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Network formation with decreasing marginal benefits of information

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Abstract
In the two-way flow connections model of the seminal paper by Bala and Goyal (2000a), the marginal benefit of obtaining the information of one more player is constant. However, it is plausible that the marginal benefit of such information is decreasing. This paper explores the consequences for the stability of networks of such decreasing marginal benefits. We start by characterizing the strict Nash networks for both the case of constant and the case of decreasing marginal benefits. Using this characterization, we next explore how the set of strict Nash networks differs for the two cases. The results and intuition tells us that long diameter networks have certain features which make them relatively more likely to be stable under decreasing marginal benefits of information as compared to short diameter networks.

Keywords: Network Formation, Concave Benefits, Two-Way Flow Model

JEL classification: C72, D85

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1 Introduction

In a seminal article Bala and Goyal (2000a), henceforth BG, introduce the two way flow model. In this model, each player has a unique private piece of information which he does not mind sharing with others. Each player can sponsor costly links to any of the other players. All links together form a network. If players are connected on this network (via any series of links, regardless of who is sponsoring[1] what link) they can all access each other's information, from which they all benefit. In particular, it is assumed that the value of the private information of each player is the same. Each player's benefits from information are assumed to be linear in information, so that any player's benefits are simply equal to the sum of the information of all the players to which he or she is connected. In this case, the center-sponsored star is the only strict Nash network. BG also treat an extension where information decays as the distance between players in the network becomes larger. In this case, the scope of strict Nash networks may be wider; on top of the periphery-sponsored star, it may include stars where the center sponsors one or more links, linked stars, and networks with a wider diameter.

Our paper extends the two-way flow BG model with information decay to the case of decreasing marginal benefits of information.[2] We argue that this is a more plausible assumption. Consider for example the potential buyer of a second-hand car. It is clear that a buyer who already received an external evaluation will be willing to pay less for extra information on the quality of a car than the buyer who has not yet received such an evaluation. Or, more formally, say that an agent can acquire i.i.d. signals about the true state of the world. The first signal typically conveys more information than the hundredth. Hence the marginal benefit of the last signal is lower than that of the first. Moreover, if we abandon the assumption that each player possesses a unique piece of information, and instead allow for the information of players to partly overlap with the information held by the other players, the marginal benefits of contacting more other agents decreases further in the number of contacts of the agent.

We formalise the following intuitions about the effect of decreasing marginal benefits of information. With constant marginal benefits, there often is a wide range of strict Nash networks, ranging from ones with a small diameter to ones with a large diameter. Which of these networks continue to be strict Nash under decreasing marginal benefits? A first intuition is that the networks survive where the sponsoring of a link means that the sponsor of that link gets access to the information of a large number of players; in this case, the fact that the marginal benefit of information may be small is compensated by the large

[1] A player is said to sponsor the link if he pays for it.

[2] Other ways in which the two-way flow model of BG has been extended include topics such as the effects of player heterogeneity, e.g. Galeotti et al. (2006) and Kamphorst and van der Laan (2007), and link failure, e.g. Bala and Goyal (2000b); and Haller et al. (2005).

change in information from any link. Thus, this argument favour the periphery-sponsored star, as each sponsored link then gives access to all other players. A second countervailing intuition is that the networks survive where players have relatively little information, whereby under decreasing marginal benefits their marginal benefit of obtaining the information of even a single player becomes larger. Under information decay, this means that dispersed networks, where players are at a large distance from one another, are relatively more likely to survive. Thus, while under constant marginal benefits of information, a range of networks is stable, under decreasing marginal benefits, there may be a gap in diameter between the stable networks. On the one hand, the periphery-sponsored star is stable. The next networks which are then stable may be networks with a relatively large diameter.

Hojman and Szeidl (2008) were the first to explore the consequences of decreasing marginal benefits of information. They show, among other things, that if (a) information can travel only a limited distance, (b) information strictly decreases in the distance traveled (decay) and (c) if the population is large enough, then the only non-empty strict Nash network[3] architectures (abbreviated by SNN, SNNs in plural) is that of a periphery-sponsored star (PSS)[4]. Even for non-strict decay they prove that each player will sponsor at most one link.

Although these and other results in Hojman and Szeidl (2008) are informative, elegant and clear, we find it worthwhile to continue this line of exploration. One of the reasons for this is that "large enough" for the results above is typically very large indeed. If information can travel only 3 steps in the network, and *doubling* your amount of information at the cost of one link becomes unattractive when you have '4 information'[5], then you need more than 16 million agents. This becomes rapidly larger in both of these two variables. Therefore, in this paper we look at the results when there is no lower limit to the population size. Our results, which we will discuss below, are quite different. This is partly due to the general population size, but in part also due to our use of the more standard but less general formulation of decay, namely the decay factor. This allows information to travel as far as needed (decaying along its way). Both these assumptions are essential for the results derived by Hojman and Szeidl (2008) and changing either one of them would probably already have changed the results significantly. In this way the combination of this paper and Hojman and Szeidl (2008) nicely illustrates the importance of any distance limits to information flows and population sizes.

[3]A Nash network is a network which constitutes a Nash equilibrium in the underlying network formation game.

[4]A PSS is a minimally connected network where one player, the player in the center of the star, is the recipient of each link. Network C in Fig. 1 is an example of a PSS (in this paper links, represented by arrows, point away from their sponsors). Networks A and B are also star networks but not (completely) periphery-sponsored.

[5]Each agent possesses one unit of information, see Hojman and Szeidl (2008) for details

Vergara-Caffarelli (2004) studies concave benefits[6] in the number of connections in the context of BGs one-way flow model and in the absence of decay. Buechel (2007) studies the pairwise-stable networks in a model of two-sided link formation. Finally, Goyal and Joshi (2006) look at a model in which the benefits may be concave not in the amount of information a player gathers, but in the number of links that player has.

The structure and results of this paper are as follows. We explore the impact of decreasing marginal benefits of information (DMBI) by comparing the set of SNNs under constant marginal benefits of information (CMBI) with that under DMBI. To do so we introduce in Section 2 the model, definitions and notation. For ease of exposition, we will follow the notation of Kamphorst and van der Laan (2007). In Section 3 we expand the characterization (in terms of necessary architectural conditions) of strict Nash networks in BG[7]. We do this in a way which for most results (all except Proposition 2) encompasses both CMBI and DMBI functions. As a result we are able to identify a class of architectures to which all strict Nash equilibria belong. Section 4 proceeds by looking explicitly at how the set of strict Nash networks changes if we move from a CMBI to a DMBI benefit function. We show that new networks may enter the set of SNNs. Such networks typically have relatively high diameters. Moreover we will show by examples and intuition, that the low diameter networks (especially stars), with the exception of the PSS, are quick to drop out of the set of SNNs. This, and the accompanying intuition suggests that, again excepting the PSS networks, low diameter networks are relatively unstable under DMBI. This is finding nicely supplements Hojman and Szeidl (2008), from which one could get the impression that DMBI is biased towards low diameter networks. Moreover examples show that in many, if not most, SNNs there are agents who maintain several links. Section 5 provides discussions on the relationship between our results and those in Hojman and Szeidl (2008), the way decay is modelled and the focus on SNNs. The section ends with a short summary.

2 The Model

This model is based on the two-way flow model in BG. The notation, however, follows more closely Kamphorst and van der Laan (2007). Consider a population of n agents, denoted by the set $\mathcal{N}$. Each player faces the choice to which of the other players he will sponsor a link. A link by player i (the sponsor) to player j (the recipient) is denoted by (i, j), or ij for short. The set of all links that a player i can possibly sponsor is given by

$$\mathcal{L}_i \equiv \{kj \in \mathcal{N} \times \mathcal{N} : k = i,\ j \neq i\}.$$

[6] In fact, the paper presents the model as having convex costs. Nonetheless, the model features decreasing marginal benefits of being connected to other agents.

