More Archimedean than Archimedes: A New Trace of Abū Sahl al-Kūhī's Work in Latin

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Abstract In 1661, Borelli and Ecchellensis published a Latin translation of a text which they called the *Lemmas* of Archimedes. The first fifteen propositions of this translation correspond to the contents of the Arabic *Book of Assumptions*, which the Arabic tradition attributes to Archimedes. The work is not found in Greek and the attribution is uncertain at best. Nevertheless, the Latin translation of the fifteen propositions was adopted as a work of Archimedes in the standard editions and translations by Heiberg, Heath, Ver Eecke and others. Our paper concerns the remaining two propositions, 16 and 17, in the Latin translation by Borelli and Ecchellensis, which are not found in the Arabic *Book of Assumptions*. Borelli and Ecchellensis believed that the Arabic *Book of Assumptions* is a mutilated version of a lost "old book" by Archimedes which is mentioned by Eutocius (ca. A.D. 500) in his commentary to Proposition 4 of Book 2 of Archimedes' *On the Sphere and Cylinder*. This proposition is about cutting a sphere by a plane in such a way that the volumes of the segments have a given ratio. Because the fifteen propositions in the Arabic *Book of Assumptions* have no connection whatsoever to this problem, Borelli and Ecchellensis "restored" two more propositions, their 16 and 17. Propositions 16 and 17 concern the problem of cutting a given line segment AG at a point X in such a way that the product $AX \cdot XG^2$ is equal to a given volume K. This problem is mentioned by Archimedes, and although he promised a solution, the solution is not found in *On the Sphere and Cylinder*. In his commentary, Eutocius presents a solution which he adapted from the "old book" of Archimedes which he had found. Proposition 17 is the synthesis of the problem by means of two conic sections, as adapted by Eutocius. Proposition 16 presents the diorismos: the problem can be solved only if $K \leqslant AB \cdot BG^2$, where point B is defined on AG such that $AB = \frac{1}{2}BG$. We will show that Borelli and Ecchellensis adapted their Proposition 16 not from the commentary by Eutocius but from the Arabic text *On Filling the Gaps in Archimedes' Sphere and Cylinder* which was written by Abū Sahl al-Kūhī in the tenth century, and which was published by Len Berggren. Borelli preferred al-Kūhī's diorismos (by elementary means) to the diorismos by means of conic sections in the commentary of Eutocius, even though Eutocius says that he had adapted it from the "old book." Just as some geometers in later Greek antiquity, Borelli and Ecchellensis believed that it is a "sin" to use conic sections in the solution of geometrical problems if elementary Euclidean means are possible. They (incorrectly) assumed that Archimedes also subscribed to this opinion, and thus they included their adaptation of al-Kūhī's proposition in their restoration of the "old book" of Archimedes.

Our paper includes the Latin text and an English translation of Propositions 16 and 17 of Borelli and Ecchellensis.

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Introduction

Abū Sahl al-Kūhī was a geometer and astronomer of Iranian origin, who had an outstanding reputation among his contemporaries. He was even called "the master of his era in the art of geometry,"[1] and one can therefore understand why the study and translation of his works has been one of Len Berggren's passions. Just as in the case of other scientists in the Eastern Islamic world, none of his works were transmitted to Europe in the 12th and 13th centuries. Al-Kūhī's name occurs for the first time in Europe in the 17th century, in connection with the *Lemmas* of (pseudo?) Archimedes.

The *Lemmas* of Archimedes have not come down to us in Greek. A text entitled *Kitāb al-maʾkhūdhāt* (Book of Assumptions) and attributed to Archimedes survives in an Arabic translation by Thābit bin Qurra (836–901).[2] All, or almost all, extant Arabic manuscripts of the work contain not the original translation by Thābit but a later edition by the famous Iranian scientist Naṣīr al-Dīn al-Ṭūsī, who died in 1274.[3] The *Book of Assumptions* was part of al-Ṭūsī's edition of a collection of mathematical texts which were called the *Middle Books*, and which were supposed to be studied between the *Elements* of Euclid and the *Almagest* of Ptolemy. Al-Ṭūsī presented his edition of the *Book of Assumptions* along with the commentary by Abu'l-Ḥasan ʿAlī ibn Aḥmad al-Nasawī (ca. 1010),[4] who had included two generalizations of Proposition 5 by Abū Sahl al-Kūhī.

The *Book of Assumptions* was first translated from Arabic into Latin by the English mathematician Greaves [1659].[5] Two years later, a much superior translation appeared in Florence [Borelli 1661]. This translation was the product of a collaboration between Abraham Ecchellensis (1605–1664), a Christian philosopher and Arabist from Northern Lebanon, and the Italian mathematician Giovanni Borelli (1608–1679) who did not know Arabic.[6] Both translations were based on the edition by al-Ṭūsī. Greaves and Ecchellensis and Borelli called the text which they translated the *Lemmas* of Archimedes, and the text has been known under this title ever since. The *Lemmas* are available in standard translations of the works of Archimedes by Heath [1912, 301–318] and Ver Eecke [1960, 523–542], although the attribution to Archimedes remains doubtful at best. Al-Kūhī's generalizations of Proposition 5 of the *Lemmas* occur in both Latin translations [Greaves 1659, 8–9; Borelli 1661, 393–395]. His name appears as "Abu Sohal Alkouhi" in Greaves [1659, 4], and as "Abusahal Alkuhi" and "Alkauhi" in Borelli [1661, 385, 393–4]. Al-Kūhī's generalizations also appear in Isaac Barrow's 1675 edition of the works of Archimedes [Barrow 1675, 269–270], where they are said to be "ad mentem Abi Sahl Cuhensis, percelebris Mathematici," and in Voogt's Dutch version of the *Elements* of Euclid of 1695. Since al-Kūhī's generalizations have been published and translated [Hogendijk 2008; Merrifeld 1866], there is no need to discuss the mathematics here.

The present paper concerns a hitherto unrecognized trace of al-Kūhī's work in 17th century Europe. The paper consists of this introduction, three sections on the Greek, Arabic and Latin traditions, and an appendix. The appendix contains the Latin text with English translations of the last two propositions, 16 and 17, of the *Lemmas* by Ecchellensis and Borelli [1661], which are not contained in the Arabic manuscripts of the *Book of Assumptions* that have been inspected hitherto, and which are therefore not considered as part of the *Lemmas* of Archimedes by modern historians. The aim of this paper is to show that Proposition 16 was adapted from al-Kūhī's work: *Filling a Lacuna in The Book by Archimedes on the Sphere and Cylinder* in the edition of al-Ṭūsī. This work was published in Arabic with English translation by Len Berggren [1996]. In the proposition in question, which we will quote below, al-Kūhī considers a straight segment ABG such that $AB = \frac{1}{2}BG$ [Berggren 1996, 201-203]. He proves that for all points $D \neq B$ on the segment AG, we have $AB \cdot BG^2 > AD \cdot DG^2$. In algebraic symbolism, putting

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1 شيخ عصره في صناعة الهندسة [Berggren 2003, 178].

2 On Thābit bin Qurra and his mathematical works, see Sezgin [1974, 264–272].

3 Al-Ṭūsī's edition survives in several dozens of manuscripts, of which Sezgin [1974, 133] gives a non-exhaustive list. It is not clear whether the *Book of Assumptions* survives in a version which predates Naṣīr al-Dīn.

