Consider instead the case where $\neg K_h^tK_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$, but $K_h^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$. Then the probability that $K_j^tK_k^{t-1}K_l^{t-2}K_m^{t-3}...\ K_1^0(b)$ is $(1-\varepsilon)^2\varepsilon/\left[\varepsilon+(1-\varepsilon)\varepsilon+(1-\varepsilon)^2\varepsilon\right] < 1/3$.

$K_h^{\sigma+1} K_g^{\sigma} ... K_k^t K_l^{t-1} K_m^{t-2} K_n^{t-3} ... K_1^0(b) \notin S_f^{\tau+1*}$. It follows that $K_g^{\sigma} .... K_k^t K_l^{t-1} K_m^{t-2} K_n^{t-3} ... K_1^0(b) \notin S_h^{(\sigma+1)*}$.

It remains to be checked that a player *i* who receives all messages in S_i^* plays *B*. Let players $\neg i$ in total need to receive *x* messages for coordinated action to take place in state *b*. Then it is a best response for player *i* to play *B* when $(1-\varepsilon)^x M - \left[1-(1-\varepsilon)^x\right]L > 0$. This is the case for sufficiently small ε. QED

Proposition 1 shows that, at one extreme, a separating equilibrium exists where arrival of $\left[(z-1)+(N-1)\right]$ messages suffices for coordinated action to take place in state *b*. In particular, for stages $t < z$, a *single* message received by a *single* player suffices. However, at stage *z*, *each* player needs to receive a confirmation of the message sent at stage $(z-1)$. At the other extreme, a separating equilibrium exists where coordinated action in state *b* requires receipt of the maximum number of messages (or a total of $\sum_{t=1}^{z}(N-1)^t$ messages).[9] In between these two extremes, a whole range of equilibria exists, which are uniquely described by the number of messages players require at stage $(z-1)$.

If *z* is a matter of design, then player $i \neq 1$ is best off in the network with $z = 1$ (a "star"[10] with player 1 at the centre). In any other network, player *i* plays *B* less often, but when she does, she is not less likely to incur cost *L*. The best possible situation for player 1 is $z = 2$, with a single message sent at stage 1 (a "star" with an uninformed player at its centre arising once the informed player has informed this uninformed player[11]). Compared to the case $z = 1$, player 1 plays *B* less often, but is less likely to incur cost *L* when she does play *B*.

3. Sequential communication network

In the communication network described in Section 2, *t*-th order knowledge is achieved at stage *t*. Consider now instead a network where, whenever receiving an e-mail, player $i \neq N$ sends an e-mail to $(i+1)$, and player *N* sends an e-mail to 1. The process starts with player 1 sending a message in state *b*, and continues until a message gets lost (which occurs with probability ε), or until stage *z* is reached (where $z \geq (N-1)$). x^{th}-order knowledge is now achieved if $x(N-1)$ messages arrive.[12]

As there is only a single path along which messages can travel, Rubinstein's (1989) result fully extends here, and coordinated action in state *b* is only achieved when each

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[9] Note that, in Rubinstein's (1989) case where $N = 2$, it is the case that $\left[(z-1)+(N-1)\right] = \sum_{t=1}^{z}(N-1)^t$: for finite *z*, there is only a single equilibrium, where the two players require the maximum number of messages.

[10] These and other names for network structures are borrowed from the network literature (see Goyal, to be published).

[11] Note that this star is defined with respect to the minimal set of messages required by players. It is likely that players receive more than one message at stage 1.