[7] We have not allowed the decay factor to be so low that some strict Nash networks are non-minimal. Such networks have different features, and fall outside the scope of this paper.

$\mathcal{L}$ is defined as the set of all possible links, meaning that $\mathcal{L} \equiv \bigcup_{i \in \mathcal{N}} \mathcal{L}_i = \{ij \in \mathcal{N} \times \mathcal{N} : i \neq j\}$.

We typically denote the strategy of player i – the set of links that he sponsors – by g_i. His strategy space $\mathcal{G}_i$, where obviously $g_i \in G_i$, is therefore the collection of all subsets of $\mathcal{L}_i$, specifically:

$$\mathcal{G}_i \equiv \{g_i \subseteq \mathcal{N} \times \mathcal{N} : g_i \subseteq \mathcal{L}_i\}.$$

All links together form a network[8], typically denoted by g, so

$$g \equiv \bigcup_{i \in \mathcal{N}} g_i.$$

The strategy space $\mathcal{G}$ is therefore the set of all possible networks, which is the collection of all subsets of the set of all possible links. Thus

$$\mathcal{G} \equiv \{g \subseteq \mathcal{N} \times \mathcal{N} : g \subseteq \mathcal{L}\},$$

Now we come to the (dis)incentives for players to sponsor links. The disincentives arise because sponsoring links is costly. The costs of a link ij are denoted by c_{ij}, and are incurred completely by the sponsor; the recipient incurs no costs. Because we wish to focus on the benefits of link formation we model the cost side as simple as possible: $c_{ij} = c$ for all $i, j \in N$. Let $N_i^S(g) \subset \mathcal{N}$ be the set of players to whom player i sponsors a link in g, so $N_i^S(g) \equiv \{j \in \mathcal{N} : ij \in g\}$. Hence the total costs for player i in network g are equal to $\left|N_i^S(g)\right| c$.

Players derive benefits from being connected to each other by a path of links. On this path, it does not matter who the sponsor of the links are. The benefits of a link 'flow in two directions'. To make this precise, we let $\overline{ij} \in g$ denote that $ij \in g$ or $ji \in g$ or both[9]. We say that in network g players i_0 and i_k are connected if there exists some subset of players $\mathcal{N}_{i_0 i_k} \subseteq \mathcal{N}$, $\mathcal{N}_{i_0 i_k} = \{i_0, ..., i_k\}$ such that for all $\ell \in \{1, ..., k\}$ we have that $\overline{i_{\ell-1} i_\ell} \in g$. When two players are connected, they exchange their private information. Let $N_i(g)$ denote the set of players to whom player i is connected in network g.

In this paper we will assume that there is decay. In other words, as this information travels through the network it becomes less accurate or less complete. We assume that what is lost at each step is independent of the path the information travels. Hence only the shortest path between any two players is relevant. The length (i.e. the number of links) of this shortest path between players i and j in network g is denoted by d_{ij}.[10] We follow the convention in assuming that every time the information is passed on a constant fraction $(1 - \delta)$,

[8] Observe that the strategy profile coincides with the network. In this paper we will refer to any strategy profile as a network. Similarly, we will refer to any (strict) Nash equilibrium as a (strict) Nash network.

[9] So $\overline{ij} \in g$ says that the intersection of $\{ij, ji\}$ and g is not empty.

[10] Note that by two-way flow we have that $d_{ij} = d_{ji}$ for all $i, j \in N$.

$\delta \in (0,1]$, of the (remaining) information is lost. Observe that decay gives players incentives to sponsor links to players to whom they are already connected for the purpose of reducing the distance between them. However, throughout this paper we assume that δ is large enough to ensure minimality of any Nash network[11], meaning that any two players are not connected by more than one path of links. So the amount of decay is limited.

To focus on the effects of DMBI, we assume that each player has one unit of private information. The value of the information does not depend on the original owner of this information, so any two units of information are *a priori* equally valuable. Players derive benefits from the information which they gathered. To model the benefits we need a few more definitions.

Let $N_i^k(g) \subset \mathcal{N}$ be the set of players at distance k from player i in network g. So $N_i^k(g) \equiv \{j \in \mathcal{N} : d_{ij} = k\}$. Due to decay, the total amount of information gathered by player i in network g is then

$$I_i(g) = \sum_{k=0}^{n-1} \left(\delta^k \left| N_i^k(g) \right| \right).$$

Note that by definition, $N_i^0(g) = 1$.

The benefits derived by player i from network g, $V_i(g)$, are an increasing function of $I_i(g)$, specifically

$$V_i(g) = f(I_i(g))$$

where $f' > 0$ and $f'' \leq 0$. In Section 3 we will characterize the set of SNN for a given value function. After that, in Section 4, we will analyze how the set of SNN changes if we move from an CMBI function, where $f'' = 0$, to a comparable DMBI function, where $f'' < 0$.

The utility which i obtains in g equals his benefits minus his costs. Formally,

$$U_i(g) = V_i(g) - \left| N_i^S(g) \right| c.$$

Define g_{-i} as all the links in g excluding the links sponsored by player i. A network g is an SNN if for each player $i \in \mathcal{N}$ and all $g_i' \in \mathcal{G}_i$, $g_i' \neq g_i$, we have

$$U_i(g) > U_i(g_{-i} \cup g_i').$$

In an SNN, by definition, every player plays his unique best reply strategy. Denote by $BR_i^f(g)$ the set of best reply strategies of player i versus network g under function f. Formally, for any benefit function f

$$BR_i^f(g^*) = \left\{ g_i \in \mathcal{G}_i : U_i\left(g_{-i}^* \cup g_i\right) \geq U_i\left(g_{-i}^* \cup g_i'\right) \text{ for all } g_i' \in \forall \mathcal{G}_i \right\}.$$

This ends Section 2.

[11] For instance BG and Lemma 3 show that there exists some $\bar{\delta} < 1$ such that for all $\delta > \bar{\delta}$ this is indeed the case.

3 Characterization

In this section we add to Propositions 5.3 and 5.4 of BG. There BG note that non-star networks may also be SNNs for a certain range of parameter values and they introduce the linked star network[12] as an example. In this section we derive necessary architectural properties of the non-star SNN networks[13]. Note that the characterization of the SNN in this section will depend nowhere on whether there are CMBI or DMBI. The only aspect of f which we will exploit is that f is increasing. Our additional characterization of the (possibly non-star) SNNs consists of three parts. *First*, we show that each SNN is connected and that there exists some $\overline{\delta}$ such that for all $\delta > \overline{\delta}$ each SNN is minimal (3.1). *Second*, we show that, in any SNN, links point away from each other, unless they point to one and the same player (3.2). *Third*, we characterize the non-empty architectures which can be an SNN(3.3). This will greatly limit the set of network architectures which can be an SNN. After that, we conclude with a closer look at the reason why the earlier necessary characterization is not necessarily efficient (3.4).

3.1 Every non-empty SNN is connected and, for δ high enough, minimal

We first introduce some additional notation and concepts. Each network g partitions the population into *components* (of g), where two players belong to the same component if and only if they are connected. Component k is denoted as $C_k(g)$. A network is *connected* if it consists of only one component. For a network g and for M, $M \subseteq N$**,** define network g_M as the set of links of network g of which both the sponsor and the recipient of the link belong to M. Formally:

$$g_M = \{ij \in g : i, j \in M\}.$$

We now define the concept of best informed player.

Definition 1 *Let $M \subseteq \mathcal{N}$ be a connected subset of players in network g. Then player i, $i \in M$, is a best informed player of M if $I_i(g_M) \geq I_j(g_M)$ for all $j \in M$.*

If in network g some player i is not part of component $C_k(g)$, then his net benefit of a link to some j, $j \in C_k(g)$, is given by $\delta f(I_j(g)) - c$. Because $f(\cdot)$ is strictly increasing, the best link which player i can have into the component is to the component's best informed player[14].

[12] We refer to BG for details on the linked star.