4 On al-Nasawī, see Sezgin [1974, 345–348].

5 On John Greaves (1602–1652), see Toomer [1996, 126-179].

6 On Borelli, see Gillispie [1973, 306–314].

$|AG| = a$, al-Kūhī's proposition implies that the maximum value of $x(a-x)^2$ for $0 < x < a$ is reached at $x = \frac{1}{3}a$.

The present paper continues with a brief section on the Greek tradition, which is about Proposition 4 of Book II of *On the Sphere and Cylinder* by Archimedes (ca. 250 BC), and the commentary by Eutocius of Ascalon (sixth century AD). In the commentary, Eutocius presents Archimedes' proof by means of conic sections that $AB \cdot BG^2 > AD \cdot DG^2$ (in al-Kūhī's notation). The next section, on the Arabic tradition, is about al-Kūhī's own proof of $AB \cdot BG^2 > AD \cdot DG^2$ without conic sections. In my final section on the Latin tradition, I argue that Proposition 16 of the *Lemmas* by Ecchellensis and Borell was adapted from al-Kūhī's proof, and I comment on the odd fact that Ecchellensis and Borelli attribute their Proposition 16 to Archimedes rather than to al-Kūhī.

The Greek Tradition: Archimedes, *On the Sphere and Cylinder* II.4, and the Commentary by Eutocius

In Proposition 4 of Book II of *On the Sphere and Cylinder*, Archimedes wants to cut a sphere into two parts by means of a plane in such a way that the volumes of the two parts are in a given ratio. He shows that this problem can be reduced to an auxiliary problem, namely the division of a given segment AB at a point O in such a way that the ratio of AO to a given length is equal to the ratio of a given area to the square of OB [Heath 1912, 62–64; Netz 2004, 202–204; Ver Eecke 1960, 101–103]; here I use lettering compatible with Proposition 17 by Ecchellensis and Borelli in the Appendix of this paper. In modern notation, the condition in the auxiliary problem boils down to $AO \cdot OB^2 = \ell \cdot \Delta$ where ℓ is the given length and Δ the given area. If we put $a = |AB|$, $x = |AO|$, $k = \ell \cdot \Delta$, the condition is equivalent to the equation $x(a-x)^2 = k$. For the special case of the sphere that has to be cut, the auxiliary problem always has a solution. Archimedes correctly states that the general auxiliary problem requires a diorism, that is, a necessary and sufficient condition for the existence of a solution. Archimedes gives no further details, but the diorism can be stated algebraically as $k \leqslant \frac{4}{27}a^3$; if $k > \frac{4}{27}a^3$ the equation has no root x with $0 < x < a$, so the geometric problem cannot be solved.

Archimedes promised to give the solution (analysis, synthesis, and diorism) of the auxiliary problem at the end, presumably at the end of Book II of *On the Sphere and Cylinder*. In the time of Eutocius, who lived seven centuries after Archimedes, the solution of the auxiliary problem was no longer found in the extant manuscripts of *On the Sphere and Cylinder*, so the Archimedean solution of the problem of cutting a sphere was left incomplete. In his commentary to *On the Sphere and Cylinder* II.4, Eutocius presents a solution of the auxiliary problem that he found in an old book, and which he attributes to Archimedes on plausible grounds [Heath 1912, 66–79; Netz 2004, 318–343; Ver Eecke 1960, 635–664]. In the solution, the required point O is constructed by means of the intersection of a parabola and a hyperbola. The solution consists successively of an analysis, a synthesis , and a diorism [Heath 1912, 67–72; Netz 2004, 319–328; Ver Eecke 1960, 636–646]. Borelli and Ecchellensis included the synthesis, with some changes in detail, in Proposition 17 of their edition of the *Lemmas*; see the appendix to this paper for the Latin text and an English translation.[7] In the diorism it is proved that the maximum of $AO \cdot OB^2$ occurs for $OB = 2AO$. The proof as presented by Eutocius uses a lengthy and ingenious argument involving the case where the parabola and a hyperbola in the synthesis are tangent. Eutocius then continues with solutions of the original problem of cutting the sphere by the Greek mathematicians Dionysodorus and Diocles. These solutions do not concern us here.

[7] In modern notation, the solution can be expressed as follows. Choose a rectangular coordinate system with origin A and positive x-axis AB. We have to construct point O on AB in such a way that $AO : \ell = \Delta : OB^2$ for a given segment ℓ and a given area Δ. Let $a = |AB|$ and describe the parabola $(a-x)^2 = \frac{\Delta}{a} \cdot (\ell - y)$ and the hyperbola $x(\ell - y) = a \cdot \ell$. Let $P(x_1, y_1)$ be a point where the hyperbola intersects the parabola. Then obviously $x_1(a-x_1)^2 = x_1 \cdot \frac{\Delta}{a} \cdot (\ell - y_1) = \frac{\Delta}{a} \cdot a \cdot \ell = \Delta \cdot \ell = k$, so if we choose point O on AB such that $|AO| = x_1$, O is a desired point. Compare Proposition 17 in the appendix to this paper, where $\ell = |AC|$, $\Delta = |BE^2| = \frac{4}{9}a^2$. $\Delta > 0$ can be chosen arbitrarily because only the product $\ell \cdot \Delta$ occurs in the equation.

The Arabic Tradition: Al-Kūhī's *Treatise on Filling the Gaps in* On the Sphere and Cylinder

On the Sphere and Cylinder of Archimedes was translated into Arabic at least twice [Sezgin 1974, 128–130]. One translation was revised by Thābit bin Qurra, who also translated the *Book of Assumptions*. The other translation was made by Isḥāq ibn Ḥunayn, who also produced an Arabic translation of the commentary by Eutocius. Naṣīr al-Dīn al-Ṭūsī had access to all these Arabic versions. He included the text by Archimedes and some (but not all) of the commentary of Eutocius in his edition of *On the Sphere and Cylinder*, which belonged to the *Middle Books* [al-Ṭūsī 1940b, 2–3]. Al-Ṭūsī presents the commentary by Eutocius immediately after the text by Archimedes to which it refers; thus Book II, Proposition 4 of *On the Sphere and Cylinder* [al-Ṭūsī 1940b, 86–89] is followed by the relevant commentary of Eutocius, including the analysis, synthesis, and diorism of the auxiliary problem [al-Ṭūsī 1940b, 89–96]. In the Arabic translation, the commentary to *Sphere and Cylinder* II.4 begins with a passage which I quote in detail for later use in the section on the Latin tradition below. The passage resembles the Greek text but is not completely equivalent to it because some of the nuances were lost in the Arabic translation. Al-Ṭūsī says [with my additions to the translation in square brackets]:

> وقد ذكر أوطوقيوس العسقلابي في شرحه لهذا الكتاب أن أرشميدس وعد بيان ذلك في كتابه هذا ولم يوجد في شيء من النسخ ما وعده ولذلك سلك كل واحد من دينوسوذورس وديفليس بعده طريقاً غير الذي سلكه هو في هذا الكتاب إلى قسمة الكرة بقسمين على نسبة مفروضة.
>
> قال وأنا وجدت في كتاب عتيق أشكالاً مستغلقة جداً لكثرة ما فيه من الخطأ وما في الأشكال من التحريف بسبب جهل الناسخين وكان فيه الفاظ من لغة ذريس التي كان أرشميدس يحب استعمالها واصطلاحات له خاصّة كما كان يعبر عن القطع المكافي والزائد بالقائم الزاوية والمنفرجة الزاوية فواظبت عليه إلى أن تقرر لي هذه المقدمة وهي هذه:
>
> اذا كان خطان معلومان عليهما اب اجـ وسطح معلوم عليه د وأردنا أن نقسم اب على ه قسمة تكون نسبة سطح د إلى مربع به كنسبة اه إلى اجـ فلنجعل كأن ذلك قد كان ...