[12] The number of messages required for increasing levels of higher-order knowledge to achieved now increases arithmetically instead of exponentially.

player receives a maximum number of messages.[13] If z is a matter of design, then the uninformed players are collectively best off with a “line”, where $z = (N-1)$. Individually, uninformed players are better off the further they are down the line. The informed player is best off with a “wheel”, where $z = N$.[14]

4. Maximal communication network, strategies in terms of *number* of required messages

The separating equilibria in Sections 2 and 3 are all vulnerable, as one message lost can stop coordinated action taking place in state b. In the game in Section 3, this is so by assumption. In the game in Section 2, however, there is plenty of redundancy in the communication process. We now show that players can exploit this redundancy and become even better off than with the sequential communication network. Adopt the modified assumption that, in order to play B, players require that they receive a certain minimum *number* of messages over different stages, regardless of the source and of the content of the message. Then it is a mutually best response for players to play B if they receive only a certain number of messages over stages $(z-1)$ and z.

Proposition 2. Denote the number of messages received by player i at stage $(z-1)$ (respectively z) as x_{z-1} (respectively x_z). Then it is a mutual best response for all players i to play B when $x = x_{z-1} + x_z$.

Proof:

Let player i receive messages such that $x_{z-1} + x_z = x$. For small ε, every other player than i is likely to receive x_{z-1} messages from player i at stage z. Let the x_z messages be received from a set of players J. Then for small ε, every player in the set $N \setminus J$ is likely also to have received x_z messages at stage z. A player j in set J from whom player i received y_z messages at stage z received at least y_z messages at stage $(z-1)$, and for small ε is likely to receive $(x_z - y_z)$ messages at stage z. It follows that the sum of messages received by other players than i over stages $(z-1)$ and z is x, meaning that these players choose action B. It follows that it is a best response for i to play B. QED

A simple example of an equilibrium in line with Propostion 2 is the case where $z = 2$, $N = 3$. Proposition 2 suggests that a separating equilibrium exists where players play B as soon as they receive a single message over stages 1 and 2.[15] It

[13] This can be shown by following the same procedure as in part (i) of the proof of Proposition 1.

[14] Note that, compared to the “star” centred around the informed player of Section 2, the “line” leaves each player at least as well off, as uninformed players down the “line” run less risk.

[15] *First*, let 1 receive a message from 2, but not from 3. Then 1 knows that 2 sent a message to 3, and that this message arrived with high probability. Therefore, if a single message received suffices for 2 and 3 to play B, then it does so for 1 as well (the same reasoning applies when 1 receives a message from 3, but not from 2). *Second*, let 2 receive a message from 1, but not from 3. Then 2 is able to send confirmations to 1 and 3. If a single message received suffices for 1 and 3 to play B, then 2 plays B (the same reasoning applies when 3 receives a message from 1, but not from 2). *Third*, let 2 receive a message from 3, but not from 1. Then 2 knows that 3 received a message from 1. 2 also knows that 3 sent a message to 1, and that this message arrived with high probability. If a single message received

remains to be shown, *first*, that uninformed players play A when not receiving any messages. This is evident, as the fact that they do not receive any messages makes it even more likely than a priori that state a occurs.[16] *Second*, it remains to be shown that player 1 plays A in state b when not receiving any messages. Clearly, for small ε, the probability that no player received any messages approaches ¼.[17] For large enough L, player 1 plays A. It should be noted that separating equilibria where players require the maximum number of messages continue to exist.

5. Conclusion

Players of the N-electronic mail game may only require a limited number of messages to achieve coordinated action. This is because they benefit from the fact, when there are more than two players, the same message can travel along different paths.

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suffices for 3 and 1 to play B, then 2 plays B (the same reasoning applies when 3 receives a message from 2, but not from 1).

[16] When an uninformed player does not receive any messages, then the probability that state a occurs is $(1-\rho)/\{(1-\rho)+\rho\varepsilon[\varepsilon+(1-\varepsilon)\varepsilon]\} > (1-\rho)$. It follows that, even in the case where all other players play A in state a and B in state b, our uninformed player prefers to play A. See also Footnote 4.

[17] The probability that no uninformed player received a message is $\varepsilon^2/[\varepsilon+(1-\varepsilon)\varepsilon]^2$. For small noise, this probability approaches ¼. Player 1 prefers to play A if ¼($-L$) + ¾M < 0. A full characterisation of these equilibria requires further conditions on L.