[13] The resulting class of networks is a basic generalization of the linked star network, as can be verified by the reader by comparing Proposition 1 with BG (page 34-35).

[14] Note that without decay (so $\delta = 1$), every player in any connected set is a most valuable player in that set. This concept can also be useful when considering heterogeneous agents.

The following two results show that any SNN satisfying the parameter conditions is connected[15].

Lemma 1 *Let network g be a non-empty SNN. Then g has no singleton component.*

Proof. We prove this by contradiction. Suppose that $ii' \in g$, and player j is unconnected to everyone (he is a singleton component). Since $I_j(g) \leq I_i(g \backslash \{ii'\})$, and since we have DMBI or CMBI, he marginal benefit for player j to sponsor a link to i' is at least as large as the marginal benefit of ii' to i. Because the marginal costs of a link are the same for each player, and ii' is part of the unique best reply of player i, we have that j prefers sponsoring ji' to sponsoring no links. Hence g is not an SNN, from which the contradiction arises. ■

Lemma 2 *Let network g be a non-empty SNN. Then g is connected.*

Proof. Suppose not. Then by Lemma 1 there is a strict Nash network g which contains multiple non-singleton components. Without loss of generality we have $ii', jj' \in g$ such that i and i' belong to one component, say $C_1(g)$, and j and j' to another, say $C_2(g)$. Because g is a strict Nash network, player i prefers to sponsor a link to i' and not to any player in $C_2(g)$. So player i' has access to more information in $g \backslash \{ii'\}$ than j' in g. Hence we obtain that $I_{i'}(g) > I_{i'}(g \backslash \{ii'\}) > I_{j'}(g) > I_{j'}(g \backslash \{jj'\})$. Because g is Nash, we also have that $I_{i'}(g) < I_{j'}(g \backslash \{jj'\})$, which gives us a contradiction. Hence any Nash network has but one component and is therefore connected. ■

So *all non-empty* SNNs are connected. Naturally, if costs are low enough the empty network is not an SNN, implying that *all* SNNs are connected. Since the marginal benefit of a link to an otherwise isolated player is at least $f(1+\delta) - f(1)$, Lemma 2 implies the following corollary.

Corollary 1 *Let $c < f(1+\delta) - f(1)$, then any network g which is an SNN is connected.*

BG pointed out that there always exists a strict Nash network. Under these more general benefit functions this remains true. Note that for $c > f(1+\delta) - f(1)$ the empty network is an SNN, while the PSS is always an SNN for $c \leq f(1+\delta) - f(1)$.

A network is *minimal* if the deletion of any link in that network will result in an increase of the number of components. A *cycle* is a set of links $\left\{\overline{j_0 j_1}, ..., \overline{j_{k-1} j_k}\right\}$ such that $j_0 = j_k$. This implies that a component (or network) is minimal if and only if it contains no cycles. Finally, a *redundant link* is a link which is part of a cycle.

[15] To let this result be applicable in the case of decreasing marginal returns, we cannot refer to BG directly, although the intuition is of course similar.

As we stated before, we will look at the case where δ is high enough to ensure that all SNNs are minimal. The following lemma shows that δ can indeed be high enough.

Lemma 3 *Consider any benefit function f and cost level c Then a level of decay $\bar{\delta}$ exists such that for all $\delta > \bar{\delta}$ no SNN contains any cycles.*

Proof. First note that if $\delta = 1$, then players have no incentive to sponsor costly links to players to which they are already connected, meaning that there can be no cycles in any SNN. By continuity of payoffs in δ, this is also true for some $\bar{\delta} < 1$ and all $\delta \in \left[\bar{\delta}, 1\right]$. ■

If such a $\bar{\delta} < 1$ exists for any one benefit function, given c, then it also exists for any two benefit functions given that c.[16]

Corollary 2 *Consider any two benefit functions, f and f', and cost level c. Then a level of decay $\bar{\delta}$ exists such that for all $\delta > \bar{\delta}$ no SNN contains any cycles for any of the two benefit functions..*

3.2 In any SNN, links point away from each other unless they point to the same player

Let $ii' \in g$. Then we denote the set of players observed by player i exclusively via link ii' by $A_{ii'}(g)$. So

$$A_{ii'}(g) = \{j \in \mathcal{N} : j \in N_i(g) \text{ and } j \notin N_i(g \backslash \{ii'\})\}.$$

Additionally, we define $A^I_{ii'}(g)$ as the set of best informed players in $A_{ii'}(g)$. So

$$A^I_{ii'}(g) = \left\{j \in A_{ii'}(g) : j \text{ is a best informed player in } g_{A_{ii'}(g)}\right\}.$$

In Section 2 we assumed that δ is high enough to ensure that each SNN is minimal. This, together with the definition of $A^I_{ii'}(g)$, gives us the following Lemma.

Lemma 4 *If g is a Nash network then $j \in A^I_{ij}(g)$ for all $ij \in g$. Furthermore, if g is a strict Nash network, then $\{j\} = A^I_{ij}(g)$ for all $ij \in g$.*

Having derived this result, we are now ready to show that if both i and j observe each other via a link that they sponsor themselves, the recipient of the link sponsored by i is the same player as the recipient of the link by j.

Lemma 5 *Let g be a strict Nash network. If $j \in A_{ii'}(g)$ and $i \in A_{jj'}(g)$ then $i' = j'$.*

[16] Define the new $\bar{\delta}$ as the maximal of the $\bar{\delta}$ of the two individual benefit functions.

Proof. We prove this by contradiction. Suppose not, so $i' \neq j'$. Note that by minimality there is one path connecting i and j, and this path goes via players i' and j'. Now observe that, with decay, the players who lose most if a link is deleted are the ones closest by. So

$$\begin{aligned} I_{i'}(g) - I_{i'}(g \backslash \{ii'\}) &> I_{j'}(g) - I_{j'}(g \backslash \{ii'\}), \text{ and} \quad (1) \\ I_{i'}(g) - I_{i'}(g \backslash \{jj'\}) &< I_{j'}(g) - I_{j'}(g \backslash \{jj'\}) \quad (2) \end{aligned}$$

By Lemma 4, i' is more informed than j' in $g_{A_{ii'}(g)}$, implying that

$$I_{i'}(g \backslash \{ii'\}) > I_{j'}(g \backslash \{ii'\})$$

Applying Eq. 1 gives:

$$I_{i'}(g) > I_{j'}(g),$$

after which Eq. 2 tells us that

$$I_{i'}(g \backslash \{jj'\}) > I_{j'}(g \backslash \{jj'\}) \quad (3)$$

However Eq. 3 implies that $j' \notin A^{I}_{jj'}(g)$, which contradicts Lemma 4. Hence a contradiction arises. ■

Verbally the proof of this lemma is that player i' receives more information via the link ii' than player j' does, as i' is at least one link closer to i and the players behind i than j' is. Similarly, player j' receives more information than i' via the link jj'. Now, if g is a SNN, then i' is more informed than j' in $g_{A_{ii'}(g)}$, and therefore i' is also more informed than j' in $g_{A_{jj'}(g)}$. However that contradicts that g is Nash, since jj' is sponsored in g.