> Eutocius of Ascalon mentioned in his commentary to this book that Archimedes promised the proof of that in this book of his, but what he promised was not found in any of the manuscripts. Therefore both Dīnūsūdhūrus [i.e., Dionysodorus] and Diyuflīs [i.e., Diocles] [who came] after him followed a method different from the method which he himself followed in this book, for the division of the sphere into two parts according to an assumed ratio.
>
> He [= Eutocius] said: I found in an old book propositions which were very obscure because of the many errors in them and the corruptions in the figures caused by the ignorance of the scribes. But in it were words from the language of Dhurīs [i.e., the Doric dialect], which Archimedes liked to use, and technical terms which were special for him; thus he called the sufficient and exceeding section [i.e., the parabola and hyperbola] the right-angled and obtuse angled [sections]. So I worked assiduously on it until this preliminary was established for me, and it is as follows: If there are two known lines AB, AG, and a known area D, and we want to divide AB at E in such a way that the ratio of area D to the square of EB is as the ratio of AE to AG, so let us assume as if that exists ... [al-Ṭūsī 1940b, 89–90].

The Arabic text then continues with a translation of Eutocius' commentary, including the analysis, synthesis and diorism, which Eutocius attributes to Archimedes, and the two solutions by Dionysodorus and Diocles.

Al-Ṭūsī does not mention his own name anywhere in the text of his edition of *On the Sphere and Cylinder*, and he mentions the name of al-Kūhī for the first time towards the end, after the last proposition of Book II, Proposition 9. There, al-Ṭūsī says: "And I say: And by Abū Sahl Wayjan ibn Rustam al-Kūhī is a treatise, and he called it: *On Filling the Gaps in the Second Book of Archimedes* [i.e., *On the Sphere and Cylinder*]."[8] Al-Ṭūsī rendered only fragments of al-Kūhī's treatise, which have been edited and translated by Len Berggren [1996], and which are also available in an uncritical edition in [al-Ṭūsī 1940b, 115-127] and in a French summary by Woepcke [1851, 103-114]. Most of these

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[8] أقول ولأبي سهل يحيا (يعني: ويجن) بن رستم القوهي رسالة وسمّها بسدّ الخلل الذي في المقالة الثانية من كتاب أرشميدس [al-Ṭūsī 1940b, 115:8–9].

fragments are related to al-Kūhī's solution to the following problem: to construct a spherical segment whose (spherical) surface area is equal to the surface area s of a given spherical segment and whose volume is equal to the volume v of a different given spherical segment [Berggren 1996, 140–141]. Al-Kūhī solves this problem by means of a parabola and a hyperbola, and he then turns to the diorism, that is the necessary and sufficient conditions for the existence of a solution. He shows that a certain ratio between two cones (which is determined only by the quantities s and v) is equal to the ratio $BD^3/BZ \cdot ZE^2$, where B, D, E, Z are points on a line segment such that $BD = 2DE$ and Z is another point on the segment. Thus the diorism boils down to the determination of the maximum value of $BZ \cdot ZE^2$. Al-Kūhī says: "But the solid of the line *BZ* by the square of *ZE* is the greatest possible when *BZ* is half of *ZE*, as was shown by (conic) sections in the account that we presented following (the method of) Eutocius. And we will give later a demonstration independent of conic sections" [Berggren 1996, 170:6–9, 196; al-Ṭūsī 1940b, 119:3–6]. Indeed, al-Kūhī first discusses the situation where the two conics in his construction are tangent, just like Archimedes (as presented by Eutocius) in the diorism of his auxiliary problem. At the end, al-Kūhī determines the maximum of $BZ \cdot ZE^2$ in a different way, without conic sections [Berggren 1996, 177:11–179:4, 202–203; al-Ṭūsī 1940b, 123:19–124:20; see also the commentary by Berggren 1996, 157–159; and the French summary by Woepcke 1851, 113–114].[9] Al-Kūhī's new determination of the maximum is of course much easier than the complicated argument by means of the tangent parabola and hyperbola, and Berggren remarks that al-Kūhī may have written his proposition "with perhaps a smile and a slight nudge in Archimedes' ribs" [Berggren 1996, 157].

I will now cite Berggren's translation because it will be essential in the next section of this paper. I have italicized the names of standard operations on ratios which are explained in Book V of the *Elements* of Euclid [Heath 1925, 2:112–137]. A summary of the proposition is to be found in the next section, where it will be compared to Proposition 16 in Borelli's edition of the *Lemmas* of Archimedes.

Al-Kūhī says (Figure 1):

> So, in order to prove it, let *AB* be half of *BG* and first let *D* be between *A* and *B*. I say that the solid of the line *AB* by the square of *BG* is greater than the solid of the line *AD* by the square of *DG*.
>
> We make *GE* equal to *GB*; hence, since the ratio of *AB* to *BG* is as the ratio of *BG* to *BE*, the surface of *AB* by *BE* equals the square of *BG*. But[10] the surface of *AB* by *BE* is larger than the surface of *AD* by *DE* because *B* is closer to the middle of *AE* than *D*; hence, the square of *BG* is greater than the surface of *AD* by *DE* and the ratio of the surface of *ED* by[11] *DB*, which is another magnitude, to the surface of *ED* by *AD*, i.e., the ratio of *BD* to *DA*, is greater than the ratio of the surface of *ED* by *DB* to the square of *BG*. So *by composition* the ratio of *BA* to *AD* is greater than the ratio of the surface of *ED* by *DB* together with the square of *BG*, i.e. the square of *DG*, to the square of *BG*. And so the solid of the line *BA* by the square of *BG* is greater than the solid of the line *AD* by the square of *DG*.
>
> Next, let *D* be between *B* and *G* and the rest as before. Then the surface of *AB* by *BE*, i.e. the square of *BG*, will be smaller than the surface of *AD* by *DE* because of *D* being nearer to the midpoint of *AE* than *B*. And the ratio of the surface of *BD* by *DE*, which is another magnitude, to the square of *BG* will be greater than its ratio to the surface of *AD* by *DE*, i.e. than the ratio of *BD* to *DA*. So, *by inversion*, the ratio of the square of *BG* to the surface of *BD* by *DE* is smaller than the ratio of *AD* to *DB*. Hence, *by separation*, the ratio of the square of *DG* to the surface of *BD* by *DE* is less than the ratio of *AB* to *BD*. And so, *by inversion*, the ratio of the surface of *BD* by *DE* to the square of *DG* is larger than the ratio of *DB* to *BA*. And *by composition* the ratio of the square on *BG* to the square of *DG* is greater than the ratio of *DA* to *AB*. And so the solid of *AB* by the square of *BG* is greater than the solid of *AD* by the square of *DG*, which is what we wanted. [Berggren 1996, 202–203]

[9] Note that the author of the demonstration must be al-Kūhī, not al-Ṭūsī, because the demonstration occurs in a long quotation from al-Kūhī that ends on the following page with "[a]nd the Shaykh Abū Sahl al-Kūhī solved the problem in another way which we shall not present" [Berggren 1996, 179:20-180:2, 203; al-Ṭūsī 1940b, 125:9]. Shortly afterwards, al-Ṭūsī concludes with the final sentence "[a]nd this is what Abū Sahl al-Kūhī presented" [Berggren 1996, 183:15, 204; al-Ṭūsī 1940b, 127:15].