3.3 Characterization

For the following proposition, recall that δ is large enough to ensure that strict Nash networks are minimal. This gives us the following characterization of strict Nash networks:

Proposition 1 *Let g be a non-empty strict Nash network and $V_i(g) = f(I_i(g))$ for all $i \in \mathcal{N}$. Moreover let δ be high enough to ensure that all SNN under function f are minimal. Then:*

1. *$A^{I}_{ij}(g) = \{j\}$ for all $ij \in g$;*
2. *there exists a player i^C, said to be the 'central player', such that between i^C and any player $j \in \mathcal{N} \setminus \{i^C\}$ there is a unique path $\{\overline{j_0 j_1}, ..., \overline{j_{k-1} j_k}\}$ where $j_0 = i^C$, and $j_k = j$. All links on this path, with the possible exception of $\overline{j_0 j_1}$, are center-sponsored; thus: for $1 < l \leq k$ we have that $j_{l-1} j_l \in g$, and $j_l j_{l-1} \notin g$.*

Proof. Part 1 is a repetition of Lemma 4. Part 2 claims two things. First that the network is connected. This follows immediately from Lemma 2. Second it tells us that there exists a central player who is connected with all other player through paths of links which contain links directed away from him and links which he receives. This we will now prove by construction. By minimality of g, there exists a player i who receives no links. By connectedness (Lemma 1) he sponsors at least one link, so $N_i^S(g)$ is non-empty. Note that if $N_i^S(g)$ contains multiple players, Lemma 5 implies directly that only one of them can receive multiple links (so a link by any player other than i). Let $i' \in N_i^S(g)$ be the player who receives the maximal number of links in $N_i^S(g)$. In the rest of this proof we will show that i' fits the definition of a central player. We can distinguish three cases. First, there is some player j, $j \neq i$, sponsoring $jj' \in g$ such that he observes i via i'. In that case, by Lemma 5 $j' = i'$. And by the same lemma any other link, say ll', either has i' as its recipient, or $i' \notin A_{ll'}(g)$, meaning that the link is directed away from i'. Given connectedness, this ensures that i' fits the requirements of a central player. Second, any sponsor j, $j \neq i$, of a link $jj' \in g$ such that $i' \in A_{jj'}(g)$ observes i' via i. Because i receives no links this connection runs through a link $ii'' \in g$, and by Lemma 5 $i'' = j'$. In that case i'' receives more links than i', which is a contradiction by construction (i' being the player who receives the maximal number of links in $N_i^S(g)$). Third, there is no player j, $j \neq i$, sponsoring a link $jj' \in g$ such that $i' \in A_{jj'}(g)$. In that case i' is connected with every other player through links directed away from him, with link ii' as the only exception. Hence i' again satisfies the condition for being a central player. Hence there exists a central player as described by Part 2. This concludes the proof. ■

Note that Part 2 of Proposition 1 does not say that there is a unique central player. It only says that there is at least one player satisfying the conditions. The characterization applies, no matter which 'central player' is considered. However we can say more based on this characterization. Part 2 of Proposition 1 implies that if a player receives two or more links, then he is the unique central player. To see this, consider some SNN, say g, and let $ij, i'j \in g$. Now suppose that some j' is a central player. Because ij and $i'j$ point towards each other, either one or both of these links point towards j'. However Part 2 of 1 directly says that then j' must be the recipient of that link. Hence j' must be j and there are no other central players.

Corollary 3 *Let g be an SNN. Then any player who receives more than one link is the unique central player.*

Now suppose that in some SNN g, there is no player who receives two or more links. Who are then the central players? By minimality $n-1$ links are received, and by construction no player receives multiple links. Hence there is a unique player who receives no links, say player i. By connectedness, the set of recipients of links by i, $N_i^S(g)$, is non-empty. Furthermore, no link is directed towards player i. If not, then the link pointed towards i should be received by a player in $N_i^S(g)$ as proved by Lemma 5. Hence that recipient would receive two links, which is ruled out by construction. However if all links point away from i, then i is a central player. And since the first link may point towards a central player, all of the players in $N_i^S(g)$ are central players too. Hence the following Corollary follows from Part 2 of Proposition 1.

Corollary 4 *Let g be an SNN where no player receives more than one link. Then the unique player to whom no links are sponsored is a central player, as are all players to which this unique player sponsors links.*

Let us discuss a few examples to clarify. Figure 1 on Page 12 show examples of networks satisfying Proposition 1. For instance, in network A in Figure 1.all players are central players. However in each of the other six networks there is but one central player. For Networks B and C it is the center of the star (the player in the middle). Also for Network F it is the player in the middle. For Networks D, E and G it is player i. In each of these networks, except for Network A, the central player received at least two links, and was unique (Corollary 3). However, note that if we would reverse the link ki in Network G (so replace the link ki by the link ik), denoting the resulting network G^R then the resulting network can still be an SNN and it would have two central players, namely player i and player k'. In Networks A and G^R no players receive more than two links, so the player who receives no links (the center in Network A, k' in Network G^R) is a central player, as well as all the recipients of his links (Corollary 4).

The following lemma says that all best informed players are central players, and therefore at least one central player is a best informed player. It does *not* imply that all central players are best informed players.

Lemma 6 *In any non-empty SNN, the best informed player is one of the central players, as defined in Proposition 1.*

Proof. We prove this by contradiction. Suppose not. Then there exists a g which is an SNN such that there is a link $ii' \in g$ through which i observes j, where $j \neq i'$ and j is the most valuable player. By the same arguments as in the proof of Lemma 5, this implies that $I_j(g \backslash \{ii'\}) > I_{i'}(g \backslash \{ii'\})$, which implies that i would benefit from replacing ii' by ij in g. Hence we have a contradiction.

■

Corollaries 3 and 4 effectively classify all non-empty SNNs into two categories on the basis of the number of central players: 1 or multiple. In the first type of equilibrium, at least two links are sponsored to a particular player. This player is then the unique central player, and is also the best informed player. In the second type of SNN, there is no player to whom more than one link is sponsored. In this case, all links are (directly or indirectly) center-sponsored. The multiple central players in this case are the unique player to whom no links

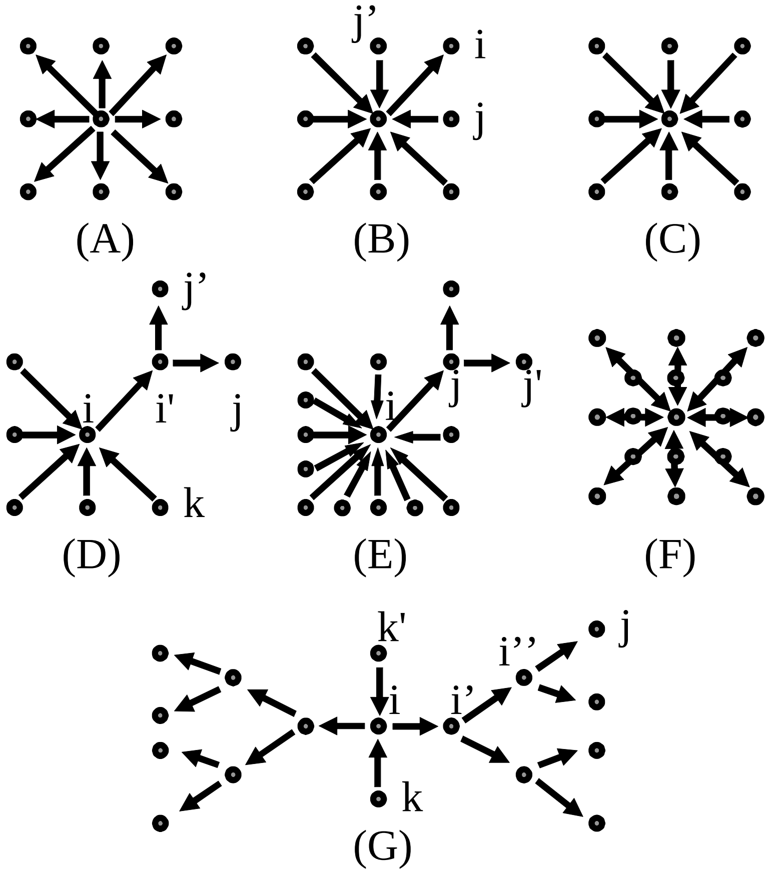

Figure 1: Example networks fitting Proposition 1.

are sponsored, and the players to whom this unique player sponsors links. At least one of these multiple central players is the best informed player.

Finally we know that in equilibrium, every recipient, say player j, of a link, say ij, has either at least two other links, or no other links. The reason is the following. If j has one other link, then player i observes at least two players through ij. If there are two players then he will be indifferent between them, hence g is not be an SNN. If there are more than two players, then j is certainly not the best informed player of the group, because he is at the complete edge. His neighbor in that group will be better informed. In this case i would strictly prefer to replace ij by some other link. This gives us the following Corollary.

Corollary 5 *Every recipient of a link in an SNN either has no other links, or at least two other links.*

As a final remark of this section, we point out that the simulation results reported by BG are all consistent with the characterization above.

3.4 Why the necessary characterization above may not be sufficient.

We conclude with a final proposition which tells us two things. First, not every network satisfying Proposition 1 is an SNN. Second, if a network satisfying Proposition 1 is not an SNN, then this is because some player has a strict preference to sponsor a redundant link (note that this is not necessarily excluded, even if δ is high enough to exclude non-minimal networks from being an SNN).

Proposition 2 *Let $c < f(1+\delta) - f(1)$ and let there be CMBI. Then any network g satisfying Proposition 1 is either an SNN or contains a player who wishes to sponsor a redundant link.*

Proof. Suppose that g satisfies Proposition 1 and is not an SNN. Then there exists at least one player, say player i, who prefers to sponsor a different, possibly empty, set of links. By minimality of g and $c < f(1+\delta) - f(1)$ we have that i does not want to sponsor less links than he currently does. By $A^I_{ij}(g) = \{j\}$ for all $ij \in g$ (see Proposition 1) he also doesn't want to replace his link to player j by a link to any player in $A_{ij}(g)$. Replacing this link by a link to any player outside $A_{ij}(g)$ is not preferrred, again by minimality of g and $c < f(1+\delta) - f(1)$. So the only option is that player i prefers to sponsor more links than before, including therefore a redundant link. ■

This Proposition does not apply to DMBI functions however. To see this, note that an end sponsor, say player i, in a network g satisfying Proposition 1 will have more information than 1. This is because g is connected and $\delta > 0$. Hence $I_i(g) > 1$. And under DMBI we have that $f_D(I_i(g)) - f_D(I_i(g \backslash \{ij\})) < f_D(1+\delta) - f_D(1)$, where j is the recipient of an end link by i. If $c > f_D(I_i(g)) - f_D(I_i(g \backslash \{ij\}))$, then g is not an SNN. Note that as this problem depends both on the information obtained through the link as well as the infomation the sponsor would have without the link, this problem can also be caused by better informed players who only sponsor non-end links. Under CMBI this consideration plays no role, since a player's incentives to sponsor a non-redundant link are independent on the amount of information without the link.

4 Decreasing marginal benefits of information

In this section we look at the effects of DMBI, formally of $f'' < 0$. To compare the two regimes we need to make the value functions comparable. We do this by assuming that $f'_D(1) = f'_C(1)$. Of course $f''_D(x) < 0$ and $f''_C(x) = 0$ for all $x \geq 1$. See Figure 2 for an illustration on the relation between f_C and f_D.

Definition 2 *For any CMBI value function $f_C(I)$ a DMBI function $f_D(I)$ is comparable if $f_D(1) = f_C(1)$ and $f'_D(1) = f'_C(1)$.*

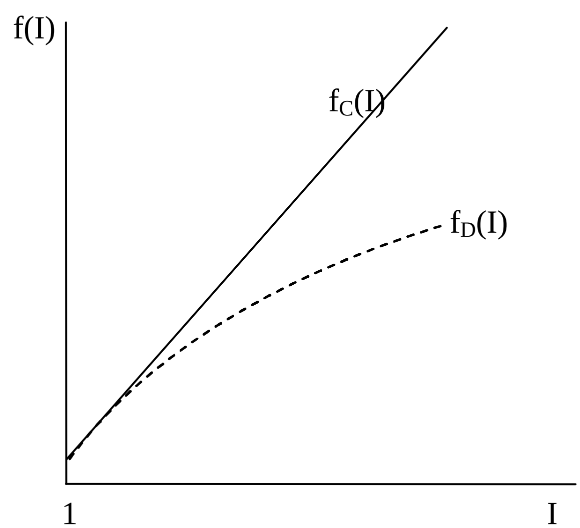

Figure 2: Relation between CMBI and DMBI.

In the previous section we derived a characterization with necessary conditions for non-empty SNNs (Proposition, provided that the benefit function is increasing. In this section we will look what changes with the set of SNN if we switch from any CMBI function, say f_C, to any comparable DMBI function, say f_D.

The switch from CMBI to a comparable case of DMBI decreases therefore the incentives of players to sponsor links. As a result some networks which were SNN under f_C, are not SNN under f_D. We will prove that if any non-minimal SNN exists, then any PSS is an SNN for any f. So if PSS is not an SNN then no non-empty networks are SNN. One could say that the PSS is the last non-empty network to 'leave' the set of SNN when changing from CMBI to DMBI. However, we also prove that any other type of star is among the first to leave the set of SNN when switching to DMBI. We continue by showing that networks with higher diameters may stay an SNN, even when shorter diameter networks, like all non-periphery-sponsored stars, are not an SNN anymore. Although the intuition for this result is clear and accessible, it also has limits, which will be discussed. Finally, we will look into the possibilities of networks which become an SNN due to the switch. Here we will also see that these networks have a relatively high diameter, in a sense which will be explained in detail there.

For optimal exposition, we define any link through which the sponsor observes only the recipient as an end link; the sponsor of this link an end sponsor and the recipient an end player. Moreover, we will look at the case in which all SNN are connected under CMBI, so with $c < f_C(1+\delta) - f_C(1)$. Although $c < f_C(1+\delta) - f_C(1)$, it is possible that $c > f_D(1+\delta) - f_D(1)$ or even $c > f_D\left(1+\delta+(n-2)\delta^2\right) - f_D(1)$, because the marginal benefits of links have decreased. As this affects the set of SNN its effects are included in the following proposition

Proposition 3 *Let $f_C(I)$ be an CMBI function, and let $f_D(I)$ be a comparable DMBI function; let $c < f_C(1+\delta) - f_C(1)$; and finally let $\mathcal{G}_C^{SNN}$ be the set of SNNs under the CMBI function, and $\mathcal{G}_D^{SNN}$ be the set of SNNs under DMBI. Then*

1.
    (a) If $c < f_D(1+\delta) - f_D(1)$, then Proposition 1 applies.
    (b) If $c > f_D\left(1+\delta+(n-2)\delta^2\right) - f_D(1)$, then $\mathcal{G}_D^{SNN}$ consists of the empty network alone.
    (c) If $f_D(1+\delta) - f_D(1) < c < f_D\left(1+\delta+(n-2)\delta^2\right) - f_D(1)$, then $\mathcal{G}_D^{SNN}$ encompasses only all PSS and the empty network.

2. If there exists any non-empty network g which is an SNN, then any PSS is also an SNN.

3.

   (a) If there exists a network $g \in \mathcal{G}_C^{SNN}$ such that $g \notin \mathcal{G}_D^{SNN}$ then the only stars which are SNN under f_D are PSS.

   (b) If $c > f_D(1 + (n-1)\delta) - f_D(1 + (n-2)\delta)$, then any star which is not a PSS is also not an SNN.

4. If any network g is SNN under DMBI but not under CMBI, so $\mathcal{G}_D^{SNN} \backslash \mathcal{G}_C^{SNN}$, then this network g is not an SNN under CMBI because it is either empty or at least one player wished to sponsor a redundant link in g under f_C.