[10] Berggren adds here "< by Premiss 9 >," and elsewhere in the translation similar references to other "premisses." Since these references occur only in the manuscript Leiden, Or. 14/25 and not in the other Arabic manuscripts which Berggren used, I assume that they were added by a later commentator to al-Ṭūsī's edition of al-Kūhī's work. I have therefore deleted the "premisses" from the translation.

[11] Berggren translates the same Arabic word *fī*, which indicates the multiplication of two quantities, as "by" but also as "in." I always use "by."

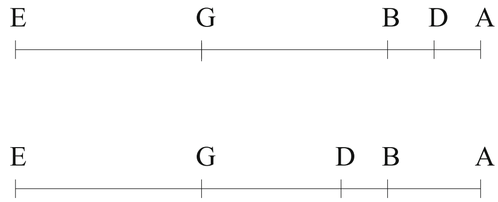

Figure 1: Al-Ṭūsī's version of al-Kūhī's lemma to *On the Sphere and Cylinder* II.4.

The Latin Tradition: Propositions 16 and 17 of the *Lemmas* of Archimedes in the Version of Borelli and Ecchellensis

Borelli and Ecchellensis begin their edition of the *Lemmas* of Archimedes with a long preface. They argue that the *Book of Assumptions*, which they translated from Arabic and which they call the *Lemmas*, is essentially the "old book" which Eutocius found and which he mentions in his commentary to *On the Sphere and Cylinder* II.4. They cite the introduction by Eutocius in which he talks about this old book, in a Latin translation based on the Greek text that had already been available since 1544.[12] Unfortunately for Borelli and Ecchellensis, the Arabic text of the *Book of Assumptions* consists of fifteen elementary geometrical theorems on straight lines and circles, which have no relationship whatsoever to *On the Sphere and Cylinder* II.4 and to Eutocius' commentary. Borelli and Ecchellensis argue that the end of the *Book of Assumptions* must have been lost in the course of time, just like the end of *On the Sphere and Cylinder*. They believe that this lost final part of the *Book of Assumptions* contained the propositions which Eutocius found in his old book, and they then decided to restore this lost final part. Thus Borelli and Ecchellensis say (the "I" who is speaking is Borelli):

"I have added at the end of this book two other propositions [i.e., 16 and 17] by Archimedes that were found by Eutocius. The last one [i.e., 17] is perhaps the same which is missing here, for Almochtasso [i.e., al-Nasawī][13] says in the preface that the propositions in this little work are sixteen, although the last one [in the extant text of the *Book of Assumptions*] is the fifteenth."[14] I note that in the Arabic manuscripts available to me, al-Nasawī says (in the edition of al-Ṭūsī) that the number of propositions is 15 and not 16, cf. [al-Ṭūsī 1940a, 2:4].

What exactly are these propositions 16 and 17 which Borelli and Ecchellensis added to the fifteen propositions of their Latin translation of the *Book of Assumptions*?

Proposition 17 is a construction of the auxiliary problem of Archimedes' *On the Sphere and Cylinder* II.4 by means of a parabola and hyperbola, The construction is essentially the same (with minor changes) as the synthesis in the commentary by Eutocius. This commentary was available to Ecchellensis and Borelli in Greek and in Arabic but a comparison between the figures shows that Proposition 17 is based on the (Latin translation of the) Greek version of *On the Sphere and Cylinder* in *Archimedis Syracusani philosophi …* [1544] (see p. 34 of the Latin translation of Eutocius' commentary), because the labels of the points in the geometrical figures are exactly the same. In the Arabic, the labels are different [al-Ṭūsī

[12] In *Archimedis Syracusani philosophi …* [1544], the Greek text is on p. 30 of *Eutokiou Askaloonitou Hypomnema*, at the end of the Greek section of the book, and a Latin translation is on p. 32 of *Commentarii Eutocii Ascalonitae* at the end of the book. The Latin translation in *Archimedis Syracusani philosophi …* is different from the translation which Ecchellensis and Borelli present in their preface [Borelli 1661, 381–382], and which had been made for them by "learned friends;" see the text in the appendix of this paper.

[13] Almochtasso is derived from the Arabic *al-mukhtaṣṣ*, meaning "the distinguished [scholar]," which al-Ṭūsī used to refer to al-Nasawī.

[14] Addidi in fine huius libris duas alias Archimedis propositiones ab Eutocio repertas quarum altera fortasse illa eadem est que hic deficit, nam Almochtasso in proemio ait, propositiones huius Opusculi sexdecim esse, cum tamen postrema sit decimaquinta [Borelli 1661, 383].

1940b, 90–91]. For example, the point that corresponds to M in Figure 4 and 5 below by Borelli and Ecchellensis is labeled *nūn* in the Arabic, so if they had derived the figure from the Arabic, one would expect N here.

In their Proposition 16, Borelli and Ecchellensis show that if point C is on line segment AB such that $AB = 3AC$, and if $D \neq C$ is another point on line segment AB, then $AC \cdot CB^2 > AD \cdot DB^2$. This is the same result as in the proposition by al-Kūhī quoted in the previous section. I will now discuss the similar structure of the proofs by Borelli/Ecchellensis and al-Kūhī by summarizing the two propositions in a uniform notation. In Figure 2, I use the same labels of points in the figure as in the Latin text of Ecchellensis and Borelli and in Berggren's translation from the Arabic, with the exception of points P and Q. In the translation of al-Kūhī, one has to read $P = B, Q = G$, and in the version of Borelli and Ecchellensis $P = C, Q = B$. Note that the orientations of the two figures of Kūhī and Borelli/Ecchellensis are opposite, as one would expect because the directions of writing in Arabic and Latin are also opposite.

Figure 2: Borelli/Ecchellensis Proposition 16 compared with al-Kūhī's lemma.

We begin with a line segment AQ with a point P on it such that $AQ = 3AP$. Choose a point $D \neq P$ on segment AQ. Required to prove that $AP \cdot PQ^2 > AD \cdot DQ^2$.

Al-Kūhī and Borelli/Ecchellensis begin by extending AQ to point E in such a way that $QE = PQ$. They then remark that $AP \cdot PE = 4AP^2 = PQ^2$. Now D can be located in two different ways. If D is between A and P, we have $AP \cdot PE > AD \cdot DE$ because, as al-Kūhī and Borelli/Ecchellensis point out, point P is closer to the midpoint of AE than point D. Similarly, if D is between P and Q, we have $AD \cdot DE > AP \cdot PE$ because in this case, D is closer to the midpoint of AE than P. Al-Kūhī and Borelli/Ecchellensis base themselves here on Euclid's *Elements* II.5 [Heath 1925, 1:382–383], without saying so. If point M (not mentioned by the authors) is the midpoint of AE, then $AD \cdot DE = MA^2 - MD^2$ and $AP \cdot PE = MA^2 - MP^2$. Thus far the two proofs agree.