Proof. 1. (a) This follows directly from Proposition 1. **(b)** The benefit to the sponsor of any link in a PSS with n players is $f_D\left(1 + \delta + (n-2)\delta^2\right) - f_D(1)$. This is also the maximal benefit any link can generate. Hence if such a link is too costly, no link is worth sponsoring and sponsoring no links is strictly prefered by every player. Consequently the empty network is the unique SNN. **(c)** Note first that Proposition 1 is independent of the value function, as long as the value function is increasing. Hence it also applies to this cost range. Second note that any non-empty SNN, say g, other than the PSS will contain an endsponsor, say i, who observes at most one player (j) through his link. However, by connectedness of g, DMBI and $f_D(1+\delta) - f_D(1) < c$ we have that $f_D(I_i(g)) - f_D(I(g \backslash \{ij\})) < f_D(1+\delta) - f_D(1) < c$. Hence the set of non-empty SNN can only contain PSS networks. The marginal benefit of the link sponsored by any peripheral player is given by $f_D\left(1 + \delta + (n-2)\delta^2\right) - f_D(1)$. As this is larger than c, any PSS is an SNN. Finally, in the empty network the marginal benefit to player i of sponsoring k links instead of zero is smaller than $k\left(f_D(1+\delta) - f_D(1)\right)$ and therefore smaller than c. Hence PSS and the empty network are SNN, and all other networks are not.

2. The maximal marginal benefit any link can yield is one in which the sponsor would access no other player without that link, and all other players at a maximal distance of two with that link. In the PSS, every sponsor receives this maximal marginal benefit for the link that he sponsors. So, if any player prefers to delete a link, no non-empty network can be stable. Replacing the link by a link to another player (so replacing a link to the star center by a link to another peripheral player), has no other effect than putting $n-3$ players at distance 3 which were at distance 2. Hence this is also not optimal. Lastly, we show that in any other non-empty network g there exists a player who has at least as much incentives to sponsor redundant links as any player in the PSS. Because g is an SNN, the benefits of this link are apparently not larger than the costs, hence the same will hold true for any PSS. Note first that in the PSS a redundant link brings a maximal increase in information of $\delta - \delta^2$ (reducing the distance to one other player from 2 to 1). At the same time, the sponsor in the PSS has a minimal information of $1 + \delta + (n-2)\delta^2$. In g there is, by minimality of g, at least one player who receives maximally $1 + \delta + (n-2)\delta^2$ information, while being able to sponsor a link to an agent who presently is at least two distance away. Concluding, if there is a non-empty g which is an SNN then any PSS is an SNN too.

3. (a). Of all the links in all the minimal networks, a center-sponsored link in a star yields the lowest possible net benefit to its sponsor. The sponsor has the maximal amount of information without that link, namely $1+(n-2)\,\delta$, while he receives the minimal possible extra information via a non-redundant link, which is δ. Therefore every other sponsor of any non-redundant link will have at least as much incentives to sponsor his links, as the center of the star has. It follows that, if any network is an SNN under CMBI but not under DMBI, non-PSS stars are not SNN under DMBI. **(b)** Consider any star network. If the center of the star, in the sense that he shares a link with each other player, sponsors any of those links, the marginal benefit on any one of those links to him is given by $f_D\left(1+(n-1)\,\delta\right)-f_D\left(1+(n-2)\,\delta\right)$. Hence for costs larger than this, there exists no star other than the PSSs which is an SNN.

4. Note first that under DMBI the marginal costs of each link are the same as under CMBI, but the marginal benefits are lower. Hence the only thing in terms of payoffs which changes from CMBI to DMBI is that the marginal benefits of links decreases. The preference ordering of any player i in network g_{-i} over the potential recipients of his links does not change, since this ordering depends only on the gain of information through a link and not on the marginal benefit which this gain would entail. It is therefore impossible that a strict best reply of player i against g under f_D contains any link which the best reply of i against g under f_C does not contain. In other words given a network g, if $g_i^c\in BR_i^{f_C}(g)$ and $g_i^D\in BR_i^{f_D}(g)$ then $g_i^D\subseteq g_i^C$. Since $\delta>\delta^C$ and $c<f_C(1+\delta)-f_C(1)$ imply that all $g\in\mathcal{G}_C^{SNN}$ are minimally connected, we have that for any $g\in\mathcal{G}_D^{SNN}\backslash\mathcal{G}_C^{CNN}$, then either g is empty (which is not a SNN under CMBI), or there exists some player j who prefers to sponsor a redundant link in g under CMBI but not anymore under DMBI. ■

Note that point 2 of Proposition 3 implies that the PSS are the 'last'[17] networks to leave the set of SNN[18] when DMBI is introduced, while point 3a implies that all other stars are the 'first' networks to leave the set of SNN when this occurs. They are least stable in this respect. As the proof indicates, this is because if the center of a star sponsors a link in the SNN, he gets relatively little new information because it will be an end link, while he already has the maximal possible information within a network where one player is isolated (unconnected to all other players). Hence his marginal benefit is minimal.

But accepting this reason, it follows that a network is less quick to leave the set if the critical sponsors (the sponsors who would be first to stop sponsoring some link) are (a) receiving more information through that link and (b) receiving less information via the rest of the network. If we follow up on this than we

[17] Obviously, they are not the last networks in the sense that by the switch to DMBI all networks leaving the set of SNN will leave at the moment of the switch. However the PSS are last in the sense that they will only leave this set if all other networks in that set leave as well. For a more literal interpretation one could consider increasing the degree of DMBI (defined in some appropriate way) in several steps. Then the PSS would be literally among the last network of the original networks in the set to leave.

[18] Or 'least likely'.

understand that a non-star network in $\mathcal{G}_C^{SNN}$ will be more stable than star networks like A and B from Figure 1, because the sponsors get at least as much information through any one of their links then the unique end sponsor in those stars, while they receive less information via the rest of the network.

As clearly stars including any sponsoring to the center are the first to drop out, this suggests two aspects that can make a network stable under DMBI. The first aspect is that a single link gives access to a large number of players. The fact that the marginal benefit is reduced under DMBI then plays less of a role. This is why PSS are always stable. The second aspect leading to stability is that players have relatively little information, and therefore still find it worth to sponsor links under DMBI. This is witnessed by the fact that degrees of DMBI exist such that the only networks that are potentially stable next to PSS are networks with a diameter of three or more.

We will now argue that higher diameter networks are relatively stable under DMBI. The intuition is that in a large diameter network players tend to have less information, giving them more incentives to maintain their links. We will first give a few examples to illustrate the intuition. After that we will present a proposition which shows that non-PSS networks of diameters 2 and 3 drop out before all networks of diameter 4 drop out. Hence a gap in diameter may arise due to DMBI, where the PSS is the only 'short diameter' network, and all other networks have a minimum diameter larger than 3. Finally, we will place a few comments to show why this relationship between diameter and stability is not absolute.

Example 1 *Assuming CMBI, network A in Figure 1 is stable if and only if $\delta > c$. With DMBI the incentive for the central player to sponsor the last of his links, say the link to player i, is reduced. He will only wish to sponsor all of these eight links if*

$$f(1+8\delta) - f(1+7\delta) > c, \tag{4}$$

and by decreasing returns we have that

$$\delta > f(1+8\delta) - f(1+7\delta). \tag{5}$$

Hence the stability condition has become more strict under decreasing returns. So for

$$f(1+8\delta) - f(1+7\delta) < c < \delta, \tag{6}$$

a CSS is stable under CMBI but not under DMBI.

Suppose that c satisfies the conditions in Eq. 6. Then the CSS is unstable, while the PSS is still stable. But what about other star networks? For example, would Network B be stable? It is easy to check that the stability condition for Network B is the same as for network A, namely the one in Eq. 4. In fact, if Eq. 6 holds, the only stable star is the PSS (see Network C).

Now consider Network D instead of networks A and B. D is the same as B except that in D one of the formerly peripheral players, namely i', sponsors *links to players j and j', whereas in network B j and j' sponsor links to player i. Network D is stable if both of the following two conditions is satisfied (for respectively links ii' and $i'j$):*

$$f\left(1+6\delta+2\delta^2\right)-f\left(1+5\delta\right) \geq c \quad (7)$$

$$f\left(1+3\delta+5\delta^2\right)-f\left(1+2\delta+5\delta^2\right) \geq c. \quad (8)$$

Eq. 7 is certainly satisfied if Eq. 8 is satisfied and $\delta > \frac{3}{5}$.[19] *More importantly, note that Eq. 8 is a weaker condition than Eq. 6. Hence, even if Network B is not stable due to decreasing returns, Network D may still be stable. But if Network D is unstable as a result of decreasing returns, then Network B is unstable too. The reason why D is stable while B is not, is because end sponsor i in Network D has less information than the end sponsor in networks A and B.*

Moreover, but probably for a very tight range of parameters, it could be possible to replace the link ki by ik and that the network is still an SNN, while in B the player in the center of the star would prefer not to sponsor any link at all. The reason is that i has less information than the center of the star in B, so that it has higher incentives to sponsor a link then the star center in B.

Example 2 *Consider Network G in Figure 1. By Proposition 1 this network can be stable under CMBI. Let this be the case. With decreasing returns it is stable if all of the following conditions (for respectively links ii', $i'i''$ and $i''j$) are satisfied:*

$$f\left(1+4\delta+4\delta^2+8\delta^3\right)-f\left(1+3\delta+2\delta^2+4\delta^3\right) > c \quad (9)$$

$$f\left(1+3\delta+6\delta^2+2\delta^3+4\delta^4\right)- f\left(1+2\delta+4\delta^2+2\delta^3+4\delta^4\right) > c \quad (10)$$

$$f\left(1+3\delta+2\delta^2+4\delta^3+2\delta^4+3\delta^5\right)- f\left(1+2\delta+2\delta^2+4\delta^3+2\delta^4+3\delta^5\right) > c \quad (11)$$

Note that for large enough δ, namely $\delta > 0.77$, again only the last condition is relevant. Now compare this to a network such as D, but with 15 players. Such a network is Network E. Now note that the two relevant conditions for stability are (for respectively the link received by i and the link ij)

$$f\left(1+12\delta+2\delta^2\right)-f\left(1+11\delta\right) > c \quad (12)$$

$$f\left(1+3\delta+11\delta^2\right)-f\left(1+2\delta+11\delta^2\right) > c \quad (13)$$

Also here only the last condition is relevant for δ large enough, specifically $\delta > \frac{9}{11}$ suffices. And by decreasing returns, the condition in Eq. 13 is more strict than that in Eq. 11.

[19] Note that $f(\cdot)$ is an increasing function in its argument, and that for $\delta \in \left(\frac{3}{5}, 1\right)$ we can rank the arguments as follows:

$$1+5\delta < 1+2\delta+5\delta^2 < 1+3\delta+5\delta^2 < 1+6\delta+2\delta^2.$$

These examples, point 3 from Proposition 3 as well as the intuition provided networks suggest that DMBI relatively favors longer diameter networks to shorter diameter networks as compared to the base CMBI function. This suggestion is further reinforced by point 4 from Proposition 3, because in any non-empty network entering the set of SNN some player wants to sponsor a redundant link under CMBI, but not under the comparable DMBI. Sponsoring a redundant link is only worthwhile if without the link sufficiently many players are far enough away which can be brought sufficiently closer by the link. Although this does not prove a large diameter, this is rather more typical for large diameter networks. So there is some evidence that low diameter networks are relatively quick to leave the set of SNN at a switch from CMBI to a comparable DMBI, while the networks which enter have typically higher diameters. Does this imply that DMBI may create a gap in diameters, where after the PSS (with diameter 2) the lowest diameter of a stable network is larger than 3? The following Proposition shows that this can indeed be the case.

Proposition 4 *Consider a network formation game with n, $n \geq 6$, players, decay level δ and a cost level c such that $1 > c > \delta + (n-4)\,\delta^2 - (n-3)\,\delta^3$. Then there exists a comparable DMBI benefit function f such that all networks with diameter lower than 4 (except for the PSS networks), are not SNNs, while there exists an SNN of diameter 4 or more. There does not exist a comparable DMBI function g such that all diameter 4 networks are not SNN while some network with diameter 2 (other than PSS) or 3 is an SNN.*

Proof. For the given limits on n, c and δ, networks g and g' which we will present below are SNNs under CMBI. We will first prove that for some f, no diameter 3 network is an SNN (by Part 3a of Proposition 3 this implies that only the PSS is stable for diameters lower than 4). We do this by considering some network g of diameter 3 which is an SNN if any diameter 3 network is SNN. We find the parameter values for which it is not an SNN anymore and then prove that for these parameters network g' of diameter 4 can still be an SNN. The reader can verify that for the cost range in the lemma, both g and g' will be SNN under CMBI (so, for example, no player wishes to sponsor a redundant link).

The most stable diameter 3 network is a periphery sponsored star with one alteration, namely that one link by a peripheral player, say j, is deleted and replaced by a link from one non-isolated peripheral player i, $i \neq j$, to player j.[20] In that case i has with link ij information equal to $1 + 2\delta + (n-3)\,\delta^2$, while without that link he has $1 + \delta + (n-3)\,\delta^2$. If

$$f\left(1 + 2\delta + (n-3)\,\delta^2\right) - f\left(1 + \delta + (n-3)\,\delta^2\right) < c \tag{14}$$

then no diameter 3 network will be an SNN.

[20] This is most stable in any diameter 3 network the end link is critical. So for stability purposes, the sponsor of the end link should have as little information as possible. This network puts most players at distance 2 (the maximal distance) from player i.

Now we will show that the incentives to sponsor the critical link for some diameter 4 network which we will name g' are strictly larger. Consider a minimal connected network g' which looks like an n-player version of Network F in Figure **1**. Specifically, let (i) player 1 sponsors no links, whilst receiving one link from $\frac{n-1}{2}$ (rounded up) distinct other players other than player 1, (ii) these distinct other players receive no links, while sponsoring at most 2 themselves. Clearly, the end links are the critical links in this network. Consider some arbitrary end link, say $i'j' \in g'$. Link $i'j'$ is optimal if[21]

$$f\left(1+2\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3\right)-f\left(1+\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3\right)>c. \quad (15)$$

So if Eq. 15 holds, then at least one network with diameter 4 exists, namely g'.

Because $1+\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3<1+\delta+(n-3)\,\delta^2$, and the additional information through the link in both cases is δ, DMBI gives us

$$\begin{gathered} f\left(1+2\delta+(n-3)\,\delta^2\right)-f\left(1+\delta+(n-3)\,\delta^2\right) \\ < \\ f\left(1+2\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3\right)-f\left(1+\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3\right) \end{gathered}$$

Hence Eq. 15 is only *violated* if Eq. 14 is *satisfied*, but the reverse is not true. Hence we can find a DMBI function f such that

$$\begin{gathered} f\left(1+2\delta+(n-3)\,\delta^2\right)-f\left(1+\delta+(n-3)\,\delta^2\right) \\ <c< \\ f\left(1+2\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3\right)-f\left(1+\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3\right). \end{gathered}$$

This concludes our proof. ■

However, this suggestion (DMBI favors higher diameter as compared to CMBI) deserves some criticism, at least because of two main reasons. First,

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[21] Player i' sponsors (without his one end link) only a link to the central player of g'. So there are $(n-3)$ players distributed over distance two and three. By construction this distribution is as uniform as possible, with, if $(n-3)$ is odd, one more player at distance two than at distance three.

This gives us respectively

$$\begin{aligned} I_{i'}\left(g'\backslash\left\{i'j'\right\}\right) &= 1+\delta+\frac{n-3}{2}\delta^2+\frac{n-3}{2}\delta^3, \text{ and} \\ I_{i'}\left(g'\backslash\left\{i'j'\right\}\right) &= 1+\delta+\frac{n-2}{2}\delta^2+\frac{n-4}{2}\delta^3. \end{aligned}$$

Because $\delta^2>\delta^3$, the latter expression is higher, so that this is the upper bound of the knowledge of player i'.

the reasoning above sidesteps the fact that having larger diameter doesn't imply that the maximal information among the set of end players is lower. As far as we know, this far from proven, and we doubt that this is true. For instance, the diameter may not directly depend on whether the most informed player is or is not an endsponsor (see Network G with link *ki* replaced by *ik* for an example). Second, it ignores that having a larger diameter may decrease the marginal information received through links. That this is not irrelevant is also shown by the stability of the PSS.