Al-Kūhī gives the two cases a separate treatment. If D is between A and P, we have

$$\frac{PD}{AD} = \frac{PD \cdot DE}{AD \cdot DE} > \frac{PD \cdot DE}{PQ^2},$$

since

$$AD \cdot DE < AP \cdot PE = PQ^2.$$

Therefore

$$\frac{AP}{AD} = 1 + \frac{PD}{AD} > 1 + \frac{PD \cdot DE}{PQ^2} = \frac{PD \cdot DE + PQ^2}{PQ^2} = \frac{DQ^2}{PQ^2};$$

hence, as required,

$$AP \cdot PQ^2 > AD \cdot DQ^2.$$

If D is between P and Q, al-Kūhī shows in a similar way that

$$\frac{PD \cdot DE}{PQ^2} > \frac{PD}{AD}.$$

He now argues, in a needlessly complicated way, via the three intermediate steps

$$\frac{PQ^2}{PD \cdot DE} < \frac{AD}{PD},$$

and

$$\frac{DQ^2}{PD \cdot DE} < \frac{AP}{PD},$$

or

$$\frac{PD \cdot DE}{DQ^2} > \frac{PD}{AP},$$

that

$$\frac{PQ^2}{DQ^2} > \frac{AD}{AP}.$$

Hence, as required,

$$AP \cdot PQ^2 > AD \cdot DQ^2.$$

Borelli and Ecchellensis shorten the argument by introducing an extra point O as follows. Let

$$\frac{PD}{DO} = \frac{PQ^2}{AD \cdot DE}.$$

Then, $DO < PD$ if D is between A and P, and $DO > PD$ if D is between P and Q. In both cases we have $AO < AP$. Now since

$$\frac{PD}{DO} = \frac{PQ^2}{AD \cdot DE},$$

also

$$\frac{AD}{DO} = \frac{PQ^2}{PD \cdot DE}.$$

If D is between A and P, then

$$\frac{AD}{AO} = \frac{AD}{DO + AD} = \frac{PQ^2}{PD \cdot DE + PQ^2} = \frac{PQ^2}{DQ^2}.$$

Also if D is between P and Q,

$$\frac{AD}{AO} = \frac{AD}{AD - DO} = \frac{PQ^2}{PQ^2 - PD \cdot DE} = \frac{PQ^2}{DQ^2}.$$

Hence, in both cases,

$$AO \cdot PQ^2 = AD \cdot DQ^2.$$

Because $AO < AP$ we conclude that $AD \cdot DQ^2 < AP \cdot PQ^2$, as required,

We note that the proofs by al-Kūhī and Ecchellensis/Borelli are not the only possible ones. The 12th-century mathematician Sharaf al-Dīn al-Ṭūsī (who is not the same as Naṣīr al-Dīn al-Ṭūsī mentioned above) proved the same theorem in a different way, but also without conic sections [Rashed 1986, vol. 2, 2-5].

In the introduction to Proposition 16 (cf. the appendix), Borelli says that the translation by Ecchellensis of the Arabic version of Eutocius' introduction was taken from "the edition of Abusahal Alkuhi." As pointed out in the previous section of this paper, al-Ṭūsī's name is nowhere mentioned in the text of the edition of *On the Sphere and Cylinder* so it is not surprising that his name does not occur in the Latin. But we also noticed that al-Kūhī's name is only mentioned (by al-Ṭūsī) after the end of *On the Sphere and Cylinder* itself, at the beginning of the treatise *On Filling the Gaps*. Thus Ecchellensis must have gone through al-Ṭūsī's entire edition of *On the Sphere and Cylinder* Book II, including al-Kūhī's work at the end. The close agreement between the first part of the proofs by al-Kūhī and Ecchellensis/Borelli now shows beyond any doubt that Proposition 16 of Ecchellensis and Borelli must be dependent on the corresponding proposition by al-Kūhī in the Arabic. As in the case of Proposition 17, some changes were made in the proof, but the basic structure remained the same. So Ecchellensis must have read and translated the proposition by al-Kūhī, and one wonders whether he also communicated to Borelli the other interesting parts of al-Kūhī's treatise.

Finally, I call attention to the end of the introduction to Proposition 16. There Borelli and Ecchellensis state that it is "not a small sin" in geometry to use conic sections in a proof if it can also be done by ruler and compass. Borelli and Ecchellensis conclude that the diorism by the tangent conic sections as presented in Eutocius' commentary cannot have been by Archimedes, but must have been mutilated, by Eutocius or by someone else. Accordingly, they present Proposition 16 as a reconstruction of a proposition by Archimedes. Thus they must have considered al-Kūhī as an editor, not as an author, and one may wonder what they thought about the authorship of al-Kūhī's construction of a spherical segment with a given spherical surface area s and a given volume v. There is no evidence that Archimedes had the same dogmatic preference for ruler-and-compass constructions and Euclidean methods as Borelli and Ecchellensis. One can find the first traces of such views in the work of the more bureaucratically minded Greek mathematician Apollonius of Perga (ca. 200 BC).[15] By attributing such opinions to Archimedes, Borelli and Ecchellensis made al-Kūhī's proposition look more Archimedean than Archimedes himself would have been. For there is no reason to deny Archimedes the authorship of the proof of the diorism by means of the tangent hyperbola and parabola that was preserved in Eutocius' commentary. Al-Kūhī might have been pleased and honored by the posthumous misattribution of his proposition to the greatest geometer of Greek antiquity.

Appendix: Texts and Translations from the Edition by Ecchellensis and Borelli [1661, 409–413].

All additions by me are in square brackets [ ]. Marginal remarks are inserted in the text for typographical reasons, and are included in pointed brackets < >. Parentheses also occur in the original Latin.

[Borelli 1661, 409] *In praefatione huius operis memini non esse omnino improbabile hunc libellum Archimedis non alium fuisse ab illo antiquo lemmatum libro ab Eutocio reperto, quod precipuè ex verbis eiusdem Eutocij in Comment. proposit. 4. lib. 2. de Sphaera & Cylindro comprobatum fuit: illa fidellisimè translata ex textu Graeco ab amicis doctissimis cum iam in praefatione excusa essent aliam translationem ex Arabico Manuscripto Serenissimi Magni Ducis misit Excell. Abrahamus Ecchellensis desumptam ex editione Abusahli Alkuhi qui pariter librum ordinationis [sic]*[16] *lemmatum Archimedis conscripsit, ut in proemio huius operis testatur Almochtasso. Verba eius sunt haec, quae paulò clarius propositum confirmare videntur;* & meminit Eutocius Ascalonita in Comment. huius libri, quod Archimedes promiserit demonstrationem huius in hoc suo libro, quod in nullo exemplari reperitur, quod promisit. Atque ita unusquisque tam Dyonisodorus, quàm Diocles post illum progressus est per aliam viam, quàm ille (scilicet Archimedes) in hoc libro in divisione Sphaerae in duas partes, quae datam habeant proportionem. Dixit, & ego reperi in [p. 410] Veteri Libro Theoremata satis obscura propter multitudinem errorum, qui in eo sunt, nec non menda, quae occurrunt in figuris propter ignorantiam amanuensium, erantque in eo Doricae dictiones, quarum usus Archimedi familiaris erat, & vocabula ipsi propria; hinc utebatur loca sectionum parabolae, & hyperbolae, rectanguli, & obtusanguli coni sectonibus quamobrem operam ipsi navavi, donec assecutus sum istam propositionem, & est ista, & c.