5 Concluding remarks

In this final section we will first relate our work to Hojman and Szeidl (2008). Then we will discuss a few modelling choices in which we followed the conventions of the literature. The advantages of following such conventions are clear. It enhances the comparability of the results in the literature and, typically, simplifies the analysis. However the way in which decay is modelled, as well as the focus on SNNs warrant some additional considerations of which the reader should be aware. Finally we will briefly summarize the main points of the paper.

5.1 Hojman and Szeidl

Why do our results differ from Hojman and Szeidl (2008)[22]? They assume that information cannot travel further than some number of links d. As a result and regardless of population size, in any SNN all players will be within a certain distance from each other, based on d. Then, if one increases the population enough, no player will ever want to sponsor more than two links[23], for if there are sufficiently many players, they will receive so much information through any links received and possibly the first link sponsored that the benefits of the second link can never warrant the costs of a link. For reasons similar to those in our paper their SNN is also minimally connected. And for reasons similar to those of Corollary 5 no recipient of a link will sponsor exactly one link himself. Since no player finds it worthwhile to sponsor two or more links, the only remaining candidate is the PSS network.

Comparing this to our previous discussion of the suggested relative instability of short diameter networks, we can ask ourselves what would happen if we take a large enough population and considering only networks with a diameter of less than some positive integer. Each end sponsor would then have so much information that he would prefer not sponsoring the end link. Hence only networks without end sponsors can be non-empty SNN, which means that PSSs are the only non-empty SNN. So we can reproduce this result by Hojman and Szeidl (2008) if we impose an upper limit on the diameter of networks. However,

[22] We will focus here on their result with strict decay, so $a_1 < a_2 < a_3$, etc.

[23] The limited distance which information can travel in Hojman and Szeidl (2008) effectively restricts the maximum diameter SNNs can have in that model.

the necessary population size could be very large indeed, as it is in Hojman and Szeidl (2008).

5.2 Decay and irrelevance of number of paths

We follow the literature by assuming that the number of paths between any pair of players i and j does not matter. Without decay this seems perfectly reasonable, if the benefits are derived from, for instance, information which is exchanged. However, when we assume that there is decay, this assumption becomes important. Decay represents that information is lost, or becomes less reliable or accurate, each time that it is passed on. However, it is not obvious that the same information is lossed (or the same noise is added) at step k along any path of length k or larger.

Abandonment of this assumption would however give players additional incentives to sponsor redundant links, which would either complicate the analysis greatly[24] or limit the applicability of the analysis to instances of more reliable transmission of benefits. We observe that many empirically documented networks do feature cycles. Burt (1992) argues that a sponsor can benefit if his recipients do not sponsor links to one another. The network is then not only more efficient, but the sponsor is then also in a stronger bargaining position, because he is bridging gaps in the network. An implication of Burt's analysis is that networks will tend not to include cycles. An opposite view is held by Coleman (1988) and Coleman (1990), who argues that redundant links are important to build up social capital in a network, in the form of reputation, trust, social norms and social control. Burt (2000) reviews mixed empirical support that corroborates both the structural hole view, and the social capital view. While Burt cites evidence on structural holes, he recognises that empirical evidence on redundancy in networks supports the social capital view. We end by noting that redundancy is in fact not incompatible with the structural hole view: players may form redundant links to avoid positional disadvantages, see e.g. Goyal and Vega-Redondo (2007). A study on BG's basic model with potentially different decay along different paths would therefore be relevant and interesting.

5.3 The focus on strict Nash networks.

In current non-cooperative models of network formation, the most common solution concept is that of SNNs. The reason for that is twofold. First, the set of solution would expand greatly, both in terms of architectures (network types) as well as in actual networks, if we would include weak Nash networks too. Second, BG found that the set of minimal curb sets[25] as well as the recurrent classes of a myopic best reply dynamic (with inertia) both coincide with the set

[24] The analysis could well become similar to models where links are not fully reliable.

[25] How and why the concept of minimal curb sets (which involves mixed strategies) can be applied to these models with pure strategies only, is discussed in Kamphorst and van der Laan (2007).

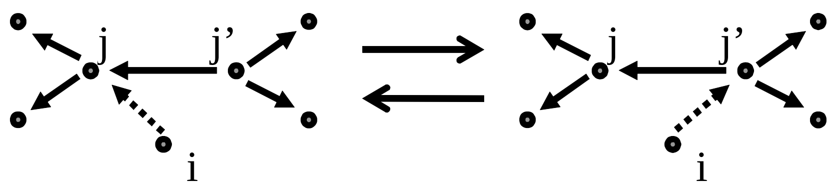

Figure 3: Weak Nash networks which survive a myopic best reply dynamic.

of SNNs. This provides ample support for focussing on SNNs, and most of the literature restricts its attention to SNNs.

However, in the presence of for instance heterogeneous players the myopic best reply dynamics (with inertia) also yields non-singleton recurrent classes which consist of multiple, equivalent, weak Nash networks. Nevertheless, the vast majority of weak Nash networks were not part of any recurrent class. Moreover the set of minimal curb sets of the game coincides with the set of recurrent classes, (see for instance Galeotti et al. (2006) and Kamphorst and van der Laan (2007)), so some minimal curb sets consisted of weak Nash networks. In this regard it is worthwhile to see whether we loose sight of certain relevant long run network architectures by focussing on SNNs. Figure **3** shows that in models with decay, this is indeed a problem too.

Consider player i. He is indifferent between sponsoring a link to j and j'. No set of best replies by the other players is affected by the choice of player i. They have a unique best reply, which is their current strategy (set of sponsored links). Thus in a myopic best reply dynamic such that a player randomises over all his best replies if indifferent, there exists a recurrent class in which all players other than i keep on sponsoring the same links as in Figure **3**, and player i occasionally replaces his link ij by link ij' and the other way around.

Our focus in this paper has stayed on SNN for reasons of simplicity and comparibility with the literature. However, an obvious and interesting extension would be to look at the recurrent classes of dynamics in which players learn which links to sponsor, and to apply the solution concept of MCS.

5.4 Conclusions

In this paper we explore what decreasing marginal benefits of information (DMBI) implies for the set of strict Nash networks[26] (SNNs) in the two-way flow con-

[26] Strict Nash networks are those networks which constitute strict Nash equilibria.

nections model by BG, when there is a small amount of decay[27]. To do so, we need a characterization of the networks which may be SNNs. Hence we start by extending the characterization of the set of non-star SNNs in BG for CMBI and DMBI. We show that any non-empty SNN will be a tree network with a central player, where the tree is semi-center sponsored in the sense that all links are center sponsored, with the possible exceptions of the links with the central player itself (see Figure 1 on page 12 for examples). The results clearly show that the results by Hojman and Szeidl (2008) do not extend to unlimited (albeit decaying) traveling of information along the links and an arbitrary population size.

After this characterization we look at how the set of SNNs changes if we move from a CMBI function to a comparable DMBI function. We find that if any non-empty SNN exists, then for sure the PSS has survived. Moreover, if there exist some networks which were SNN under CMBI but not under DMBI, then no star which is not a PSS will be an SNN under DMBI. Finally, we show that the results, examples and intuition suggest that DMBI may favor networks with larger diameters, as compared to CMBI. However, despite the strong examples and intuition, this suggestion is rather tentative, as we argue in the final paragraph of Section 4.

We conclude in this section by discussing several modelling choices.

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[27]In case of decay, it matters how far information has to travel through the network. Each time the information is passed on, so for each link it passes, the information becomes less accurate or complete, i.e. it looses some of its value.

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