Modo quia in praedicto libro antiquo ab Eutocio reperto recensentur due propositiones, quarum unam promiserat se demonstraturum Archimedes, & utraque in nostro opusculo iniuria temporum deficit; earum altera forsan erit 16. illa propositio in proemio ab Almochtasso numerata ubi ait propositiones huius opusculi sexdecim esse, cum tamen postrema sit 15. quare inutile forsan non erit eas hic reponere, praecipuè quia Eutocius non rite eas restituit, nec omninò repurgavit a mendis, quibus scatebat exemplar antiquum ab ipso

[15] See *Conics* V.51–52, where Apollonius constructs two minimal straight lines from a given point to a conic section by intersecting it by a second conic section, but if there is only one minimum straight line and the conics would be tangent, he omits the second conic section and presents a more elementary proof, cf. [Toomer 1990, 144–172].

[16] "librum ornationis" would be a correct translation of the Arabic, but Ecchellensis and Borelli [1661, 385] also printed "ordinatio" (rather than "ornatio") in their translation of the preface to the *Lemmas*.

inventum. Et primo noto, quod Eutocius eas vocat theoremata, cum potius problemata sint, & sic etiam ad eodem Eutocio postmodum appellantur. Forsan hoc accidit, quia in libro illo antiquo in formam theorematum scripta erant, sed Eutocius ut ad propositionem Archimedis ea accomodaret [sic], forma problematica ea exposuit. Rursus Eutocius primum theorema se expositurum pollicetur, ut deinde analysi problematis Archimedei accomodetur [sic]. Unde conijcere licet alterum theorema additum, vel alteratum ab Eutocio, vel ab aliquo alio fuisse, in quo proponit, quod, si aliqua recta linea secta sit in duo segmenta, quorum unum duplum sit alterius, solidum parallelepipedum rectangulum contentum sub quadrato maioris, & sub minore segmento maximum erit omnium similium solidorum, quae ex divisione euisdem recta linea in quolibet alio eius puncto consurgunt. Et hoc quidem ostenditur per sectiones conicas, contra artis praecepta; peccatum enim est non parvum apud Geometras, problema planum per conicas sectiones resolvere cum via plana absolvi posse, hoc autem preclari nonnulli viri pariter adnotarunt, & praestiterunt, ut nuper accepi,

Translation:

In the preface of this book, I have mentioned that it is not altogether improbable that this booklet by Archimedes [i.e., the *Lemmas*] is the same as the old book of lemmas which was found by Eutocius. This is particularly confirmed by the words of the same Eutocius in the Commentary on Proposition 4 of Book 2 of *On the Sphere and Cylinder*. Since these [words] have already been presented in the preface in a most faithful translation from the Greek text by learned friends, the most excellent Abraham Ecchellensis made another translation from the Arabic Manuscript of the Most Exalted Great Duke,[17] taken from the edition of Abusahal Alkuhi who also wrote the book on the Arrangement[18] of the *Lemmas* of Archimedes, as Almochtasso [i.e., al-Nasawī] states in the preface of that work. His words are as follows, and they seem to make a little clearer what has been proposed [by me]:

And Eutocius of Ascalon mentioned in the Commentary to this work, that Archimedes promised a demonstration of this in this his book, [and] that what he promised was found in no copy. Then Dionysodorus as well as Diocles proceeded in another way than he (that is, Archimedes) in this book in [solving] the division of the sphere into two parts which have a given ratio. He [Eutocius] said: And I have found in an old book theorems which were quite unclear because of the multitude of errors in it, and the mistakes which occur in the figures because of the ignorance of the scribes. And in it were Doric expressions, which Archimedes commonly used, and terminology proper to him; thus instead of "sections of the parabola and hyperbola" he used "sections of a right-angled and obtuse-angled cone." For this reason, I zealously studied this work for myself until I grasped the proposition. It is as follows: etc.

But because in the above-mentioned old book which was found by Eutocius two propositions were recorded, and Archimedes had promised that he would demonstrate one of them, and each of them is missing in our little work [i.e. the *Lemmas*] because of the vicissitudes of time, [therefore] the second of them will perhaps be that 16th proposition which was enumerated by Almochtasso [= al-Nasawī] in the preface [of the *Lemmas*], where he says that the propositions of this work are sixteen, although the last one is nr. 15. This is why it will perhaps not be useless to place them here, especially because Eutocius did not restore them properly, and did not completely clean them from the errors which were abundant in the ancient book that he had found. First, I note that Eutocius calls them theorems, although they are better [called] problems, and this is how they are called by Eutocius later on. Perhaps this happened because they were written in the form of theorems in the old book, but in order to adapt them to the proposition of Archimedes, Eutocius presented them in the form of problems. Secondly, Eutocius declared that he would present the first theorem, in order to adapt it later to the analysis of the problem of Archimedes [i.e., the division of the sphere]. Here it is possible to conjecture that the other theorem that was added [i.e., the diorism], was changed either by Eutocius or by someone else; [this is

[17] The reference is to Fernando II de' Medici, 1610–1670.

[18] The Latin "ordinatio" is a mistake for "ornatio," meaning embellishment, which is the equivalent of the Arabic *tazyīn*.

the theorem] in which he stated the proposition that if some straight line is divided into two segments such that the one [segment] is twice the other, the rectangular parallelepipedal solid contained by the square of the greater [segment] and by the lesser [segment] is the maximum of all solids of the same kind, which emerge from the division of the same straight line in an arbitrary other point of it. And this is indeed shown [i.e., in Eutocius' text] by conic sections, against the rules of the art. For it is not a small sin among geometers to solve a plane problem by means of conic sections, since it can be solved in a plane way [i.e., by ruler and compass]. Several distinguished men have unanimously noted and expressed this [opinion], as I have learned recently.

Propositio XVI

Si recta linea *AB* sit tripla *AC*, non vero tripla ipsius *AD*; Dico parallelepipedum rectangulũ contentum sub quadrato *CB* in *AC* maius esse parallelepipedo sub quadrato *DB* in *AD*.

Producatur *AB* in *E*, ut sit *BE* aequalis *BC*. Quoniam *BC* dupla erat ipsius *AC*, erit *EC* quadrupla ipsius *AC*, & propterea rectangulum *ACE* aequale erit quadruplo quadrati *AC*, scilicet aequale erit quadrato *CB*: Est vero in primo casu, rectangulum *ADE* maius rectangulo *ACE*, in secundo vero minus, (eo quod punctum *D* in primo casu propinquius est semipartitioni totius *AE*, quàm *C*, in secundo verò remotius); igitur si fiat *CD* ad *DO*, ut quadratum *CB* ad rectangulum [p. 411] *ADE*, erit in primo casu *DO* maior, quàm *CD*, in secundo vero minor; & propterea *AO* minor erit, quam *AC* in utroque casu. Et quia quadratum *CB* ad rectangulum *ADE* est ut *CD* ad *DO*, igitur solida parallelepipeda reciproca erunt aequalia, scilicet solidum quadrato *CB* in *DO* ducto aequale erit solido, cuius basis rectangulum *ADE*, altitudo vero *CD*, seu potius aequale erit solido, cuius basis rectangulum *EDC*, altitudo vero *AD*, & propterea ut quadratum *BC* ad rectangulum *EDC*, ita erit reciproce *AD* ad *DO*, & comparando antecedentes ad terminorum differentias in primo casu, & ad eorundem summas in secundo casu, erit quadratum *BC* ad quadratum *DB* ut *AD* ad *AO*, & denuo solidum parallelepipedum rectangulum contentum sub quadrato *BC* in *AO* aequale erit ei, cuius basis quadratum *DB*, altitudo vera *AD*: Est vero *AO* ostensa minor, quàm *AC* in utroque casu, igitur parallelepipedum, cuius basis quadratum *BC*, altitudo *AC* maius est eo, cuius basis est idem quadratum *BC*, altitudo *AO*; ideoque parallelepipedum, cuius basis quadratum *BC*, altitudo *AC* maius est quolibet parallelepipedo, cuius basis quadratum *BD*, altitudo *AD*: quare patet propositum.

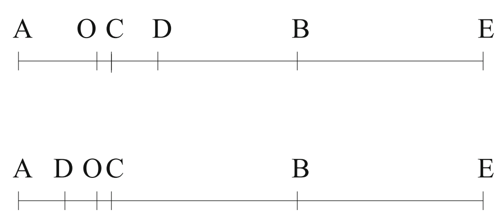

Figure 3: Borelli/Ecchellensis Proposition 16.

Translation:

Proposition 16

If the straight line *AB* is three times *AC*, but not three times *AD*, I say that the rectangular parallelepiped contained by the square of *CB* in *AC* is greater than the parallelepiped [contained] by the square of *DB* in *AD*.

Let *AB* be produced to *E* such that *BE* is equal to *BC*. Since *BC* was [assumed to be] twice *AC*, *EC* will be four times *AC*, and therefore the rectangle *ACE*[19] will be equal to four times the square of *AC*, that is to say, equal to the square of *CB*.

In the first case, the rectangle *ADE* is greater than the rectangle *ACE*, in the second case it is less (because point *D* is in the first case closer to the midpoint of the whole *AE* than *C*, but in the second case farther away). So if *CD* is made to *DO* as the square of *CB* to the rectangle *ADE*, *DO* will be greater than *CD* in the first case and less in the second case. And therefore *AO* will be less than *AC* in both cases. And since the square of *CB* is to the rectangle *ADE* as *CD* to *DO*, thus the reciprocal parallelepipedal solids will be equal, that is to say, the solid of the square of *CB* multiplied by *DO* will be equal to the solid whose base is the rectangle *ADE* and whose altitude is *CD*, and moreover, it will be equal to the solid whose base is the rectangle *EDC* and whose altitude is *AD*. And therefore, the square of *BC* will be to the rectangle *EDC*, reciprocally, as *AD* to *DO*. By comparing the antecedent terms to the differences of the terms in the first case, and to their sums in the second case, the square of *BC* will be to the square of *DB* as *AD* to *AO*, and again, the rectangular parallelepipedal solid contained by the square of *BC* and *AO* will be equal to the [solid], whose base is the square of *DB* and whose altitude is *AD*.

But *AO* was shown to be less than *AC* in both cases. Therefore the parallelepiped, whose base is the square of *BC* and whose altitude is *AC* is greater than the [solid] whose basis is the same square *BC* and whose altitude is *AO*; and therefore the parallelepiped whose basis is the square of *BC* and whose altitude is *AC* is greater than any parallelepiped whose base is the square of *BD* and whose altitude is *AD*. Thus the proposition is evident.

Propositio XVII

Sit *AB* tripla ipsius *AE*, maior vero quàm tripla alterius *CA*, secari debet eadem *AB* citra, & ultra *E*, in *O*, ita ut parallelepipedum, cuius basis quadratum *OB*, altitudo *OA* aequale sit parallelepipedo, cuius basis quadratum *EB*, altitudo *AC*.

Fiat rectangulum *ACBF*, & producantur latera *CA*, *FB* & fiat rectangulum *CFN* aequale quadrato *EB*, & ducta diametro *CEG* compleantur [p. 412] parallelogramma rectangula *AL*, *AK*, *LB*, *BK*, atque axe *FG*, latere recto *FN*[20] describatur parabole < Prop. 52. lib. 1. >[21] *FM* secans *HG* in *M*; erit igitur in parabola quadratum *MG* aequale rectangulo *GFN* sub abscissa, & latere recto contento < Prop. 11. lib. 1. > ideoque idem quadratum *FG* ad rectangulum *NFG*, atque ad quadratum *MG* eandem proportionem habebit: est vero quadratum *FG* ad rectangulum *NFG*, ut *FG* ad *FN*, cum *FG* sit illorum altitudo communis, nec non ut *CFG* ad *CFN* sumpta nimirum CF communi altitudine, ergo rectangulum *CFG* ad *CFN* eandem proportionem habebit, quàm quadratum *FG* ad quadratum *MG*, & permutando rectangulum *CFG* ad quadratum *FG* erit ut rectangulum *CFN* ad quadratum *GM*, sed ut rectangulum *CFG* ad quadratum *FG*, ita est *CF* ad *FG*, & *EA* ad *AC*, igitur *EA* ad *AC* erit ut rectangulum *CFN* ad quadratum *GM*, seu ut quadratum *EB*, vel *KG* ad quadratum *GM*: est vero *AC* minor, quàm *AE*, quae triens est totius *AB*, igitur *MG* minor est, quàm *GK*. Postea per *B* circa asymptotos *ACF* describatur hyperbole *BK*, quae transibit per punctum *K* < Prop. 4. & 12. lib. 2. >, cum parallelogramma *AF*, & *CK* aequalia sint propter diagonalem *CEG*, quare punctum *M* paraboles cadet intra hyperbolem *BK*, sed parabole *FM* occurrit asymptoto *CF* in vertice *F*, & occurrit etiam asymptoto *CA* in aliquo alio puncto < Prop. 26. lib. 1. >, cum *CA* sit parallela axi *FG* paraboles, & hyperbole semper intra asymptotos incedat < ex 1. & 2. lib. 2 >, igitur parabola *FM* bis hyperbole occurrit supra, & infra punctum *M*: sint occursus *X*, a quibus ductis parallellis ad asymptotos compleantur parallelogramma *RP*, & *AF*, quae erunt aequalia inter asymptotos, & hyperbolen constituta < Prop. 12. lib. 2. >, & propterea *COS* parallelogrammorum diameter erit, & una linea recta: & quia *OA* ad *AC* est ut *CF*

[19] By the rectangle *ACE*, Borelli means the rectangle contained by sides equal to *AC* and *CE*, that is in modern terms, $AC \cdot CE$.

[20] The printed text has *PN*.

[21] The references to the *Conics* of Apollonius, which Borelli put in the margin, appear in the text for typographical reasons.

ad *FS*, sive ut rectangulum *CFN* ad rectangulum *SFN*: erat autem quadratum *EB* aequale rectangulo *CFN* ex constructione, & quadratum [p. 413] *OB*, sive *XS* in parabola aequale est rectangulo *SFN*, ergo *AO* ad *AC* est ut quadratum *EB* ad quadratum *OB*, & propterea parallelepipedum, cuius basis quadratum *OB*, altitudo *OA* aequale erit parallelepipedo base quadrato *EB*, altitudine *AC* contento, quod erat propositum.

Note: Borelli adds some notes on the tangent *XI* in the figure, which do not concern us here.

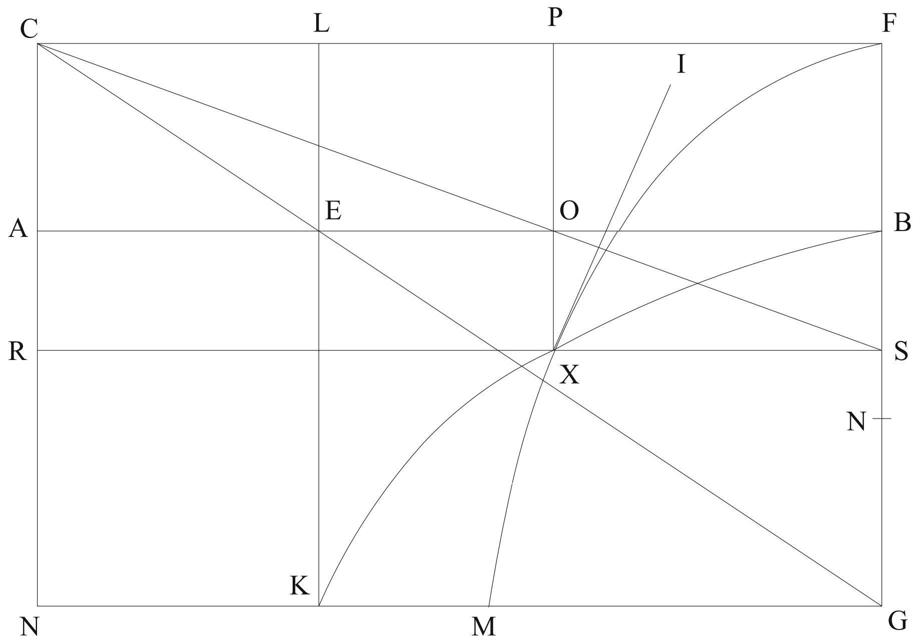

Figure 4: Borelli/Ecchellensis Proposition 17, part 1.

Translation:

Proposition 17

Let *AB* be three times *AE* but greater than three times another [segment] *AC*, then *AB* has to be divided on both sides of *E* at *O* in such a way that the parallelepiped whose base is the square of *OB* and whose altitude is *AO* is equal to the parallelepiped whose base is the square of *EB* and whose altitude is *AC*.

Let the rectangle *ACBF* be made, and let the sides *CA*, *FB* be extended, and let the rectangle *CFN* be made equal to the square of *EB*. After the diameter *CEG* has been drawn, let the right-angled parallelograms *AL*, *AK*, *LB*, *BK* be completed. With axis *FG* and latus rectum *FN*[22] let the parabola *FM* be described < Prop. 52 of book 1 >,[23] intersecting *HG* at *M*. Then in the parabola, the square of *MG* is equal to the rectangle *GFN* which is contained by the abscissa [*GF*] and by the latus rectum [*FN*] < Prop. 11 of book 1 >. Thus, the same square of *FG* has the same proportion to the rectangle *NFG* and to the square of *MG*. But the square of *FG* is to the rectangle *NFG* as *FG* to *FN*, since *FG* is a common altitude of them, and also as *CFG* to *CFN*, if *CF* is of course taken as common altitude, so the rectangle *CGF* has the same proportion to *CFN* as the square of *FG* to the square of *MG*.

[22] The printed text has *PN*.

[23] Borelli's references are to the *Conics* of Apollonius.

Permutando, the rectangle *CFG* will be to the square of *FG* as the rectangle *CFN* to the square of *GM*. But as the rectangle *CFG* is to the square of *FG*, so is *CF* to *FG*, and *EA* to *AC*. Therefore *EA* will be to *AC* as the rectangle *CFN* to the square of *GM*, or as the square of *EB*, that is *KG*, to the square of *GM*. But *AC* is less than *AE*, which is one-third of the whole *AB*. Therefore *MG* is less than *GK*.

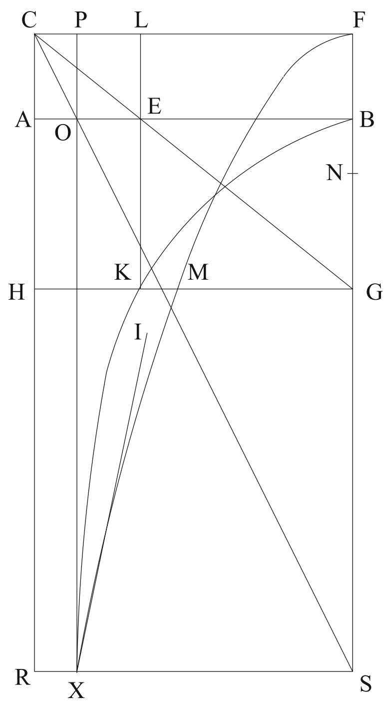

Figure 5: Borelli/Ecchellensis Proposition 17, part 2.

After this, let through *B* with asymptotes *ACF* the hyperbola *BK* be described, which will pass through point *K* < Prop. 4 and 12 of book 2 >, since the parallelograms *AF* and *CK* are equal because of the diagonal *CEG*, so point *M* of the parabola is located inside the hyperbola *BK*. But the parabola *FM* meets the asymptote *CF* in the vertex *F*, and it also meets the asymptote *CA* in some other point < Prop. 26 of book 1 >, since *CA* is parallel to the axis *FG* of the parabola, and the hyperbola is always inside its asymptotes < by Prop. 1 and 2 of book 2 >. Therefore, the parabola *FM* intersects the hyperbola twice, above and under point *M*. Let the intersections be point *X*, from which, after the parallels to the asymptotes have been drawn, parallograms *RP* and *AF* are completed, which [parallelograms] will be equal since they are constituted between the hyperbola and its asymptotes < Prop. 12 of book 2 >. Therefore *COS* will be the diameter of the [two] parallelograms, and it is one straight line. And since *OA* is to *AC* as *CF* to *FS*, that is, as the rectangle *CFN* to the rectangle *SFN*, and the square of *EB* was [assumed to be] equal to the rectangle *CFN* by construction, and the square of *OB*, that is *XS* in the parabola, is equal to the rectangle *SFN*, therefore *AO* is to *AC* as the square of *EB* to the square of *OB*. Therefore the parallelepiped whose base is the square of *OB* and whose altitude is *OA* will be equal to the parallelepiped contained by the square of *EB* as base and by the altitude *AC*, which is what was proposed.

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